HAESE MATHEMATICS

Mathematics

Core Topics HL 1

for use with
Mathematics: Analysis and Approaches HL
Mathematics: Applications and Interpretation HL

Michael Haese
Mark Humphries
Chris Sangwin
Ngoc Vo

for use with

IB Diploma Programme

WORKED SOLUTIONS

Bradley Steventon
Joseph Small
Charlotte Frost
Michael Mampusti

MATHEMATICS: CORE TOPICS HL WORKED SOLUTIONS

Bradley Steventon B.Ma.Sc.
Charlotte Frost B.Sc.
Joseph Small B.Ma.Sc.
Michael Mampusti B.Ma.Adv.(Hons.)

Haese Mathematics
152 Richmond Road, Marleston, SA 5033, AUSTRALIA
Telephone: +61 8 8210 4666, Fax: +61 8 8354 1238
Email: info@haesemathematics.com
Web: www.haesemathematics.com

National Library of Australia Card Number & ISBN 978-1-925489-85-9

Published by Haese Mathematics.
152 Richmond Road, Marleston, SA 5033, AUSTRALIA

First Edition 2019

Artwork by Brian Houston and Charlotte Frost.

Typeset in Australia by Charlotte Frost and Deanne Gallasch. Typeset in Times Roman 10.

This book is available on Snowflake only.

Acknowledgements: While every attempt has been made to trace and acknowledge copyright, the authors and publishers apologise for any accidental infringement where copyright has proved untraceable. They would be pleased to come to a suitable agreement with the rightful owner.

FOREWORD

This book gives you fully worked solutions for every question in Exercises, Review Sets, Activities, and Investigations (which do not involve student experimentation) in each chapter of our textbook *Mathematics: Core Topics HL*.

Correct answers can sometimes be obtained by different methods. In this book, where applicable, each worked solution is modelled on the worked example in the textbook.

Be aware of the limitations of calculators and computer modelling packages. Understand that when your calculator gives an answer that is different from the answer you find in the book, you have not necessarily made a mistake, but the book may not be wrong either.

We have a list of errata for our books on our website. Please contact us if you notice any errors in this book.

BS CF JS MM

e-mail: info@haesemathematics.com
web: www.haesemathematics.com

TABLE OF CONTENTS

Chapter 1

STRAIGHT LINES

EXERCISE 1A

1 **a** $y = 3x + 7$ has gradient $m = 3$ and y-intercept $c = 7$.

b $y = -2x - 5$ has gradient $m = -2$ and y-intercept $c = -5$.

c $y = \frac{2}{3}x - \frac{1}{3}$ has gradient $m = \frac{2}{3}$ and y-intercept $c = -\frac{1}{3}$.

d $y = \dfrac{7x+2}{9} = \frac{7}{9}x + \frac{2}{9}$ has gradient $m = \frac{7}{9}$ and y-intercept $c = \frac{2}{9}$.

e $y = \dfrac{2x-3}{6} = \frac{1}{3}x - \frac{1}{2}$ has gradient $m = \frac{1}{3}$ and y-intercept $c = -\frac{1}{2}$.

f $y = \dfrac{3-5x}{8} = \frac{3}{8} - \frac{5}{8}x$ has gradient $m = -\frac{5}{8}$ and y-intercept $c = \frac{3}{8}$.

2 **a** The equation of the line is

$$\begin{aligned} y - 1 &= 3(x-4) \\ \therefore\ y - 1 &= 3x - 12 \\ \therefore\ y &= 3x - 11 \end{aligned}$$

b The equation of the line is

$$\begin{aligned} y - 5 &= -2(x - (-3)) \\ \therefore\ y - 5 &= -2(x+3) \\ \therefore\ y - 5 &= -2x - 6 \\ \therefore\ y &= -2x - 1 \end{aligned}$$

c The equation of the line is

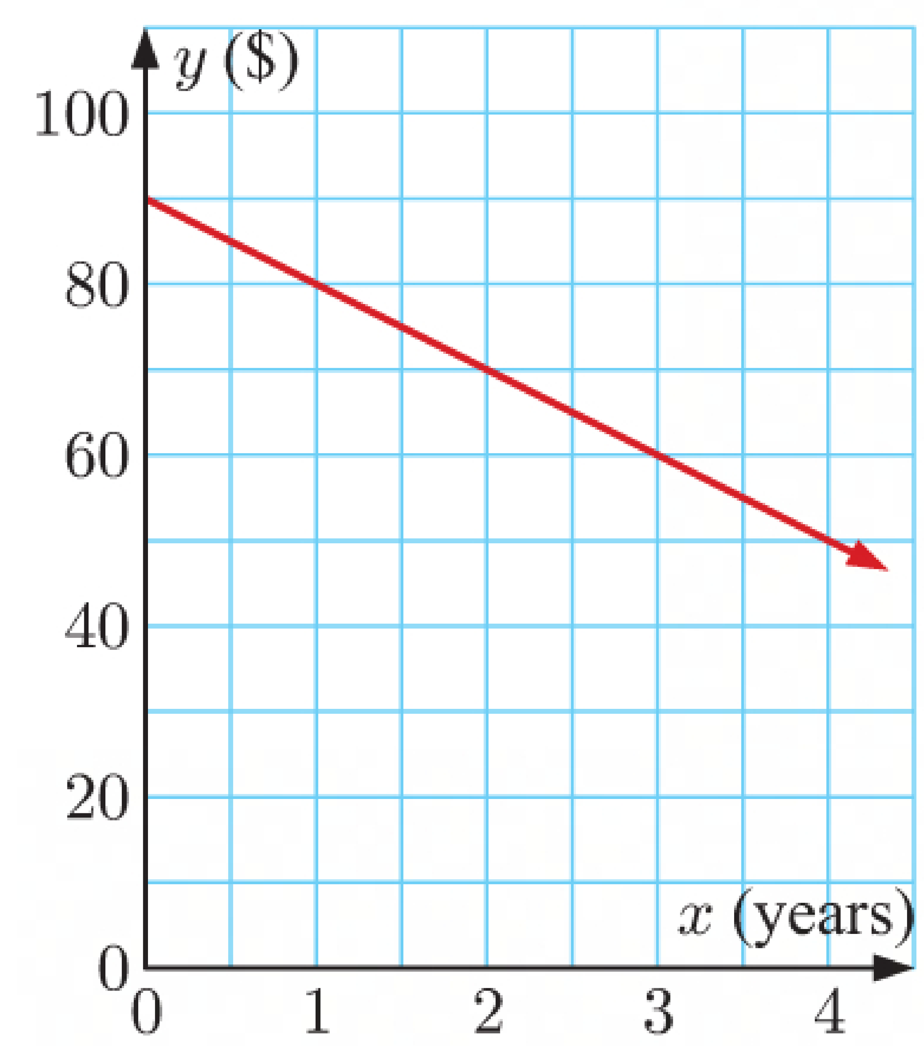
$$\begin{aligned} y - (-3) &= \tfrac{1}{4}(x-4) \\ \therefore\ y + 3 &= \tfrac{1}{4}x - 1 \\ \therefore\ y &= \tfrac{1}{4}x - 4 \end{aligned}$$

d The equation of the line is $y = -\frac{3}{4}x + 4$.

3 **a** The line passes through $(0, 90)$ and $(1, 80)$, so the gradient is $\dfrac{80-90}{1-0} = -10$.

This means that the balance in the account decreases by \$10 each year.

The y-intercept is 90. This means that the initial balance was \$90.



b The gradient is -10 and the y-intercept is 90, so the equation is $y = -10x + 90$.

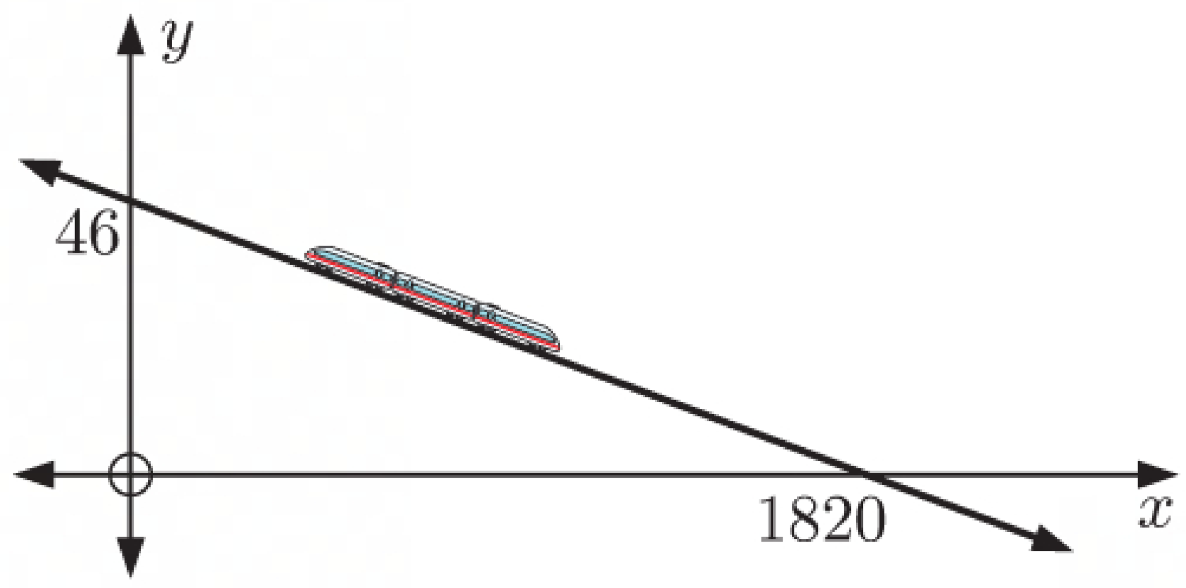
c The account runs out of money when $y = 0$

$$\begin{aligned} \therefore\ -10x + 90 &= 0 \\ \therefore\ 10x &= 90 \\ \therefore\ x &= 9 \end{aligned}$$

The account will run out of money after 9 years.

4 **a** The line passes through $(0, 46)$ and $(1820, 0)$,

so the gradient is $\dfrac{0-46}{1820-0} = -\dfrac{23}{910}$.



b The gradient is $-\frac{23}{910}$ and the y-intercept is 46, so the equation is $y = -\frac{23}{910}x + 46$.

5 **a** When $t = 0$, $H = 150 + 120(0)$

$= 150$

The helicopter took off from a height of 150 m.

b The height of the helicopter above sea level increases by 120 m each minute after taking off.

c When $t = 2$, $H = 150 + 120(2)$

$= 390$

The helicopter is 390 m above sea level after 2 minutes.

d When the helicopter is 650 m above sea level, $H = 650$

$\therefore\ 150 + 120t = 650$

$\therefore\ 120t = 500$

$\therefore\ t = \dfrac{500}{120} = 4\frac{1}{6}$

The helicopter is 650 m above sea level after $4\frac{1}{6}$ minutes, or 4 minutes 10 seconds.

6 **a** $y = -4x + 6$

$\therefore\ 4x + y = 6$ {adding $4x$ to both sides}

b $y = 5x - 3$

$\therefore\ -5x + y = -3$ {subtracting $5x$ from both sides}

$\therefore\ 5x - y = 3$ {multiplying both sides by -1}

c $y = -\frac{3}{4}x + \frac{5}{4}$

$\therefore\ 4y = -3x + 5$ {multiplying both sides by 4}

$\therefore\ 3x + 4y = 5$ {adding $3x$ to both sides}

d $y = \frac{3}{5}x - \frac{1}{5}$

$\therefore\ 5y = 3x - 1$ {multiplying both sides by 5}

$\therefore\ -3x + 5y = -1$ {subtracting $3x$ from both sides}

$\therefore\ 3x - 5y = 1$ {multiplying both sides by -1}

7 **a** $5x + y = 2$

$\therefore\ y = -5x + 2$ {subtracting $5x$ from both sides}

b $3x + 7y = -2$

$\therefore\ 7y = -3x - 2$ {subtracting $3x$ from both sides}

$\therefore\ y = -\frac{3}{7}x - \frac{2}{7}$ {dividing both sides by 7}

c $2x - y = 6$

$\therefore\ -y = -2x + 6$ {subtracting $2x$ from both sides}

$\therefore\ y = 2x - 6$ {multiplying both sides by -1}

d $3x - 13y = -4$

$\therefore\ -13y = -3x - 4$ {subtracting $3x$ from both sides}

$\therefore\ y = \frac{3}{13}x + \frac{4}{13}$ {dividing both sides by -13}

8 $ax + by = d$

$\therefore\ by = -ax + d$

$\therefore\ y = -\dfrac{a}{b}x + \dfrac{d}{b}$ which has the form $y = mx + c$

The gradient of the line is $-\dfrac{a}{b}$.

9 **a** Since the line has gradient -4, the general form of its equation is $4x + y = d$

Using the point $(1, 2)$, the equation is $4x + y = 4(1) + 2$

which is $4x + y = 6$.

b Since the line has gradient $\frac{1}{2}$, the general form of its equation is $x - 2y = d$

Using the point $(3, -5)$, the equation is $x - 2y = 3 - 2(-5)$

which is $x - 2y = 13$.

c Since the line has gradient $-\frac{5}{3}$, the general form of its equation is $5x + 3y = d$

Using the point $(-2, 6)$, the equation is $5x + 3y = 5(-2) + 3(6)$

which is $5x + 3y = 8$.

d Since the line has gradient $\frac{7}{6}$, the general form of its equation is $7x - 6y = d$

Using the point $(-1, -4)$, the equation is $7x - 6y = 7(-1) - 6(-4)$

which is $7x - 6y = 17$.

10 **a** The line has gradient $\dfrac{11 - 1}{3 - (-2)} = \dfrac{10}{5} = 2$, and passes through the point $\mathrm{A}(-2, 1)$.

$\therefore$ the equation of the line is $y - 1 = 2(x - (-2))$

$\therefore\ y - 1 = 2x + 4$

$\therefore\ y = 2x + 5$

b The line has gradient $\dfrac{5 - 2}{4 - 7} = \dfrac{3}{-3} = -1$, and passes through the point $\mathrm{A}(7, 2)$.

$\therefore$ the equation of the line is $y - 2 = -(x - 7)$

$\therefore\ y - 2 = -x + 7$

$\therefore\ y = -x + 9$

c The line has gradient $\dfrac{2 - (-5)}{3 - (-2)} = \dfrac{7}{5}$, and passes through the point $\mathrm{M}(-2, -5)$.

$\therefore$ the equation of the line is $y - (-5) = \frac{7}{5}(x - (-2))$

$\therefore\ y + 5 = \frac{7}{5}(x + 2)$

$\therefore\ y + 5 = \frac{7}{5}x + \frac{14}{5}$

$\therefore\ y = \frac{7}{5}x - \frac{11}{5}$

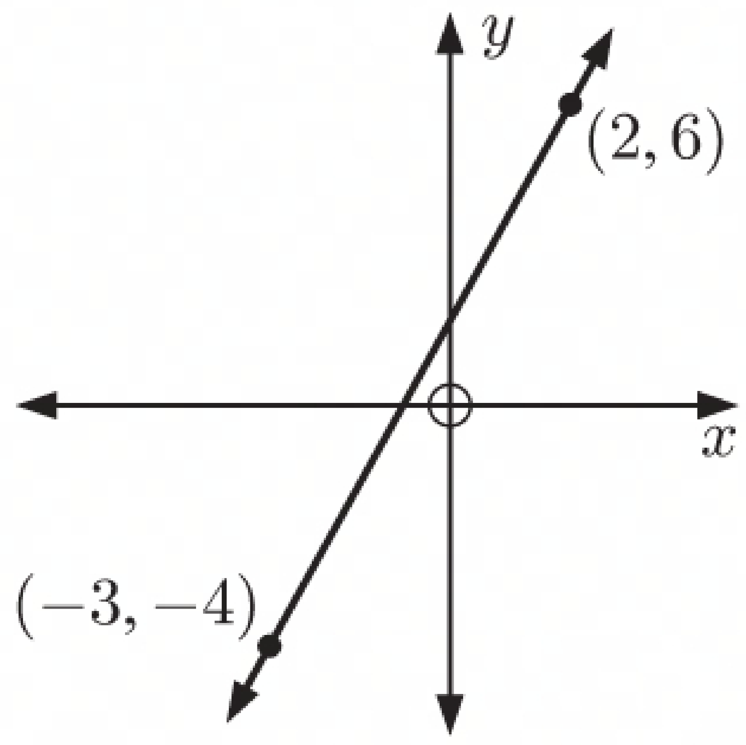
d The line has gradient $\dfrac{9-(-1)}{-7-5} = \dfrac{10}{-12} = -\dfrac{5}{6}$, and passes through the point R$(5, -1)$.

$\therefore$ the equation of the line is $y-(-1) = -\frac{5}{6}(x-5)$

$$\therefore\ y+1 = -\tfrac{5}{6}x + \tfrac{25}{6}$$

$$\therefore\ y = -\tfrac{5}{6}x + \tfrac{19}{6}$$

11 **a** The line has gradient $\dfrac{6-(-4)}{2-(-3)} = \dfrac{10}{5} = 2.$

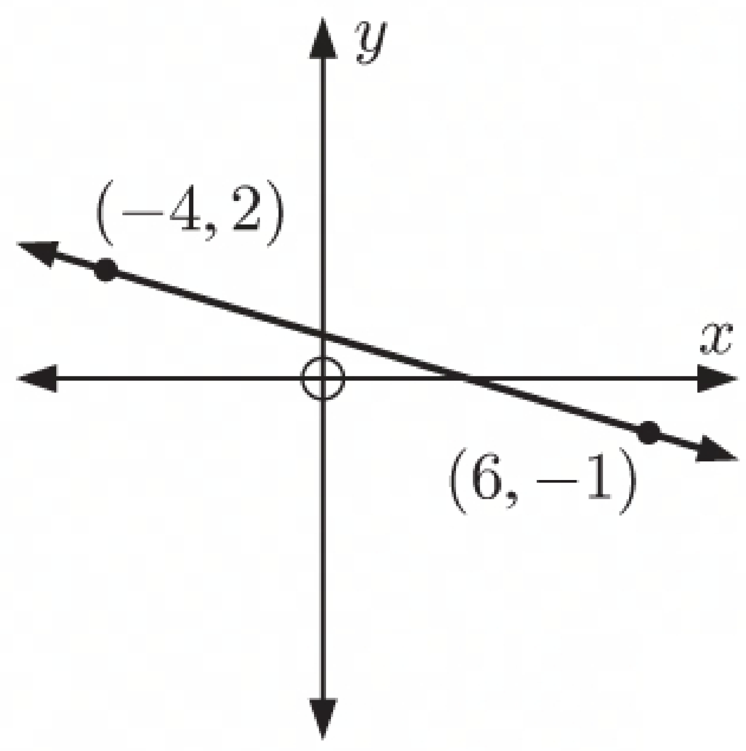


Since the line has gradient 2, the general form of its equation is $2x - y = d$.

Using the point $(-3, -4)$, $2x - y = 2(-3) - (-4)$

$$\therefore\ 2x - y = -2$$

b The line has gradient $\dfrac{-1-2}{6-(-4)} = -\dfrac{3}{10}.$

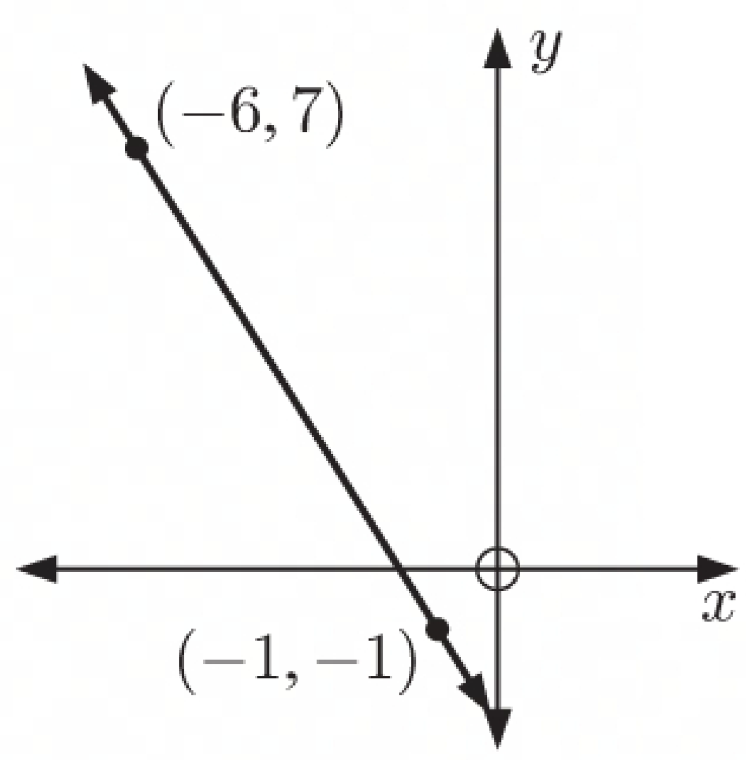


Since the line has gradient $-\frac{3}{10}$, the general form of its equation is $3x + 10y = d$.

Using the point $(-4, 2)$, $3x + 10y = 3(-4) + 10(2)$

$$\therefore\ 3x + 10y = 8$$

c The line has gradient $\dfrac{-1-7}{-1-(-6)} = -\dfrac{8}{5}.$

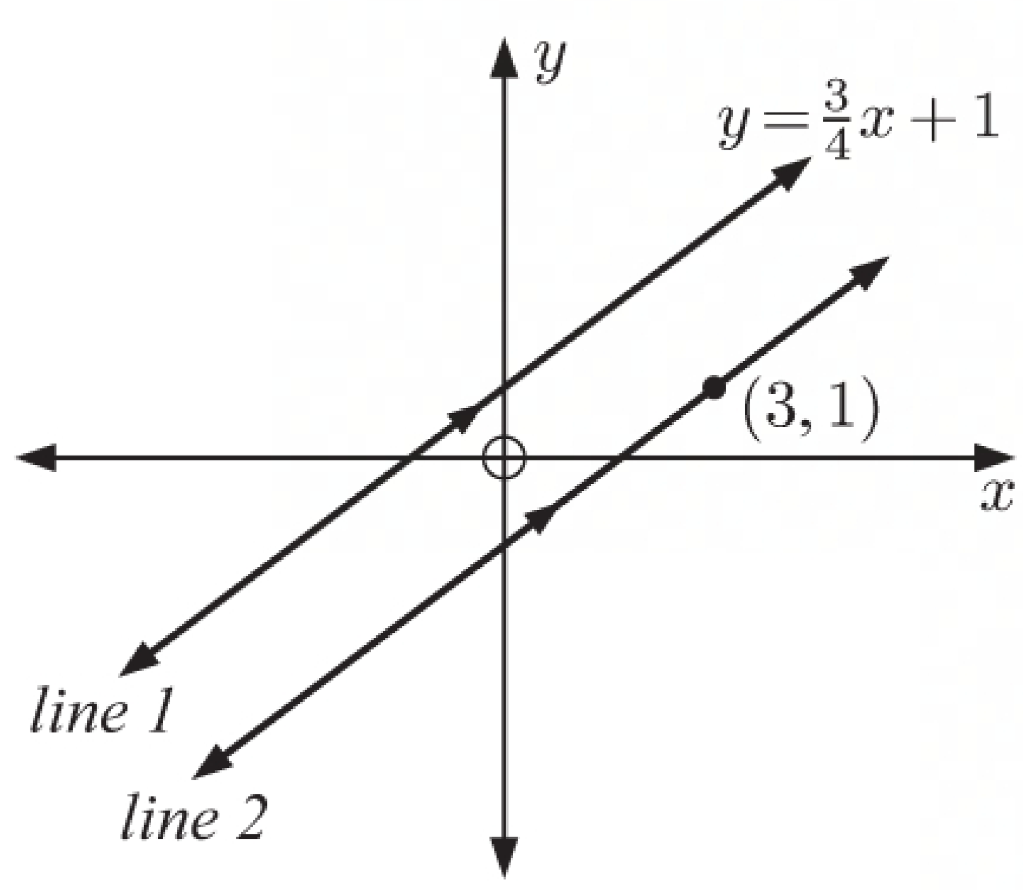


Since the line has gradient $-\frac{8}{5}$, the general form of its equation is $8x + 5y = d$.

Using the point $(-6, 7)$, $8x + 5y = 8(-6) + 5(7)$

$$\therefore\ 8x + 5y = -13$$

12 **a** *Line 2* is parallel to $y = \frac{3}{4}x + 1$, which has gradient $\frac{3}{4}$.



$\therefore$ *line 2* has gradient $\frac{3}{4}$ and passes through $(3, 1)$.

$\therefore$ *line 2* has equation $y - 1 = \frac{3}{4}(x-3)$

$$\therefore\ y - 1 = \tfrac{3}{4}x - \tfrac{9}{4}$$

$$\therefore\ y = \tfrac{3}{4}x - \tfrac{5}{4}$$

b The y-intercept of *line 2* is $-\frac{5}{4}$.

13 **a** The line is parallel to $y = 3x - 2$, which has gradient 3.

$\therefore$ the line has gradient 3 and passes through $(1, 4)$.

$\therefore$ the equation of the line is $y - 4 = 3(x - 1)$

$$\therefore\ y - 4 = 3x - 3$$
$$\therefore\ y = 3x + 1$$

b The line is parallel to $2x - y = -3$, which has gradient $-\dfrac{2}{-1} = 2$.

$\therefore$ the line has gradient 2 and passes through $(3, -1)$.

$\therefore$ the equation of the line is $y - (-1) = 2(x - 3)$

$$\therefore\ y + 1 = 2x - 6$$
$$\therefore\ 2x - y = 7$$

c The line is perpendicular to $y = -2x + 1$, which has gradient -2.

$\therefore$ the line has gradient $\frac{1}{2}$ and passes through $(-1, 5)$.

$\therefore$ the equation of the line is $y - 5 = \frac{1}{2}(x - (-1))$

$$\therefore\ y - 5 = \tfrac{1}{2}(x + 1)$$
$$\therefore\ y - 5 = \tfrac{1}{2}x + \tfrac{1}{2}$$
$$\therefore\ y = \tfrac{1}{2}x + \tfrac{11}{2}$$

d The line is perpendicular to $x + 2y = 6$, which has gradient $-\frac{1}{2}$.

$\therefore$ the line has gradient 2 and passes through $(-2, -1)$.

$\therefore$ the equation of the line is $y - (-1) = 2(x - (-2))$

$$\therefore\ y + 1 = 2(x + 2)$$
$$\therefore\ y + 1 = 2x + 4$$
$$\therefore\ 2x - y = -3$$

14 *Line 1* has gradient $\dfrac{3 - (-1)}{4 - (-2)} = \dfrac{4}{6} = \dfrac{2}{3}$, and passes through the point $B(4, 3)$.

$\therefore$ the equation of *line 1* is $y - 3 = \frac{2}{3}(x - 4)$

$$\therefore\ y - 3 = \tfrac{2}{3}x - \tfrac{8}{3}$$
$$\therefore\ y = \tfrac{2}{3}x + \tfrac{1}{3}$$

Line 2 is perpendicular to $y = \frac{2}{3}x + \frac{1}{3}$, which has gradient $\frac{2}{3}$.

$\therefore$ *line 2* has gradient $-\frac{3}{2}$ and passes through $A(-2, -1)$.

$\therefore$ the equation of *line 2* is $y - (-1) = -\frac{3}{2}(x - (-2))$

$$\therefore\ y + 1 = -\tfrac{3}{2}(x + 2)$$
$$\therefore\ y + 1 = -\tfrac{3}{2}x - 3$$
$$\therefore\ y = -\tfrac{3}{2}x - 4$$

15 **a** When $x = 3$, we have

$$y = 4(3) - 1$$
$$= 11 \ \checkmark$$

So, $(3, 11)$ does lie on the line.

b When $x = -6$, we have

$$y = \tfrac{2}{3}(-6) - 6$$
$$= -4 - 6$$
$$= -10 \ \times$$

So, $(-6, -2)$ does *not* lie on the line.

c Substituting $x = -4$ and $y = -8$ into the LHS gives
$$\begin{aligned} 7(-4) - 3(-8) &= -28 + 24 \\ &= -4 \quad \checkmark \end{aligned}$$
So, $(-4, -8)$ does lie on the line.

d Substituting $x = -\frac{1}{2}$ and $y = 2$ into the LHS gives
$$\begin{aligned} 6(-\tfrac{1}{2}) + 10(2) &= -3 + 20 \\ &= 17 \quad \checkmark \end{aligned}$$
So, $(-\frac{1}{2}, 2)$ does lie on the line.

16 **a** Substituting $x = 2$ and $y = 15$ into the equation gives $15 = 4(2) + c$
$$\therefore\ c + 8 = 15$$
$$\therefore\ c = 7$$

b Substituting $x = \frac{1}{2}$ and $y = 3$ into the equation gives $3 = m(\frac{1}{2}) - \frac{5}{2}$
$$\therefore\ \frac{m}{2} - \frac{5}{2} = 3$$
$$\therefore\ m - 5 = 6$$
$$\therefore\ m = 11$$

c Substituting $x = t$ and $y = 4$ into the equation gives $4 = \frac{2}{3}t - \frac{4}{3}$
$$\therefore\ \tfrac{2}{3}t = \tfrac{16}{3}$$
$$\therefore\ t = 8$$

17 **a** Substituting $x = 6$ and $y = -3$ into the equation gives
$$2(6) + 5(-3) = k$$
$$\therefore\ k = 12 - 15$$
$$\therefore\ k = -3$$

b Substituting $x = -8$ and $y = -5$ into the equation gives
$$7(-8) - (-5) = k$$
$$\therefore\ k = -56 + 5$$
$$\therefore\ k = -51$$

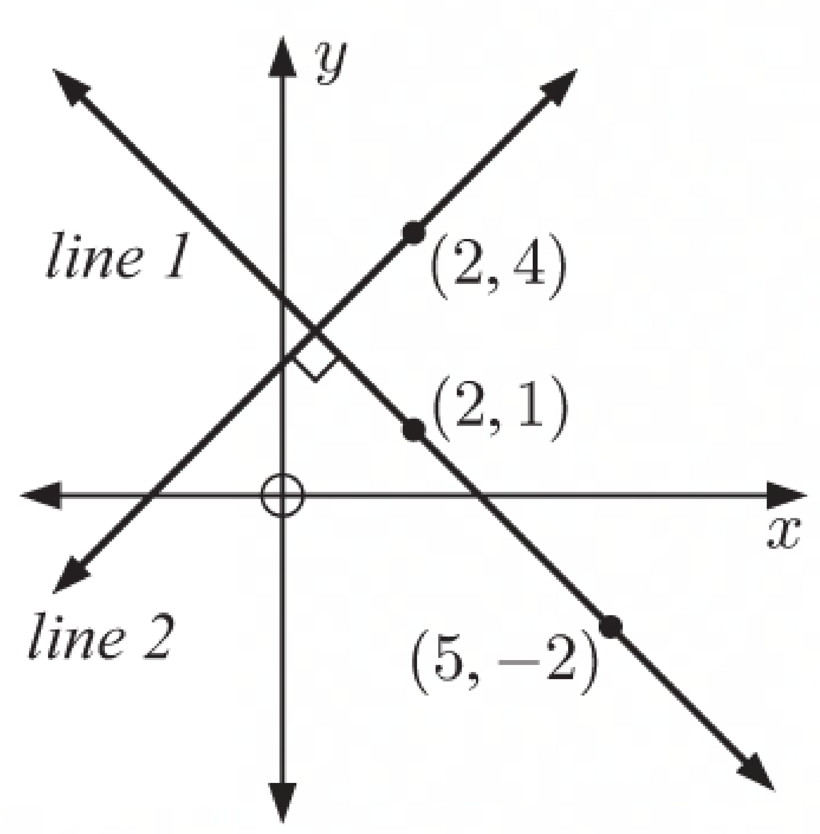
c Substituting $x = k$ and $y = 0$ into the equation gives
$$3k - 4(0) = -36$$
$$\therefore\ 3k = -36$$
$$\therefore\ k = -12$$

18 **a** *Line 1* has gradient $\dfrac{-2-1}{5-2} = \dfrac{-3}{3} = -1$.

Line 2 is perpendicular to *line 1*, so its gradient is 1.

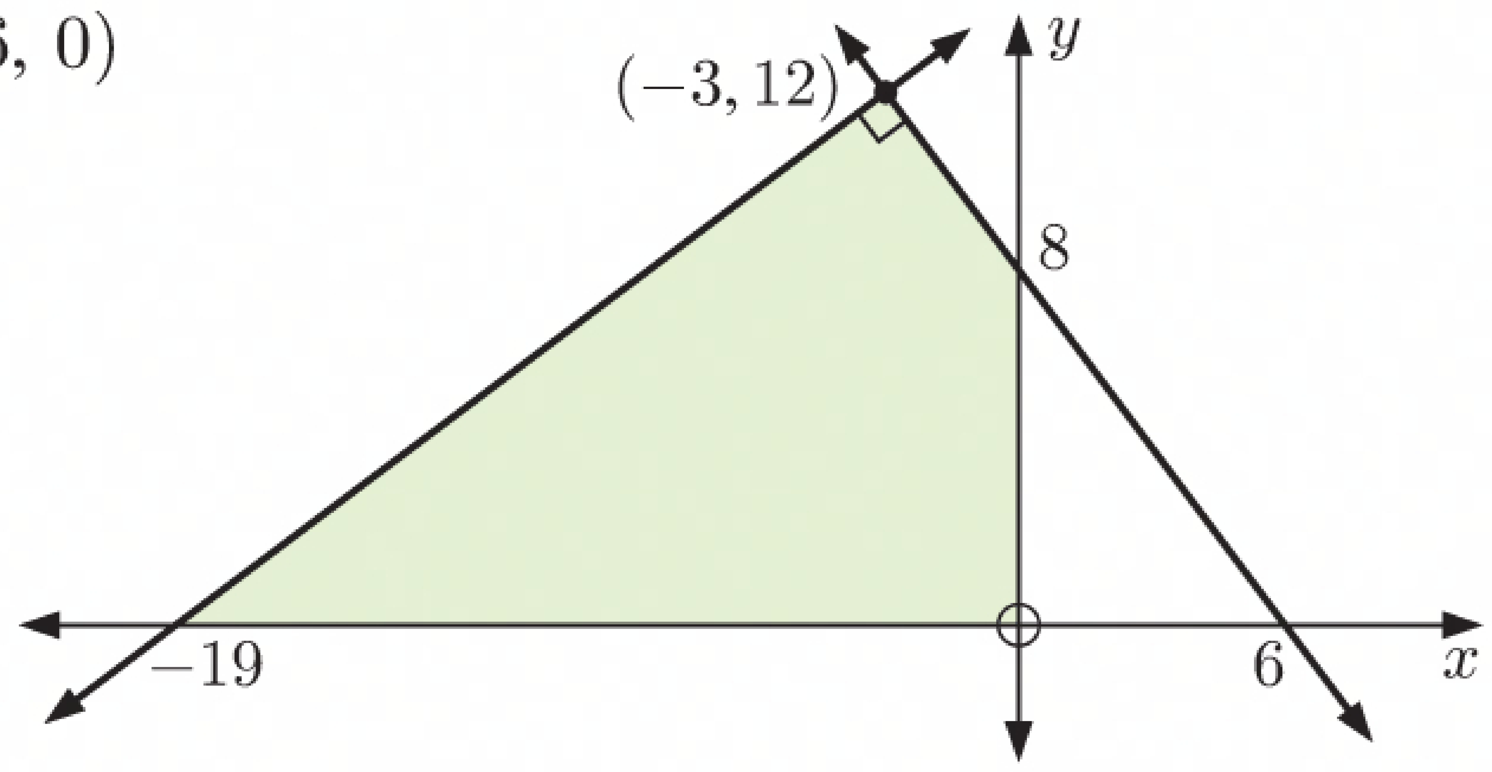
$\therefore$ *line 2* has gradient 1 and passes through $(2, 4)$.

$\therefore$ *line 2* has equation $y - 4 = x - 2$
$$\therefore\ x - y + 2 = 0$$



b When $y = 0$, $x - 0 + 2 = 0$
$$\therefore\ x = -2$$
$\therefore$ the x-intercept of *line 2* is -2.

19 The line which passes through $(-3, 12)$ and $(6, 0)$ has gradient $\dfrac{0-12}{6-(-3)} = \dfrac{-12}{9} = -\dfrac{4}{3}$.



The equation of this line is

$$y - 0 = -\tfrac{4}{3}(x-6)$$
$$\therefore\ y = -\tfrac{4}{3}x + 8$$

which has y-intercept 8.

The line which is perpendicular to $y = -\frac{4}{3}x + 8$ has gradient $\frac{3}{4}$ and passes through $(-3, 12)$,

so it has equation $y - 12 = \frac{3}{4}(x - (-3))$

$$\therefore\ y - 12 = \tfrac{3}{4}(x+3)$$
$$\therefore\ y - 12 = \tfrac{3}{4}x + \tfrac{9}{4}$$
$$\therefore\ y = \tfrac{3}{4}x + \tfrac{57}{4}$$

This line cuts the x-axis when $y = \frac{3}{4}x + \frac{57}{4} = 0$

$$\therefore\ \tfrac{3}{4}x = -\tfrac{57}{4}$$
$$\therefore\ x = -19$$

So, the triangle formed by the two lines and the x-axis has height 12 units and base length $6 - (-19) = 25$ units.

The y-intercept of the line $y = -\frac{4}{3}x + 8$ is 8. So, the unshaded triangle has height 8 units and base length 6 units.

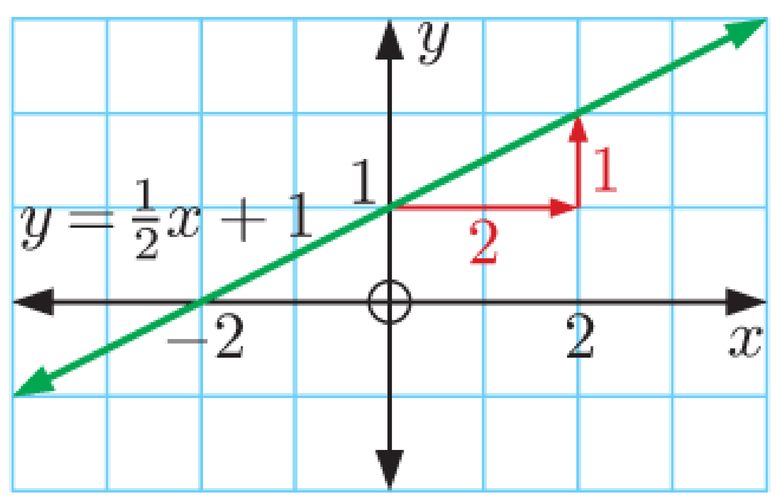
Shaded area = area of large triangle − area of unshaded triangle

$$= \tfrac{1}{2} \times 25 \times 12 - \tfrac{1}{2} \times 6 \times 8$$
$$= 150 - 24$$
$$= 126 \text{ units}^2$$

EXERCISE 1B

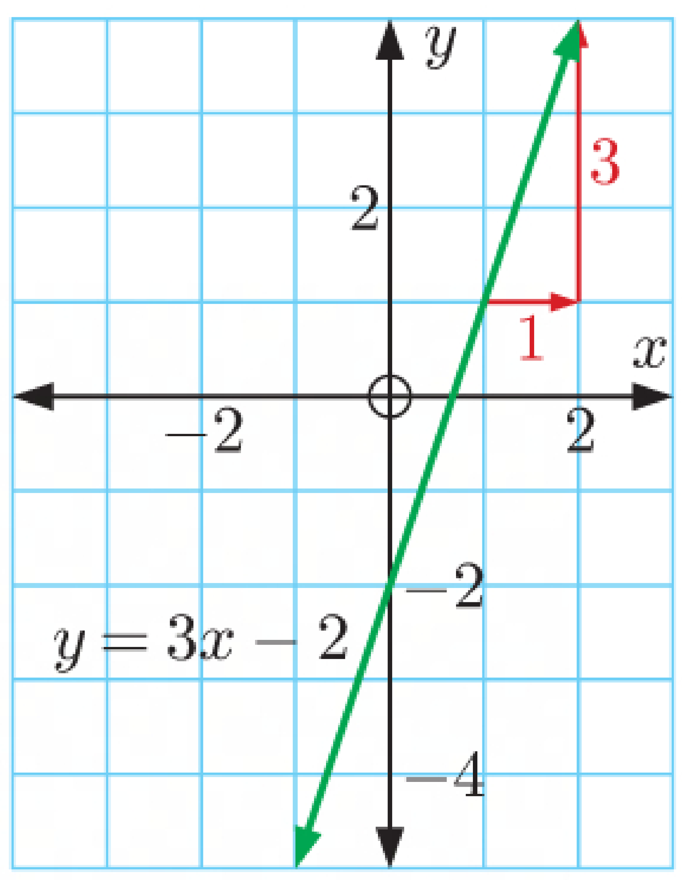
1 **a** For $y = \frac{1}{2}x + 1$:

- the y-intercept is $c = 1$
- the gradient is $m = \frac{1}{2}$



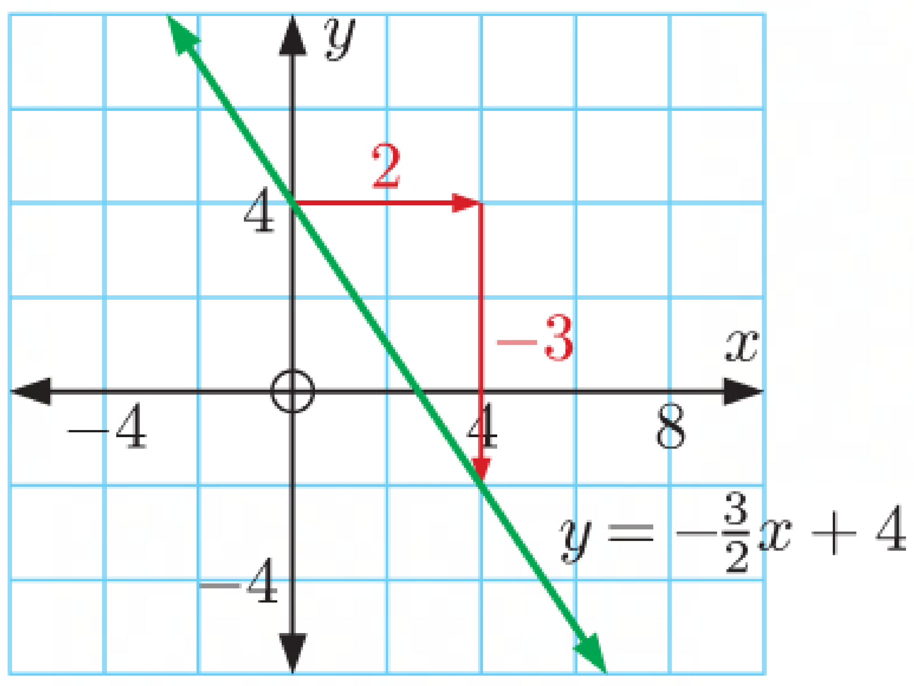
b For $y = 3x - 2$:

- the y-intercept is $c = -2$
- the gradient is $m = 3 = \frac{3}{1}$



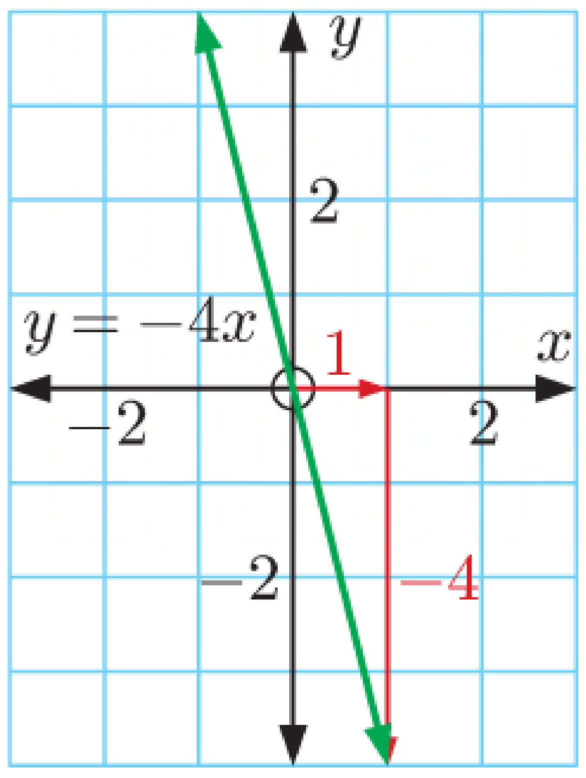
c For $y = -\frac{3}{2}x + 4$:

- the y-intercept is $c = 4$
- the gradient is $m = -\frac{3}{2} = \frac{-3}{2}$



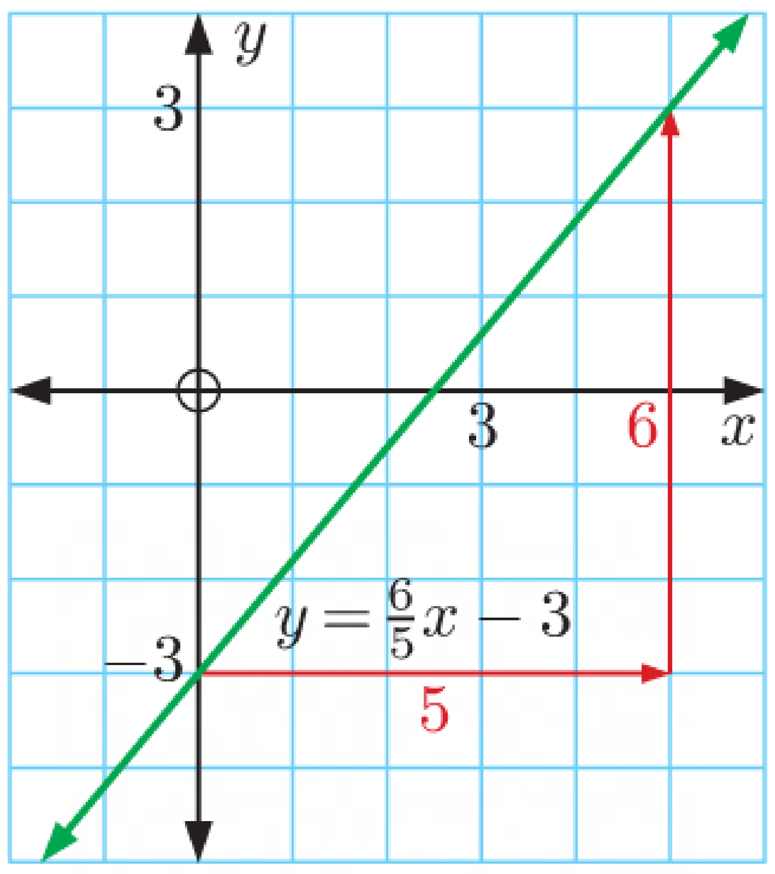
d For $y = -4x$:

- the y-intercept is $c = 0$
- the gradient is $m = -4 = \frac{-4}{1}$



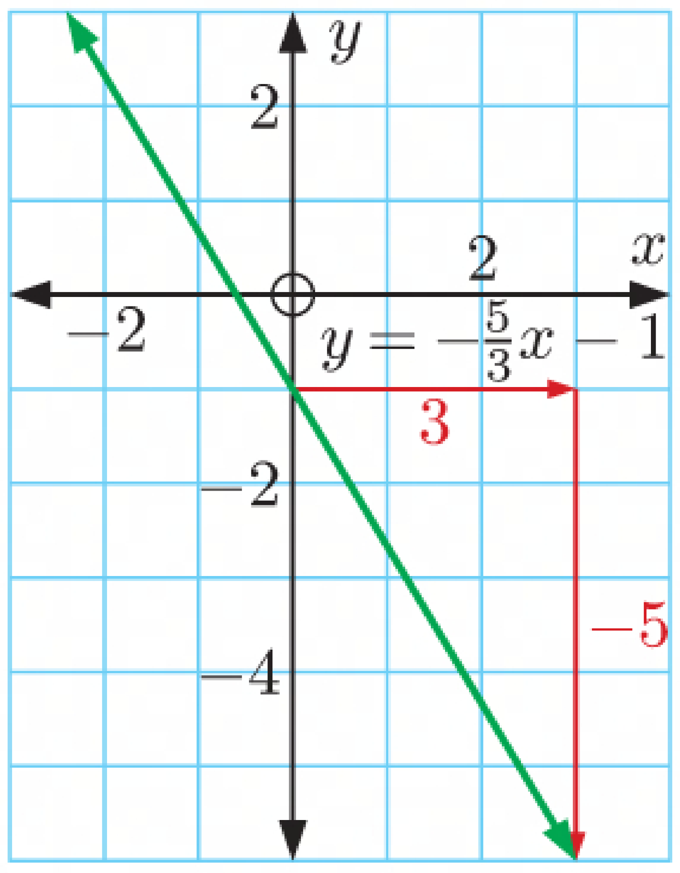
e For $y = \frac{6}{5}x - 3$:

- the y-intercept is $c = -3$
- the gradient is $m = \frac{6}{5}$



f For $y = -\frac{5}{3}x - 1$:

- the y-intercept is $c = -1$
- the gradient is $m = -\frac{5}{3} = \frac{-5}{3}$



2 **a** For $3x + 2y = 12$:

When $x = 0$, $2y = 12$

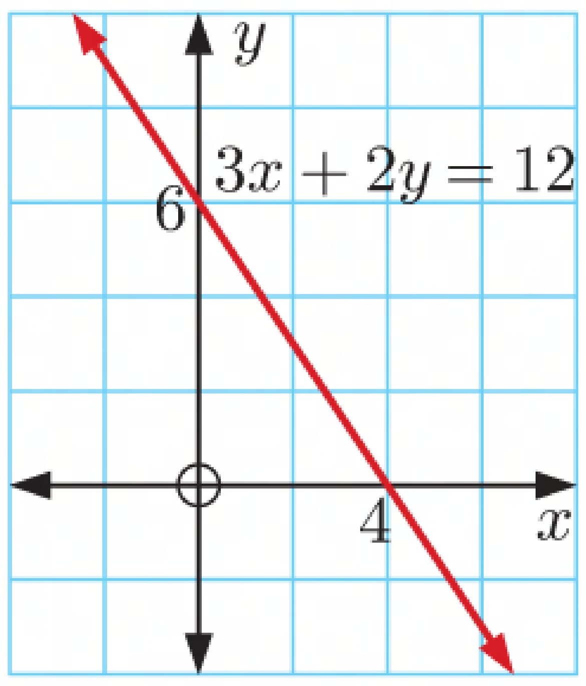
$\therefore y = 6$

So, the y-intercept is 6.

When $y = 0$, $3x = 12$

$\therefore x = 4$

So, the x-intercept is 4.



b For $4x - y = 8$:

When $x = 0$, $-y = 8$

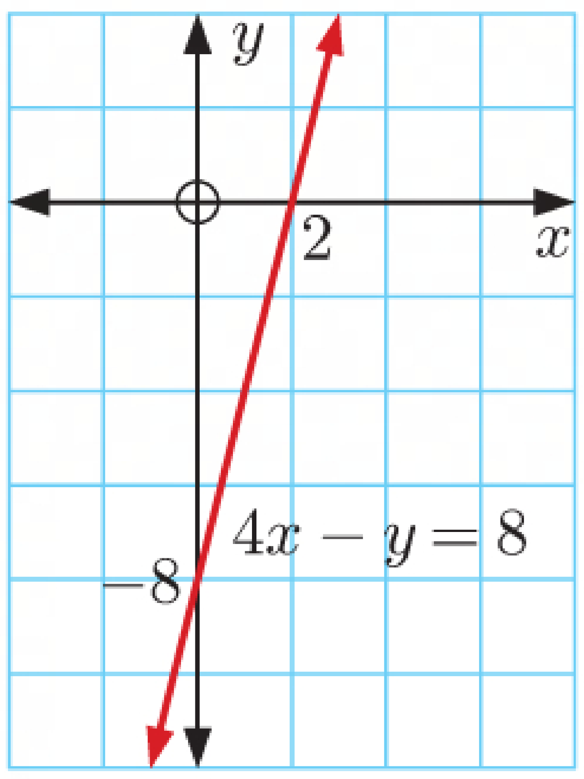
$\therefore\ y = -8$

So, the y-intercept is -8.

When $y = 0$, $4x = 8$

$\therefore\ x = 2$

So, the x-intercept is 2.



c For $3x - 4y = -24$:

When $x = 0$, $-4y = -24$

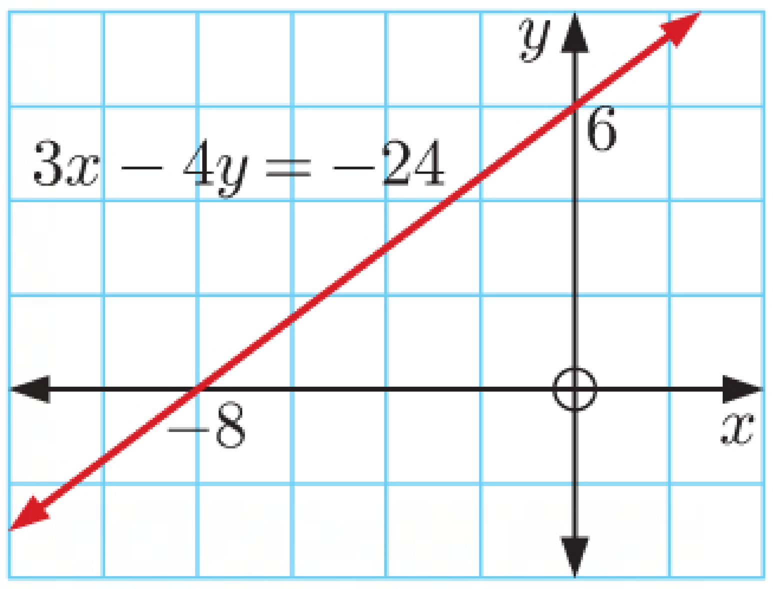
$\therefore\ y = 6$

So, the y-intercept is 6.

When $y = 0$, $3x = -24$

$\therefore\ x = -8$

So, the x-intercept is -8.



d For $2x + 5y = 15$:

When $x = 0$, $5y = 15$

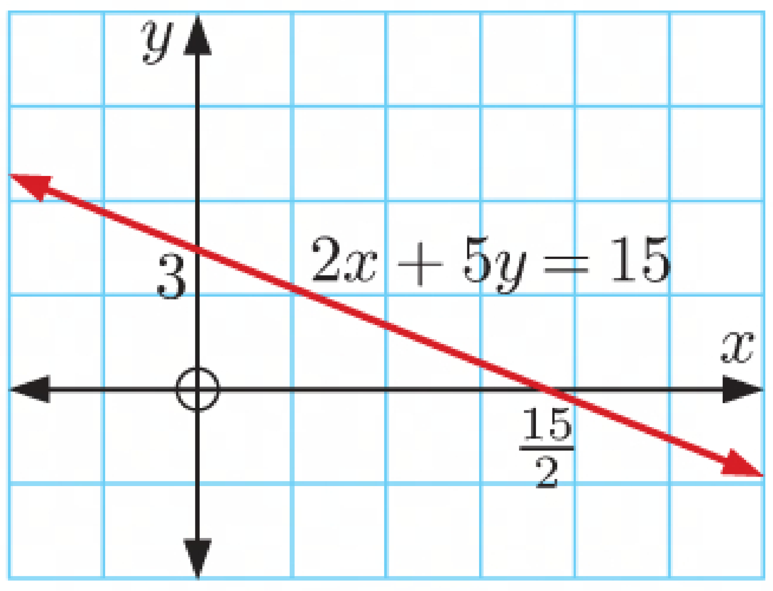
$\therefore\ y = 3$

So, the y-intercept is 3.

When $y = 0$, $2x = 15$

$\therefore\ x = \frac{15}{2}$

So, the x-intercept is $\frac{15}{2}$.



e For $6x + 4y = -36$:

When $x = 0$, $4y = -36$

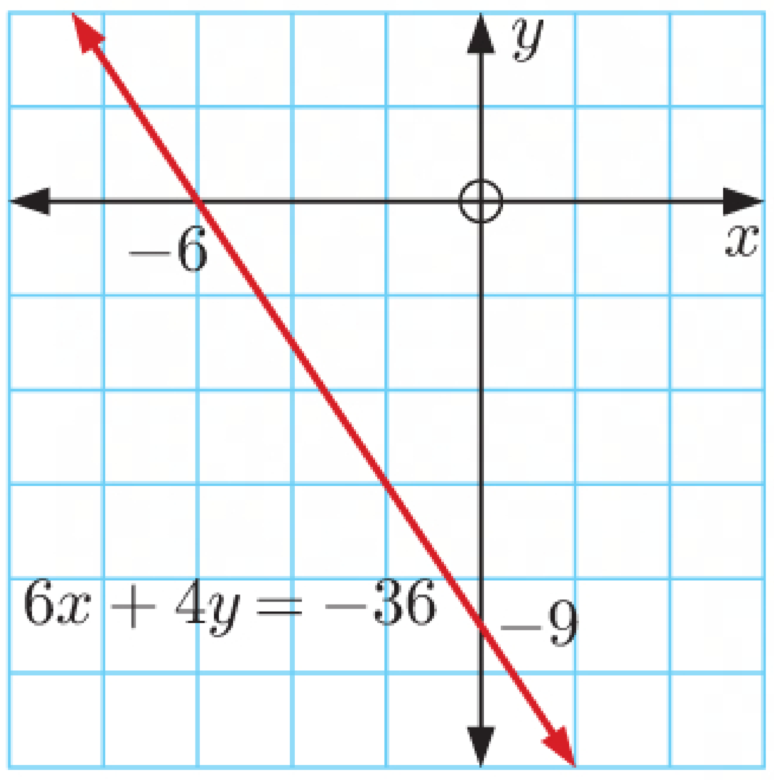
$\therefore\ y = -9$

So, the y-intercept is -9.

When $y = 0$, $6x = -36$

$\therefore\ x = -6$

So, the x-intercept is -6.



f For $7x + 4y = 42$:

When $x = 0$, $4y = 42$

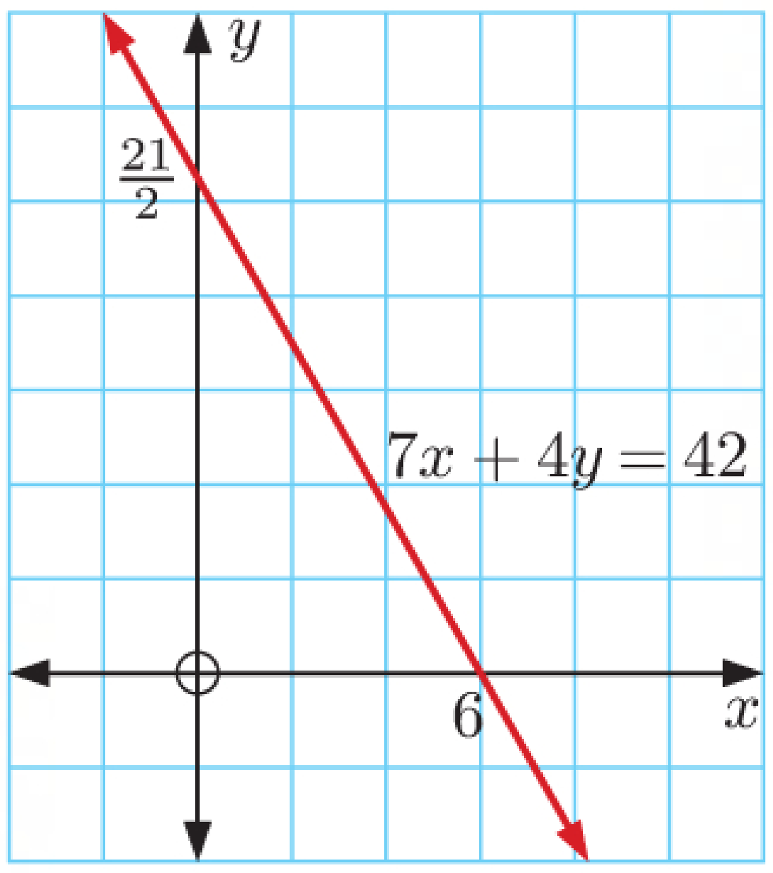
$\therefore\ y = \frac{21}{2}$

So, the y-intercept is $\frac{21}{2}$.

When $y = 0$, $7x = 42$

$\therefore\ x = 6$

So, the x-intercept is 6.



3 **a** $y = -\frac{3}{4}x + 2$ has gradient $m = -\frac{3}{4}$ and y-intercept $c = 2$.

b **i** When $x = 8$, we have

$$\begin{aligned} y &= -\tfrac{3}{4}(8) + 2 \\ &= -6 + 2 \\ &= -4 \quad \checkmark \end{aligned}$$

So, $(8, -4)$ does lie on the line.

ii When $x = 1$, we have

$$\begin{aligned} y &= -\tfrac{3}{4}(1) + 2 \\ &= -\tfrac{3}{4} + 2 \\ &= \tfrac{5}{4} \quad \times \end{aligned}$$

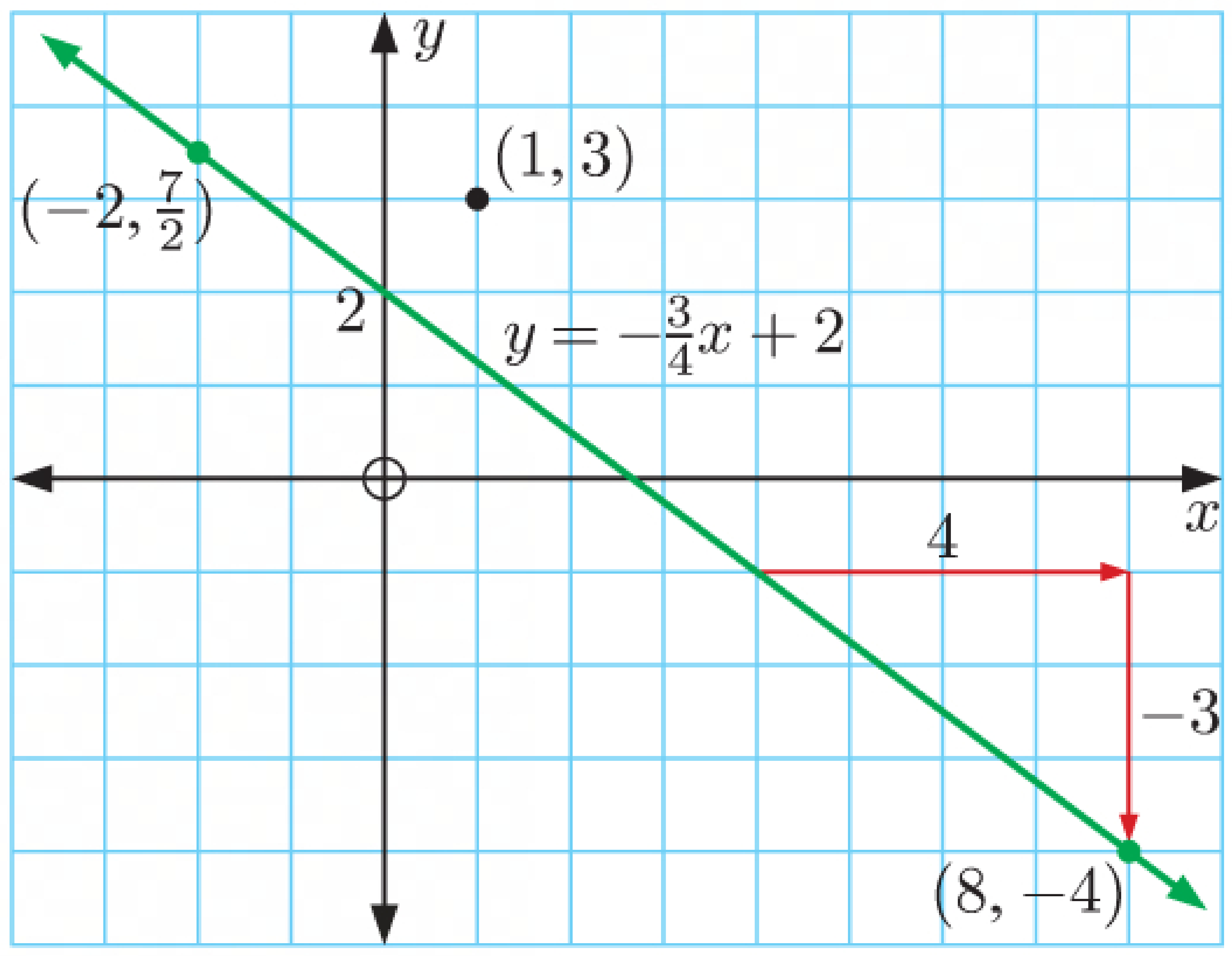
So, $(1, 3)$ does *not* lie on the line.

iii When $x = -2$, we have

$$\begin{aligned} y &= -\tfrac{3}{4}(-2) + 2 \\ &= \tfrac{7}{2} \quad \checkmark \end{aligned}$$

So, $(-2, \frac{7}{2})$ does lie on the line.

c



4 **a** For $2x - 3y = 18$:

When $x = 0$, $-3y = 18$

$\therefore\ y = -6$

So, the y-intercept is -6.

When $y = 0$, $2x = 18$

$\therefore\ x = 9$

So, the x-intercept is 9.

b **i** Substituting $x = 3$ and $y = -4$ into the LHS gives

$$\begin{aligned} &2(3) - 3(-4) \\ &= 6 + 12 \\ &= 18 \quad \checkmark \end{aligned}$$

So, $(3, -4)$ does lie on the line.

ii Substituting $x = 7$ and $y = -2$ into the LHS gives

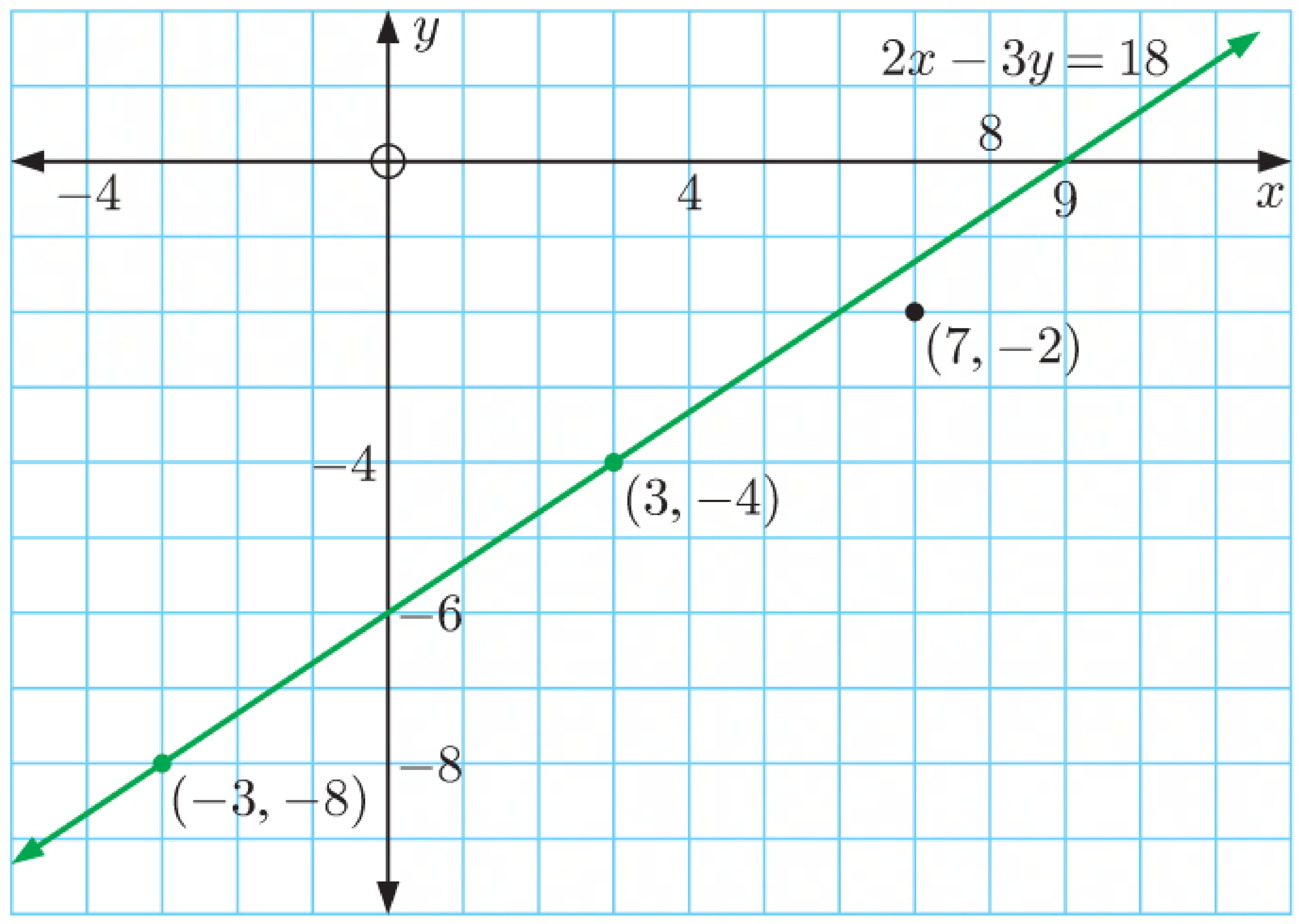
$$\begin{aligned} &2(7) - 3(-2) \\ &= 14 + 6 \\ &= 20 \quad \times \end{aligned}$$

So, $(7, -2)$ does *not* lie on the line.

c If $(-3, c)$ lies on the line then

$$\begin{aligned} 2(-3) - 3c &= 18 \\ \therefore\ -6 - 3c &= 18 \\ \therefore\ -3c &= 24 \\ \therefore\ c &= -8 \end{aligned}$$

d



5 **a** x serves of nigiri at \$4.50 each and y serves of sashimi at \$9 each adds up to a total of \$45.

$\therefore\ 4.5x + 9y = 45$

b When $x = 4$, $4.5(4) + 9y = 45$

$\therefore\ 18 + 9y = 45$

$\therefore\ 9y = 27$

$\therefore\ y = 3$

$\therefore$ Hiroko bought 3 serves of sashimi.

c When $y = 1$, $4.5x + 9(1) = 45$

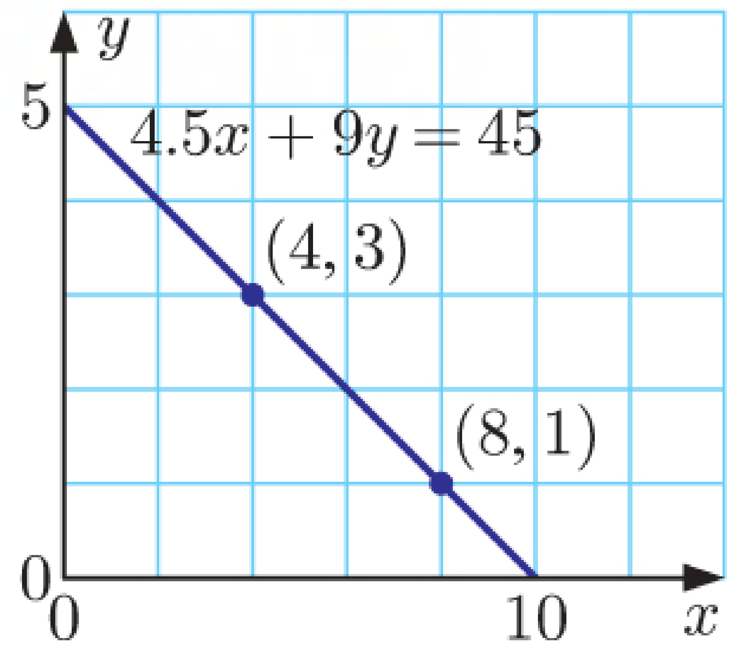
$\therefore\ 4.5x + 9 = 45$

$\therefore\ 4.5x = 36$

$\therefore\ x = 8$

$\therefore$ Hiroko bought 8 serves of nigiri.

d



EXERCISE 1C

1 **a** The midpoint M of [AB] is $\left(\dfrac{3+5}{2}, \dfrac{1+7}{2}\right)$ or $(4, 4)$.

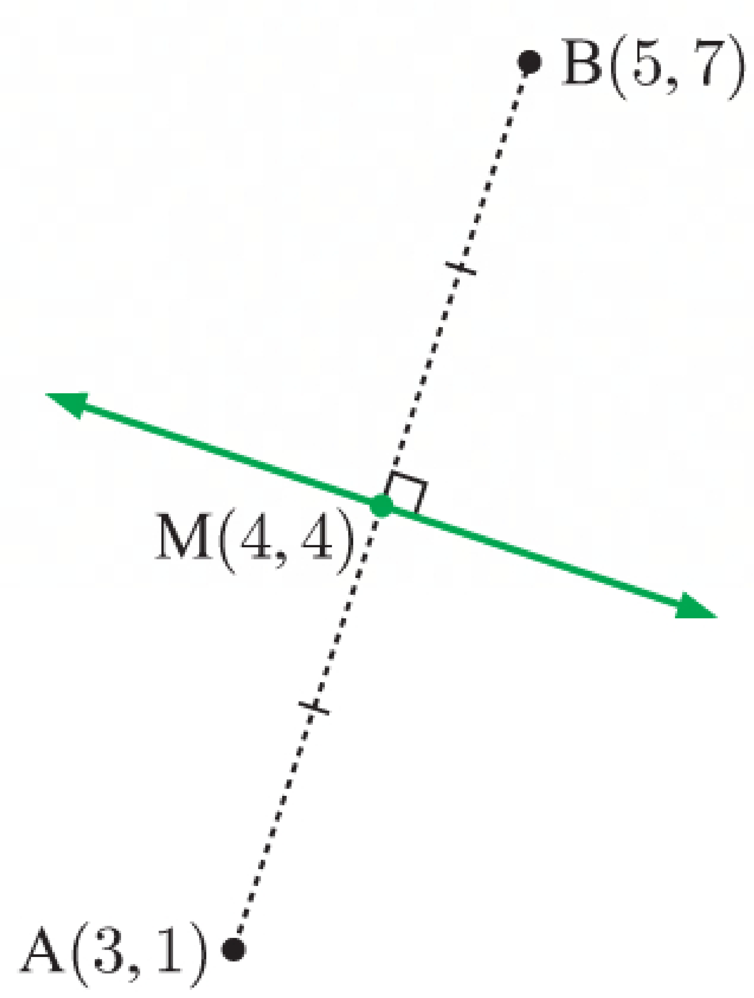
b The gradient of [AB] is $\dfrac{7-1}{5-3} = \dfrac{6}{2} = 3$.

c The gradient of the perpendicular bisector is $-\frac{1}{3}$, the negative reciprocal of the gradient of [AB].

d The perpendicular bisector has gradient $-\frac{1}{3}$ and passes through $(4, 4)$.

$\therefore$ its equation is $x + 3y = 4 + 3(4)$

which is $x + 3y = 16$.



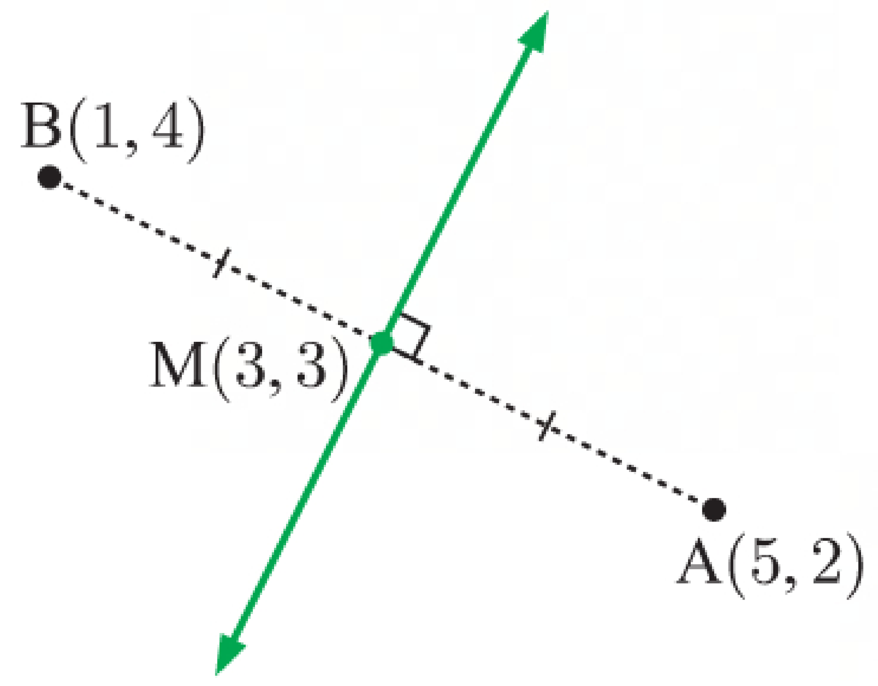
2 **a** The midpoint M of [AB] is $\left(\frac{5+1}{2}, \frac{2+4}{2}\right)$ or $(3, 3)$.

The gradient of [AB] is $\frac{4-2}{1-5} = \frac{2}{-4} = -\frac{1}{2}$

$\therefore$ the gradient of the perpendicular bisector is 2.

$\therefore$ the equation of the perpendicular bisector is $2x - y = 2(3) - 3$

which is $2x - y = 3$.



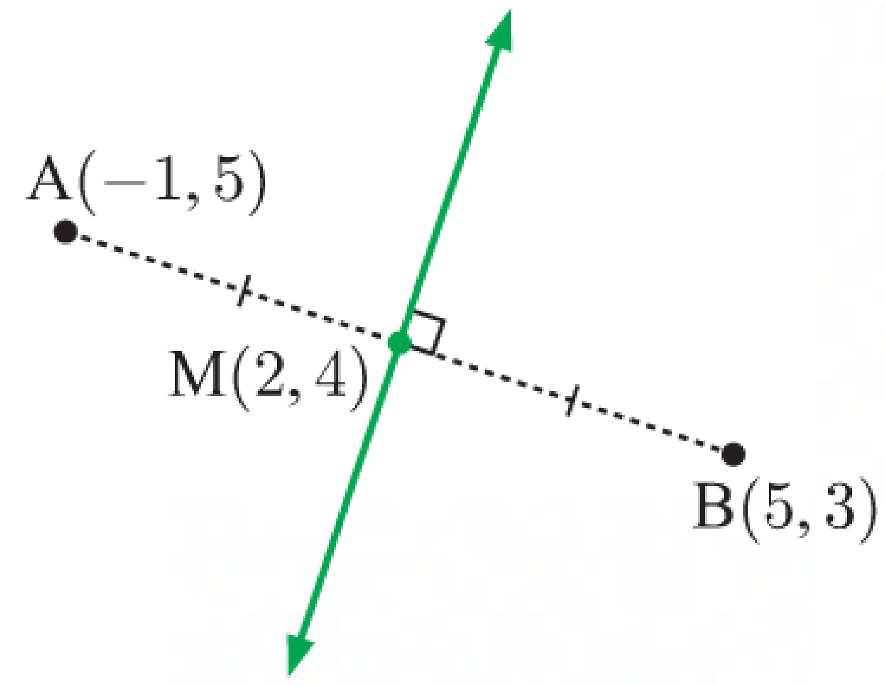
b The midpoint M of [AB] is $\left(\frac{-1+5}{2}, \frac{5+3}{2}\right)$ or $(2, 4)$.

The gradient of [AB] is $\frac{3-5}{5-(-1)} = \frac{-2}{6} = -\frac{1}{3}$

$\therefore$ the gradient of the perpendicular bisector is 3.

$\therefore$ the equation of the perpendicular bisector is $3x - y = 3(2) - 4$

which is $3x - y = 2$.



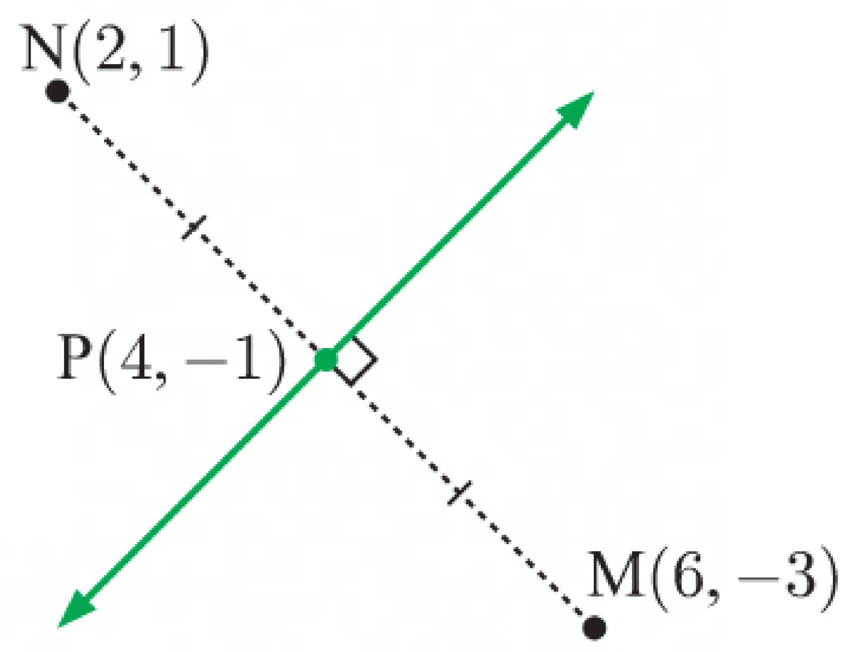
c The midpoint P of [MN] is $\left(\frac{6+2}{2}, \frac{-3+1}{2}\right)$ or $(4, -1)$.

The gradient of [MN] is $\frac{1-(-3)}{2-6} = \frac{4}{-4} = -1$

$\therefore$ the gradient of the perpendicular bisector is 1.

$\therefore$ the equation of the perpendicular bisector is $x - y = 4 - (-1)$

which is $x - y = 5$.



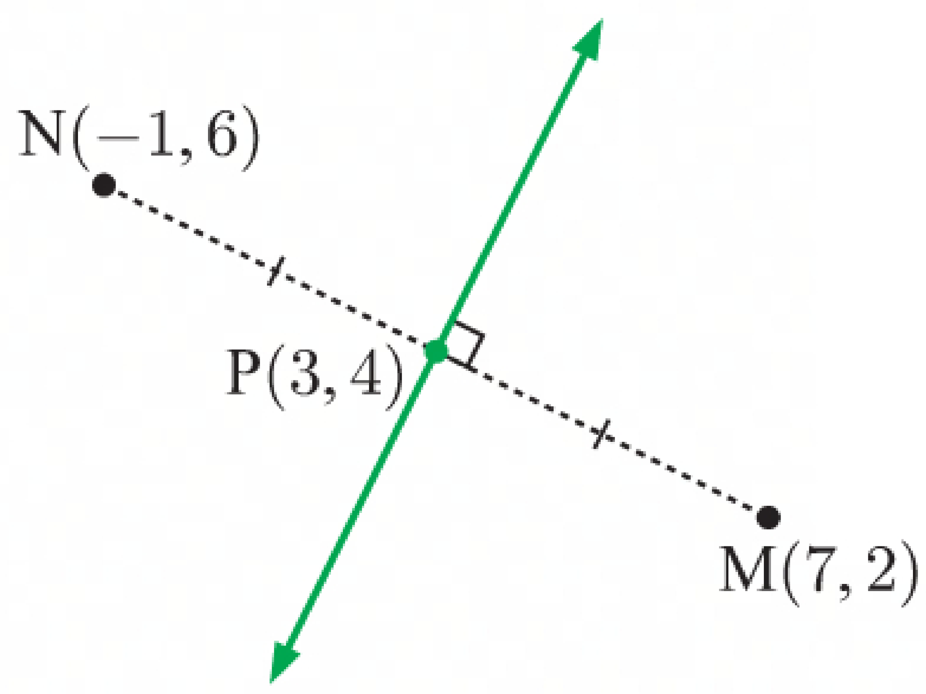
d The midpoint P of [MN] is $\left(\frac{7+(-1)}{2}, \frac{2+6}{2}\right)$ or $(3, 4)$.

The gradient of [MN] is $\frac{6-2}{-1-7} = \frac{4}{-8} = -\frac{1}{2}$

$\therefore$ the gradient of the perpendicular bisector is 2.

$\therefore$ the equation of the perpendicular bisector is $2x - y = 2(3) - 4$

which is $2x - y = 2$.

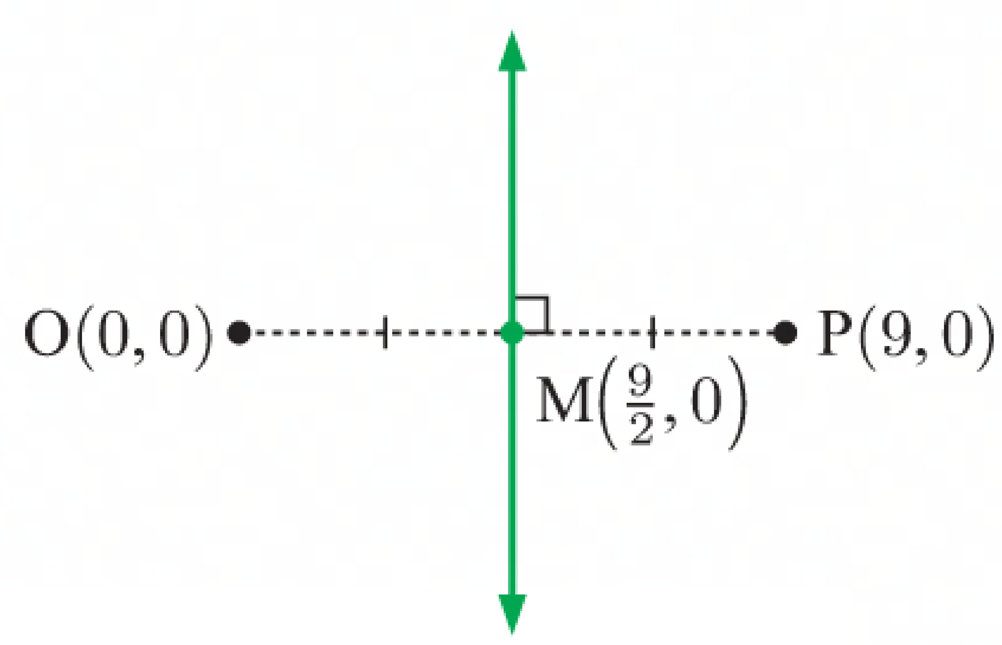


e The midpoint M of [OP] is $\left(\frac{0+9}{2}, \frac{0+0}{2}\right)$ or $\left(\frac{9}{2}, 0\right)$.

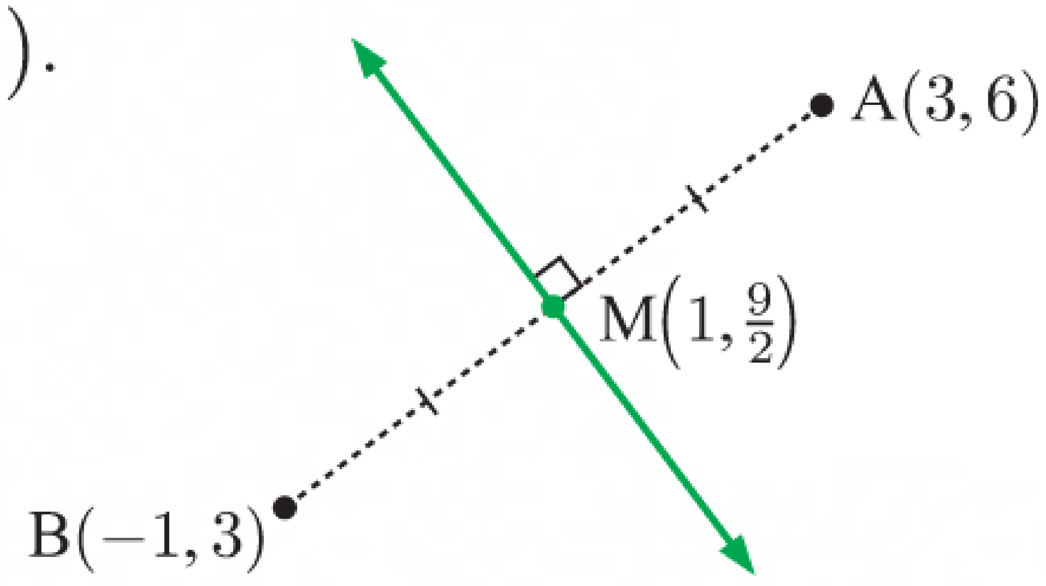
The gradient of [OP] is $\frac{0-0}{9-0} = 0$.

So, [OP] is horizontal, and the perpendicular bisector is the vertical line passing through $\left(\frac{9}{2}, 0\right)$.

$\therefore$ the equation of the perpendicular bisector is $x = \frac{9}{2}$.



f The midpoint M of [AB] is $\left(\frac{3+(-1)}{2}, \frac{6+3}{2}\right)$ or $\left(1, \frac{9}{2}\right)$.



The gradient of [AB] is $\frac{3-6}{-1-3} = \frac{-3}{-4} = \frac{3}{4}$

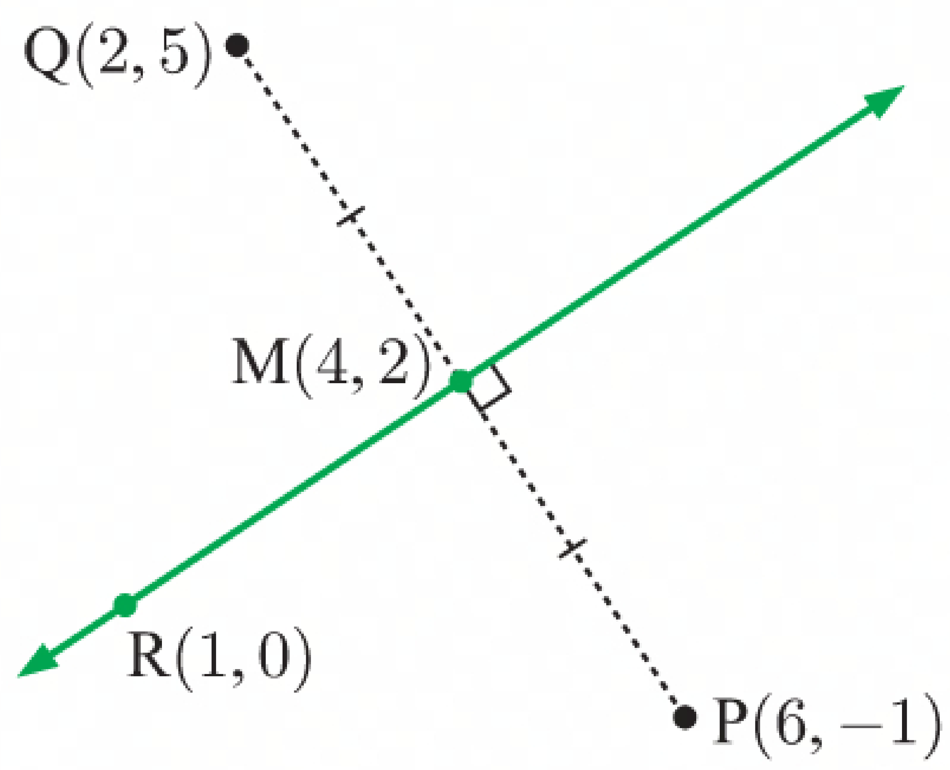
$\therefore$ the gradient of the perpendicular bisector is $-\frac{4}{3}$.

$\therefore$ the equation of the perpendicular bisector is $4x + 3y = 4(1) + 3(\frac{9}{2})$

which is $4x + 3y = \frac{35}{2}$

or $8x + 6y = 35$.

3 **a** The midpoint M of [PQ] is $\left(\frac{6+2}{2}, \frac{-1+5}{2}\right)$ or $(4, 2)$.



The gradient of [PQ] is $\frac{5-(-1)}{2-6} = \frac{6}{-4} = -\frac{3}{2}$

$\therefore$ the gradient of the perpendicular bisector is $\frac{2}{3}$.

$\therefore$ the equation of the perpendicular bisector is $2x - 3y = 2(4) - 3(2)$

which is $2x - 3y = 2$.

b Substituting $x = 1$ and $y = 0$ into the LHS gives $2(1) - 3(0) = 2$ ✓

So, R(1, 0) does lie on the line.

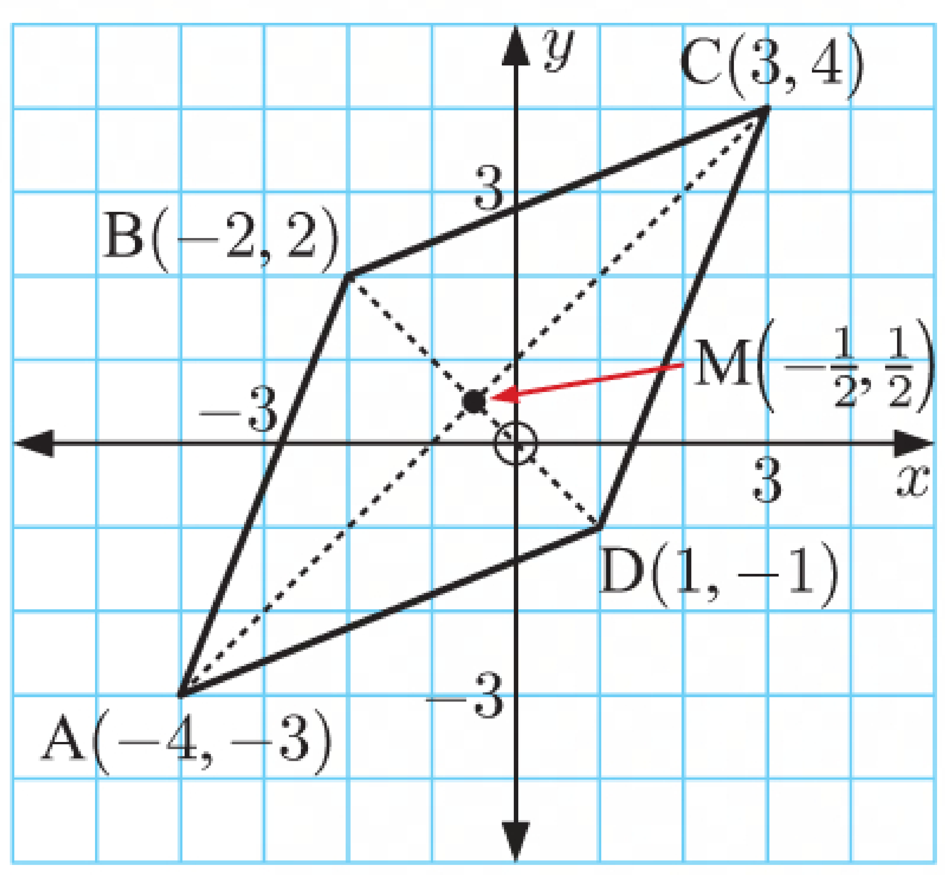
c $\text{PR} = \sqrt{(1-6)^2 + (0-(-1))^2}$
$= \sqrt{(-5)^2 + 1^2}$
$= \sqrt{26}$ units

$\text{QR} = \sqrt{(1-2)^2 + (0-5)^2}$
$= \sqrt{(-1)^2 + (-5)^2}$
$= \sqrt{26}$ units

$\text{PR} = \text{QR} = \sqrt{26}$ units

$\therefore$ R is equidistant from P and Q.

4 **a** $\text{AB} = \sqrt{(-2-(-4))^2 + (2-(-3))^2}$
$= \sqrt{2^2 + 5^2}$
$= \sqrt{29}$ units



$\text{BC} = \sqrt{(3-(-2))^2 + (4-2)^2}$
$= \sqrt{5^2 + 2^2}$
$= \sqrt{29}$ units

$\text{CD} = \sqrt{(1-3)^2 + (-1-4)^2}$
$= \sqrt{(-2)^2 + (-5)^2}$
$= \sqrt{29}$ units

$\text{AD} = \sqrt{(1-(-4))^2 + (-1-(-3))^2}$
$= \sqrt{5^2 + 2^2}$
$= \sqrt{29}$ units

All side lengths are equal, $\therefore$ ABCD is a rhombus.

b The midpoint M of [AC] is $\left(\dfrac{-4+3}{2}, \dfrac{-3+4}{2}\right)$ or $\left(-\frac{1}{2}, \frac{1}{2}\right)$.

The gradient of [AC] is $\dfrac{4-(-3)}{3-(-4)} = 1$

$\therefore$ the gradient of the perpendicular bisector is -1.

$\therefore$ the equation of the perpendicular bisector is $x + y = -\frac{1}{2} + \frac{1}{2}$

which is $x + y = 0$

or $y = -x$.

c B: $2 = -(-2)$ ✓ D: $-1 = -(1)$ ✓

5 **a** **i** $3x - 2y + 1 = 0$

$\therefore\ 2y = 3x + 1$

$\therefore\ y = \frac{3}{2}x + \frac{1}{2}$ has gradient $\frac{3}{2}$

ii The perpendicular bisector has gradient $-\frac{2}{3}$.

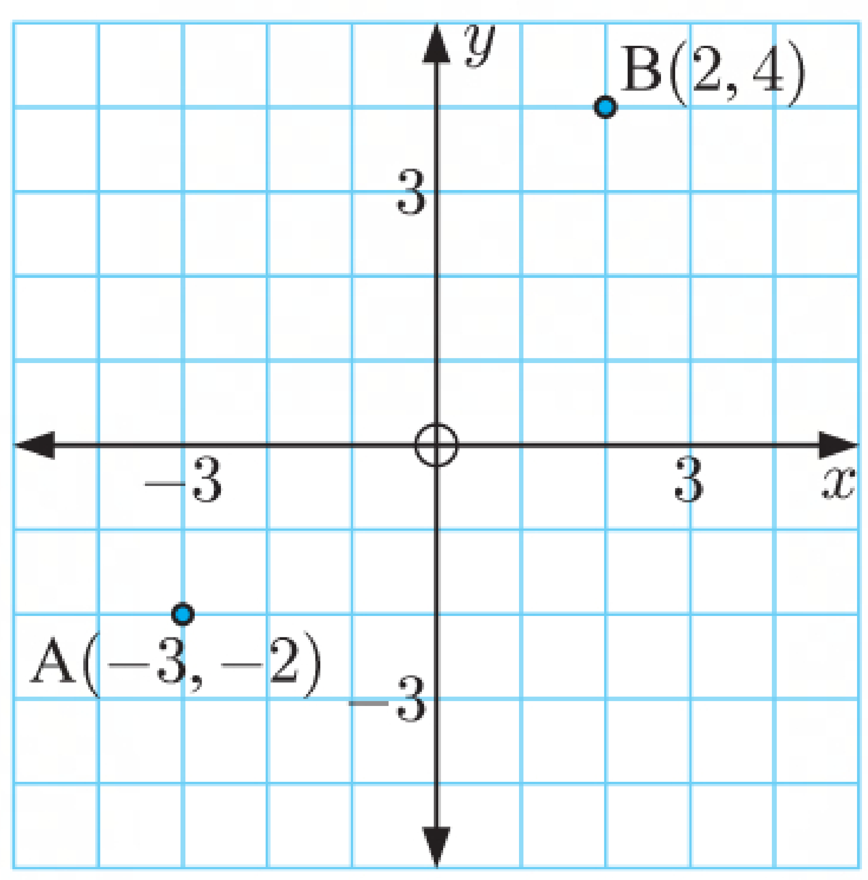
b The equation of the perpendicular bisector is $2x + 3y = 2(3) + 3(5)$

which is $2x + 3y = 21$

or $2x + 3y - 21 = 0$.

6 **a** The midpoint between the hospitals is $\left(\dfrac{-3+2}{2}, \dfrac{-2+4}{2}\right)$

or $\left(-\frac{1}{2}, 1\right)$.



b The gradient of [AB] is $\dfrac{4-(-2)}{2-(-3)} = \dfrac{6}{5}$

$\therefore$ the gradient of the perpendicular bisector is $-\frac{5}{6}$.

$\therefore$ the equation of the perpendicular bisector is

$5x + 6y = 5(-\frac{1}{2}) + 6(1)$

which is $5x + 6y = \frac{7}{2}$

or $10x + 12y = 7$

The perpendicular bisector of the line joining the two hospitals divides the town into two regions. An ambulance crew should be sent from A to locations below this line, and from B to locations above this line.

7 **a** The midpoint of [AB] is $\left(\dfrac{x_1+x_2}{2}, \dfrac{y_1+y_2}{2}\right)$.

The gradient of [AB] is $\dfrac{y_2-y_1}{x_2-x_1}$

$\therefore$ the gradient of the perpendicular bisector is $-\dfrac{x_2-x_1}{y_2-y_1}$.

$\therefore$ the equation of the perpendicular bisector is

$$(x_2-x_1)x + (y_2-y_1)y = (x_2-x_1)\frac{x_1+x_2}{2} + (y_2-y_1)\frac{y_1+y_2}{2}$$

$$(x_2-x_1)x + (y_2-y_1)y = \frac{{x_2}^2-{x_1}^2}{2} + \frac{{y_2}^2-{y_1}^2}{2}$$

$$(x_2-x_1)x + (y_2-y_1)y = \frac{\left({x_2}^2+{y_2}^2\right)-\left({x_1}^2+{y_1}^2\right)}{2}$$

b We can find the perpendicular bisector of any two points $A(x_1, y_1)$ and $B(x_2, y_2)$ by substituting in the values of x_1, x_2, y_1, and y_2.

8 **a** **i** The midpoint of [AB] is $\left(\frac{1+4}{2}, \frac{2+5}{2}\right)$ or $\left(\frac{5}{2}, \frac{7}{2}\right)$.

The gradient of [AB] is $\frac{5-2}{4-1} = 1$

$\therefore$ the gradient of the perpendicular bisector is -1.

$\therefore$ the equation of the perpendicular bisector is $x + y = \frac{5}{2} + \frac{7}{2}$

which is $x + y = 6$.

ii The midpoint of [AC] is $\left(\frac{1+2}{2}, \frac{2+(-1)}{2}\right)$ or $\left(\frac{3}{2}, \frac{1}{2}\right)$.

The gradient of [AC] is $\frac{-1-2}{2-1} = -3$

$\therefore$ the gradient of the perpendicular bisector is $\frac{1}{3}$.

$\therefore$ the equation of the perpendicular bisector is $x - 3y = \frac{3}{2} - 3(\frac{1}{2})$

which is $x - 3y = 0$.

iii The midpoint of [BC] is $\left(\frac{4+2}{2}, \frac{5+(-1)}{2}\right)$ or $(3, 2)$.

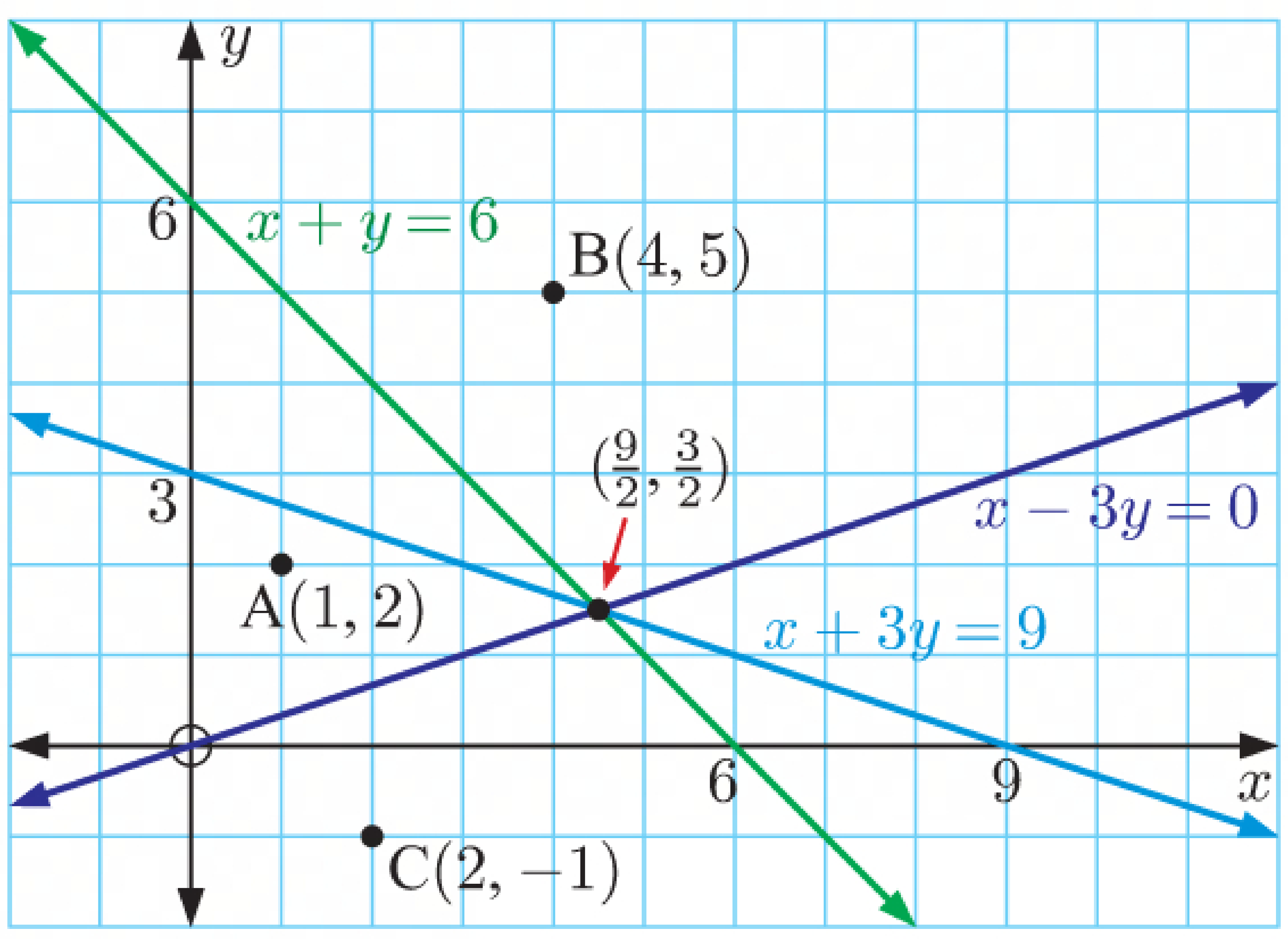
The gradient of [BC] is $\frac{-1-5}{2-4} = 3$

$\therefore$ the gradient of the perpendicular bisector is $-\frac{1}{3}$.

$\therefore$ the equation of the perpendicular bisector is $x + 3y = 3 + 3(2)$

which is $x + 3y = 9$.

b



The perpendicular bisectors all intersect at $\left(\frac{9}{2}, \frac{3}{2}\right)$.

A, B, and C are all equidistant from this point.

c The perpendicular bisectors of each pair of points will meet at a single point. As the three points are equidistant from the point of intersection, a circle centred at the point of intersection that passes through one of them will pass through all of them.

9 **a** **i** The midpoint of [PQ] is $\left(\frac{-8+1}{2}, \frac{-6+5}{2}\right)$ or $\left(-\frac{7}{2}, -\frac{1}{2}\right)$.

The gradient of [PQ] is $\frac{5-(-6)}{1-(-8)} = \frac{11}{9}$

$\therefore$ the gradient of the perpendicular bisector is $-\frac{9}{11}$.

$\therefore$ the equation of the perpendicular bisector is $9x + 11y = 9\left(-\frac{7}{2}\right) + 11\left(-\frac{1}{2}\right)$

which is $9x + 11y = -37$.

ii The midpoint of [PR] is $\left(\dfrac{-8+4}{2}, \dfrac{-6+(-2)}{2}\right)$ or $(-2, -4)$.

The gradient of [PR] is $\dfrac{-2-(-6)}{4-(-8)} = \dfrac{1}{3}$

$\therefore$ the gradient of the perpendicular bisector is -3.

$\therefore$ the equation of the perpendicular bisector is $3x + y = 3(-2) + (-4)$

which is $3x + y = -10$.

iii The midpoint of [QR] is $\left(\dfrac{1+4}{2}, \dfrac{5+(-2)}{2}\right)$ or $\left(\frac{5}{2}, \frac{3}{2}\right)$.

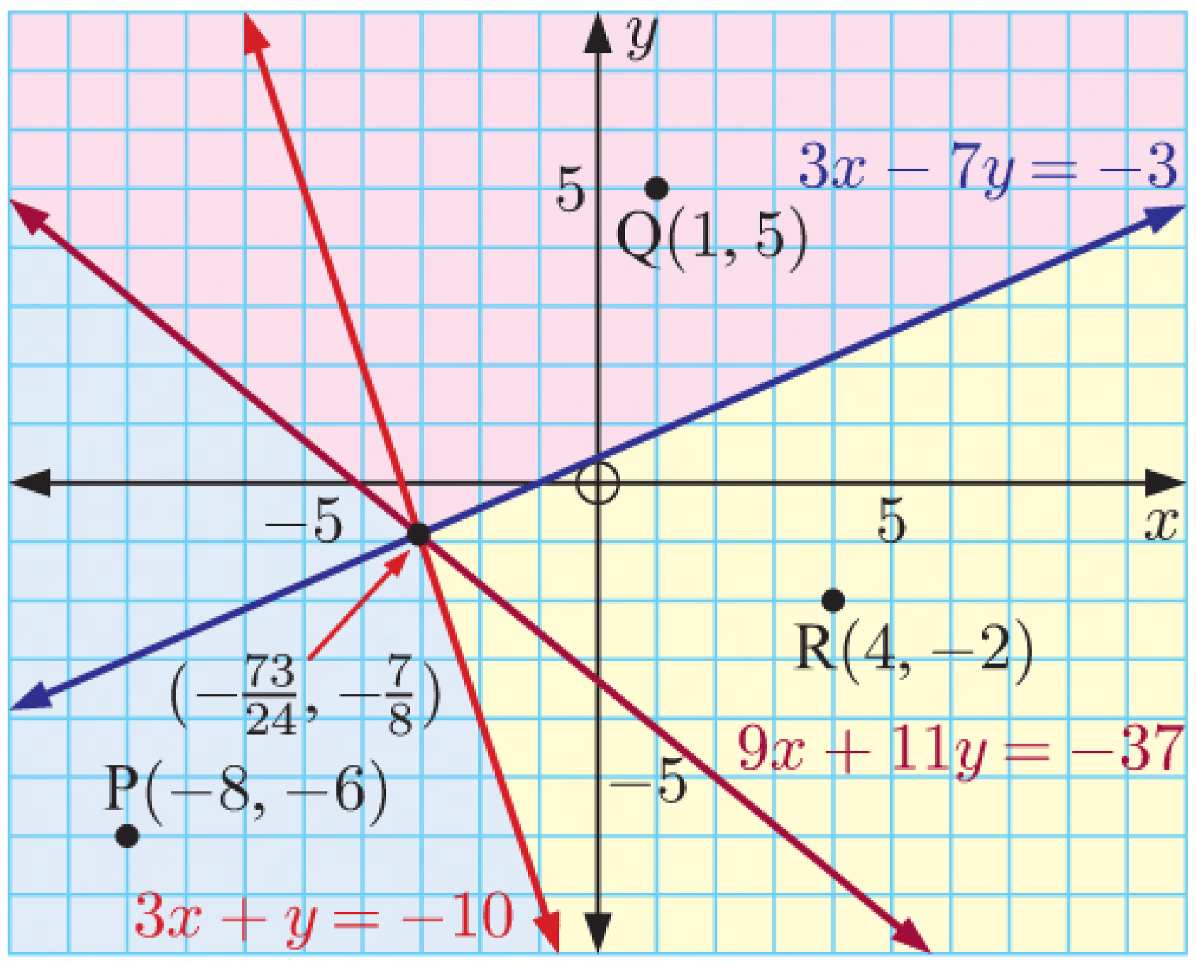
The gradient of [QR] is $\dfrac{-2-5}{4-1} = -\dfrac{7}{3}$

$\therefore$ the gradient of the perpendicular bisector is $\frac{3}{7}$.

$\therefore$ the equation of the perpendicular bisector is $3x - 7y = 3\left(\frac{5}{2}\right) - 7\left(\frac{3}{2}\right)$

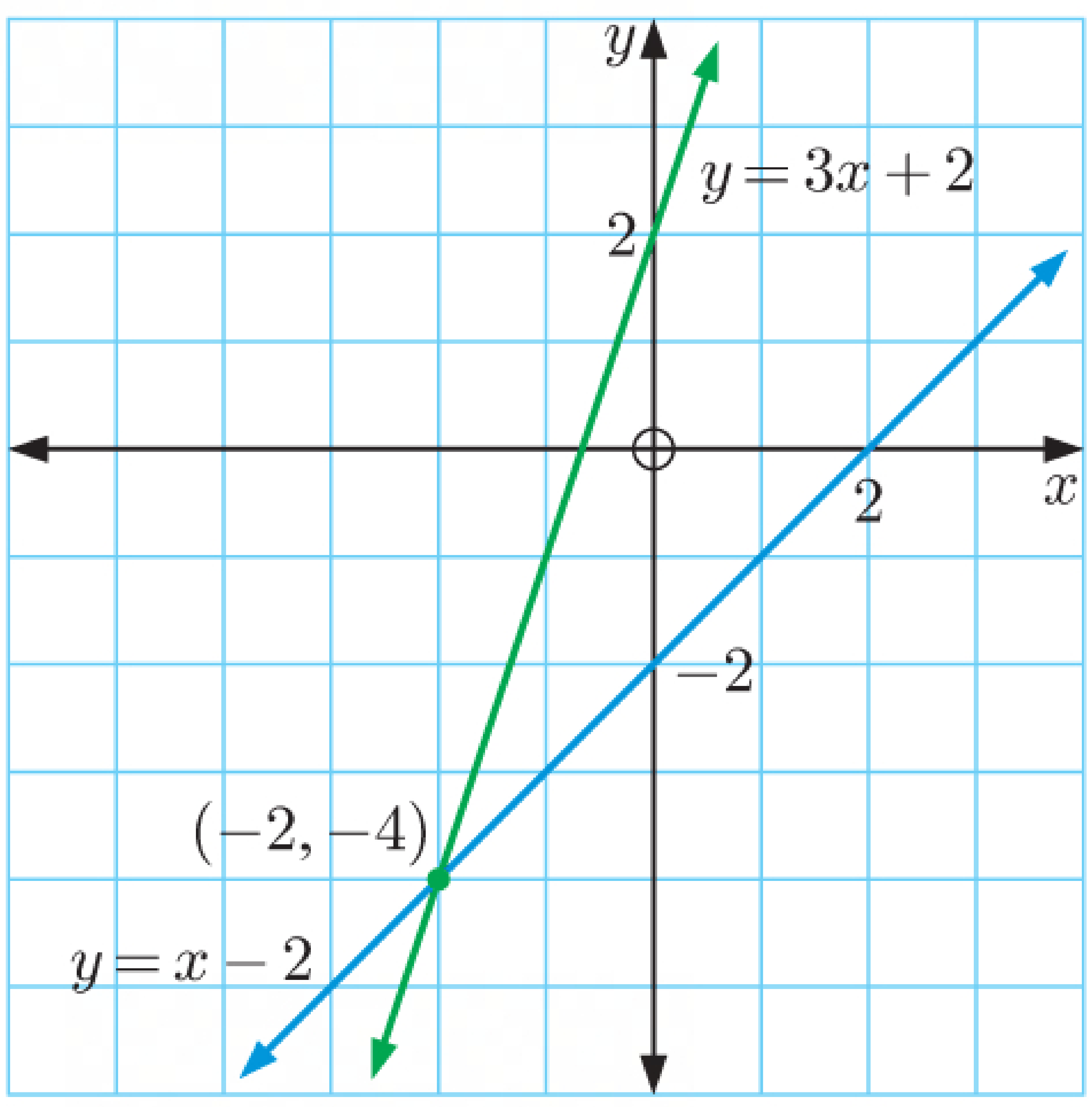
which is $3x - 7y = -3$.

b



EXERCISE 1D

1 **a**



We draw the graphs of $y = 3x + 2$ and $y = x - 2$ on the same set of axes.

The graphs meet at the point $(-2, -4)$.

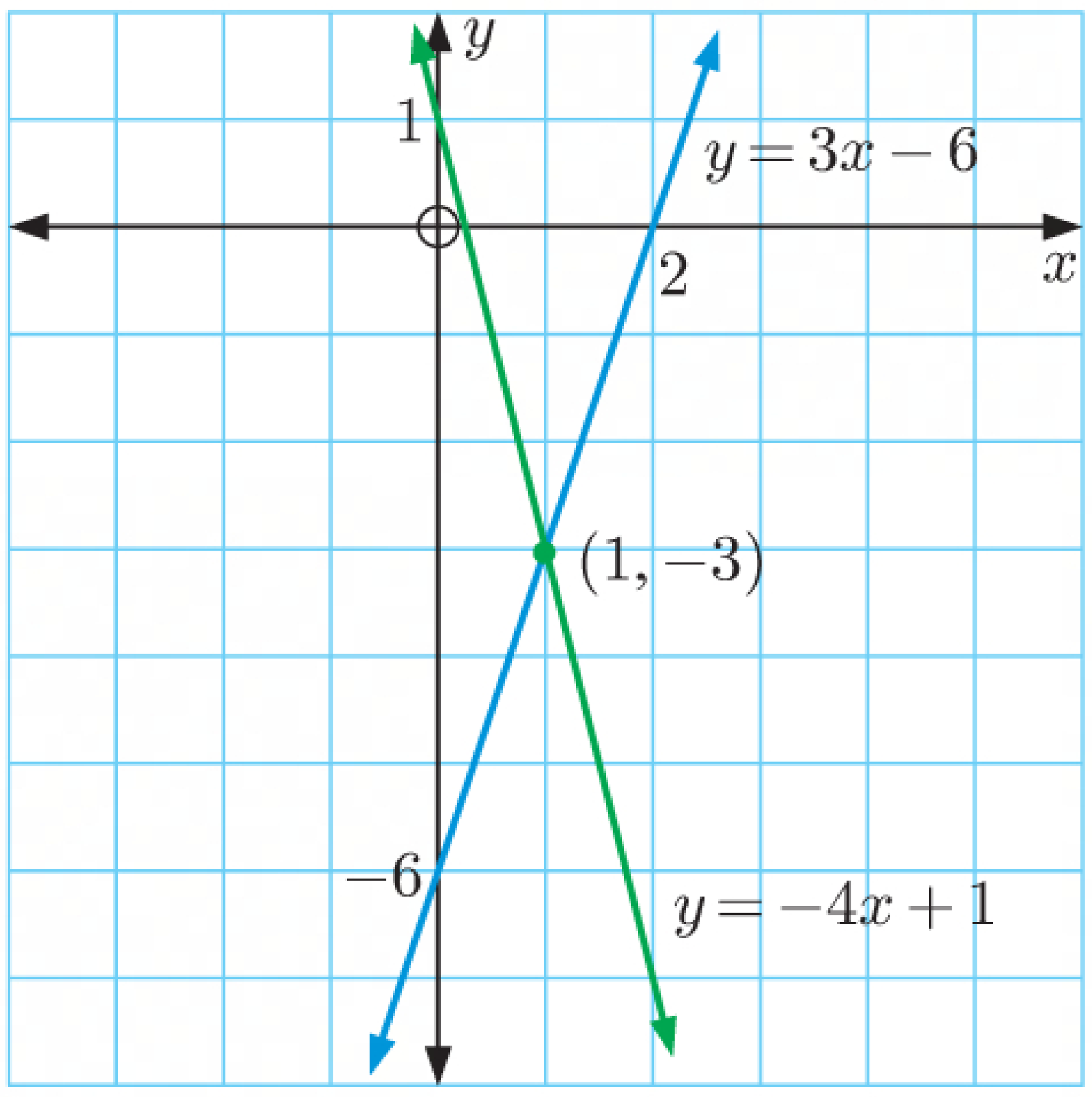
$\therefore$ the solution is $x = -2$, $y = -4$.

Check:

Substituting these values into:

- $y = 3x + 2$ gives $-4 = 3(-2) + 2$ ✓
- $y = x - 2$ gives $-4 = -2 - 2$ ✓

b

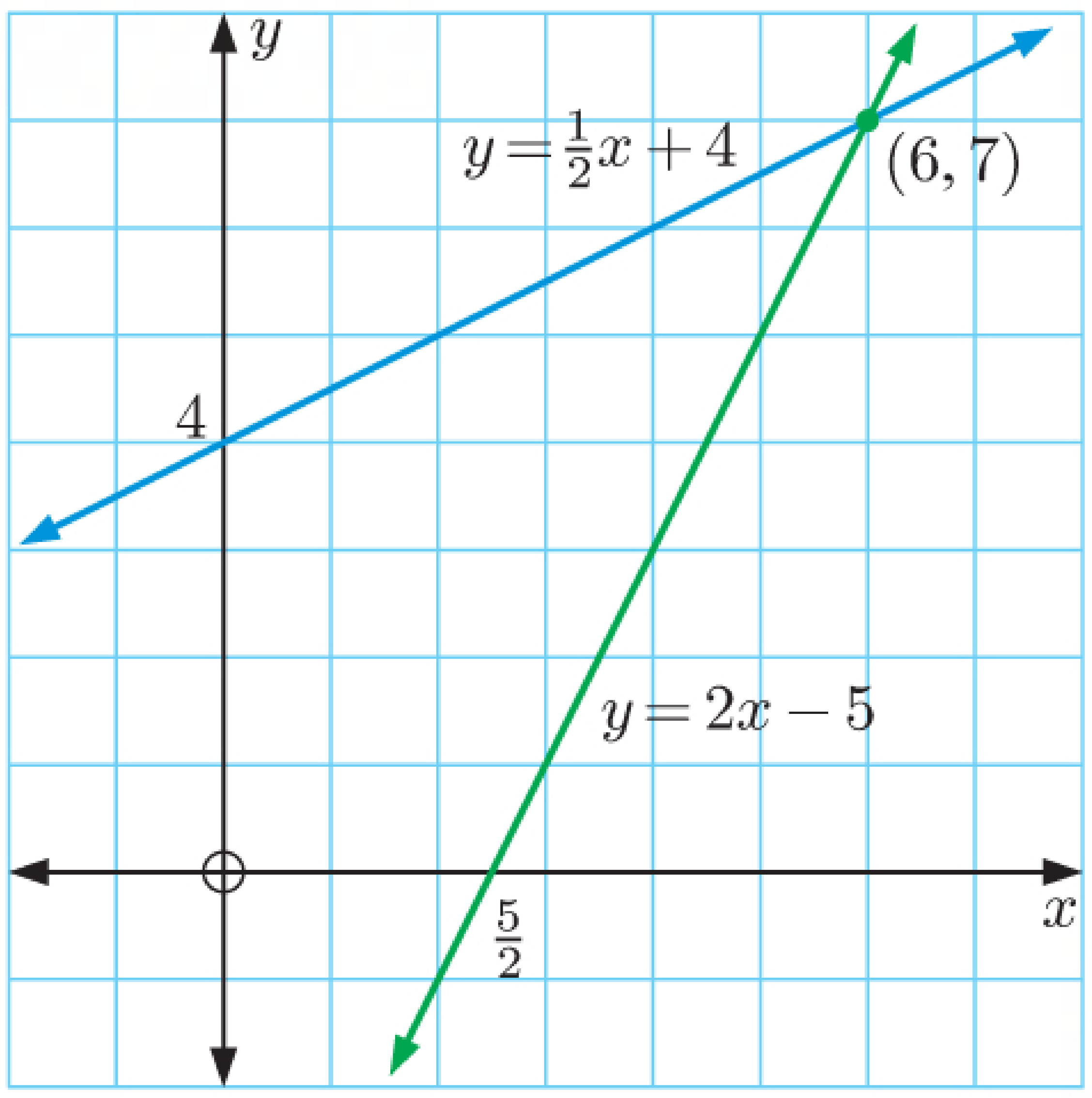


We draw the graphs of $y = -4x + 1$ and $y = 3x - 6$ on the same set of axes.
The graphs meet at the point $(1, -3)$.
$\therefore$ the solution is $x = 1$, $y = -3$.

Check:
Substituting these values into:

- $y = -4x + 1$ gives $-3 = -4(1) + 1$ ✓
- $y = 3x - 6$ gives $-3 = 3(1) - 6$ ✓

c

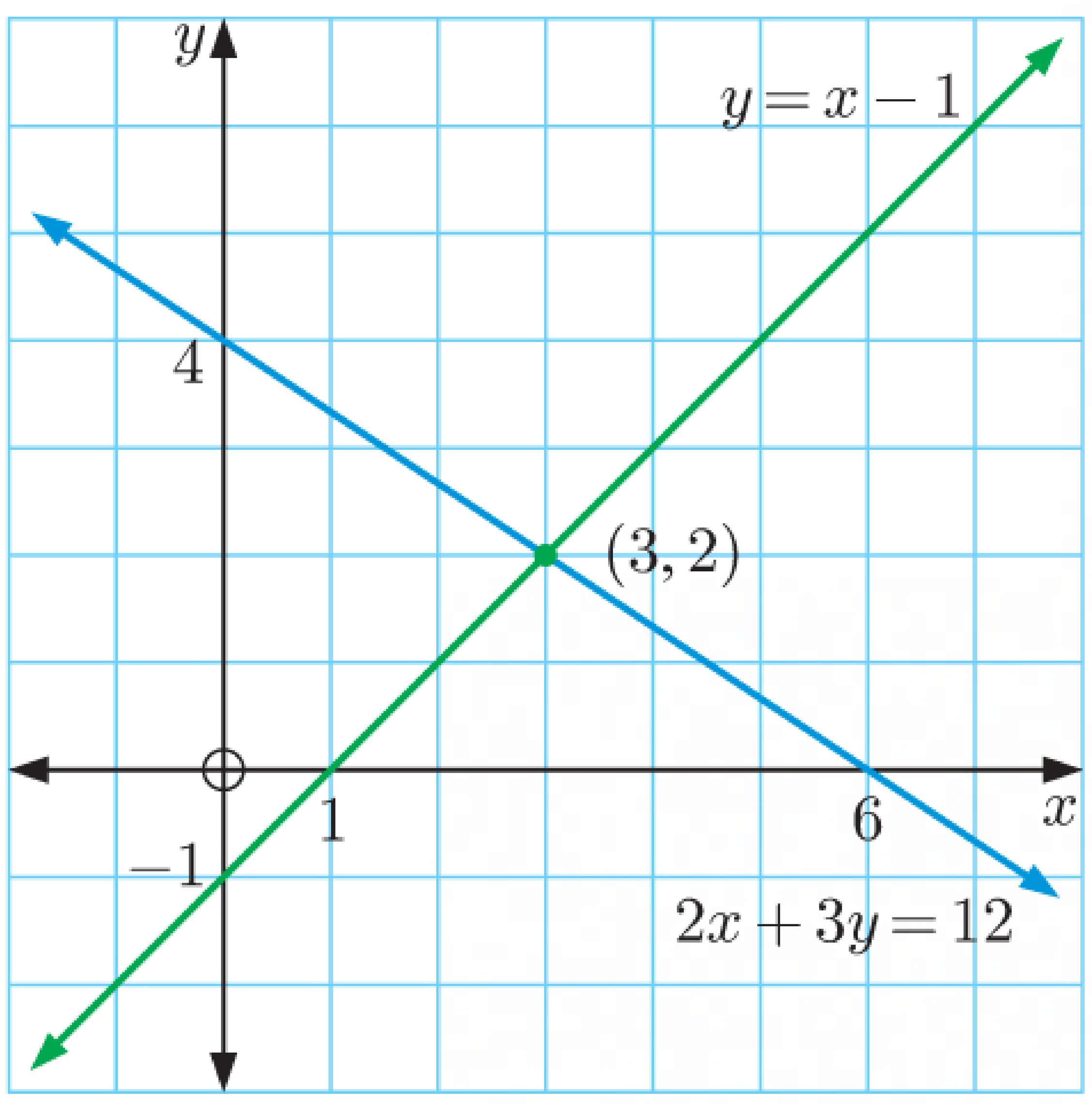


We draw the graphs of $y = 2x - 5$ and $y = \frac{1}{2}x + 4$ on the same set of axes.
The graphs meet at the point $(6, 7)$.
$\therefore$ the solution is $x = 6$, $y = 7$.

Check:
Substituting these values into:

- $y = 2x - 5$ gives $7 = 2(6) - 5$ ✓
- $y = \frac{1}{2}x + 4$ gives $7 = \frac{1}{2}(6) + 4$ ✓

d

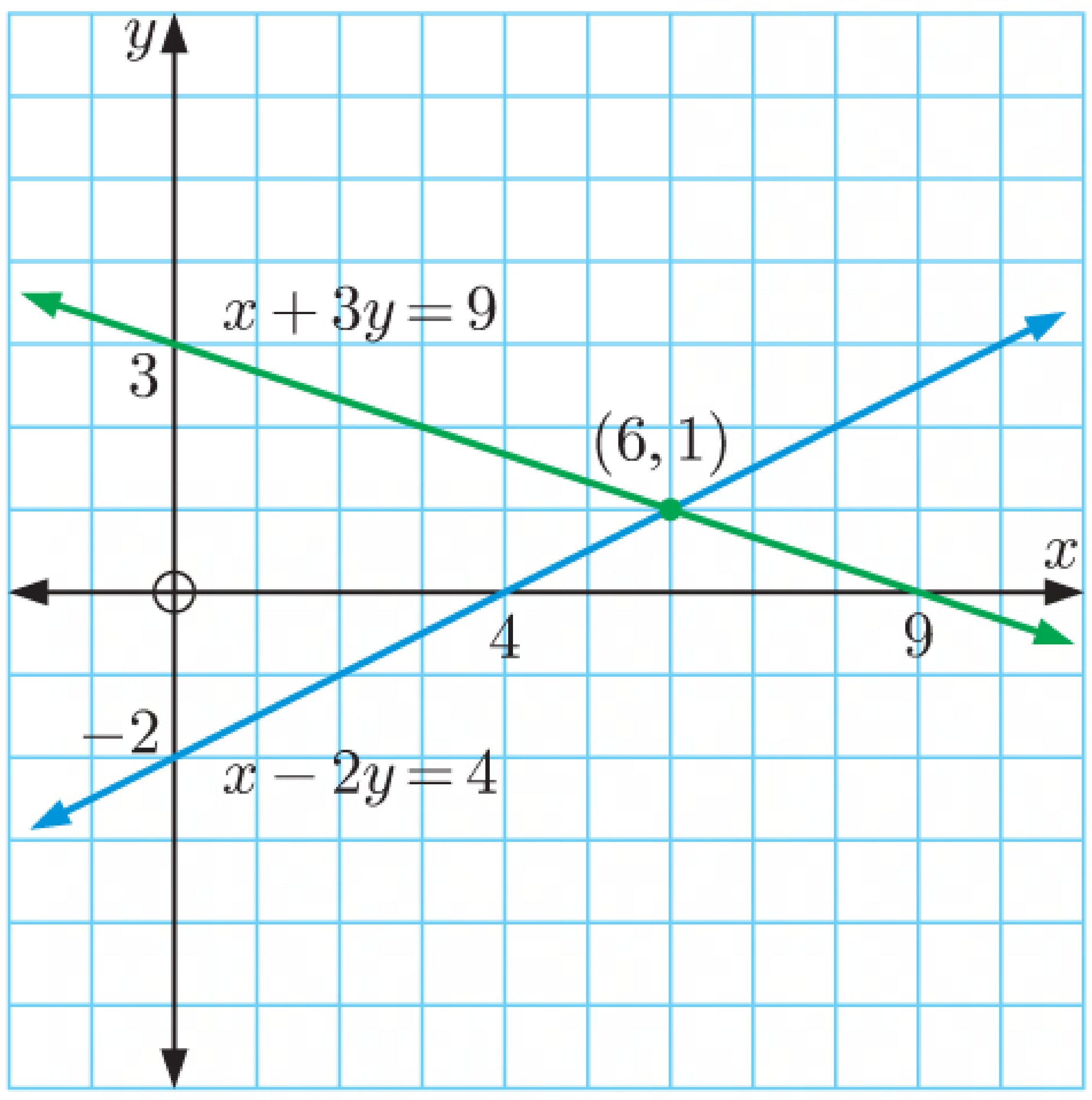


We draw the graphs of $y = x - 1$ and $2x + 3y = 12$ on the same set of axes.
The graphs meet at the point $(3, 2)$.
$\therefore$ the solution is $x = 3$, $y = 2$.

Check:
Substituting these values into:

- $y = x - 1$ gives $2 = 3 - 1$ ✓
- $2x + 3y = 12$ gives $2(3) + 3(2) = 12$ ✓

e

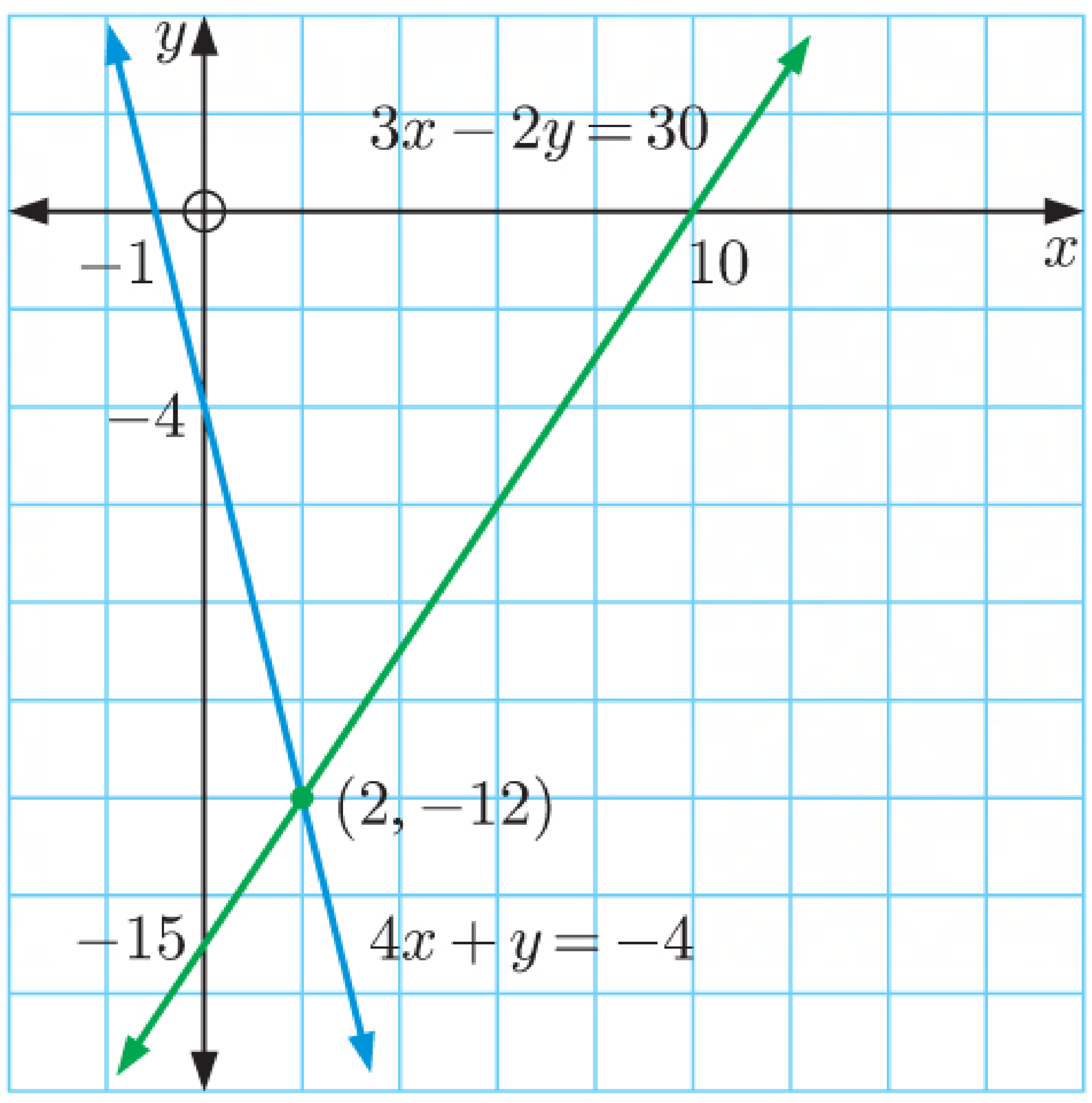


We draw the graphs of $x + 3y = 9$ and $x - 2y = 4$ on the same set of axes.
The graphs meet at the point $(6, 1)$.
$\therefore$ the solution is $x = 6$, $y = 1$.

Check:
Substituting these values into:

- $x + 3y = 9$ gives $6 + 3(1) = 9$ ✓
- $x - 2y = 4$ gives $6 - 2(1) = 4$ ✓

f



We draw the graphs of $3x - 2y = 30$ and $4x + y = -4$ on the same set of axes.
The graphs meet at the point $(2, -12)$.
$\therefore$ the solution is $x = 2$, $y = -12$.

Check:
Substituting these values into:

- $3x - 2y = 30$ gives $3(2) - 2(-12) = 30$ ✓
- $4x + y = -4$ gives $4(2) + (-12) = -4$ ✓

2 **a** $y = x + 2$ .... (1)
$2x + 3y = 21$ .... (2)

Substituting (1) into (2) gives $2x + 3(x + 2) = 21$

$$\therefore\ 2x + 3x + 6 = 21$$
$$\therefore\ 5x = 15$$
$$\therefore\ x = 3$$

Substituting $x = 3$ into (1) gives $y = 3 + 2$

$$\therefore\ y = 5$$

The solution is $x = 3$, $y = 5$.

Check: (1) $5 = 3 + 2$ ✓
(2) $2(3) + 3(5) = 6 + 15 = 21$ ✓

b $y = 2x - 3 \quad .... \ (1)$

$4x - 3y = 7 \quad .... \ (2)$

Substituting (1) into (2) gives $4x - 3(2x - 3) = 7$

$\therefore \ 4x - 6x + 9 = 7$

$\therefore \ -2x = -2$

$\therefore \ x = 1$

Substituting $x = 1$ into (1) gives $y = 2(1) - 3$

$\therefore \ y = -1$

The solution is $x = 1, \ y = -1$.

Check: (1) $-1 = 2(1) - 3 = 2 - 3$ ✓

(2) $4(1) - 3(-1) = 4 + 3 = 7$ ✓

c $5x + 3y = 19 \quad .... \ (1)$

$y = 6 - 2x \quad .... \ (2)$

Substituting (2) into (1) gives $5x + 3(6 - 2x) = 19$

$\therefore \ 5x + 18 - 6x = 19$

$\therefore \ -x = 1$

$\therefore \ x = -1$

Substituting $x = -1$ into (2) gives $y = 6 - 2(-1)$

$\therefore \ y = 8$

The solution is $x = -1, \ y = 8$.

Check: (1) $5(-1) + 3(8) = -5 + 24 = 19$ ✓

(2) $8 = 6 - 2(-1) = 6 + 2$ ✓

d $x = y - 3 \quad .... \ (1)$

$5x - 2y = 9 \quad .... \ (2)$

Substituting (1) into (2) gives $5(y - 3) - 2y = 9$

$\therefore \ 5y - 15 - 2y = 9$

$\therefore \ 3y = 24$

$\therefore \ y = 8$

Substituting $y = 8$ into (1) gives $x = 8 - 3$

$\therefore \ x = 5$

The solution is $x = 5, \ y = 8$.

Check: (1) $5 = 8 - 3$ ✓

(2) $5(5) - 2(8) = 25 - 16 = 9$ ✓

e $3x + 4y = -13$ .... (1)

$x = 8y - 2$ .... (2)

Substituting (2) into (1) gives $3(8y - 2) + 4y = -13$

$\therefore\ 24y - 6 + 4y = -13$

$\therefore\ 28y = -7$

$\therefore\ y = -\frac{1}{4}$

Substituting $y = -\frac{1}{4}$ into (2) gives $x = 8\left(-\frac{1}{4}\right) - 2$

$\therefore\ x = -2 - 2$

$\therefore\ x = -4$

The solution is $x = -4,\ y = -\frac{1}{4}$.

Check: (1) $3(-4) + 4\left(-\frac{1}{4}\right) = -12 - 1 = -13$ ✓

(2) $-4 = 8\left(-\frac{1}{4}\right) - 2 = -2 - 2$ ✓

f $x = -5y - 2$ .... (1)

$7x + 4y = -10$ .... (2)

Substituting (1) into (2) gives $7(-5y - 2) + 4y = -10$

$\therefore\ -35y - 14 + 4y = -10$

$\therefore\ -31y = 4$

$\therefore\ y = -\frac{4}{31}$

Substituting $y = -\frac{4}{31}$ into (1) gives $x = -5\left(-\frac{4}{31}\right) - 2$

$\therefore\ x = \frac{20}{31} - 2$

$\therefore\ x = -1\frac{11}{31}$

The solution is $x = -1\frac{11}{31},\ y = -\frac{4}{31}$.

Check: (1) $-1\frac{11}{31} = -5\left(-\frac{4}{31}\right) - 2 = \frac{20}{31} - 2 = -\frac{42}{31}$ ✓

(2) $7\left(-1\frac{11}{31}\right) + 4\left(-\frac{4}{31}\right) = 7\left(-\frac{42}{31}\right) + 4\left(-\frac{4}{31}\right) = -\frac{310}{31} = -10$ ✓

g $y = \frac{1}{2}x + 5$ .... (1)

$3x + 4y = 5$ .... (2)

Substituting (1) into (2) gives $3x + 4\left(\frac{1}{2}x + 5\right) = 5$

$\therefore\ 3x + 2x + 20 = 5$

$\therefore\ 5x = -15$

$\therefore\ x = -3$

Substituting $x = -3$ into (1) gives $y = \frac{1}{2}(-3) + 5$

$\therefore\ y = -\frac{3}{2} + 5$

$\therefore\ y = 3\frac{1}{2}$

The solution is $x = -3,\ y = 3\frac{1}{2}$.

Check: (1) $3\frac{1}{2} = \frac{1}{2}(-3) + 5 = -\frac{3}{2} + 5 = \frac{7}{2}$ ✓

(2) $3(-3) + 4\left(3\frac{1}{2}\right) = -9 + 4\left(\frac{7}{2}\right) = -9 + 14 = 5$ ✓

h $x = -\frac{3}{4}y \quad .... (1)$

$4x - 5y = -24 \quad .... (2)$

Substituting (1) into (2) gives $4\left(-\frac{3}{4}y\right) - 5y = -24$

$$\therefore \ -3y - 5y = -24$$
$$\therefore \ -8y = -24$$
$$\therefore \ y = 3$$

Substituting $y = 3$ into (1) gives $x = -\frac{3}{4}(3)$

$$\therefore \ x = -2\tfrac{1}{4}$$

The solution is $x = -2\frac{1}{4},\ y = 3.$

Check: (1) $-2\frac{1}{4} = -\frac{3}{4}(3) = -\frac{9}{4}$ ✓

(2) $4\left(-2\frac{1}{4}\right) - 5(3) = 4\left(-\frac{9}{4}\right) - 15 = -9 - 15 = -24$ ✓

i $3x + 7y = 6 \quad .... (1)$

$x = \frac{5}{3}y - 1 \quad .... (2)$

Substituting (2) into (1) gives $3\left(\frac{5}{3}y - 1\right) + 7y = 6$

$$\therefore \ 5y - 3 + 7y = 6$$
$$\therefore \ 12y = 9$$
$$\therefore \ y = \tfrac{3}{4}$$

Substituting $y = \frac{3}{4}$ into (2) gives $x = \frac{5}{3}\left(\frac{3}{4}\right) - 1$

$$\therefore \ x = \tfrac{5}{4} - 1$$
$$\therefore \ x = \tfrac{1}{4}$$

The solution is $x = \frac{1}{4},\ y = \frac{3}{4}.$

Check: (1) $3\left(\frac{1}{4}\right) + 7\left(\frac{3}{4}\right) = \frac{3}{4} + \frac{21}{4} = 6$ ✓ (2) $\frac{1}{4} = \frac{5}{3}\left(\frac{3}{4}\right) - 1 = \frac{5}{4} - 1$ ✓

3 **a** $\begin{cases} 3x - y = 5 \\ 4x + y = 9 \end{cases}$

The coefficients of y are the same size but opposite in sign.
We *add* the LHSs and the RHSs to get an equation which contains x only.

$$\begin{array}{rll} & 3x - y = 5 & .... (1) \\ & \underline{4x + y = 9} & .... (2) \\ \text{Adding,} & 7x \quad\ = 14 & \\ & \therefore \ x = 2 & \end{array}$$

Substituting $x = 2$ into (1) gives $3(2) - y = 5$

$$\therefore \ 6 - y = 5$$
$$\therefore \ -y = -1$$
$$\therefore \ y = 1$$

The solution is $x = 2,\ y = 1.$

Check: In (2): $4(2) + 1 = 8 + 1 = 9$ ✓

b $\begin{cases} 5x - 2y = 17 \\ 3x + 2y = 7 \end{cases}$

The coefficients of y are the same size but opposite in sign.
We *add* the LHSs and the RHSs to get an equation which contains x only.

$$\begin{aligned} 5x - 2y &= 17 \quad \text{.... (1)} \\ 3x + 2y &= 7 \quad \text{.... (2)} \end{aligned}$$

Adding, $8x = 24$

$\therefore \; x = 3$

Substituting $x = 3$ into (1) gives $5(3) - 2y = 17$

$\therefore \; 15 - 2y = 17$

$\therefore \; -2y = 2$

$\therefore \; y = -1$

The solution is $x = 3, \; y = -1$.

Check: In (2): $3(3) + 2(-1) = 9 - 2 = 7$ ✓

c $3x + y = 16$ .... (1)

$7x - 2y = 7$ .... (2)

To make the coefficients of y the same size but opposite in sign, we multiply (1) by 2.

$$\begin{aligned} \therefore \; 6x + 2y &= 32 \quad \{(1) \times 2\} \\ 7x - 2y &= 7 \end{aligned}$$

Adding, $13x = 39$

$\therefore \; x = 3$

Substituting $x = 3$ into (1) gives $3(3) + y = 16$

$\therefore \; 9 + y = 16$

$\therefore \; y = 7$

The solution is $x = 3, \; y = 7$.

Check: In (2): $7(3) - 2(7) = 21 - 14 = 7$ ✓

d $3x - 7y = -27$ .... (1)

$-6x + 5y = 18$ .... (2)

To make the coefficients of x the same size but opposite in sign, we multiply (1) by 2.

$$\begin{aligned} \therefore \; 6x - 14y &= -54 \quad \{(1) \times 2\} \\ -6x + 5y &= 18 \end{aligned}$$

Adding, $-9y = -36$

$\therefore \; y = 4$

Substituting $y = 4$ into (1) gives $3x - 7(4) = -27$

$\therefore \; 3x - 28 = -27$

$\therefore \; 3x = 1$

$\therefore \; x = \frac{1}{3}$

The solution is $x = \frac{1}{3}, \; y = 4$.

Check: In (2): $-6\left(\frac{1}{3}\right) + 5(4) = -2 + 20 = 18$ ✓

e $3x - 7y = -8$ .... (1)
$9x + 11y = 16$ .... (2)

To make the coefficients of x the same size but opposite in sign, we multiply (1) by -3.

$$\begin{aligned} \therefore\ -9x + 21y &= 24 \quad \{(1) \times -3\} \\ 9x + 11y &= 16 \end{aligned}$$

Adding, $32y = 40$

$\therefore\ y = 1\frac{1}{4}$

Substituting $y = 1\frac{1}{4}$ into (1) gives $3x - 7\left(1\frac{1}{4}\right) = -8$

$\therefore\ 3x - 7\left(\frac{5}{4}\right) = -8$

$\therefore\ 3x - \frac{35}{4} = -8$

$\therefore\ 3x = \frac{3}{4}$

$\therefore\ x = \frac{1}{4}$

The solution is $x = \frac{1}{4},\ y = 1\frac{1}{4}$.

Check: In (2): $9\left(\frac{1}{4}\right) + 11\left(1\frac{1}{4}\right) = \frac{9}{4} + 11\left(\frac{5}{4}\right) = \frac{9}{4} + \frac{55}{4} = \frac{64}{4} = 16$ ✓

f $4x + 3y = 14$ .... (1)
$3x - 4y = 23$ .... (2)

To make the coefficients of y the same size but opposite in sign, we multiply (1) by 4 and (2) by 3.

$$\begin{aligned} \therefore\ 16x + 12y &= 56 \quad \{(1) \times 4\} \\ 9x - 12y &= 69 \quad \{(2) \times 3\} \end{aligned}$$

Adding, $25x \qquad = 125$

$\therefore\ x = 5$

Substituting $x = 5$ into (1) gives $4(5) + 3y = 14$

$\therefore\ 20 + 3y = 14$

$\therefore\ 3y = -6$

$\therefore\ y = -2$

The solution is $x = 5,\ y = -2$.

Check: In (2): $3(5) - 4(-2) = 15 + 8 = 23$ ✓

g $2x - 3y = 6$ .... (1)
$5x - 4y = 1$ .... (2)

To make the coefficients of y the same size but opposite in sign, we multiply (1) by -4 and (2) by 3.

$$\begin{aligned} \therefore\ -8x + 12y &= -24 \quad \{(1) \times -4\} \\ 15x - 12y &= 3 \quad \{(2) \times 3\} \end{aligned}$$

Adding, $7x \qquad = -21$

$\therefore\ x = -3$

Substituting $x = -3$ into (1) gives $2(-3) - 3y = 6$

$\therefore\ -6 - 3y = 6$

$\therefore\ -3y = 12$

$\therefore\ y = -4$

The solution is $x = -3,\ y = -4$.

Check: In (2): $5(-3) - 4(-4) = -15 + 16 = 1$ ✓

h $4x + 2y = -23$ .... (1)

$5x - 7y = -5$ .... (2)

To make the coefficients of x the same size but opposite in sign, we multiply (1) by 5 and (2) by -4.

$$\begin{array}{rll} \therefore & 20x + 10y = -115 & \{(1) \times 5\} \\ & -20x + 28y = 20 & \{(2) \times -4\} \\ \hline \text{Adding,} & 38y = -95 & \\ & \therefore\ y = -2\frac{1}{2} & \end{array}$$

Substituting $y = -2\frac{1}{2}$ into (1) gives $4x + 2\left(-2\frac{1}{2}\right) = -23$

$$\therefore\ 4x + 2\left(-\tfrac{5}{2}\right) = -23$$

$$\therefore\ 4x - 5 = -23$$

$$\therefore\ 4x = -18$$

$$\therefore\ x = -4\tfrac{1}{2}$$

The solution is $x = -4\frac{1}{2}$, $y = -2\frac{1}{2}$.

Check: In (2): $5\left(-4\frac{1}{2}\right) - 7\left(-2\frac{1}{2}\right) = 5\left(-\frac{9}{2}\right) - 7\left(-\frac{5}{2}\right) = -\frac{45}{2} + \frac{35}{2} = -\frac{10}{2} = -5$ ✓

i $4x - 7y = 9$ .... (1)

$5x - 8y = -2$ .... (2)

To make the coefficients of x the same size but opposite in sign, we multiply (1) by 5 and (2) by -4.

$$\begin{array}{rll} \therefore & 20x - 35y = 45 & \{(1) \times 5\} \\ & -20x + 32y = 8 & \{(2) \times -4\} \\ \hline \text{Adding,} & -3y = 53 & \\ & \therefore\ y = -17\frac{2}{3} & \end{array}$$

Substituting $y = -17\frac{2}{3}$ into (1) gives $4x - 7\left(-17\frac{2}{3}\right) = 9$

$$\therefore\ 4x - 7\left(-\tfrac{53}{3}\right) = 9$$

$$\therefore\ 4x + \tfrac{371}{3} = 9$$

$$\therefore\ 4x = -\tfrac{344}{3}$$

$$\therefore\ x = -28\tfrac{2}{3}$$

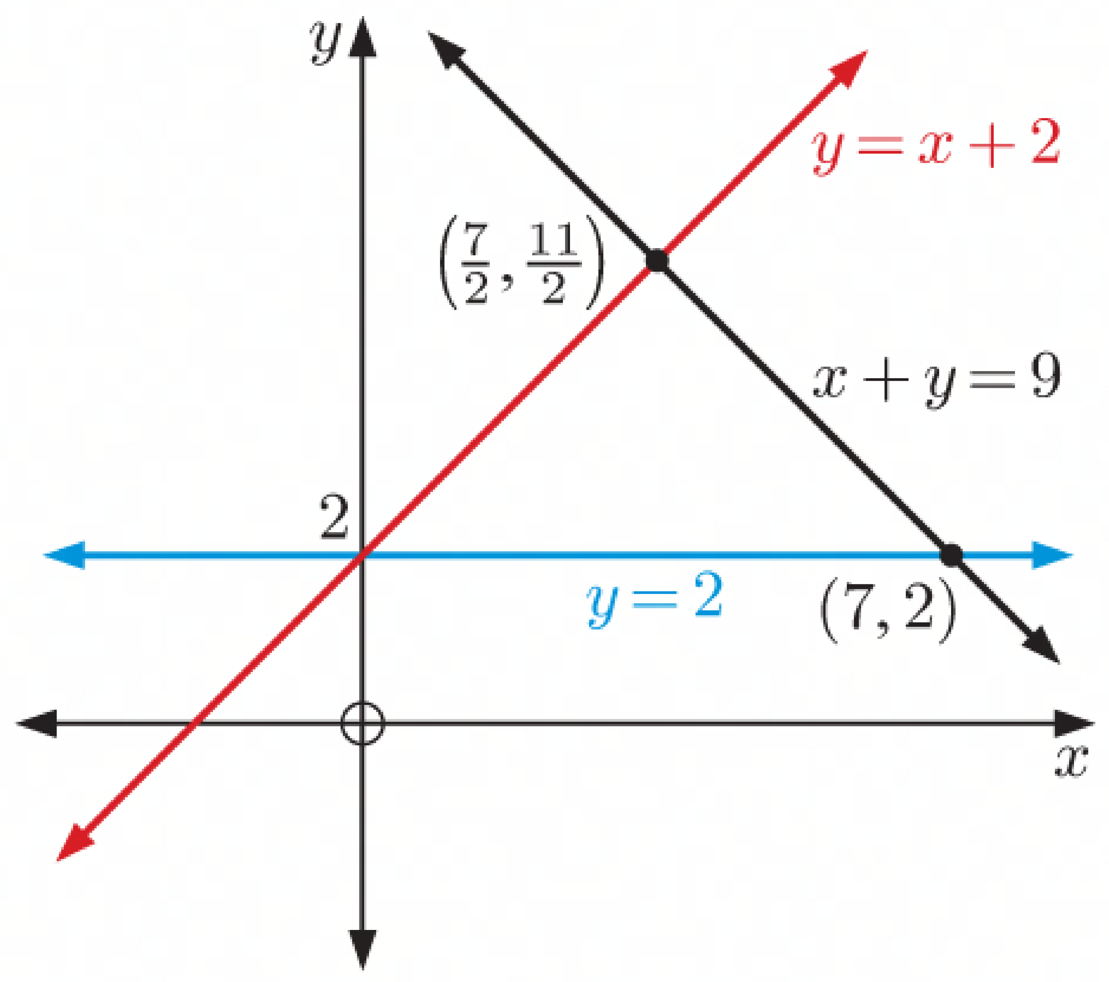
The solution is $x = -28\frac{2}{3}$, $y = -17\frac{2}{3}$.

Check:

In (2): $5\left(-28\frac{2}{3}\right) - 8\left(-17\frac{2}{3}\right) = 5\left(-\frac{86}{3}\right) - 8\left(-\frac{53}{3}\right) = -\frac{430}{3} + \frac{424}{3} = -\frac{6}{3} = -2$ ✓

4 **a**

$$y = x + 2 \quad .... \ (1)$$
$$x + y = 9 \quad .... \ (2)$$
$$y = 2 \quad .... \ (3)$$



Substituting (1) into (2): $x + (x + 2) = 9$

$\therefore \ 2x + 2 = 9$

$\therefore \ 2x = 7$

$\therefore \ x = \frac{7}{2}$

Substituting $x = \frac{7}{2}$ into (1): $y = \frac{7}{2} + 2 = \frac{11}{2}$

So, (1) and (2) intersect at $\left(\frac{7}{2}, \frac{11}{2}\right)$.

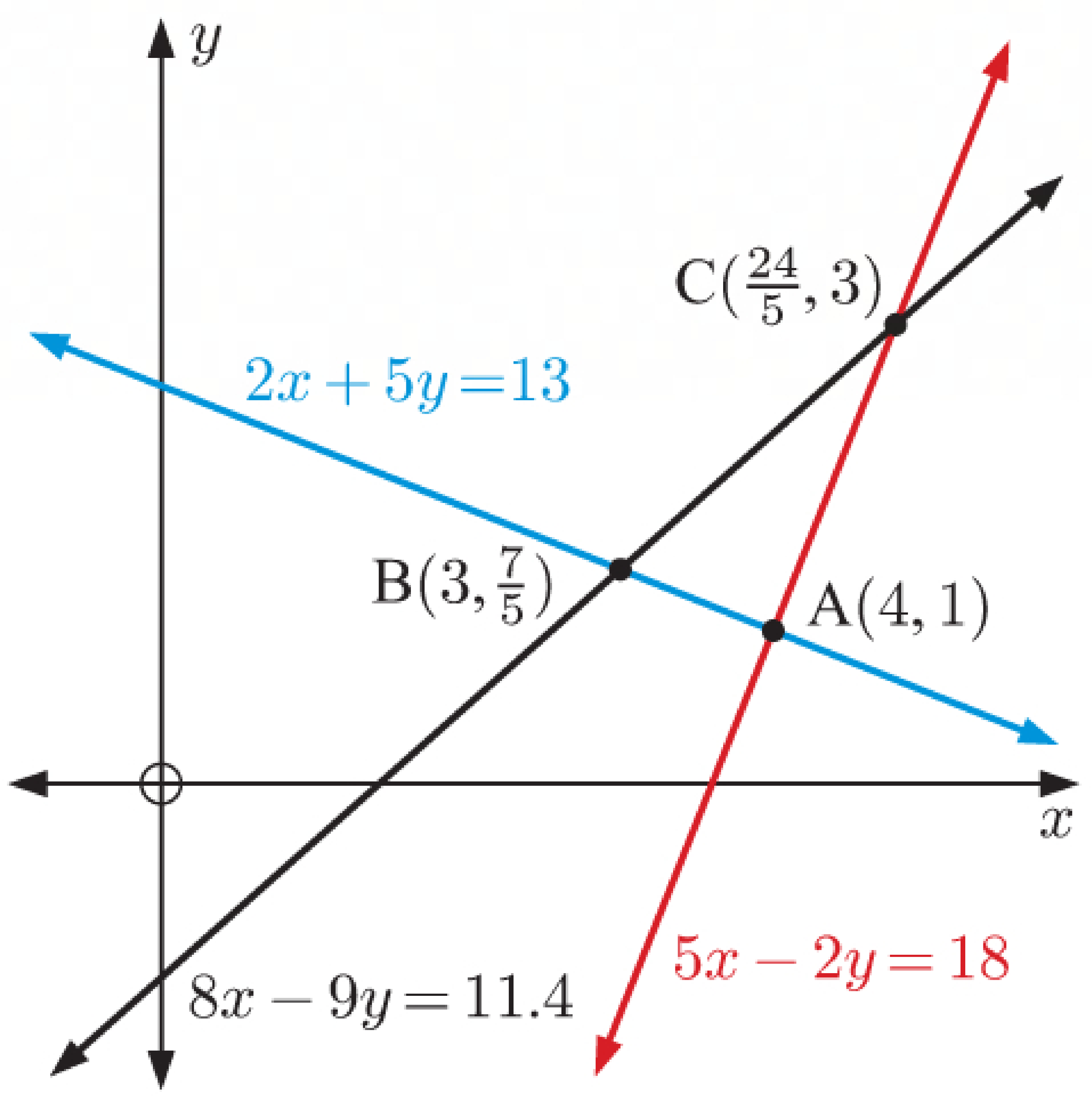
Substituting $y = 2$ into (1) and (2) gives us the remaining points of intersection $(0, 2)$ and $(7, 2)$.

The triangle has base length $7 - 0 = 7$ units and height $\frac{11}{2} - 2 = \frac{7}{2}$ units.

$$\therefore \ \text{area} = \frac{1}{2} \times 7 \times \frac{7}{2} = \frac{49}{4} = 12\tfrac{1}{4} \text{ units}^2$$

b

$$5x - 2y = 18 \quad .... \ (1)$$
$$2x + 5y = 13 \quad .... \ (2)$$
$$8x - 9y = 11.4 \quad .... \ (3)$$



$$\therefore \ 25x - 10y = 90 \quad \{(1) \times 5\}$$
$$4x + 10y = 26 \quad \{(2) \times 2\}$$

Adding, $29x = 116$

$\therefore \ x = 4$

Substituting $x = 4$ into (1) gives $5(4) - 2y = 18$

$\therefore \ 20 - 2y = 18$

$\therefore \ -2y = -2$

$\therefore \ y = 1$

So, (1) and (2) intersect at $A(4, 1)$.

Also, $-8x - 20y = -52 \quad \{(2) \times -4\}$

$8x - 9y = 11.4$

Adding, $-29y = -40.6$

$\therefore \ y = \frac{7}{5}$

Substituting $y = \frac{7}{5}$ into (2) gives $2x + 5\left(\frac{7}{5}\right) = 13$

$\therefore \ 2x + 7 = 13$

$\therefore \ 2x = 6$

$\therefore \ x = 3$

So, (2) and (3) intersect at $B\left(3, \frac{7}{5}\right)$.

Also, $45x - 18y = 162 \quad \{(1) \times 9\}$

$-16x + 18y = -22.8 \quad \{(3) \times -2\}$

Adding, $29x = 139.2$

$\therefore \ x = \frac{24}{5} = 4.8$

Substituting $x = 4.8$ into (3) gives $8(4.8) - 9y = 11.4$

$\therefore\ 38.4 - 9y = 11.4$

$\therefore\ -9y = -27$

$\therefore\ y = 3$

So, (1) and (3) intersect at $C(\frac{24}{5}, 3)$.

Now, $2x + 5y = 13$ has gradient $-\frac{2}{5}$ and $5x - 2y = 18$ has gradient $\frac{5}{2}$, so [AB] and [AC] are perpendicular.

$\therefore$ ABC is a right angled triangle.

$$AB = \sqrt{(3-4)^2 + \left(\tfrac{7}{5} - 1\right)^2} = \sqrt{(-1)^2 + \left(\tfrac{2}{5}\right)^2} = \sqrt{\tfrac{29}{25}} = \frac{\sqrt{29}}{5}$$

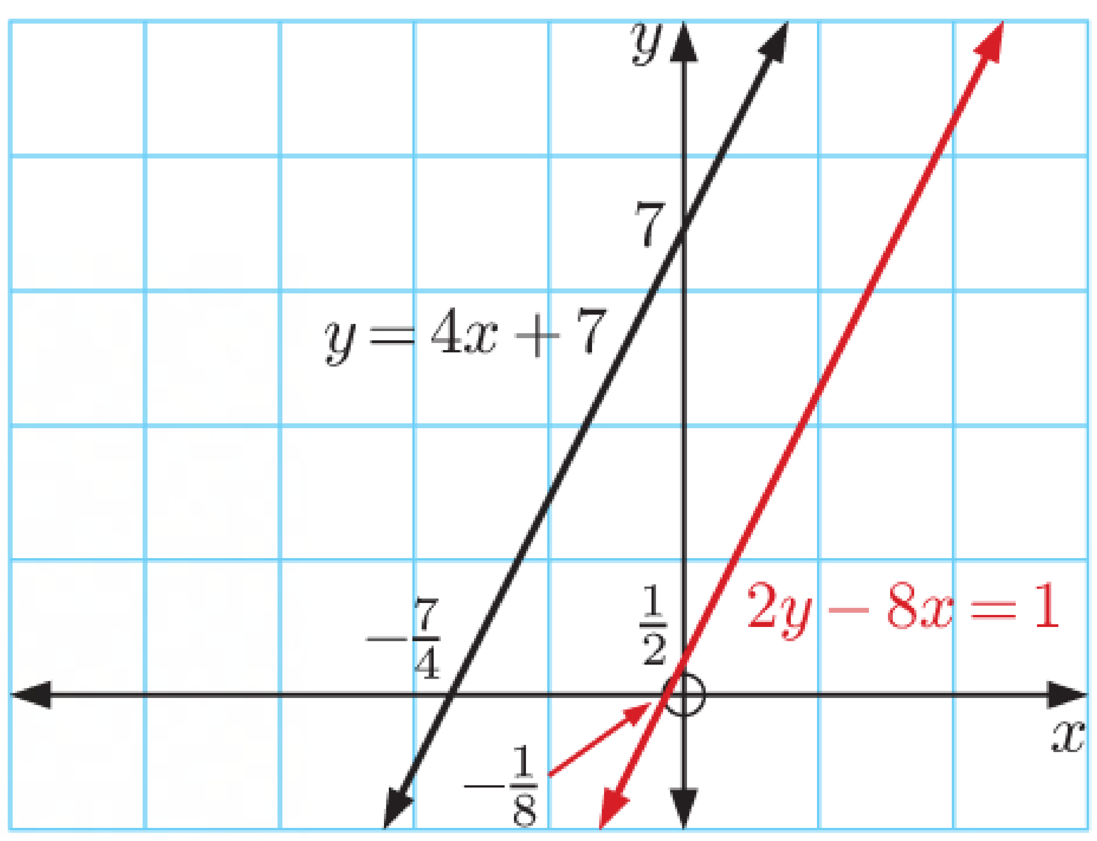
$$AC = \sqrt{\left(\tfrac{24}{5} - 4\right)^2 + (3-1)^2} = \sqrt{\left(\tfrac{4}{5}\right)^2 + 2^2} = \sqrt{\tfrac{116}{25}} = \frac{2\sqrt{29}}{5}$$

$\therefore$ area of $\triangle ABC = \frac{1}{2} \times \frac{\sqrt{29}}{5} \times \frac{2\sqrt{29}}{5}$

$= \frac{29}{25} = 1\frac{4}{25}$ units^2

5 **a** We draw the graphs of $y = 4x + 7$ and $2y - 8x = 1$ on the same set of axes.

We can see that the lines are parallel and therefore do not meet.



b **i** $y = 4x + 7$ .... (1)

$2y - 8x = 1$ .... (2)

Substituting (1) into (2) gives $2(4x + 7) - 8x = 1$

$\therefore\ 8x + 14 - 8x = 1$

$\therefore\ 14 = 1$ which is false

$\therefore$ there are no solutions.

ii We rearrange the second equation, so the system is now: $y = 4x + 7$ .... (1)

$2y = 8x + 1$ .... (2)

To make the coefficients of y the same size but opposite in sign, we multiply (1) by -2.

$\therefore\ -2y = -8x - 14$ $\{(1) \times -2\}$

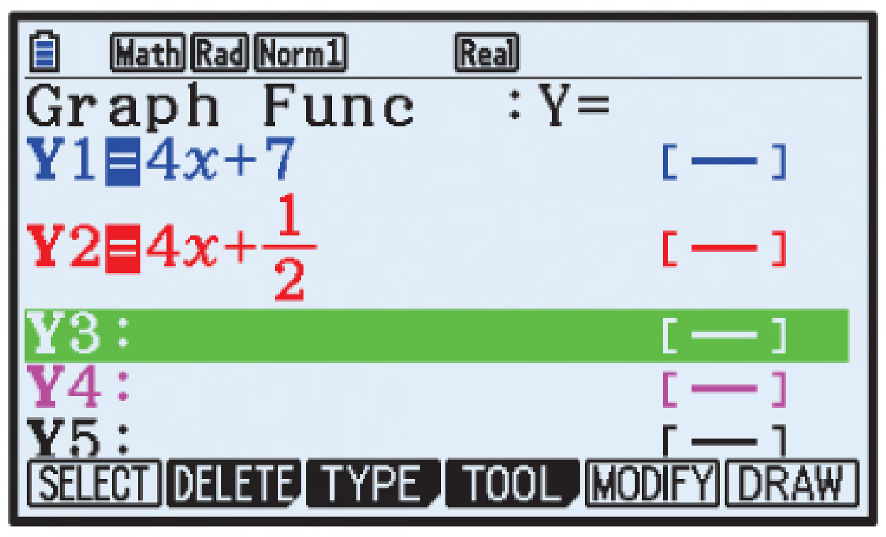
$2y = 8x + 1$

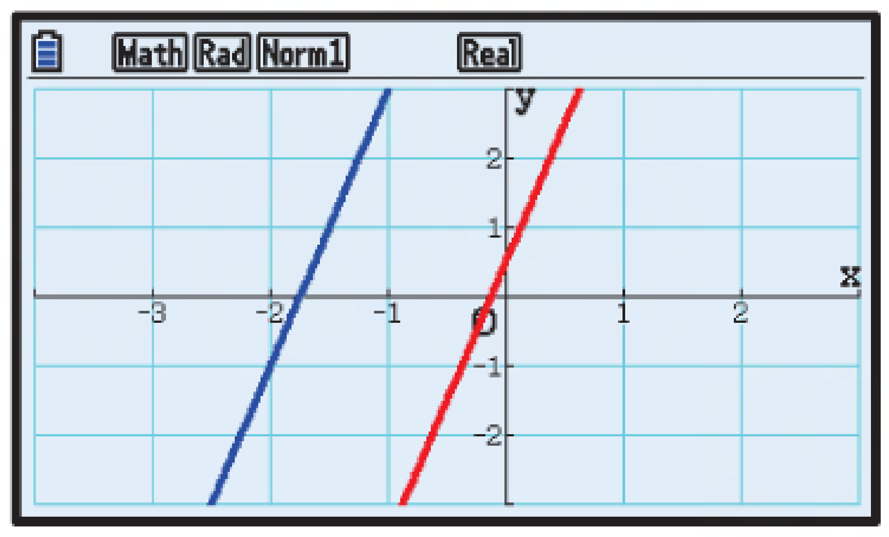
Adding, $0 = -13$ which is false

$\therefore$ there are no solutions.

iii We rearrange the second equation, so the system is now:

$$\begin{cases} y = 4x + 7 \\ y = 4x + \frac{1}{2} \end{cases}$$



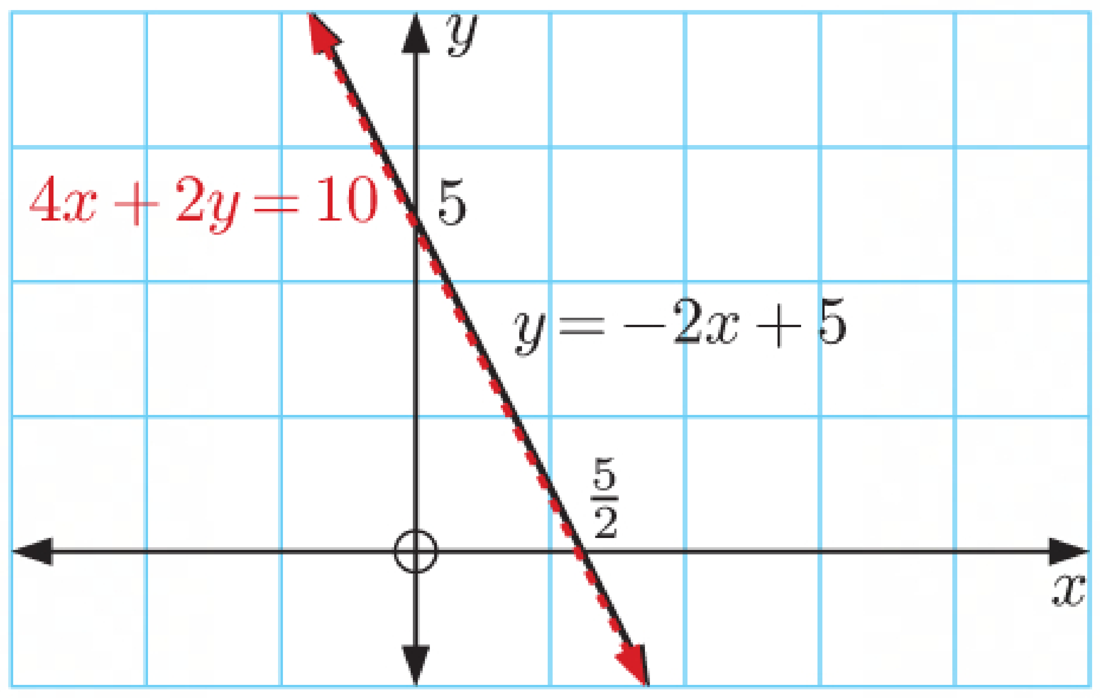


Since the equations have the same gradient, the lines are parallel.

$\therefore$ there are no solutions.

c This system of simultaneous equations has no solutions.

6 **a** We draw the graphs of $y = -2x + 5$ and $4x + 2y = 10$ on the same set of axes.
We can see that the lines are coincident.



b **i** $y = -2x + 5$ .... (1)

$4x + 2y = 10$ .... (2)

Substituting (1) into (2) gives $4x + 2(-2x + 5) = 10$

$\therefore 4x - 4x + 10 = 10$

$\therefore 10 = 10$ which is always true

$\therefore$ there are infinitely many solutions.

ii We rearrange the second equation, so the system is now: $y = -2x + 5$ .... (1)

$2y = -4x + 10$ .... (2)

To make the coefficients of y the same size but opposite in sign, we multiply (1) by -2.

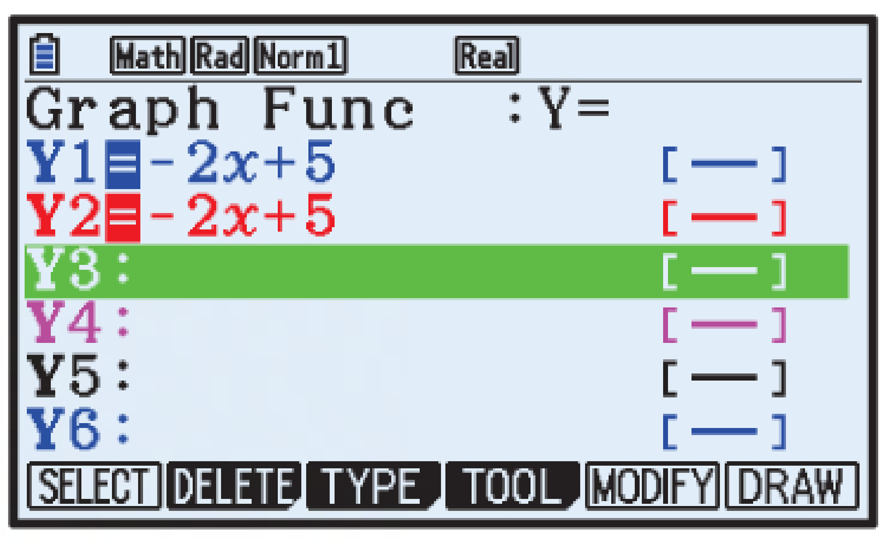
$$\therefore \begin{aligned} -2y &= 4x - 10 \quad \{(1) \times -2\} \\ 2y &= -4x + 10 \\ \hline 0 &= 0 \end{aligned}$$

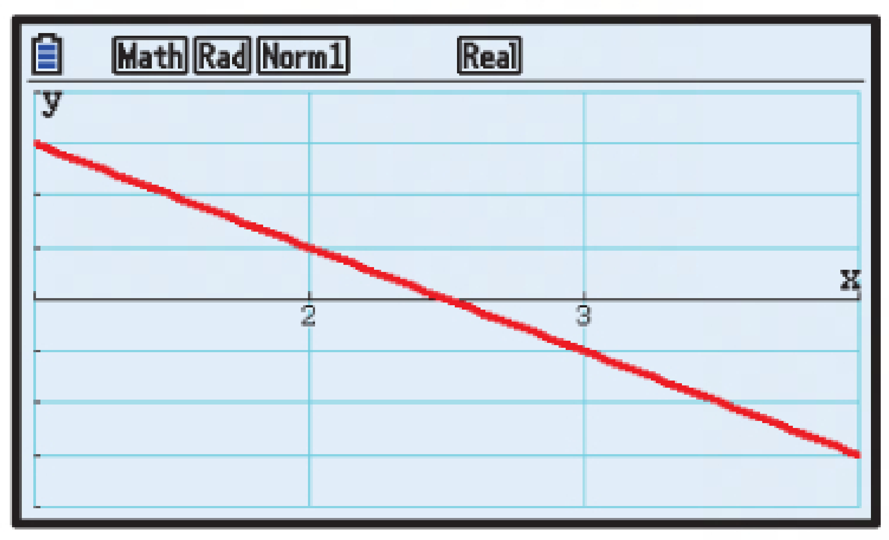
Adding, $0 = 0$ which is always true

$\therefore$ there are infinitely many solutions.

iii We rearrange the second equation, so the system is now:

$$\begin{cases} y = -2x + 5 \\ y = -2x + 5 \end{cases}$$





Since the equations are identical, the lines are coincident.

$\therefore$ there are infinitely many solutions.

c This system of simultaneous equations has infinitely many solutions.

7 $3x - 2y = 12 \quad .... (1)$
$y = mx - 6 \quad .... (2)$

a The gradient of the line $3x - 2y = 12$ is $-\frac{3}{-2} = \frac{3}{2}$.
The gradient of the line $y = mx - 6$ is m.

b If $m = \frac{3}{2}$ then the lines are parallel as their gradients are equal.
The line $y = mx - 6$ thus becomes $y = \frac{3}{2}x - 6$ which can be rearranged to $3x - 2y = 12$.
The equations will therefore be identical, so the lines are coincident.
$\therefore$ there are infinitely many solutions in this case.

c Suppose $m \neq \frac{3}{2}$.
Substituting (2) into (1) gives

$$\begin{aligned} 3x - 2(mx - 6) &= 12 \\ \therefore\ 3x - 2mx + 12 &= 12 \\ \therefore\ (3 - 2m)x &= 0 \\ \therefore\ x &= 0 \quad \{\text{as } m \neq \tfrac{3}{2}\} \end{aligned}$$

Substituting $x = 0$ into (2) gives

$$\begin{aligned} y &= m(0) - 6 \\ y &= -6 \end{aligned}$$

So the unique solution is $x = 0$, $y = -6$.
Check: In (1): $3(0) - 2(-6) = 0 + 12$ ✓ In (2): $-6 = m(0) - 6 = 0 - 6$ ✓

8 $\begin{cases} 12 = 4x - cy \\ 2x + 6 = 3y \end{cases}$

a The gradient of the line $12 = 4x - cy$ is $-\dfrac{4}{-c} = \dfrac{4}{c}$.
The gradient of the line $2x + 6 = 3y$ is $\frac{2}{3}$.

b If $\dfrac{4}{c} = \frac{2}{3}$, then $c = 6$, and the lines will be parallel as their gradients are equal.
The line $12 = 4x - cy$ thus becomes $12 = 4x - 6y$ which can be rearranged to $2x - 6 = 3y$.
The equations are therefore not equal, so the lines are parallel but not coincident.
$\therefore$ there are no solutions.

c Suppose $c \neq 6$.
We rearrange the second equation, so the system is now:

$$\begin{aligned} 12 &= 4x - cy \quad .... (1) \\ 6 &= -2x + 3y \quad .... (2) \end{aligned}$$

To make the coefficients of x the same size but opposite in sign, we multiply (2) by 2.

$$\begin{aligned} \therefore\ 12 &= \ \ 4x - cy \\ 12 &= -4x + 6y \quad \{(2) \times 2\} \\ \hline \text{Adding,}\quad 24 &= (6 - c)y \\ \therefore\ y &= \frac{24}{6 - c} \quad \{\text{as } c \neq 6\} \end{aligned}$$

Substituting $y = \dfrac{24}{6-c}$ into (2) gives $6 = -2x + 3\left(\dfrac{24}{6-c}\right)$

$$\therefore\ 6 = -2x + \frac{72}{6-c}$$

$$\therefore\ 2x = \frac{72 - 6(6-c)}{6-c} = \frac{72 - 36 + 6c}{6-c} = \frac{36+6c}{6-c} = \frac{2(18+3c)}{6-c}$$

$$\therefore\ x = \frac{18+3c}{6-c}$$

So the unique solution is $x = \dfrac{18+3c}{6-c}$, $y = \dfrac{24}{6-c}$, $c \neq 6$.

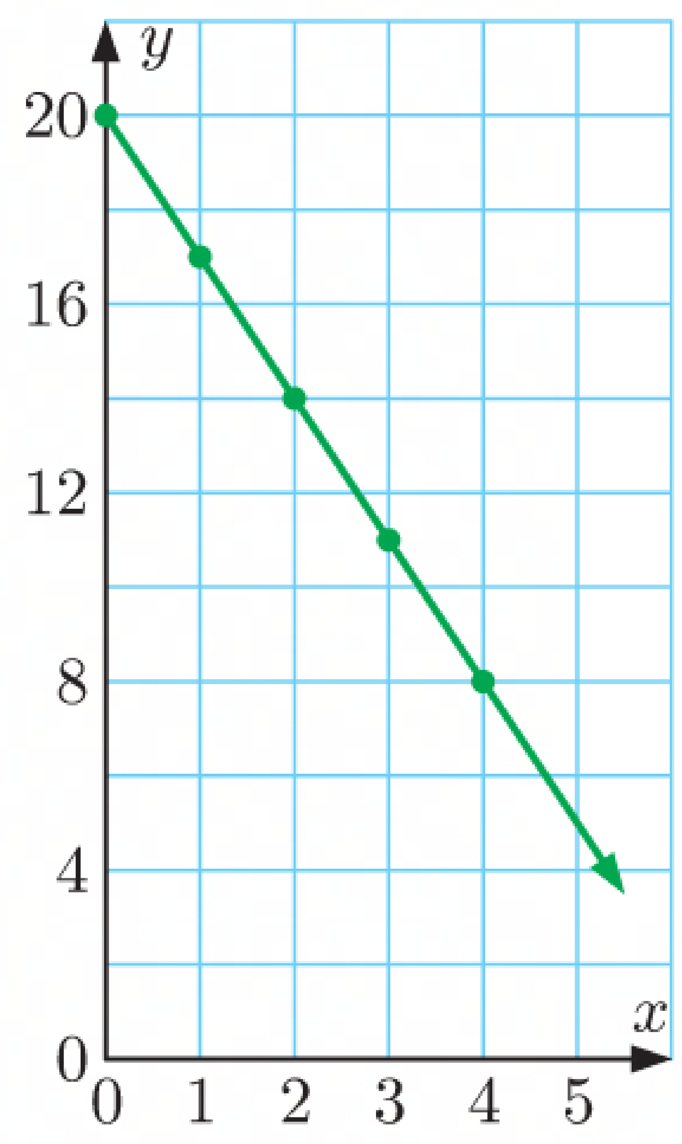
Check:

In (1): $12 = 4\left(\dfrac{18+3c}{6-c}\right) - c\left(\dfrac{24}{6-c}\right) = \dfrac{72+12c-24c}{6-c} = \dfrac{72-12c}{6-c} = \dfrac{12(6-c)}{6-c}$ ✓

REVIEW SET 1A

1 **a**

x	0	1	2	3	4
y	20	17	14	11	8



b Yes, the variables are linearly related as the points all lie on a straight line.

c The line passes through (0, 20) and (1, 17), so the gradient is $\dfrac{17-20}{1-0} = -3$.
The y-intercept is 20.

d The gradient is -3 and the y-intercept is 20, so the equation is $y = -3x + 20$.

e When $x = 7$, $y = -3(7) + 20$
$= -1$

2 **a** The equation of the line is

$$y - 2 = -\tfrac{1}{3}(x - 6)$$
$$\therefore\ y - 2 = -\tfrac{1}{3}x + 2$$
$$\therefore\ y = -\tfrac{1}{3}x + 4$$

b

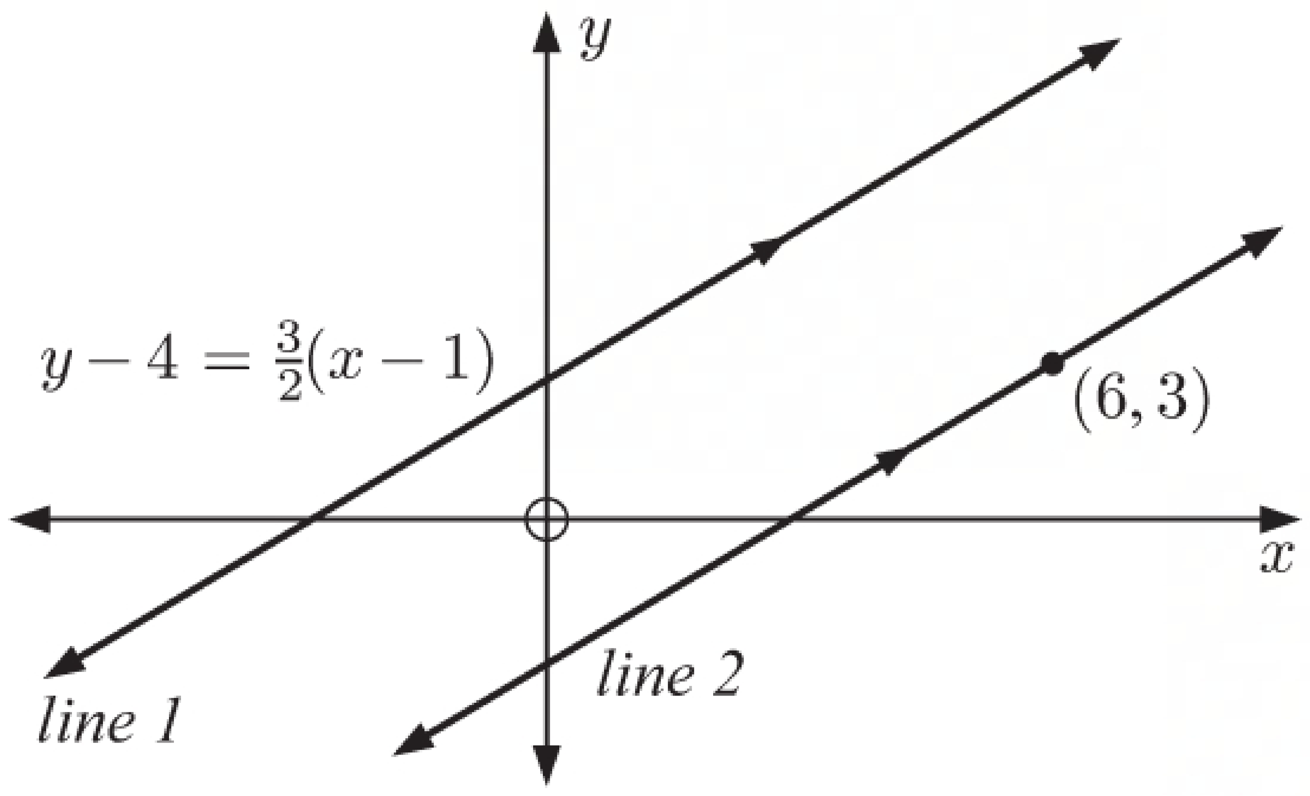
$$y = -\tfrac{1}{3}x + 4$$
$$\therefore\ 3y = -x + 12$$
$$\therefore\ x + 3y - 12 = 0$$

3 **a** *Line 2* is parallel to $y - 4 = \frac{3}{2}(x-1)$, which has gradient $\frac{3}{2}$.

$\therefore$ *line 2* has gradient $\frac{3}{2}$ and passes through $(6, 3)$.

$\therefore$ *line 2* has equation $y - 3 = \frac{3}{2}(x - 6)$

$$\therefore\ y - 3 = \tfrac{3}{2}x - 9$$
$$\therefore\ y = \tfrac{3}{2}x - 6$$
$$\therefore\ 2y = 3x - 12$$
$$\therefore\ 3x - 2y = 12$$



b When $y = 0$, $3x - 2(0) = 12$

$$\therefore\ 3x = 12$$
$$\therefore\ x = 4$$

$\therefore$ the x-intercept of *line 2* is 4.

4 **a** When $x = 5$, we have

$$y = -5 + 3$$
$$= -2 \quad \checkmark$$

So, $(5, -2)$ does lie on the line.

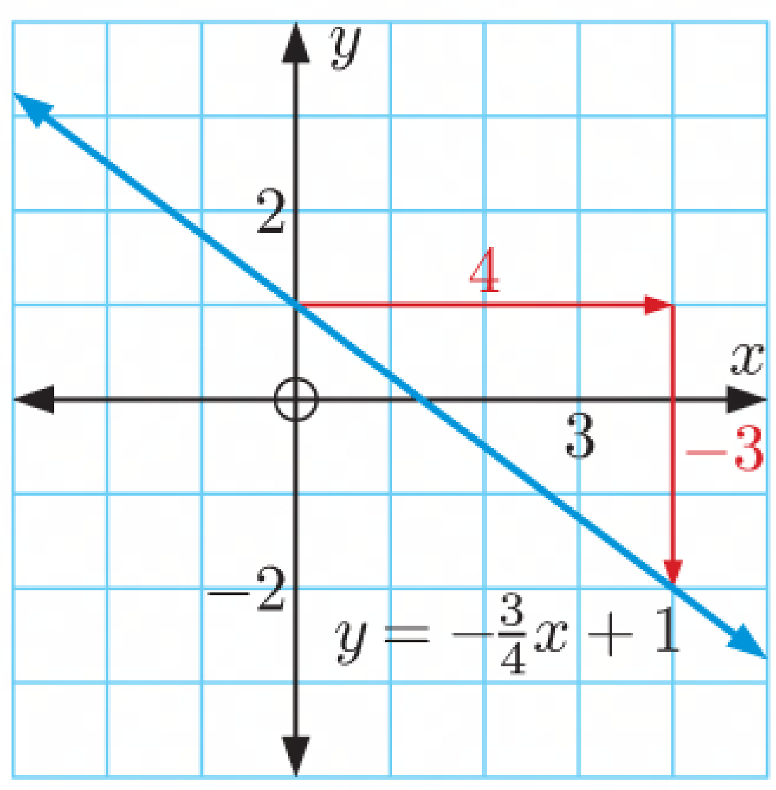
b When $x = -3$, we have

$$3(-3) + 8y = -5$$
$$\therefore\ -9 + 8y = -5$$
$$\therefore\ 8y = 4$$
$$\therefore\ y = \tfrac{1}{2} \quad \checkmark$$

So, $(-3, \frac{1}{2})$ does lie on the line.

5 **a** For $y = -\frac{3}{4}x + 1$:

- the y-intercept is $c = 1$
- the gradient is $m = -\frac{3}{4} = \frac{-3}{4}$



b For $3x - 4y = 72$:

When $x = 0$, $-4y = 72$

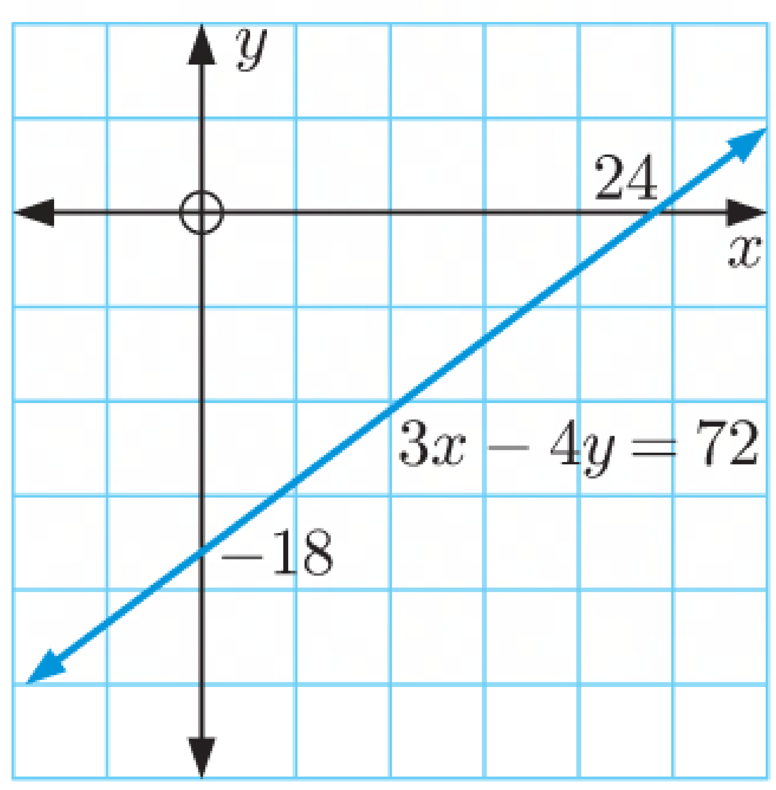
$$\therefore\ y = -18$$

So, the y-intercept is -18.

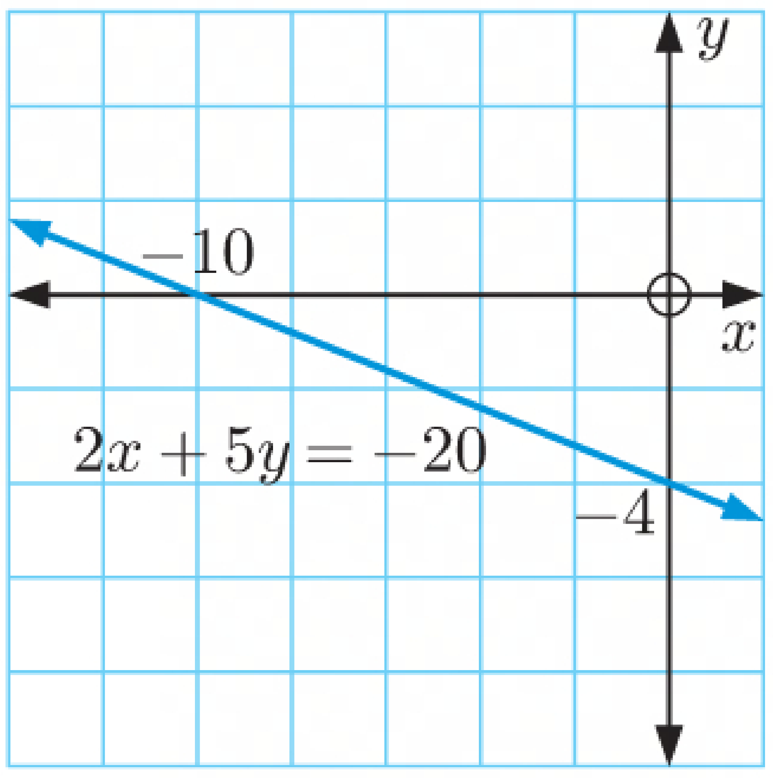
When $y = 0$, $3x = 72$

$$\therefore\ x = 24$$

So, the x-intercept is 24.



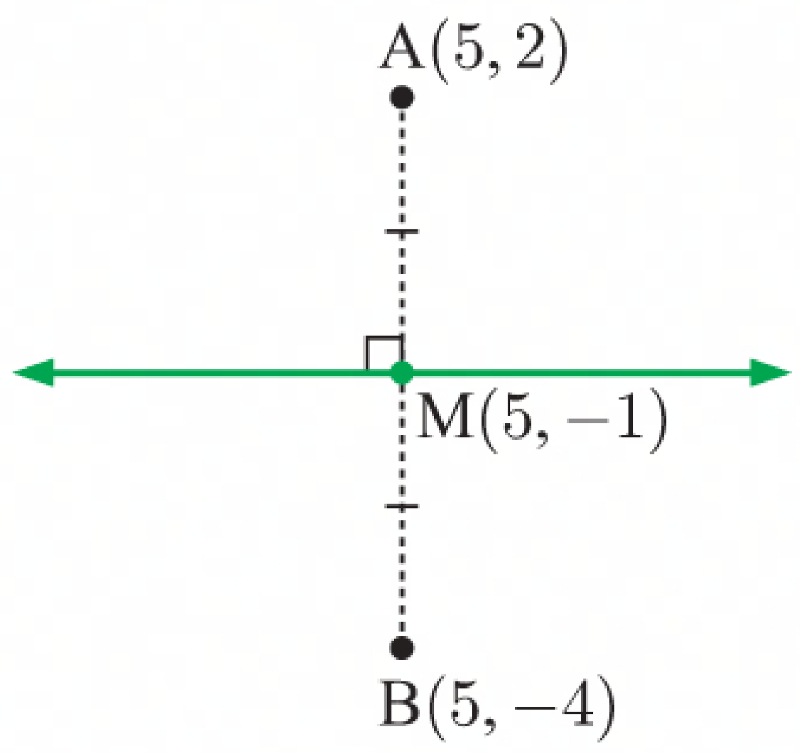
c For $2x + 5y = -20$:
When $x = 0$, $5y = -20$
$\therefore\ y = -4$
So, the y-intercept is -4.
When $y = 0$, $2x = -20$
$\therefore\ x = -10$
So, the x-intercept is -10.



6 **a** The midpoint M of [AB] is $\left(\frac{5+5}{2}, \frac{2+(-4)}{2}\right)$ or $(5, -1)$.

The gradient of [AB] is $\frac{-4-2}{5-5} = \frac{-6}{0}$ which is undefined.

So, [AB] is a vertical line, and hence the perpendicular bisector of [AB] is a horizontal line through $(5, -1)$.
$\therefore$ the equation of the perpendicular bisector is $y = -1$.

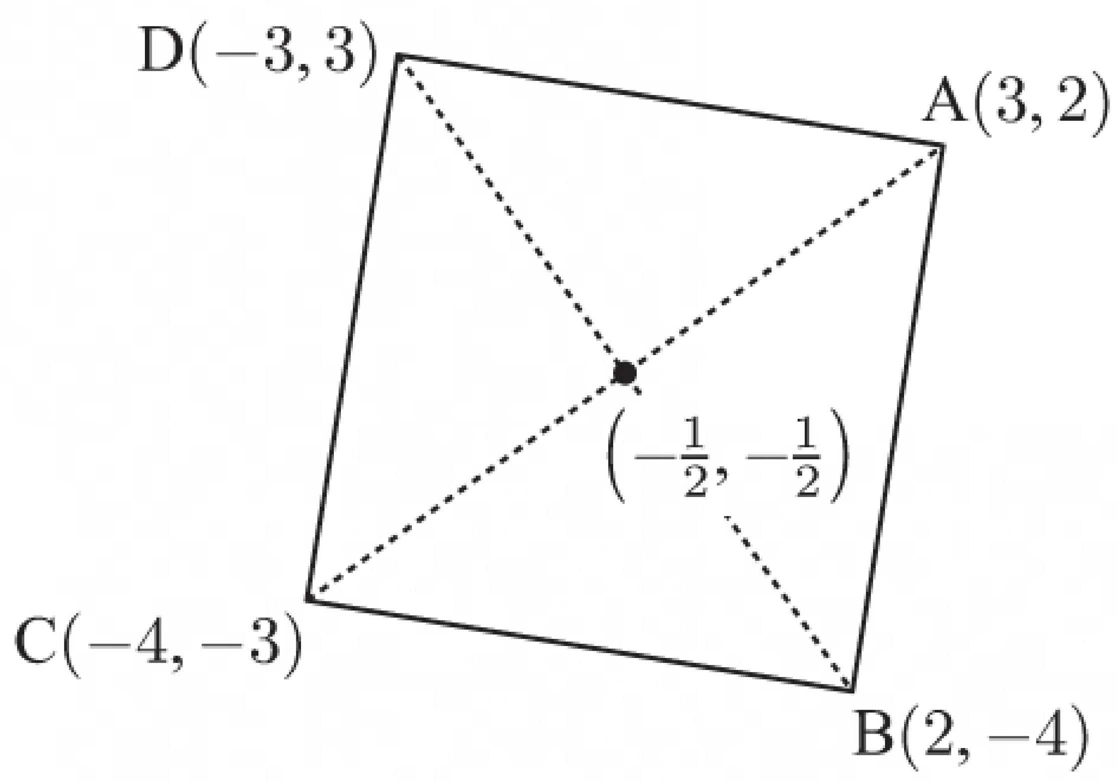


b The midpoint M of [AB] is $\left(\frac{8+2}{2}, \frac{1+5}{2}\right)$ or $(5, 3)$.

The gradient of [AB] is $\frac{5-1}{2-8} = \frac{4}{-6} = -\frac{2}{3}$

$\therefore$ the gradient of the perpendicular bisector is $\frac{3}{2}$.
$\therefore$ the equation of the perpendicular bisector is $3x - 2y = 3(5) - 2(3)$
which is $3x - 2y = 9$.

7 **a** **i** The midpoint of [AC] is $\left(\frac{3+(-4)}{2}, \frac{2+(-3)}{2}\right)$
or $\left(-\frac{1}{2}, -\frac{1}{2}\right)$.
The gradient of [AC] is $\frac{-3-2}{-4-3} = \frac{-5}{-7} = \frac{5}{7}$
$\therefore$ the gradient of the perpendicular bisector is $-\frac{7}{5}$.
$\therefore$ the equation of the perpendicular
bisector is $7x + 5y = 7\left(-\frac{1}{2}\right) + 5\left(-\frac{1}{2}\right)$
which is $7x + 5y = -6$.



ii The midpoint of [BD] is $\left(\frac{2+(-3)}{2}, \frac{-4+3}{2}\right)$ or $\left(-\frac{1}{2}, -\frac{1}{2}\right)$.

The gradient of [BD] is $\frac{3-(-4)}{-3-2} = \frac{7}{-5} = -\frac{7}{5}$

$\therefore$ the gradient of the perpendicular bisector is $\frac{5}{7}$.
$\therefore$ the equation of the perpendicular bisector is $5x - 7y = 5\left(-\frac{1}{2}\right) - 7\left(-\frac{1}{2}\right)$
which is $5x - 7y = 1$.

b $\text{AC} = \sqrt{(-4-3)^2+(-3-2)^2}$
$= \sqrt{(-7)^2+(-5)^2}$
$= \sqrt{49+25}$
$= \sqrt{74}$ units

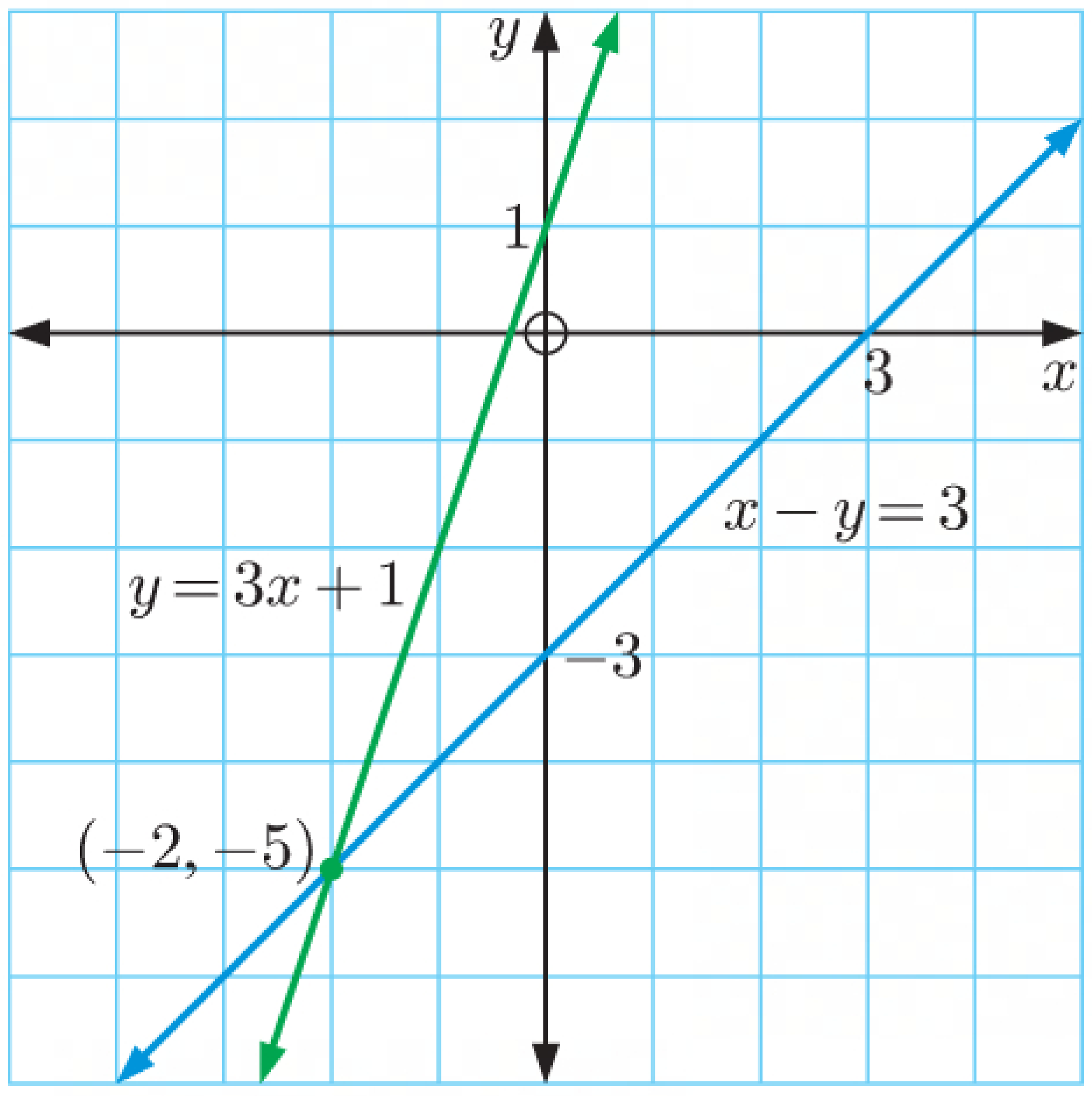
$\text{BD} = \sqrt{(-3-2)^2+(3-(-4))^2}$
$= \sqrt{(-5)^2+7^2}$
$= \sqrt{25+49}$
$= \sqrt{74}$ units

Now, [AC] and [BD] both have midpoint $\left(-\frac{1}{2}, -\frac{1}{2}\right)$, and the gradients of [AC] and [BD] are negative reciprocals of each other.

$\therefore$ [AC] and [BD] are perpendicular bisectors of each other, and equal in length.

$\therefore$ ABCD is a square with diagonals [AC] and [BD].

8 **a**



We draw the graphs of $y = 3x + 1$ and $x - y = 3$ on the same set of axes.

The graphs meet at the point $(-2, -5)$.

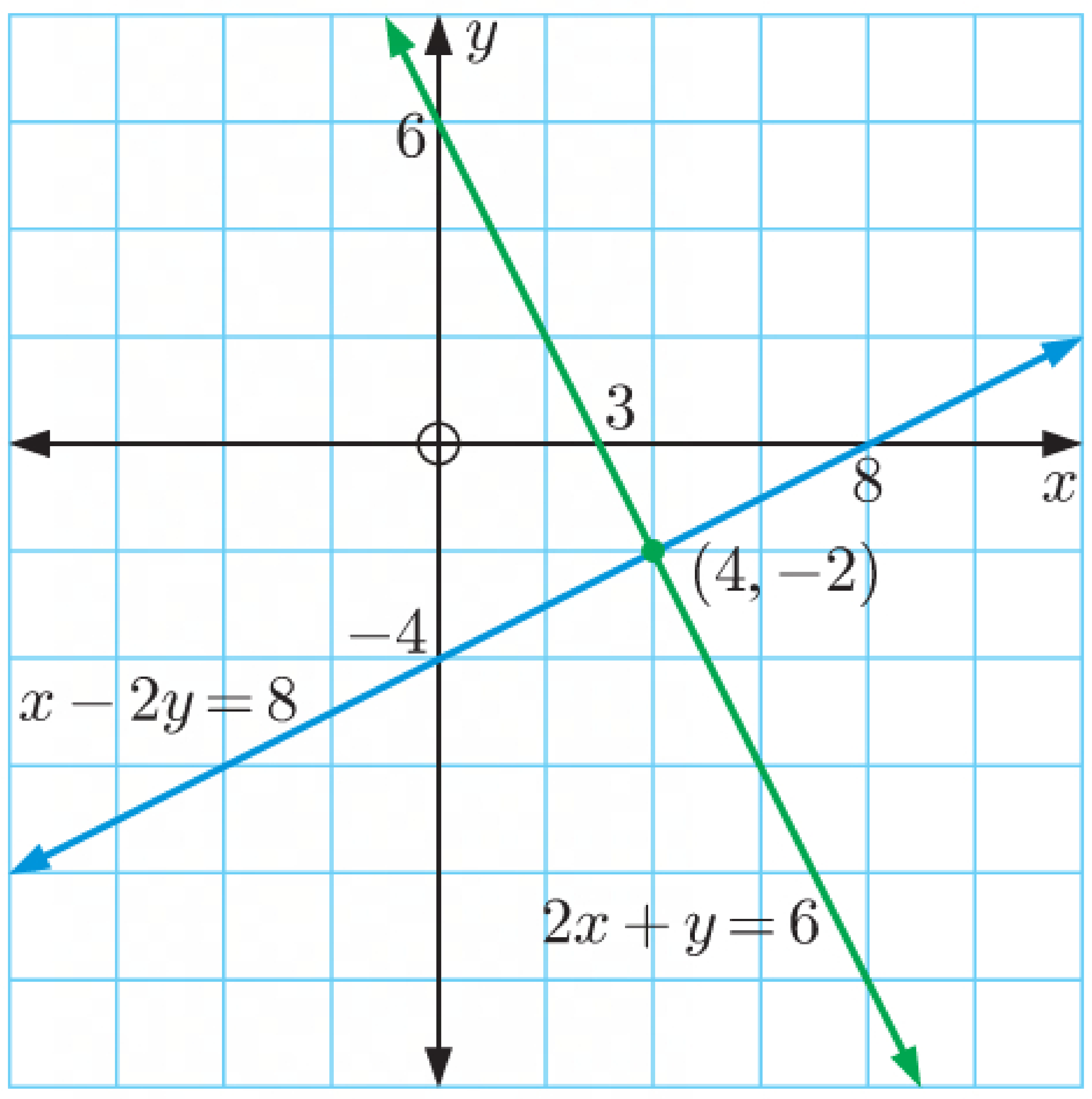
$\therefore$ the solution is $x = -2$, $y = -5$.

Check:

Substituting these values into:

- $y = 3x + 1$ gives
  $-5 = 3(-2) + 1 = -6 + 1$ ✓
- $x - y = 3$ gives
  $-2 - (-5) = -2 + 5 = 3$ ✓

b



We draw the graphs of $2x + y = 6$ and $x - 2y = 8$ on the same set of axes.

The graphs meet at the point $(4, -2)$.

$\therefore$ the solution is $x = 4$, $y = -2$.

Check:

Substituting these values into:

- $2x + y = 6$ gives
  $2(4) + (-2) = 8 - 2 = 6$ ✓
- $x - 2y = 8$ gives
  $4 - 2(-2) = 4 + 4 = 8$ ✓

9 **a** $y = 3x + 4$ .... (1)
$2x - y = -5$ .... (2)

Substituting (1) into (2) gives $2x - (3x + 4) = -5$
$\therefore\ 2x - 3x - 4 = -5$
$\therefore\ -x = -1$
$\therefore\ x = 1$

Substituting $x = 1$ into (1) gives $y = 3(1) + 4$
$\therefore\ y = 7$

The solution is $x = 1,\ y = 7$.

Check: (1) $7 = 3(1) + 4 = 3 + 4$ ✓
(2) $2(1) - 7 = 2 - 7 = -5$ ✓

b $x = 2y - 5$ .... (1)
$3x + 4y = 5$ .... (2)

Substituting (1) into (2) gives $3(2y - 5) + 4y = 5$
$\therefore\ 6y - 15 + 4y = 5$
$\therefore\ 10y = 20$
$\therefore\ y = 2$

Substituting $y = 2$ into (1) gives $x = 2(2) - 5$
$\therefore\ x = -1$

The solution is $x = -1,\ y = 2$.

Check: (1) $-1 = 2(2) - 5 = 4 - 5$ ✓
(2) $3(-1) + 4(2) = -3 + 8 = 5$ ✓

10 **a** $\begin{cases} 3x + 2y = 7 \\ 5x - 2y = 17 \end{cases}$

The coefficients of y are the same size but opposite in sign.
We *add* the LHSs and the RHSs to get an equation which contains x only.

$$\begin{array}{rl} 3x + 2y = 7 & \text{.... (1)} \\ \underline{5x - 2y = 17} & \text{.... (2)} \end{array}$$

Adding, $8x \quad = 24$
$\therefore\ x = 3$

Substituting $x = 3$ into (1) gives $3(3) + 2y = 7$
$\therefore\ 9 + 2y = 7$
$\therefore\ 2y = -2$
$\therefore\ y = -1$

The solution is $x = 3,\ y = -1$.

Check: In (2): $5(3) - 2(-1) = 15 + 2 = 17$ ✓

b $2x + 7y = 13 \quad .... (1)$
$-4x + 3y = 25 \quad .... (2)$

To make the coefficients of x the same size but opposite in sign, we multiply (1) by 2.

$$\therefore \quad 4x + 14y = 26 \quad \{(1) \times 2\}$$
$$-4x + \ \ 3y = 25$$

Adding, $\quad 17y = 51$
$$\therefore \quad y = 3$$

Substituting $y = 3$ into (1) gives $2x + 7(3) = 13$
$$\therefore \quad 2x + 21 = 13$$
$$\therefore \quad 2x = -8$$
$$\therefore \quad x = -4$$

The solution is $x = -4, \ y = 3$.

Check: In (2): $-4(-4) + 3(3) = 16 + 9 = 25$ ✓

11 $x = k - 2y \quad .... (1)$
$y = -\frac{1}{2}x + 2 \quad .... (2)$

a The gradient of the line $x = k - 2y$ is $-\frac{1}{2}$.
The gradient of the line $y = -\frac{1}{2}x + 2$ is $-\frac{1}{2}$.

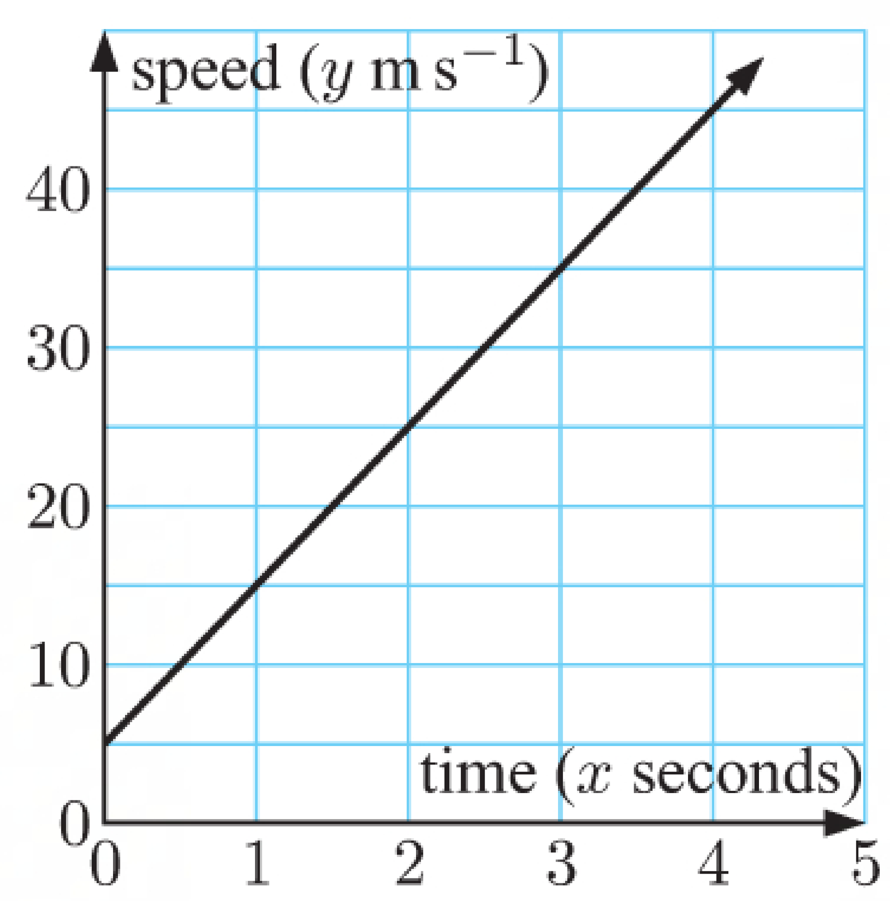
b Substituting (1) into (2) gives $y = -\frac{1}{2}(k - 2y) + 2$
$$= -\frac{1}{2}k + y + 2$$
$$\therefore \quad \frac{1}{2}k = 2$$
$$\therefore \quad k = 4$$

If $k = 4$ then equation (1) becomes $x = 4 - 2y$ which can be rearranged to $y = -\frac{1}{2}x + 2$.
The equations will therefore be identical, so the lines are coincident.
If $k \neq 4$ then the lines will still have the same gradient but they will not be identical.

i There are no solutions when $k \neq 4$ as the lines will be parallel but not coincident.
ii There are infinitely many solutions when $k = 4$ as the lines will be coincident.

REVIEW SET 1B

1 **a** The graph passes through $(0, 5)$ and $(1, 15)$, so the gradient is $\dfrac{15-5}{1-0} = 10$. This means that the speed of the pebble increases by 10 m s^{-1} each second.
The y-intercept is 5. This means that the initial speed was 5 m s^{-1}.



b The gradient is 10 and the y-intercept is 5, so the equation is $y = 10x + 5$.

c When $x = 8$, $y = 10(8) + 5$
$$= 85$$

The speed of the pebble after 8 seconds is 85 m s^{-1}.

2 **a** The line is parallel to $y = 3x - 8$, which has gradient 3.

$\therefore$ the line has equation $y = 3x + c$.

Substituting $x = 2$, $y = 7$, we get $7 = 3(2) + c$

$\therefore\ c = 1$

$\therefore$ the line has equation $y = 3x + 1$.

b $2x + 5y = 7$

or $y = \frac{7}{5} - \frac{2}{5}x$ has gradient $-\frac{2}{5}$.

$\therefore$ the line perpendicular to $2x + 5y = 7$ has gradient $\frac{5}{2}$.

Since the line has gradient $\frac{5}{2}$, the general form of its equation is $5x - 2y = d$.

Using the point $(-1, -1)$, the equation is $5x - 2y = 5(-1) - 2(-1)$

which is $5x - 2y = -3$.

3 **a** Substituting $x = 2$, $y = k$ into the equation gives $k = 5(2) - 3$

$\therefore\ k = 7$

b Substituting $x = \frac{1}{2}$, $y = -\frac{3}{2}$ into the equation gives $5\left(\frac{1}{2}\right) + 9\left(-\frac{3}{2}\right) = k$

$\therefore\ k = \frac{5}{2} - \frac{27}{2}$

$\therefore\ k = -11$

4 **a** For $3x + 2y = 30$:

When $x = 0$, $2y = 30$

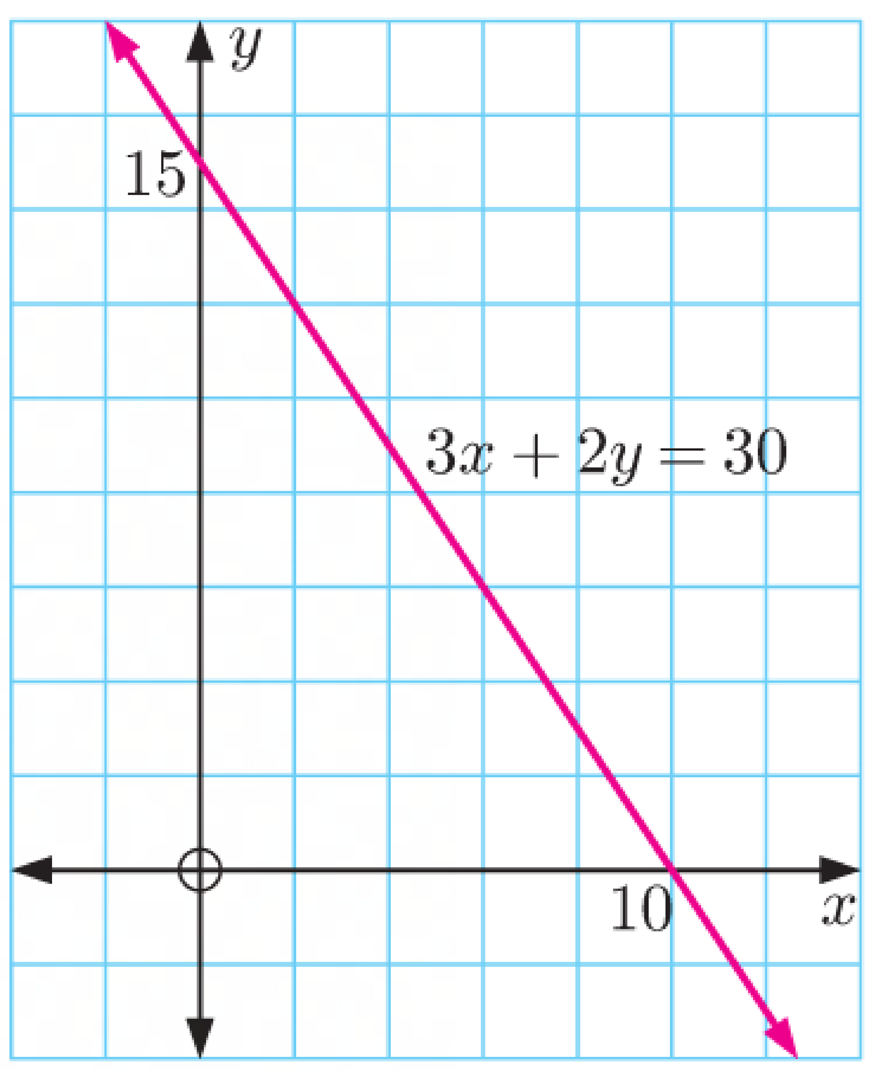
$\therefore\ y = 15$

So, the y-intercept is 15.

When $y = 0$, $3x = 30$

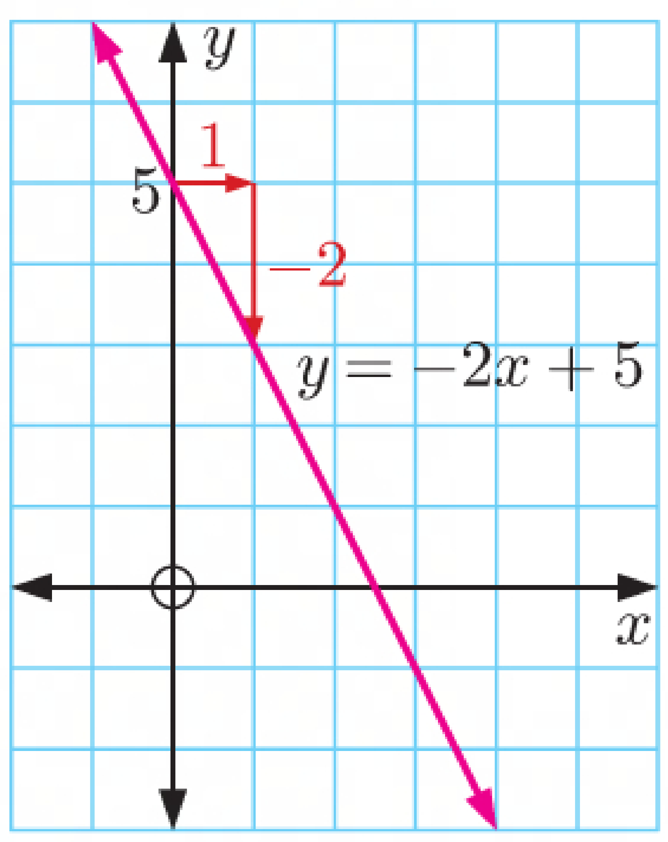
$\therefore\ x = 10$

So, the x-intercept is 10.



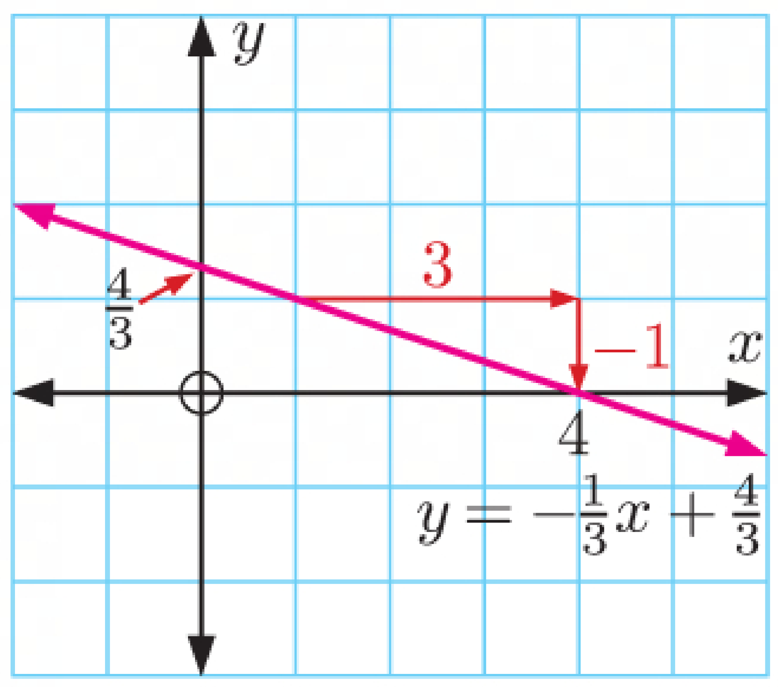
b For $y = -2x + 5$:

- the y-intercept is $c = 5$
- the gradient is $m = -2 = \frac{-2}{1}$



c For $y = -\frac{1}{3}x + \frac{4}{3}$:

- the y-intercept is $c = \frac{4}{3}$
- the gradient is $m = -\frac{1}{3} = \frac{-1}{3}$



5 **a** $y = \frac{2}{3}x - \frac{8}{3}$ has gradient $m = \frac{2}{3}$

b **i** When $x = -2$, we have

$$\begin{aligned} y &= \tfrac{2}{3}(-2) - \tfrac{8}{3} \\ &= -\tfrac{4}{3} - \tfrac{8}{3} \\ &= -4 \quad \checkmark \end{aligned}$$

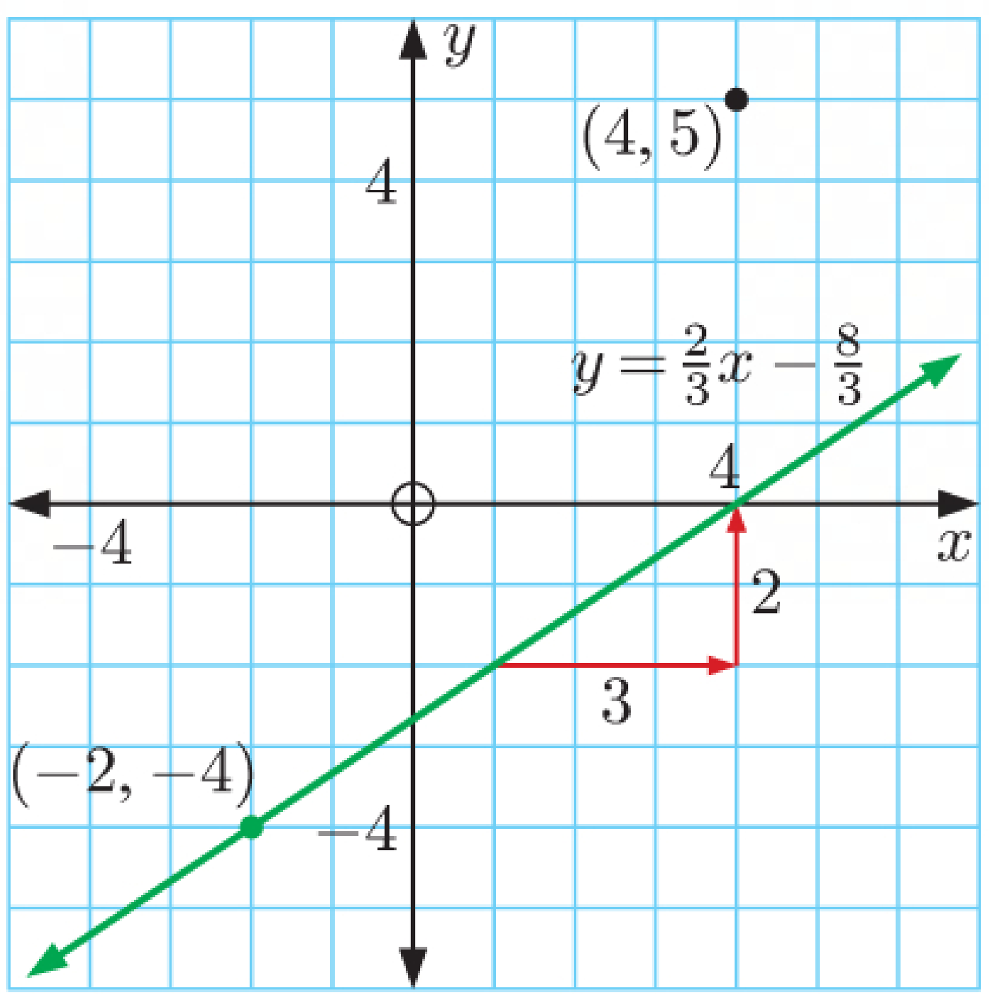
So, $(-2, -4)$ lies on the line.

ii When $x = 4$, we have

$$\begin{aligned} y &= \tfrac{2}{3}(4) - \tfrac{8}{3} \\ &= \tfrac{8}{3} - \tfrac{8}{3} \\ &= 0 \quad \times \end{aligned}$$

So, $(4, 5)$ does *not* lie on the line.

c

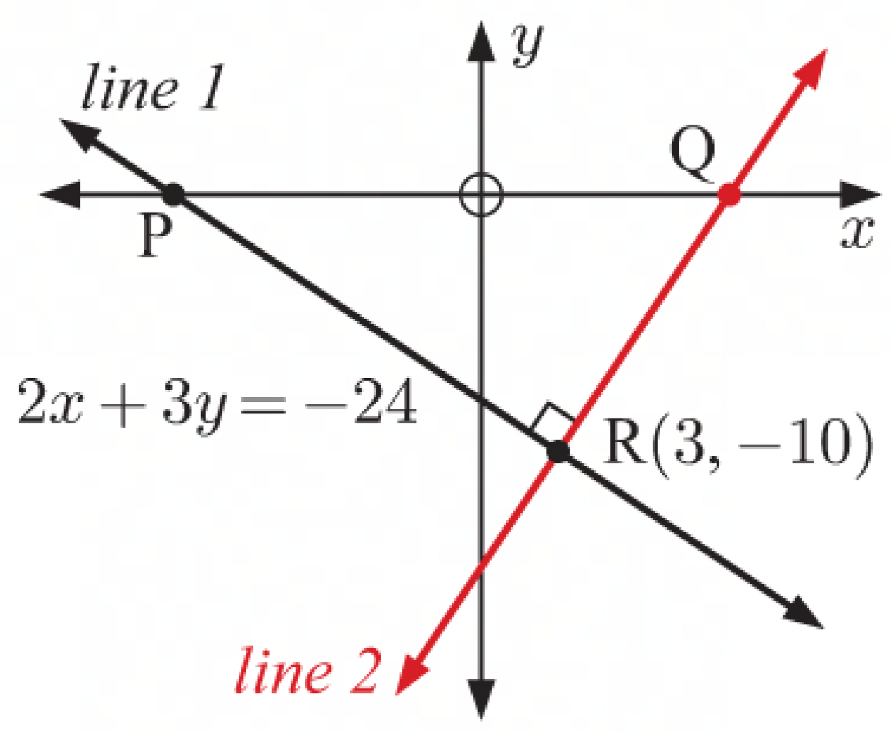


6 *Line 2* is perpendicular to $2x + 3y = -24$ which has gradient $-\frac{2}{3}$.

$\therefore$ *line 2* has gradient $\frac{3}{2}$ and passes through $R(3, -10)$.

$\therefore$ the equation of *line 2* is

$$\begin{aligned} y - (-10) &= \tfrac{3}{2}(x - 3) \\ \therefore\ y + 10 &= \tfrac{3}{2}x - \tfrac{9}{2} \\ \therefore\ \tfrac{3}{2}x - y &= \tfrac{29}{2} \\ \therefore\ 3x - 2y &= 29 \end{aligned}$$



Line 1 meets the x-axis when $y = 0$.

$$\begin{aligned} \therefore\ 2x + 3(0) &= -24 \\ \therefore\ 2x &= -24 \\ \therefore\ x &= -12 \end{aligned}$$

So, *line 1* meets the x-axis at $P(-12, 0)$.

Line 2 meets the x-axis when $y = 0$.

$$\begin{aligned} \therefore\ 3x - 2(0) &= 29 \\ \therefore\ 3x &= 29 \\ \therefore\ x &= \tfrac{29}{3} \end{aligned}$$

So, *line 2* meets the x-axis at $Q(\frac{29}{3}, 0)$.

The base of triangle PQR is $\frac{29}{3} - (-12) = \frac{65}{3}$ units, and the height of triangle PQR is 10 units.

$\therefore$ area of triangle PQR $= \frac{1}{2} \times \frac{65}{3} \times 10$

$= \frac{325}{3} = 108\frac{1}{3}$ units^2

7 **a** **i** $x - 5y + 6 = 0$

$\therefore \; 5y = x + 6$

$\therefore \; y = \frac{1}{5}x + \frac{6}{5}$ which has gradient $\frac{1}{5}$.

ii $x - 5y + 6 = 0$ has gradient $\frac{1}{5}$, so its perpendicular bisector has gradient -5.

b The perpendicular bisector has gradient -5, so the general form of its equation is $5x + y = d$.

Using the point $(4, 2)$, the equation is $5x + y = 5(4) + 2$

which is $5x + y = 22$.

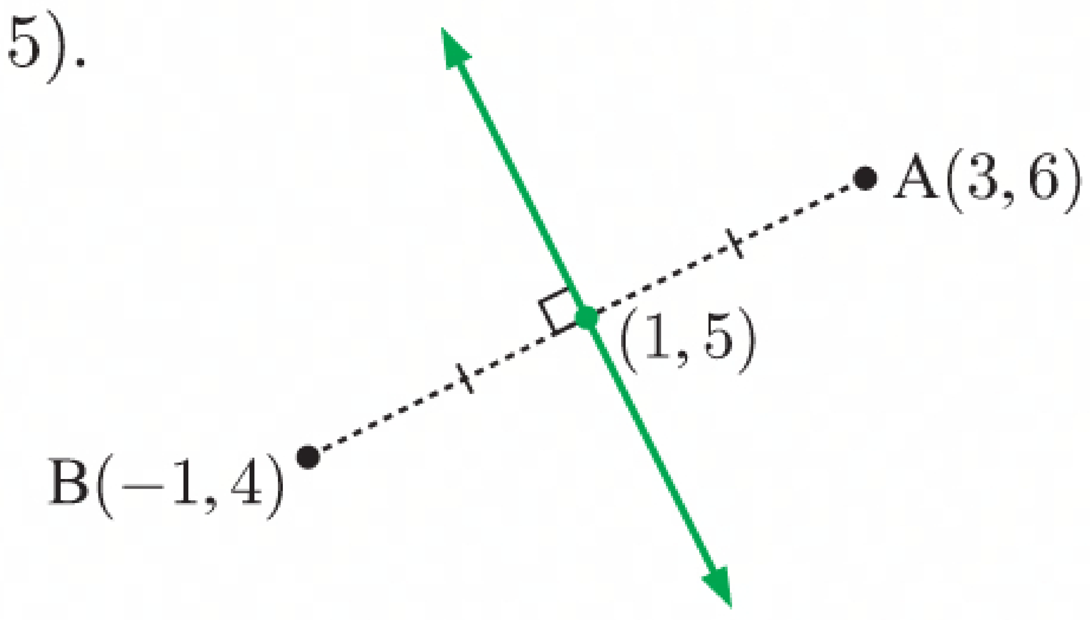
8 **a** **i** The midpoint of [AB] is $\left(\frac{3+(-1)}{2}, \frac{6+4}{2}\right)$ or $(1, 5)$.

The gradient of [AB] is $\frac{4-6}{-1-3} = \frac{-2}{-4} = \frac{1}{2}$

$\therefore$ the gradient of the perpendicular bisector is -2.

$\therefore$ the equation of the perpendicular bisector is $2x + y = 2(1) + 5$

which is $2x + y = 7$.



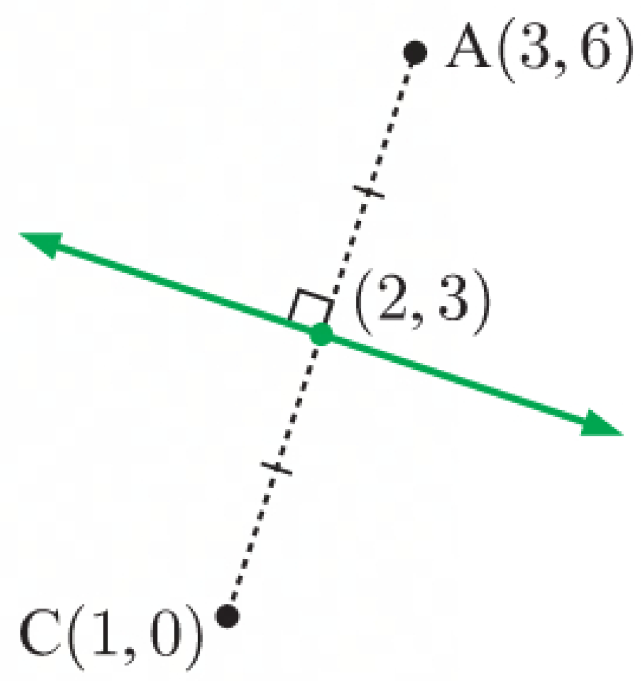
ii The midpoint of [AC] is $\left(\frac{3+1}{2}, \frac{6+0}{2}\right)$ or $(2, 3)$.

The gradient of [AC] is $\frac{0-6}{1-3} = \frac{-6}{-2} = 3$

$\therefore$ the gradient of the perpendicular bisector is $-\frac{1}{3}$.

$\therefore$ the equation of the perpendicular bisector is $x + 3y = 2 + 3(3)$

which is $x + 3y = 11$.



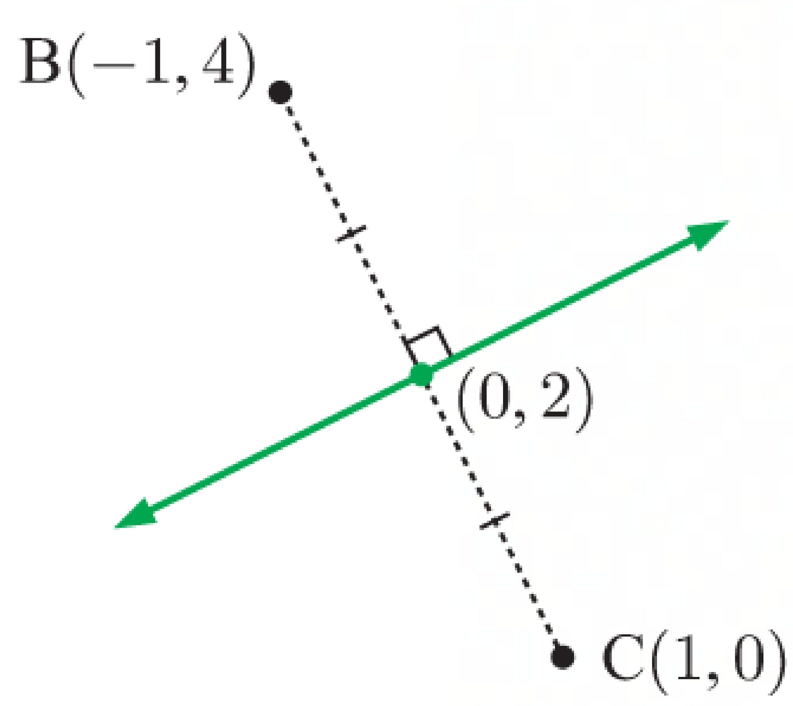
iii The midpoint of [BC] is $\left(\frac{-1+1}{2}, \frac{4+0}{2}\right)$ or $(0, 2)$.

The gradient of [BC] is $\frac{0-4}{1-(-1)} = \frac{-4}{2} = -2$

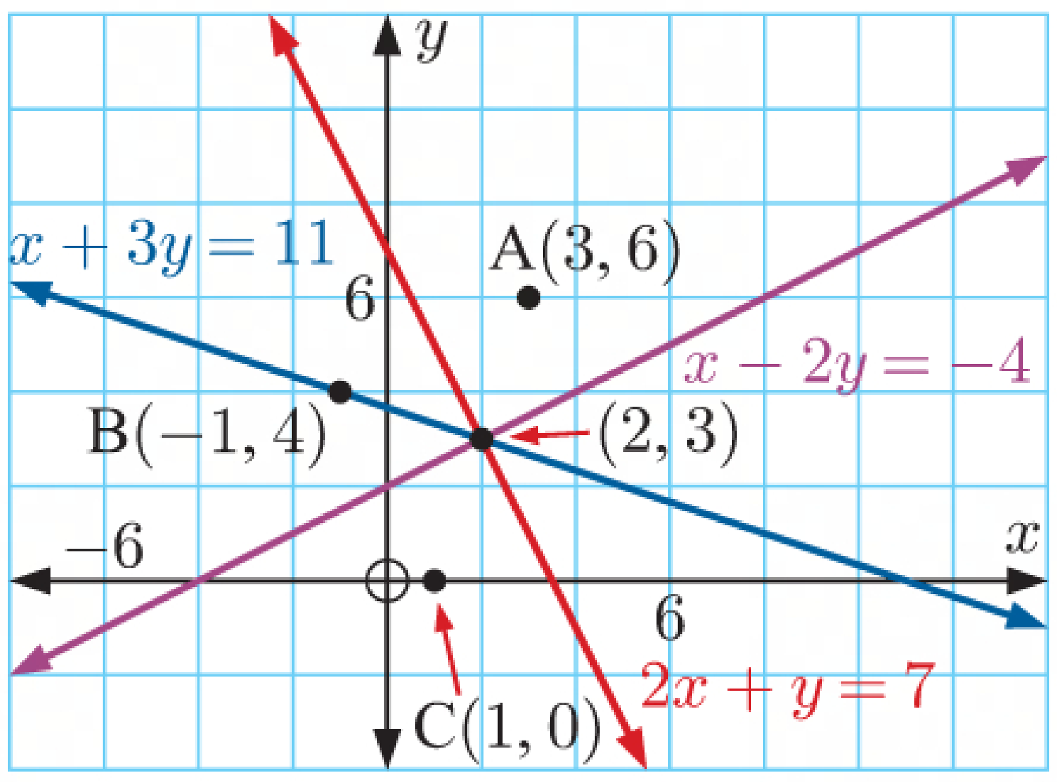
$\therefore$ the gradient of the perpendicular bisector is $\frac{1}{2}$.

$\therefore$ the equation of the perpendicular bisector is $x - 2y = 0 - 2(2)$

which is $x - 2y = -4$.



b



All three perpendicular bisectors intersect at $(2, 3)$.
A, B, and C are all equidistant from this point.
B lies on the perpendicular bisector of [AC], which means $AB = BC$.
So, triangle ABC is isosceles.

9 **a** $y = 6x + 2$ .... (1)
$3x - 2y = -7$ .... (2)

Substituting (1) into (2) gives $3x - 2(6x + 2) = -7$

$\therefore\ 3x - 12x - 4 = -7$

$\therefore\ -9x = -3$

$\therefore\ x = \frac{1}{3}$

Substituting $x = \frac{1}{3}$ into (1) gives $y = 6(\frac{1}{3}) + 2$

$\therefore\ y = 4$

The solution is $x = \frac{1}{3}$, $y = 4$.

Check: (1) $4 = 6(\frac{1}{3}) + 2 = 2 + 2$ ✓

(2) $3(\frac{1}{3}) - 2(4) = 1 - 8 = -7$ ✓

b $y = \frac{1}{2}x + 5$ .... (1)
$4x + 3y = 4$ .... (2)

Substituting (1) into (2) gives $4x + 3(\frac{1}{2}x + 5) = 4$

$\therefore\ 4x + \frac{3}{2}x + 15 = 4$

$\therefore\ \frac{11}{2}x = -11$

$\therefore\ x = -2$

Substituting $x = -2$ into (1) gives $y = \frac{1}{2}(-2) + 5$

$\therefore\ y = 4$

The solution is $x = -2$, $y = 4$.

Check: (1) $4 = \frac{1}{2}(-2) + 5 = -1 + 5$ ✓

(2) $4(-2) + 3(4) = -8 + 12 = 4$ ✓

10 **a** $3x + 2y = 8$ .... (1)
$5x - 4y = 17$ .... (2)

To make the coefficients of y the same size but opposite in sign, we multiply (1) by 2.

$$\begin{array}{lrl} & \therefore\ 6x + 4y = 16 & \{(1) \times 2\} \\ & 5x - 4y = 17 & \\ \hline \text{Adding,} & 11x \quad\quad = 33 & \\ & \therefore\ x = 3 & \end{array}$$

Substituting $x = 3$ into (1) gives $3(3) + 2y = 8$
$\therefore\ 9 + 2y = 8$
$\therefore\ 2y = -1$
$\therefore\ y = -\frac{1}{2}$

The solution is $x = 3$, $y = -\frac{1}{2}$.

Check: In (2): $5(3) - 4(-\frac{1}{2}) = 15 + 2 = 17$ ✓

b $4x + 6y = -15$ .... (1)
$3x - 5y = 22$ .... (2)

To make the coefficients of y the same size but opposite in sign, we multiply (1) by 5 and (2) by 6.

$$\begin{array}{lrl} & \therefore\ 20x + 30y = -75 & \{(1) \times 5\} \\ & 18x - 30y = 132 & \{(2) \times 6\} \\ \hline \text{Adding,} & 38x \quad\quad = 57 & \\ & \therefore\ x = 1\frac{1}{2} & \end{array}$$

Substituting $x = 1\frac{1}{2}$ into (1) gives $4(1\frac{1}{2}) + 6y = -15$
$\therefore\ 6 + 6y = -15$
$\therefore\ 6y = -21$
$\therefore\ y = -3\frac{1}{2}$

The solution is $x = 1\frac{1}{2}$, $y = -3\frac{1}{2}$.

Check: In (2): $3(1\frac{1}{2}) - 5(-3\frac{1}{2}) = 4\frac{1}{2} + 17\frac{1}{2} = 22$ ✓

11 $\begin{cases} ax + 4y = 6 \\ x - 2y = -2 \end{cases}$

a The gradient of $ax + 4y = 6$ is $-\frac{a}{4}$.

The gradient of $x - 2y = -2$ is $-\frac{1}{-2} = \frac{1}{2}$.

b If $-\frac{a}{4} = \frac{1}{2}$, then $a = -2$, and the lines will be parallel as their gradients are equal.

The line $ax + 4y = 6$ thus becomes $-2x + 4y = 6$, which can be rewritten as $x - 2y = -3$.
The equations are therefore not equal, so the lines are parallel but not coincident.
$\therefore$ there are no solutions.

c Suppose $a \neq -2$.

$ax + 4y = 6$ .... (1)

$x - 2y = -2$ .... (2)

To make the coefficients of y the same size but opposite in sign, we multiply (2) by 2.

$$\begin{array}{lrl} & \therefore \quad ax + 4y &= 6 \\ & 2x - 4y &= -4 \qquad \{(2) \times 2\} \\ \hline \text{Adding,} & (a+2)x &= 2 \end{array}$$

$$\therefore \quad x = \frac{2}{a+2} \qquad \{\text{as } a \neq -2\}$$

Substituting $x = \dfrac{2}{a+2}$ into (2) gives $\dfrac{2}{a+2} - 2y = -2$

$$\begin{aligned} \therefore \quad 2y &= \frac{2}{a+2} + 2 \\ &= \frac{2 + 2(a+2)}{a+2} \\ &= \frac{2a+6}{a+2} \\ &= \frac{2(a+3)}{a+2} \\ \therefore \quad y &= \frac{a+3}{a+2} \end{aligned}$$

So, the solution is $x = \dfrac{2}{a+2}$, $y = \dfrac{a+3}{a+2}$, $a \neq -2$.

Check: In (1): $a\left(\dfrac{2}{a+2}\right) + 4\left(\dfrac{a+3}{a+2}\right) = \dfrac{2a + 4a + 12}{a+2} = \dfrac{6a+12}{a+2} = \dfrac{6(a+2)}{a+2} = 6$ ✓

Chapter 2

SETS AND VENN DIAGRAMS

EXERCISE 2A

1 **a** $A = \{\text{factors of } 8\}$
$= \{1, 2, 4, 8\}$
$n(A) = 4$

b $A = \{\text{composite numbers less than } 20\}$
$= \{4, 6, 8, 9, 10, 12, 14, 15, 16, 18\}$
$n(A) = 10$

c $A = \{\text{letters in the word AARDVARK}\}$
$= \{\text{A, R, D, V, K}\}$
$n(A) = 5$

d $A = \{\text{prime numbers between } 40 \text{ and } 50\}$
$= \{41, 43, 47\}$
$n(A) = 3$

2 **a** $\{\text{factors of } 10\} = \{1, 2, 5, 10\}$
There are 4 elements in this set.
$\therefore$ this is a finite set.

b $\{\text{multiples of } 10\} = \{...., -20, -10, 0, 10, 20, ...\}$
This set has an endless number of elements.
$\therefore$ this is an infinite set.

c $\{\text{perfect squares}\} = \{1, 4, 9, 16, 25, ...\}$
This set has an endless number of elements.
$\therefore$ this is an infinite set.

3 $S = \{1, 2, 4, 5, 9, 12\}$ and $T = \{2, 5, 9\}$

a **i** $n(S) = 6$ **ii** $n(T) = 3$

b **i** 4 is an element of S.
$\therefore$ $4 \in S$ is a true statement.

ii 4 is not an element of T.
$\therefore$ $4 \in T$ is a false statement.

iii 1 is not an element of T.
$\therefore$ $1 \notin T$ is a true statement.

iv Every element of T is also an element of S.
$\therefore$ $T \subseteq S$ is a true statement.

v $T \subseteq S$ {from **iv**}
However, 1, 4, and 12 are all elements of S but not elements of T, so $T \neq S$.
$\therefore$ $T \subset S$ is a true statement.

4 $S = \{1, 2\}$ and $T = \{1, 2, 3\}$

a subsets of S: $\varnothing$, $\{1\}$, $\{2\}$, $\{1, 2\}$
subsets of T: $\varnothing$, $\{1\}$, $\{2\}$, $\{3\}$, $\{1, 2\}$, $\{1, 3\}$, $\{2, 3\}$, $\{1, 2, 3\}$

b Yes, every subset of S is also a subset of T.

c There are 4 subsets of S and 8 subsets of T.

$\therefore$ the fraction of the subsets of T which are also subsets of $S = \frac{4}{8}$

$= \frac{1}{2}$

5 Each of the four elements of $\{p, q, r, s\}$ can be *included* or *not included* in a subset.

$\therefore$ the set has $2 \times 2 \times 2 \times 2 = 16$ subsets.

6 $A = \{2, 4, 6, x\}$ and $B = \{2, 3, 5, 6, x+1\}$

$A \subseteq B$ which means every element of A must also be an element of B. The elements 2 and 6 are in both A and B, so the element 4 in A must be represented by $x+1$ in B.

$\therefore$ $4 = x + 1$

$\therefore$ $x = 3$

7 If $A \subseteq B$, then all elements of A are in B.

If $B \subseteq A$, then all elements of B are in A.

This is only possible if A and B contain exactly the same elements.

$\therefore$ $A = B$

EXERCISE 2B

1 **a** $A = \{6, 7, 9, 11, 12\}$ and $B = \{5, 8, 10, 13, 9\}$

i $A \cap B = \{9\}$ since 9 is the element common to both sets.

ii Every element which is in either A or B is in the union of A and B.

$\therefore$ $A \cup B = \{5, 6, 7, 8, 9, 10, 11, 12, 13\}$

b $A = \{1, 2, 3, 4\}$ and $B = \{5, 6, 7, 8\}$

i $A \cap B = \varnothing$ since there are no elements which are common to both sets.

ii Every element which is in either A or B is in the union of A and B.

$\therefore$ $A \cup B = \{1, 2, 3, 4, 5, 6, 7, 8\}$

c $A = \{1, 3, 5, 7\}$ and $B = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$

i $A \cap B = \{1, 3, 5, 7\}$ since 1, 3, 5, and 7 are the elements common to both sets.

ii Every element which is in either A or B is in the union of A and B.

$\therefore$ $A \cup B = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$

d $A = \{0, 3, 5, 8, 14\}$ and $B = \{1, 4, 5, 8, 11, 13\}$

i $A \cap B = \{5, 8\}$ since 5 and 8 are the elements common to both sets.

ii Every element which is in either A or B is in the union of A and B.

$\therefore$ $A \cup B = \{0, 1, 3, 4, 5, 8, 11, 13, 14\}$

2 **a** $A = \{3, 5, 7, 9\}$ and $B = \{2, 4, 6, 8\}$

Sets A and B have no elements in common.

$\therefore$ these sets are disjoint.

b $P = \{3, 5, 6, 7, 8, 10\}$ and $Q = \{4, 9, 10\}$

The element 10 is common to both sets.

$\therefore$ these sets are not disjoint.

3 **a** True, $R \cap S = \varnothing$ tells us that R and S have no elements in common, and hence are disjoint.

b True, every element of $A \cap B$ is an element of A, and every element of $A \cap B$ is an element of B.

$\therefore \quad n(A \cap B) \leqslant n(A) \quad$ and $\quad n(A \cap B) \leqslant n(B)$.

c True, if $A \cap B = A \cup B$ then there are no elements that are in only A or only B.

$\therefore \quad A = B$.

d False, consider $A = \{1, 2, 3, ...\}$ and $B = \{-1, -2, -3, ...\}$ which are infinite but $A \cap B = \varnothing$ which is finite.

4 If A and B are disjoint sets, and B and C are disjoint sets, then A and C are not necessarily disjoint sets.

Consider the sets $A = \{1, 2\}$, $B = \{3, 4\}$, and $C = \{1, 6\}$.

$A \cap B = \varnothing$ and $B \cap C = \varnothing$, but $A \cap C = \{1\}$, so A and C are not disjoint sets.

5 $n(A) = 8$ and $n(B) = 11$

a $n(A \cap B) = 0, 1, 2, 3, 4, 5, 6, 7,$ or 8

b $n(A \cup B) = 11, 12, 13, 14, 15, 16, 17, 18,$ or 19

6 Each element in $A \cup B$ must be in A or B, or both.

It is not possible that $n(A \cup B) > n(A) + n(B)$

$\therefore \quad n(A \cup B) \leqslant n(A) + n(B)$

EXERCISE 2C

1 $U = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$

a The complement of $A = \{2, 3, 6, 7, 8\}$ is the set of all elements of U that are not elements of A.

$\therefore \quad A' = \{1, 4, 5, 9\}$

b $P = \{\text{prime numbers}\}$
$\quad = \{2, 3, 5, 7\}$

$\{\text{composite numbers}\} = \{4, 6, 8, 9\}$

P' is the set of all elements of U that are not elements of P.

$\therefore \quad P' = \{1, 4, 6, 8, 9\} \neq \{\text{composite numbers}\}$ as 1 is neither prime nor composite.

2 $U = \{\text{whole numbers between 10 and 20 inclusive}\}$
$\quad = \{10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20\}$

a $A = \{\text{factors of } 120\}$
$\quad = \{10, 12, 15, 20\}$

b $B = \{\text{multiples of } 3\}$
$\quad = \{12, 15, 18\}$

c A' is the set of all elements of U that are not elements of A.

$\therefore \quad A' = \{11, 13, 14, 16, 17, 18, 19\}$

d B' is the set of all elements of U that are not elements of B.

$\therefore \quad B' = \{10, 11, 13, 14, 16, 17, 19, 20\}$

e $A \cap B = \{12, 15\}$ since 12 and 15 are the elements common to both sets.

f Every element which is in either A or B is in the union of A and B.

$\therefore \quad A \cup B = \{10, 12, 15, 18, 20\}$

g $A' \cap B = \{18\}$ since 18 is the element common to both sets.

h Every element which is in either A' or B is in the union of A' and B.
$\therefore \ A' \cup B = \{11, 12, 13, 14, 15, 16, 17, 18, 19\}$

i $A \cap B' = \{10, 20\}$ since 10 and 20 are the elements common to both sets.

j Every element which is in either A or B' is in the union of A and B'.
$\therefore \ A \cup B' = \{10, 11, 12, 13, 14, 15, 16, 17, 19, 20\}$

k $A' \cap B' = \{11, 13, 14, 16, 17, 19\}$ since these are the elements common to both sets.

l Every element which is in either A' or B' is in the union of A' and B'.
$\therefore \ A' \cup B' = \{10, 11, 13, 14, 16, 17, 18, 19, 20\}$

3 **a** $U = \{2, 3, 4, 5, 6, 7, 8\}$, $A = \{3, 5, 7\}$, $B = \{2, 4, 7, 8\}$

i $n(U) = 7$

ii $n(A) = 3$

iii $n(A') = n(U) - n(A)$
$= 7 - 3$
$= 4$

iv $n(B) = 4$

v $n(B') = n(U) - n(B)$
$= 7 - 4$
$= 3$

b For any set S within a universal set U, $n(S) + n(S') = n(U)$.

4 $n(U) = 15$, $n(P) = 6$, $n(Q') = 4$

a $n(P) + n(P') = n(U)$
$\therefore \ 6 + n(P') = 15$
$\therefore \ n(P') = 9$

b $n(Q) + n(Q') = n(U)$
$\therefore \ n(Q) + 4 = 15$
$\therefore \ n(Q) = 11$

5 As $P \subseteq Q$, then all elements of P are in Q.
$\therefore$ if an element is not in Q then it is not in P.
$\therefore \ Q' \subseteq P'$

6 Let $U = \{2, 3, 4, ...\}$ and $P = \{\text{primes}\}$.
$P' = \{\text{composites}\}$ which is an infinite set.
Let $U = \{0, 1, 2, 3, ...\}$ and $P = \{1, 2, 3, ...\}$
$P' = \{0\}$ which is a finite set.

EXERCISE 2D

1 First notice that $\mathbb{N} \subseteq \mathbb{Z}$, $\mathbb{Z} \subseteq \mathbb{Q}$, and $\mathbb{Q} \subseteq \mathbb{R}$.

$-\frac{3}{8} \notin \mathbb{N}$ or $\mathbb{Z}$ as it is not an integer. $-\frac{3}{8} \in \mathbb{Q}$ since -3 and 8 are integers, $\therefore \ -\frac{3}{8} \in \mathbb{R}$.

$1.8 \notin \mathbb{N}$ or $\mathbb{Z}$ as it is not an integer. $1.8 \in \mathbb{Q}$ since $1.8 = \frac{18}{10}$, 18 and 10 are integers,
$\therefore \ 1.8 \in \mathbb{R}$.

$1.\overline{8} \notin \mathbb{N}$ or $\mathbb{Z}$ as it is not an integer. $1.\overline{8} \in \mathbb{Q}$ since $1.\overline{8} = \frac{17}{9}$, 17 and 9 are integers,
$\therefore \ 1.\overline{8} \in \mathbb{R}$.

$-17 \notin \mathbb{N}$ as $-17 < 0$. $-17 \in \mathbb{Z}$ as it is an integer, $\therefore \ -17 \in \mathbb{Q}$ and $\mathbb{R}$.

$\sqrt{64} \in \mathbb{N}$ as $\sqrt{64} = 8$, which is an integer, $\therefore \ \sqrt{64} \in \mathbb{Z}$, $\mathbb{Q}$, and $\mathbb{R}$.

$\frac{\pi}{2} \notin \mathbb{Q}$ as π is irrational, $\therefore \ \frac{\pi}{2} \notin \mathbb{Z}$ or $\mathbb{N}$. $\frac{\pi}{2} \in \mathbb{R}$ as it can be placed on the number line.

$\sqrt{-3} \notin \mathbb{R}$ as it cannot be placed on the number line, $\therefore$ $\sqrt{-3} \notin \mathbb{Q}$, $\mathbb{Z}$, or $\mathbb{N}$.
$-\sqrt{3} \notin \mathbb{Q}$ as $\sqrt{3}$ is irrational, $\therefore$ $-\sqrt{3} \notin \mathbb{Z}$ or $\mathbb{N}$. $-\sqrt{3} \in \mathbb{R}$ as it can be placed on the number line.

So, the table is:

Number	$\mathbb{N}$	$\mathbb{Z}$	$\mathbb{Q}$	$\mathbb{R}$
6	✓	✓	✓	✓
$-\frac{3}{8}$	✗	✗	✓	✓
1.8	✗	✗	✓	✓
$1.\overline{8}$	✗	✗	✓	✓
-17	✗	✓	✓	✓
$\sqrt{64}$	✓	✓	✓	✓
$\frac{\pi}{2}$	✗	✗	✗	✓
$\sqrt{-3}$	✗	✗	✗	✗
$-\sqrt{3}$	✗	✗	✗	✓

2 **a** $-7 \in \mathbb{Z}$ but $-7 \notin \mathbb{Z}^+$.
$\therefore$ $-7 \in \mathbb{Z}^+$ is a false statement.

b $\frac{2}{3}$ is not an integer.
$\therefore$ $\frac{2}{3} \notin \mathbb{Z}$ is a true statement.

c $\sqrt{3}$ is irrational.
$\therefore$ $\sqrt{3} \in \mathbb{Q}$ is a false statement.

d $\frac{7}{9} \in \mathbb{Q}$ since 7 and 9 are both integers.
$\therefore$ $\frac{7}{9} \in \mathbb{Q}$ is a true statement.

e 0.201 is not an integer.
$\therefore$ $0.201 \in \mathbb{Z}$ is a false statement.

f $\dfrac{7}{0.31} = \dfrac{700}{31}$, so $\dfrac{7}{0.31} \in \mathbb{Q}$ since 700 and 31 are integers.
$\therefore$ $\dfrac{7}{0.31} \in \mathbb{Q}$ is a true statement.

g $\sqrt{|-1|} = \sqrt{1} = 1$ which is real.
$\therefore$ $\sqrt{|-1|} \in \mathbb{R}$ is a true statement.

h $\sqrt{-9}$ is not real.
$\therefore$ $\sqrt{-9} \in \mathbb{R}$ is a false statement.

3 **a** Every element of $\mathbb{Z}^+$ is also an element of $\mathbb{N}$.
$\therefore$ $\mathbb{Z}^+ \subseteq \mathbb{N}$ is a true statement.

b $\mathbb{N} \subseteq \mathbb{Z}$ since every element of $\mathbb{N}$ is also an element of $\mathbb{Z}$.
However -1, -2, -3, ... are all elements of $\mathbb{Z}$ but not elements of $\mathbb{N}$, so $\mathbb{N} \neq \mathbb{Z}$.
$\therefore$ $\mathbb{N} \subset \mathbb{Z}$ is a true statement.

c $\mathbb{N} \neq \mathbb{Z}^+$ as $0 \in \mathbb{N}$ but $0 \notin \mathbb{Z}^+$.
$\therefore$ $\mathbb{N} = \mathbb{Z}^+$ is a false statement.

d $\mathbb{Z}^- \subseteq \mathbb{Z}$ since every element of $\mathbb{Z}^-$ is also an element of $\mathbb{Z}$.
$\therefore$ $\mathbb{Z}^- \subseteq \mathbb{Z}$ is a true statement.

e $\mathbb{Q}$ contains fractions such as $\frac{1}{3}$, which are not integers.
$\therefore$ $\mathbb{Q} \subset \mathbb{Z}$ is a false statement.

f $\{0\}$ is the set containing the element 0 only, which is in $\mathbb{Z}$.
$\therefore$ $\{0\} \subseteq \mathbb{Z}$ is a true statement.

g $\mathbb{Z} \subseteq \mathbb{Q}$ since every element of $\mathbb{Z}$ is also an element of $\mathbb{Q}$.
$\therefore$ $\mathbb{Z} \subseteq \mathbb{Q}$ is a true statement.

h Every element which is in either $\mathbb{Z}^+$ or $\mathbb{Z}^-$ is in the union of $\mathbb{Z}^+$ and $\mathbb{Z}^-$. 0 however is in $\mathbb{Z}$ but not in $\mathbb{Z}^+$ nor $\mathbb{Z}^-$.
$\therefore$ $\mathbb{Z}^+ \cup \mathbb{Z}^- = \mathbb{Z}$ is a false statement.

4 **a** The set of integers between 10 and 20 $= \{11, 12, 13, 14, 15, 16, 17, 18, 19\}$.
The number of elements is a particular defined value.
$\therefore$ this set is finite.

b The set of integers greater than 5 $= \{6, 7, 8, 9, 10, ...\}$.
This set has an endless number of elements.
$\therefore$ this set is infinite.

c The set of all rational numbers between 0 and 1 $= \{\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, ...\}$.
This set has an endless number of elements.
$\therefore$ this set is infinite.

d The set of all irrational numbers between 0 and 1 $= \{\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{3}, \frac{\sqrt{2}}{4}, ...\}$
This set has an endless number of elements.
$\therefore$ this set is infinite.

5 $U = \mathbb{Z} = \{0, \pm 1, \pm 2, \pm 3, \pm 4, ...\}$
$\mathbb{Z}^+ = \{1, 2, 3, 4, ...\}$
The complement of $\mathbb{Z}^+$ is the set of all elements of U that are not elements of $\mathbb{Z}^+$. This includes $\{0\}$ and $\mathbb{Z}^-$.
$\therefore$ the complement of $\mathbb{Z}^+$ is $\mathbb{Z}^- \cup \{0\}$.

EXERCISE 2E

1 **a** $A = \{x \in \mathbb{Z} \mid -1 \leqslant x \leqslant 7\}$

i The set of all x such that x is an integer between -1 and 7, including -1 and 7.

ii $A = \{-1, 0, 1, 2, 3, 4, 5, 6, 7\}$ **iii** $n(A) = 9$

b $A = \{x \in \mathbb{N} \mid -2 < x < 8\}$

i The set of all x such that x is a natural number between -2 and 8.

ii $A = \{0, 1, 2, 3, 4, 5, 6, 7\}$ **iii** $n(A) = 8$

c $A = \{x \mid 0 \leqslant x \leqslant 1\}$

i The set of all real x such that x is greater than or equal to 0, and less than or equal to 1.

ii It is not possible to list the elements of A.

iii A is an infinite set, so $n(A)$ is undefined.

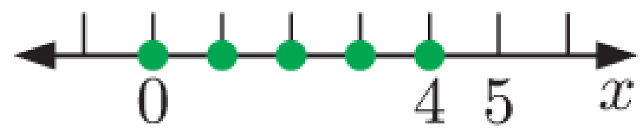
d $A = \{x \in \mathbb{Q} \mid 5 \leqslant x \leqslant 6\}$

i The set of all x such that x is a rational number greater than or equal to 5, and less than or equal to 6.

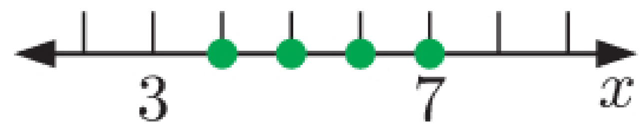
ii It is not possible to list the elements of A.

iii A is an infinite set, so $n(A)$ is undefined.

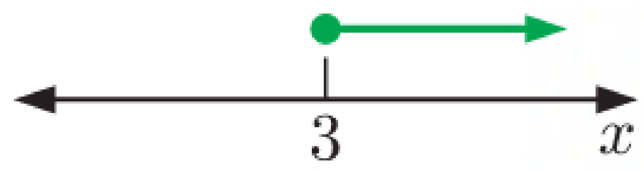
2 **a** $\{x \in \mathbb{N} \mid x < 5\}$ can be represented by:



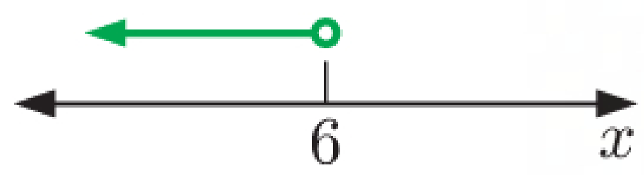
b $\{x \in \mathbb{Z}^+ \mid 3 < x \leqslant 7\}$ can be represented by:



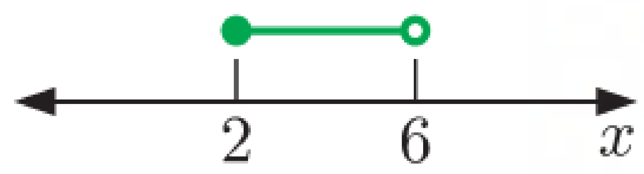
c $\{x \in \mathbb{R} \mid x \geqslant 3\}$ can be represented by:



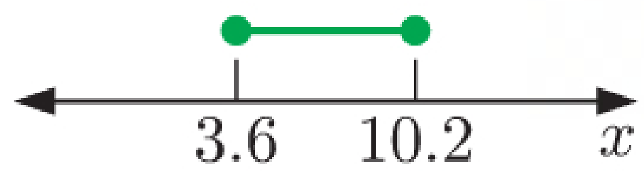
d $\{x \in \mathbb{R} \mid x < 6\}$ can be represented by:



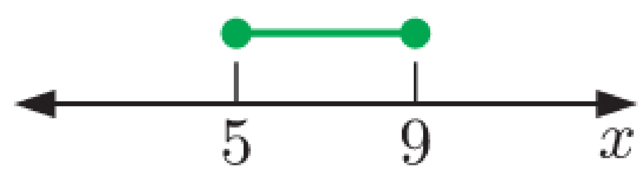
e $\{x \in \mathbb{R} \mid 2 \leqslant x < 6\}$ can be represented by:



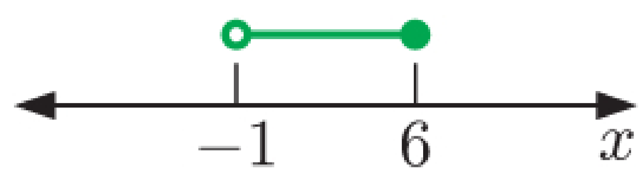
f $\{x \mid 3.6 \leqslant x \leqslant 10.2\}$ can be represented by:



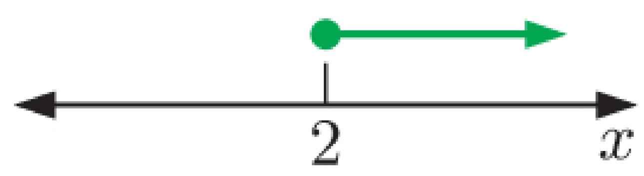
g $x \in [5, 9]$ can be represented by:



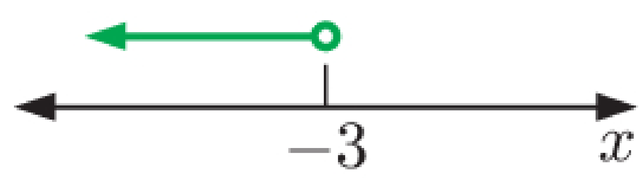
h $x \in\ ]-1, 6]$ can be represented by:



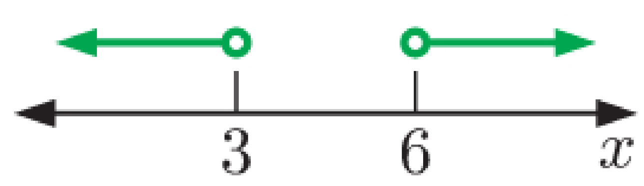
i $x \in [2, \infty[$ can be represented by:



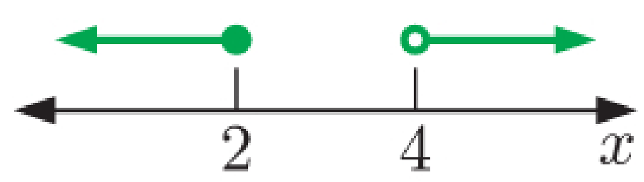
j $x \in\ ]-\infty, -3[$ can be represented by:



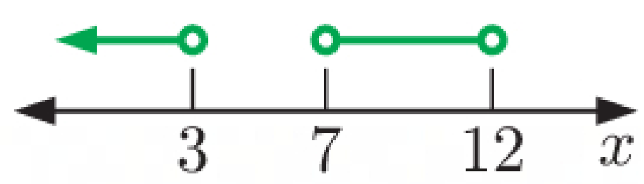
k $\{x \mid x < 3\} \cup \{x \mid x > 6\}$ can be represented by:



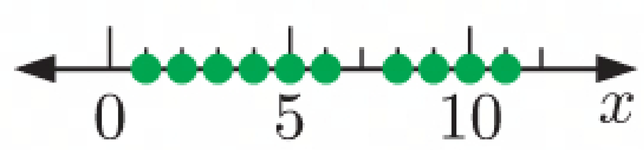
l $\{x \mid x \leqslant 2\} \cup \{x \mid x > 4\}$ can be represented by:



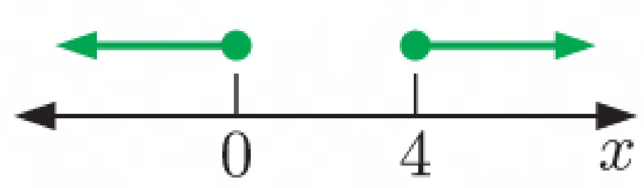
m $\{x \mid x < 3\} \cup \{x \mid 7 < x < 12\}$ can be represented by:



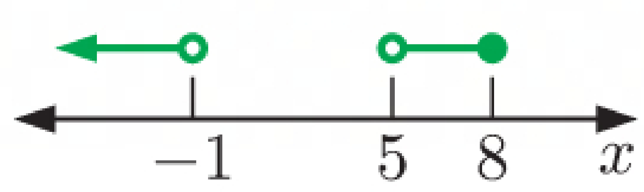
n $\{x \in \mathbb{Z}^+ \mid x \leqslant 6\} \cup \{x \in \mathbb{Z}^+ \mid 8 \leqslant x \leqslant 11\}$ can be represented by:



o $x \in\ ]-\infty, 0] \cup x \in [4, \infty[$ can be represented by:



p $x \in\ ]-\infty, 1[\cup x \in\ ]5, 8]$ can be represented by:

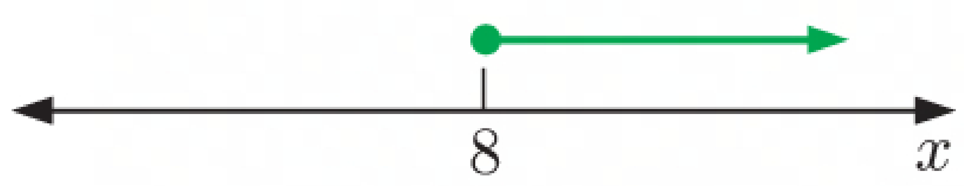


3 **a** The set of all integers between -100 and 100 can be represented by $\{x \in \mathbb{Z} \mid -100 < x < 100\}$.

b The set of all real numbers greater than 1000 can be represented by $\{x \in \mathbb{R} \mid x > 1000\}$.

c The set of all rational numbers between 2 and 3 inclusive can be represented by $\{x \in \mathbb{Q} \mid 2 \leqslant x \leqslant 3\}$.

4 **a**

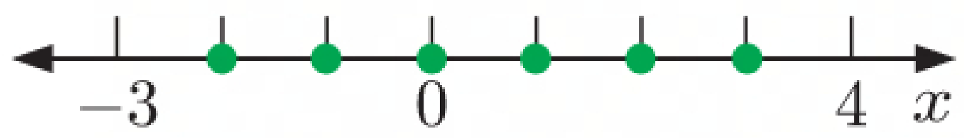


can be represented by $\{x \mid x \geqslant 8\}$.

b

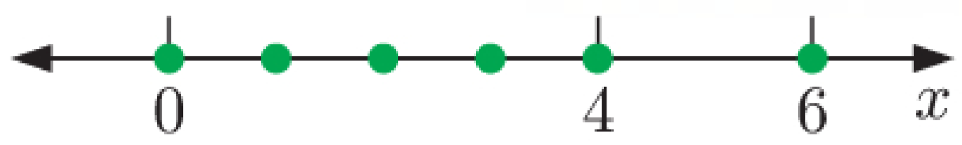
can be represented by $\{x \mid -1 \leqslant x < 4\}$.

c



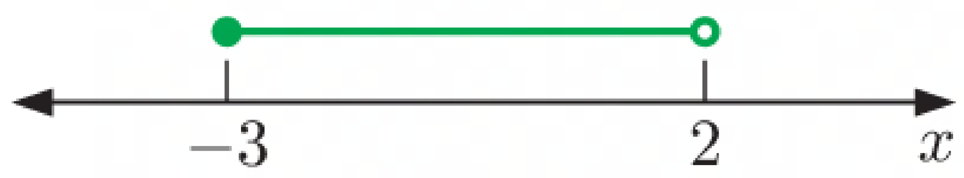
can be represented by
$\{x \in \mathbb{Z} \mid -3 < x < 4\}$ or
$\{x \in \mathbb{Z} \mid -2 \leqslant x \leqslant 3\}$.

d



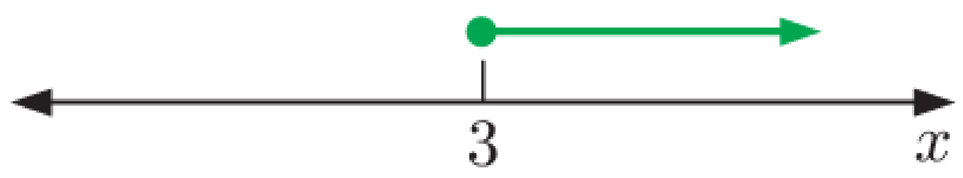
can be represented by
$\{x \in \mathbb{N} \mid x \leqslant 4\} \cup \{x \in \mathbb{N} \mid x = 6\}$.

5 **a**



can be represented by $x \in [-3, 2[$.

b

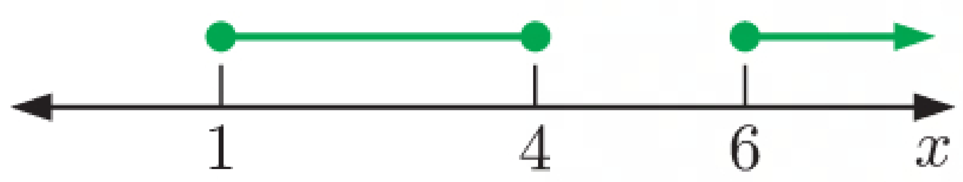


can be represented by $x \in [3, \infty[$.

c

can be represented by $x \in]0, 2[$.

d



can be represented by
$x \in [1, 4] \cup [6, \infty[$.

6 **a** $A = \varnothing, \quad B = \{2, 5, 7, 9\}$
The empty set $\varnothing$ is a subset of all other sets.
$\therefore \quad A \subseteq B$

b $A = \{2, 5, 8, 9\}, \quad B = \{8, 9\}$
Only two elements of A are in B.
$\therefore \quad A \nsubseteq B$

c $A = \{x \in \mathbb{R} \mid 2 \leqslant x \leqslant 3\}, \quad B = \{x \in \mathbb{R}\}$
Every element of A is in B.
$\therefore \quad A \subseteq B$

d $A = \{x \in \mathbb{Q} \mid 3 \leqslant x \leqslant 9\}, \quad B = \{x \in \mathbb{R} \mid 0 \leqslant x \leqslant 10\}$
Every element of A is in B.
$\therefore \quad A \subseteq B$

e $A = \{x \in \mathbb{Z} \mid -10 \leqslant x \leqslant 10\}, \quad B = \{z \in \mathbb{Z} \mid 0 \leqslant z \leqslant 5\}$
The elements $-10, -9, -8, \ldots., -1$, and $6, 7, 8, 9, 10$ are in A but not in B.
$\therefore \quad A \nsubseteq B$

f $A = \{x \in \mathbb{Q} \mid 0 \leqslant x \leqslant 1\}, \quad B = \{y \in \mathbb{Q} \mid 0 < y \leqslant 2\}$
The element 0 is in A but not in B.
$\therefore \quad A \nsubseteq B$

7 $U = \{x \in \mathbb{Z} \mid 0 \leqslant x \leqslant 8\}, \quad A = \{x \in \mathbb{Z} \mid 2 \leqslant x \leqslant 7\}, \quad B = \{x \in \mathbb{Z} \mid 5 \leqslant x \leqslant 8\}$

a $A = \{2, 3, 4, 5, 6, 7\}$ **b** $A' = \{0, 1, 8\}$ **c** $B = \{5, 6, 7, 8\}$

d $B' = \{0, 1, 2, 3, 4\}$ **e** $A \cap B = \{5, 6, 7\}$ **f** $A \cup B = \{2, 3, 4, 5, 6, 7, 8\}$

g $A \cap B' = \{2, 3, 4\}$

8 $U = \{x \in \mathbb{Z} \mid 0 \leqslant x \leqslant 40\}, \quad P = \{\text{factors of } 28\}, \quad Q = \{\text{factors of } 40\}$

a $P = \{1, 2, 4, 7, 14, 28\}, \quad Q = \{1, 2, 4, 5, 8, 10, 20, 40\}$

b $P \cap Q = \{1, 2, 4\}$

c $P \cup Q = \{1, 2, 4, 5, 7, 8, 10, 14, 20, 28, 40\}$

d $n(P \cup Q) = 11$ and $n(P) + n(Q) - n(P \cap Q) = 6 + 8 - 3 = 11$
$\therefore\ n(P \cup Q) = n(P) + n(Q) - n(P \cap Q)$

9 $U = \mathbb{Z}$, $C = \{y \in \mathbb{Z} \mid -4 \leqslant y \leqslant -1\}$, $D = \{y \in \mathbb{Z} \mid -7 \leqslant y < 0\}$

a $C = \{-4, -3, -2, -1\}$, $D = \{-7, -6, -5, -4, -3, -2, -1\}$

b $C \cap D = \{-4, -3, -2, -1\}$

c $C \cup D = \{-7, -6, -5, -4, -3, -2, -1\}$

d $n(C \cup D) = 7$ and $n(C) + n(D) - n(C \cap D) = 4 + 7 - 4 = 7$
$\therefore\ n(C \cup D) = n(C) + n(D) - n(C \cap D)$

10 $U = \{x \in \mathbb{Z}^+ \mid x < 31\}$, $A = \{\text{multiples of 6 less than 31}\}$, $B = \{\text{factors of 30}\}$,
$C = \{\text{primes} < 30\}$

a $A = \{6, 12, 18, 24, 30\}$, $B = \{1, 2, 3, 5, 6, 10, 15, 30\}$,
$C = \{2, 3, 5, 7, 11, 13, 17, 19, 23, 29\}$

b **i** $A \cap B = \{6, 30\}$ **ii** $B \cap C = \{2, 3, 5\}$

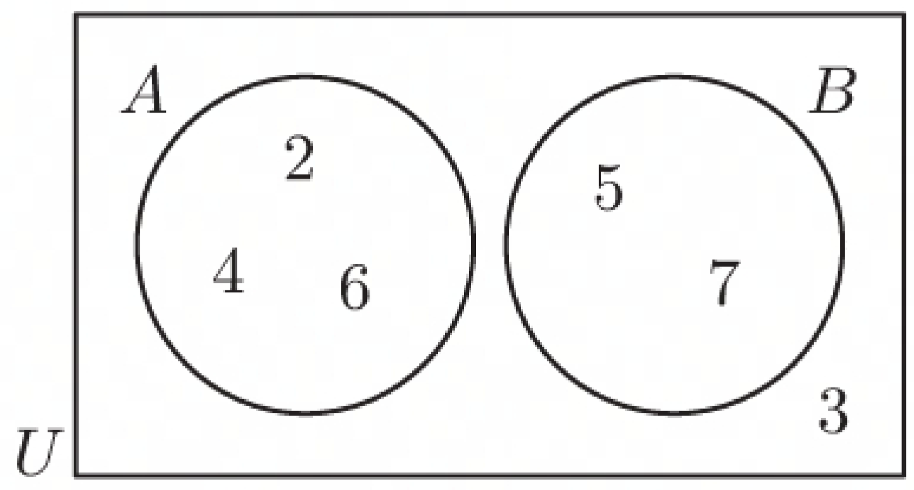
iii $A \cap C = \varnothing$ **iv** $A \cap B \cap C = \varnothing$

v $A \cup B \cup C = \{1, 2, 3, 5, 6, 7, 10, 11, 12, 13, 15, 17, 18, 19, 23, 24, 29, 30\}$

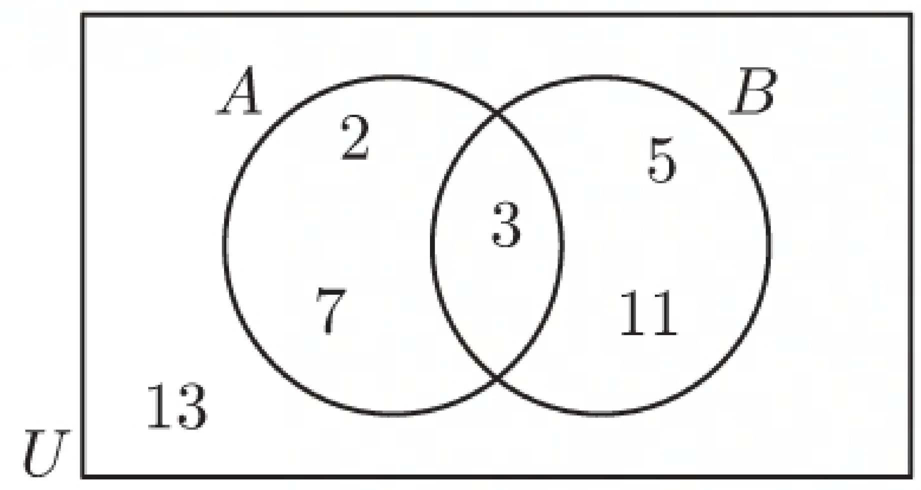
c $n(A \cup B \cup C) = 18$ and
$n(A) + n(B) + n(C) - n(A \cap B) - n(B \cap C) - n(A \cap C) + n(A \cap B \cap C)$
$= 5 + 8 + 10 - 2 - 3 - 0 + 0$
$= 18$
$\therefore\ n(A \cup B \cup C) = n(A) + n(B) + n(C) - n(A \cap B) - n(B \cap C) - n(A \cap C) + n(A \cap B \cap C)$

EXERCISE 2F

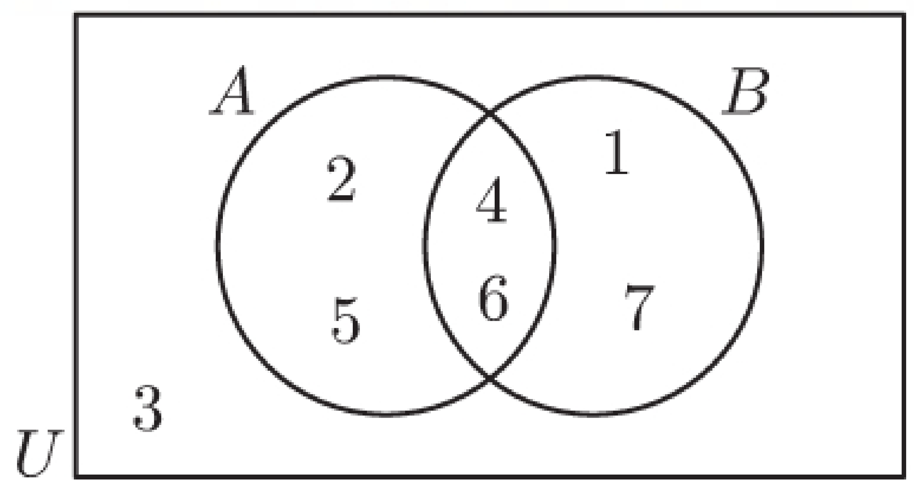
1 **a** $U = \{2, 3, 4, 5, 6, 7\}$,
$A = \{2, 4, 6\}$, $B = \{5, 7\}$
$A \cap B = \varnothing$



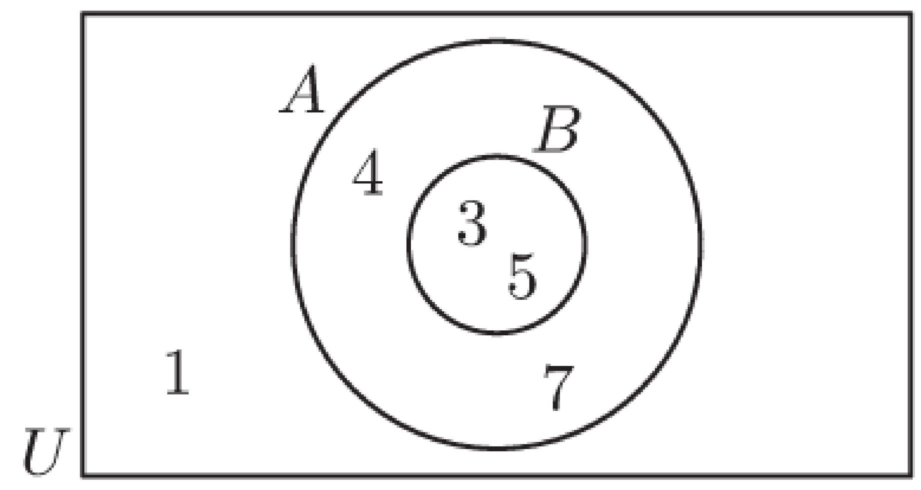
b $U = \{2, 3, 5, 7, 11, 13\}$,
$A = \{2, 3, 7\}$, $B = \{3, 5, 11\}$
$A \cap B = \{3\}$



c $U = \{1, 2, 3, 4, 5, 6, 7\}$,
$A = \{2, 4, 5, 6\}$, $B = \{1, 4, 6, 7\}$
$A \cap B = \{4, 6\}$



d $U = \{1, 3, 4, 5, 7\}$,
$A = \{3, 4, 5, 7\}$, $B = \{3, 5\}$
$A \cap B = \{3, 5\} = B$ and $B \neq A$,
so $B \subset A$.

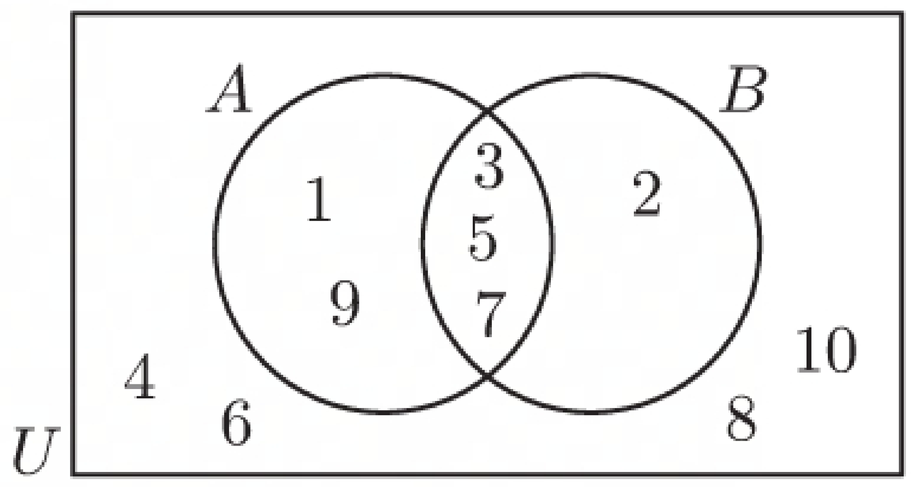


2 $U = \{x \in \mathbb{Z} \mid 1 \leqslant x \leqslant 10\}$, $A = \{\text{odd numbers} < 10\}$, $B = \{\text{primes} < 10\}$

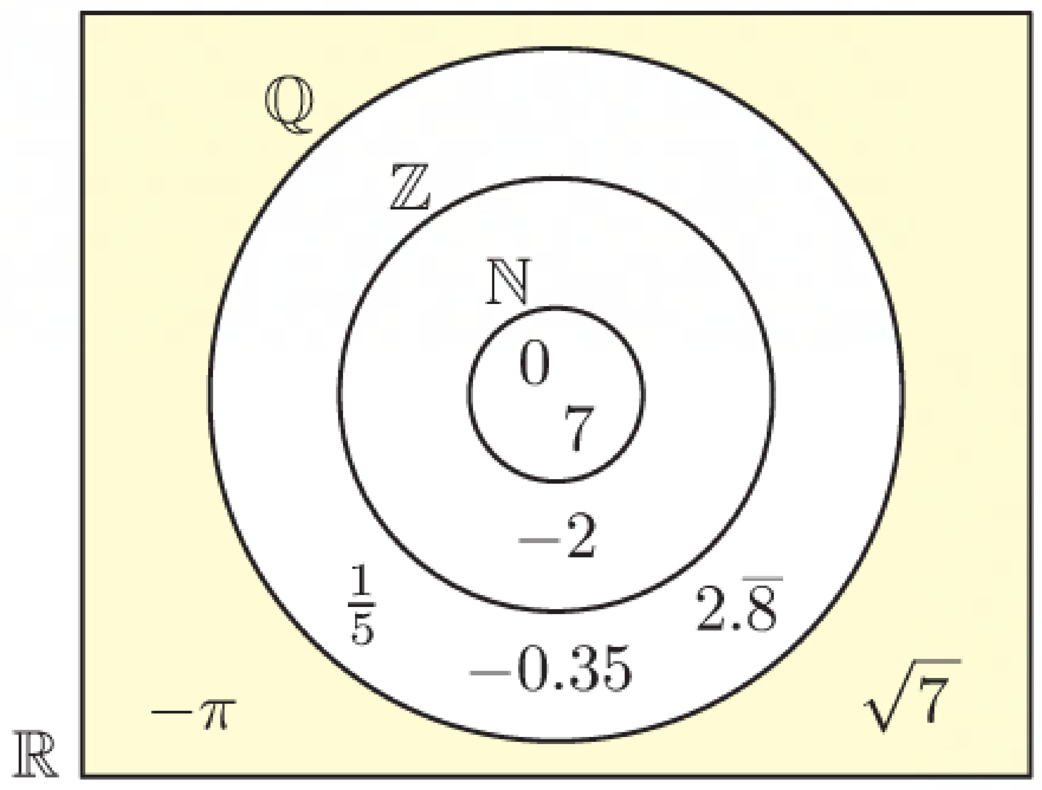
a $A = \{1, 3, 5, 7, 9\}$, $B = \{2, 3, 5, 7\}$

b $A \cap B = \{3, 5, 7\}$, $A \cup B = \{1, 2, 3, 5, 7, 9\}$

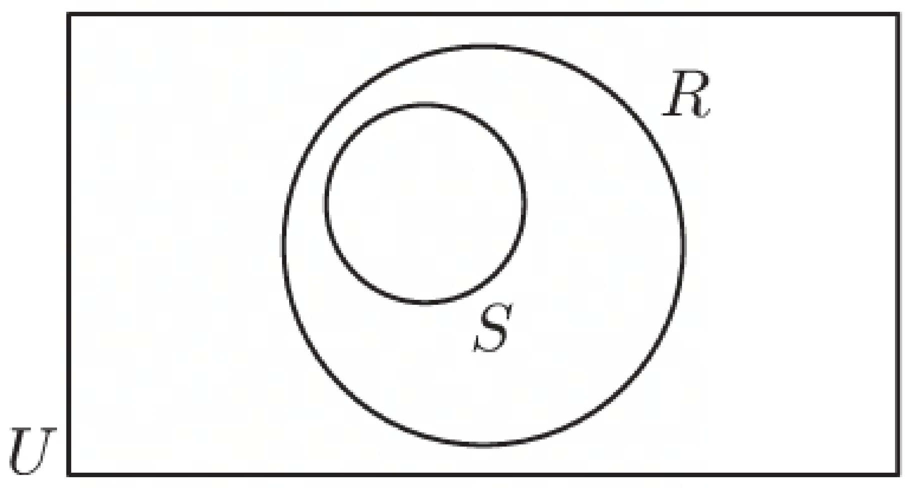
c



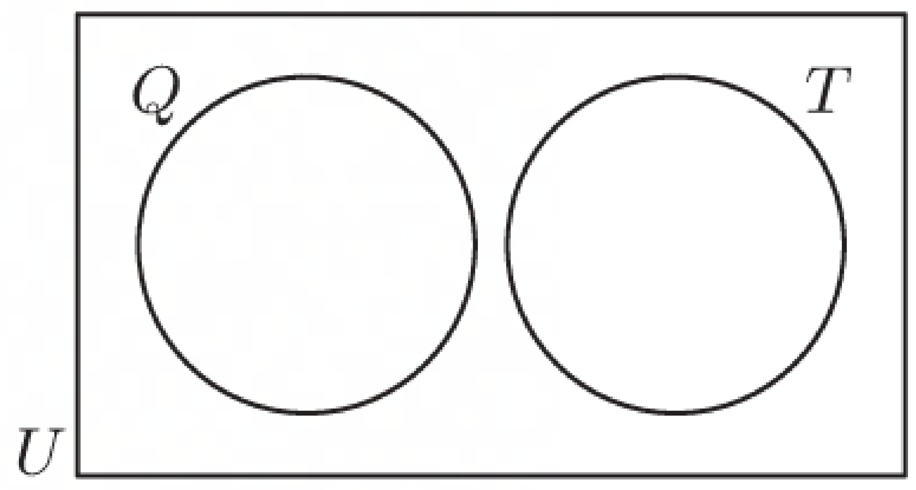
3



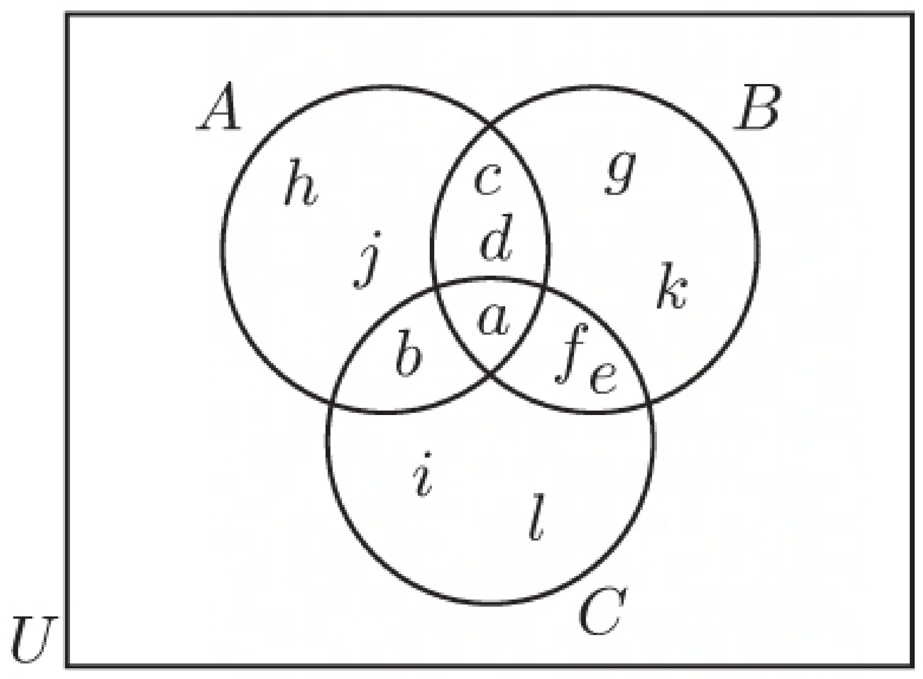
4 **a** $U = \{\text{parallelograms}\}$,
$R = \{\text{rectangles}\}$, $S = \{\text{squares}\}$
$R \cap S = S$ and $S \neq R$, so $S \subset R$.



b $U = \{\text{polygons}\}$,
$Q = \{\text{quadrilaterals}\}$, $T = \{\text{triangles}\}$
$Q \cap T = \varnothing$



5



a **i** $A = \{a, b, c, d, h, j\}$

ii $B = \{a, c, d, e, f, g, k\}$

iii $C = \{a, b, e, f, i, l\}$ **iv** $A \cap B = \{a, c, d\}$

v $A \cup B = \{a, b, c, d, e, f, g, h, j, k\}$

vi $B \cap C = \{a, e, f\}$ **vii** $A \cap B \cap C = \{a\}$

viii $A \cup B \cup C = \{a, b, c, d, e, f, g, h, i, j, k, l\}$

b $n(A \cap C) = 2$ and $n(A \cup B \cup C) = 12$ and

$$n(A) + n(B) + n(C) - n(A \cap B) - n(A \cap C) - n(B \cap C) + n(A \cap B \cap C)$$
$$= 6 + 7 + 6 - 3 - 2 - 3 + 1$$
$$= 12$$

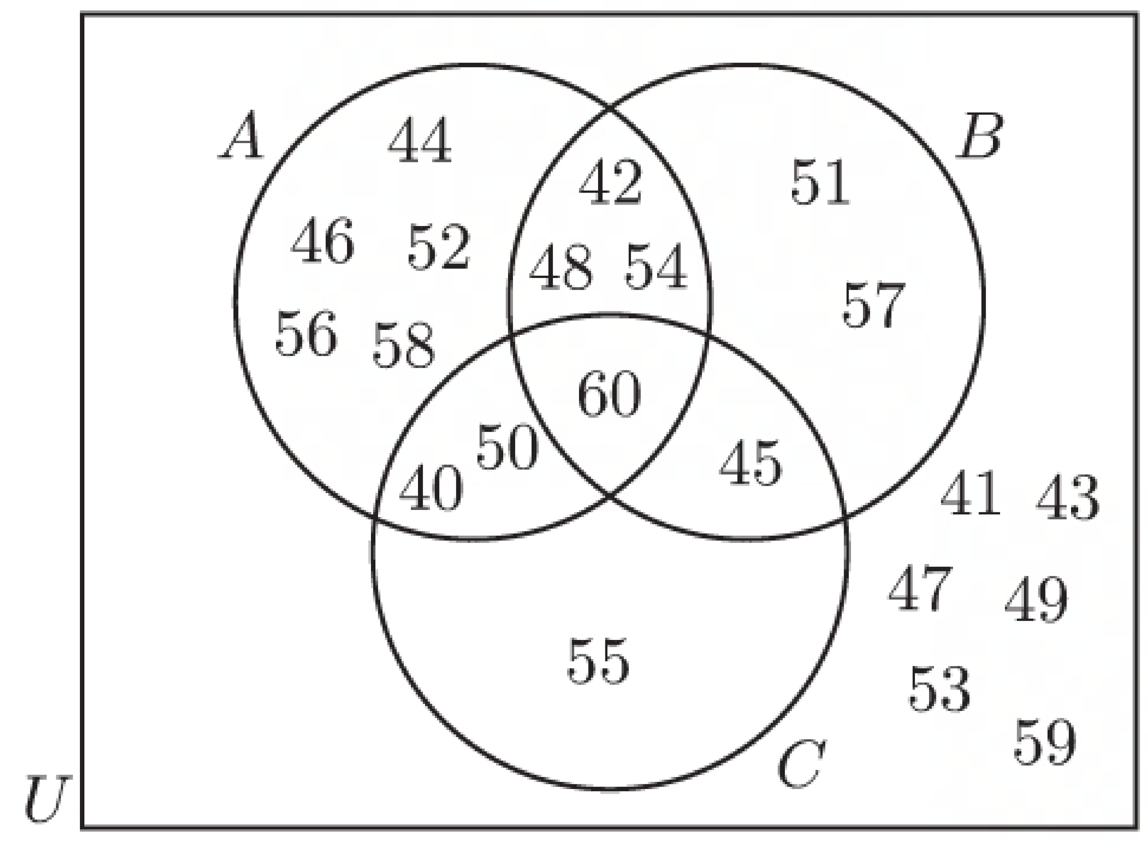
$$\therefore\ n(A \cup B \cup C) = n(A) + n(B) + n(C) - n(A \cap B) - n(A \cap C) - n(B \cap C) + n(A \cap B \cap C)$$

6 $U = \{x \in \mathbb{Z}^+ \mid 40 \leqslant x \leqslant 60\}$

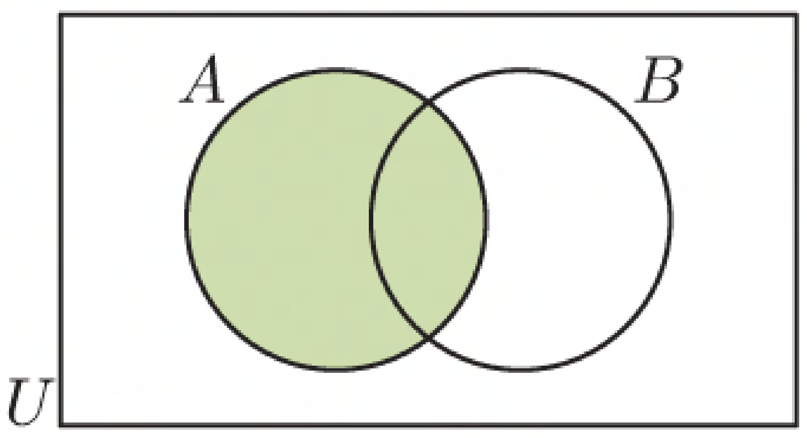
$A = \{\text{multiples of } 2\}$
$= \{40, 42, 44, 46, 48, 50, 52, 54, 56, 58, 60\}$

$B = \{\text{multiples of } 3\}$
$= \{42, 45, 48, 51, 54, 57, 60\}$

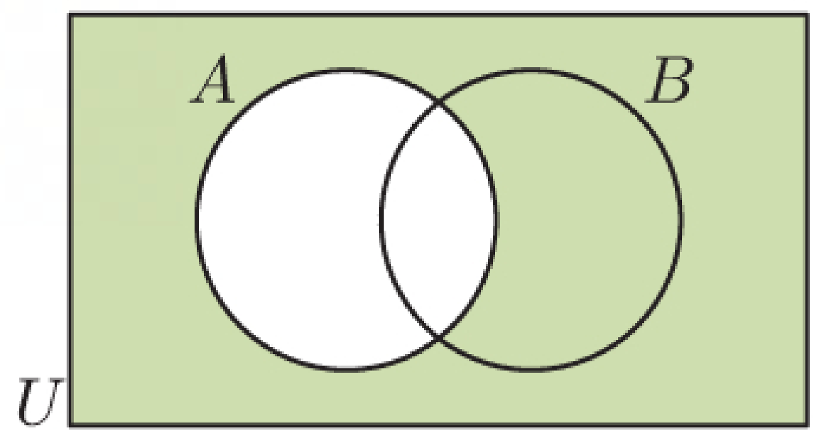
$C = \{\text{multiples of } 5\}$
$= \{40, 45, 50, 55, 60\}$



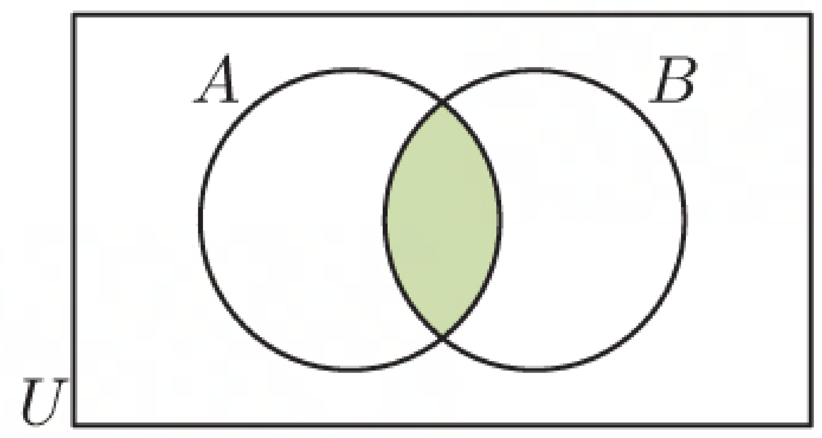
7 **a** A is shaded



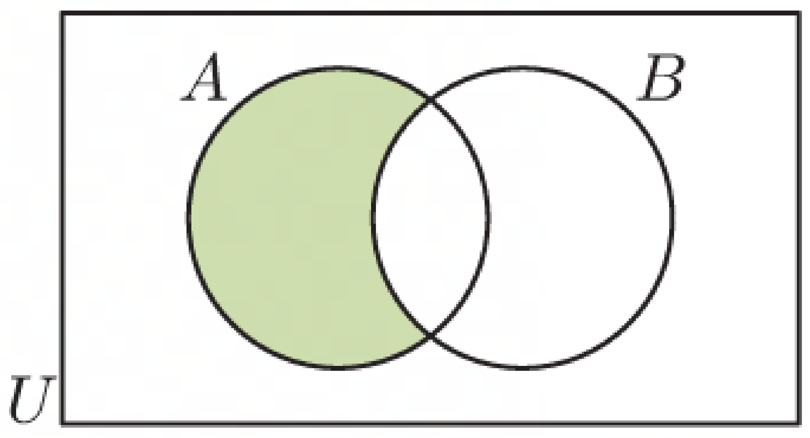
b A' is shaded



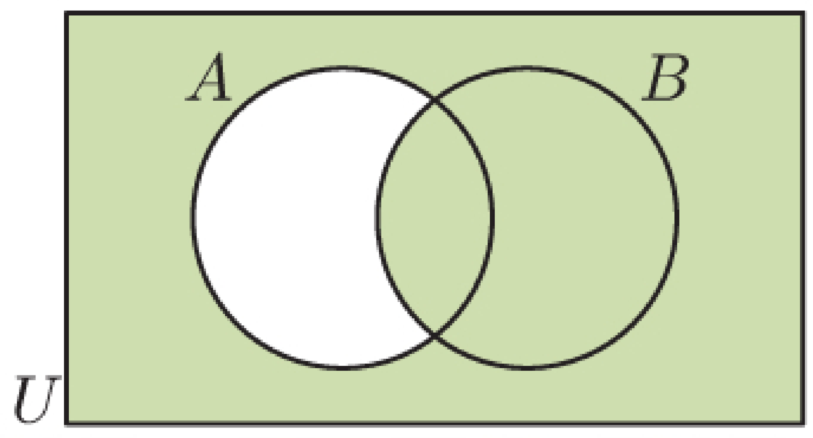
c $A \cap B$ is shaded



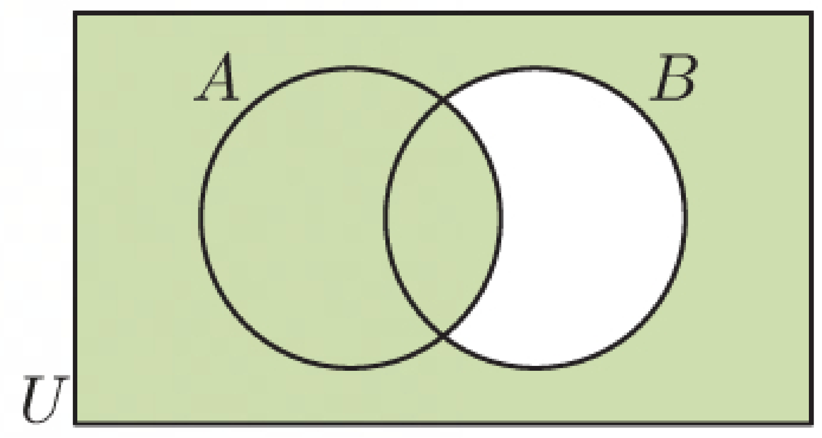
d $A \cap B'$ is shaded



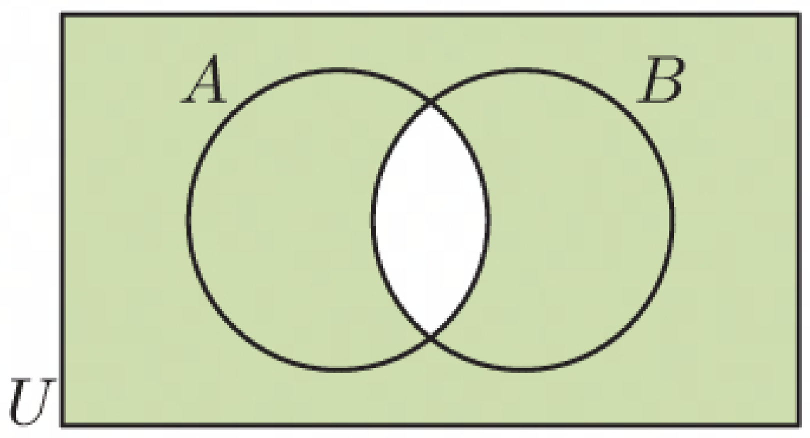
e $A' \cup B$ is shaded



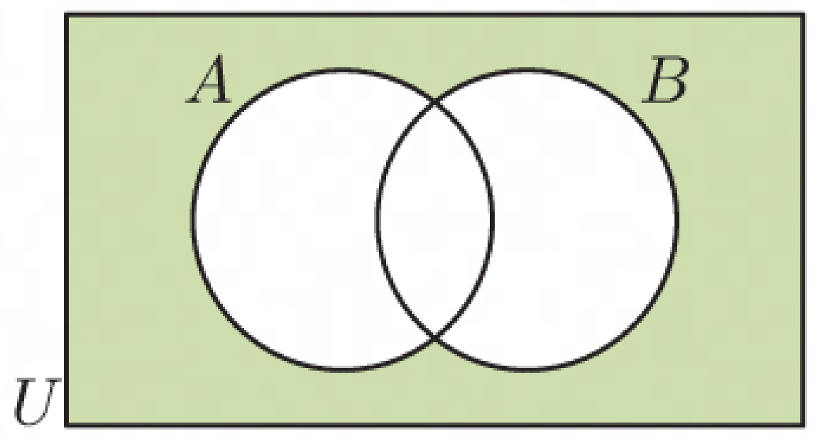
f $A \cup B'$ is shaded



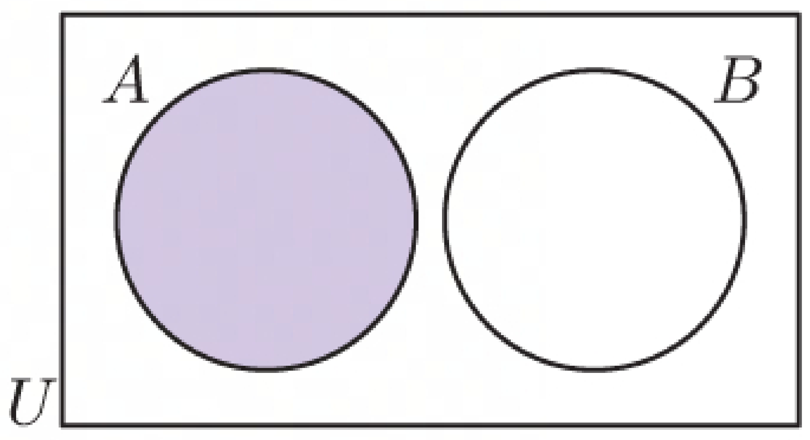
g $(A \cap B)'$ is shaded



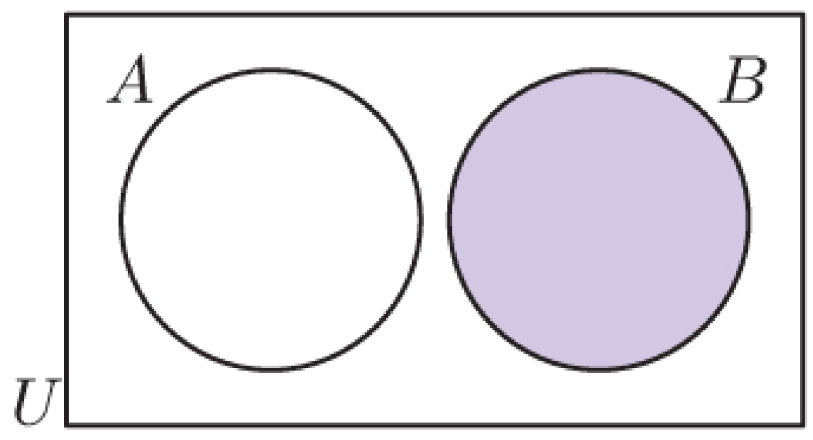
h $(A \cup B)'$ is shaded



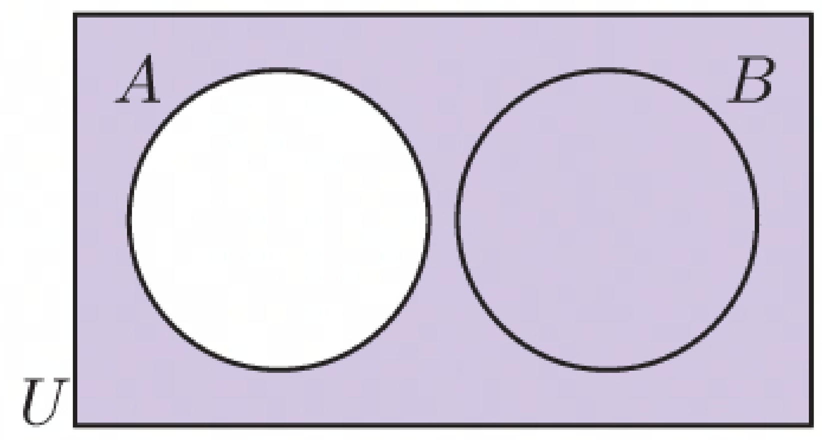
8 **a** A is shaded



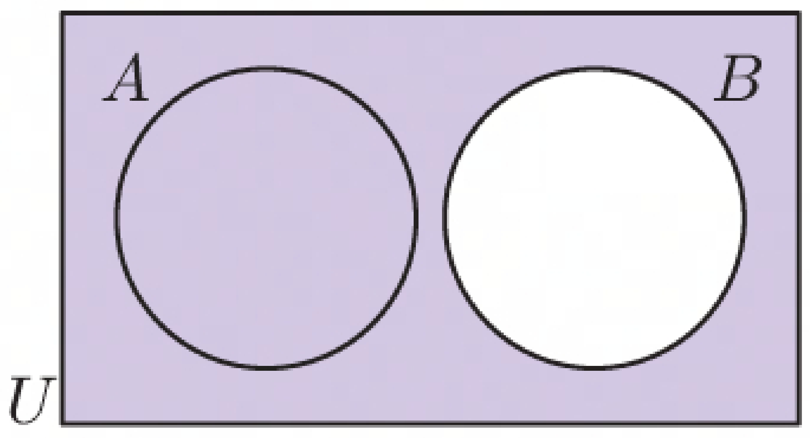
b B is shaded



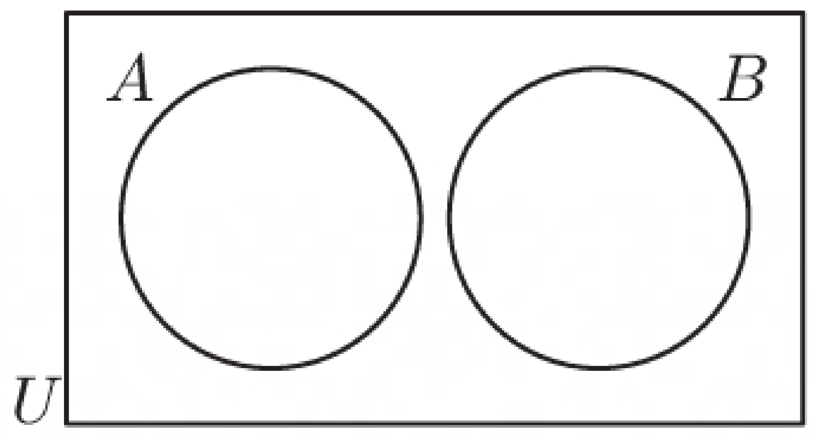
c A' is shaded



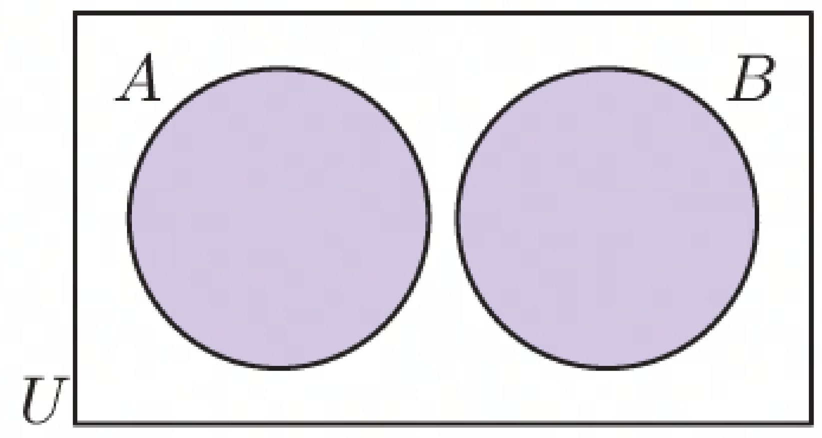
d B' is shaded



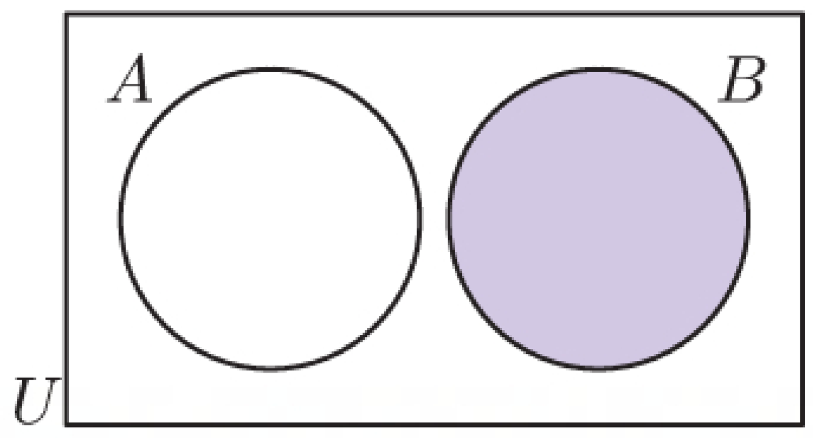
e $A \cap B$ is shaded



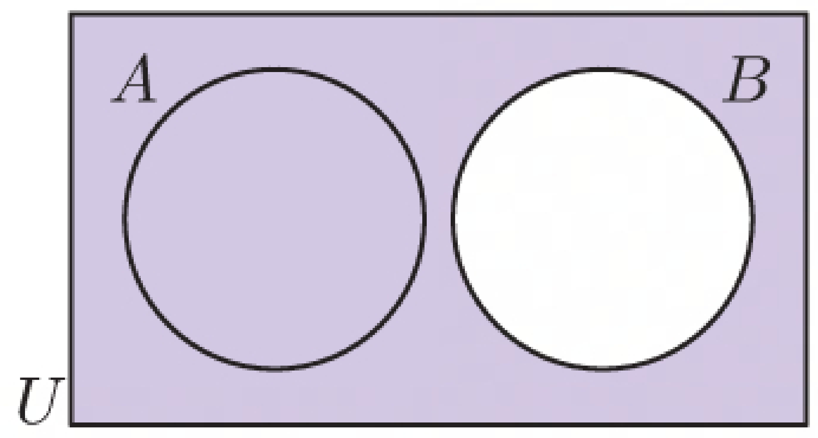
f $A \cup B$ is shaded



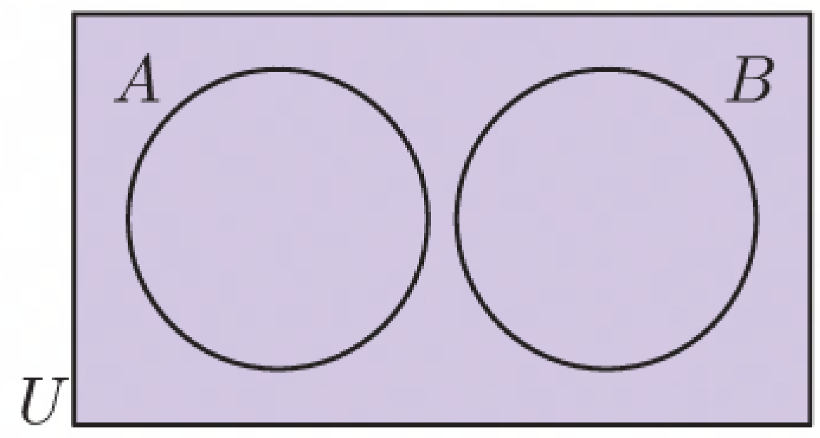
g $A' \cap B$ is shaded



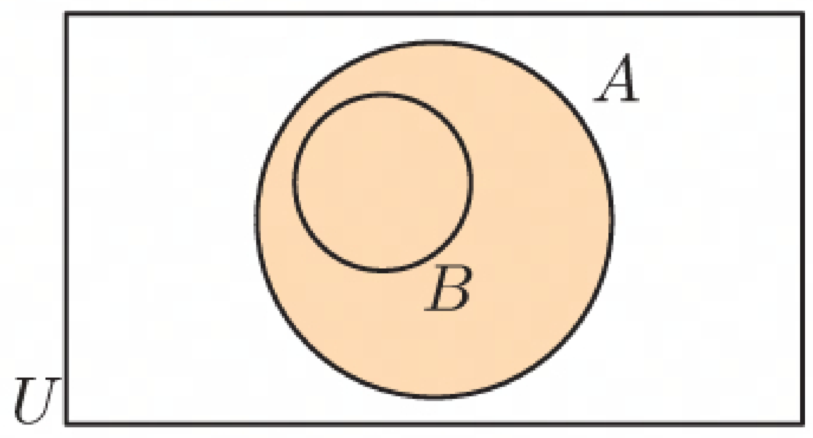
h $A \cup B'$ is shaded



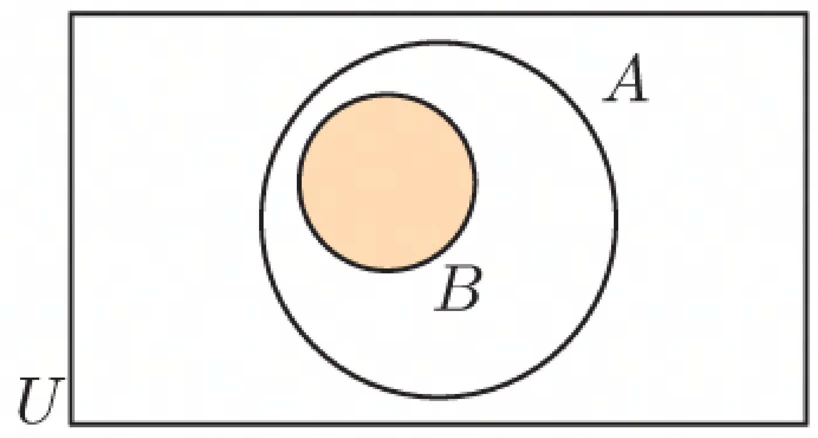
i $(A \cap B)'$ is shaded



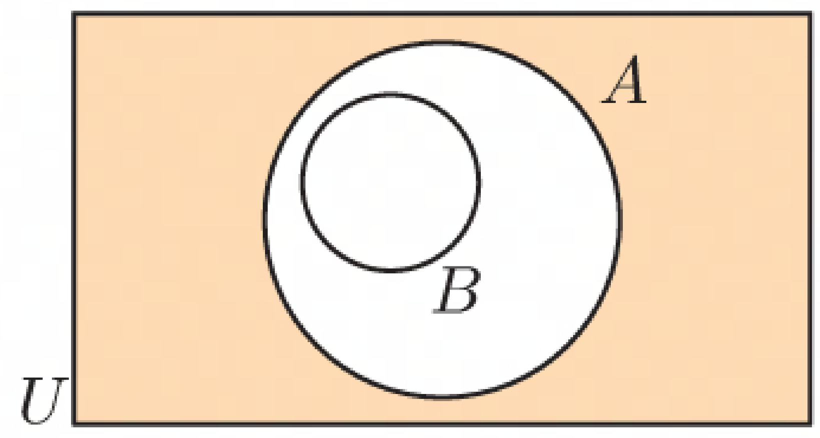
9 **a** A is shaded



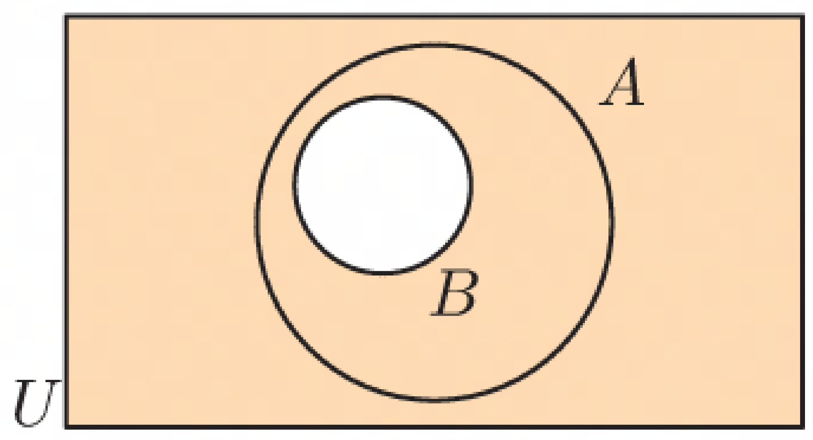
b B is shaded



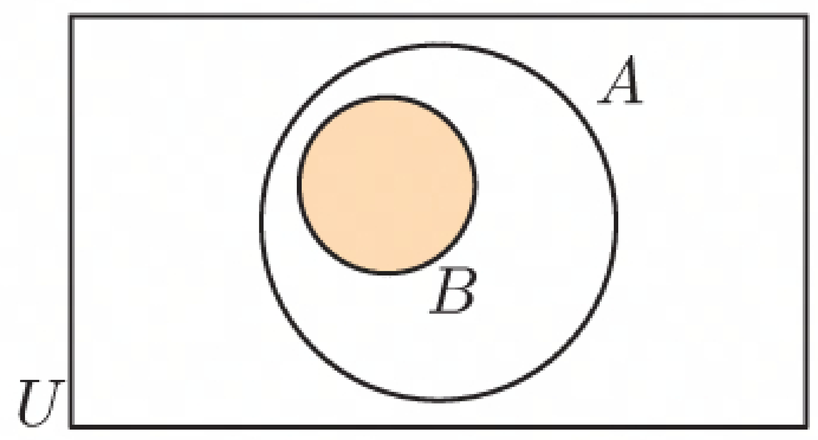
c A' is shaded



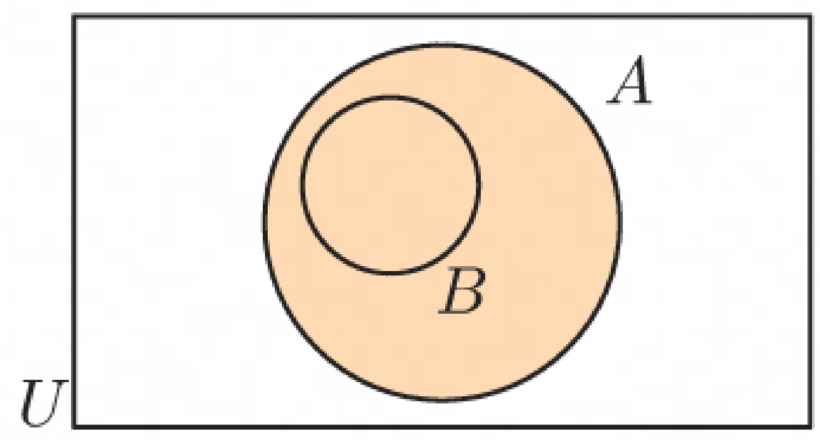
d B' is shaded



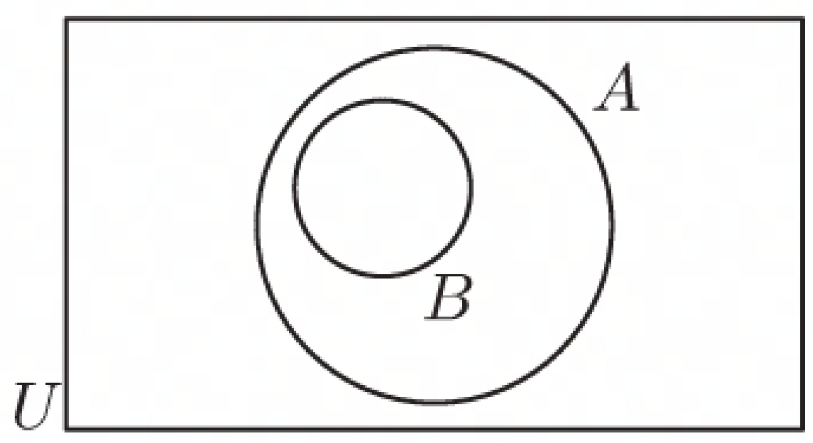
e $A \cap B$ is shaded



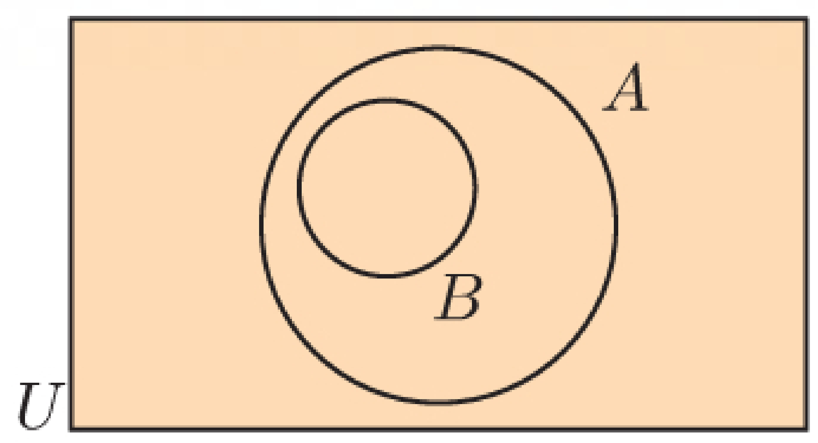
f $A \cup B$ is shaded



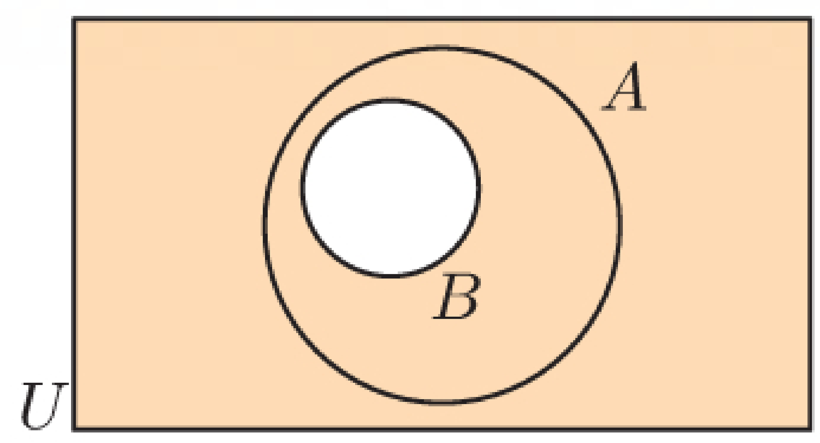
g $A' \cap B$ is shaded



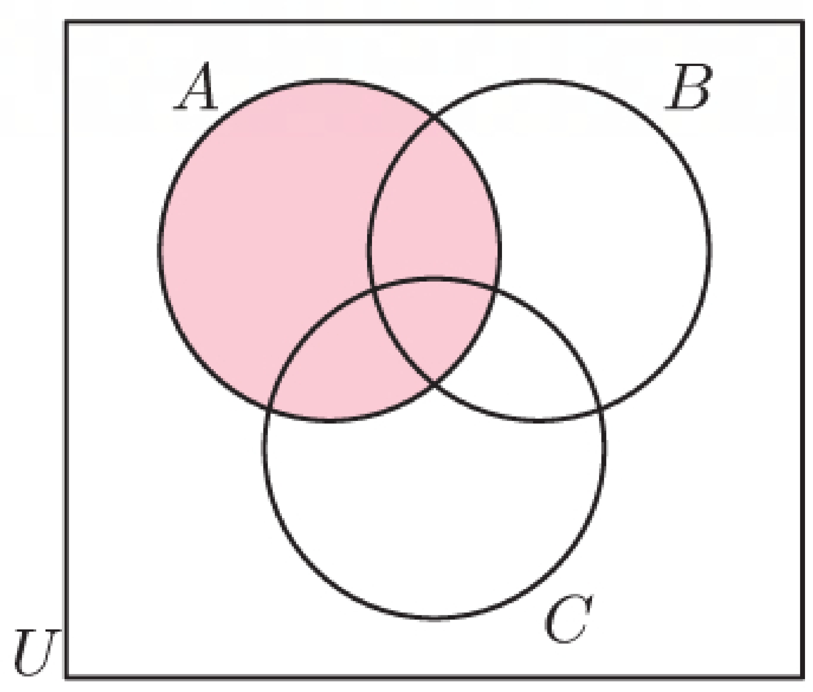
h $A \cup B'$ is shaded



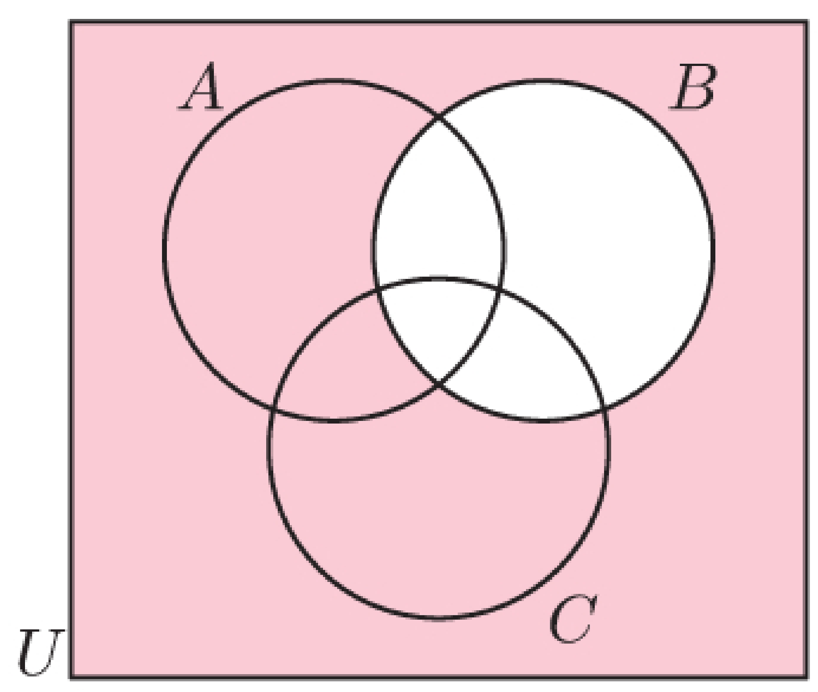
i $(A \cap B)'$ is shaded



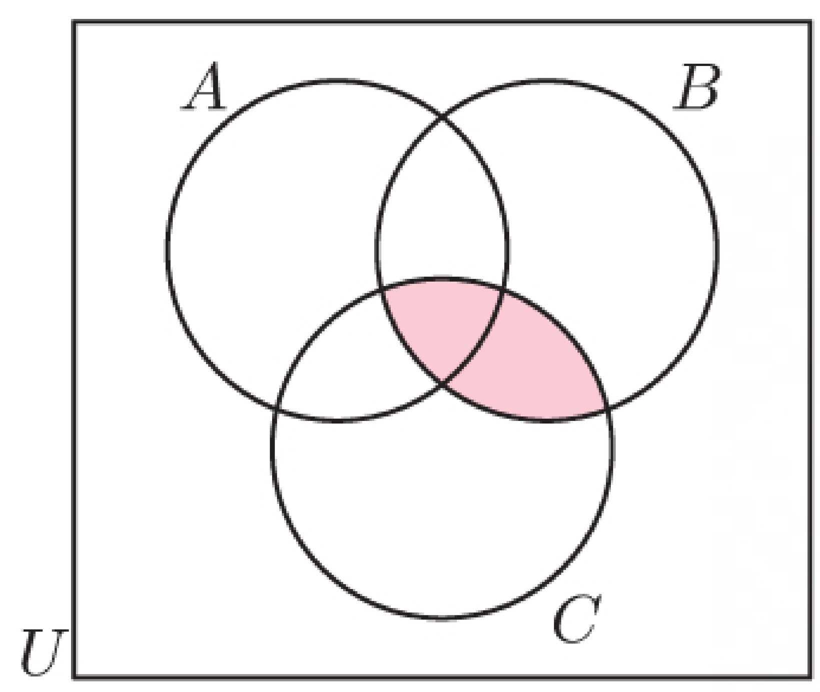
10 **a** A is shaded



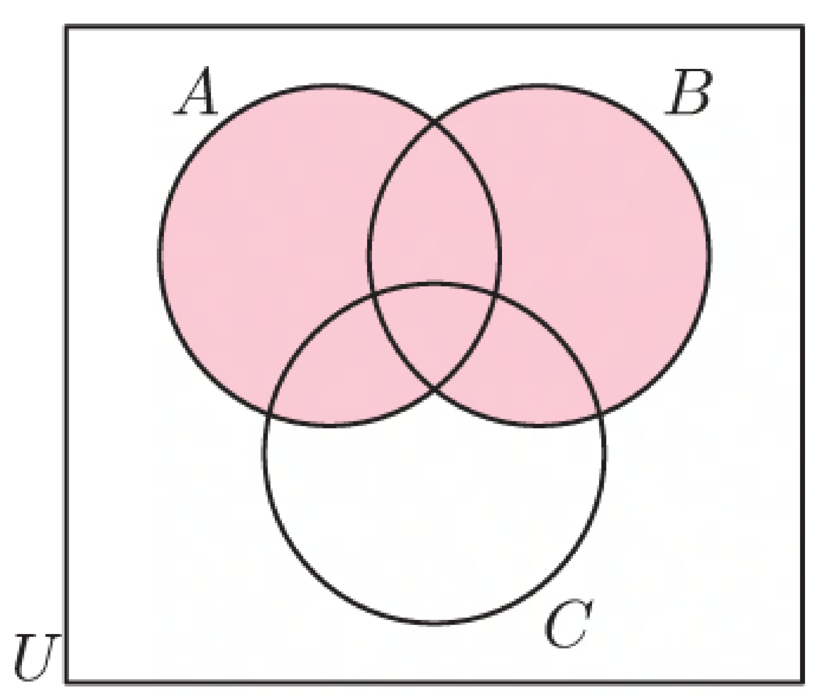
b B' is shaded



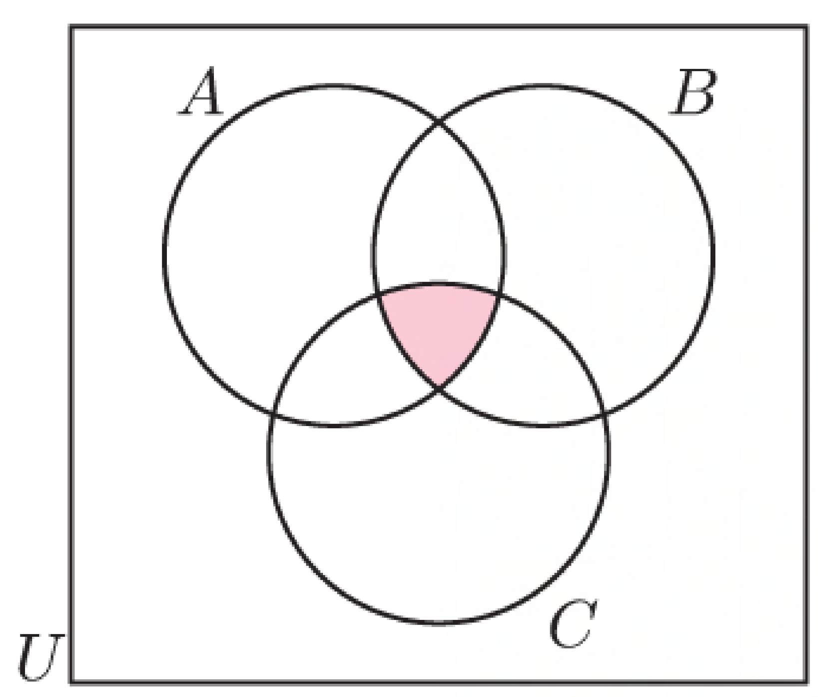
c $B \cap C$ is shaded



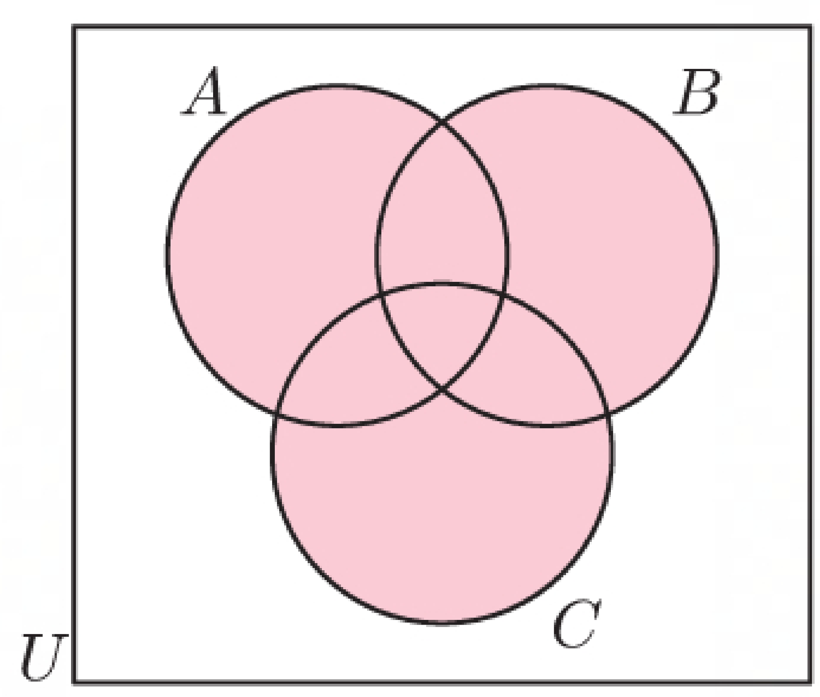
d $A \cup B$ is shaded



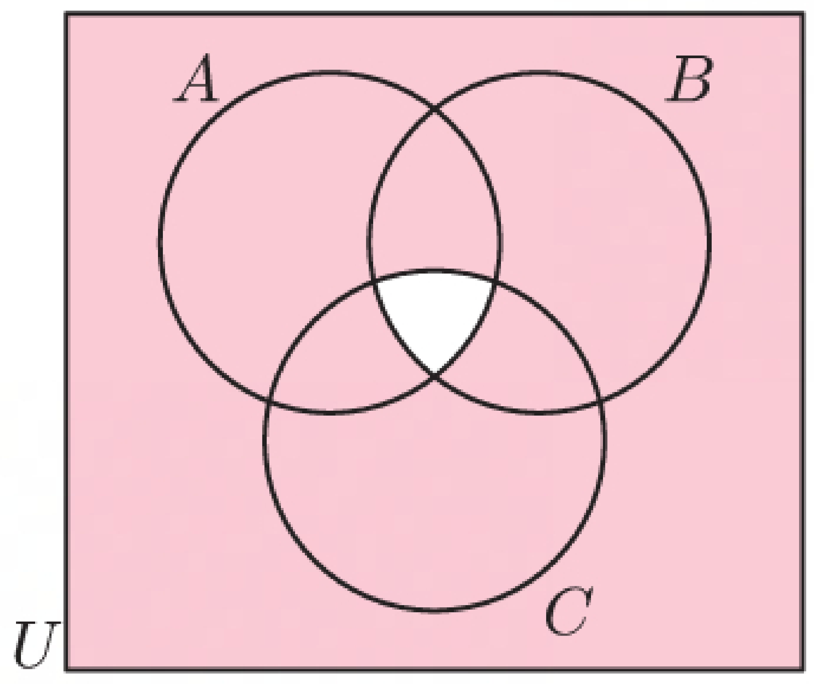
e $A \cap B \cap C$ is shaded



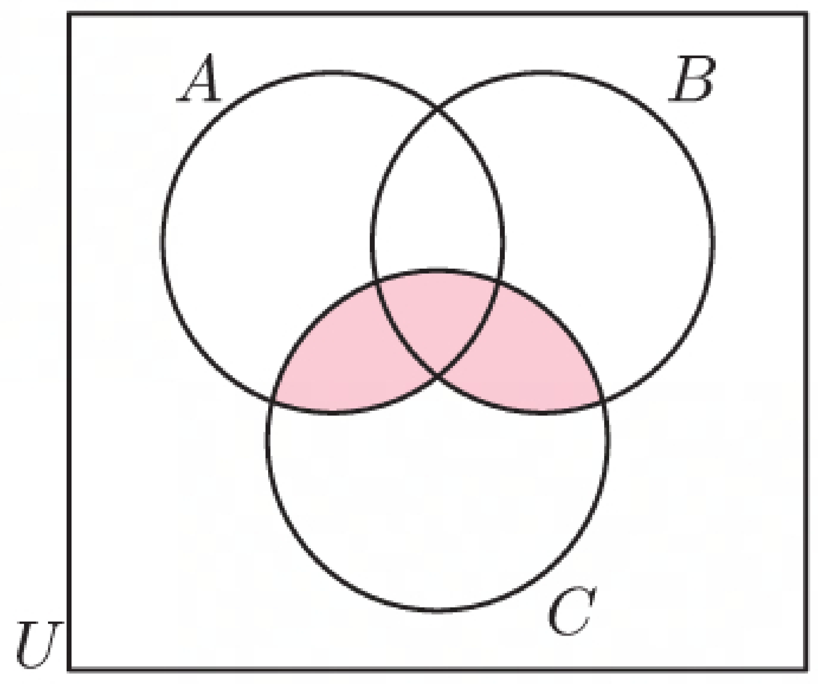
f $A \cup B \cup C$ is shaded



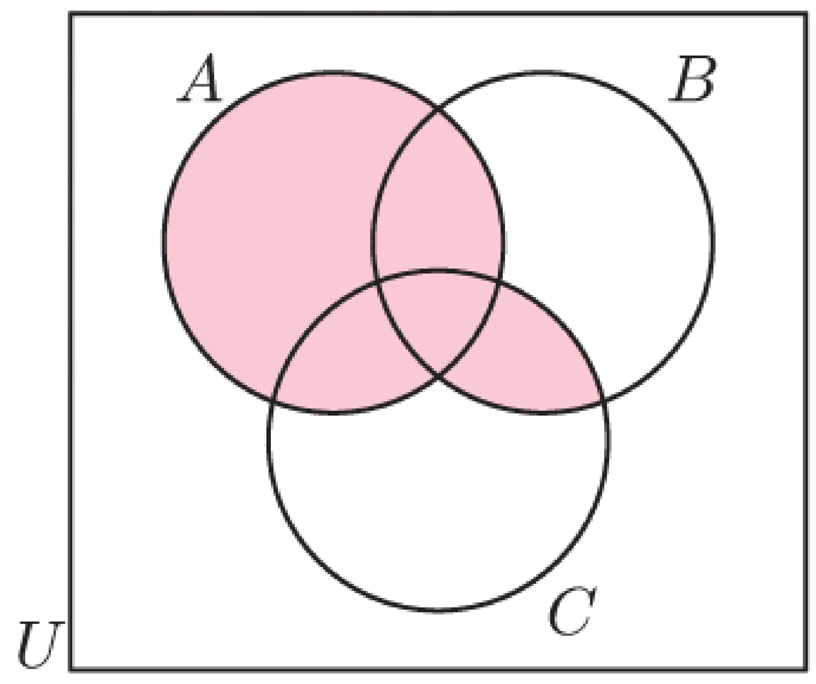
g $(A \cap B \cap C)'$ is shaded



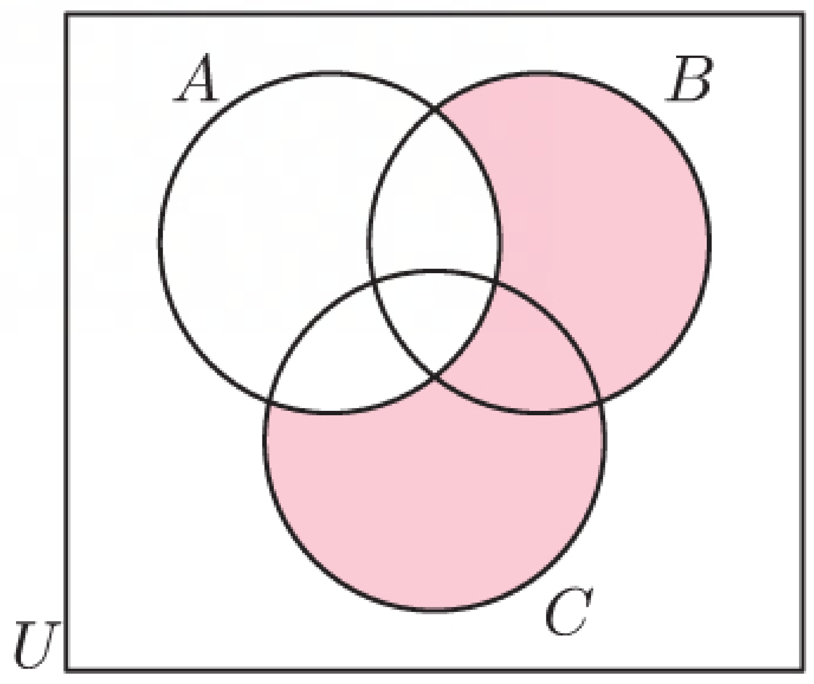
h $(A \cup B) \cap C$ is shaded



i $(B \cap C) \cup A$ is shaded

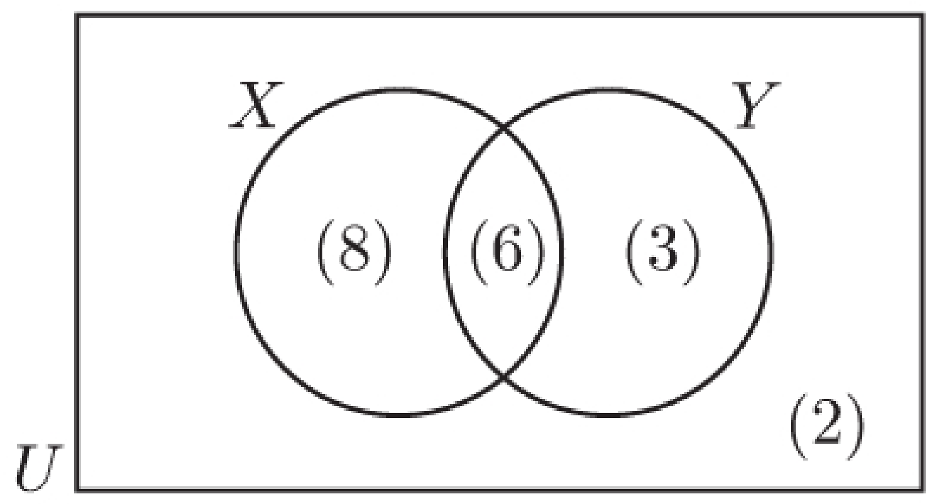


j $A' \cap (B \cup C)$ is shaded



EXERCISE 2G

1



a $n(X') = 3 + 2 = 5$ **b** $n(X \cap Y) = 6$

c $n(X \cup Y) = 8 + 6 + 3 = 17$

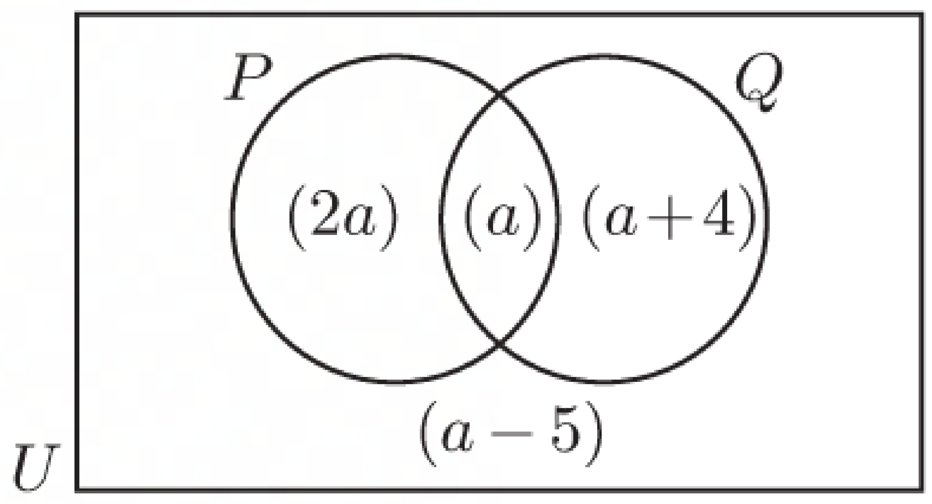
d $n(X, \text{ but not } Y) = 8$ **e** $n(Y, \text{ but not } X) = 3$

f $n(\text{neither } X \text{ nor } Y) = 2$

2 **a** **i** $n(P \cap Q) = a$

ii $n(P) = 2a + a = 3a$

iii $n(Q) = a + a + 4$
$= 2a + 4$



iv $n(P \cup Q) = 2a + a + a + 4$
$= 4a + 4$

v $n(Q') = 2a + (a - 5)$
$= 3a - 5$

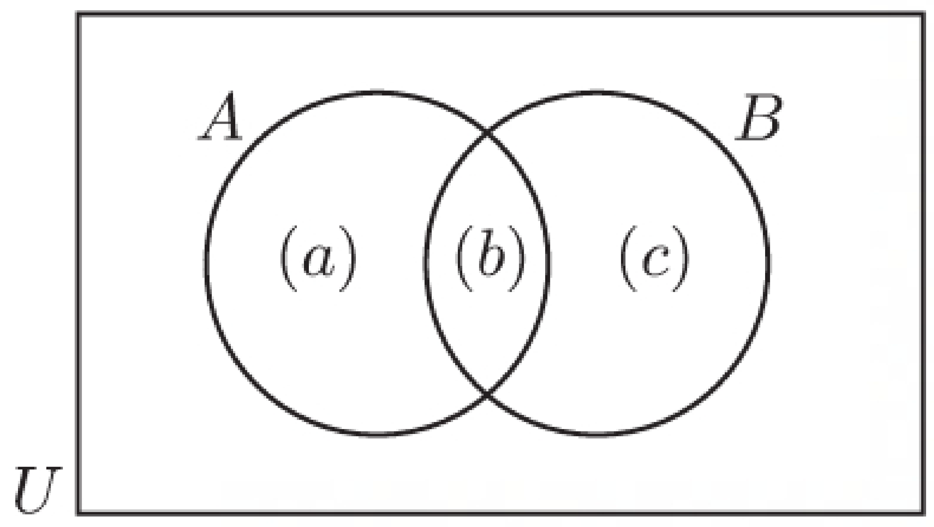
vi $n(U) = 2a + a + a + 4 + (a - 5)$
$= 5a - 1$

b **i** $n(U) = 29$
$\therefore\ 5a - 1 = 29$
$\therefore\ 5a = 30$
$\therefore\ a = 6$

ii $n(U) = 31$
$\therefore\ 5a - 1 = 31$
$\therefore\ 5a = 32$
$\therefore\ a = \frac{32}{5} = 6.4$

It is not possible to have a non-integer number of elements, as we have in **ii**.
$\therefore$ $n(U)$ can be equal to 29, but not equal to 31.

3



a $n(A \cup B) = a + b + c$ and

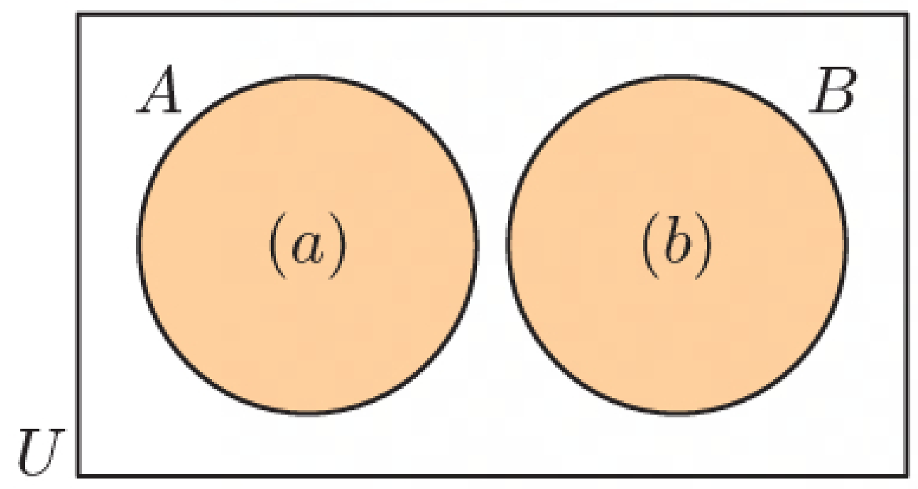
$$n(A) + n(B) - n(A \cap B) = a + b + b + c - b = a + b + c$$

$\therefore \quad n(A \cup B) = n(A) + n(B) - n(A \cap B)$

b $n(A \cap B') = a$ and $n(A) - n(A \cap B) = a + b - b = a$

$\therefore \quad n(A \cap B') = n(A) - n(A \cap B)$

4



$n(A \cup B) = a + b$ and $n(A) + n(B) = a + b$

$\therefore \quad n(A \cup B) = n(A) + n(B)$ for disjoint sets A and B.

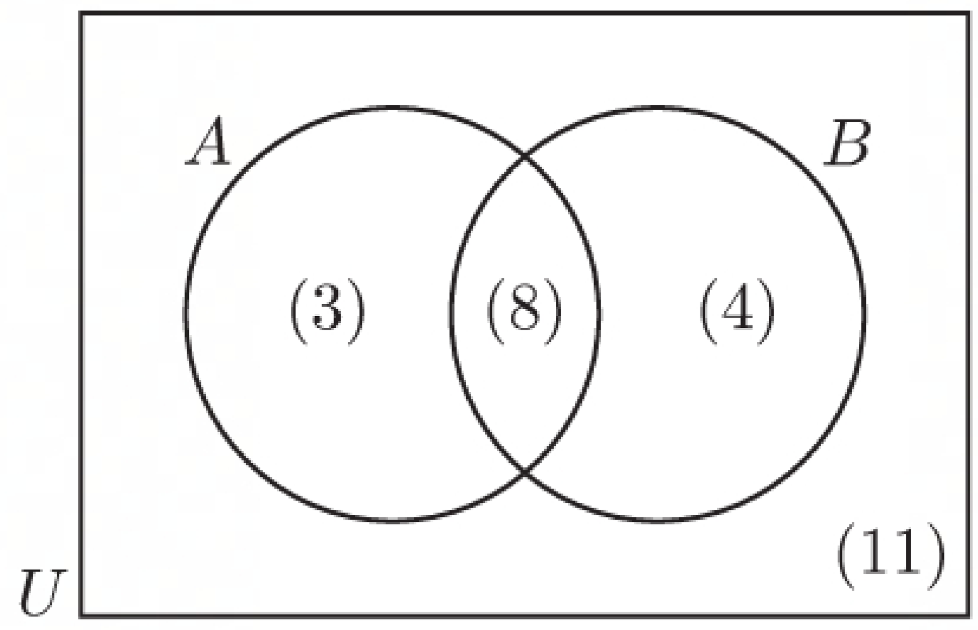
5 $n(U) = 26, \quad n(A) = 11, \quad n(B) = 12, \quad n(A \cap B) = 8$

We are given $n(A \cap B) = 8$

$\therefore \quad n(A \cap B') = 11 - 8 = 3$

and $n(A' \cap B) = 12 - 8 = 4$

$\therefore \quad n(A' \cap B') = 26 - 8 - 3 - 4 = 11$



a $n(A \cup B) = 3 + 8 + 4 = 15$

b $n(B$, but not $A) = 4$

6 $n(U) = 32, \quad n(M) = 13, \quad n(M \cap N) = 5, \quad n(M \cup N) = 26$

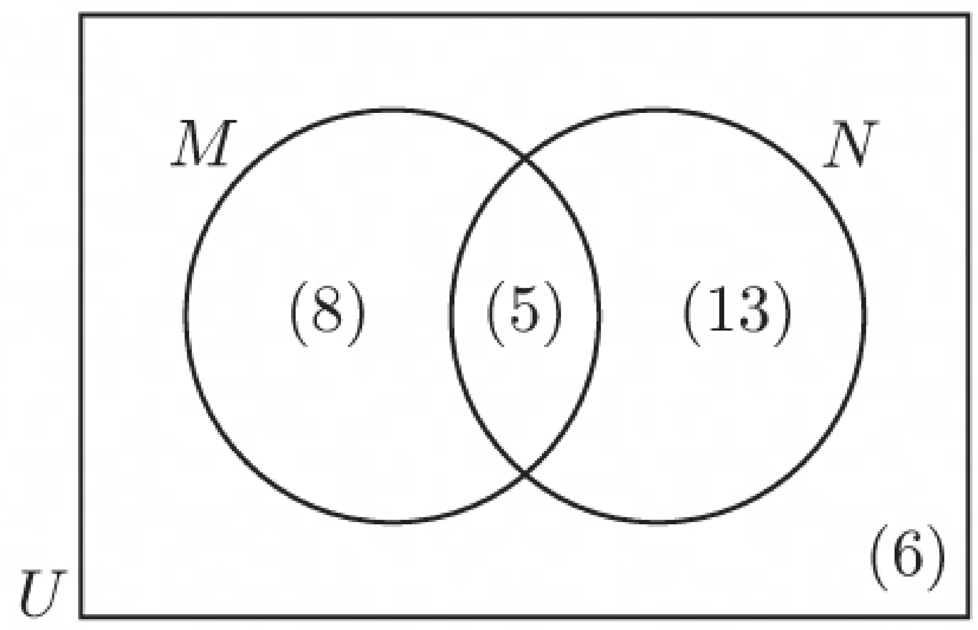
We are given $n(M \cap N) = 5$

$\therefore \quad n(M \cap N') = 13 - 5 = 8$

Also, $n(M \cup N) = 26$

$\therefore \quad n(M' \cap N) = 26 - 5 - 8 = 13$

$n(M' \cap N') = 32 - 26 = 6$



a $n(N) = 5 + 13 = 18$

b $n((M \cup N)') = 6$

7 $n(U) = 50, \quad n(S) = 30, \quad n(R) = 25, \quad n(R \cup S) = 48$

We are given $n(R \cup S) = 48$

$\therefore \quad n(R' \cap S') = 50 - 48 = 2$

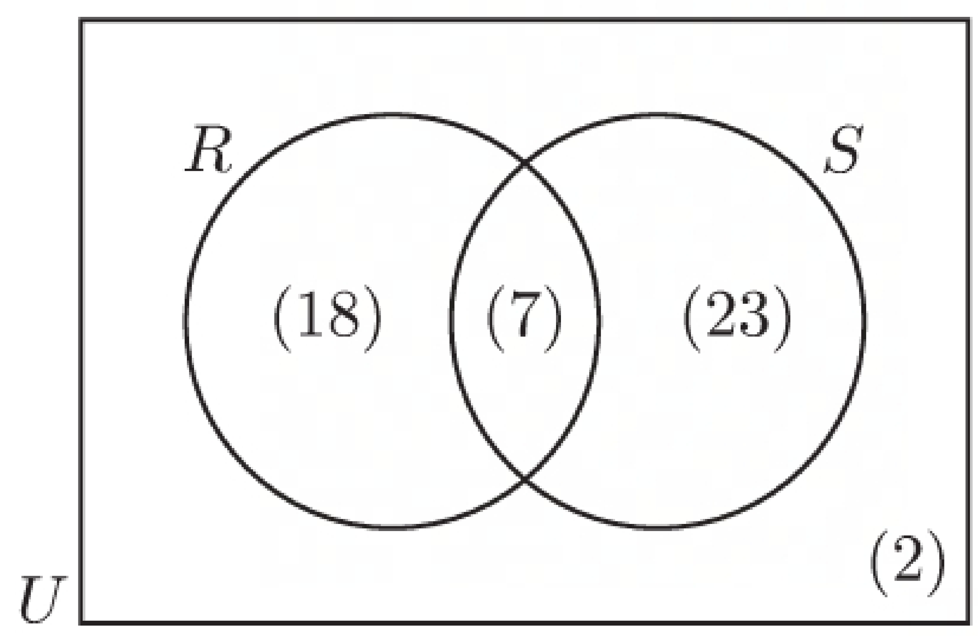
Also, $n(R \cup S) = n(R) + n(S) - n(R \cap S)$

$\therefore \quad 48 = 25 + 30 - n(R \cap S)$

$\therefore \quad n(R \cap S) = 7$

$n(R \cap S') = 25 - 7 = 18$

and $n(R' \cap S) = 30 - 7 = 23$

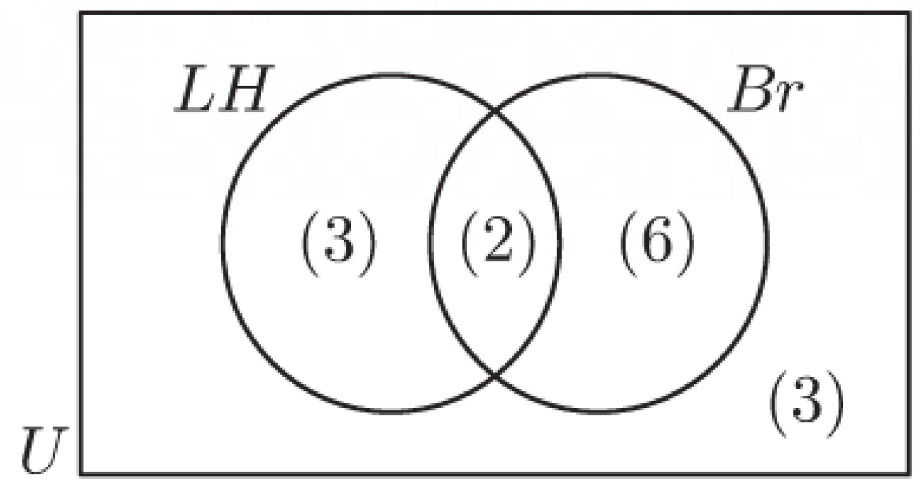


a $n(R \cap S) = 7$

b $n(S$, but not $R) = 23$

EXERCISE 2H

1 **a** Let LH represent those with long hair and Br represent those that are brown.



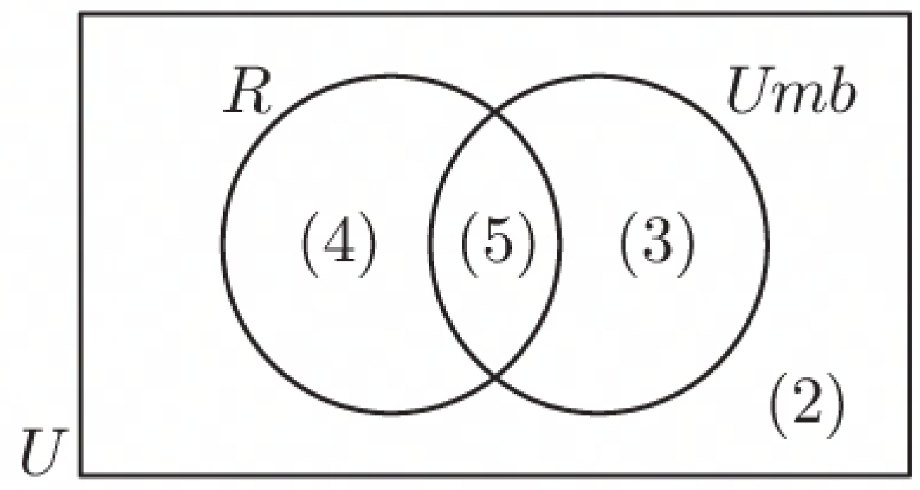
$$n(LH \cap Br) = 2$$
$$\therefore \; n(LH \cap Br') = 5 - 2 = 3$$
and $$n(LH' \cap Br) = 8 - 2 = 6$$
$$\therefore \; n(LH' \cap Br') = 14 - 2 - 3 - 6 = 3$$

b **i** $n(LH') = 6 + 3 = 9$
9 cavies do not have long hair.

ii $n(LH \cap Br') = 3$
3 cavies have long hair and are not brown.

iii $n((LH \cup Br)') = 3$
3 cavies are neither long-haired nor brown.

2 **a** Let R represent those days on which it rained and Umb represent those days on which Murielle took her umbrella.

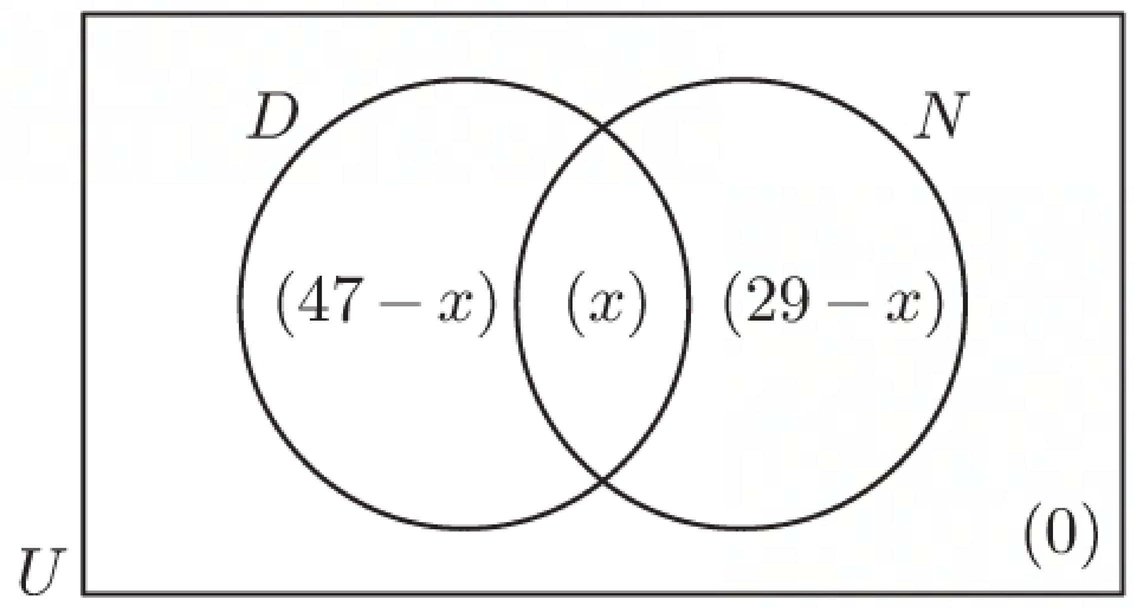


$$n(R \cap Umb) = 5$$
$$\therefore \; n(R \cap Umb') = 9 - 5 = 4$$
and $$n(R' \cap Umb) = 8 - 5 = 3$$
$$\therefore \; n(R' \cap Umb') = 14 - 5 - 4 - 3 = 2$$

b **i** $n(R \cap Umb') = 4$
Murielle did not take her umbrella and it rained on 4 days.

ii $n(R' \cap Umb') = 2$
Murielle did not take her umbrella and it did not rain on 2 days.

3 Let D represent those who work day shifts and
N represent those who work night shifts.



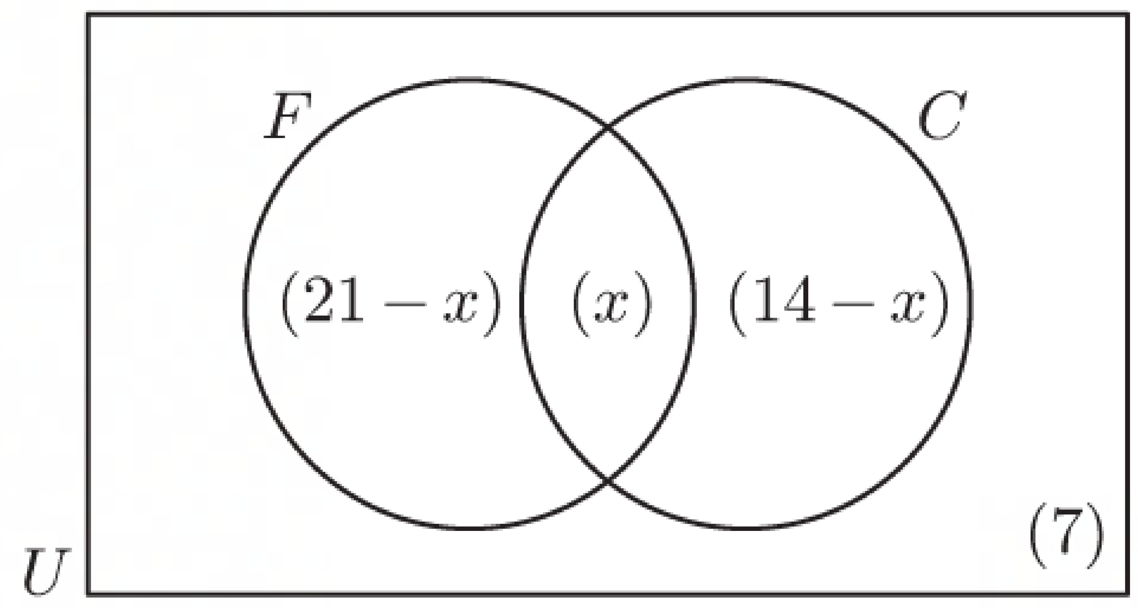
Let $n(D \cap N) = x$
$\therefore \; n(D \cap N') = 47 - x$ and $n(D' \cap N) = 29 - x$

$n(D' \cap N') = 0$ since every person in the factory must work either day shifts or night shifts.

But $n(U) = 56$, so $(47 - x) + x + (29 - x) = 56$
$$\therefore \; 76 - x = 56$$
$$\therefore \; x = 20$$

20 people work both day shifts and night shifts.

4 Let F represent the stalls which sell food and
C represent the stalls which sell craft.



Let $n(F \cap C) = x$

$\therefore \; n(F \cap C') = 21 - x$ and $n(F' \cap C) = 14 - x$

But $n(U) = 38$, so $(21 - x) + x + (14 - x) + 7 = 38$

$$\therefore \; 42 - x = 38$$
$$\therefore \; x = 4$$

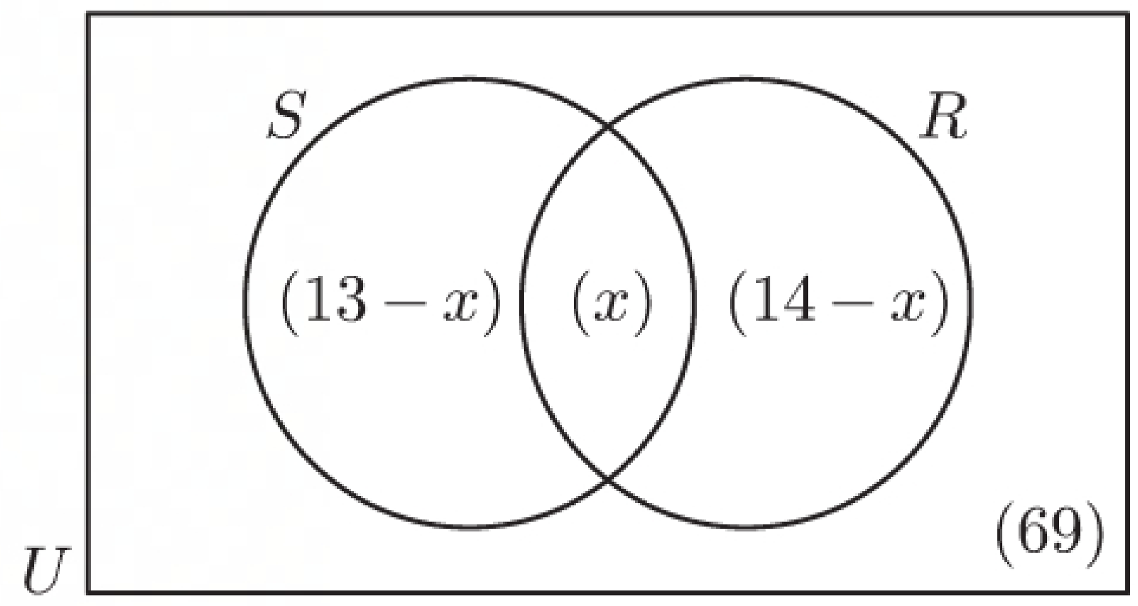
a $n(F \cap C) = x = 4$

4 stalls sell both food and craft.

b
$$\begin{aligned} n(F \cap C') + n(F' \cap C) &= (21 - x) + (14 - x) \\ &= 35 - 2x \\ &= 35 - 2(4) \qquad \{\text{as } x = 4\} \\ &= 27 \end{aligned}$$

27 stalls sell food or craft but not both.

5 Let S represent the movies seen by Sandra and
R represent the movies seen by Robert.



Let $n(S \cap R) = x$

$\therefore \; n(S \cap R') = 13 - x$ and $n(S' \cap R) = 14 - x$

But $n(U) = 86$, so $(13 - x) + x + (14 - x) + 69 = 86$

$$\therefore \; 96 - x = 86$$
$$\therefore \; x = 10$$

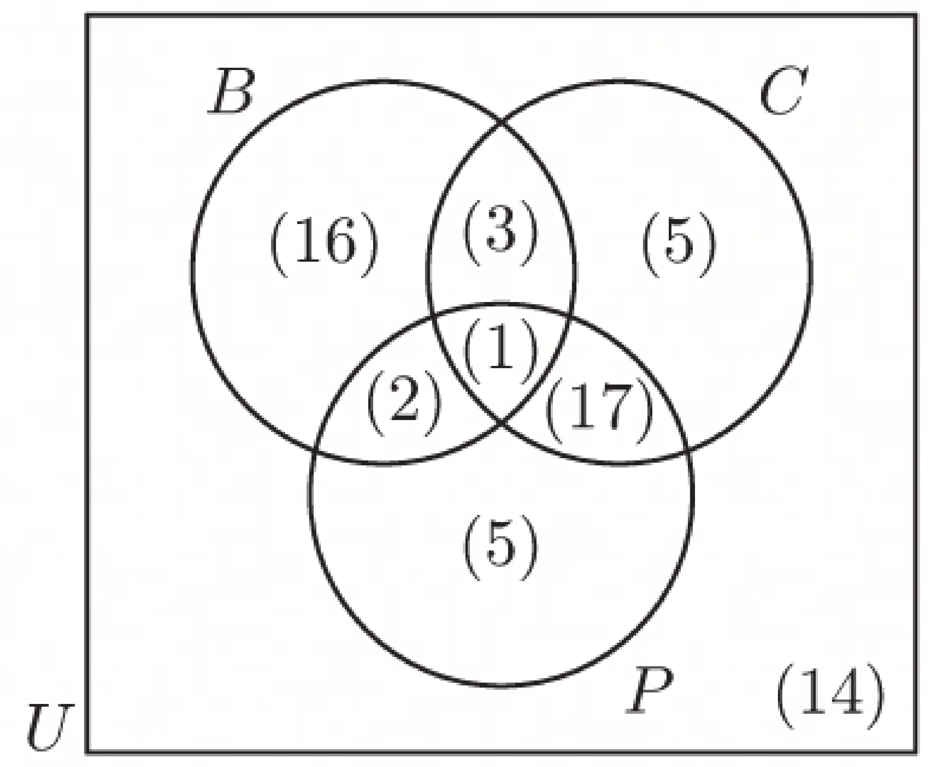
a $n(S \cap R) = x = 10$

10 movies have been seen by both Sandra and Robert.

b
$$\begin{aligned} n(S' \cap R) &= 14 - x \\ &= 14 - 10 \\ &= 4 \end{aligned}$$

4 movies have been seen by Robert but not Sandra.

6 **a** Let B, C, and P represent the students studying Biology, Chemistry, and Physics respectively.



$n(B \cap C \cap P) = 1$

$n(P \cap C) = 18$

$\therefore \; n(P \cap C \cap B') = 18 - 1 = 17$

$n(B \cap C) = 4$

$\therefore \; n(B \cap C \cap P') = 4 - 1 = 3$

$n(P \cap B) = 3$

$\therefore \; n(P \cap B \cap C') = 3 - 1 = 2$

Also, $n(B) = 22$, so $n(B \cap C' \cap P') = 22 - 3 - 1 - 2 = 16$

$n(C) = 26$, so $n(C \cap B' \cap P') = 26 - 3 - 1 - 17 = 5$

$n(P) = 25$, so $n(P \cap B' \cap C') = 25 - 2 - 1 - 17 = 5$

Now $n(U) = 63$

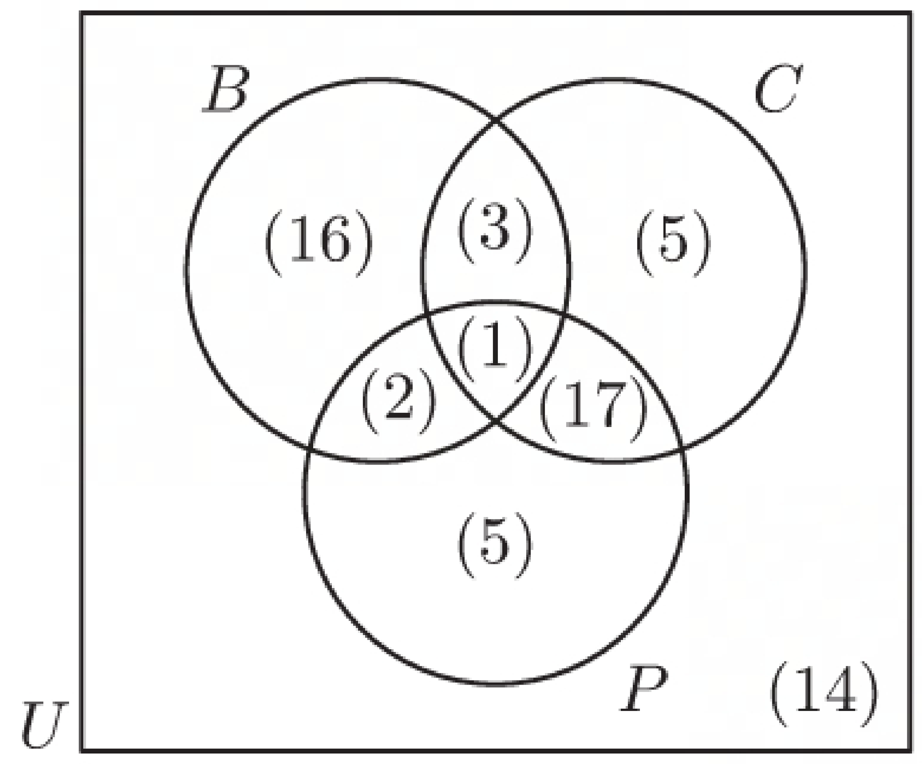
$\therefore \; n((B \cup C \cup P)') = 63 - 16 - 3 - 1 - 2 - 5 - 17 - 5 = 14$

b **i** $n(B \cap C' \cap P') = 16$
16 students study Biology only.

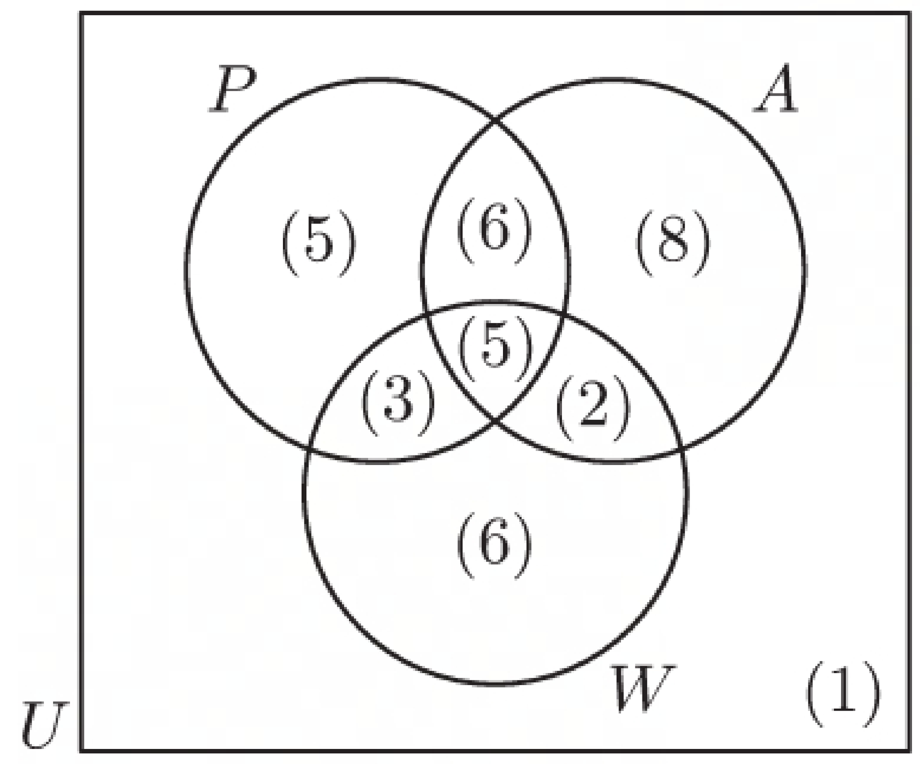
ii $n(P \cup C) = 5 + 2 + 1 + 17 + 3 + 5 = 33$
33 students study Physics or Chemistry.

iii $n((B \cup P \cup C)') = 14$
14 students study none of Biology, Physics, or Chemistry.

iv $n(P \cap C') = 5 + 2 = 7$
7 students study Physics but not Chemistry.



7 Let P, A, and W represent the students who went paragliding, abseiling, and white water rafting respectively.



$$n(P \cap A \cap W) = 5$$
$$n(A \cap W) = 7$$
$$\therefore\ n(A \cap W \cap P') = 7 - 5 = 2$$
$$n(P \cap W) = 8$$
$$\therefore\ n(P \cap W \cap A') = 8 - 5 = 3$$
$$n(P \cap A) = 11$$
$$\therefore\ n(P \cap A \cap W') = 11 - 5 = 6$$

Also, $n(P) = 19$, so $n(P \cap A' \cap W') = 19 - 6 - 5 - 3 = 5$
$n(A) = 21$, so $n(A \cap P' \cap W') = 21 - 6 - 5 - 2 = 8$
and $n(W) = 16$, so $n(W \cap P' \cap A') = 16 - 3 - 5 - 2 = 6$

Now $n(U) = 36$
$$\therefore\ n((P \cup A \cup W)') = 36 - 5 - 6 - 5 - 3 - 8 - 2 - 6 = 1$$

a $n(P \cup A) = 5 + 6 + 5 + 3 + 8 + 2 = 29$
29 students went paragliding or abseiling.

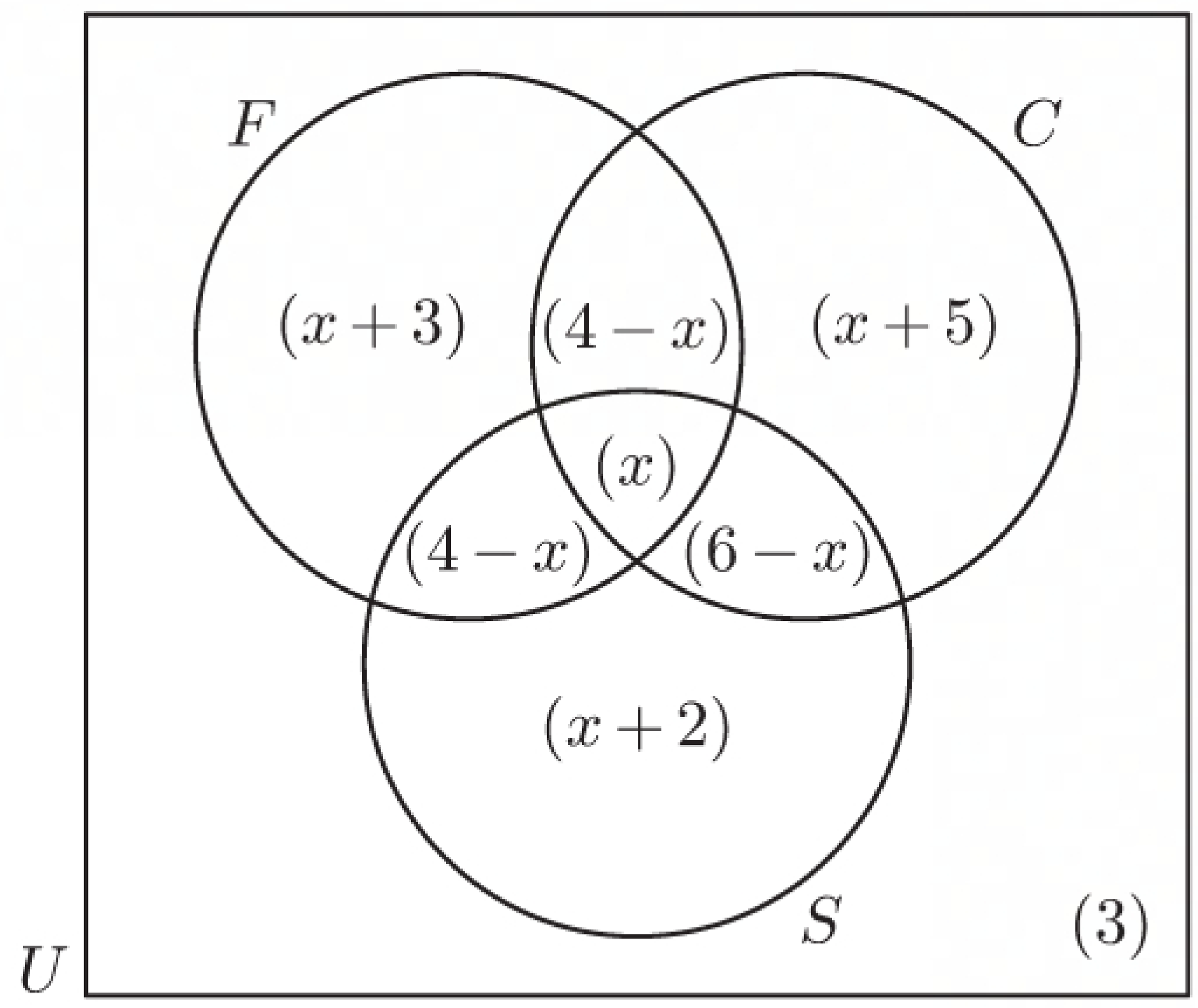
b $n(W \cap P' \cap A') = 6$
6 students only went white water rafting.

c $n((P \cup A \cup W)') = 1$
1 student did not participate in any of the activities mentioned.

d $n(P \cap A \cap W') + n(A \cap W \cap P') + n(P \cap W \cap A') = 6 + 2 + 3 = 11$
11 students did exactly two of the activities mentioned.

8 **a** Let F, C, and S represent the students who can play the flute, clarinet, and saxophone respectively.

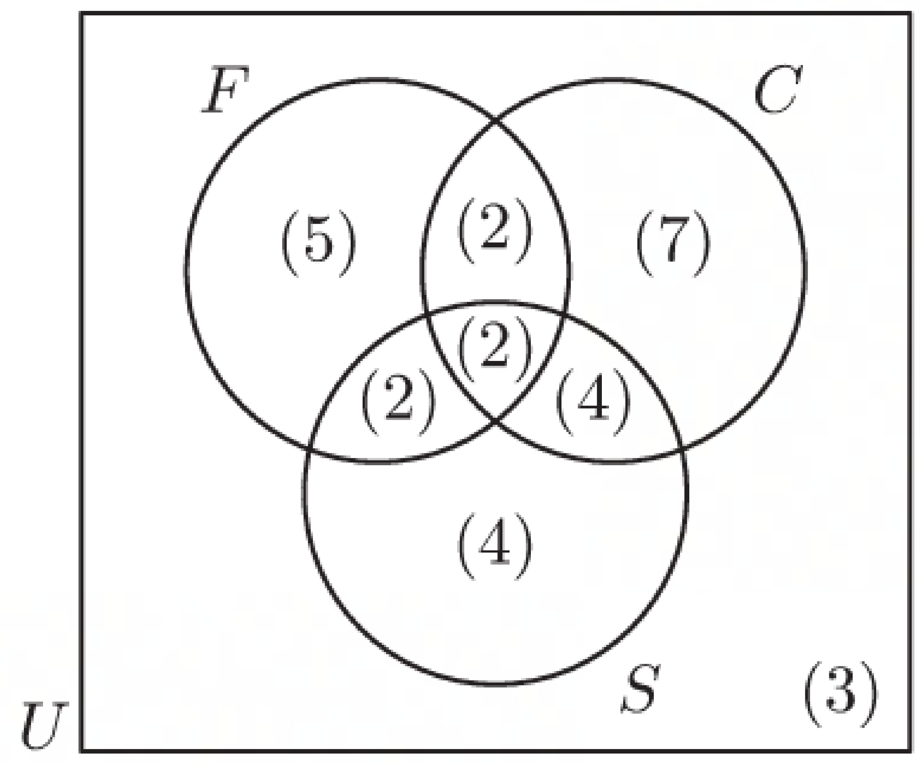
Let $n(F \cap C \cap S) = x$
$$\therefore\ n(F \cap C \cap S') = 4 - x$$
$$n(F \cap C' \cap S) = 4 - x$$
and $n(F' \cap C \cap S) = 6 - x$



$$\begin{aligned}
\text{Now}\quad n(F) = 11, &\quad \text{so}\quad n(F \cap C' \cap S') = 11 - x - (4 - x) - (4 - x) = x + 3 \\
n(C) = 15, &\quad \text{so}\quad n(F' \cap C \cap S') = 15 - x - (4 - x) - (6 - x) = x + 5 \\
\text{and}\quad n(S) = 12, &\quad \text{so}\quad n(F' \cap C' \cap S) = 12 - x - (4 - x) - (6 - x) = x + 2
\end{aligned}$$

$$\text{Now}\quad n(U) = 29$$
$$\therefore\quad (x+3) + (4-x) + x + (4-x) + (x+5) + (6-x) + (x+2) + 3 = 29$$
$$\therefore\quad x + 27 = 29$$
$$\therefore\quad x = 2$$

So, the Venn diagram is:



b **i** $n(F \cap C \cap S) = 2$

2 students can play all of the instruments mentioned.

ii $n(F' \cap C' \cap S) = 4$

4 students can play only the saxophone.

iii $n(F' \cap C \cap S) = 4$

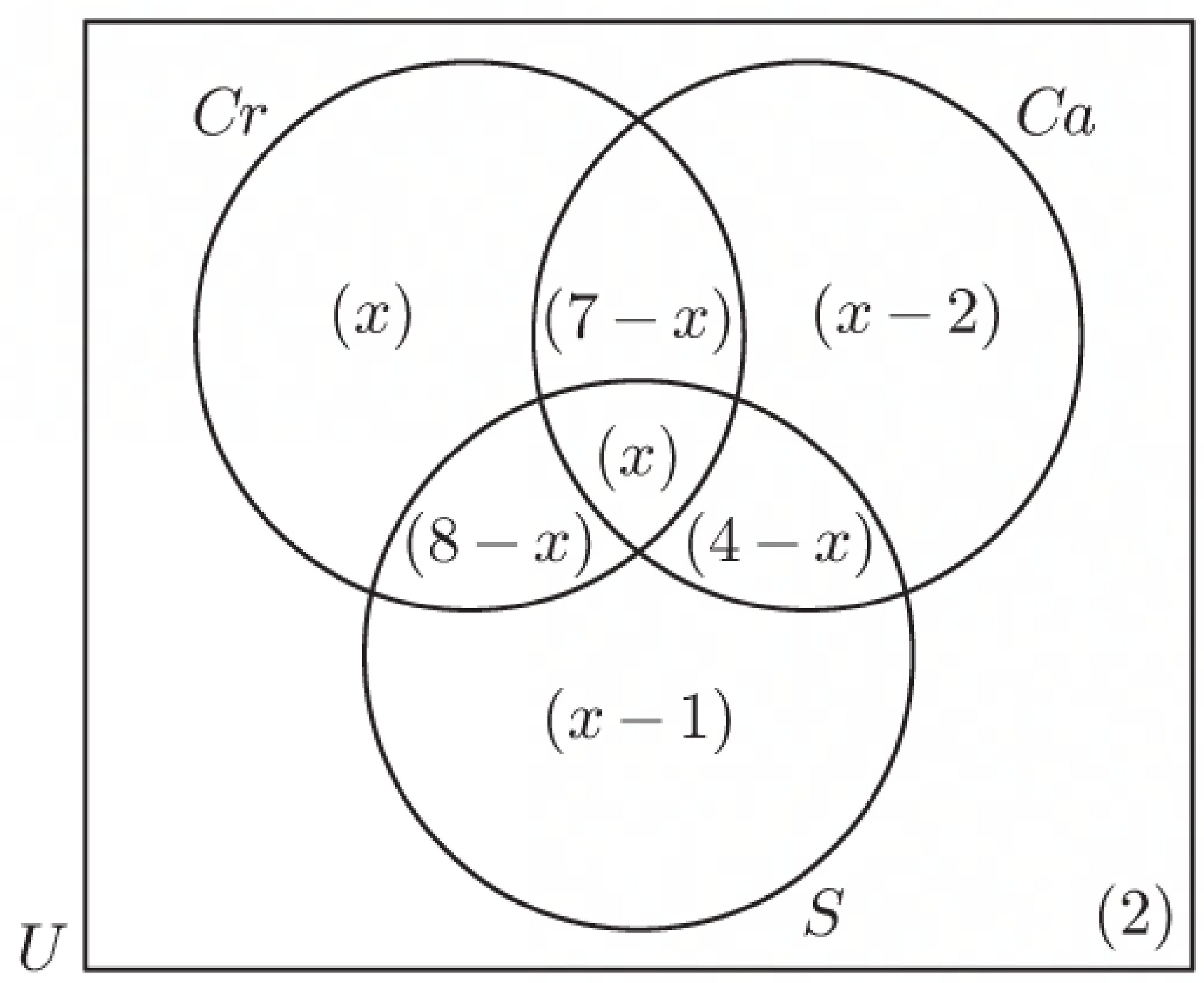
4 students can play the saxophone and the clarinet, but not the flute.

iv $n(F \cap C' \cap S') + n(F' \cap C \cap S') + n(F' \cap C' \cap S) = 5 + 7 + 4 = 16$

16 students can play exactly one of the clarinet, saxophone, or flute.

9 **a** Let Cr, Ca, and S represent the farms which have crops, cattle, and sheep respectively.

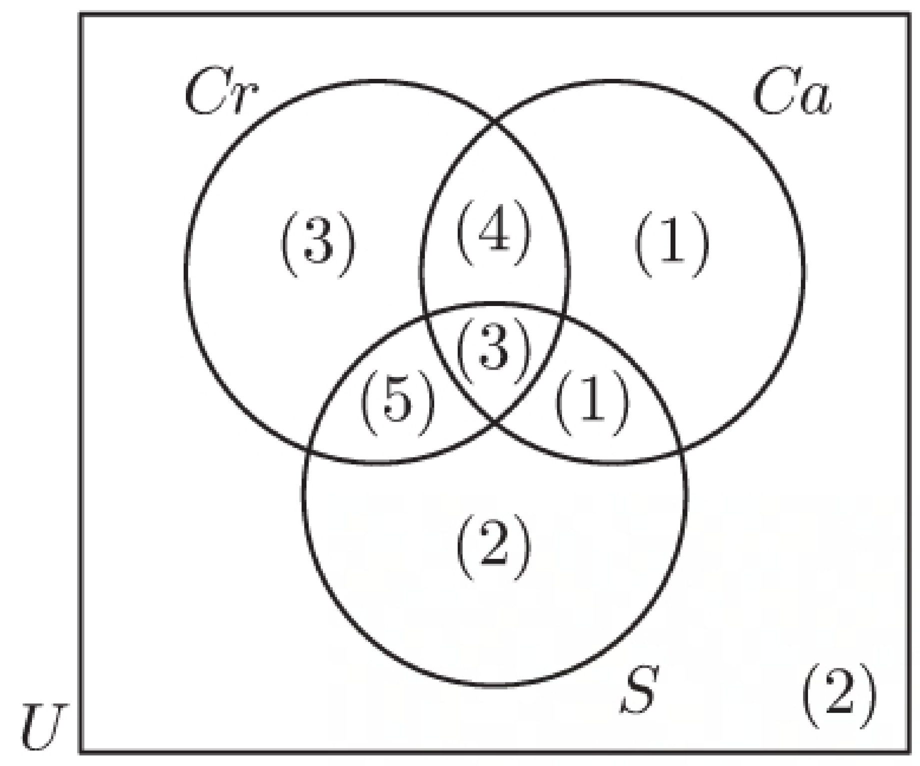
$$\begin{aligned}
\text{Let}\quad n(Cr \cap Ca \cap S) &= x \\
\therefore\quad n(Cr \cap Ca \cap S') &= 7 - x \\
n(Cr \cap Ca' \cap S) &= 8 - x \\
\text{and}\quad n(Cr' \cap Ca \cap S) &= 4 - x
\end{aligned}$$



$$\begin{aligned}
\text{Now}\quad n(Cr) = 15, &\quad \text{so}\quad n(Cr \cap Ca' \cap S') = 15 - x - (7 - x) - (8 - x) = x \\
n(Ca) = 9, &\quad \text{so}\quad n(Cr' \cap Ca \cap S') = 9 - x - (7 - x) - (4 - x) = x - 2 \\
\text{and}\quad n(S) = 11, &\quad \text{so}\quad n(Cr' \cap Ca' \cap S) = 11 - x - (8 - x) - (4 - x) = x - 1
\end{aligned}$$

$$\text{Now}\quad n(U) = 21$$
$$\therefore\quad x + (7-x) + x + (8-x) + (x-2) + (4-x) + (x-1) + 2 = 21$$
$$\therefore\quad x + 18 = 21$$
$$\therefore\quad x = 3$$

So, the Venn diagram is:



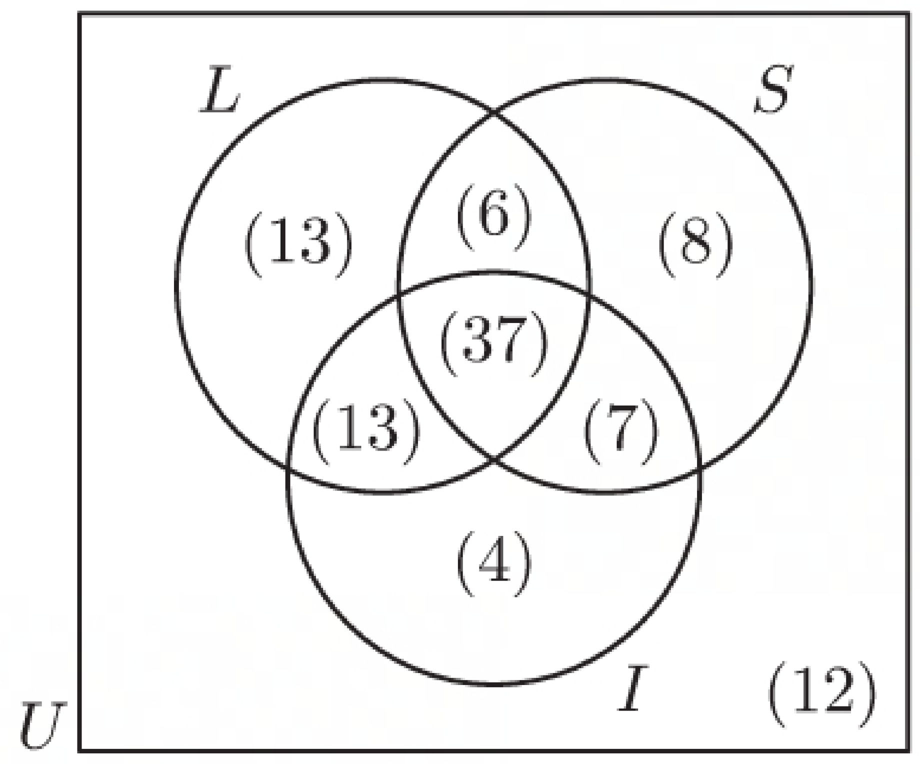
b **i** $n(Cr \cap Ca' \cap S') = 3$
3 farms have only crops.

ii $n(Cr' \cap (Ca \cup S)) = 1 + 1 + 2 = 4$
4 farms have only animals.

iii $n(Cr \cap Ca \cap S') + n(Cr \cap Ca' \cap S) = 4 + 5 = 9$
9 farms have exactly one type of animal, and crops.

10 **a** Let L represent the nations with a life expectancy of more than 75 years, S represent the nations with mean years of schooling greater than 10, and I represent the nations with a gross national income more than \$18 000 USD per capita.

$n(L \cap S \cap I) = 37$
$n(L \cap I) = 50$
$\therefore\ n(L \cap S' \cap I) = 50 - 37 = 13$
$n(S \cap I) = 44$
$\therefore\ n(L' \cap S \cap I) = 44 - 37 = 7$
$n(L \cap S) = 43$
$\therefore\ n(L \cap S \cap I') = 43 - 37 = 6$



Also, $n(L) = 69$, so $n(L \cap S' \cap I') = 69 - 6 - 37 - 13 = 13$
$n(S) = 58$, so $n(L' \cap S \cap I') = 58 - 6 - 37 - 7 = 8$
and $n(I) = 61$, so $n(L' \cap S' \cap I) = 61 - 13 - 37 - 7 = 4$

Now $n(U) = 100$
$\therefore\ n((L \cup S \cup I)') = 100 - 13 - 6 - 37 - 13 - 8 - 7 - 4 = 12$

b $n((L \cup S \cup I)') = 12$
12 nations were not in any of L, S, or I.

c **i** $n(L' \cap S \cap I') = 8$
8 nations were in S only.

ii $n((L \cup I) \cap S') = 13 + 13 + 4 = 30$
30 nations were in L or I but not S.

iii $n((S \cap I) \cap L') = 7$
7 nations were in S and I but not L.

REVIEW SET 2A

1 **a** $A = \{$letters in the word VENN$\}$
$= \{$V, E, N$\}$

$B = \{$letters in the word DIAGRAM$\}$
$= \{$D, I, A, G, R, M$\}$

b $n(A) = 3, \quad n(B) = 6$

c $A \cap B = \varnothing$, 'VENN' and 'DIAGRAM' have no letters in common.

d **i** V $\in A$
$\therefore$ V $\notin A$ is a false statement.

ii G $\in B$
$\therefore$ G $\in B$ is a true statement.

iii $A \cup B = \{$V, E, N, D, I, A, G, R, M$\}$
$\therefore \; n(A \cup B) = 9$
$n(A) + n(B) = 3 + 6 = 9 = n(A \cup B)$
$\therefore \; n(A \cup B) = n(A) + n(B)$ is a true statement.

2 $U = \{$multiples of 6 less than 70$\}$
$= \{6, 12, 18, 24, 30, 36, 42, 48, 54, 60, 66\}$
$A = \{6, 6^2, 66\} = \{6, 36, 66\}$
A' is the set of all elements of U that are not elements of A.
$\therefore \; A' = \{12, 18, 24, 30, 42, 48, 54, 60\}$

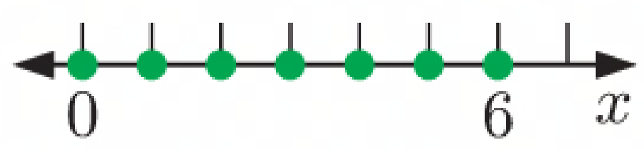
3 **a** $\mathbb{N} \subseteq \mathbb{Q}$ since every element of $\mathbb{N}$ is also an element of $\mathbb{Q}$.
However $\frac{1}{2}, \frac{1}{4}, \frac{2}{3}, \frac{3}{5}, ...$ are all elements of $\mathbb{Q}$ but not elements of $\mathbb{N}$, so $\mathbb{N} \neq \mathbb{Q}$.
$\therefore \; \mathbb{N} \subset \mathbb{Q}$ is a true statement.

b $0 \in \mathbb{Z}$ but $0 \notin \mathbb{Z}^+$
$\therefore \; 0 \in \mathbb{Z}^+$ is a false statement.

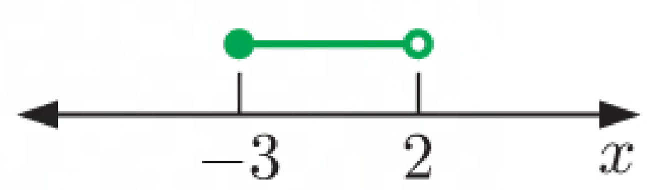
c 0 can be written as $\frac{0}{3}$ or $\frac{0}{7}$, and so on, and 0, 3, and 7 are integers.
$\therefore \; 0 \in \mathbb{Q}$ is a true statement.

d $\mathbb{R}$ contains irrational numbers such as π and $\sqrt{5}$ which are not in $\mathbb{Q}$.
$\therefore \; \mathbb{R} \subseteq \mathbb{Q}$ is a false statement.

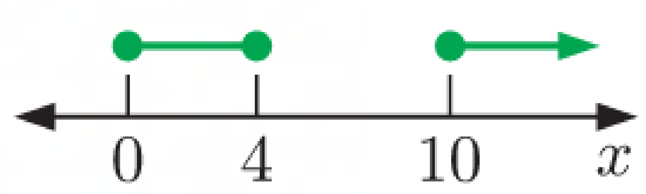
4 **a** $\{x \in \mathbb{N} \mid x \leqslant 6\}$ can be represented by:



b $\{x \in \mathbb{R} \mid -3 \leqslant x < 2\}$ can be represented by:



c $x \in [0, 4] \; \cup \; x \in [10, \infty[$ can be represented by:

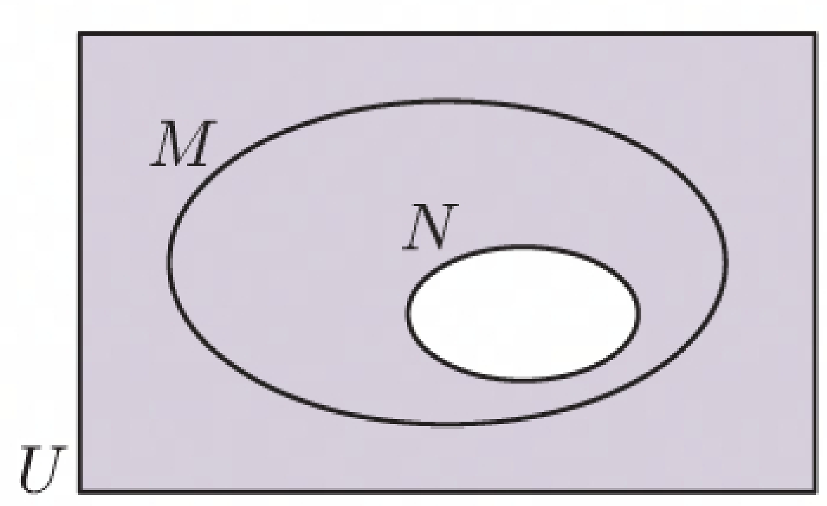


5 $U = \{x \in \mathbb{Z}^+ \mid x \leqslant 30\}$, $P = \{$factors of 24$\}$, $Q = \{$factors of 30$\}$

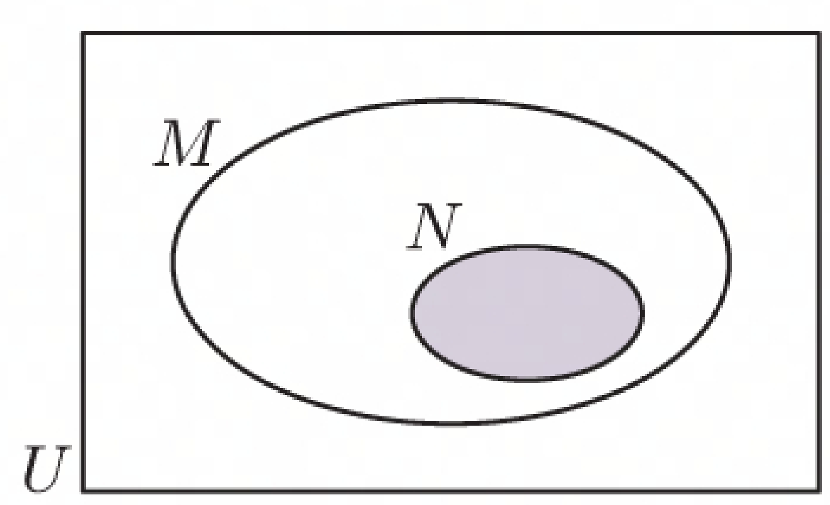
a **i** $P = \{1, 2, 3, 4, 6, 8, 12, 24\}$
ii $Q = \{1, 2, 3, 5, 6, 10, 15, 30\}$
iii $P \cap Q = \{1, 2, 3, 6\}$
iv $P \cup Q = \{1, 2, 3, 4, 5, 6, 8, 10, 12, 15, 24, 30\}$

b $n(P \cup Q) = 12$ and $n(P) + n(Q) - n(P \cap Q) = 8 + 8 - 4 = 12$
$\therefore \; n(P \cup Q) = n(P) + n(Q) - n(P \cap Q)$

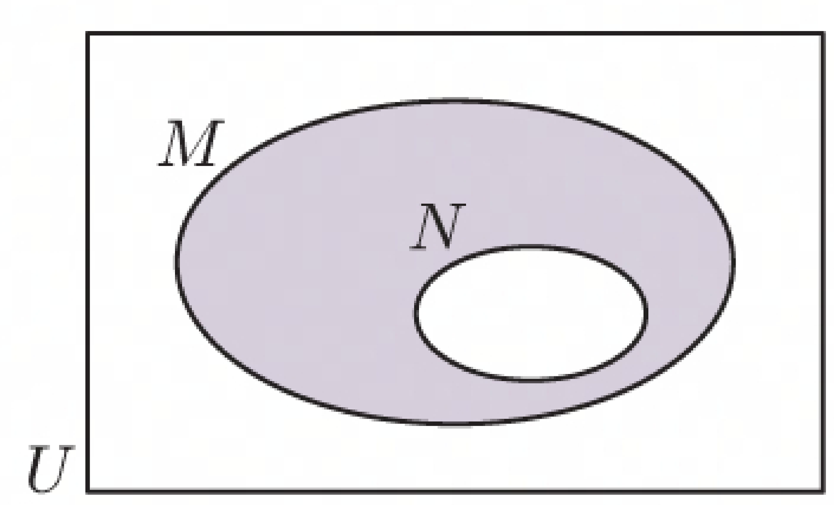
6 **a** N' is shaded



b $M \cap N$ is shaded



c $M \cap N'$ is shaded



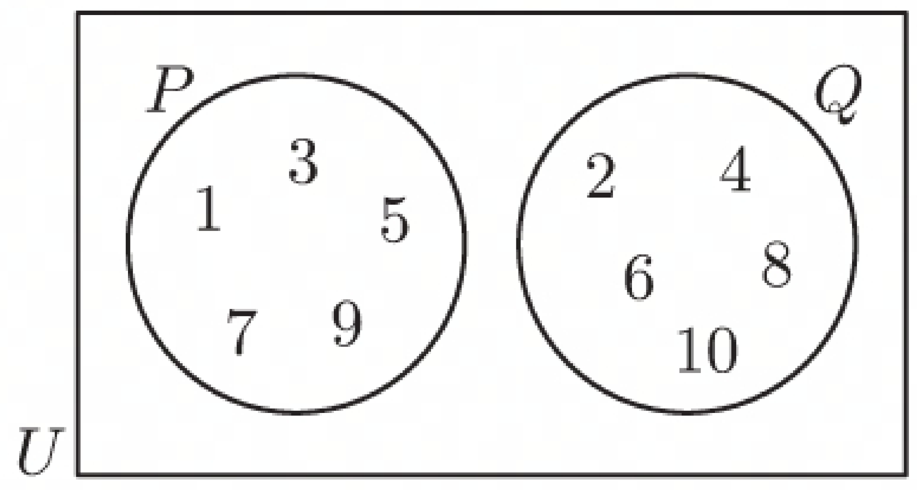
7 As $P \subseteq Q$, then all elements of P are also in Q.
An element which is not in Q must not be in P either.
$\therefore$ $P \cap Q' = \varnothing$, P and Q' are disjoint.

8 $U = \{x \in \mathbb{Z}^+ \mid x \leqslant 10\}$, $P = \{\text{odd numbers less than } 10\}$, $Q = \{\text{even numbers less than } 11\}$

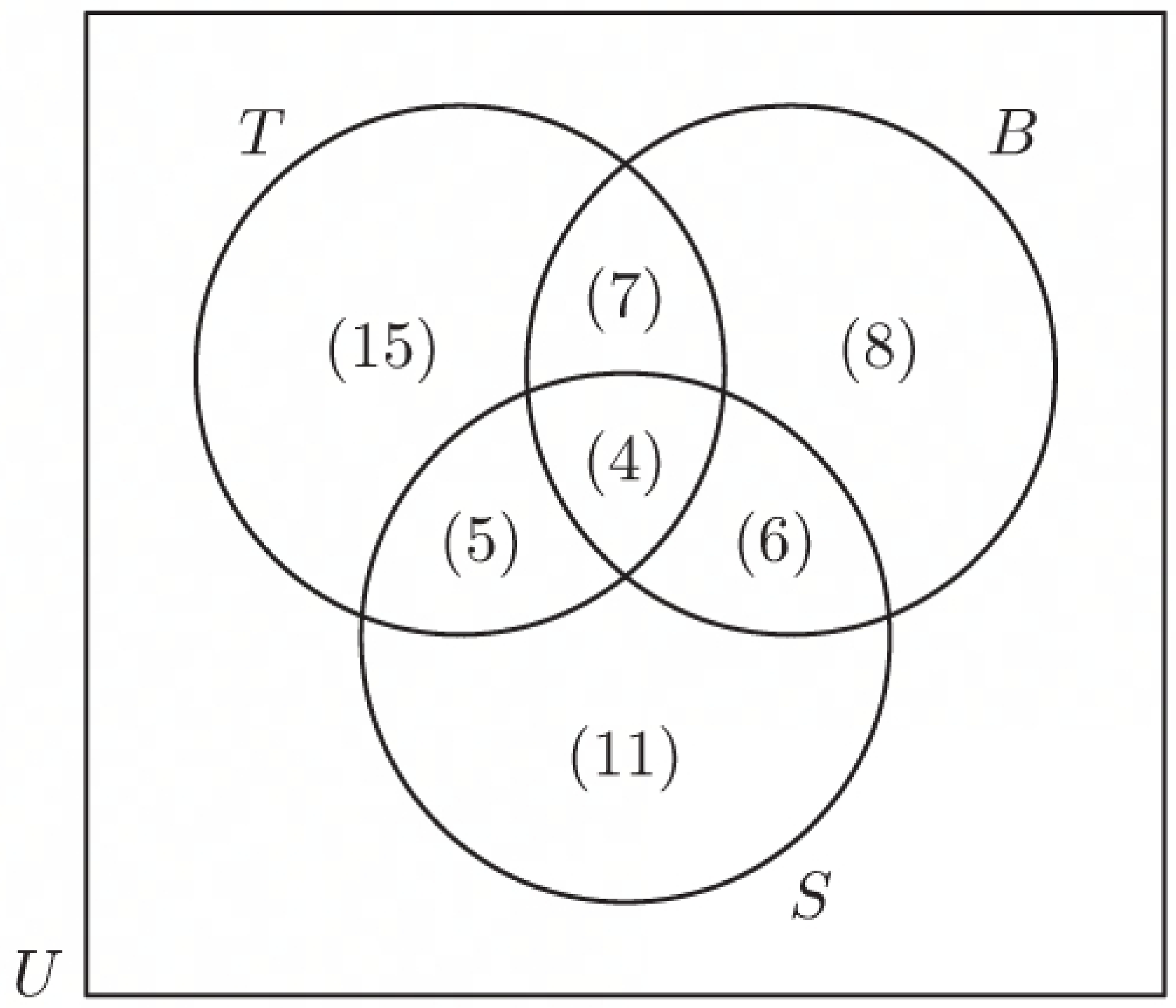
a $P = \{1, 3, 5, 7, 9\}$, $Q = \{2, 4, 6, 8, 10\}$

b P and Q are disjoint, as P and Q have no elements in common.

c



9



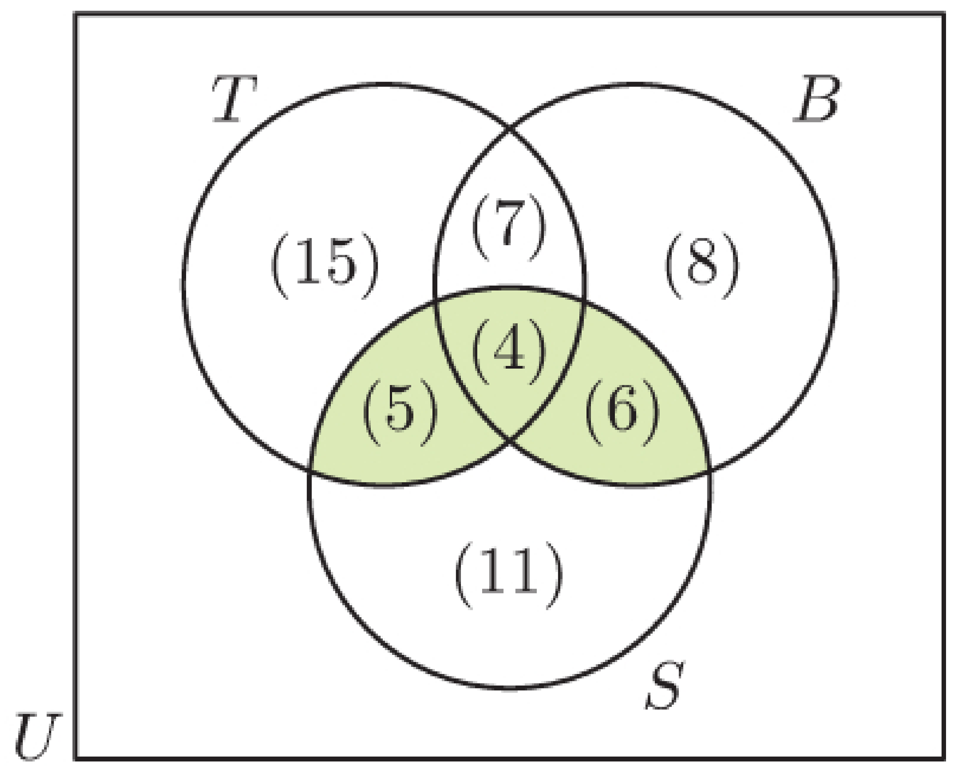
a Number of members in the club
$= 15 + 7 + 4 + 5 + 8 + 6 + 11$
$= 56$ members

b **i** Number of members who only play badminton
$= 8$ members

ii Number of members who do not play tennis
$= 8 + 6 + 11$
$= 25$ members

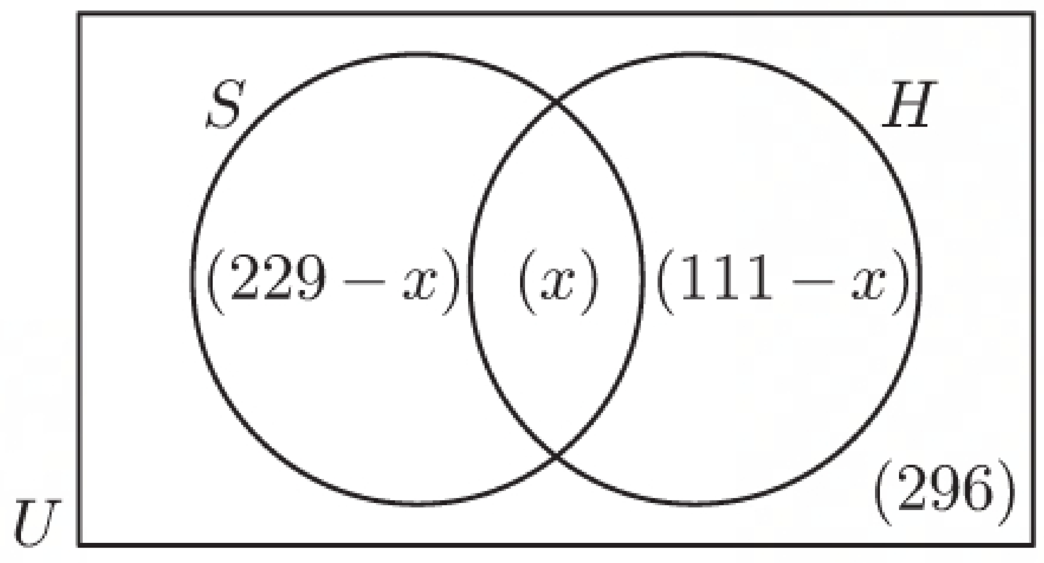
iii Number of members who play both tennis and squash, but not badminton $= 5$ members

c $S \cap (T \cup B)$ is shaded



d $n(S \cap (T \cup B)') = 11$ members

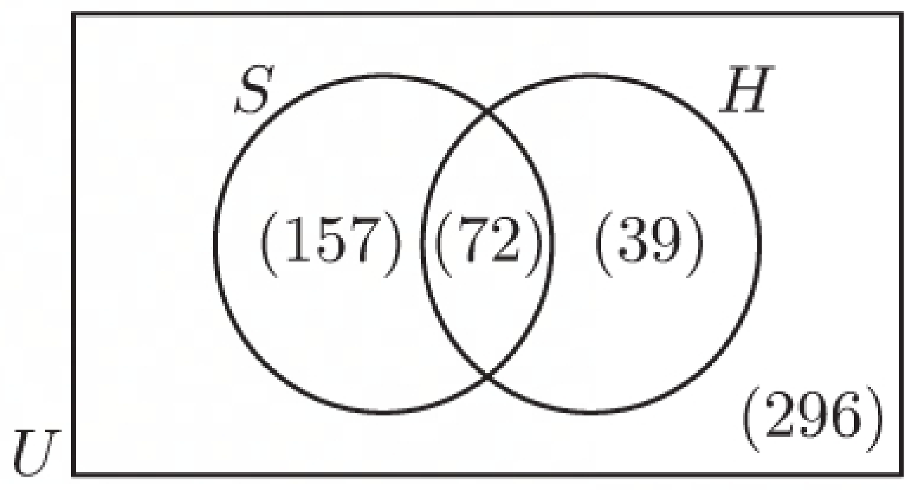
10 **a** Let S represent those who were absent for at least one day due to sickness, and H represent those who missed some school because of family holidays.



Let $n(S \cap H) = x$
$\therefore$ $n(S \cap H') = 229 - x$
and $n(S' \cap H) = 111 - x$

But $n(U) = 564$, so $(229 - x) + x + (111 - x) + 296 = 564$
$\therefore$ $636 - x = 564$
$\therefore$ $x = 72$

So, the Venn diagram is:



b **i** $n(S \cap H) = 72$

72 students missed school for both illness and holidays.

ii $n(S' \cap H) = 39$

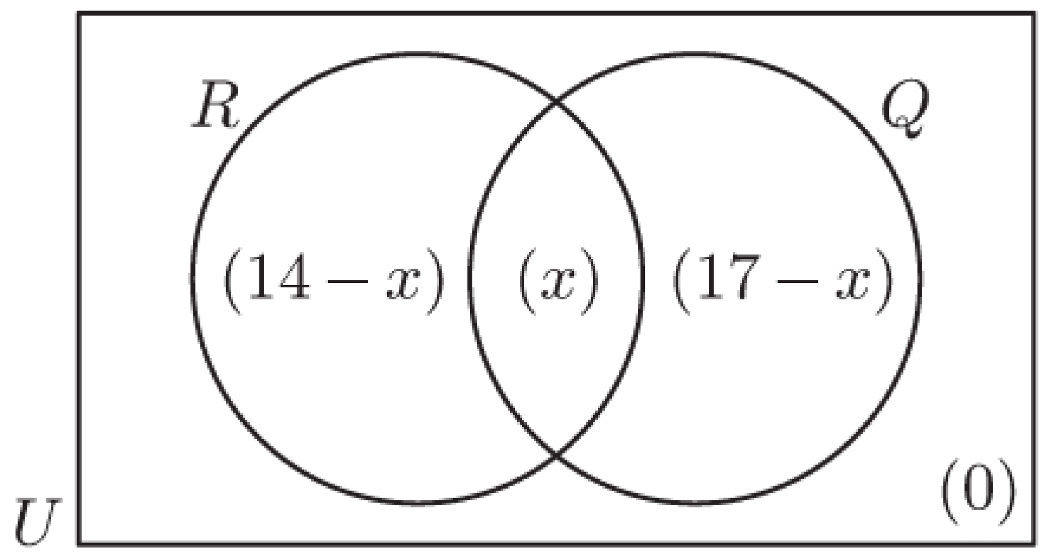
39 students were away for holidays but not sickness.

iii $n(S \cup H) = 157 + 72 + 39$

$= 268$

268 students were absent during Term 1 for any reason.

11 Let R represent the meals which contain rice and
O represent the meals which contain onion.



Let $n(R \cap O) = x$

$\therefore\ n(R \cap O') = 14 - x$

and $n(R' \cap O) = 17 - x$

$n(R' \cap O') = 0$ since every main course contains rice or onion.

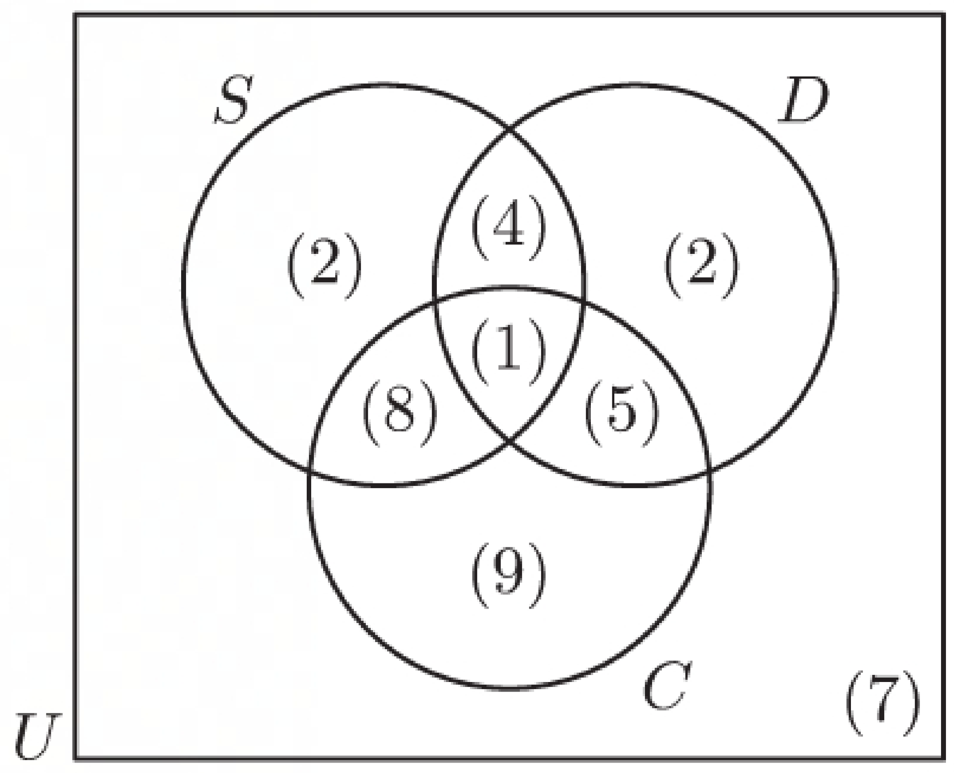
But $n(U) = 23$, so $(14 - x) + x + (17 - x) = 23$

$\therefore\ 31 - x = 23$

$\therefore\ x = 8$

8 dishes contain both rice and onion.

12 Let S, D, and C represent the students who could swim, drive, and cook respectively.



$n(S \cap D \cap C) = 1$

$n(S \cap C) = 9$

$\therefore\ n(S \cap D' \cap C) = 9 - 1 = 8$

$n(S \cap D) = 5$

$\therefore\ n(S \cap D \cap C') = 5 - 1 = 4$

$n(D \cap C) = 6$

$\therefore\ n(S' \cap D \cap C) = 6 - 1 = 5$

Also, $n(S) = 15$, so $n(S \cap D' \cap C') = 15 - 4 - 1 - 8 = 2$

$n(D) = 12$, so $n(S' \cap D \cap C') = 12 - 4 - 1 - 5 = 2$

and $n(C) = 23$, so $n(S' \cap D' \cap C) = 23 - 8 - 1 - 5 = 9$

Now $n(U) = 38$

$\therefore\ n((S \cup D \cup C)') = 38 - 2 - 4 - 1 - 8 - 2 - 5 - 9 = 7$

a $n(S' \cap D' \cap C) = 9$

9 students could only cook.

b $n((S \cup D \cup C)') = 7$

7 students could not do any of these things.

c $n(S \cap D \cap C') + n(S' \cap D \cap C) + n(S \cap D' \cap C) = 4 + 5 + 8 = 17$

17 students had exactly two of these life skills.

REVIEW SET 2B

1 The subsets of $\{1, 3, 5\}$ are $\varnothing$, $\{1\}$, $\{3\}$, $\{5\}$, $\{1, 3\}$, $\{1, 5\}$, $\{3, 5\}$, $\{1, 3, 5\}$.

2 **a** S and T are disjoint, so they do not have any common elements.
$\therefore\ S \cap T = \varnothing$

b $n(S) = s$ and $n(T) = t$, where S and T are disjoint.
$\therefore\ n(S \cup T) = s + t$

3 **a** The set of real numbers between 5 and 12 can be represented by $\{x \in \mathbb{R} \mid 5 < x < 12\}$. This set has an endless number of elements, so it is an infinite set.

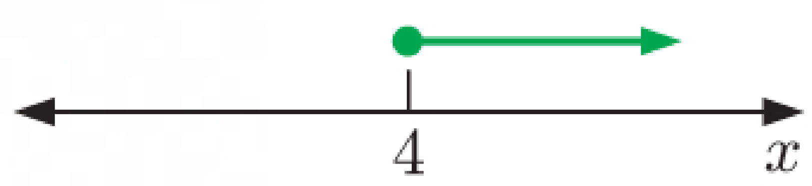
b The set of integers between -4 and 7, including -4, can be represented by $\{x \in \mathbb{Z} \mid -4 \leqslant x < 7\}$. The number of elements in this set is a particular defined value, so it is a finite set.

c The set of natural numbers greater than 45 can be represented by $\{x \in \mathbb{N} \mid x > 45\}$. This set has an endless number of elements, so it is an infinite set.

4 **a**

can be represented by $x \in \,]2, 5]$.

b



can be represented by $x \in [4, \infty[$.

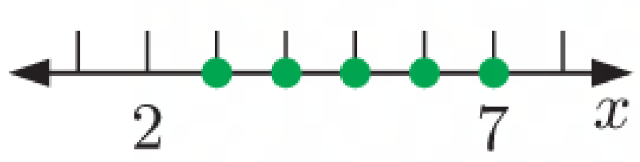
c

can be represented by $x \in \,]-\infty, -3] \cup [1, \infty[$.

5 $S = \{x \in \mathbb{Z} \mid 2 < x \leqslant 7\}$

a $S = \{3, 4, 5, 6, 7\}$

b



c $n(S) = 5$

6 **a** $A = \{2, 4, 6, 8\}$ and $B = \{x \in \mathbb{Z} \mid 0 < x < 10\} = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$
Every element of A is also an element of B.
$\therefore\ A \subseteq B$

b $A = \varnothing$ and $B = \{x \mid 2 < x < 3\}$
The empty set $\varnothing$ is a subset of all other sets.
$\therefore\ A \subseteq B$

c $A = \{x \in \mathbb{Q} \mid 2 < x \leqslant 4\}$ and $B = \{x \in \mathbb{R} \mid 0 \leqslant x < 4\}$
The element 4 is in A but not in B.
$\therefore\ A \nsubseteq B$

d $A = \{x \mid x < 3\}$ and $B = \{x \mid x \leqslant 4\}$
Every element of A is also an element of B.
$\therefore\ A \subseteq B$

7 **a** $U = \{\text{the 7 colours of the rainbow}\}$
$= \{\text{red, orange, yellow, green, blue, indigo, violet}\}$
$X = \{\text{red, indigo, violet}\}$
X' is the set of all elements of U that are not elements of X.
$\therefore \quad X' = \{\text{orange, yellow, green, blue}\}$

b $U = \{x \in \mathbb{Z} \mid -5 \leqslant x \leqslant 5\}$
$= \{-5, -4, -3, -2, -1, 0, 1, 2, 3, 4, 5\}$
$X = \{-4, -1, 3, 4\}$
X' is the set of all elements of U that are not elements of X.
$\therefore \quad X' = \{-5, -3, -2, 0, 1, 2, 5\}$

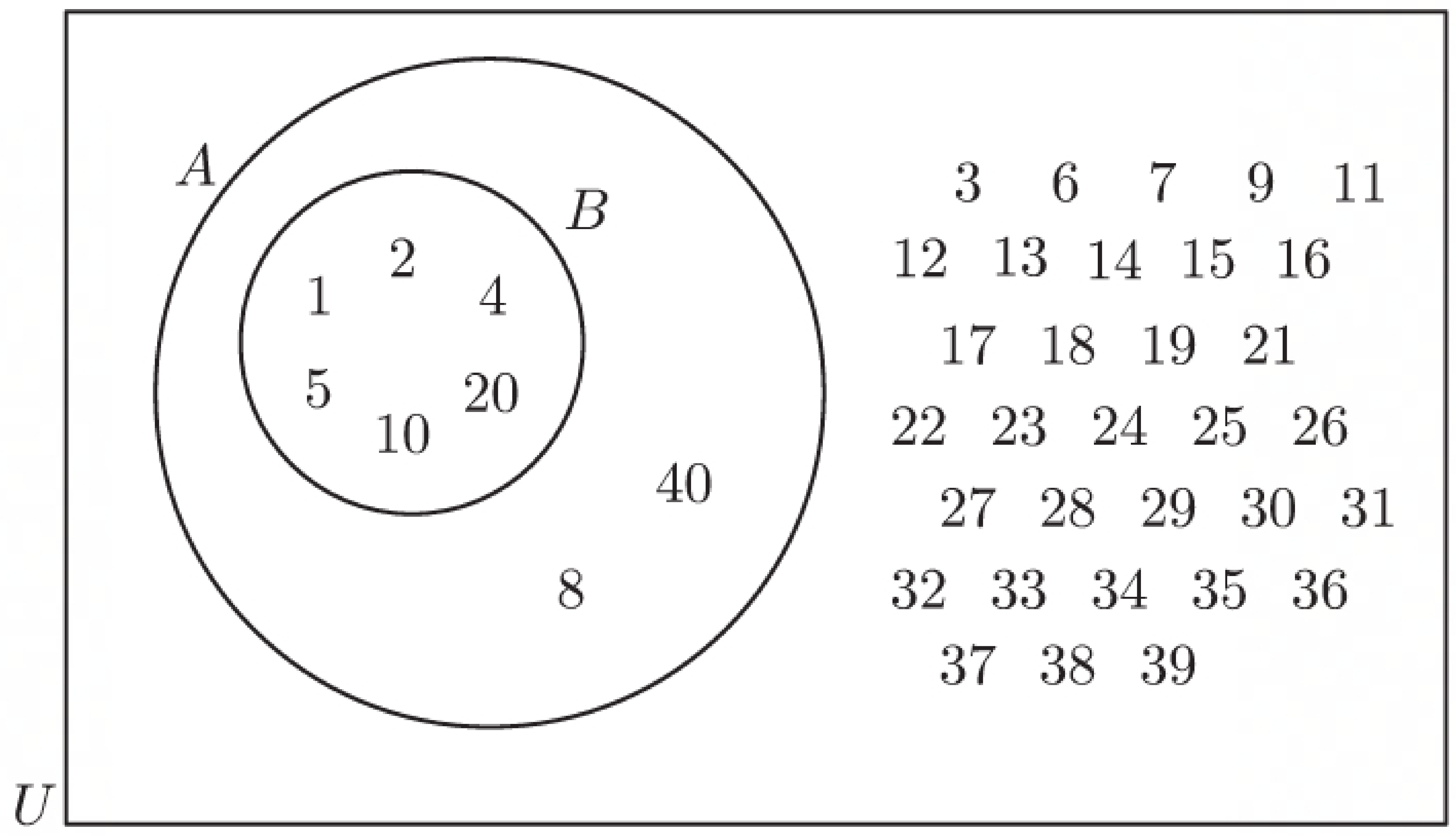
c $U = \{x \in \mathbb{Q}\}$ and $X = \{x \in \mathbb{Q} \mid x < -8\}$
X' is the set of all elements of U that are not elements of X.
$\therefore \quad X' = \{x \in \mathbb{Q} \mid x \geqslant -8\}$

8 $U = \{x \in \mathbb{Z}^+ \mid x \leqslant 40\}$, $A = \{\text{factors of } 40\}$, $B = \{\text{factors of } 20\}$

a $A = \{1, 2, 4, 5, 8, 10, 20, 40\}$, $B = \{1, 2, 4, 5, 10, 20\}$

b $B \subseteq A$ since every element of B is also an element of A.
However, 8 and 40 are both elements of A but not elements of B, so $B \neq A$.
$\therefore \quad B \subset A$

c



9 $P = \{x \in \mathbb{Z} \mid 3 \leqslant x < 10\}$, $Q = \{2, 9, 15\}$, $R = \{\text{multiples of 3 less than 12}\}$

a $P = \{3, 4, 5, 6, 7, 8, 9\}$

b $n(P) = 7$

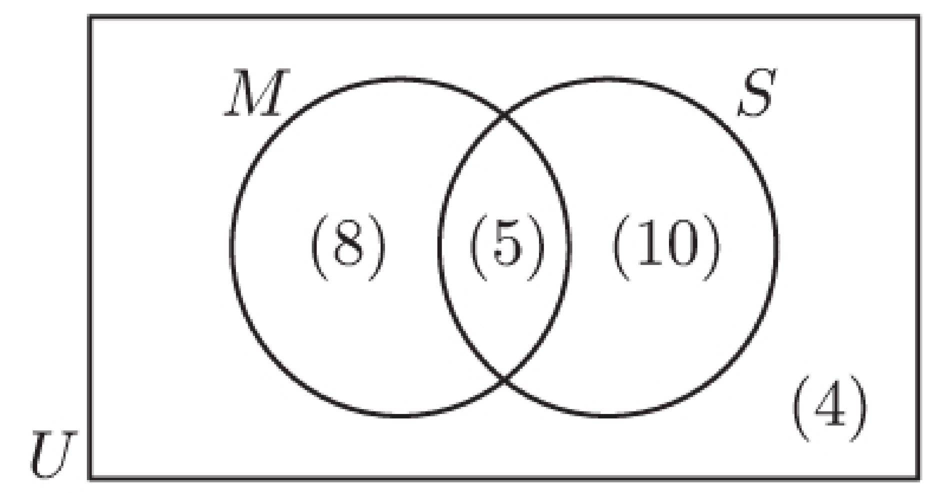
c The number of elements in P is a particular defined value, so P is a finite set.

d **i** The elements 2 and 15 are in Q but not in P.
$\therefore \quad Q \nsubseteq P$ which also means $Q \not\subset P$.

ii $R = \{3, 6, 9\}$ and all of these elements are also in P.
However, 4, 5, 7, and 8 are all elements of P but not elements of R, so $R \neq P$.
$\therefore \quad R \subset P$

e **i** $P \cap Q = \{9\}$ **ii** $R \cap Q = \{9\}$ **iii** $R \cup Q = \{2, 3, 6, 9, 15\}$

10 **a** Let M represent those who drive a manual car and
S represent those who have a car with a sunroof.



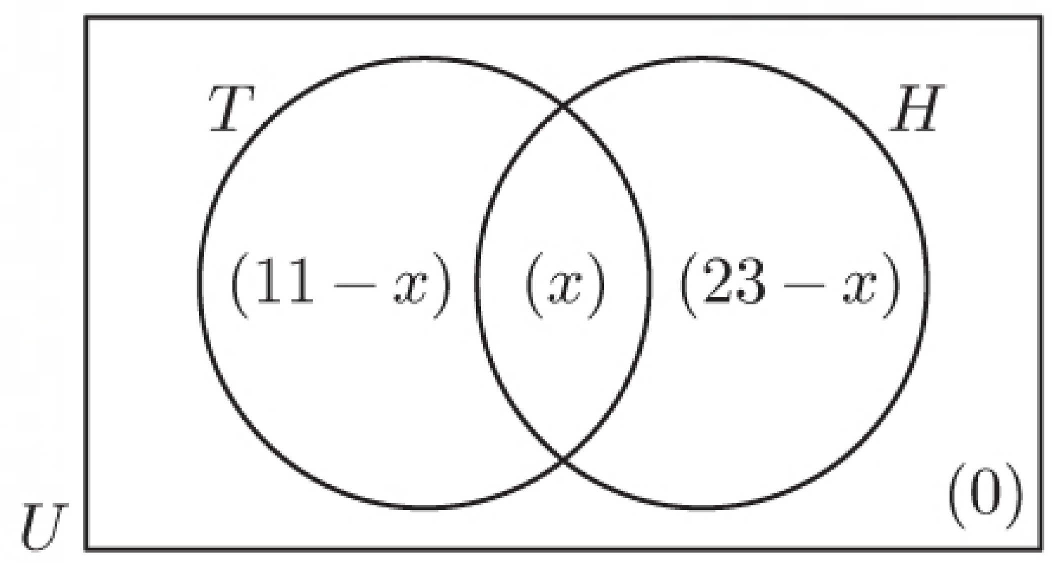
$n(M \cap S) = 5$
$\therefore\ n(M \cap S') = 13 - 5 = 8$
and $n(M' \cap S) = 15 - 5 = 10$
$n((M \cup S)') = 4$

b **i** Number of members in the club $= 8 + 5 + 10 + 4$
$= 27$ members

ii $n(M \cap S') = 8$
8 members drive a manual car without a sunroof.

iii $n(M') = 10 + 4 = 14$
14 members do not drive a manual car.

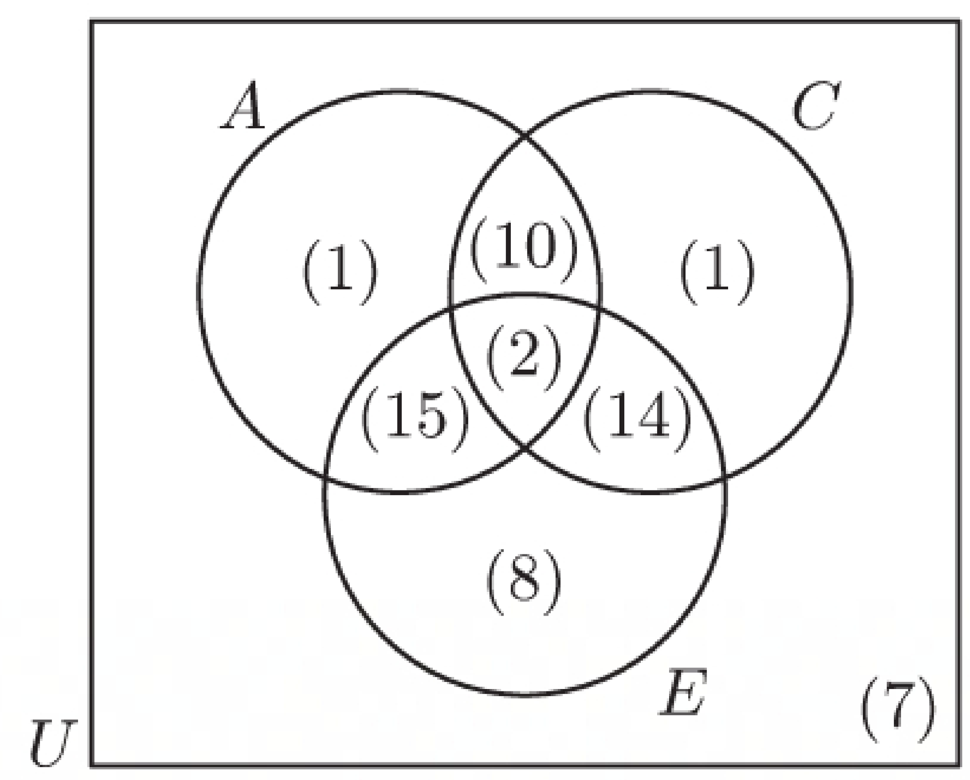
11 Let T represent those who forgot their towel and
H represent those who forgot their hat.



Let $n(T \cap H) = x$
$\therefore\ n(T \cap H') = 11 - x$
and $n(T' \cap H) = 23 - x$
$n(T' \cap H') = 0$ since every student left something at home.
$n(U) = 30$, so $(11 - x) + x + (23 - x) = 30$
$\therefore\ 34 - x = 30$
$\therefore\ x = 4$

4 students had neither a hat nor a towel.

12 Let A, C, and E represent the delegates who could speak Arabic, Chinese, and English respectively.



$n(A \cap C \cap E) = 2$
$n(A \cap C) = 12$
$\therefore\ n(A \cap C \cap E') = 12 - 2 = 10$
$n(C \cap E) = 16$
$\therefore\ n(A' \cap C \cap E) = 16 - 2 = 14$
$n(A \cap E) = 17$
$\therefore\ n(A \cap C' \cap E) = 17 - 2 = 15$

Also, $n(A) = 28$, so $n(A \cap C' \cap E') = 28 - 10 - 2 - 15 = 1$
$n(C) = 27$, so $n(A' \cap C \cap E') = 27 - 10 - 2 - 14 = 1$
and $n(E) = 39$, so $n(A' \cap C' \cap E) = 39 - 15 - 2 - 14 = 8$
Now $n(U) = 58$
$\therefore\ n((A \cup C \cup E)') = 58 - 1 - 10 - 2 - 15 - 1 - 14 - 8 = 7$

a $n(A' \cap C \cap E') = 1$
1 delegate speaks Chinese only.

b $n((A \cup C \cup E)') = 7$
7 delegates speak none of these languages.

c $n((A \cup C)') = 8 + 7 = 15$
15 delegates speak neither Arabic nor Chinese.

Chapter 3

SURDS AND EXPONENTS

INVESTIGATION — PROPERTIES OF RADICALS

1 **a** $(\sqrt{2\times 3})^2 = (\sqrt{6})^2$
$= 6$
$= 2\times 3$ ✓

$2\times 3 = (\sqrt{2}\times\sqrt{2})\times(\sqrt{3}\times\sqrt{3})$ ✓ {since $2=\sqrt{2}\times\sqrt{2}$ and $3=\sqrt{3}\times\sqrt{3}$}

$(\sqrt{2}\times\sqrt{2})\times(\sqrt{3}\times\sqrt{3}) = (\sqrt{2}\times\sqrt{3})\times(\sqrt{2}\times\sqrt{3})$ ✓

$(\sqrt{2}\times\sqrt{3})\times(\sqrt{2}\times\sqrt{3}) = (\sqrt{2}\times\sqrt{3})^2$ ✓

$\therefore$ every step of this argument is valid, and we can deduce that $\sqrt{2\times 3}=\sqrt{2}\times\sqrt{3}$.

b $(\sqrt{a\times b})^2 = a\times b$ {definition of square root}
$= (\sqrt{a}\times\sqrt{a})\times(\sqrt{b}\times\sqrt{b})$ {definition of square root}
$= (\sqrt{a}\times\sqrt{b})\times(\sqrt{a}\times\sqrt{b})$ {changing order of multiplication}
$= (\sqrt{a}\times\sqrt{b})^2$ {definition of perfect square}

$\therefore$ since square roots are non-negative, $\sqrt{a\times b}=\sqrt{a}\times\sqrt{b}$ for any $a\geqslant 0,\ b\geqslant 0$.

2 **a** $\left(\sqrt{\frac{2}{3}}\right)^2 = \frac{2}{3}$ ✓

$\frac{2}{3} = \dfrac{\sqrt{2}\times\sqrt{2}}{\sqrt{3}\times\sqrt{3}}$ ✓ {since $2=\sqrt{2}\times\sqrt{2}$ and $3=\sqrt{3}\times\sqrt{3}$}

$\dfrac{\sqrt{2}\times\sqrt{2}}{\sqrt{3}\times\sqrt{3}} = \dfrac{\sqrt{2}}{\sqrt{3}}\times\dfrac{\sqrt{2}}{\sqrt{3}}$ ✓

$\dfrac{\sqrt{2}}{\sqrt{3}}\times\dfrac{\sqrt{2}}{\sqrt{3}} = \left(\dfrac{\sqrt{2}}{\sqrt{3}}\right)^2$ ✓

$\therefore$ every step of this argument is valid, and we can deduce that $\sqrt{\frac{2}{3}} = \frac{\sqrt{2}}{\sqrt{3}}$.

b $\left(\sqrt{\frac{a}{b}}\right)^2 = \frac{a}{b}$ {definition of square root}
$= \dfrac{\sqrt{a}\times\sqrt{a}}{\sqrt{b}\times\sqrt{b}}$ {definition of square root}
$= \dfrac{\sqrt{a}}{\sqrt{b}}\times\dfrac{\sqrt{a}}{\sqrt{b}}$ {multiplication of fractions}
$= \left(\dfrac{\sqrt{a}}{\sqrt{b}}\right)^2$

$\therefore$ since square roots are non-negative, $\sqrt{\dfrac{a}{b}} = \dfrac{\sqrt{a}}{\sqrt{b}}$ for any $a\geqslant 0,\ b>0$.

EXERCISE 3A

1 **a** $\sqrt{11}\times\sqrt{11}$
$= 11$

b $\sqrt{3}\times\sqrt{5}$
$= \sqrt{3\times 5}$
$= \sqrt{15}$

c $\sqrt{5}\times\sqrt{6}$
$= \sqrt{5\times 6}$
$= \sqrt{30}$

d $3\sqrt{7}\times 2\sqrt{7}$
$= 3\times 2\times\sqrt{7}\times\sqrt{7}$
$= 6\times 7$
$= 42$

e $(3\sqrt{5})^2$
$= 3\sqrt{5}\times 3\sqrt{5}$
$= 3\times 3\times\sqrt{5}\times\sqrt{5}$
$= 9\times 5$
$= 45$

f $3\sqrt{2}\times\sqrt{5}$
$= 3\times\sqrt{2}\times\sqrt{5}$
$= 3\times\sqrt{2\times 5}$
$= 3\sqrt{10}$
$= \sqrt{3^2}\times\sqrt{10}$
$= \sqrt{9}\times\sqrt{10}$
$= \sqrt{9\times 10}$
$= \sqrt{90}$

g $-2\sqrt{3}\times 3\sqrt{5}$
$= -2\times 3\times\sqrt{3}\times\sqrt{5}$
$= -6\times\sqrt{3\times 5}$
$= -6\sqrt{15}$
$= -\sqrt{6^2}\times\sqrt{15}$
$= -\sqrt{36}\times\sqrt{15}$
$= -\sqrt{36\times 15}$
$= -\sqrt{540}$

h $2\sqrt{6}\times\sqrt{12}$
$= 2\times\sqrt{6}\times\sqrt{12}$
$= 2\times\sqrt{6\times 12}$
$= 2\sqrt{72}$
$= \sqrt{2^2}\times\sqrt{72}$
$= \sqrt{4}\times\sqrt{72}$
$= \sqrt{4\times 72}$
$= \sqrt{288}$

2 **a** $\dfrac{\sqrt{12}}{\sqrt{2}}$
$= \sqrt{\frac{12}{2}}$
$= \sqrt{6}$

b $\dfrac{\sqrt{18}}{\sqrt{3}}$
$= \sqrt{\frac{18}{3}}$
$= \sqrt{6}$

c $\dfrac{\sqrt{20}}{\sqrt{5}}$
$= \sqrt{\frac{20}{5}}$
$= \sqrt{4}$
$= 2$

d $\dfrac{\sqrt{3}}{\sqrt{12}}$
$= \sqrt{\frac{3}{12}}$
$= \sqrt{\frac{1}{4}}$
$= \dfrac{\sqrt{1}}{\sqrt{4}}$
$= \frac{1}{2}$

e $\dfrac{\sqrt{6}}{\sqrt{18}}$
$= \sqrt{\frac{6}{18}}$
$= \sqrt{\frac{1}{3}}$

f $\dfrac{\sqrt{6}\times\sqrt{10}}{\sqrt{12}}$
$= \dfrac{\sqrt{6\times 10}}{\sqrt{12}}$
$= \dfrac{\sqrt{60}}{\sqrt{12}}$
$= \sqrt{\frac{60}{12}}$
$= \sqrt{5}$

g
$$\frac{\sqrt{3}}{\sqrt{6}\times\sqrt{8}}$$
$$=\frac{\sqrt{3}}{\sqrt{6\times 8}}$$
$$=\frac{\sqrt{3}}{\sqrt{48}}$$
$$=\sqrt{\tfrac{3}{48}}$$
$$=\sqrt{\tfrac{1}{16}}$$
$$=\frac{\sqrt{1}}{\sqrt{16}}$$
$$=\tfrac{1}{4}$$

h
$$\frac{\sqrt{5}}{2\sqrt{10}\times\sqrt{2}}$$
$$=\frac{\sqrt{5}}{\sqrt{2^2}\times\sqrt{10}\times\sqrt{2}}$$
$$=\frac{\sqrt{5}}{\sqrt{4}\times\sqrt{10}\times\sqrt{2}}$$
$$=\frac{\sqrt{5}}{\sqrt{4\times 10\times 2}}$$
$$=\frac{\sqrt{5}}{\sqrt{80}}$$
$$=\sqrt{\tfrac{5}{80}}$$
$$=\sqrt{\tfrac{1}{16}}$$
$$=\frac{\sqrt{1}}{\sqrt{16}}$$
$$=\tfrac{1}{4}$$

3 a
$$\sqrt{12}$$
$$=\sqrt{4\times 3}$$
$$=\sqrt{4}\times\sqrt{3}$$
$$=2\sqrt{3}$$

b
$$\sqrt{20}$$
$$=\sqrt{4\times 5}$$
$$=\sqrt{4}\times\sqrt{5}$$
$$=2\sqrt{5}$$

c
$$\sqrt{27}$$
$$=\sqrt{9\times 3}$$
$$=\sqrt{9}\times\sqrt{3}$$
$$=3\sqrt{3}$$

d
$$\sqrt{54}$$
$$=\sqrt{9\times 6}$$
$$=\sqrt{9}\times\sqrt{6}$$
$$=3\sqrt{6}$$

e
$$\sqrt{50}$$
$$=\sqrt{25\times 2}$$
$$=\sqrt{25}\times\sqrt{2}$$
$$=5\sqrt{2}$$

f
$$\sqrt{80}$$
$$=\sqrt{16\times 5}$$
$$=\sqrt{16}\times\sqrt{5}$$
$$=4\sqrt{5}$$

g
$$\sqrt{96}$$
$$=\sqrt{16\times 6}$$
$$=\sqrt{16}\times\sqrt{6}$$
$$=4\sqrt{6}$$

h
$$\sqrt{108}$$
$$=\sqrt{36\times 3}$$
$$=\sqrt{36}\times\sqrt{3}$$
$$=6\sqrt{3}$$

4 a
$$2\sqrt{2}+3\sqrt{2}$$
$$=5\sqrt{2}$$

b
$$2\sqrt{2}-3\sqrt{2}$$
$$=-\sqrt{2}$$

c
$$5\sqrt{5}-3\sqrt{5}$$
$$=2\sqrt{5}$$

d
$$5\sqrt{5}+3\sqrt{5}$$
$$=8\sqrt{5}$$

e
$$3\sqrt{5}-5\sqrt{5}$$
$$=-2\sqrt{5}$$

f
$$7\sqrt{3}+2\sqrt{3}$$
$$=9\sqrt{3}$$

g
$$9\sqrt{6}-12\sqrt{6}$$
$$=-3\sqrt{6}$$

h
$$\sqrt{2}+\sqrt{2}+\sqrt{2}$$
$$=3\sqrt{2}$$

5 **a**
$$\begin{aligned}&4\sqrt{3}-\sqrt{12}\\&=4\sqrt{3}-\sqrt{4\times 3}\\&=4\sqrt{3}-2\times\sqrt{3}\\&=4\sqrt{3}-2\sqrt{3}\\&=2\sqrt{3}\end{aligned}$$

b
$$\begin{aligned}&3\sqrt{2}+\sqrt{50}\\&=3\sqrt{2}+\sqrt{25\times 2}\\&=3\sqrt{2}+5\times\sqrt{2}\\&=3\sqrt{2}+5\sqrt{2}\\&=8\sqrt{2}\end{aligned}$$

c
$$\begin{aligned}&3\sqrt{6}+\sqrt{24}\\&=3\sqrt{6}+\sqrt{4\times 6}\\&=3\sqrt{6}+2\times\sqrt{6}\\&=3\sqrt{6}+2\sqrt{6}\\&=5\sqrt{6}\end{aligned}$$

d
$$\begin{aligned}&2\sqrt{27}+2\sqrt{12}\\&=2\sqrt{9\times 3}+2\sqrt{4\times 3}\\&=2\times 3\times\sqrt{3}+2\times 2\times\sqrt{3}\\&=6\sqrt{3}+4\sqrt{3}\\&=10\sqrt{3}\end{aligned}$$

e
$$\begin{aligned}&\sqrt{75}-\sqrt{12}\\&=\sqrt{25\times 3}-\sqrt{4\times 3}\\&=5\times\sqrt{3}-2\times\sqrt{3}\\&=5\sqrt{3}-2\sqrt{3}\\&=3\sqrt{3}\end{aligned}$$

f
$$\begin{aligned}&\sqrt{2}+\sqrt{8}-\sqrt{32}\\&=\sqrt{2}+\sqrt{4\times 2}-\sqrt{16\times 2}\\&=\sqrt{2}+2\times\sqrt{2}-4\times\sqrt{2}\\&=\sqrt{2}+2\sqrt{2}-4\sqrt{2}\\&=-\sqrt{2}\end{aligned}$$

6 **a**
$$\begin{aligned}&\sqrt{2}(3-\sqrt{2})\\&=\sqrt{2}\times 3+\sqrt{2}\times(-\sqrt{2})\\&=3\sqrt{2}-2\end{aligned}$$

b
$$\begin{aligned}&2\sqrt{6}(\sqrt{6}-7)\\&=2\sqrt{6}\times\sqrt{6}+2\sqrt{6}\times(-7)\\&=2\times 6-14\sqrt{6}\\&=12-14\sqrt{6}\end{aligned}$$

c
$$\begin{aligned}&-\sqrt{8}(\sqrt{8}-5)\\&=-\sqrt{8}\times\sqrt{8}+(-\sqrt{8})\times(-5)\\&=-8+5\sqrt{8}\\&=-8+10\sqrt{2}\end{aligned}$$

d
$$\begin{aligned}&-3\sqrt{2}(4-6\sqrt{2})\\&=-3\sqrt{2}\times 4+(-3\sqrt{2})\times(-6\sqrt{2})\\&=-12\sqrt{2}+3\times 6\times\sqrt{2}\times\sqrt{2}\\&=-12\sqrt{2}+18\times 2\\&=-12\sqrt{2}+36\end{aligned}$$

e
$$\begin{aligned}&(5+\sqrt{2})(4+\sqrt{2})\\&=20+5\sqrt{2}+\sqrt{2}(4)+\sqrt{2}(\sqrt{2})\\&=20+5\sqrt{2}+4\sqrt{2}+2\\&=22+9\sqrt{2}\end{aligned}$$

f
$$\begin{aligned}&(9-\sqrt{7})(4+2\sqrt{7})\\&=36+9(2\sqrt{7})-\sqrt{7}(4)-\sqrt{7}(2\sqrt{7})\\&=36+18\sqrt{7}-4\sqrt{7}-14\\&=22+14\sqrt{7}\end{aligned}$$

g
$$\begin{aligned}&(\sqrt{3}+1)(2-3\sqrt{3})\\&=\sqrt{3}(2)+\sqrt{3}(-3\sqrt{3})+2-3\sqrt{3}\\&=2\sqrt{3}-9+2-3\sqrt{3}\\&=-7-\sqrt{3}\end{aligned}$$

h
$$\begin{aligned}&(\sqrt{8}-6)(2\sqrt{8}-3)\\&=\sqrt{8}(2\sqrt{8})+\sqrt{8}(-3)-6(2\sqrt{8})-6(-3)\\&=16-3\sqrt{8}-12\sqrt{8}+18\\&=34-15\sqrt{8}\\&=34-30\sqrt{2}\end{aligned}$$

i
$$\begin{aligned}&(2\sqrt{5}-7)(1-4\sqrt{5})\\&=2\sqrt{5}+2\sqrt{5}(-4\sqrt{5})-7-7(-4\sqrt{5})\\&=2\sqrt{5}-8\times 5-7+28\sqrt{5}\\&=30\sqrt{5}-40-7\\&=30\sqrt{5}-47\end{aligned}$$

7 **a**
$$\begin{aligned}&(3+\sqrt{2})^2\\&=3^2+2(3)(\sqrt{2})+(\sqrt{2})^2\\&=9+6\sqrt{2}+2\\&=11+6\sqrt{2}\end{aligned}$$

b
$$\begin{aligned}&(6-\sqrt{3})^2\\&=6^2+2(6)(-\sqrt{3})+(\sqrt{3})^2\\&=36-12\sqrt{3}+3\\&=39-12\sqrt{3}\end{aligned}$$

c
$$\begin{aligned}&(3\sqrt{5}+1)^2\\&=(3\sqrt{5})^2+2(3\sqrt{5})(1)+1^2\\&=(9\times 5)+6\sqrt{5}+1\\&=45+6\sqrt{5}+1\\&=46+6\sqrt{5}\end{aligned}$$

d
$$\begin{aligned}&(7-2\sqrt{10})^2\\&=7^2+2(7)(-2\sqrt{10})+(2\sqrt{10})^2\\&=49-28\sqrt{10}+(4\times 10)\\&=49-28\sqrt{10}+40\\&=89-28\sqrt{10}\end{aligned}$$

e
$$\begin{aligned}&(3+\sqrt{7})(3-\sqrt{7})\\&=3^2-(\sqrt{7})^2\\&=9-7\\&=2\end{aligned}$$

f
$$\begin{aligned}&(\sqrt{2}+5)(\sqrt{2}-5)\\&=(\sqrt{2})^2-5^2\\&=2-25\\&=-23\end{aligned}$$

g
$$\begin{aligned}&(2\sqrt{2}+1)(2\sqrt{2}-1)\\&=(2\sqrt{2})^2-1^2\\&=(4\times 2)-1\\&=8-1\\&=7\end{aligned}$$

h
$$\begin{aligned}&(9\sqrt{3}-5)(9\sqrt{3}+5)\\&=(9\sqrt{3})^2-5^2\\&=(81\times 3)-25\\&=243-25\\&=218\end{aligned}$$

i
$$\begin{aligned}(\sqrt{3}+1)^3&=(\sqrt{3}+1)(\sqrt{3}+1)^2\\&=(\sqrt{3}+1)((\sqrt{3})^2+2(\sqrt{3})(1)+1^2)\\&=(\sqrt{3}+1)(3+2\sqrt{3}+1)\\&=(\sqrt{3}+1)(2\sqrt{3}+4)\\&=\sqrt{3}(2\sqrt{3})+\sqrt{3}(4)+1(2\sqrt{3})+4\\&=6+4\sqrt{3}+2\sqrt{3}+4\\&=10+6\sqrt{3}\end{aligned}$$

EXERCISE 3B

1 **a**
$$\begin{aligned}&\frac{1}{\sqrt{3}}\\&=\frac{1}{\sqrt{3}}\times\frac{\sqrt{3}}{\sqrt{3}}\\&=\frac{\sqrt{3}}{3}\end{aligned}$$

b
$$\begin{aligned}&\frac{11}{\sqrt{3}}\\&=\frac{11}{\sqrt{3}}\times\frac{\sqrt{3}}{\sqrt{3}}\\&=\frac{11\sqrt{3}}{3}\end{aligned}$$

c
$$\begin{aligned}&\frac{\sqrt{2}}{3\sqrt{3}}\\&=\frac{\sqrt{2}}{3\sqrt{3}}\times\frac{\sqrt{3}}{\sqrt{3}}\\&=\frac{\sqrt{2\times 3}}{3\times 3}\\&=\frac{\sqrt{6}}{9}\end{aligned}$$

d
$$\begin{aligned}&\frac{12}{\sqrt{2}}\\&=\frac{12}{\sqrt{2}}\times\frac{\sqrt{2}}{\sqrt{2}}\\&=\frac{12\sqrt{2}}{2}\\&=6\sqrt{2}\end{aligned}$$

e $\dfrac{\sqrt{3}}{\sqrt{2}}$
$= \dfrac{\sqrt{3}}{\sqrt{2}} \times \dfrac{\sqrt{2}}{\sqrt{2}}$
$= \dfrac{\sqrt{3 \times 2}}{2}$
$= \dfrac{\sqrt{6}}{2}$

f $\dfrac{1}{4\sqrt{2}}$
$= \dfrac{1}{4\sqrt{2}} \times \dfrac{\sqrt{2}}{\sqrt{2}}$
$= \dfrac{\sqrt{2}}{4 \times 2}$
$= \dfrac{\sqrt{2}}{8}$

g $\dfrac{15}{\sqrt{5}}$
$= \dfrac{15}{\sqrt{5}} \times \dfrac{\sqrt{5}}{\sqrt{5}}$
$= \dfrac{15\sqrt{5}}{5}$
$= 3\sqrt{5}$

h $\dfrac{-3}{\sqrt{5}}$
$= \dfrac{-3}{\sqrt{5}} \times \dfrac{\sqrt{5}}{\sqrt{5}}$
$= \dfrac{-3\sqrt{5}}{5}$
$= -\dfrac{3\sqrt{5}}{5}$

i $\dfrac{1}{3\sqrt{5}}$
$= \dfrac{1}{3\sqrt{5}} \times \dfrac{\sqrt{5}}{\sqrt{5}}$
$= \dfrac{\sqrt{5}}{3 \times 5}$
$= \dfrac{\sqrt{5}}{15}$

j $\dfrac{21}{\sqrt{7}}$
$= \dfrac{21}{\sqrt{7}} \times \dfrac{\sqrt{7}}{\sqrt{7}}$
$= \dfrac{21\sqrt{7}}{7}$
$= 3\sqrt{7}$

k $\dfrac{2}{\sqrt{11}}$
$= \dfrac{2}{\sqrt{11}} \times \dfrac{\sqrt{11}}{\sqrt{11}}$
$= \dfrac{2\sqrt{11}}{11}$

l $\dfrac{1}{(\sqrt{3})^3}$
$= \dfrac{1}{\sqrt{3} \times \sqrt{3} \times \sqrt{3}}$
$= \dfrac{1}{3\sqrt{3}}$
$= \dfrac{1}{3\sqrt{3}} \times \dfrac{\sqrt{3}}{\sqrt{3}}$
$= \dfrac{\sqrt{3}}{3 \times 3}$
$= \dfrac{\sqrt{3}}{9}$

2 **a** $\dfrac{1}{3+\sqrt{2}} = \left(\dfrac{1}{3+\sqrt{2}}\right)\left(\dfrac{3-\sqrt{2}}{3-\sqrt{2}}\right)$
$= \dfrac{3-\sqrt{2}}{3^2-(\sqrt{2})^2}$
$= \dfrac{3-\sqrt{2}}{9-2}$
$= \dfrac{3-\sqrt{2}}{7}$

b $\dfrac{2}{3-\sqrt{2}} = \left(\dfrac{2}{3-\sqrt{2}}\right)\left(\dfrac{3+\sqrt{2}}{3+\sqrt{2}}\right)$
$= \dfrac{2(3+\sqrt{2})}{3^2-(\sqrt{2})^2}$
$= \dfrac{6+2\sqrt{2}}{9-2}$
$= \dfrac{6+2\sqrt{2}}{7}$

c $\dfrac{10}{\sqrt{6}-1} = \left(\dfrac{10}{\sqrt{6}-1}\right)\left(\dfrac{\sqrt{6}+1}{\sqrt{6}+1}\right)$
$= \dfrac{10(\sqrt{6}+1)}{(\sqrt{6})^2-1^2}$
$= \dfrac{10(\sqrt{6}+1)}{6-1}$
$= \dfrac{10(\sqrt{6}+1)}{5}$
$= 2(\sqrt{6}+1)$
$= 2\sqrt{6}+2$

d $\dfrac{\sqrt{3}}{\sqrt{7}+2} = \left(\dfrac{\sqrt{3}}{\sqrt{7}+2}\right)\left(\dfrac{\sqrt{7}-2}{\sqrt{7}-2}\right)$
$= \dfrac{\sqrt{3}(\sqrt{7}-2)}{(\sqrt{7})^2-2^2}$
$= \dfrac{\sqrt{3 \times 7}-2\sqrt{3}}{7-4}$
$= \dfrac{\sqrt{21}-2\sqrt{3}}{3}$

e $$\frac{1+\sqrt{2}}{1-\sqrt{2}} = \left(\frac{1+\sqrt{2}}{1-\sqrt{2}}\right)\left(\frac{1+\sqrt{2}}{1+\sqrt{2}}\right) = \frac{(1+\sqrt{2})^2}{1^2-(\sqrt{2})^2} = \frac{1^2+2(1)(\sqrt{2})+(\sqrt{2})^2}{1-2} = \frac{1+2\sqrt{2}+2}{-1} = \frac{3+2\sqrt{2}}{-1} = -3-2\sqrt{2}$$

f $$\frac{-2\sqrt{2}}{1-\sqrt{2}} = \left(\frac{-2\sqrt{2}}{1-\sqrt{2}}\right)\left(\frac{1+\sqrt{2}}{1+\sqrt{2}}\right) = \frac{-2\sqrt{2}(1+\sqrt{2})}{1^2-(\sqrt{2})^2} = \frac{-2\sqrt{2}-(2\times 2)}{1-2} = \frac{-2\sqrt{2}-4}{-1} = 2\sqrt{2}+4$$

g $$\frac{1+\sqrt{5}}{2-\sqrt{5}} = \left(\frac{1+\sqrt{5}}{2-\sqrt{5}}\right)\left(\frac{2+\sqrt{5}}{2+\sqrt{5}}\right) = \frac{2+\sqrt{5}+2\sqrt{5}+5}{2^2-(\sqrt{5})^2} = \frac{7+3\sqrt{5}}{4-5} = \frac{7+3\sqrt{5}}{-1} = -7-3\sqrt{5}$$

h $$\frac{\sqrt{10}-7}{\sqrt{10}+4} = \left(\frac{\sqrt{10}-7}{\sqrt{10}+4}\right)\left(\frac{\sqrt{10}-4}{\sqrt{10}-4}\right) = \frac{10-4\sqrt{10}-7\sqrt{10}+28}{10-16} = \frac{38-11\sqrt{10}}{-6} = \frac{-38+11\sqrt{10}}{6}$$

i $$\frac{3+\sqrt{5}}{4+\sqrt{5}} = \left(\frac{3+\sqrt{5}}{4+\sqrt{5}}\right)\left(\frac{4-\sqrt{5}}{4-\sqrt{5}}\right) = \frac{12-3\sqrt{5}+4\sqrt{5}-5}{4^2-(\sqrt{5})^2} = \frac{7+\sqrt{5}}{16-5} = \frac{7+\sqrt{5}}{11}$$

j $$\frac{6-\sqrt{2}}{5-\sqrt{2}} = \left(\frac{6-\sqrt{2}}{5-\sqrt{2}}\right)\left(\frac{5+\sqrt{2}}{5+\sqrt{2}}\right) = \frac{30+6\sqrt{2}-5\sqrt{2}-2}{5^2-(\sqrt{2})^2} = \frac{28+\sqrt{2}}{25-2} = \frac{28+\sqrt{2}}{23}$$

k $$\frac{\sqrt{7}+5}{\sqrt{7}-2} = \left(\frac{\sqrt{7}+5}{\sqrt{7}-2}\right)\left(\frac{\sqrt{7}+2}{\sqrt{7}+2}\right) = \frac{7+2\sqrt{7}+5\sqrt{7}+10}{(\sqrt{7})^2-2^2} = \frac{17+7\sqrt{7}}{7-4} = \frac{17+7\sqrt{7}}{3}$$

l $$\frac{\sqrt{11}-3}{4-\sqrt{11}} = \left(\frac{\sqrt{11}-3}{4-\sqrt{11}}\right)\left(\frac{4+\sqrt{11}}{4+\sqrt{11}}\right) = \frac{4\sqrt{11}+11-12-3\sqrt{11}}{4^2-(\sqrt{11})^2} = \frac{\sqrt{11}-1}{16-11} = \frac{\sqrt{11}-1}{5}$$

3 **a**
$$\frac{\sqrt{2}-1}{\sqrt{2}+1} = \left(\frac{\sqrt{2}-1}{\sqrt{2}+1}\right)\left(\frac{\sqrt{2}-1}{\sqrt{2}-1}\right)$$
$$= \frac{(\sqrt{2}-1)^2}{(\sqrt{2})^2-1^2}$$
$$= \frac{(\sqrt{2})^2+2(\sqrt{2})(-1)+1^2}{2-1}$$
$$= \frac{2-2\sqrt{2}+1}{1}$$
$$= 3-2\sqrt{2}$$

b
$$\frac{5-\sqrt{2}}{6-\sqrt{2}} = \left(\frac{5-\sqrt{2}}{6-\sqrt{2}}\right)\left(\frac{6+\sqrt{2}}{6+\sqrt{2}}\right)$$
$$= \frac{30+5\sqrt{2}-6\sqrt{2}-2}{6^2-(\sqrt{2})^2}$$
$$= \frac{28-\sqrt{2}}{36-2}$$
$$= \frac{28-\sqrt{2}}{34}$$
$$= \frac{28}{34}-\frac{\sqrt{2}}{34}$$
$$= \tfrac{14}{17}-\tfrac{1}{34}\sqrt{2}$$

c
$$\frac{1}{(\sqrt{2}+1)^2} = \frac{1}{(\sqrt{2})^2+2(\sqrt{2})(1)+1^2}$$
$$= \frac{1}{2+2\sqrt{2}+1}$$
$$= \frac{1}{3+2\sqrt{2}}$$
$$= \left(\frac{1}{3+2\sqrt{2}}\right)\left(\frac{3-2\sqrt{2}}{3-2\sqrt{2}}\right)$$
$$= \frac{3-2\sqrt{2}}{3^2-(2\sqrt{2})^2}$$
$$= \frac{3-2\sqrt{2}}{9-(4\times 2)}$$
$$= \frac{3-2\sqrt{2}}{1}$$
$$= 3-2\sqrt{2}$$

d
$$\frac{1}{(3-\sqrt{2})^2} = \frac{1}{3^2+2(3)(-\sqrt{2})+(\sqrt{2})^2}$$
$$= \frac{1}{9-6\sqrt{2}+2}$$
$$= \frac{1}{11-6\sqrt{2}}$$
$$= \left(\frac{1}{11-6\sqrt{2}}\right)\left(\frac{11+6\sqrt{2}}{11+6\sqrt{2}}\right)$$
$$= \frac{11+6\sqrt{2}}{11^2-(6\sqrt{2})^2}$$
$$= \frac{11+6\sqrt{2}}{121-(36\times 2)}$$
$$= \frac{11+6\sqrt{2}}{121-72}$$
$$= \frac{11+6\sqrt{2}}{49}$$
$$= \tfrac{11}{49}+\tfrac{6}{49}\sqrt{2}$$

e
$$\frac{1}{3+2\sqrt{2}} = \left(\frac{1}{3+2\sqrt{2}}\right)\left(\frac{3-2\sqrt{2}}{3-2\sqrt{2}}\right)$$
$$= \frac{3-2\sqrt{2}}{3^2-(2\sqrt{2})^2}$$
$$= \frac{3-2\sqrt{2}}{9-(4\times 2)}$$
$$= \frac{3-2\sqrt{2}}{1}$$
$$= 3-2\sqrt{2}$$

f
$$\frac{1}{2\sqrt{2}-7} = \left(\frac{1}{2\sqrt{2}-7}\right)\left(\frac{2\sqrt{2}+7}{2\sqrt{2}+7}\right)$$
$$= \frac{2\sqrt{2}+7}{(2\sqrt{2})^2-7^2}$$
$$= \frac{2\sqrt{2}+7}{(4\times 2)-49}$$
$$= \frac{2\sqrt{2}+7}{8-49}$$
$$= \frac{2\sqrt{2}+7}{-41}$$
$$= -\tfrac{7}{41}-\tfrac{2}{41}\sqrt{2}$$

g

$$\frac{\sqrt{2}+1}{(5+\sqrt{2})^2}$$

$$= \frac{\sqrt{2}+1}{5^2+2(5)(\sqrt{2})+(\sqrt{2})^2}$$

$$= \frac{\sqrt{2}+1}{25+10\sqrt{2}+2}$$

$$= \frac{\sqrt{2}+1}{27+10\sqrt{2}}$$

$$= \left(\frac{\sqrt{2}+1}{27+10\sqrt{2}}\right)\left(\frac{27-10\sqrt{2}}{27-10\sqrt{2}}\right)$$

$$= \frac{27\sqrt{2}-\sqrt{2}(10\sqrt{2})+27-10\sqrt{2}}{27^2-(10\sqrt{2})^2}$$

$$= \frac{17\sqrt{2}-20+27}{729-200}$$

$$= \frac{7+17\sqrt{2}}{529}$$

$$= \tfrac{7}{529}+\tfrac{17}{529}\sqrt{2}$$

h

$$\frac{1}{(3-\sqrt{2})^3}$$

$$= \frac{1}{(3-\sqrt{2})(3-\sqrt{2})^2}$$

$$= \frac{1}{(3-\sqrt{2})(3^2+2(3)(-\sqrt{2})+(\sqrt{2})^2)}$$

$$= \frac{1}{(3-\sqrt{2})(9-6\sqrt{2}+2)}$$

$$= \frac{1}{(3-\sqrt{2})(11-6\sqrt{2})}$$

$$= \frac{1}{33-18\sqrt{2}-11\sqrt{2}+\sqrt{2}(6\sqrt{2})}$$

$$= \frac{1}{33-29\sqrt{2}+12}$$

$$= \frac{1}{45-29\sqrt{2}}$$

$$= \left(\frac{1}{45-29\sqrt{2}}\right)\left(\frac{45+29\sqrt{2}}{45+29\sqrt{2}}\right)$$

$$= \frac{45+29\sqrt{2}}{45^2-(29\sqrt{2})^2}$$

$$= \frac{45+29\sqrt{2}}{2025-(841\times 2)}$$

$$= \frac{45+29\sqrt{2}}{2025-1682}$$

$$= \frac{45+29\sqrt{2}}{343}$$

$$= \tfrac{45}{343}+\tfrac{29}{343}\sqrt{2}$$

EXERCISE 3C

1 **a** $2^1=2,\quad 2^2=4,\quad 2^3=8,\quad 2^4=16,\quad 2^5=32,\quad 2^6=64$

b $3^1=3,\quad 3^2=9,\quad 3^3=27,\quad 3^4=81,\quad 3^5=243,\quad 3^6=729$

c $4^1=4,\quad 4^2=16,\quad 4^3=64,\quad 4^4=256,\quad 4^5=1024,\quad 4^6=4096$

2 **a**

$$(-1)^5$$
$$=(-1)\times(-1)\times(-1)\times(-1)\times(-1)$$
$$=1\times 1\times(-1)$$
$$=-1$$

b

$$(-1)^6$$
$$=(-1)^5\times(-1)$$
$$=(-1)\times(-1)$$
$$=1$$

c

$$(-1)^{14}$$
$$=1$$

d

$$(-1)^{19}$$
$$=-1$$

e

$$(-1)^8$$
$$=1$$

f

$$-1^8$$
$$=-(1^8)$$
$$=-1$$

g

$$-(-1)^8$$
$$=-(1)$$
$$=-1$$

h $(-2)^5$
$= (-2) \times (-2) \times (-2) \times (-2) \times (-2)$
$= 4 \times 4 \times (-2)$
$= -32$

i -2^5
$= -(2^5)$
$= -32$

j $-(-2)^6$
$= -(-2)^5 \times (-2)$
$= 32 \times (-2)$
$= -64$

k $(-5)^4$
$= (-5) \times (-5) \times (-5) \times (-5)$
$= 25 \times 25$
$= 625$

l $-(-5)^4$
$= -(-5) \times (-5) \times (-5) \times (-5)$
$= -25 \times 25$
$= -625$

3 **a** $4^7 = 16\,384$ **b** $7^4 = 2401$ **c** $-5^5 = -3125$
d $(-5)^5 = -3125$ **e** $8^6 = 262\,144$ **f** $(-8)^6 = 262\,144$
g $-8^6 = -262\,144$ **h** $2.13^9 \approx 902.436\,039\,6$ **i** $-2.13^9 \approx -902.436\,039\,6$
j $(-2.13)^9 \approx -902.436\,039\,6$

4 **a** $9^{-1} = 0.\overline{1}$ and $\dfrac{1}{9^1} = 0.\overline{1}$ **b** $6^{-2} = 0.02\overline{7}$ and $\dfrac{1}{6^2} = 0.02\overline{7}$

c $3^{-4} = 0.\overline{012\,345\,679}$ and $\dfrac{1}{3^4} = 0.\overline{012\,345\,679}$ **d** $17^0 = 1$ and $(0.366)^0 = 1$

We notice that $a^{-n} = \dfrac{1}{a^n}$ and $a^0 = 1$ for $a \neq 0$.

5 $3^1 = 3$, $3^2 = 9$, $3^3 = 27$, $3^4 = 81$, $3^5 = 243$, $3^6 = 729$, $3^7 = 2187$, $3^8 = 6561$, ...
So, the last digit of the powers of 3 follow the pattern 3, 9, 7, 1, 3, 9, 7, 1, ...

$$3^{101} = \underbrace{3^4 \times 3^4 \times 3^4 \times .... \times 3^4}_{\text{25 of these}} \times 3^1$$

But $3^4 = 81$ which ends in a 1

$\therefore$ $\underbrace{3^4 \times 3^4 \times 3^4 \times .... \times 3^4}_{\text{25 of these}}$ ends in a 1

$\therefore$ 3^{101} ends in a 3

6 $7^1 = 7$, $7^2 = 49$, $7^3 = 343$, $7^4 = 2401$, $7^5 = 16\,807$, $7^6 = 117\,649$, $7^7 = 823\,543$, $7^8 = 5\,764\,801$, ...
So, the last digit of the powers of 7 follow the pattern 7, 9, 3, 1, 7, 9, 3, 1, ...

$$7^{217} = \underbrace{7^4 \times 7^4 \times 7^4 \times .... \times 7^4}_{\text{54 of these}} \times 7^1$$

But $7^4 = 2401$ which ends in a 1

$\therefore$ $\underbrace{7^4 \times 7^4 \times 7^4 \times .... \times 7^4}_{\text{54 of these}}$ ends in a 1

$\therefore$ 7^{217} ends in a 7

7

$$
\begin{aligned}
1 &= 1^3\\
3+5 = 8 &= 2^3\\
7+9+11 = 27 &= 3^3\\
13+15+17+19 = 64 &= 4^3\\
21+23+25+27+29 = 125 &= 5^3\\
31+33+35+37+39+41 = 216 &= 6^3\\
43+45+47+49+51+53+55 = 343 &= 7^3\\
57+59+61+63+65+67+69+71 = 512 &= 8^3\\
73+75+77+79+81+83+85+87+89 = 729 &= 9^3\\
91+93+95+97+99+101+103+105+107+109 = 1000 &= 10^3\\
111+113+115+117+119+121+123+125+127+129+131 = 1331 &= 11^3\\
133+135+137+139+141+143+145+147+149+151+153+155 = 1728 &= 12^3
\end{aligned}
$$

a $5^3 = 21+23+25+27+29$

b $7^3 = 43+45+47+49+51+53+55$

c $12^3 = 133+135+137+139+141+143+145+147+149+151+153+155$

EXERCISE 3D

1 **a** $k^4 \times k^2$
$= k^{4+2}$
$= k^6$

b $c^8 \times c^m$
$= c^{8+m}$

c $r^2 \times r^5 \times r^4$
$= r^{2+5} \times r^4$
$= r^{2+5+4}$
$= r^{11}$

d $\dfrac{7^8}{7^3}$
$= 7^{8-3}$
$= 7^5$

e $\dfrac{m^{10}}{m^4}$
$= m^{10-4}$
$= m^6$

f $\dfrac{x^{3a}}{x^2}$
$= x^{3a-2}$

g $(7^6)^d$
$= 7^{6\times d}$
$= 7^{6d}$

h $(m^3)^t$
$= m^{3\times t}$
$= m^{3t}$

i $(11^x)^{2y}$
$= 11^{x\times 2y}$
$= 11^{2xy}$

j $\dfrac{7^6}{7^n}$
$= 7^{6-n}$

k $(x^{2s})^3$
$= x^{2s\times 3}$
$= x^{6s}$

l $3^2 \times 3^7 \times 3^4$
$= 3^{2+7} \times 3^4$
$= 3^{2+7+4}$
$= 3^{13}$

m $(j^4)^{3x}$
$= j^{4\times 3x}$
$= j^{12x}$

n $\dfrac{z^7}{z^{4t}}$
$= z^{7-4t}$

o $(13^c)^{5d}$
$= 13^{c\times 5d}$
$= 13^{5cd}$

p $w^{7p} \div w$
$= \dfrac{w^{7p}}{w}$
$= \dfrac{w^{7p}}{w^1}$
$= w^{7p-1}$

q $k^{5t} \div k^3$
$= \dfrac{k^{5t}}{k^3}$
$= k^{5t-3}$

r $\dfrac{(x^m)^3}{x^n}$
$= \dfrac{x^{m\times 3}}{x^n}$
$= \dfrac{x^{3m}}{x^n}$
$= x^{3m-n}$

2 **a** $\dfrac{4b^5}{b^2}$
$= 4 \times b^{5-2}$
$= 4b^3$

b $2w^4 \times 3w$
$= 2 \times 3 \times w^4 \times w$
$= 6 \times w^{4+1}$
$= 6w^5$

c $\dfrac{12p^4}{3p^2}$
$= \frac{12}{3} \times p^{4-2}$
$= 4p^2$

d $5c^7 \times 6c^4$
$= 5 \times 6 \times c^7 \times c^4$
$= 30 \times c^{7+4}$
$= 30c^{11}$

e $\dfrac{d^2 \times d^7}{d^5}$
$= \dfrac{d^{2+7}}{d^5}$
$= \dfrac{d^9}{d^5}$
$= d^4$

f $\dfrac{18a^2b^3}{6ab}$
$= \dfrac{18}{6} \times \dfrac{a^2}{a} \times \dfrac{b^3}{b}$
$= 3 \times a^{2-1} \times b^{3-1}$
$= 3ab^2$

g $\dfrac{24m^2n^4}{6m^2n}$
$= \dfrac{24}{6} \times \dfrac{m^2}{m^2} \times \dfrac{n^4}{n}$
$= 4 \times m^{2-2} \times n^{4-1}$
$= 4 \times m^0 \times n^3$
$= 4 \times 1 \times n^3$
$= 4n^3$

h $\dfrac{t^5 \times t^8}{(t^2)^3}$
$= \dfrac{t^{5+8}}{t^{2\times 3}}$
$= \dfrac{t^{13}}{t^6}$
$= t^7$

i $5s^2t \times 4t^3$
$= 5 \times 4 \times s^2 \times t^{1+3}$
$= 20s^2t^4$

j $\dfrac{(k^4)^5}{k^3 \times k^6}$
$= \dfrac{k^{4\times 5}}{k^{3+6}}$
$= \dfrac{k^{20}}{k^9}$
$= k^{11}$

k $\dfrac{12x^2y^5}{8xy^2}$
$= \dfrac{12}{8} \times \dfrac{x^2}{x} \times \dfrac{y^5}{y^2}$
$= \frac{3}{2} \times x^{2-1} \times y^{5-2}$
$= \dfrac{3xy^3}{2}$

l $\dfrac{(b^3)^4 \times b^5}{b^2 \times b^6} = \dfrac{b^{3\times 4} \times b^5}{b^{2+6}}$
$= \dfrac{b^{12} \times b^5}{b^8}$
$= \dfrac{b^{12+5}}{b^8}$
$= \dfrac{b^{17}}{b^8}$
$= b^9$

3 **a** $4 = 2 \times 2$
$= 2^2$

b $\dfrac{1}{4} = \dfrac{1}{2^2}$
$= 2^{-2}$

c $8 = 2 \times 2 \times 2$
$= 2^3$

d $\dfrac{1}{8} = \dfrac{1}{2^3}$
$= 2^{-3}$

e $32 = 2 \times 2 \times 2 \times 2 \times 2$
$= 2^5$

f $\dfrac{1}{32} = \dfrac{1}{2^5}$
$= 2^{-5}$

g $2 = 2^1$

h $\frac{1}{2} = \frac{1}{2^1}$
$= 2^{-1}$

i $64 = 32 \times 2$
$= 2^5 \times 2^1$
$= 2^6$

j $\frac{1}{64} = \frac{1}{2^6}$
$= 2^{-6}$

k $128 = 64 \times 2$
$= 2^6 \times 2^1$
$= 2^7$

l $\frac{1}{128} = \frac{1}{2^7}$
$= 2^{-7}$

4 **a** $9 = 3 \times 3$
$= 3^2$

b $\frac{1}{9} = \frac{1}{3^2}$
$= 3^{-2}$

c $27 = 3 \times 3 \times 3$
$= 3^3$

d $\frac{1}{27} = \frac{1}{3^3}$
$= 3^{-3}$

e $3 = 3^1$

f $\frac{1}{3} = \frac{1}{3^1}$
$= 3^{-1}$

g $81 = 3 \times 3 \times 3 \times 3$
$= 3^4$

h $\frac{1}{81} = \frac{1}{3^4}$
$= 3^{-4}$

i $1 = 3^0$

j $243 = 81 \times 3$
$= 3^4 \times 3^1$
$= 3^5$

k $\frac{1}{243} = \frac{1}{3^5}$
$= 3^{-5}$

5 **a** $2 \times 2^a = 2^1 \times 2^a$
$= 2^{1+a}$

b $4 \times 2^b = 2^2 \times 2^b$
$= 2^{2+b}$

c $8 \times 2^t = 2^3 \times 2^t$
$= 2^{3+t}$

d $(2^{x+1})^2 = 2^{2(x+1)}$
$= 2^{2x+2}$

e $\left(2^{1-n}\right)^{-1} = 2^{-(1-n)}$
$= 2^{n-1}$

f $\frac{2^c}{4} = \frac{2^c}{2^2}$
$= 2^{c-2}$

g $\frac{2^m}{2^{-m}} = 2^{m-(-m)}$
$= 2^{2m}$

h $\frac{4}{2^{1-n}} = \frac{2^2}{2^{1-n}}$
$= 2^{2-(1-n)}$
$= 2^{1+n}$

i $\frac{2^{x+1}}{2^x} = 2^{x+1-x}$
$= 2^1$

j $\frac{4^x}{2^{1-x}} = \frac{(2^2)^x}{2^{1-x}}$
$= 2^{2x-(1-x)}$
$= 2^{3x-1}$

6 **a** $9 \times 3^p = 3^2 \times 3^p$
$= 3^{2+p}$

b $27^a = (3^3)^a$
$= 3^{3a}$

c $3 \times 9^n = 3^1 \times (3^2)^n$
$= 3^1 \times 3^{2n}$
$= 3^{1+2n}$

d $27 \times 3^d = 3^3 \times 3^d$
$= 3^{3+d}$

e $9 \times 27^t = 3^2 \times (3^3)^t$
$= 3^2 \times 3^{3t}$
$= 3^{2+3t}$

f $\frac{3^y}{3} = \frac{3^y}{3^1}$
$= 3^{y-1}$

g $\frac{3}{3^y} = \frac{3^1}{3^y}$
$= 3^{1-y}$

h $\frac{9}{27^t} = \frac{3^2}{(3^3)^t}$
$= \frac{3^2}{3^{3t}}$
$= 3^{2-3t}$

i $\frac{9^a}{3^{1-a}} = \frac{(3^2)^a}{3^{1-a}}$
$= \frac{3^{2a}}{3^{1-a}}$
$= 3^{2a-(1-a)}$
$= 3^{3a-1}$

j $\frac{9^{n+1}}{3^{2n-1}} = \frac{(3^2)^{n+1}}{3^{2n-1}}$
$= \frac{3^{2n+2}}{3^{2n-1}}$
$= 3^{2n+2-(2n-1)}$
$= 3^3$

7 **a** 32
$= 2^5$

b 25^3
$= (5^2)^3$
$= 5^{2\times 3}$
$= 5^6$

c 16^p
$= (2^4)^p$
$= 2^{4\times p}$
$= 2^{4p}$

d $5^a \times 25$
$= 5^a \times 5^2$
$= 5^{a+2}$

e $4^n \times 8^n$
$= (2^2)^n \times (2^3)^n$
$= 2^{2n} \times 2^{3n}$
$= 2^{2n+3n}$
$= 2^{5n}$

f $\frac{8^m}{16^n}$
$= \frac{(2^3)^m}{(2^4)^n}$
$= \frac{2^{3m}}{2^{4n}}$
$= 2^{3m-4n}$

g $\frac{25^p}{5^4}$
$= \frac{(5^2)^p}{5^4}$
$= \frac{5^{2p}}{5^4}$
$= 5^{2p-4}$

h 9^{t+2}
$= (3^2)^{t+2}$
$= 3^{2(t+2)}$
$= 3^{2t+4}$

i 32^{2-r}
$= (2^5)^{2-r}$
$= 2^{5(2-r)}$
$= 2^{10-5r}$

j $\frac{81}{3^{y+1}}$
$= \frac{3^4}{3^{y+1}}$
$= 3^{4-(y+1)}$
$= 3^{4-y-1}$
$= 3^{3-y}$

k $\frac{16^k}{4^k}$
$= \frac{(2^4)^k}{(2^2)^k}$
$= \frac{2^{4k}}{2^{2k}}$
$= 2^{4k-2k}$
$= 2^{2k}$

l $\frac{5^{a+1} \times 125}{25^{2a}}$
$= \frac{5^{a+1} \times 5^3}{(5^2)^{2a}}$
$= \frac{5^{a+1+3}}{5^{2(2a)}}$
$= \frac{5^{a+4}}{5^{4a}}$
$= 5^{a+4-4a}$
$= 5^{4-3a}$

8 **a** 4^0
$= 1$

b 3^{-1}
$= \frac{1}{3}$

c 7^{-2}
$= \frac{1}{7^2}$
$= \frac{1}{49}$

d x^{-3}
$= \frac{1}{x^3}$

e $5^0 + 5^{-1}$
$= 1 + \frac{1}{5}$
$= \frac{6}{5} \quad (= 1\frac{1}{5})$

f $\left(\frac{5}{3}\right)^0$
$= 1$

g $\left(\frac{7}{4}\right)^{-1}$
$= \left(\frac{4}{7}\right)^1$
$= \frac{4}{7}$

h $\left(\frac{1}{6}\right)^{-1}$
$= \left(\frac{6}{1}\right)^1$
$= 6$

i $\left(\frac{4}{3}\right)^{-2} = \left(\frac{3}{4}\right)^2 = \frac{3^2}{4^2} = \frac{9}{16}$

j $2^1 + 2^{-1} = 2 + \frac{1}{2} = \frac{5}{2} \quad (= 2\frac{1}{2})$

k $\left(1\frac{2}{3}\right)^{-3} = \left(\frac{5}{3}\right)^{-3} = \left(\frac{3}{5}\right)^3 = \frac{3^3}{5^3} = \frac{27}{125}$

l $5^2 + 5^1 + 5^{-1} = 25 + 5 + \frac{1}{5} = 30 + \frac{1}{5} = \frac{151}{5} \quad (= 30\frac{1}{5})$

9 **a** $\frac{1}{9} = \frac{1}{3^2} = 3^{-2}$

b $\frac{1}{16} = \frac{1}{2^4} = 2^{-4}$

c $\frac{1}{125} = \frac{1}{5^3} = 5^{-3}$

d $\frac{3}{5} = 3 \times \frac{1}{5} = 3^1 \times 5^{-1}$

e $\frac{4}{27} = \frac{2^2}{3^3} = 2^2 \times 3^{-3}$

f $\frac{2^c}{8 \times 9} = \frac{2^c}{2^3 \times 3^2} = 2^{c-3} \times 3^{-2}$

g $\frac{9^k}{10} = \frac{(3^2)^k}{2 \times 5} = 3^{2k} \times 2^{-1} \times 5^{-1}$

h $\frac{6^p}{75} = \frac{(2 \times 3)^p}{3 \times 5^2} = \frac{2^p \times 3^p}{3 \times 5^2} = 2^p \times 3^{p-1} \times 5^{-2}$

10 **a** $(2a)^2 = 2^2 \times a^2 = 4a^2$

b $(3n)^2 = 3^2 \times n^2 = 9n^2$

c $(5m)^3 = 5^3 \times m^3 = 125m^3$

d $(mn)^3 = m^3n^3$

e $\left(\frac{a}{2}\right)^3 = \frac{a^3}{2^3} = \frac{a^3}{8}$

f $\left(\frac{3}{m}\right)^2 = \frac{3^2}{m^2} = \frac{9}{m^2}$

g $\left(\frac{p}{q}\right)^4 = \frac{p^4}{q^4}$

h $\left(\frac{t}{5}\right)^2 = \frac{t^2}{5^2} = \frac{t^2}{25}$

11 **a** $(2ab)^2 = 2^2 \times a^2 \times b^2 = 4a^2b^2$

b $(-2a)^2 = (-2)^2 \times a^2 = 4a^2$

c $(6b^2)^2 = 6^2 \times (b^2)^2 = 36b^4$

d $(-2a)^3 = (-2)^3 \times a^3 = -8a^3$

e $(-3m^2n^2)^3 = (-3)^3 \times (m^2)^3 \times (n^2)^3 = -27m^6n^6$

f $(-2ab^4)^4 = (-2)^4 \times a^4 \times (b^4)^4 = 16a^4b^{16}$

g $\left(\frac{2a}{b}\right)^0 = 1, \quad \text{provided } a \neq 0, \ b \neq 0$

h $\left(\frac{m}{3n}\right)^4 = \frac{m^4}{3^4 \times n^4} = \frac{m^4}{81n^4}$

i $\left(\frac{xy}{2}\right)^3 = \frac{x^3y^3}{2^3}$
$= \frac{x^3y^3}{8}$

j $\left(\frac{-2a^2}{b^2}\right)^3 = \frac{(-2)^3 \times (a^2)^3}{(b^2)^3}$
$= -\frac{8a^6}{b^6}$

k $\left(\frac{-4a^3}{b}\right)^2 = \frac{(-4)^2 \times (a^3)^2}{b^2}$
$= \frac{16a^6}{b^2}$

l $\left(\frac{-3p^2}{q^3}\right)^2 = \frac{(-3)^2 \times (p^2)^2}{(q^3)^2}$
$= \frac{9p^4}{q^6}$

12 **a** $x^2(x^3 + x)$
$= x^2 \times x^3 + x^2 \times x^1$
$= x^{2+3} + x^{2+1}$
$= x^5 + x^3$

b $x^2(x^2 - 2x + 3)$
$= x^2 \times x^2 + x^2 \times (-2x) + x^2 \times 3$
$= x^{2+2} - 2x^{2+1} + 3x^2$
$= x^4 - 2x^3 + 3x^2$

c $x(x^2 + 1)(x^2 - 1)$
$= x[(x^2)^2 - 1^2]$
$= x(x^4 - 1)$
$= x \times x^4 - x$
$= x^{1+4} - x$
$= x^5 - x$

d $(x^3 - x^2)(x^2 + 2)$
$= x^3 \times x^2 + 2x^3 - x^2 \times x^2 - 2x^2$
$= x^{3+2} + 2x^3 - x^{2+2} - 2x^2$
$= x^5 - x^4 + 2x^3 - 2x^2$

e $(x^3 - x)^2$
$= (x^3)^2 + 2(x^3)(-x) + x^2$
$= x^{3\times 2} - 2x^{3+1} + x^2$
$= x^6 - 2x^4 + x^2$

f $x^2(x - 2 + x^{-1})$
$= x^2 \times x - 2x^2 + x^2 \times x^{-1}$
$= x^{2+1} - 2x^2 + x^{2-1}$
$= x^3 - 2x^2 + x$

g $x^{-1}(x^3 + x^2 - x)$
$= x^{-1} \times x^3 + x^{-1} \times x^2 + x^{-1}(-x)$
$= x^{-1+3} + x^{-1+2} - x^{-1+1}$
$= x^2 + x - x^0$
$= x^2 + x - 1$

h $(x^2 + x^{-1})^2$
$= (x^2)^2 + 2(x^2)(x^{-1}) + (x^{-1})^2$
$= x^{2\times 2} + 2x^{2-1} + x^{-1\times 2}$
$= x^4 + 2x + x^{-2}$

i $(x^2 + x^{-1})(x^2 - x^{-1})$
$= (x^2)^2 - (x^{-1})^2$
$= x^{2\times 2} - x^{-1\times 2}$
$= x^4 - x^{-2}$

13 **a** ab^{-2}
$= a \times \frac{1}{b^2}$
$= \frac{a}{b^2}$

b $(ab)^{-2}$
$= \frac{1}{(ab)^2}$
$= \frac{1}{a^2b^2}$

c $(2ab^{-1})^2$
$= 2^2 \times a^2 \times (b^{-1})^2$
$= 4a^2 \times b^{-1\times 2}$
$= 4a^2 \times b^{-2}$
$= 4a^2 \times \frac{1}{b^2}$
$= \frac{4a^2}{b^2}$

d $(5m^2)^{-2}$
$= \dfrac{1}{(5m^2)^2}$
$= \dfrac{1}{5^2 \times m^{2\times 2}}$
$= \dfrac{1}{25m^4}$

e $(3a^{-2}b)^2$
$= 3^2 \times (a^{-2})^2 \times b^2$
$= 9 \times a^{-2\times 2} \times b^2$
$= 9 \times a^{-4} \times b^2$
$= 9 \times \dfrac{1}{a^4} \times b^2$
$= \dfrac{9b^2}{a^4}$

f $(3xy^4)^{-3}$
$= \dfrac{1}{(3xy^4)^3}$
$= \dfrac{1}{3^3 \times x^3 \times y^{4\times 3}}$
$= \dfrac{1}{27x^3y^{12}}$

g $\dfrac{a^2b^{-1}}{c^2}$
$= \dfrac{a^2}{c^2} \times b^{-1}$
$= \dfrac{a^2}{c^2} \times \dfrac{1}{b^1}$
$= \dfrac{a^2}{bc^2}$

h $\dfrac{a^2b^{-1}}{c^{-2}}$
$= a^2 \times b^{-1} \times \dfrac{1}{c^{-2}}$
$= a^2 \times \dfrac{1}{b} \times c^2$
$= \dfrac{a^2c^2}{b}$

i $\dfrac{1}{a^{-3}} = a^3$

j $\dfrac{a^{-2}}{b^{-3}}$
$= a^{-2} \times \dfrac{1}{b^{-3}}$
$= \dfrac{1}{a^2} \times b^3$
$= \dfrac{b^3}{a^2}$

k $\dfrac{2a^{-1}}{d^2}$
$= \dfrac{2}{d^2} \times a^{-1}$
$= \dfrac{2}{d^2} \times \dfrac{1}{a}$
$= \dfrac{2}{ad^2}$

l $\dfrac{12a}{m^{-3}}$
$= 12a \times \dfrac{1}{m^{-3}}$
$= 12a \times m^3$
$= 12am^3$

14 **a** $\dfrac{1}{a^n} = a^{-n}$

b $\dfrac{5}{a^m} = 5a^{-m}$

c $\dfrac{1}{b^{-n}} = b^n$

d $\dfrac{1}{2^{n-3}} = 2^{-(n-3)}$
$= 2^{-n+3}$
$= 2^{3-n}$

e $\dfrac{1}{3^{2-n}} = 3^{-(2-n)}$
$= 3^{-2+n}$
$= 3^{n-2}$

f $\dfrac{3}{a^{4-m}} = 3a^{-(4-m)}$
$= 3a^{-4+m}$
$= 3a^{m-4}$

g $\dfrac{a^n}{b^{-m}} = a^n \times b^m$
$= a^nb^m$

h $\dfrac{a^{-n}}{a^{2+n}} = a^{-n-(2+n)}$
$= a^{-n-2-n}$
$= a^{-2n-2}$

15 **a** $\dfrac{1}{x^2} = x^{-2}$

b $\dfrac{2}{x} = 2x^{-1}$

c $x + \dfrac{1}{x} = x + x^{-1}$

d $x^2 - \dfrac{2}{x^3} = x^2 - 2x^{-3}$

e $\dfrac{1}{x} + \dfrac{3}{x^2} = x^{-1} + 3x^{-2}$

f $\dfrac{4}{x} - \dfrac{5}{x^3} = 4x^{-1} - 5x^{-3}$

g $7x - \dfrac{4}{x} + \dfrac{5}{x^2} = 7x - 4x^{-1} + 5x^{-2}$

h $\dfrac{3}{x} - \dfrac{2}{x^2} + \dfrac{5}{x^4} = 3x^{-1} - 2x^{-2} + 5x^{-4}$

16 **a** $\dfrac{x+3}{x}$
$= \dfrac{x}{x} + \dfrac{3}{x}$
$= 1 + 3x^{-1}$

b $\dfrac{3-2x}{x}$
$= \dfrac{3}{x} - \dfrac{2x}{x}$
$= 3x^{-1} - 2$

c $\dfrac{5-x}{x^2}$
$= \dfrac{5}{x^2} - \dfrac{x}{x^2}$
$= 5x^{-2} - x^{1-2}$
$= 5x^{-2} - x^{-1}$

d $\dfrac{x+2}{x^3}$
$= \dfrac{x}{x^3} + \dfrac{2}{x^3}$
$= x^{1-3} + 2x^{-3}$
$= x^{-2} + 2x^{-3}$

e $\dfrac{x^2+5}{x}$
$= \dfrac{x^2}{x} + \dfrac{5}{x}$
$= x + 5x^{-1}$

f $\dfrac{x^2+x-2}{x}$
$= \dfrac{x^2}{x} + \dfrac{x}{x} - \dfrac{2}{x}$
$= x + 1 - 2x^{-1}$

g $\dfrac{2x^2-3x+4}{x}$
$= \dfrac{2x^2}{x} - \dfrac{3x}{x} + \dfrac{4}{x}$
$= 2x - 3 + 4x^{-1}$

h $\dfrac{x^3-3x+5}{x^2}$
$= \dfrac{x^3}{x^2} - \dfrac{3x}{x^2} + \dfrac{5}{x^2}$
$= x - 3x^{-1} + 5x^{-2}$

i $\dfrac{5-x-x^2}{x}$
$= \dfrac{5}{x} - \dfrac{x}{x} - \dfrac{x^2}{x}$
$= 5x^{-1} - 1 - x$

j $\dfrac{8+5x-2x^3}{x}$
$= \dfrac{8}{x} + \dfrac{5x}{x} - \dfrac{2x^3}{x}$
$= 8x^{-1} + 5 - 2x^2$

k $\dfrac{16-3x+x^3}{x^2}$
$= \dfrac{16}{x^2} - \dfrac{3x}{x^2} + \dfrac{x^3}{x^2}$
$= 16x^{-2} - 3x^{-1} + x$

l $\dfrac{5x^4-3x^2+x+6}{x^2}$
$= \dfrac{5x^4}{x^2} - \dfrac{3x^2}{x^2} + \dfrac{x}{x^2} + \dfrac{6}{x^2}$
$= 5x^2 - 3 + x^{-1} + 6x^{-2}$

17 **a** $\dfrac{4+2x}{x^{-1}}$
$= \dfrac{4}{x^{-1}} + \dfrac{2x}{x^{-1}}$
$= 4x + 2x^{1-(-1)}$
$= 4x + 2x^2$

b $\dfrac{5-4x}{x^{-2}}$
$= \dfrac{5}{x^{-2}} - \dfrac{4x}{x^{-2}}$
$= 5x^2 - 4x^{1-(-2)}$
$= 5x^2 - 4x^3$

c $\dfrac{6+3x}{x^{-3}}$
$= \dfrac{6}{x^{-3}} + \dfrac{3x}{x^{-3}}$
$= 6x^3 + 3x^{1-(-3)}$
$= 6x^3 + 3x^4$

d $\dfrac{x^2+3}{x^{-1}}$
$= \dfrac{x^2}{x^{-1}} + \dfrac{3}{x^{-1}}$
$= x^{2-(-1)} + 3x$
$= x^3 + 3x$

e $\dfrac{x^2+x-4}{x^{-2}}$
$= \dfrac{x^2}{x^{-2}} + \dfrac{x}{x^{-2}} - \dfrac{4}{x^{-2}}$
$= x^{2-(-2)} + x^{1-(-2)} - 4x^2$
$= x^4 + x^3 - 4x^2$

f $\dfrac{x^3-3x+6}{x^{-3}}$
$= \dfrac{x^3}{x^{-3}} - \dfrac{3x}{x^{-3}} + \dfrac{6}{x^{-3}}$
$= x^{3-(-3)} - 3x^{1-(-3)} + 6x^3$
$= x^6 - 3x^4 + 6x^3$

g $\dfrac{x^3-6x+10}{x^{-2}}$
$= \dfrac{x^3}{x^{-2}} - \dfrac{6x}{x^{-2}} + \dfrac{10}{x^{-2}}$
$= x^{3-(-2)} - 6x^{1-(-2)} + 10x^2$
$= x^5 - 6x^3 + 10x^2$

h $\dfrac{x^2+4+x^{-1}}{x^{-3}}$

$= \dfrac{x^2}{x^{-3}} + \dfrac{4}{x^{-3}} + \dfrac{x^{-1}}{x^{-3}}$

$= x^{2-(-3)} + 4x^3 + x^{-1-(-3)}$

$= x^5 + 4x^3 + x^2$

EXERCISE 3E

1 **C** is not in scientific notation.

0.3×10^5
$= 3 \times 10^{-1} \times 10^5$
$= 3 \times 10^4$

It should be 3×10^4.

D is not in scientific notation.

21×10^{11}
$= 2.1 \times 10^1 \times 10^{11}$
$= 2.1 \times 10^{12}$

It should be 2.1×10^{12}.

2 **a** $259 = 2.59 \times 100$
$= 2.59 \times 10^2$

b $259\,000 = 2.59 \times 100\,000$
$= 2.59 \times 10^5$

c $2\,590\,000\,000 = 2.59 \times 1\,000\,000\,000$
$= 2.59 \times 10^9$

d $2.59 = 2.59 \times 1$
$= 2.59 \times 10^0$

e $0.259 = 2.59 \div 10$
$= 2.59 \times 10^{-1}$

f $0.000\,259 = 2.59 \div 10\,000$
$= 2.59 \times 10^{-4}$

g $40.7 = 4.07 \times 10$
$= 4.07 \times 10^1$

h $4070 = 4.07 \times 1000$
$= 4.07 \times 10^3$

i $0.0407 = 4.07 \div 100$
$= 4.07 \times 10^{-2}$

j $407\,000 = 4.07 \times 100\,000$
$= 4.07 \times 10^5$

k $407\,000\,000 = 4.07 \times 100\,000\,000$
$= 4.07 \times 10^8$

l $0.000\,040\,7 = 4.07 \div 100\,000$
$= 4.07 \times 10^{-5}$

3 **a** $47\,450\,000 = 4.745 \times 10\,000\,000$
$= 4.745 \times 10^7$ kg

b $0.003 = 3 \div 1000$
$= 3 \times 10^{-3}$ m

c $2\,599\,000 = 2.599 \times 1\,000\,000$
$= 2.599 \times 10^6$ hands

d $0.000\,000\,47 = 4.7 \div 10\,000\,000$
$= 4.7 \times 10^{-7}$ m

4 **a** 4×10^3
$= 4.000 \times 1000$
$= 4000$

b 3.8×10^5
$= 3.800\,00 \times 100\,000$
$= 380\,000$

c 8.6×10^1
$= 8.6 \times 10$
$= 86$

d 4.33×10^7
$= 4.330\,000\,0 \times 10\,000\,000$
$= 43\,300\,000$

e 4×10^{-3}
$= 0004. \div 10^3$
$= 0.004$

f 3.8×10^{-5}
$= 000\,003.8 \div 10^5$
$= 0.000\,038$

g 8.6×10^{-1}

$= 08.6 \div 10^1$

$= 0.86$

h 4.33×10^{-7}

$= 00\,000\,004.33 \div 10^7$

$= 0.000\,000\,433$

5 **a** 7.4×10^9

$= 7.400\,000\,000 \times 1\,000\,000\,000$

$= 7\,400\,000\,000$ people

b 1.12×10^{-2}

$= 001.12 \div 10^2$

$= 0.0112$ kg

c 5×10^{-7}

$= 00\,000\,005. \div 10^7$

$= 0.000\,000\,5$ m

d 7.3×10^6

$= 7.300\,000 \times 1\,000\,000$

$= 7\,300\,000$ kg

6 **a** $\boxed{\texttt{4.5E07}}$ can be represented

as 4.5×10^7.

4.5×10^7

$= 4.500\,000\,0 \times 10\,000\,000$

$= 45\,000\,000$

b $\boxed{\texttt{3.8E-04}}$ can be represented

as 3.8×10^{-4}.

3.8×10^{-4}

$= 00\,003.8 \div 10^4$

$= 0.000\,38$

c $\boxed{\texttt{2.1E05}}$ can be represented

as 2.1×10^5.

2.1×10^5

$= 2.100\,00 \times 100\,000$

$= 210\,000$

d $\boxed{\texttt{4.0E-03}}$ can be represented

as 4×10^{-3}.

4×10^{-3}

$= 0004. \div 10^3$

$= 0.004$

e $\boxed{\texttt{6.1E03}}$ can be represented

as 6.1×10^3.

6.1×10^3

$= 6.100 \times 1000$

$= 6100$

f $\boxed{\texttt{1.6E-06}}$ can be represented

as 1.6×10^{-6}.

1.6×10^{-6}

$= 0\,000\,001.6 \div 10^6$

$= 0.000\,001\,6$

g $\boxed{\texttt{3.9E04}}$ can be represented

as 3.9×10^4.

3.9×10^4

$= 3.9000 \times 10\,000$

$= 39\,000$

h $\boxed{\texttt{6.7E-02}}$ can be represented

as 6.7×10^{-2}.

6.7×10^{-2}

$= 006.7 \div 10^2$

$= 0.067$

7 Using technology:

a $680\,000 \times 73\,000\,000$

$= 4.964 \times 10^{13}$

b $0.0006 \div 15\,000$

$= 4 \times 10^{-8}$

c $(0.0007)^3$

$= 3.43 \times 10^{-10}$

d $(3.42 \times 10^5) \times (4.8 \times 10^4)$

$= 1.6416 \times 10^{10}$

e $(6.42 \times 10^{-2})^2$

$= 4.121\,64 \times 10^{-3}$

f $\dfrac{3.16 \times 10^{-10}}{6 \times 10^{7}}$

$\approx 5.27 \times 10^{-18}$

g $(9.8 \times 10^{-4}) \div (7.2 \times 10^{-6})$

$\approx 1.36 \times 10^{2}$

h $\dfrac{1}{3.8 \times 10^{5}}$

$\approx 2.63 \times 10^{-6}$

i $(1.2 \times 10^{3})^{3}$

$= 1.728 \times 10^{9}$

8 Using technology, $\dfrac{6 \times 10^{4}}{8 \times 10^{-4}} = 7.5 \times 10^{7}$ peanuts.

9 Using technology, $4.6 \times 10^{-7} + 2.15 \times 10^{-6} = 2.61 \times 10^{-6}$ m.

10 **a** Minimum distance to travel from Earth to Venus, then to Mercury

$= 3.8 \times 10^{9} + 7.7 \times 10^{9}$

$= 1.15 \times 10^{10}$ m

b We have assumed that we will always be on the side of the planet that is closest to the next planet, at the time when the planets are closest. It could take a very long time for these ideal conditions to occur.

11 **a** **i** Distance travelled $=$ speed $\times$ time

$= 2.9979 \times 10^{8}\ \mathrm{m\,s^{-1}} \times 60\ \mathrm{s}$ {1 minute = 60 seconds}

$= 1.798\,74 \times 10^{10}$ m

$\approx 1.80 \times 10^{10}$ m

ii Distance travelled

$=$ speed $\times$ time

$= 2.9979 \times 10^{8}\ \mathrm{m\,s^{-1}} \times (60 \times 60 \times 24)\ \mathrm{s}$ {1 day $= 60 \times 60 \times 24$ seconds}

$\approx 2.59 \times 10^{13}$ m

b 1 year $\approx$ 365.25 days

One light-year $=$ speed $\times$ time

$= 2.9979 \times 10^{8}\ \mathrm{m\,s^{-1}} \times (60 \times 60 \times 24 \times 365.25)\ \mathrm{s}$

$\approx 9.46 \times 10^{15}$ m

c 4.22 light-years $\approx 4.22 \times 9.46 \times 10^{15}$

$\approx 3.99 \times 10^{16}$ m

d

Galaxy	*Diameter* (light-years)
Milky Way	100 000
M87	980 000
Hercules A	1 500 000

i 980 000 light-years $\approx 980\,000 \times 9.46 \times 10^{15}$ m

$\approx 9.27 \times 10^{21}$ m

The diameter of M87 is approximately 9.27×10^{21} m.

ii 1 500 000 light-years $\approx 1\,500\,000 \times 9.46 \times 10^{15}$ m

$\approx 1.42 \times 10^{22}$ m

The diameter of Hercules A is approximately 1.42×10^{22} m.

Scale factor of diagram $\approx \dfrac{1.42 \times 10^{22} \text{ m}}{0.26 \text{ m}}$ $\{26 \text{ cm} \equiv 0.26 \text{ m}\}$

$\approx 5.46 \times 10^{22}$

So, Hercules A is about 5.46×10^{22} times wider than the diagram.

iii 100 000 light-years $\approx 100\,000 \times 9.46 \times 10^{15}$ m

$\approx 9.46 \times 10^{20}$ m

The diameter of the Milky Way is approximately 9.46×10^{20} m.

Time taken to cross the Milky Way

$= \dfrac{\text{distance}}{\text{speed}}$

$\approx \dfrac{9.46 \times 10^{20} \text{ m}}{100\,000\,000 \text{ m h}^{-1}}$ $\{100\,000 \text{ km h}^{-1} \equiv 100\,000\,000 \text{ m h}^{-1}\}$

$\approx 9.46 \times 10^{12}$ hours

$\approx 9.46 \times 10^{12} \div 24 \div 365.25$ years

$\approx 1.08 \times 10^{9}$ years

≈ 1.08 billion years

12

Particle	*Mass* (kg)	*Charge* (coulombs)
electron	$9.109\,383\,56 \times 10^{-31}$	$-1.602\,176\,620\,8 \times 10^{-19}$
proton	$1.672\,621\,898 \times 10^{-27}$	$+1.602\,176\,620\,8 \times 10^{-19}$
neutron	$1.674\,927\,471 \times 10^{-27}$	0

a Writing numbers in a form involving a power of 10 allows us to write very small numbers without having to write and count lots of zeros.

b **i** $\dfrac{\text{mass of one neutron}}{\text{mass of one electron}} = \dfrac{1.674\,927\,471 \times 10^{-27} \text{ kg}}{9.109\,383\,56 \times 10^{-31} \text{ kg}}$

$\approx 1.839 \times 10^{3}$

≈ 1839

A neutron is approximately 1839 times more massive than an electron.

ii $\dfrac{\text{mass of one proton}}{\text{mass of one electron}} = \dfrac{1.672\,621\,898 \times 10^{-27} \text{ kg}}{9.109\,383\,56 \times 10^{-31} \text{ kg}}$

$\approx 1.836 \times 10^{3}$

≈ 1836

A proton is approximately 1836 times more massive than an electron.

iii $\dfrac{\text{mass of one neutron}}{\text{mass of one proton}} = \dfrac{1.674\,927\,471 \times 10^{-27} \text{ kg}}{1.672\,621\,898 \times 10^{-27} \text{ kg}}$

≈ 1.001

A neutron is approximately 1.001 times more massive than a proton.

c An atom of silver has 47 protons and has no charge

$\therefore$ it must have the same number of electrons as protons

$\therefore$ it has 47 electrons.

Total mass of neutrons

$= $ total mass of atom $-$ mass of 47 protons $-$ mass of 47 electons

$= 1.791\,193\,4 \times 10^{-25} - 47 \times 1.672\,621\,898 \times 10^{-27} - 47 \times 9.109\,383\,56 \times 10^{-31}$

$= 1.791\,193\,4 \times 10^{-25} - 7.861\,322\,920\,6 \times 10^{-26} - 4.281\,410\,273\,2 \times 10^{-29}$

$= 1.004\,632\,966\,912\,68 \times 10^{-25}$ kg

$$\text{Number of neutrons} = \frac{\text{mass of total number of neutrons}}{\text{mass of one neutron}}$$

$$= \frac{1.004\,632\,966\,912\,68 \times 10^{-25}\text{ kg}}{1.674\,927\,471 \times 10^{-27}\text{ kg}}$$

$$\approx 59.98$$

$$\approx 60 \text{ neutrons}$$

So, the atom of silver has 47 electrons and 60 neutrons.

d $\dfrac{350 \text{ coulombs}}{\text{charge of one electron}} = \dfrac{350}{-1.602\,176\,620\,8 \times 10^{-19}}$

$\approx -2.18 \times 10^{21}$

$\therefore$ $\approx 2.18 \times 10^{21}$ electrons are transferred by 350 coulombs of charge.

REVIEW SET 3A

1 **a** $7\sqrt{5} - 3\sqrt{5}$
$= 4\sqrt{5}$

b $2\sqrt{6} - \sqrt{54}$
$= 2\sqrt{6} - \sqrt{9 \times 6}$
$= 2\sqrt{6} - (\sqrt{9} \times \sqrt{6})$
$= 2\sqrt{6} - 3\sqrt{6}$
$= -\sqrt{6}$

c $5\sqrt{3}(4 - \sqrt{3})$
$= 5\sqrt{3} \times 4 - 5\sqrt{3} \times \sqrt{3}$
$= 20\sqrt{3} - 5 \times 3$
$= 20\sqrt{3} - 15$

d $(1 + \sqrt{2})(2 + \sqrt{2})$
$= 2 + \sqrt{2} + 2\sqrt{2} + (\sqrt{2})^2$
$= 2 + 3\sqrt{2} + 2$
$= 4 + 3\sqrt{2}$

e $(6 - 5\sqrt{2})^2$
$= 6^2 - 2(6)(5\sqrt{2}) + (5\sqrt{2})^2$
$= 36 - 60\sqrt{2} + 25 \times 2$
$= 36 - 60\sqrt{2} + 50$
$= 86 - 60\sqrt{2}$

f $(3 + \sqrt{5})(3 - \sqrt{5})$
$= 3^2 - (\sqrt{5})^2$
$= 9 - 5$
$= 4$

2 **a** $-(-1)^{10} = -1$

b $-(-3)^3 = -(-27)$
$= 27$

c $3^0 - 3^{-1} = 1 - \frac{1}{3}$
$= \frac{2}{3}$

3 **a** $x^4 \times x^2 = x^{4+2} = x^6$

b $(2^{-1})^7 = 2^{-1\times 7} = 2^{-7}$ *or* $(2^{-1})^7 = (\frac{1}{2})^7 = \frac{1}{2^7} = \frac{1}{128}$

c $(ab^3)^6 = a^6 \times b^{3\times 6} = a^6 b^{18}$

4 **a** $3^{-3} = \frac{1}{3^3} = \frac{1}{27}$

b $x^{-1}y = \frac{1}{x} \times y = \frac{y}{x}$

c $\left(\frac{a}{b}\right)^{-1} = \left(\frac{b}{a}\right)^1 = \frac{b}{a}$

5 **a** $27 = 3^3$

b $9^t = (3^2)^t = 3^{2t}$

c $\frac{4}{2^{m-1}} = \frac{2^2}{2^{m-1}} = 2^{2-(m-1)} = 2^{3-m}$

6 **a** $\frac{15xy^2}{3y^4} = \frac{15}{3} \times x \times \frac{y^2}{y^4} = 5 \times x \times y^{2-4} = 5xy^{-2} = \frac{5x}{y^2}$

b $\frac{j^6}{j^5 \times j^8} = \frac{j^6}{j^{5+8}} = \frac{j^6}{j^{13}} = j^{6-13} = j^{-7} = \frac{1}{j^7}$

c $\frac{36g^3h^5}{12h^2} = \frac{36}{12} \times g^3 \times \frac{h^5}{h^2} = 3 \times g^3 \times h^{5-2} = 3g^3h^3$

7 **a** $\left(\frac{t}{4s}\right)^3 = \frac{t^3}{4^3 s^3} = \frac{t^3}{64s^3}$

b $\left(\frac{m^2}{5n}\right)^0 = 1$

provided $m \neq 0,\ \ n \neq 0$

c $(5p^3q)^2 = 5^2 \times (p^3)^2 \times q^2 = 25 \times p^{3\times 2} \times q^2 = 25p^6q^2$

8 **a** $\frac{x^2+8}{x} = \frac{x^2}{x} + \frac{8}{x} = x + 8x^{-1}$

b $\frac{4+x+x^3}{x^{-2}} = \frac{4}{x^{-2}} + \frac{x}{x^{-2}} + \frac{x^3}{x^{-2}} = 4x^2 + x^{1-(-2)} + x^{3-(-2)} = 4x^2 + x^3 + x^5$

c $\frac{k^{-x}}{k^{x+6}} = k^{-x-(x+6)} = k^{-2x-6}$

9 **a** $a^4b^5 \times a^2b^2 = a^4 \times a^2 \times b^5 \times b^2 = a^{4+2} \times b^{5+2} = a^6b^7$

b $6xy^5 \div 9x^2y^5 = \frac{6xy^5}{9x^2y^5} = \frac{6}{9} \times x^{1-2} \times y^{5-5} = \frac{2}{3}x^{-1}y^0 = \frac{2}{3x}$

c $\frac{5(x^2y)^2}{(5x^2)^2} = \frac{5 \times (x^2)^2 \times y^2}{5^2 \times (x^2)^2} = \frac{5x^4y^2}{25x^4} = \frac{1}{5} \times x^{4-4} \times y^2 = \frac{1}{5}x^0y^2 = \frac{y^2}{5}$

10 **a** 4.6×10^{11}
$= 4.600\,000\,000\,00 \times 100\,000\,000\,000$
$= 460\,000\,000\,000$

b 1.9×10^{0}
$= 1.9 \times 1$
$= 1.9$

c 3.2×10^{-3}
$= 0003.2 \div 10^{3}$
$= 0.0032$

11 **a** 12.74 million metres $= 12.74 \times 10^{6}$ m
$= 1.274 \times 10^{7}$ m

b $0.000\,12$ m $= 1.2 \div 10\,000$ m
$= 1.2 \times 10^{-4}$ m

12 Number of sheets of paper required $= \dfrac{\text{height of pile}}{\text{thickness of one sheet}}$

$$= \frac{0.1 \text{ m}}{3.2 \times 10^{-4} \text{ m}} \qquad \{10 \text{ cm} \equiv 0.1 \text{ m}\}$$
$$= 3.125 \times 10^{2}$$
$$= 312.5$$

So we would require 313 sheets of paper.

13 $\dfrac{\text{distance from Earth to Neptune}}{\text{distance from Earth to Saturn}} = \dfrac{4.3 \times 10^{9} \text{ km}}{1.5 \times 10^{9} \text{ km}}$
≈ 2.87

Neptune is approximately 2.87 times further from Earth than Saturn is from Earth.

REVIEW SET 3B

1 **a** $4\sqrt{11} - 5\sqrt{11}$
$= -\sqrt{11}$

b $\sqrt{32} - 3\sqrt{2}$
$= \sqrt{16 \times 2} - 3\sqrt{2}$
$= \sqrt{16} \times \sqrt{2} - 3\sqrt{2}$
$= 4\sqrt{2} - 3\sqrt{2}$
$= \sqrt{2}$

c $(7 + 2\sqrt{3})(5 - 3\sqrt{3})$
$= 7 \times 5 - 7 \times 3\sqrt{3} + 2\sqrt{3} \times 5 - 2\sqrt{3} \times 3\sqrt{3}$
$= 35 - 21\sqrt{3} + 10\sqrt{3} - 6 \times 3$
$= 35 - 11\sqrt{3} - 18$
$= 17 - 11\sqrt{3}$

d $(6 + 2\sqrt{2})(6 - 2\sqrt{2})$
$= 6^{2} - (2\sqrt{2})^{2}$
$= 36 - (4 \times 2)$
$= 36 - 8$
$= 28$

2 **a** $\dfrac{2}{\sqrt{3}} = \dfrac{2}{\sqrt{3}} \times \dfrac{\sqrt{3}}{\sqrt{3}}$
$= \dfrac{2\sqrt{3}}{3}$

b $\dfrac{\sqrt{7}}{\sqrt{5}} = \dfrac{\sqrt{7}}{\sqrt{5}} \times \dfrac{\sqrt{5}}{\sqrt{5}}$
$= \dfrac{\sqrt{7 \times 5}}{5}$
$= \dfrac{\sqrt{35}}{5}$

c $\dfrac{3}{\sqrt{3}+2} = \left(\dfrac{3}{\sqrt{3}+2}\right)\left(\dfrac{\sqrt{3}-2}{\sqrt{3}-2}\right)$
$= \dfrac{3(\sqrt{3}-2)}{(\sqrt{3})^2-2^2}$
$= \dfrac{3\sqrt{3}-6}{3-4}$
$= \dfrac{3\sqrt{3}-6}{-1}$
$= 6-3\sqrt{3}$

d $\dfrac{1}{4+\sqrt{7}} = \left(\dfrac{1}{4+\sqrt{7}}\right)\left(\dfrac{4-\sqrt{7}}{4-\sqrt{7}}\right)$
$= \dfrac{4-\sqrt{7}}{4^2-(\sqrt{7})^2}$
$= \dfrac{4-\sqrt{7}}{16-7}$
$= \dfrac{4-\sqrt{7}}{9}$

3 **a** $\dfrac{m^9}{m^5} = m^{9-5}$
$= m^4$

b $y^0 = 1$
provided $y \neq 0$

c $\left(\dfrac{7z}{w}\right)^{-2} = \left(\dfrac{w}{7z}\right)^2$
$= \dfrac{w^2}{(7z)^2}$
$= \dfrac{w^2}{7^2z^2}$
$= \dfrac{w^2}{49z^2}$

4 **a** $\dfrac{k^x}{k^2} = k^{x-2}$

b $11^r \times 11^{-4} = 11^{r-4}$

c $9 \times 3^b = 3^2 \times 3^b$
$= 3^{2+b}$

5 **a** $\dfrac{a}{b^2} = ab^{-2}$

b $\dfrac{jk^4}{l^a} = jk^4l^{-a}$

c $\dfrac{1}{x^3} - \dfrac{2}{x^5} = x^{-3} - 2x^{-5}$

6 **a** $\dfrac{1}{16} = \dfrac{1}{2^4}$
$= 2^{-4}$

b $3^k \times 81 = 3^k \times 3^4$
$= 3^{k+4}$

c $\dfrac{125^a}{5^b} = \dfrac{(5^3)^a}{5^b}$
$= 5^{3a} \times 5^{-b}$
$= 5^{3a-b}$

7 **a** $x^{-1}(x^2-5x+6)$
$= x^{-1} \times x^2 + x^{-1} \times (-5x) + 6x^{-1}$
$= x^{-1+2} - 5x^{-1+1} + \dfrac{6}{x}$
$= x - 5x^0 + \dfrac{6}{x}$
$= x - 5 + \dfrac{6}{x}$

b $(x^3+x^{-1})^2$
$= (x^3)^2 + 2(x^3)(x^{-1}) + (x^{-1})^2$
$= x^{3\times 2} + 2x^{3-1} + x^{-1\times 2}$
$= x^6 + 2x^2 + x^{-2}$
$= x^6 + 2x^2 + \dfrac{1}{x^2}$

c $(x^2-3)(x+x^{-1})$
$= x^2 \times x + x^2 \times x^{-1} - 3x - 3x^{-1}$
$= x^{2+1} + x^{2-1} - 3x - \dfrac{3}{x}$
$= x^3 + x - 3x - \dfrac{3}{x}$
$= x^3 - 2x - \dfrac{3}{x}$

8 **a** $\left(\dfrac{2a^6}{8b^2}\right)^3 = \dfrac{2^3 \times a^{6\times3}}{8^3 \times b^{2\times3}}$
$= \dfrac{8a^{18}}{512b^6}$
$= \dfrac{a^{18}}{64b^6}$

b $(5d \times d^{-5})^2$
$= (5d^{1+(-5)})^2$
$= (5d^{-4})^2$
$= 5^2 d^{-4\times2}$
$= 25d^{-8}$
$= \dfrac{25}{d^8}$

c $\dfrac{16z^2 \times z^5}{(2z)^3} = \dfrac{16z^{2+5}}{2^3 \times z^3}$
$= \dfrac{16z^7}{8z^3}$
$= \frac{16}{8} z^{7-3}$
$= 2z^4$

9 **a** $x^{-2} \times x^{-3}$
$= x^{-2+(-3)}$
$= x^{-5}$
$= \dfrac{1}{x^5}$

b $2(ab)^{-2}$
$= 2 \times \dfrac{1}{(ab)^2}$
$= \dfrac{2}{a^2b^2}$

c $2ab^{-2}$
$= 2a \times \left(\dfrac{1}{b^2}\right)$
$= \dfrac{2a}{b^2}$

10 **a** $\dfrac{27}{9^a}$
$= \dfrac{3^3}{(3^2)^a}$
$= \dfrac{3^3}{3^{2a}}$
$= 3^{3-2a}$

b $81^{1-x} \times 9^{1-2x}$
$= (3^4)^{1-x} \times (3^2)^{1-2x}$
$= 3^{4-4x} \times 3^{2-4x}$
$= 3^{4-4x+(2-4x)}$
$= 3^{6-8x}$

11 **a** 1.43×10^5 km
$= 1.430\,00 \times 100\,000$ km
$= 143\,000$ km

b 8.2×10^{-8} m
$= 000\,000\,008.2 \div 10^8$ m
$= 0.000\,000\,082$ m

12 **a** Time taken $= \dfrac{\text{distance}}{\text{speed}}$
$= \dfrac{3740 \text{ m}}{1.91 \times 10^8 \text{ m s}^{-1}}$
$\approx 1.96 \times 10^{-5}$ s

b Time taken $= \dfrac{\text{distance}}{\text{speed}}$
$= \dfrac{2.1 \times 10^6 \text{ m}}{1.91 \times 10^8 \text{ m s}^{-1}}$
$\approx 1.10 \times 10^{-2}$ s
≈ 0.0110 s

13 Number of sheets of gold leaf required $= \dfrac{\text{height of dime}}{\text{thickness of one sheet}}$
$= \dfrac{1.35 \times 10^{-3} \text{ m}}{1.8 \times 10^{-7} \text{ m}}$
$= 7.5 \times 10^3$
$= 7500$ sheets

Chapter 4

EQUATIONS

EXERCISE 4A

1 **a** $3x^2 = 48$
$\therefore\ x^2 = 16$ {dividing both sides by 3}
$\therefore\ x = \pm\sqrt{16}$
$\therefore\ x = \pm 4$

b $5x^2 = 35$
$\therefore\ x^2 = 7$ {dividing both sides by 5}
$\therefore\ x = \pm\sqrt{7}$

c $2x^2 = -10$
$\therefore\ x^2 = -5$ {dividing both sides by 2}
which has no real solutions as x^2 cannot be negative.

d $6x^2 = 0$
$\therefore\ x^2 = 0$ {dividing both sides by 6}
$\therefore\ x = \pm\sqrt{0}$
$\therefore\ x = 0$

e $4x^2 - 5 = 15$
$\therefore\ 4x^2 = 20$ {adding 5 to both sides}
$\therefore\ x^2 = 5$ {dividing both sides by 4}
$\therefore\ x = \pm\sqrt{5}$

f $7 - 3x^2 = 19$
$\therefore\ -3x^2 = 12$ {subtracting 7 from both sides}
$\therefore\ x^2 = -4$ {dividing both sides by -3}
which has no real solutions as x^2 cannot be negative.

2 **a** $x^3 = 27$
$\therefore\ x = \sqrt[3]{27}$
$\therefore\ x = 3$

b $x^4 = 16$
$\therefore\ x = \sqrt[4]{16}$
$\therefore\ x = \pm 2$

c $x^6 = -10$
has no real solutions as x^6 cannot be negative.

d $x^5 = -13$
$\therefore\ x = \sqrt[5]{-13}$

e $x^3 + 8 = 0$
$\therefore\ x^3 = -8$
$\therefore\ x = \sqrt[3]{-8}$
$\therefore\ x = -2$

f $2x^3 = 14$
$\therefore\ x^3 = 7$
$\therefore\ x = \sqrt[3]{7}$

g $5x^4 = 30$
$\therefore\ x^4 = 6$
$\therefore\ x = \pm\sqrt[4]{6}$

h $x^3 = \frac{8}{27}$
$\therefore\ x = \sqrt[3]{\frac{8}{27}}$
$\therefore\ x = \frac{\sqrt[3]{8}}{\sqrt[3]{27}}$
$\therefore\ x = \frac{2}{3}$

i $x^4 = \frac{1}{16}$
$\therefore\ x = \pm\sqrt[4]{\frac{1}{16}}$
$\therefore\ x = \pm\frac{1}{2}$

j $3x^5 = 1$
$\therefore\ x^5 = \frac{1}{3}$
$\therefore\ x = \sqrt[5]{\frac{1}{3}}$

k $4x^3 + 5 = -19$
$\therefore\ 4x^3 = -24$
$\therefore\ x^3 = -6$
$\therefore\ x = \sqrt[3]{-6}$

l $2x^4 - 55 = 107$
$\therefore\ 2x^4 = 162$
$\therefore\ x^4 = 81$
$\therefore\ x = \pm\sqrt[4]{81}$
$\therefore\ x = \pm 3$

3 **a** $(x-3)^2 = 16$
$\therefore\ x-3 = \pm\sqrt{16}$
$\therefore\ x-3 = \pm 4$
$\therefore\ x = 3 \pm 4$
$\therefore\ x = 7$ or -1

b $(x+4)^2 = -25$
has no real solutions as $(x+4)^2$ cannot be negative.

c $(x+4)^2 = 13$
$\therefore\ x+4 = \pm\sqrt{13}$
$\therefore\ x = -4 \pm \sqrt{13}$

d $(x-7)^2 = 0$
$\therefore\ x-7 = \pm\sqrt{0}$
$\therefore\ x-7 = 0$
$\therefore\ x = 7$

e $(2x-3)^2 = 25$
$\therefore\ 2x-3 = \pm\sqrt{25}$
$\therefore\ 2x-3 = \pm 5$
$\therefore\ 2x = 3 \pm 5$
$\therefore\ 2x = 8$ or -2
$\therefore\ x = 4$ or -1

f $\frac{1}{2}(3x+1)^2 = 7$
$\therefore\ (3x+1)^2 = 14$
$\therefore\ 3x+1 = \pm\sqrt{14}$
$\therefore\ 3x = -1 \pm \sqrt{14}$
$\therefore\ x = \dfrac{-1 \pm \sqrt{14}}{3}$

g $(x-\sqrt{2})^2 = 2$
$\therefore\ x-\sqrt{2} = \pm\sqrt{2}$
$\therefore\ x = \sqrt{2} \pm \sqrt{2}$
$\therefore\ x = 2\sqrt{2}$ or 0

h $(2x-\sqrt{3})^2 = 2$
$\therefore\ 2x-\sqrt{3} = \pm\sqrt{2}$
$\therefore\ 2x = \sqrt{3} \pm \sqrt{2}$
$\therefore\ x = \dfrac{\sqrt{3} \pm \sqrt{2}}{2}$

i $(2x+1)^2 = 7$
$\therefore\ 2x+1 = \pm\sqrt{7}$
$\therefore\ 2x = -1 \pm \sqrt{7}$
$\therefore\ x = \dfrac{-1 \pm \sqrt{7}}{2}$

4 **a** $(x-1)^3 = 17$
$\therefore\ x-1 = \sqrt[3]{17}$
$\therefore\ x = 1 + \sqrt[3]{17}$

b $(x+3)^5 = -1$
$\therefore\ x+3 = \sqrt[5]{-1}$
$\therefore\ x+3 = -1$
$\therefore\ x = -4$

c $(x-2)^4 = 20$
$\therefore\ x-2 = \pm\sqrt[4]{20}$
$\therefore\ x = 2 \pm \sqrt[4]{20}$

d $(x+5)^4 = -16$
has no real solutions as $(x+5)^4$ cannot be negative.

e $2(x+4)^5 = -24$
$\therefore\ (x+4)^5 = -12$
$\therefore\ x+4 = \sqrt[5]{-12}$
$\therefore\ x = -4 + \sqrt[5]{-12}$

f $(3x-1)^6 = 1$
$\therefore\ 3x-1 = \pm\sqrt[6]{1}$
$\therefore\ 3x-1 = \pm 1$
$\therefore\ 3x = 1 \pm 1$
$\therefore\ 3x = 2$ or 0
$\therefore\ x = \frac{2}{3}$ or 0

g $(2x-3)^4 = 15$
$\therefore\ 2x-3 = \pm\sqrt[4]{15}$
$\therefore\ 2x = 3 \pm \sqrt[4]{15}$
$\therefore\ x = \dfrac{3 \pm \sqrt[4]{15}}{2}$

h $\frac{1}{4}(1-x)^5 = 8$
$\therefore\ (1-x)^5 = 32$
$\therefore\ 1-x = \sqrt[5]{32}$
$\therefore\ 1-x = 2$
$\therefore\ x = -1$

i $14 = 3 - \frac{1}{2}(4-x)^3$
$\therefore\ \frac{1}{2}(4-x)^3 = -11$
$\therefore\ (4-x)^3 = -22$
$\therefore\ 4-x = \sqrt[3]{-22}$
$\therefore\ -x = -4 + \sqrt[3]{-22}$
$\therefore\ x = 4 - \sqrt[3]{-22}$

5 **a** If $3x^3 - 24 = 0$
then $3x^3 = 24$
$\therefore\ x^3 = 8$
$\therefore\ x = \sqrt[3]{8}$
$\therefore\ x = 2$
$\therefore$ the zero of $3x^3 - 24$ is 2.

b If $(x-1)^4 - 11 = 0$
then $(x-1)^4 = 11$
$\therefore\ x-1 = \pm\sqrt[4]{11}$
$\therefore\ x = 1 \pm \sqrt[4]{11}$
$\therefore$ the zeros of $(x-1)^4 - 11$ are $1 \pm \sqrt[4]{11}$.

c If $(2x+3)^5 + 1 = 0$
then $(2x+3)^5 = -1$
$\therefore\ 2x+3 = \sqrt[5]{-1}$
$\therefore\ 2x+3 = -1$
$\therefore\ 2x = -4$
$\therefore\ x = -2$
$\therefore$ the zero of $(2x+3)^5 + 1$ is -2.

6 **a** $x^{-1} = \frac{1}{6}$
$\therefore\ x = 6$

b $x^{-2} = \frac{1}{9}$
$\therefore\ x^2 = 9$
$\therefore\ x = \pm\sqrt{9}$
$\therefore\ x = \pm 3$

c $x^{-3} = -\frac{1}{27}$
$\therefore\ x^3 = -27$
$\therefore\ x = \sqrt[3]{-27}$
$\therefore\ x = -3$

d $x^{-2} = 49$
$\therefore\ x^2 = \frac{1}{49}$
$\therefore\ x = \pm\sqrt{\frac{1}{49}}$
$\therefore\ x = \pm\frac{1}{7}$

e $x^{-4} = \frac{1}{16}$
$\therefore\ x^4 = 16$
$\therefore\ x = \pm\sqrt[4]{16}$
$\therefore\ x = \pm 2$

f $x^{-3} = -64$
$\therefore\ x^3 = -\frac{1}{64}$
$\therefore\ x = \sqrt[3]{-\frac{1}{64}}$
$\therefore\ x = -\frac{1}{4}$

g $(x+1)^{-2} = -4$
$\therefore\ (x+1)^2 = -\frac{1}{4}$
which has no real solutions as $(x+1)^2$ cannot be negative.

h $(2x-5)^{-3} = \frac{1}{5}$
$\therefore\ (2x-5)^3 = 5$
$\therefore\ 2x - 5 = \sqrt[3]{5}$
$\therefore\ 2x = 5 + \sqrt[3]{5}$
$\therefore\ x = \dfrac{5+\sqrt[3]{5}}{2}$

EXERCISE 4B

1 **a** $4x = 0$
$\therefore\ x = 0$

b $ab = 0$
$\therefore\ a = 0$ or $b = 0$

c $2xy = 0$
$\therefore\ x = 0$ or $y = 0$

d $3abc = 0$
$\therefore\ abc = 0$
$\therefore\ a = 0$ or $b = 0$ or $c = 0$

2 **a** $x(x-5) = 0$
$\therefore\ x = 0$ or $x - 5 = 0$
$\therefore\ x = 0$ or 5

b $2x(x+3) = 0$
$\therefore\ 2x = 0$ or $x + 3 = 0$
$\therefore\ x = 0$ or -3

c $(x+1)(x-3) = 0$
$\therefore\ x + 1 = 0$ or $x - 3 = 0$
$\therefore\ x = -1$ or 3

d $3x(7-x) = 0$
$\therefore\ 3x = 0$ or $7 - x = 0$
$\therefore\ x = 0$ or 7

e $4(x+6)(2x-3) = 0$
$\therefore\ (x+6)(2x-3) = 0$
$\therefore\ x + 6 = 0$ or $2x - 3 = 0$
$\therefore\ x = -6$ or $2x = 3$
$\therefore\ x = -6$ or $\frac{3}{2}$

f $(2x+1)(2x-1) = 0$
$\therefore\ 2x + 1 = 0$ or $2x - 1 = 0$
$\therefore\ 2x = -1$ or $2x = 1$
$\therefore\ x = -\frac{1}{2}$ or $\frac{1}{2}$

3 **a** $x^2(x+5) = 0$
$\therefore\ x^2 = 0$ or $x + 5 = 0$
$\therefore\ x = 0$ or -5

b $4(5-x)^2 = 0$
$\therefore\ (5-x)^2 = 0$
$\therefore\ 5 - x = 0$ {null factor law}
$\therefore\ x = 5$

c
$$-3(3x-1)^2=0$$
$$\therefore\ (3x-1)^2=0$$
$$\therefore\ 3x-1=0 \quad \{\text{null factor law}\}$$
$$\therefore\ 3x=1$$
$$\therefore\ x=\tfrac{1}{3}$$

d
$$-6(x-5)(3x+2)^2=0$$
$$\therefore\ (x-5)(3x+2)^2=0$$
$$\therefore\ x-5=0 \text{ or } (3x+2)^2=0$$
$$\therefore\ x-5=0 \text{ or } 3x+2=0$$
$$\therefore\ x=5 \text{ or } 3x=-2$$
$$\therefore\ x=5 \text{ or } -\tfrac{2}{3}$$

e
$$x(x+1)(x-2)=0$$
$$\therefore\ x=0 \text{ or } x+1=0 \text{ or } x-2=0$$
$$\therefore\ x=0,\ -1, \text{ or } 2$$

f
$$3(x+2)(x+4)(2x-1)=0$$
$$\therefore\ (x+2)(x+4)(2x-1)=0$$
$$\therefore\ x+2=0 \text{ or } x+4=0 \text{ or } 2x-1=0$$
$$\therefore\ x=-2 \text{ or } -4 \text{ or } 2x=1$$
$$\therefore\ x=-2,\ -4, \text{ or } \tfrac{1}{2}$$

4 **a**
$$\frac{a}{b}=0$$
$$\therefore\ a=0,\ b\neq 0$$

b
$$\frac{3xy}{z}=0$$
$$\therefore\ 3xy=0,\ z\neq 0$$
$$\therefore\ xy=0,\ z\neq 0$$
$$\therefore\ x=0 \text{ or } y=0,\ z\neq 0$$

c
$$\frac{2}{xy}=0$$
$$\therefore\ 2=0 \quad \text{which is impossible}$$
$\therefore$ there are no solutions.

d
$$-\frac{x}{2y}=0$$
$$\therefore\ -x=0,\ y\neq 0$$
$$\therefore\ x=0,\ y\neq 0$$

EXERCISE 4C.1

1 **a**
$$4x^2+7x=0$$
$$\therefore\ x(4x+7)=0$$
$$\therefore\ x=0 \text{ or } 4x+7=0$$
$$\therefore\ x=0 \text{ or } -\tfrac{7}{4}$$

b
$$6x^2+2x=0$$
$$\therefore\ 2x(3x+1)=0$$
$$\therefore\ 2x=0 \text{ or } 3x+1=0$$
$$\therefore\ x=0 \text{ or } -\tfrac{1}{3}$$

c
$$3x^2-7x=0$$
$$\therefore\ x(3x-7)=0$$
$$\therefore\ x=0 \text{ or } 3x-7=0$$
$$\therefore\ x=0 \text{ or } \tfrac{7}{3}$$

d
$$2x^2-11x=0$$
$$\therefore\ x(2x-11)=0$$
$$\therefore\ x=0 \text{ or } 2x-11=0$$
$$\therefore\ x=0 \text{ or } \tfrac{11}{2}$$

e
$$3x^2=8x$$
$$\therefore\ 3x^2-8x=0$$
$$\therefore\ x(3x-8)=0$$
$$\therefore\ x=0 \text{ or } 3x-8=0$$
$$\therefore\ x=0 \text{ or } \tfrac{8}{3}$$

f
$$9x=6x^2$$
$$\therefore\ 6x^2-9x=0$$
$$\therefore\ 3x(2x-3)=0$$
$$\therefore\ 3x=0 \text{ or } 2x-3=0$$
$$\therefore\ x=0 \text{ or } \tfrac{3}{2}$$

2 **a** $x^2 - 5x + 6 = 0$
$\therefore (x-2)(x-3) = 0$
$\therefore x = 2 \text{ or } 3$

b $x^2 - 2x + 1 = 0$
$\therefore (x-1)^2 = 0$
$\therefore x = 1$

c $x^2 + 2x - 8 = 0$
$\therefore (x+4)(x-2) = 0$
$\therefore x = -4 \text{ or } 2$

d $x^2 + 7x + 12 = 0$
$\therefore (x+3)(x+4) = 0$
$\therefore x = -3 \text{ or } -4$

e $x^2 = 2x + 8$
$\therefore x^2 - 2x - 8 = 0$
$\therefore (x+2)(x-4) = 0$
$\therefore x = -2 \text{ or } 4$

f $x^2 + 21 = 10x$
$\therefore x^2 - 10x + 21 = 0$
$\therefore (x-3)(x-7) = 0$
$\therefore x = 3 \text{ or } 7$

g $9 + x^2 = 6x$
$\therefore x^2 - 6x + 9 = 0$
$\therefore (x-3)^2 = 0$
$\therefore x = 3$

h $x^2 + x = 12$
$\therefore x^2 + x - 12 = 0$
$\therefore (x+4)(x-3) = 0$
$\therefore x = -4 \text{ or } 3$

i $x^2 + 8x = 33$
$\therefore x^2 + 8x - 33 = 0$
$\therefore (x+11)(x-3) = 0$
$\therefore x = -11 \text{ or } 3$

j $3x^2 + 9x = 12$
$\therefore 3x^2 + 9x - 12 = 0$
$\therefore 3(x^2 + 3x - 4) = 0$
$\therefore 3(x+4)(x-1) = 0$
$\therefore (x+4)(x-1) = 0$
$\therefore x = -4 \text{ or } 1$

k $4x = 70 - 2x^2$
$\therefore 2x^2 + 4x - 70 = 0$
$\therefore 2(x^2 + 2x - 35) = 0$
$\therefore 2(x+7)(x-5) = 0$
$\therefore (x+7)(x-5) = 0$
$\therefore x = -7 \text{ or } 5$

l $50 - 5x^2 = -15x$
$\therefore 5x^2 - 15x - 50 = 0$
$\therefore 5(x^2 - 3x - 10) = 0$
$\therefore 5(x+2)(x-5) = 0$
$\therefore (x+2)(x-5) = 0$
$\therefore x = -2 \text{ or } 5$

3 **a** $9x^2 - 12x + 4 = 0$
$\therefore (3x-2)^2 = 0$
$\therefore x = \frac{2}{3}$

b $2x^2 - 13x - 7 = 0$
$\therefore (2x+1)(x-7) = 0$
$\therefore x = -\frac{1}{2} \text{ or } 7$

c $3x^2 = 16x + 12$
$\therefore 3x^2 - 16x - 12 = 0$
$\therefore (3x+2)(x-6) = 0$
$\therefore x = -\frac{2}{3} \text{ or } 6$

d $3x^2 + 5x = 2$
$\therefore 3x^2 + 5x - 2 = 0$
$\therefore (3x-1)(x+2) = 0$
$\therefore x = \frac{1}{3} \text{ or } -2$

e $2x^2 + 3 = 5x$
$\therefore 2x^2 - 5x + 3 = 0$
$\therefore (2x-3)(x-1) = 0$
$\therefore x = \frac{3}{2} \text{ or } 1$

f $3x^2 + 8x + 4 = 0$
$\therefore (3x+2)(x+2) = 0$
$\therefore x = -\frac{2}{3} \text{ or } -2$

g
$$\begin{aligned} 3x^2 &= 10x + 8 \\ \therefore\ 3x^2 - 10x - 8 &= 0 \\ \therefore\ (3x+2)(x-4) &= 0 \\ \therefore\ x &= -\tfrac{2}{3} \text{ or } 4 \end{aligned}$$

h
$$\begin{aligned} 4x^2 + 4x &= 3 \\ \therefore\ 4x^2 + 4x - 3 &= 0 \\ \therefore\ (2x-1)(2x+3) &= 0 \\ \therefore\ x &= \tfrac{1}{2} \text{ or } -\tfrac{3}{2} \end{aligned}$$

i
$$\begin{aligned} 4x^2 &= 11x + 3 \\ \therefore\ 4x^2 - 11x - 3 &= 0 \\ \therefore\ (4x+1)(x-3) &= 0 \\ \therefore\ x &= -\tfrac{1}{4} \text{ or } 3 \end{aligned}$$

j
$$\begin{aligned} 12x^2 &= 11x + 15 \\ \therefore\ 12x^2 - 11x - 15 &= 0 \\ \therefore\ (4x+3)(3x-5) &= 0 \\ \therefore\ x &= -\tfrac{3}{4} \text{ or } \tfrac{5}{3} \end{aligned}$$

k
$$\begin{aligned} 7x^2 + 6x &= 1 \\ \therefore\ 7x^2 + 6x - 1 &= 0 \\ \therefore\ (7x-1)(x+1) &= 0 \\ \therefore\ x &= \tfrac{1}{7} \text{ or } -1 \end{aligned}$$

l
$$\begin{aligned} 15x^2 + 2x &= 56 \\ \therefore\ 15x^2 + 2x - 56 &= 0 \\ \therefore\ (x+2)(15x-28) &= 0 \\ \therefore\ x &= -2 \text{ or } \tfrac{28}{15} \end{aligned}$$

4 **a**
$$\begin{aligned} (x+1)^2 &= 2x^2 - 5x + 11 \\ \therefore\ x^2 + 2x + 1 &= 2x^2 - 5x + 11 \\ \therefore\ x^2 - 7x + 10 &= 0 \\ \therefore\ (x-2)(x-5) &= 0 \\ \therefore\ x &= 2 \text{ or } 5 \end{aligned}$$

b
$$\begin{aligned} (x+2)(1-x) &= -4 \\ \therefore\ x - x^2 + 2 - 2x &= -4 \\ \therefore\ x^2 + x - 6 &= 0 \\ \therefore\ (x+3)(x-2) &= 0 \\ \therefore\ x &= -3 \text{ or } 2 \end{aligned}$$

c
$$\begin{aligned} 5 - 4x^2 &= 3(2x+1) + 2 \\ \therefore\ 5 - 4x^2 &= 6x + 3 + 2 \\ \therefore\ 4x^2 + 6x &= 0 \\ \therefore\ 2x(2x+3) &= 0 \\ \therefore\ 2x = 0 &\text{ or } 2x + 3 = 0 \\ \therefore\ x = 0 &\text{ or } -\tfrac{3}{2} \end{aligned}$$

d
$$\begin{aligned} x + \frac{2}{x} &= 3 \\ \therefore\ x^2 + 2 &= 3x \\ \therefore\ x^2 - 3x + 2 &= 0 \\ \therefore\ (x-1)(x-2) &= 0 \\ \therefore\ x &= 1 \text{ or } 2 \end{aligned}$$

e
$$\begin{aligned} 2x - \frac{1}{x} &= -1 \\ \therefore\ 2x^2 - 1 &= -x \\ \therefore\ 2x^2 + x - 1 &= 0 \\ \therefore\ (2x-1)(x+1) &= 0 \\ \therefore\ x &= \tfrac{1}{2} \text{ or } -1 \end{aligned}$$

f
$$\begin{aligned} \frac{x+3}{1-x} &= -\frac{9}{x} \\ \therefore\ x(x+3) &= -9(1-x) \\ \therefore\ x^2 + 3x &= -9 + 9x \\ \therefore\ x^2 - 6x + 9 &= 0 \\ \therefore\ (x-3)^2 &= 0 \\ \therefore\ x &= 3 \end{aligned}$$

g
$$\begin{aligned} (x+3)(2-x) &= 4 \\ \therefore\ 2x - x^2 + 6 - 3x &= 4 \\ \therefore\ x^2 + x - 2 &= 0 \\ \therefore\ (x+2)(x-1) &= 0 \\ \therefore\ x &= -2 \text{ or } 1 \end{aligned}$$

h
$$\begin{aligned} (x-4)(x+2) &= 16 \\ \therefore\ x^2 - 2x - 8 &= 16 \\ \therefore\ x^2 - 2x - 24 &= 0 \\ \therefore\ (x+4)(x-6) &= 0 \\ \therefore\ x &= -4 \text{ or } 6 \end{aligned}$$

i
$$\begin{aligned} (x-5)(x+3) &= 20 \\ \therefore\ x^2-2x-15 &= 20 \\ \therefore\ x^2-2x-35 &= 0 \\ \therefore\ (x+5)(x-7) &= 0 \\ \therefore\ x &= -5 \text{ or } 7 \end{aligned}$$

j
$$\begin{aligned} (4x-5)(4x-3) &= 143 \\ \therefore\ 16x^2-32x+15 &= 143 \\ \therefore\ 16x^2-32x-128 &= 0 \\ \therefore\ 16(x^2-2x-8) &= 0 \\ \therefore\ 16(x+2)(x-4) &= 0 \\ \therefore\ (x+2)(x-4) &= 0 \\ \therefore\ x &= -2 \text{ or } 4 \end{aligned}$$

EXERCISE 4C.2

1 **a**
$$\begin{aligned} x^2-4x+1 &= 0 \\ \therefore\ x^2-4x &= -1 \\ \therefore\ x^2-4x+(-2)^2 &= -1+(-2)^2 \\ \therefore\ (x-2)^2 &= 3 \\ \therefore\ x-2 &= \pm\sqrt{3} \\ \therefore\ x &= 2\pm\sqrt{3} \end{aligned}$$

b
$$\begin{aligned} x^2+6x+2 &= 0 \\ \therefore\ x^2+6x &= -2 \\ \therefore\ x^2+6x+3^2 &= -2+3^2 \\ \therefore\ (x+3)^2 &= 7 \\ \therefore\ x+3 &= \pm\sqrt{7} \\ \therefore\ x &= -3\pm\sqrt{7} \end{aligned}$$

c
$$\begin{aligned} x^2-14x+46 &= 0 \\ \therefore\ x^2-14x &= -46 \\ \therefore\ x^2-14x+(-7)^2 &= -46+(-7)^2 \\ \therefore\ (x-7)^2 &= 3 \\ \therefore\ x-7 &= \pm\sqrt{3} \\ \therefore\ x &= 7\pm\sqrt{3} \end{aligned}$$

d
$$\begin{aligned} x^2 &= 4x+3 \\ \therefore\ x^2-4x &= 3 \\ \therefore\ x^2-4x+(-2)^2 &= 3+(-2)^2 \\ \therefore\ (x-2)^2 &= 7 \\ \therefore\ x-2 &= \pm\sqrt{7} \\ \therefore\ x &= 2\pm\sqrt{7} \end{aligned}$$

e
$$\begin{aligned} x^2+6x+7 &= 0 \\ \therefore\ x^2+6x &= -7 \\ \therefore\ x^2+6x+3^2 &= -7+3^2 \\ \therefore\ (x+3)^2 &= 2 \\ \therefore\ x+3 &= \pm\sqrt{2} \\ \therefore\ x &= -3\pm\sqrt{2} \end{aligned}$$

f
$$\begin{aligned} x^2 &= 2x+6 \\ \therefore\ x^2-2x &= 6 \\ \therefore\ x^2-2x+(-1)^2 &= 6+(-1)^2 \\ \therefore\ (x-1)^2 &= 7 \\ \therefore\ x-1 &= \pm\sqrt{7} \\ \therefore\ x &= 1\pm\sqrt{7} \end{aligned}$$

g
$$\begin{aligned} x^2+6x &= 2 \\ \therefore\ x^2+6x+3^2 &= 2+3^2 \\ \therefore\ (x+3)^2 &= 11 \\ \therefore\ x+3 &= \pm\sqrt{11} \\ \therefore\ x &= -3\pm\sqrt{11} \end{aligned}$$

h
$$\begin{aligned} x^2+10 &= 8x \\ \therefore\ x^2-8x &= -10 \\ \therefore\ x^2-8x+(-4)^2 &= -10+(-4)^2 \\ \therefore\ (x-4)^2 &= 6 \\ \therefore\ x-4 &= \pm\sqrt{6} \\ \therefore\ x &= 4\pm\sqrt{6} \end{aligned}$$

i
$$\begin{aligned} x^2+6x &= -11 \\ \therefore\ x^2+6x+3^2 &= -11+3^2 \\ \therefore\ (x+3)^2 &= -2 \end{aligned}$$

which has no real solutions, as $(x+3)^2$ cannot be negative.

2 **a**
$$\begin{aligned} 2x^2+4x+1&=0\\ \therefore\ x^2+2x+\tfrac{1}{2}&=0\\ \therefore\ x^2+2x&=-\tfrac{1}{2}\\ \therefore\ x^2+2x+1^2&=-\tfrac{1}{2}+1^2\\ \therefore\ (x+1)^2&=\tfrac{1}{2}\\ \therefore\ x+1&=\pm\sqrt{\tfrac{1}{2}}\\ \therefore\ x+1&=\pm\tfrac{1}{\sqrt{2}}\\ \therefore\ x&=-1\pm\tfrac{1}{\sqrt{2}} \end{aligned}$$

b
$$\begin{aligned} 2x^2-10x+3&=0\\ \therefore\ x^2-5x+\tfrac{3}{2}&=0\\ \therefore\ x^2-5x&=-\tfrac{3}{2}\\ \therefore\ x^2-5x+(-\tfrac{5}{2})^2&=-\tfrac{3}{2}+(-\tfrac{5}{2})^2\\ \therefore\ (x-\tfrac{5}{2})^2&=-\tfrac{3}{2}+\tfrac{25}{4}\\ \therefore\ (x-\tfrac{5}{2})^2&=\tfrac{19}{4}\\ \therefore\ x-\tfrac{5}{2}&=\pm\sqrt{\tfrac{19}{4}}\\ \therefore\ x-\tfrac{5}{2}&=\pm\tfrac{\sqrt{19}}{2}\\ \therefore\ x&=\tfrac{5}{2}\pm\tfrac{\sqrt{19}}{2} \end{aligned}$$

c
$$\begin{aligned} 3x^2+12x+5&=0\\ \therefore\ x^2+4x+\tfrac{5}{3}&=0\\ \therefore\ x^2+4x&=-\tfrac{5}{3}\\ \therefore\ x^2+4x+2^2&=-\tfrac{5}{3}+2^2\\ \therefore\ (x+2)^2&=\tfrac{7}{3}\\ \therefore\ x+2&=\pm\sqrt{\tfrac{7}{3}}\\ \therefore\ x&=-2\pm\sqrt{\tfrac{7}{3}} \end{aligned}$$

d
$$\begin{aligned} 3x^2&=6x+4\\ \therefore\ x^2&=2x+\tfrac{4}{3}\\ \therefore\ x^2-2x&=\tfrac{4}{3}\\ \therefore\ x^2-2x+(-1)^2&=\tfrac{4}{3}+(-1)^2\\ \therefore\ (x-1)^2&=\tfrac{7}{3}\\ \therefore\ x-1&=\pm\sqrt{\tfrac{7}{3}}\\ \therefore\ x&=1\pm\sqrt{\tfrac{7}{3}} \end{aligned}$$

e
$$\begin{aligned} 5x^2-15x+2&=0\\ \therefore\ x^2-3x+\tfrac{2}{5}&=0\\ \therefore\ x^2-3x&=-\tfrac{2}{5}\\ \therefore\ x^2-3x+(-\tfrac{3}{2})^2&=-\tfrac{2}{5}+(-\tfrac{3}{2})^2\\ \therefore\ (x-\tfrac{3}{2})^2&=-\tfrac{2}{5}+\tfrac{9}{4}\\ \therefore\ (x-\tfrac{3}{2})^2&=\tfrac{37}{20}\\ \therefore\ x-\tfrac{3}{2}&=\pm\sqrt{\tfrac{37}{20}}\\ \therefore\ x&=\tfrac{3}{2}\pm\sqrt{\tfrac{37}{20}} \end{aligned}$$

f
$$\begin{aligned} 4x^2+4x&=5\\ \therefore\ x^2+x&=\tfrac{5}{4}\\ \therefore\ x^2+x+(\tfrac{1}{2})^2&=\tfrac{5}{4}+(\tfrac{1}{2})^2\\ \therefore\ (x+\tfrac{1}{2})^2&=\tfrac{6}{4}\\ \therefore\ x+\tfrac{1}{2}&=\pm\sqrt{\tfrac{6}{4}}\\ \therefore\ x+\tfrac{1}{2}&=\pm\tfrac{\sqrt{6}}{2}\\ \therefore\ x&=-\tfrac{1}{2}\pm\tfrac{\sqrt{6}}{2} \end{aligned}$$

3 **a**

$$\begin{aligned} 3x - \frac{2}{x} &= 4 \\ \therefore\ 3x^2 - 2 &= 4x \\ \therefore\ x^2 - \tfrac{2}{3} &= \tfrac{4}{3}x \\ \therefore\ x^2 - \tfrac{4}{3}x &= \tfrac{2}{3} \\ \therefore\ x^2 - \tfrac{4}{3}x + (-\tfrac{2}{3})^2 &= \tfrac{2}{3} + (-\tfrac{2}{3})^2 \\ \therefore\ (x - \tfrac{2}{3})^2 &= \tfrac{2}{3} + \tfrac{4}{9} = \tfrac{10}{9} \\ \therefore\ x - \tfrac{2}{3} &= \pm\sqrt{\tfrac{10}{9}} \\ \therefore\ x - \tfrac{2}{3} &= \pm\tfrac{\sqrt{10}}{3} \\ \therefore\ x &= \tfrac{2}{3} \pm \tfrac{\sqrt{10}}{3} \end{aligned}$$

b

$$\begin{aligned} 1 - \frac{1}{x} &= -5x \\ \therefore\ x - 1 &= -5x^2 \\ \therefore\ 5x^2 + x &= 1 \\ \therefore\ x^2 + \tfrac{1}{5}x &= \tfrac{1}{5} \\ \therefore\ x^2 + \tfrac{1}{5}x + (\tfrac{1}{10})^2 &= \tfrac{1}{5} + (\tfrac{1}{10})^2 \\ \therefore\ (x + \tfrac{1}{10})^2 &= \tfrac{1}{5} + \tfrac{1}{100} = \tfrac{21}{100} \\ \therefore\ x + \tfrac{1}{10} &= \pm\sqrt{\tfrac{21}{100}} \\ \therefore\ x + \tfrac{1}{10} &= \pm\tfrac{\sqrt{21}}{10} \\ \therefore\ x &= -\tfrac{1}{10} \pm \tfrac{\sqrt{21}}{10} \end{aligned}$$

c

$$\begin{aligned} 3 + \frac{1}{x^2} &= -\frac{5}{x} \\ \therefore\ 3x^2 + 1 &= -5x \\ \therefore\ x^2 + \tfrac{1}{3} &= -\tfrac{5}{3}x \\ \therefore\ x^2 + \tfrac{5}{3}x &= -\tfrac{1}{3} \\ \therefore\ x^2 + \tfrac{5}{3}x + (\tfrac{5}{6})^2 &= -\tfrac{1}{3} + (\tfrac{5}{6})^2 \\ \therefore\ (x + \tfrac{5}{6})^2 &= -\tfrac{1}{3} + \tfrac{25}{36} = \tfrac{13}{36} \\ \therefore\ x + \tfrac{5}{6} &= \pm\sqrt{\tfrac{13}{36}} \\ \therefore\ x + \tfrac{5}{6} &= \pm\tfrac{\sqrt{13}}{6} \\ \therefore\ x &= -\tfrac{5}{6} \pm \tfrac{\sqrt{13}}{6} \end{aligned}$$

4

$$\begin{aligned} ax^2 + bx + c &= 0 \\ \therefore\ x^2 + \frac{b}{a}x + \frac{c}{a} &= 0 \\ \therefore\ x^2 + \frac{b}{a}x &= -\frac{c}{a} \\ \therefore\ x^2 + \frac{b}{a}x + \left(\frac{b}{2a}\right)^2 &= -\frac{c}{a} + \left(\frac{b}{2a}\right)^2 \\ \therefore\ \left(x + \frac{b}{2a}\right)^2 &= -\frac{c}{a} + \frac{b^2}{4a^2} \\ \therefore\ \left(x + \frac{b}{2a}\right)^2 &= \frac{b^2 - 4ac}{4a^2} \\ \therefore\ x + \frac{b}{2a} &= \pm\sqrt{\frac{b^2 - 4ac}{4a^2}} \\ \therefore\ x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \end{aligned}$$

EXERCISE 4C.3

1 **a** $x^2 - 4x - 3 = 0$

has $a = 1,\ b = -4,\ c = -3$

$$\therefore\ x = \frac{-(-4) \pm \sqrt{(-4)^2 - 4(1)(-3)}}{2(1)}$$

$$\therefore\ x = \frac{4 \pm \sqrt{28}}{2}$$

$$\therefore\ x = \frac{4 \pm 2\sqrt{7}}{2}$$

$$\therefore\ x = 2 \pm \sqrt{7}$$

b $x^2 + 6x + 7 = 0$

has $a = 1,\ b = 6,\ c = 7$

$$\therefore\ x = \frac{-6 \pm \sqrt{6^2 - 4(1)(7)}}{2(1)}$$

$$\therefore\ x = \frac{-6 \pm \sqrt{8}}{2}$$

$$\therefore\ x = \frac{-6 \pm 2\sqrt{2}}{2}$$

$$\therefore\ x = -3 \pm \sqrt{2}$$

c $x^2 + 1 = 4x$

$\therefore\ x^2 - 4x + 1 = 0$

which has $a = 1,\ b = -4,\ c = 1$

$$\therefore\ x = \frac{-(-4) \pm \sqrt{(-4)^2 - 4(1)(1)}}{2(1)}$$

$$\therefore\ x = \frac{4 \pm \sqrt{12}}{2}$$

$$\therefore\ x = \frac{4 \pm 2\sqrt{3}}{2}$$

$$\therefore\ x = 2 \pm \sqrt{3}$$

d $x^2 + 4x = 1$

$\therefore\ x^2 + 4x - 1 = 0$

which has $a = 1,\ b = 4,\ c = -1$

$$\therefore\ x = \frac{-4 \pm \sqrt{4^2 - 4(1)(-1)}}{2(1)}$$

$$\therefore\ x = \frac{-4 \pm \sqrt{20}}{2}$$

$$\therefore\ x = \frac{-4 \pm 2\sqrt{5}}{2}$$

$$\therefore\ x = -2 \pm \sqrt{5}$$

e $x^2 - 4x + 2 = 0$

has $a = 1,\ b = -4,\ c = 2$

$$\therefore\ x = \frac{-(-4) \pm \sqrt{(-4)^2 - 4(1)(2)}}{2(1)}$$

$$\therefore\ x = \frac{4 \pm \sqrt{8}}{2}$$

$$\therefore\ x = \frac{4 \pm 2\sqrt{2}}{2}$$

$$\therefore\ x = 2 \pm \sqrt{2}$$

f $2x^2 - 2x - 3 = 0$

has $a = 2,\ b = -2,\ c = -3$

$$\therefore\ x = \frac{-(-2) \pm \sqrt{(-2)^2 - 4(2)(-3)}}{2(2)}$$

$$\therefore\ x = \frac{2 \pm \sqrt{28}}{4}$$

$$\therefore\ x = \frac{2 \pm 2\sqrt{7}}{4}$$

$$\therefore\ x = \frac{1 \pm \sqrt{7}}{2}$$

g $3x^2 - 5x - 1 = 0$

has $a = 3,\ b = -5,\ c = -1$

$$\therefore\ x = \frac{-(-5) \pm \sqrt{(-5)^2 - 4(3)(-1)}}{2(3)}$$

$$\therefore\ x = \frac{5 \pm \sqrt{37}}{6}$$

h $-x^2 + 4x + 6 = 0$

has $a = -1,\ b = 4,\ c = 6$

$$\therefore\ x = \frac{-4 \pm \sqrt{4^2 - 4(-1)(6)}}{2(-1)}$$

$$\therefore\ x = \frac{-4 \pm \sqrt{40}}{-2}$$

$$\therefore\ x = \frac{-4 \pm 2\sqrt{10}}{-2}$$

$$\therefore\ x = 2 \pm \sqrt{10}$$

i $-2x^2 + 7x - 2 = 0$

has $a = -2,\ b = 7,\ c = -2$

$$\therefore\ x = \frac{-7 \pm \sqrt{7^2 - 4(-2)(-2)}}{2(-2)}$$

$$\therefore\ x = \frac{-7 \pm \sqrt{33}}{-4}$$

$$\therefore\ x = \frac{7 \pm \sqrt{33}}{4}$$

2 **a** $(x+2)(x-1) = 2 - 3x$

$\therefore\ x^2 - x + 2x - 2 = 2 - 3x$

$\therefore\ x^2 + 4x - 4 = 0$

which has $a = 1,\ b = 4,\ c = -4$

$$\therefore\ x = \frac{-4 \pm \sqrt{4^2 - 4(1)(-4)}}{2(1)}$$

$$\therefore\ x = \frac{-4 \pm \sqrt{32}}{2}$$

$$\therefore\ x = \frac{-4 \pm 4\sqrt{2}}{2}$$

$$\therefore\ x = -2 \pm 2\sqrt{2}$$

b $(2x+1)^2 = 3 - x$

$\therefore\ 4x^2 + 4x + 1 = 3 - x$

$\therefore\ 4x^2 + 5x - 2 = 0$

which has $a = 4,\ b = 5,\ c = -2$

$$\therefore\ x = \frac{-5 \pm \sqrt{5^2 - 4(4)(-2)}}{2(4)}$$

$$\therefore\ x = \frac{-5 \pm \sqrt{57}}{8}$$

c $(x-2)^2 = 1 + x$

$\therefore\ x^2 - 4x + 4 = 1 + x$

$\therefore\ x^2 - 5x + 3 = 0$

which has $a = 1,\ b = -5,\ c = 3$

$$\therefore\ x = \frac{-(-5) \pm \sqrt{(-5)^2 - 4(1)(3)}}{2(1)}$$

$$\therefore\ x = \frac{5 \pm \sqrt{13}}{2}$$

d $(3x+1)^2 = -2x$

$\therefore\ 9x^2 + 6x + 1 = -2x$

$\therefore\ 9x^2 + 8x + 1 = 0$

which has $a = 9,\ b = 8,\ c = 1$

$$\therefore\ x = \frac{-8 \pm \sqrt{8^2 - 4(9)(1)}}{2(9)}$$

$$\therefore\ x = \frac{-8 \pm \sqrt{28}}{18}$$

$$\therefore\ x = \frac{-8 \pm 2\sqrt{7}}{18}$$

$$\therefore\ x = \frac{-4 \pm \sqrt{7}}{9}$$

e $(x+3)(2x+1) = 9$

$\therefore\ 2x^2 + x + 6x + 3 = 9$

$\therefore\ 2x^2 + 7x - 6 = 0$

which has $a = 2,\ b = 7,\ c = -6$

$$\therefore\ x = \frac{-7 \pm \sqrt{7^2 - 4(2)(-6)}}{2(2)}$$

$$\therefore\ x = \frac{-7 \pm \sqrt{97}}{4}$$

f $(2x+3)(2x-3) = x$

$\therefore\ 4x^2 - 9 = x$

$\therefore\ 4x^2 - x - 9 = 0$

which has $a = 4,\ b = -1,\ c = -9$

$$\therefore\ x = \frac{-(-1) \pm \sqrt{(-1)^2 - 4(4)(-9)}}{2(4)}$$

$$\therefore\ x = \frac{1 \pm \sqrt{145}}{8}$$

g
$$\frac{x-1}{2-x} = 2x+1$$
$$\therefore\ x-1 = (2x+1)(2-x)$$
$$\therefore\ x-1 = 4x-2x^2+2-x$$
$$\therefore\ 2x^2-2x-3=0$$
which has $a=2,\ b=-2,\ c=-3$
$$\therefore\ x = \frac{-(-2)\pm\sqrt{(-2)^2-4(2)(-3)}}{2(2)}$$
$$\therefore\ x = \frac{2\pm\sqrt{28}}{4}$$
$$\therefore\ x = \frac{2\pm 2\sqrt{7}}{4}$$
$$\therefore\ x = \frac{1\pm\sqrt{7}}{2}$$

h
$$x-\frac{1}{x} = 1$$
$$\therefore\ x^2-1 = x$$
$$\therefore\ x^2-x-1=0$$
which has $a=1,\ b=-1,\ c=-1$
$$\therefore\ x = \frac{-(-1)\pm\sqrt{(-1)^2-4(1)(-1)}}{2(1)}$$
$$\therefore\ x = \frac{1\pm\sqrt{5}}{2}$$

i
$$2x-\frac{1}{x} = 3$$
$$\therefore\ 2x^2-1 = 3x$$
$$\therefore\ 2x^2-3x-1=0$$
which has $a=2,\ b=-3,\ c=-1$
$$\therefore\ x = \frac{-(-3)\pm\sqrt{(-3)^2-4(2)(-1)}}{2(2)}$$
$$\therefore\ x = \frac{3\pm\sqrt{17}}{4}$$

EXERCISE 4C.4

1 $x^2-7x+9=0$ has $a=1,\ b=-7,\ c=9$

a
$$\Delta = b^2-4ac$$
$$= (-7)^2-4(1)(9)$$
$$= 13$$

b Since $\Delta > 0$, but 13 is not a square, there are 2 distinct irrational roots.

c
$$x = \frac{-b\pm\sqrt{\Delta}}{2a}$$
$$\therefore\ x = \frac{-(-7)\pm\sqrt{13}}{2(1)}$$
$$\therefore\ x = \frac{7\pm\sqrt{13}}{2}$$
$$\therefore\ x = \tfrac{7}{2}\pm\tfrac{\sqrt{13}}{2}$$
So there are 2 distinct irrational roots as expected.

2 $4x^2 - 4x + 1 = 0$ has $a = 4,\ b = -4,\ c = 1$

a $\Delta = b^2 - 4ac$
$= (-4)^2 - 4(4)(1)$
$= 0$

b Since $\Delta = 0$, there is one repeated root.

c $x = \dfrac{-b \pm \sqrt{\Delta}}{2a}$

$\therefore\ x = \dfrac{-(-4) \pm 0}{2(4)}$

$\therefore\ x = \frac{4}{8} = \frac{1}{2}$

So, there is one repeated root as expected.

3 **a** $x^2 + 5 \neq 0$ for any real value of x, since $x^2 \neq -5$ for any real x.
$\therefore\ x^2 + 5 = 0$ has no real roots.

b $x^2 + 5 = 0$ has $a = 1,\ b = 0,\ c = 5$
$\Delta = b^2 - 4ac$
$= 0^2 - 4(1)(5)$
$= -20$ which is < 0 ✓

4 **a** $x^2 + 7x - 3 = 0$
has $a = 1,\ b = 7,\ c = -3$
$\Delta = b^2 - 4ac$
$= 7^2 - 4(1)(-3)$
$= 61$
Since $\Delta > 0$, but 61 is not a square, there are 2 distinct irrational roots.

b $x^2 - 3x + 2 = 0$
has $a = 1,\ b = -3,\ c = 2$
$\Delta = b^2 - 4ac$
$= (-3)^2 - 4(1)(2)$
$= 1$
Since $\Delta > 0$, and 1 is a square, there are 2 distinct rational roots.

c $3x^2 + 2x - 1 = 0$
has $a = 3,\ b = 2,\ c = -1$
$\Delta = b^2 - 4ac$
$= 2^2 - 4(3)(-1)$
$= 16$
Since $\Delta > 0$, and 16 is a square, there are 2 distinct rational roots.

d $5x^2 + 4x - 3 = 0$
has $a = 5,\ b = 4,\ c = -3$
$\Delta = b^2 - 4ac$
$= 4^2 - 4(5)(-3)$
$= 76$
Since $\Delta > 0$, but 76 is not a square, there are 2 distinct irrational roots.

e $x^2 + x + 5 = 0$
has $a = 1,\ b = 1,\ c = 5$
$\Delta = b^2 - 4ac$
$= 1^2 - 4(1)(5)$
$= -19$
Since $\Delta < 0$, there are no real roots.

f $16x^2 - 8x + 1 = 0$
has $a = 16,\ b = -8,\ c = 1$
$\Delta = b^2 - 4ac$
$= (-8)^2 - 4(16)(1)$
$= 0$
Since $\Delta = 0$, there is one repeated root.

5 **a** $6x^2 - 5x - 6 = 0$
has $a = 6$, $b = -5$, $c = -6$
$\Delta = b^2 - 4ac$
$= (-5)^2 - 4(6)(-6)$
$= 169$
Since $\Delta > 0$, and 169 is a square, there are 2 distinct rational roots which can be found by factorisation.

b $2x^2 - 7x - 5 = 0$
has $a = 2$, $b = -7$, $c = -5$
$\Delta = b^2 - 4ac$
$= (-7)^2 - 4(2)(-5)$
$= 89$
Since $\Delta > 0$, but 89 is not a square, there are 2 distinct irrational roots.

c $3x^2 + 4x + 1 = 0$
has $a = 3$, $b = 4$, $c = 1$
$\Delta = b^2 - 4ac$
$= 4^2 - 4(3)(1)$
$= 4$
Since $\Delta > 0$, and 4 is a square, there are 2 distinct rational roots which can be found by factorisation.

d $6x^2 - 47x - 8 = 0$
has $a = 6$, $b = -47$, $c = -8$
$\Delta = b^2 - 4ac$
$= (-47)^2 - 4(6)(-8)$
$= 2401$
Since $\Delta > 0$, and 2401 is a square, there are 2 distinct rational roots which can be found by factorisation.

e $4x^2 - 3x + 2 = 0$
has $a = 4$, $b = -3$, $c = 2$
$\Delta = b^2 - 4ac$
$= (-3)^2 - 4(4)(2)$
$= -23$
Since $\Delta < 0$, there are no real roots.

f $8x^2 + 2x - 3 = 0$
has $a = 8$, $b = 2$, $c = -3$
$\Delta = b^2 - 4ac$
$= 2^2 - 4(8)(-3)$
$= 100$
Since $\Delta > 0$, and 100 is a square, there are 2 distinct rational roots which can be found by factorisation.

6 **a** $x^2 + 4x + m = 0$ has $a = 1$, $b = 4$, $c = m$
$\Delta = b^2 - 4ac$
$= 4^2 - 4(1)(m)$
$= 16 - 4m$

i For a repeated root,
$\Delta = 0$
$\therefore\ 16 - 4m = 0$
$\therefore\ -4m = -16$
$\therefore\ m = 4$

ii For two distinct real roots,
$\Delta > 0$
$\therefore\ 16 - 4m > 0$
$\therefore\ -4m > -16$
$\therefore\ m < 4$

iii For no real roots,
$\Delta < 0$
$\therefore\ 16 - 4m < 0$
$\therefore\ -4m < -16$
$\therefore\ m > 4$

b $mx^2 + 3x + 2 = 0$, $m \neq 0$ has $a = m$, $b = 3$, $c = 2$
$\Delta = b^2 - 4ac$
$= 3^2 - 4(m)(2)$
$= 9 - 8m$

i For a repeated root,
$\Delta = 0$
$\therefore\ 9 - 8m = 0$
$\therefore\ -8m = -9$
$\therefore\ m = \frac{9}{8}$

ii For two distinct real roots,
$\Delta > 0$
$\therefore\ 9 - 8m > 0$
$\therefore\ -8m > -9$
$\therefore\ m < \frac{9}{8}$, $m \neq 0$

iii For no real roots,
$\Delta < 0$
$\therefore\ 9 - 8m < 0$
$\therefore\ -8m < -9$
$\therefore\ m > \frac{9}{8}$

c $mx^2 - 3x + 1 = 0, \quad m \neq 0$ has $a = m, \; b = -3, \; c = 1$

$$\begin{aligned}\Delta &= b^2 - 4ac \\ &= (-3)^2 - 4(m)(1) \\ &= 9 - 4m\end{aligned}$$

i For a repeated root,
$$\Delta = 0$$
$$\therefore \; 9 - 4m = 0$$
$$\therefore \; -4m = -9$$
$$\therefore \; m = \tfrac{9}{4}$$

ii For two distinct real roots,
$$\Delta > 0$$
$$\therefore \; 9 - 4m > 0$$
$$\therefore \; -4m > -9$$
$$\therefore \; m < \tfrac{9}{4}, \; m \neq 0$$

iii For no real roots,
$$\Delta < 0$$
$$\therefore \; 9 - 4m < 0$$
$$\therefore \; -4m < -9$$
$$\therefore \; m > \tfrac{9}{4}$$

7 $4x^2 + kx + (3 - k) = 0$ has $a = 4, \; b = k, \; c = 3 - k$

$$\begin{aligned}\Delta &= b^2 - 4ac \\ &= k^2 - 4(4)(3 - k) \\ &= k^2 - 48 + 16k\end{aligned}$$

For a repeated root, $\Delta = 0$

$$\therefore \; k^2 - 48 + 16k = 0$$
$$\therefore \; k^2 + 16k = 48$$
$$\therefore \; k^2 + 16k + 8^2 = 48 + 8^2 \qquad \{\text{completing the square}\}$$
$$\therefore \; (k+8)^2 = 112$$
$$\therefore \; k + 8 = \pm\sqrt{112}$$
$$\therefore \; k = -8 \pm \sqrt{112}$$
$$\therefore \; k = -8 \pm 4\sqrt{7}$$

So the repeated root is $x = \dfrac{-k \pm \sqrt{\Delta}}{2(4)} = \dfrac{-k \pm \sqrt{0}}{8} = \dfrac{-k}{8}$.

When $k = -8 + 4\sqrt{7}$, $x = \dfrac{-(-8 + 4\sqrt{7})}{8} = \dfrac{8 - 4\sqrt{7}}{8} = 1 - \frac{\sqrt{7}}{2}$

When $k = -8 - 4\sqrt{7}$, $x = \dfrac{-(-8 - 4\sqrt{7})}{8} = \dfrac{8 + 4\sqrt{7}}{8} = 1 + \frac{\sqrt{7}}{2}$

EXERCISE 4C.5

1 a i $x^2 + 4x - 21 = 0$ has $a = 1, \; b = 4, \; c = -21$

If α and β are the roots then $\alpha + \beta = -\dfrac{b}{a} = -\dfrac{4}{1} = -4$

and $\alpha\beta = \dfrac{c}{a} = \dfrac{-21}{1} = -21$

So, the sum of the roots is -4, and the product of the roots is -21.

ii $x^2 + 4x - 21 = 0$ has roots

$$\frac{-4 \pm \sqrt{4^2 - 4(1)(-21)}}{2(1)} = \frac{-4 \pm \sqrt{100}}{2} = \frac{-4 \pm 10}{2} = -2 \pm 5 = 3 \text{ and } -7$$

These have sum $= 3 + (-7) = -4$ ✓

and product $= 3(-7) = -21$ ✓

b **i** $x^2 - 5x + 5 = 0$ has $a = 1,\ b = -5,\ c = 5$

If α and β are the roots then $\alpha + \beta = -\dfrac{b}{a} = -\dfrac{(-5)}{1} = 5$

and $\alpha\beta = \dfrac{c}{a} = \dfrac{5}{1} = 5$

So, the sum of the roots is 5, and the product of the roots is 5.

ii $x^2 - 5x + 5 = 0$ has roots

$$\frac{-(-5) \pm \sqrt{(-5)^2 - 4(1)(5)}}{2(1)} = \frac{5 \pm \sqrt{5}}{2}$$

These have sum $= \dfrac{5+\sqrt{5}}{2} + \dfrac{5-\sqrt{5}}{2} = \dfrac{10}{2} = 5$ ✓

and product $= \left(\dfrac{5+\sqrt{5}}{2}\right)\left(\dfrac{5-\sqrt{5}}{2}\right) = \dfrac{25-5}{4} = \dfrac{20}{4} = 5$ ✓

c **i** $4x^2 - 12x + 5 = 0$ has $a = 4,\ b = -12,\ c = 5$

If α and β are the roots then $\alpha + \beta = -\dfrac{b}{a} = -\dfrac{(-12)}{4} = 3$

and $\alpha\beta = \dfrac{c}{a} = \dfrac{5}{4}$

So, the sum of the roots is 3, and the product of the roots is $\frac{5}{4}$.

ii $4x^2 - 12x + 5 = 0$ has roots

$$\frac{-(-12) \pm \sqrt{(-12)^2 - 4(4)(5)}}{2(4)} = \frac{12 \pm \sqrt{64}}{8} = \frac{12 \pm 8}{8} = \frac{3 \pm 2}{2} = \tfrac{5}{2} \text{ and } \tfrac{1}{2}$$

These have sum $= \frac{5}{2} + \frac{1}{2} = \frac{6}{2} = 3$ ✓

and product $= (\frac{5}{2})(\frac{1}{2}) = \frac{5}{4}$ ✓

d **i** $3x^2 - 4x - 2 = 0$ has $a = 3,\ b = -4,\ c = -2$

If α and β are the roots then $\alpha + \beta = -\dfrac{b}{a} = -\dfrac{(-4)}{3} = \dfrac{4}{3}$

and $\alpha\beta = \dfrac{c}{a} = \dfrac{-2}{3} = -\dfrac{2}{3}$

So, the sum of the roots is $\frac{4}{3}$, and the product of the roots is $-\frac{2}{3}$.

ii $3x^2 - 4x - 2 = 0$ has roots

$$\frac{-(-4) \pm \sqrt{(-4)^2 - 4(3)(-2)}}{2(3)} = \frac{4 \pm \sqrt{40}}{6} = \frac{4 \pm 2\sqrt{10}}{6} = \frac{2 \pm \sqrt{10}}{3}$$

These have sum $= \dfrac{2+\sqrt{10}}{3} + \dfrac{2-\sqrt{10}}{3} = \dfrac{4}{3}$ ✓

and product $= \left(\dfrac{2+\sqrt{10}}{3}\right)\left(\dfrac{2-\sqrt{10}}{3}\right) = \dfrac{4-10}{9} = -\dfrac{6}{9} = -\dfrac{2}{3}$ ✓

2 $kx^2 - (1+k)x + (3k+2) = 0$ has $a = k$, $b = -(1+k)$, $c = 3k+2$

If α and β are the roots then $\alpha + \beta = -\frac{b}{a} = -\frac{-(1+k)}{k} = \frac{1+k}{k}$

and $\alpha\beta = \frac{c}{a} = \frac{3k+2}{k}$

Since the sum of the roots is twice their product, then

$$\frac{1+k}{k} = 2 \times \frac{3k+2}{k}$$
$$\therefore \frac{1+k}{k} = \frac{6k+4}{k}$$
$$\therefore 1+k = 6k+4 \quad \{\text{equating numerators}\}$$
$$\therefore -3 = 5k$$
$$\therefore k = -\tfrac{3}{5}$$

The equation is $-\frac{3}{5}x^2 - \left(1 + (-\frac{3}{5})\right)x + 3(-\frac{3}{5}) + 2 = 0$

$$\therefore -\tfrac{3}{5}x^2 - \tfrac{2}{5}x + \tfrac{1}{5} = 0$$
$$\therefore 3x^2 + 2x - 1 = 0 \quad \{\text{multiplying both sides by } -5\}$$
$$\therefore (3x-1)(x+1) = 0$$
$$\therefore x = \tfrac{1}{3} \text{ or } -1$$

So, the two roots are -1 and $\frac{1}{3}$.

3 **a** $ax^2 - 6x + a - 2 = 0$, $a \neq 0$ has $a = a$, $b = -6$, $c = a - 2$

If α and 2α are the roots, then $\alpha + 2\alpha = -\frac{b}{a} = -\frac{(-6)}{a} = \frac{6}{a}$

$$\therefore 3\alpha = \frac{6}{a}$$

and $\alpha(2\alpha) = \frac{c}{a} = \frac{a-2}{a}$

$$\therefore 2\alpha^2 = \frac{a-2}{a}$$

b $3\alpha = \frac{6}{a}$ and $2\alpha^2 = \frac{a-2}{a}$

$\therefore \alpha = \frac{2}{a}$ $\qquad \therefore 2\left(\frac{2}{a}\right)^2 = \frac{a-2}{a}$

$$\therefore \frac{8}{a^2} = \frac{a-2}{a}$$
$$\therefore 8 = a(a-2) \quad \{\text{multiplying both sides by } a^2\}$$
$$\therefore 8 = a^2 - 2a$$
$$\therefore a^2 - 2a - 8 = 0$$
$$\therefore (a+2)(a-4) = 0$$
$$\therefore a = -2 \text{ or } 4$$

When $a = -2$, $\alpha = \frac{2}{-2} = -1$

and $2\alpha = 2(-1) = -2$

So, the two roots are -1 and -2.

When $a = 4$, $\alpha = \frac{2}{4} = \frac{1}{2}$

and $2\alpha = 2(\frac{1}{2}) = 1$

So, the two roots are $\frac{1}{2}$ and 1.

4 $kx^2+(k-8)x+(1-k)=0,\quad k\neq 0$ has $a=k,\quad b=k-8,\quad c=1-k$

If α and $\alpha+2$ are the roots, then $\alpha+(\alpha+2)=-\dfrac{b}{a}=-\dfrac{(k-8)}{k}=\dfrac{8-k}{k}$

$$\therefore\ 2\alpha+2=\frac{8-k}{k}\quad \text{.... (1)}$$

$$\text{and}\quad \alpha(\alpha+2)=\frac{c}{a}=\frac{1-k}{k}$$

$$\therefore\ \alpha^2+2\alpha=\frac{1-k}{k}\quad \text{.... (2)}$$

$$\begin{aligned}\text{Using (1)},\quad 2\alpha&=\frac{8-k}{k}-2\\&=\frac{8-k-2k}{k}\\&=\frac{8-3k}{k}\\ \therefore\ \alpha&=\frac{8-3k}{2k}\end{aligned}$$

Substituting $\alpha=\dfrac{8-3k}{2k}$ into (2) gives

$$\left(\frac{8-3k}{2k}\right)^2+2\left(\frac{8-3k}{2k}\right)=\frac{1-k}{k}$$

$$\therefore\ \frac{64-48k+9k^2}{4k^2}+\frac{8-3k}{k}=\frac{1-k}{k}$$

$$\therefore\ 64-48k+9k^2+4k(8-3k)=4k(1-k)\qquad\{\text{multiplying both sides by }4k^2\}$$

$$\begin{aligned}\therefore\ 64-48k+9k^2+32k-12k^2&=4k-4k^2\\ \therefore\ -3k^2-16k+64&=-4k^2+4k\\ \therefore\ k^2-20k+64&=0\\ \therefore\ (k-4)(k-16)&=0\\ \therefore\ k&=4\ \text{ or }\ 16\end{aligned}$$

When $k=4$,
$$\begin{aligned}\alpha&=\frac{8-3(4)}{2(4)}\\&=\frac{-4}{8}\\&=-\tfrac{1}{2}\end{aligned}$$
$$\therefore\ \alpha+2=-\tfrac{1}{2}+2=\tfrac{3}{2}$$

So, the two roots are $-\frac{1}{2}$ and $\frac{3}{2}$.

When $k=16$,
$$\begin{aligned}\alpha&=\frac{8-3(16)}{2(16)}\\&=\frac{-40}{32}\\&=-\tfrac{5}{4}\end{aligned}$$
$$\therefore\ \alpha+2=-\tfrac{5}{4}+2=\tfrac{3}{4}$$

So, the two roots are $-\frac{5}{4}$ and $\frac{3}{4}$.

5 **a** For the quadratic equation with roots 3 and -5,

the sum of the roots $=3+(-5)=-2$

and the product of the roots $=3(-5)=-15$

So, we have $-\dfrac{b}{a}=-2$ and $\dfrac{c}{a}=-15$.

The simplest solution is $a=1,\ b=2,\ c=-15$.

$\therefore$ the quadratic equation is $x^2+2x-15=0$.

b For the quadratic equation with roots $2+\sqrt{3}$ and $2-\sqrt{3}$,

the sum of the roots $= (2+\sqrt{3}) + (2-\sqrt{3}) = 4$

and the product of the roots $= (2+\sqrt{3})(2-\sqrt{3}) = 4 - 3 = 1$

So, we have $-\dfrac{b}{a} = 4$ and $\dfrac{c}{a} = 1$.

The simplest solution is $a = 1,\ b = -4,\ c = 1$.

$\therefore$ the quadratic equation is $x^2 - 4x + 1 = 0$.

6 If α and β are the roots of $3x^2 + 2x - 4 = 0$, then $\alpha + \beta = -\frac{2}{3}$ and $\alpha\beta = -\frac{4}{3}$.

a For the quadratic equation with roots $-\alpha$ and $-\beta$,

the sum of the roots $= -\alpha + (-\beta) = -(\alpha+\beta) = -(-\frac{2}{3}) = \frac{2}{3}$

and the product of the roots $= (-\alpha)(-\beta) = \alpha\beta = -\frac{4}{3}$

So, we have $-\dfrac{b}{a} = \dfrac{2}{3}$ and $\dfrac{c}{a} = -\dfrac{4}{3}$.

The simplest solution is $a = 3,\ b = -2,\ c = -4$.

$\therefore$ the quadratic equation is $3x^2 - 2x - 4 = 0$.

b For the quadratic equation with roots 2α and 2β,

the sum of the roots $= 2\alpha + 2\beta = 2(\alpha+\beta) = 2(-\frac{2}{3}) = -\frac{4}{3}$

and the product of the roots $= (2\alpha)(2\beta) = 4\alpha\beta = 4(-\frac{4}{3}) = -\frac{16}{3}$

So, we have $-\dfrac{b}{a} = -\dfrac{4}{3}$ and $\dfrac{c}{a} = -\dfrac{16}{3}$.

The simplest solution is $a = 3,\ b = 4,\ c = -16$.

$\therefore$ the quadratic equation is $3x^2 + 4x - 16 = 0$.

7 If α and β are the roots of $-2x^2 + 5x + 1 = 0$, then $\alpha + \beta = -\dfrac{5}{(-2)} = \dfrac{5}{2}$ and $\alpha\beta = \dfrac{1}{-2} = -\dfrac{1}{2}$.

a For the quadratic equation with roots $\alpha + 2$ and $\beta + 2$,

the sum of the roots $= (\alpha+2) + (\beta+2) = (\alpha+\beta) + 4 = \frac{5}{2} + 4 = \frac{13}{2}$

and the product of the roots $= (\alpha+2)(\beta+2) = \alpha\beta + 2\alpha + 2\beta + 4 = \alpha\beta + 2(\alpha+\beta) + 4 = -\frac{1}{2} + 2(\frac{5}{2}) + 4 = \frac{17}{2}$

So, we have $-\dfrac{b}{a} = \dfrac{13}{2}$ and $\dfrac{c}{a} = \dfrac{17}{2}$.

The simplest solution is $a = 2,\ b = -13,\ c = 17$.

$\therefore$ the quadratic equation is $2x^2 - 13x + 17 = 0$.

b For all quadratic equations with roots $\dfrac{\alpha}{2}$ and $\dfrac{\beta}{2}$,

the sum of the roots $= \dfrac{\alpha}{2} + \dfrac{\beta}{2}$ and the product of the roots $= \left(\dfrac{\alpha}{2}\right)\left(\dfrac{\beta}{2}\right)$

$$= \frac{\alpha+\beta}{2} \qquad\qquad = \frac{\alpha\beta}{4}$$

$$= \frac{\frac{5}{2}}{2} \qquad\qquad = \frac{-\frac{1}{2}}{4}$$

$$= \tfrac{5}{4} \qquad\qquad = -\tfrac{1}{8}$$

So, we have $-\dfrac{b}{a} = \dfrac{5}{4} = \dfrac{10}{8}$ and $\dfrac{c}{a} = -\dfrac{1}{8}$.

The simplest solution is $a = 8, \quad b = -10, \quad c = -1$.

$\therefore$ all quadratic equations with roots $\dfrac{\alpha}{2}$ and $\dfrac{\beta}{2}$ are of the form

$k(8x^2 - 10x - 1) = 0, \quad k \in \mathbb{R}, \quad k \neq 0$.

8 If α and β are the roots of $x^2 - 4x - 11 = 0$, then $\alpha + \beta = -\dfrac{(-4)}{1} = 4$ and $\alpha\beta = \dfrac{-11}{1} = -11$.

For all quadratic equations with roots $\alpha\beta$ and $\alpha + \beta$,

the sum of the roots $= \alpha\beta + (\alpha + \beta)$ and the product of the roots $= (\alpha\beta)(\alpha + \beta)$

$$= -11 + 4 \qquad\qquad = -11(4)$$

$$= -7 \qquad\qquad = -44$$

So, we have $-\dfrac{b}{a} = -7$ and $\dfrac{c}{a} = -44$.

The simplest solution is $a = 1, \; b = 7, \; c = -44$.

$\therefore$ all quadratic equations with roots $\alpha\beta$ and $\alpha + \beta$ are of the form

$k(x^2 + 7x - 44) = 0, \quad k \in \mathbb{R}, \quad k \neq 0$.

9 If α and β are the roots of $2x^2 + 5x - 9 = 0$, then $\alpha + \beta = -\frac{5}{2}$ and $\alpha\beta = -\frac{9}{2}$.

For all quadratic equations with roots α^2 and β^2,

the sum of the roots $= \alpha^2 + \beta^2$ and the product of the roots $= \alpha^2\beta^2$

$$= \alpha^2 + 2\alpha\beta + \beta^2 - 2\alpha\beta \qquad\qquad = (\alpha\beta)^2$$

$$= (\alpha + \beta)^2 - 2\alpha\beta \qquad\qquad = (-\tfrac{9}{2})^2$$

$$= (-\tfrac{5}{2})^2 - 2(-\tfrac{9}{2}) \qquad\qquad = \tfrac{81}{4}$$

$$= \tfrac{25}{4} + 9$$

$$= \tfrac{61}{4}$$

So, we have $-\dfrac{b}{a} = \dfrac{61}{4}$ and $\dfrac{c}{a} = \dfrac{81}{4}$.

The simplest solution is $a = 4, \; b = -61, \; c = 81$.

$\therefore$ all quadratic equations with roots α^2 and β^2 are of the form

$k(4x^2 - 61x + 81) = 0, \quad k \in \mathbb{R}, \quad k \neq 0$.

10 If α and β are the roots of $x^2-6x+7=0$, then $\alpha+\beta=-\frac{(-6)}{1}=6$ and $\alpha\beta=\frac{7}{1}=7$.

For the quadratic equation with roots $\alpha+\frac{1}{\beta}$ and $\beta+\frac{1}{\alpha}$,

the sum of the roots $=\left(\alpha+\frac{1}{\beta}\right)+\left(\beta+\frac{1}{\alpha}\right)$

$$=\alpha+\beta+\frac{\alpha+\beta}{\alpha\beta}$$
$$=6+\frac{6}{7}$$
$$=\tfrac{48}{7}$$

and the product of the roots $=\left(\alpha+\frac{1}{\beta}\right)\left(\beta+\frac{1}{\alpha}\right)$

$$=\alpha\beta+1+1+\frac{1}{\alpha\beta}$$
$$=7+2+\tfrac{1}{7}$$
$$=\tfrac{64}{7}$$

So, we have $-\frac{b}{a}=\frac{48}{7}$ and $\frac{c}{a}=\frac{64}{7}$.

The simplest solution is $a=7$, $b=-48$, $c=64$.

$\therefore$ the quadratic equation is $7x^2-48x+64=0$.

11 If p and q are the roots of $2x^2-3x-5=0$, then $p+q=-\frac{(-3)}{2}=\frac{3}{2}$ and $pq=-\frac{5}{2}$.

For all quadratic equations with roots p^2+q and q^2+p,

the sum of the roots $=p^2+q+q^2+p$

$$=(p+q)+(p+q)^2-2pq$$
$$=\tfrac{3}{2}+(\tfrac{3}{2})^2-2(-\tfrac{5}{2})$$
$$=\tfrac{3}{2}+\tfrac{9}{4}+5$$
$$=\tfrac{35}{4}$$

and the product of the roots $=(p^2+q)(q^2+p)$

$$=p^2q^2+p^3+q^3+pq$$
$$=(pq)^2+(p+q)^3-3p^2q-3pq^2+pq$$
$$=(pq)^2+(p+q)^3-3pq(p+q)+pq$$
$$=(-\tfrac{5}{2})^2+(\tfrac{3}{2})^3-3(-\tfrac{5}{2})(\tfrac{3}{2})+(-\tfrac{5}{2})$$
$$=\tfrac{25}{4}+\tfrac{27}{8}+\tfrac{45}{4}-\tfrac{5}{2}$$
$$=\tfrac{147}{8}$$

So, we have $-\frac{b}{a}=\frac{35}{4}=\frac{70}{8}$ and $\frac{c}{a}=\frac{147}{8}$.

The simplest solution is $a=8$, $b=-70$, $c=147$.

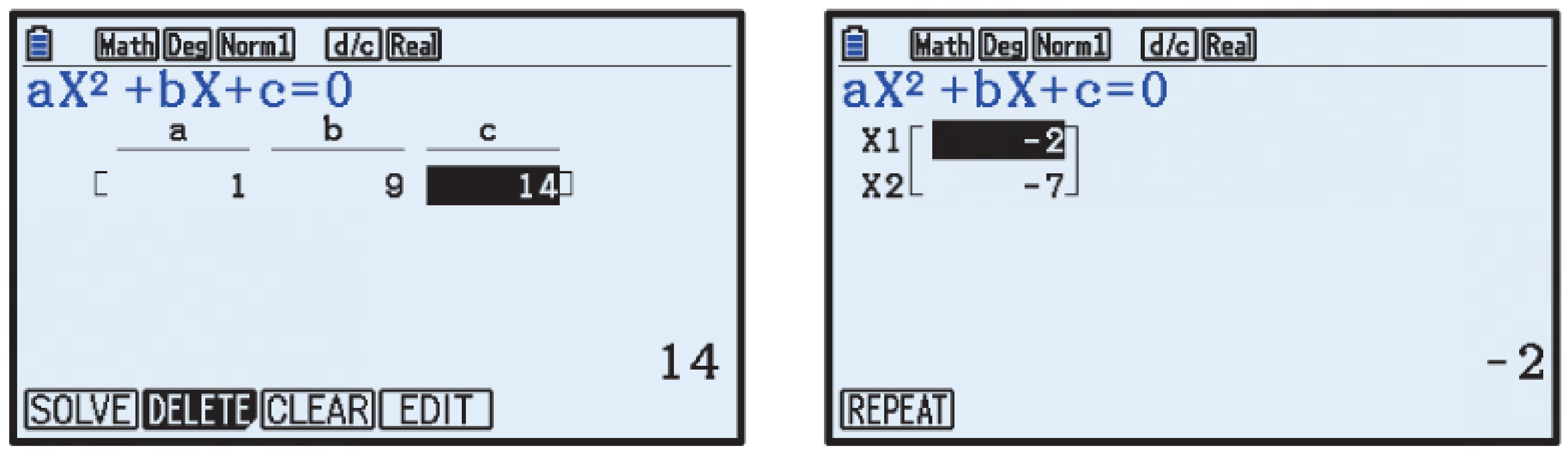
$\therefore$ all quadratic equations with roots p^2+q and q^2+p are of the form $k(8x^2-70x+147)=0$, $k\in\mathbb{R}$, $k\neq 0$.

EXERCISE 4D

1 **a** $x^2 + 9x + 14 = 0$

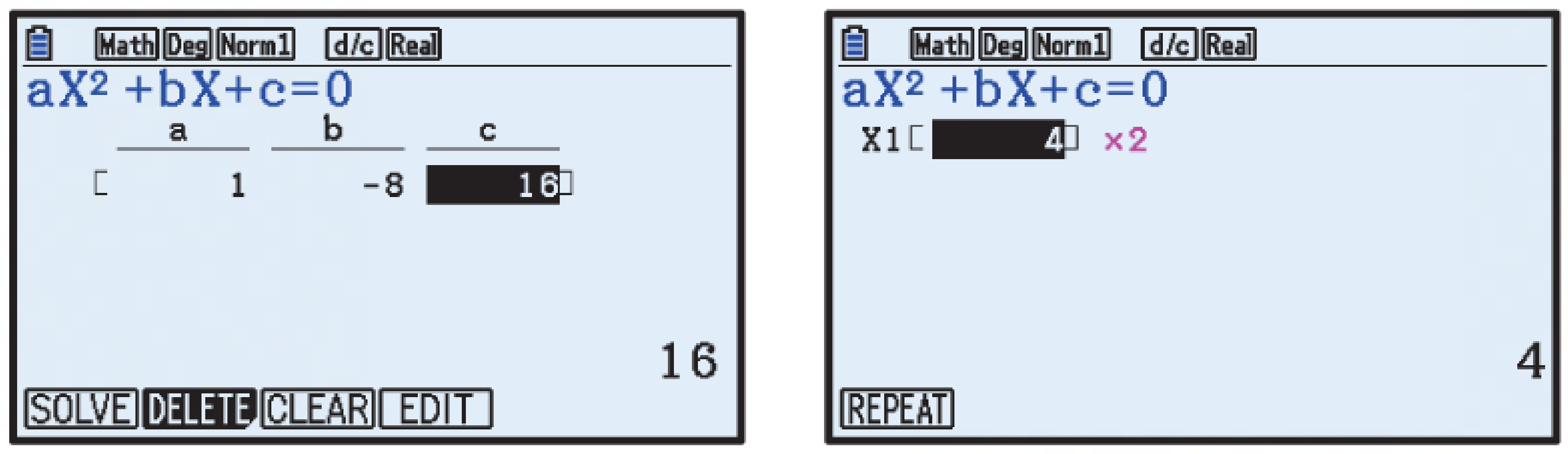
Using technology,

$x = -2$ or -7



b $x^2 - 8x + 16 = 0$

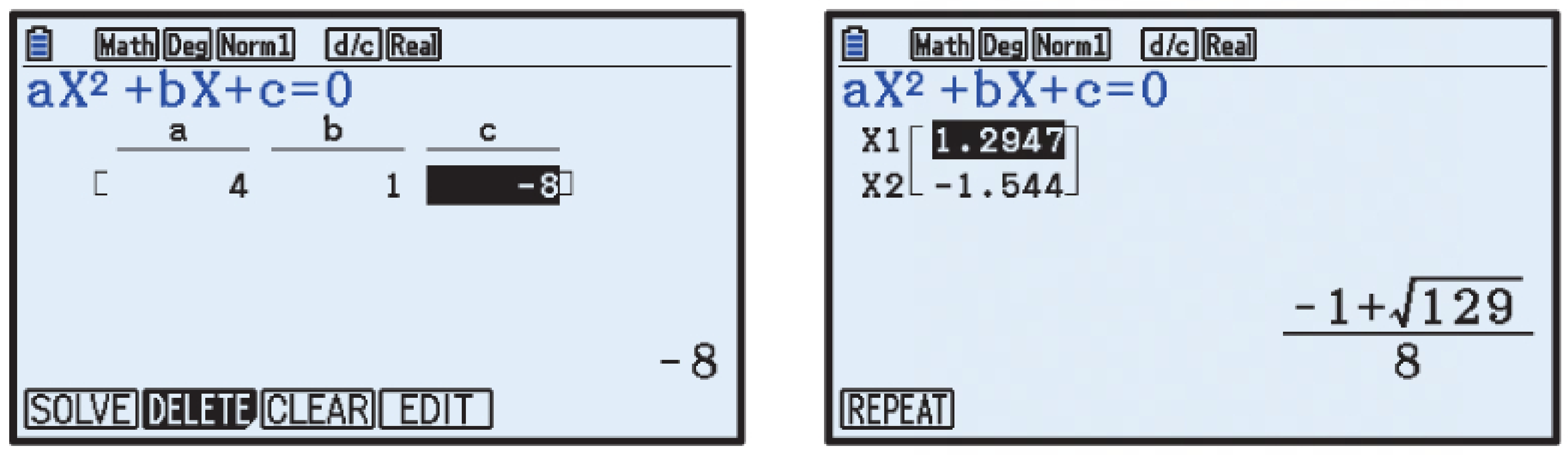
Using technology, $x = 4$



c $4x^2 + x - 8 = 0$

Using technology,

$x \approx 1.29$ or -1.54

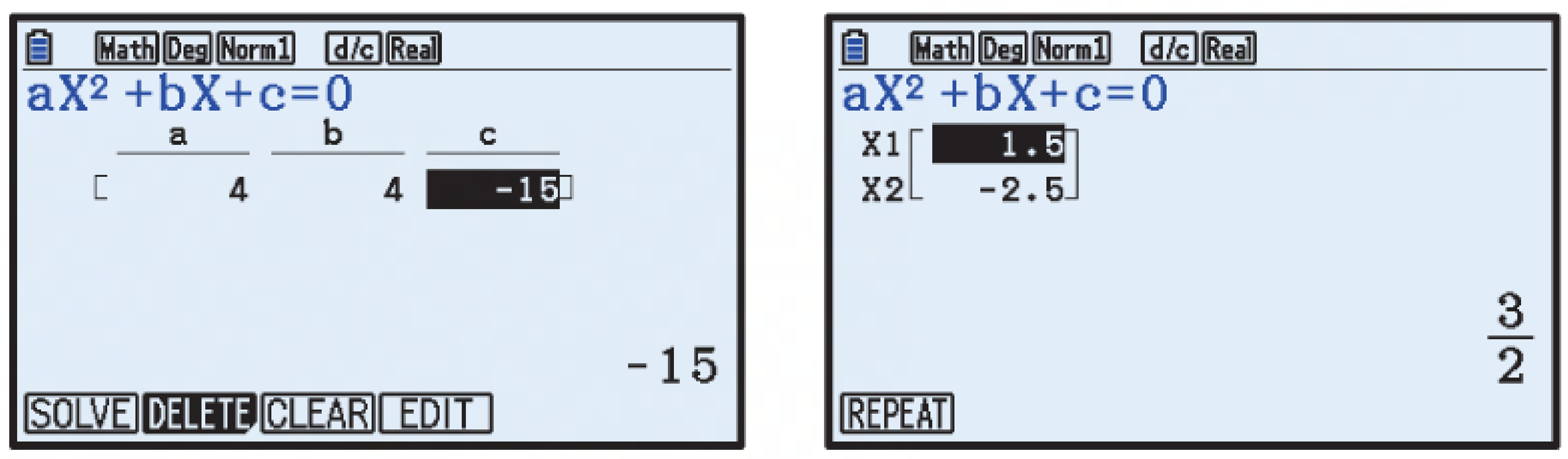


d $4x^2 + 4x = 15$

$\therefore\ 4x^2 + 4x - 15 = 0$

Using technology,

$x = 1.5$ or -2.5

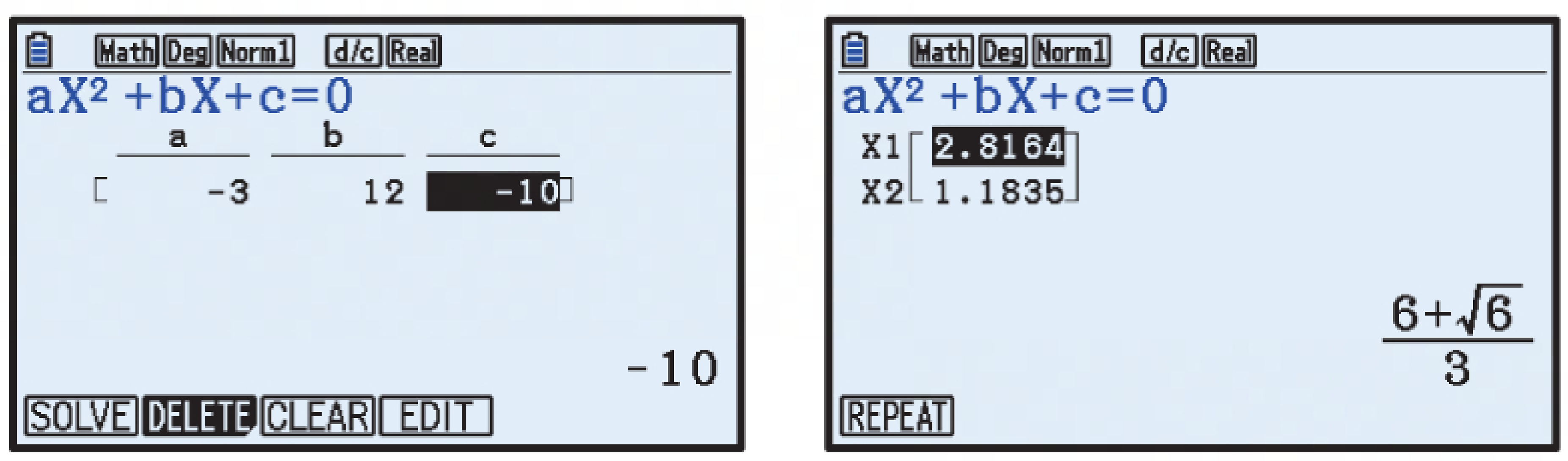


e $-3x^2 + 12x = 10$

$\therefore\ -3x^2 + 12x - 10 = 0$

Using technology,

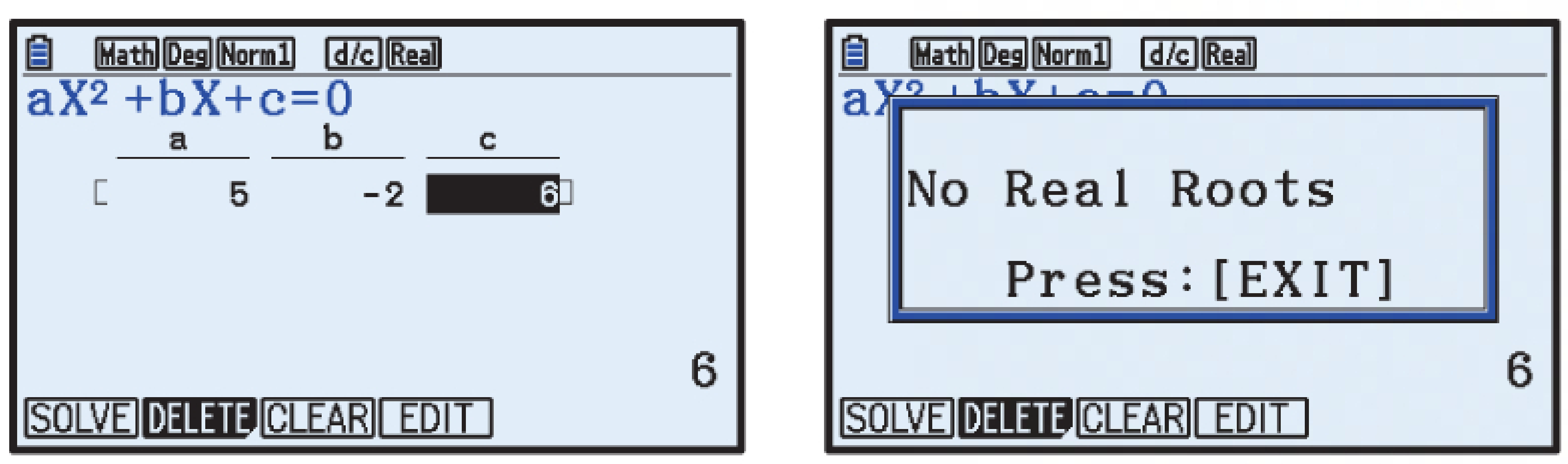
$x \approx 2.82$ or 1.18



f $6 = 2x - 5x^2$

$\therefore\ 5x^2 - 2x + 6 = 0$

Using technology, there are no real solutions.

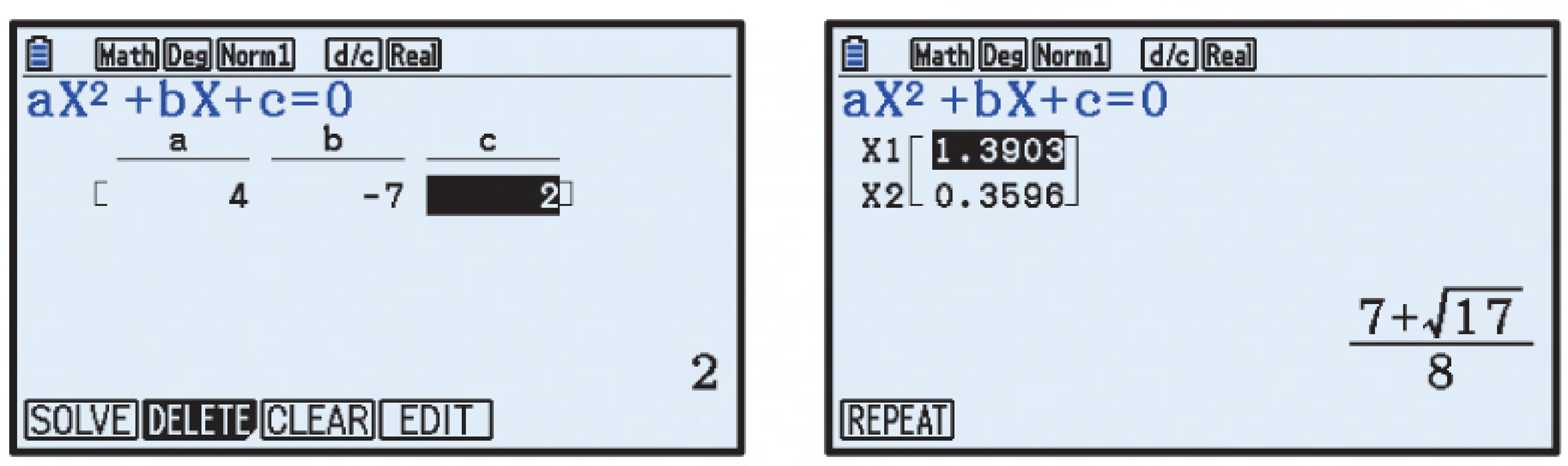


g $7x - 2 = 4x^2$

$\therefore\ 4x^2 - 7x + 2 = 0$

Using technology,

$x \approx 1.39$ or 0.360

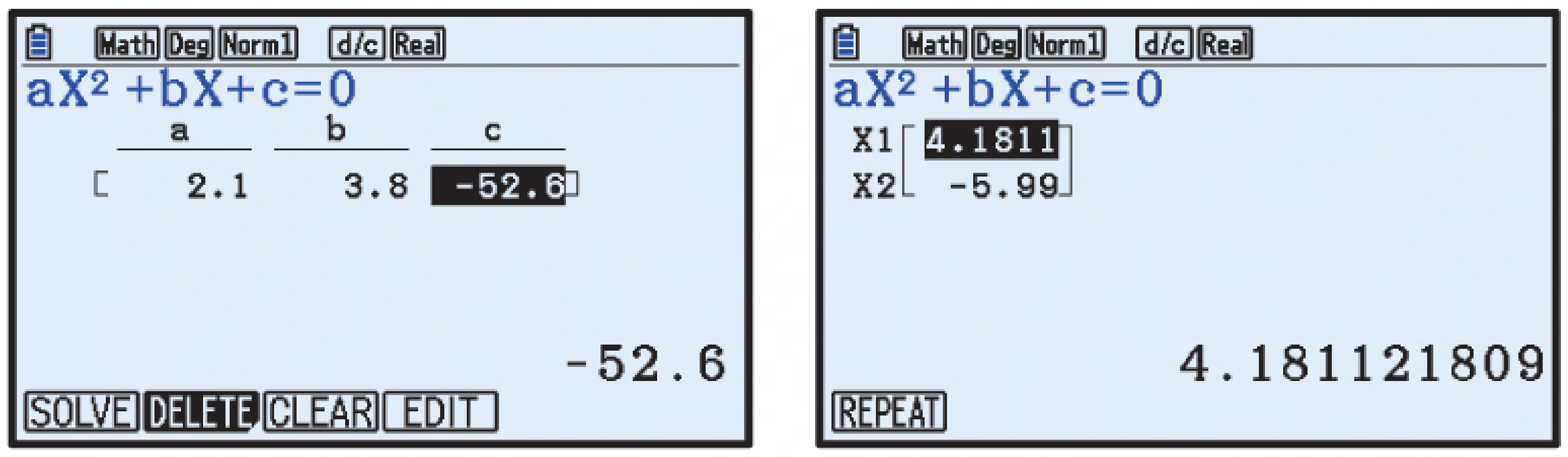


h $\quad 3.8x + 2.1x^2 = 52.6$

$\therefore \; 2.1x^2 + 3.8x - 52.6 = 0$

Using technology,

$x \approx 4.18$ or -5.99



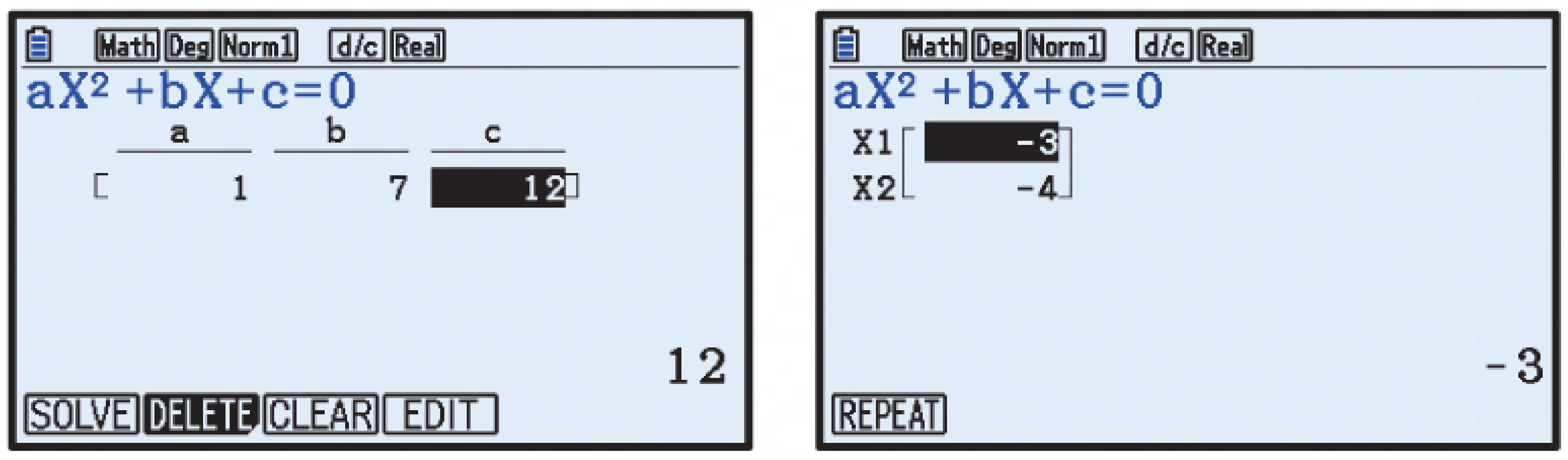
2 **a** $\quad x(x+5) + 2(x+6) = 0$

$\therefore \; x^2 + 5x + 2x + 12 = 0$

$\therefore \; x^2 + 7x + 12 = 0$

Using technology,

$x = -3$ or -4



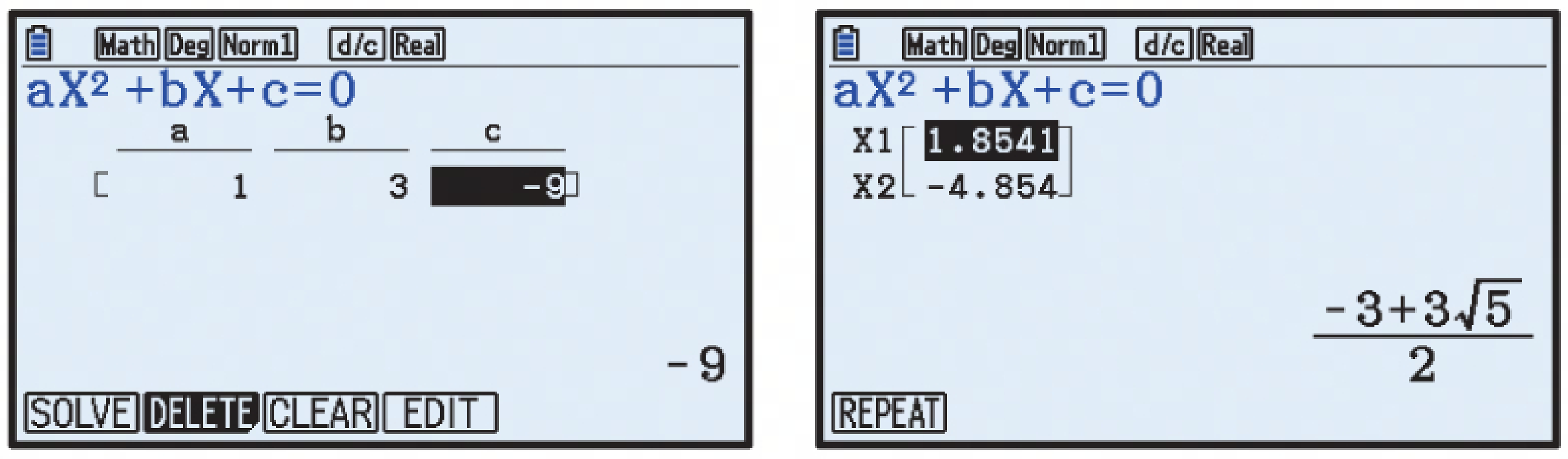
b $\quad (x-1)(x+9) = 5x$

$\therefore \; x^2 + 8x - 9 - 5x = 0$

$\therefore \; x^2 + 3x - 9 = 0$

Using technology,

$x \approx 1.85$ or -4.85



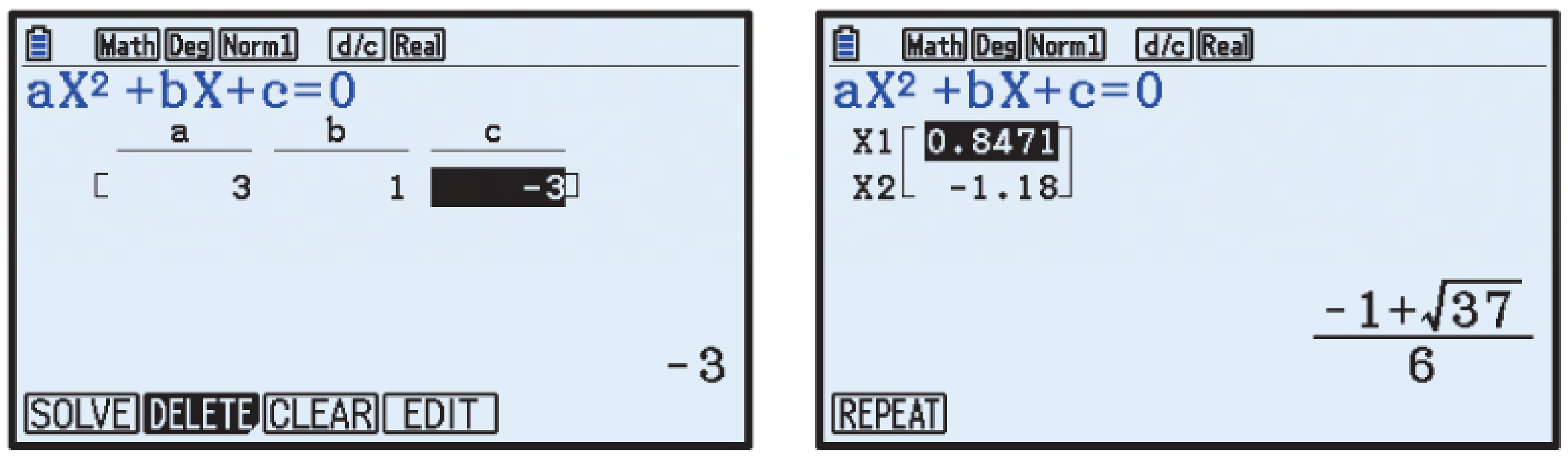
c $\quad 3x(x+2) - 5(x-3) = 18$

$\therefore \; 3x^2 + 6x - 5x + 15 - 18 = 0$

$\therefore \; 3x^2 + x - 3 = 0$

Using technology,

$x \approx 0.847$ or -1.18

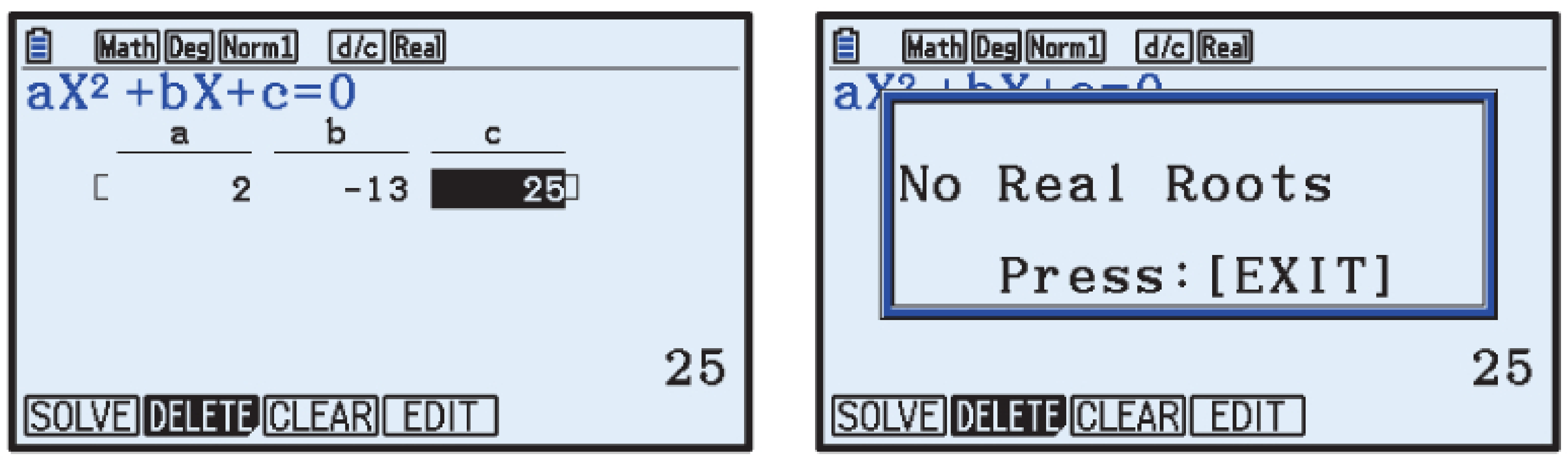


d $\quad 2x(x-6) = x - 25$

$\therefore \; 2x^2 - 12x - x + 25 = 0$

$\therefore \; 2x^2 - 13x + 25 = 0$

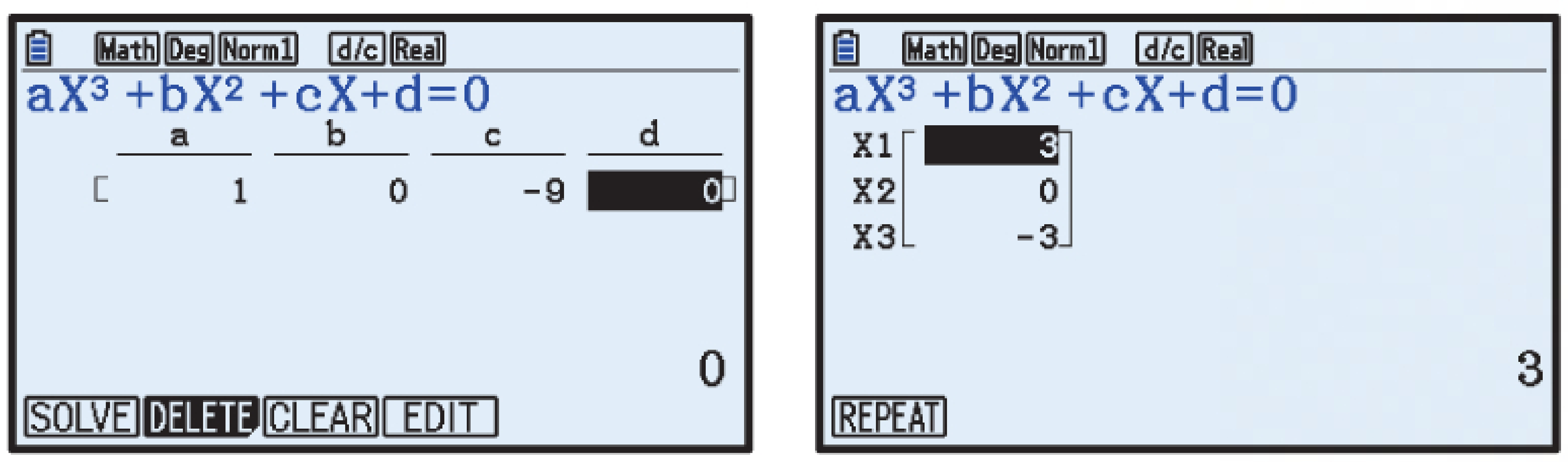
Using technology, there are no real solutions.



3 **a** $\quad x^3 - 9x = 0$

Using technology,

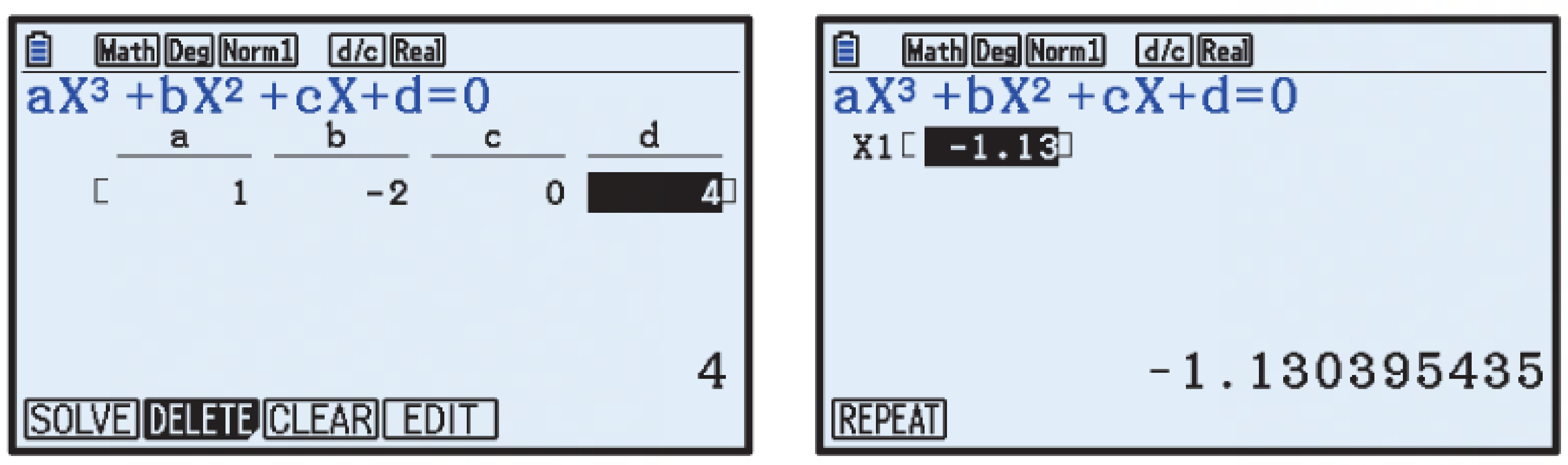
$x = 3, 0,$ or -3



b $\quad x^3 - 2x^2 + 4 = 0$

Using technology,

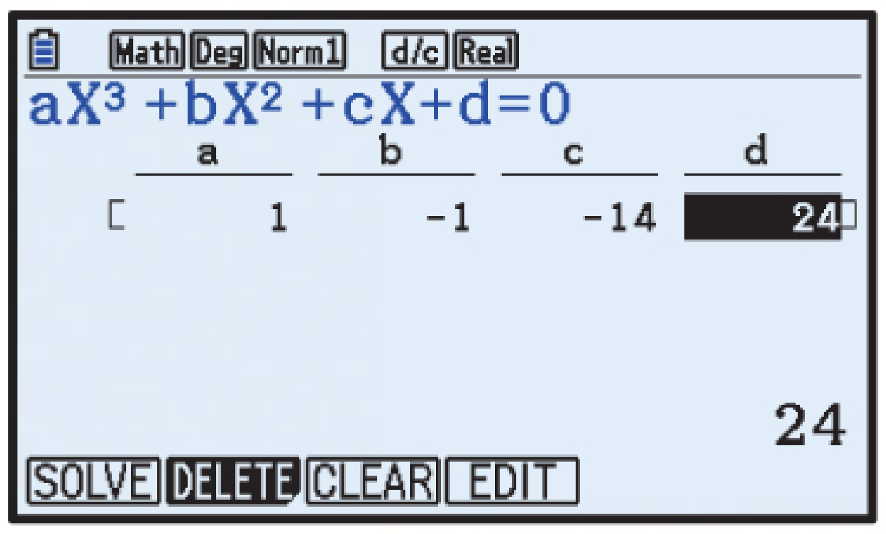
$x \approx -1.13$

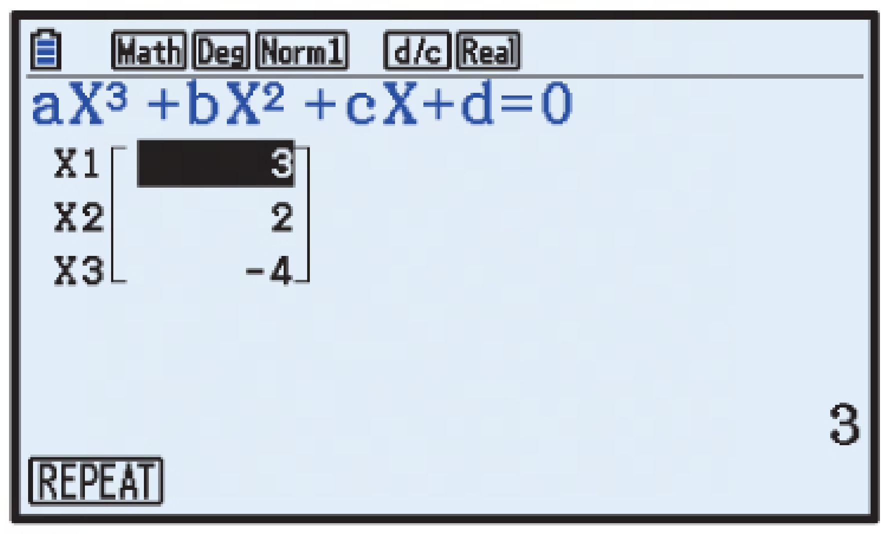


c $x^3 - x^2 - 14x + 24 = 0$

Using technology,

$x = 3,\ 2,$ or -4

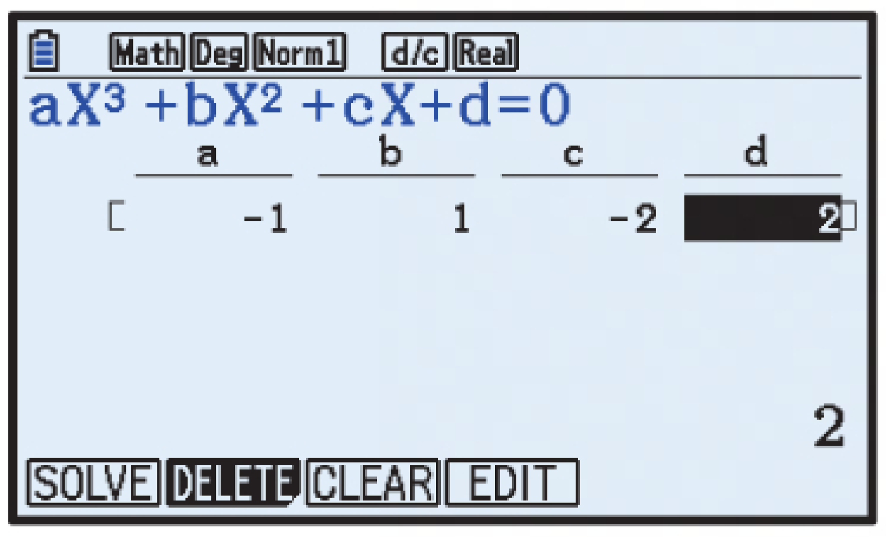


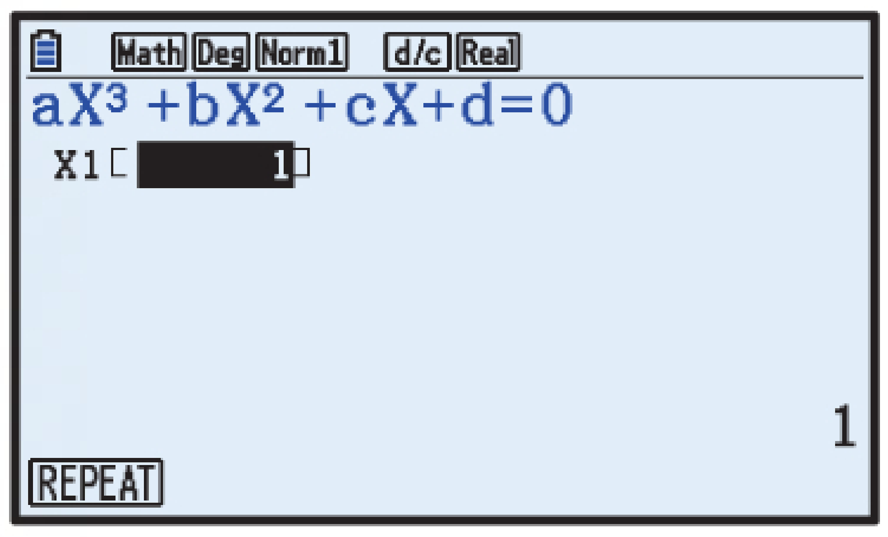


d $-x^3 + 2 = 2x - x^2$

$\therefore\ -x^3 + x^2 - 2x + 2 = 0$

Using technology, $x = 1$



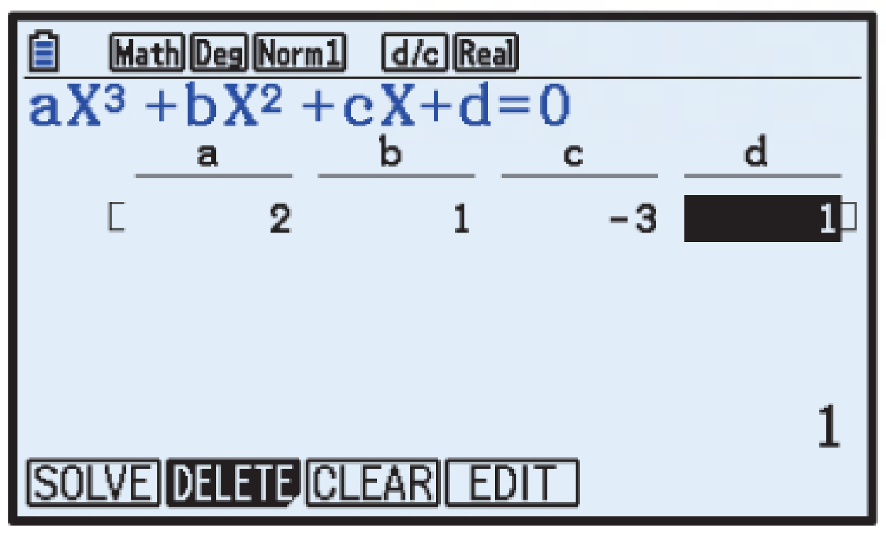


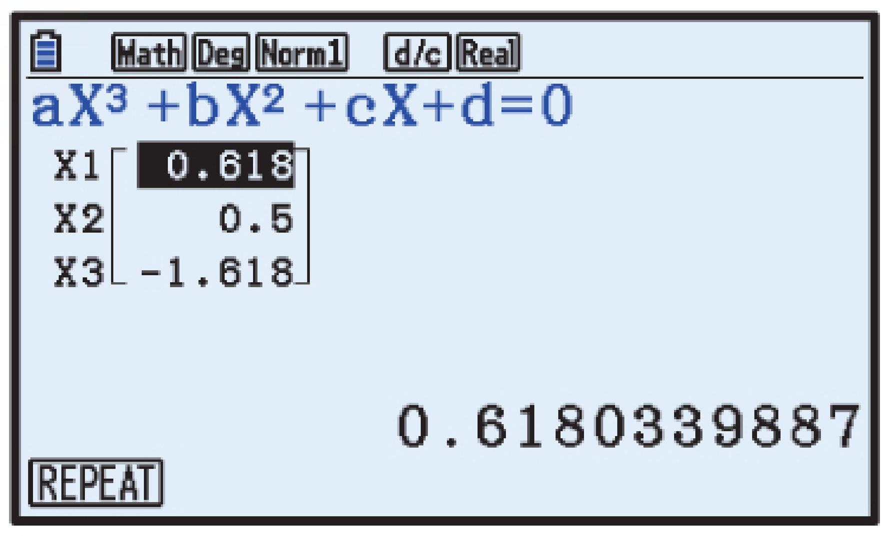
e $2x^3 + x^2 = 3x - 1$

$\therefore\ 2x^3 + x^2 - 3x + 1 = 0$

Using technology,

$x = 0.5, \approx 0.618,$ or -1.62



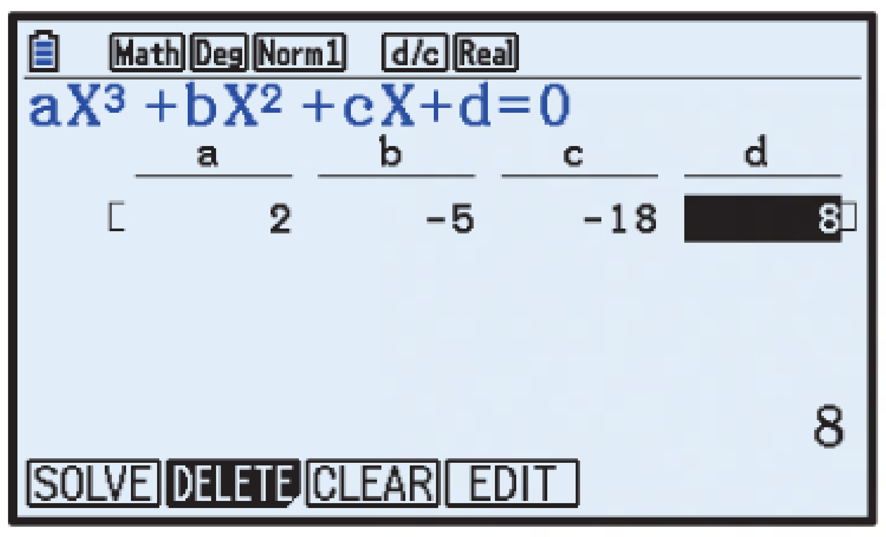


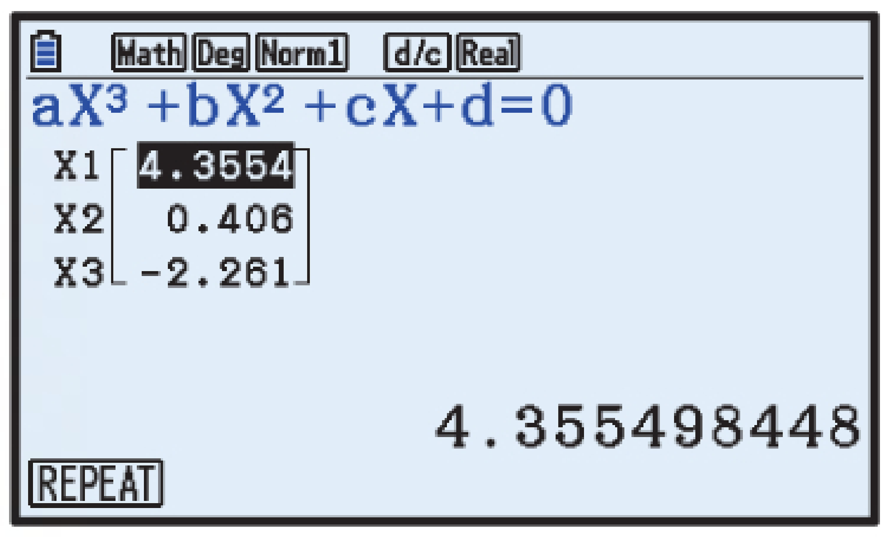
f $2x^3 + 8 = 5x^2 + 18x$

$\therefore\ 2x^3 - 5x^2 - 18x + 8 = 0$

Using technology,

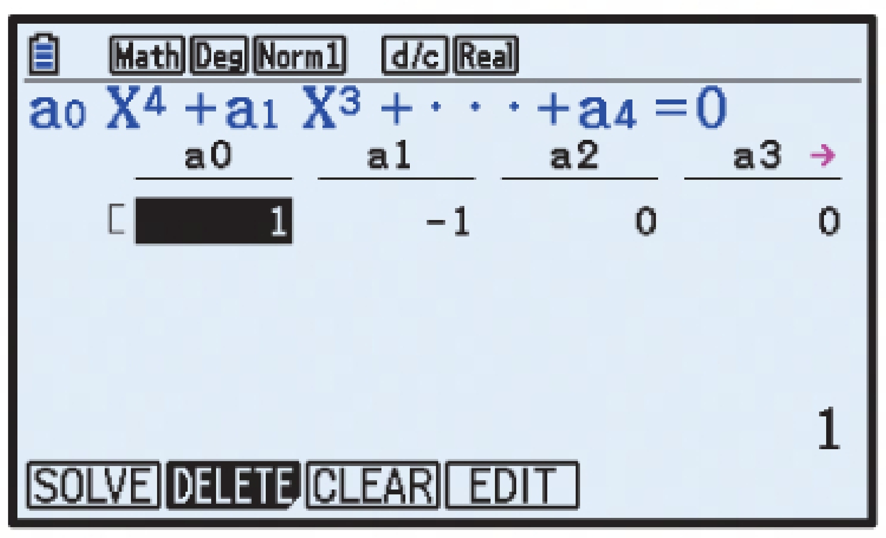
$x \approx 4.36,\ 0.406,$ or -2.26

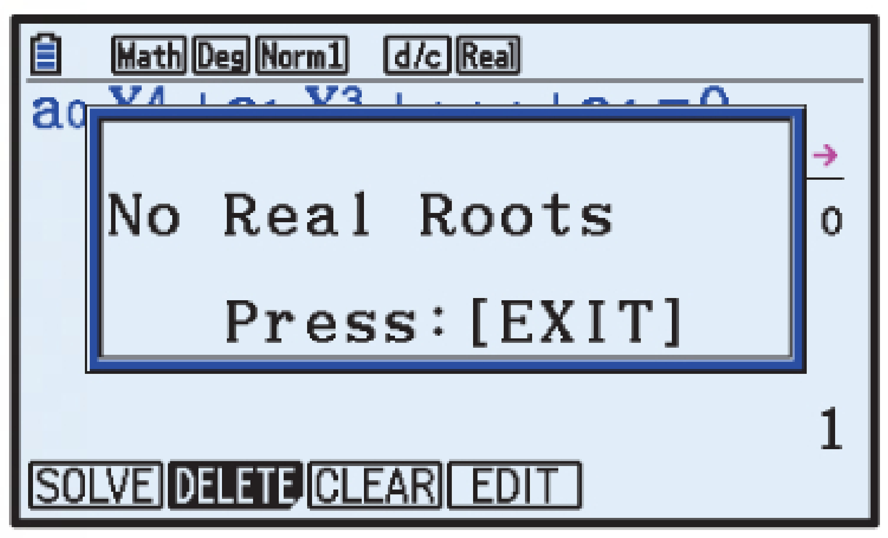




4 **a** $x^4 - x^3 + 2 = 0$

Using technology, there are no real solutions.

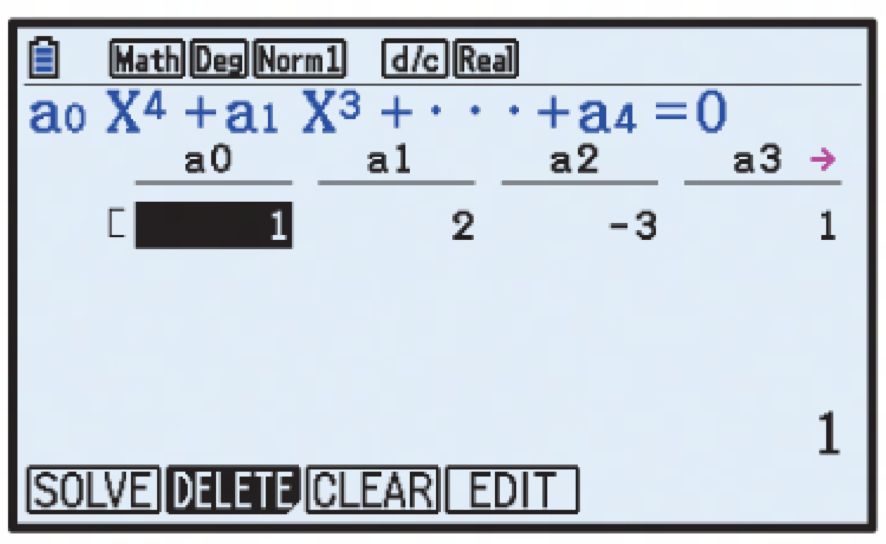


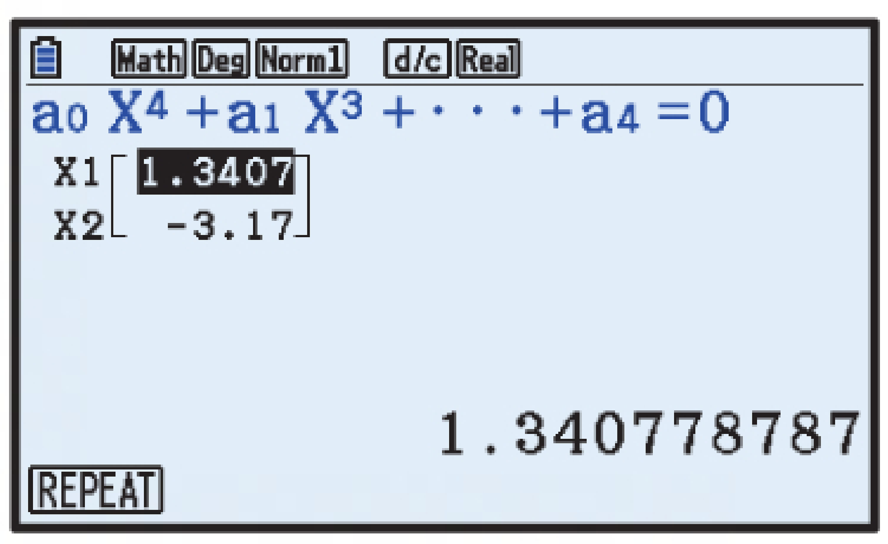


b $x^4 + 2x^3 - 3x^2 + x - 4 = 0$

Using technology,

$x \approx 1.34$ or -3.17

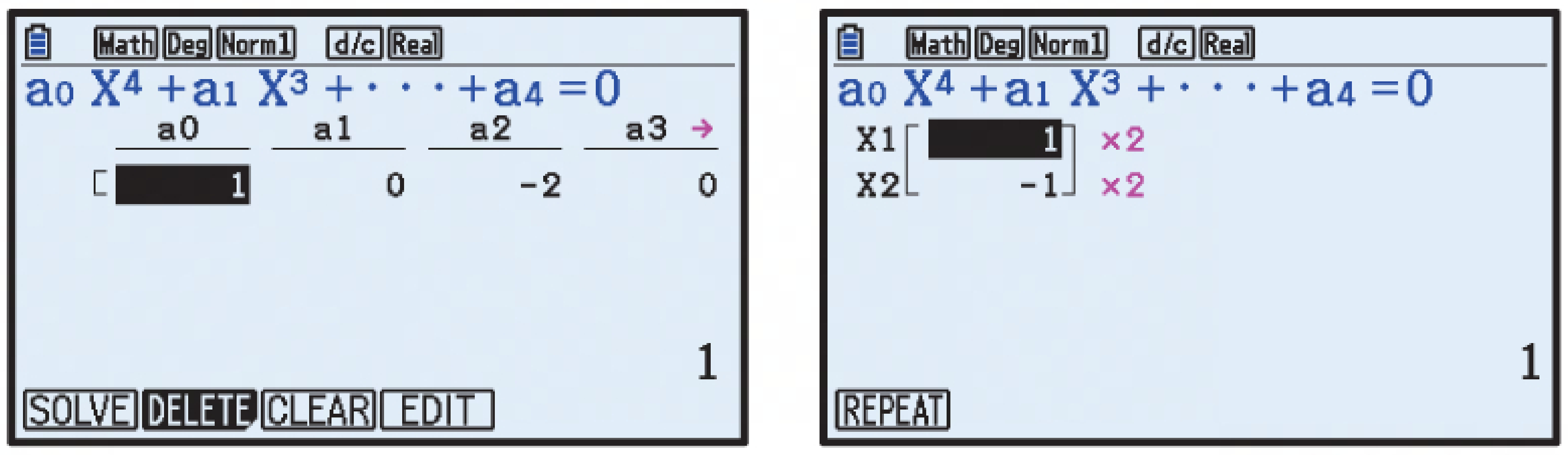




c $x^4 - 2x^2 + 1 = 0$

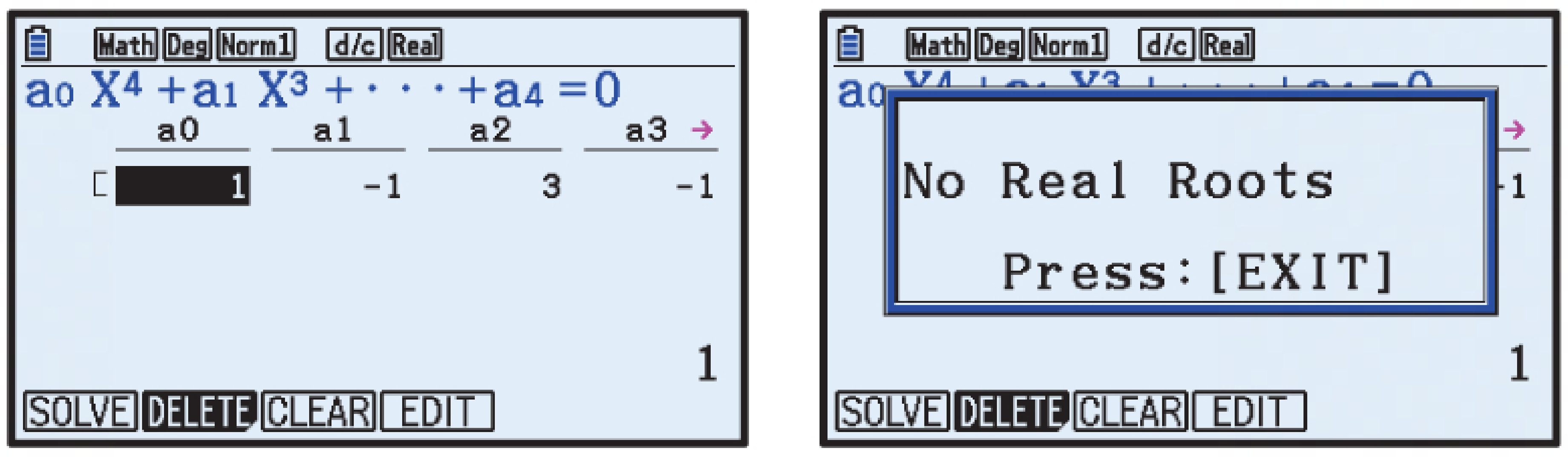
Using technology,

$x = 1$ or -1



d $x^4 - x^3 + 3x^2 - x + 6 = 0$

Using technology, there are no real solutions.

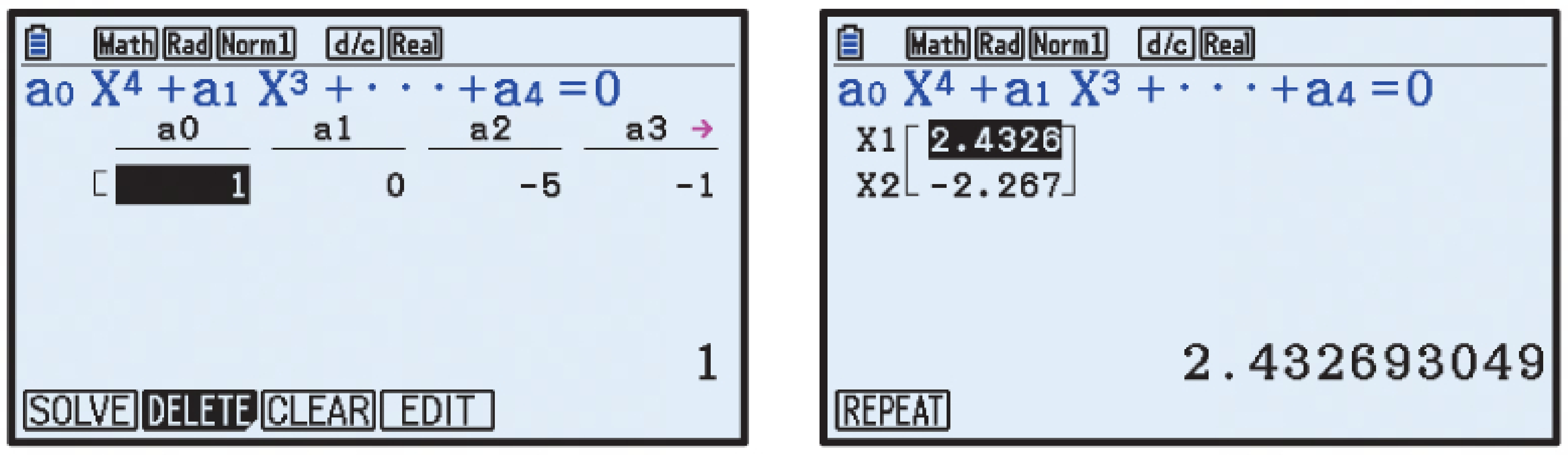


e $x^4 - x = 5x^2 + 3$

$\therefore \; x^4 - 5x^2 - x - 3 = 0$

Using technology,

$x \approx 2.43$ or -2.27



f $x^4 + 2x^3 + 3 = x(9x + 14)$

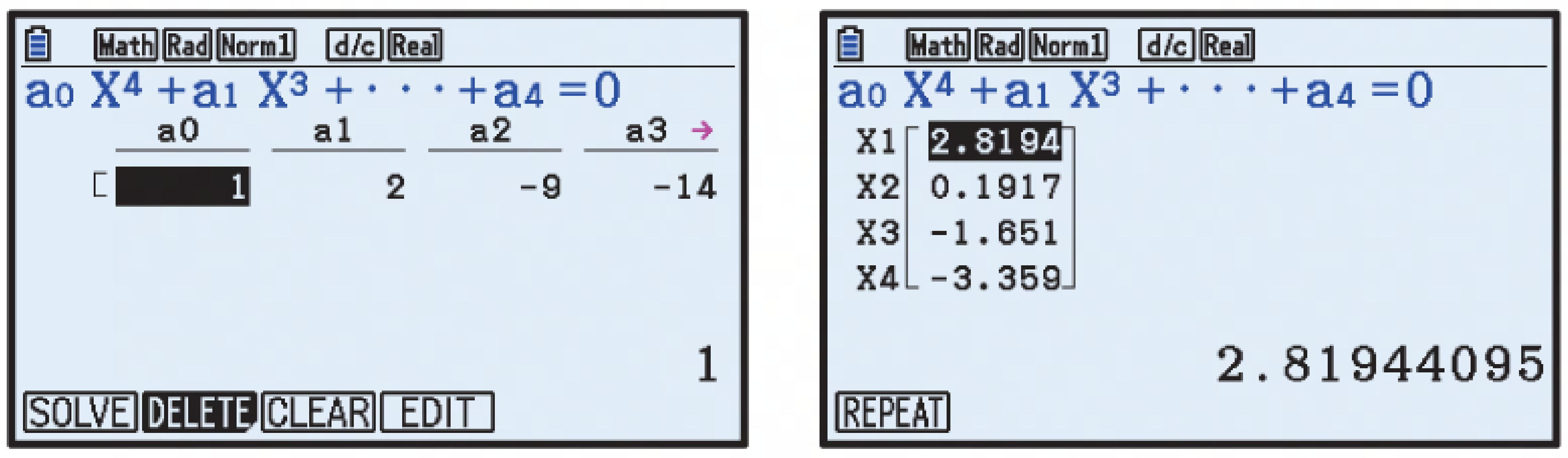
$\therefore \; x^4 + 2x^3 + 3 = 9x^2 + 14x$

$\therefore \; x^4 + 2x^3 - 9x^2 - 14x + 3 = 0$

Using technology,

$x \approx 2.82, 0.192,$

-1.65, or -3.36



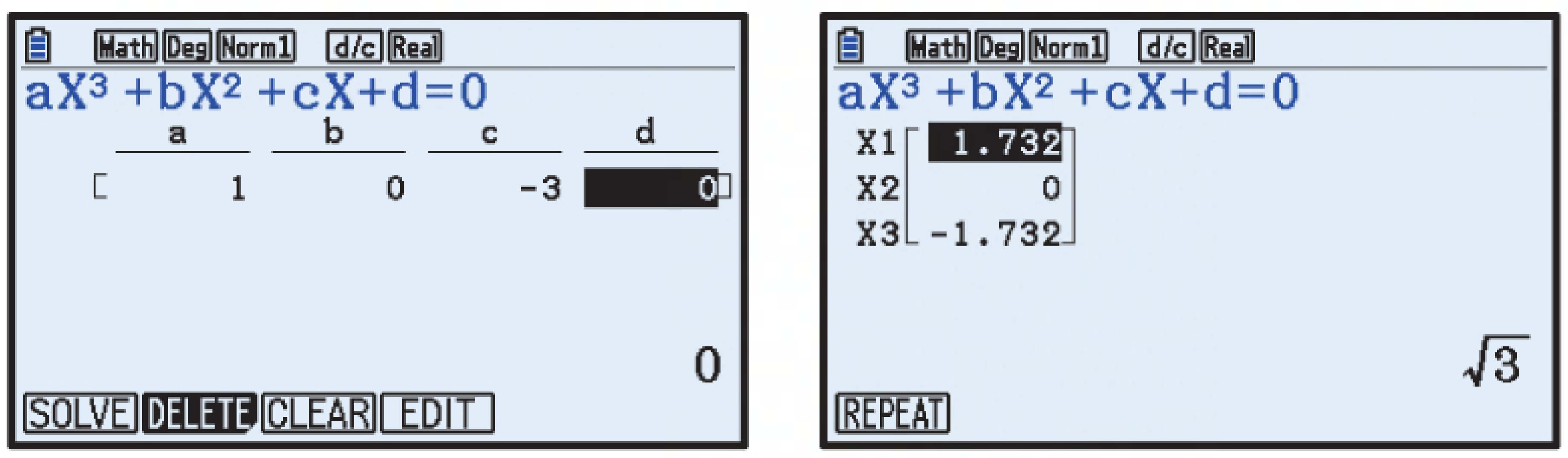
5 **a** $x(x^2 - 1) = 2x$

$\therefore \; x^3 - x - 2x = 0$

$\therefore \; x^3 - 3x = 0$

Using technology,

$x = 0, \approx 1.73$, or -1.73



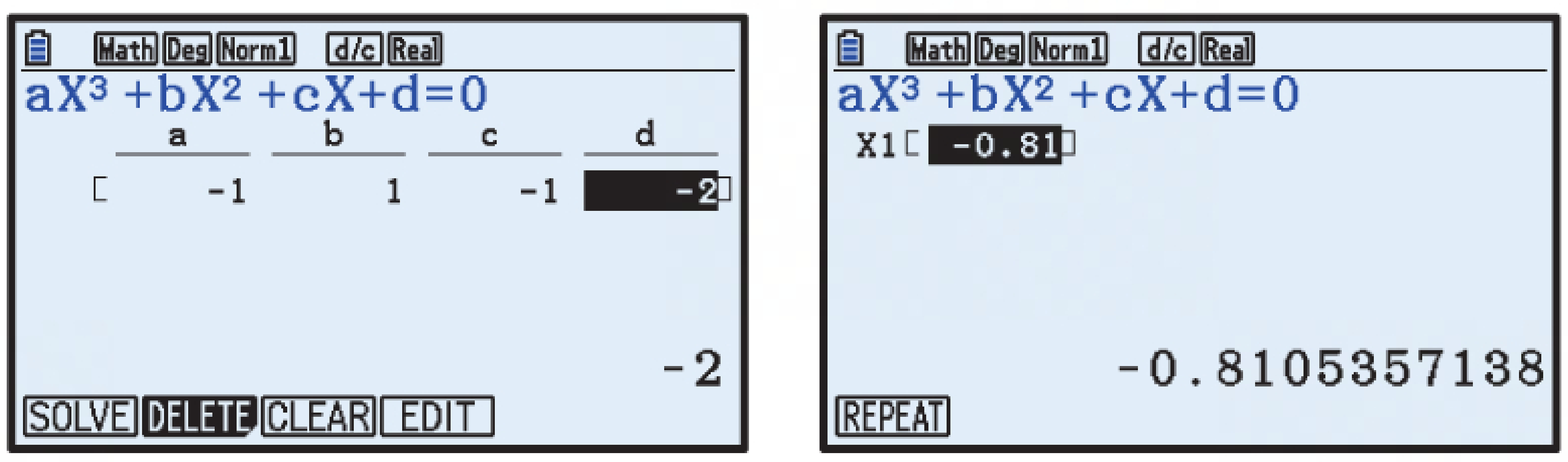
b $(x - 2)(x + 1) = x^3$

$\therefore \; x^2 - x - 2 - x^3 = 0$

$\therefore \; -x^3 + x^2 - x - 2 = 0$

Using technology,

$x \approx -0.811$



6 **a** $$\frac{x}{x-3} - \frac{2}{x^2} = 2$$

$\therefore\ x \times x^2 - 2(x-3) = 2x^2(x-3)$ {multiplying both sides by $x^2(x-3)$}

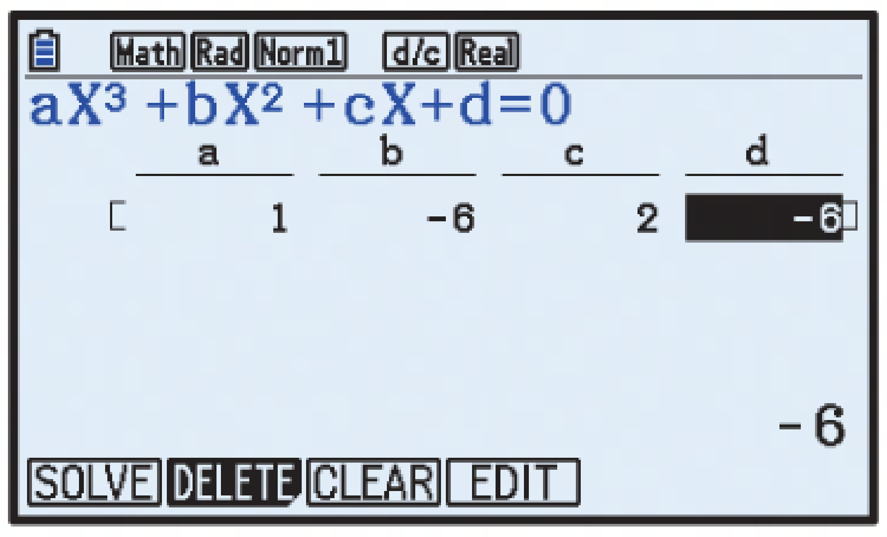
$\therefore\ x^3 - 2x + 6 = 2x^3 - 6x^2$

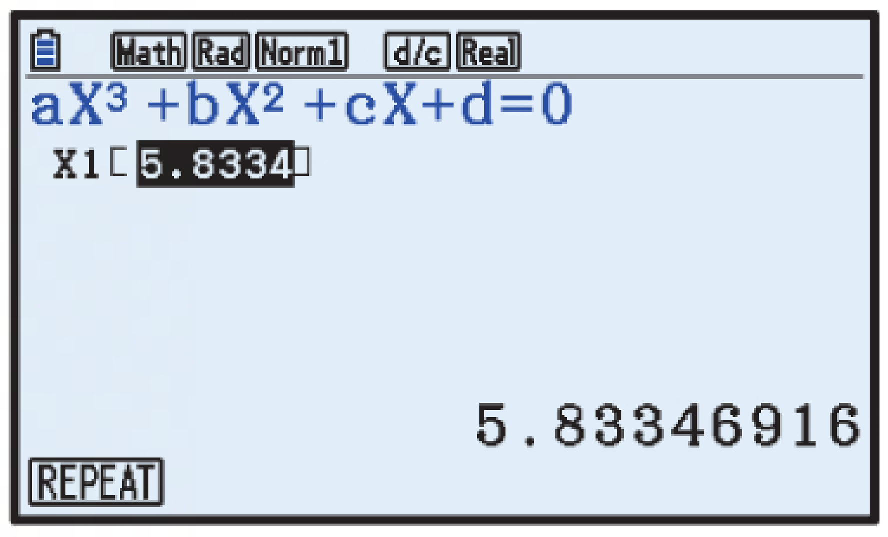
$\therefore\ x^3 - 6x^2 + 2x - 6 = 0$

b $x^3 - 6x^2 + 2x - 6 = 0$

Using technology,

$x \approx 5.83$





EXERCISE 4E

1 **a** $$x^2 - 5 = x + 1$$

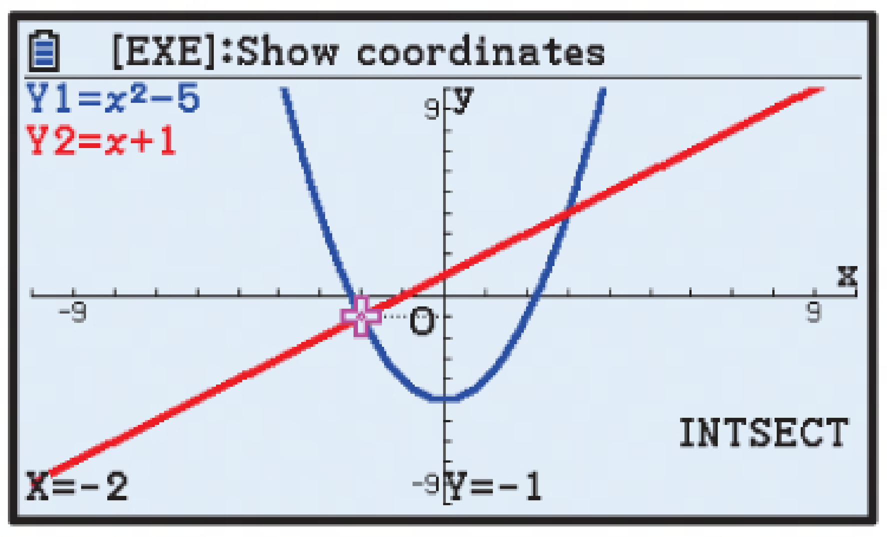
$\therefore\ x^2 - 5 - x - 1 = 0$

$\therefore\ x^2 - x - 6 = 0$

$\therefore\ (x+2)(x-3) = 0$

$\therefore\ x = -2 \text{ or } 3$

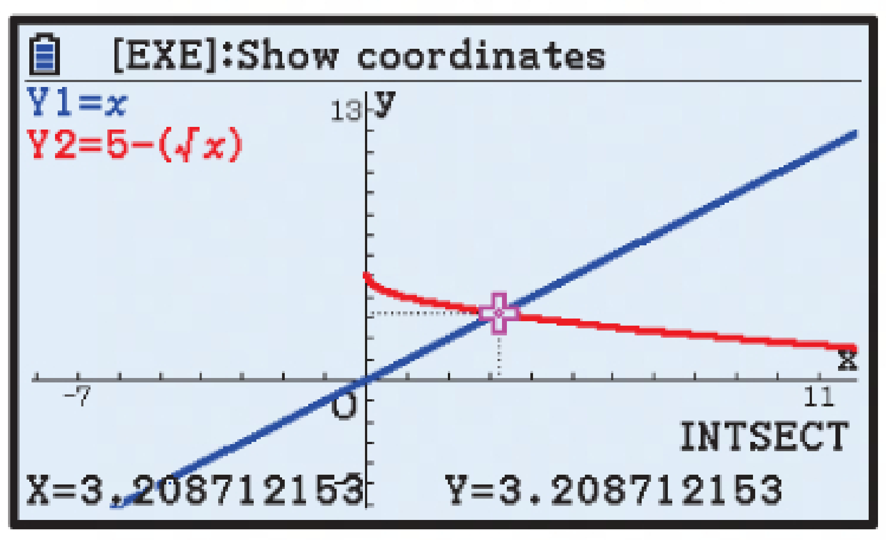
b We graph $y = x^2 - 5$ and $y = x + 1$ on the same set of axes.



The graphs intersect at $(-2,\ -1)$ and $(3,\ 4)$.

$\therefore$ the solutions are $x = -2$ or 3.

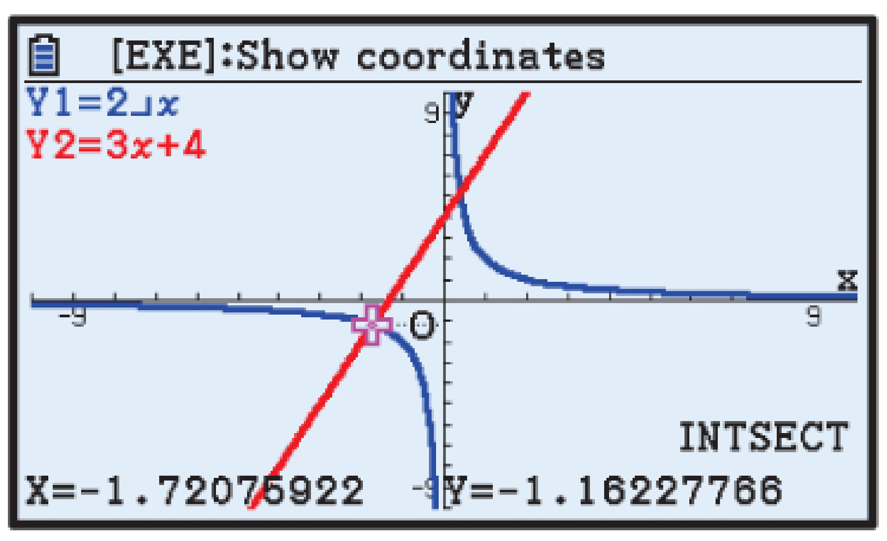
2 **a** We graph $y = x$ and $y = 5 - \sqrt{x}$ on the same set of axes.



The graphs intersect at $(3.21,\ 3.21)$.

$\therefore$ the solution is $x \approx 3.21$.

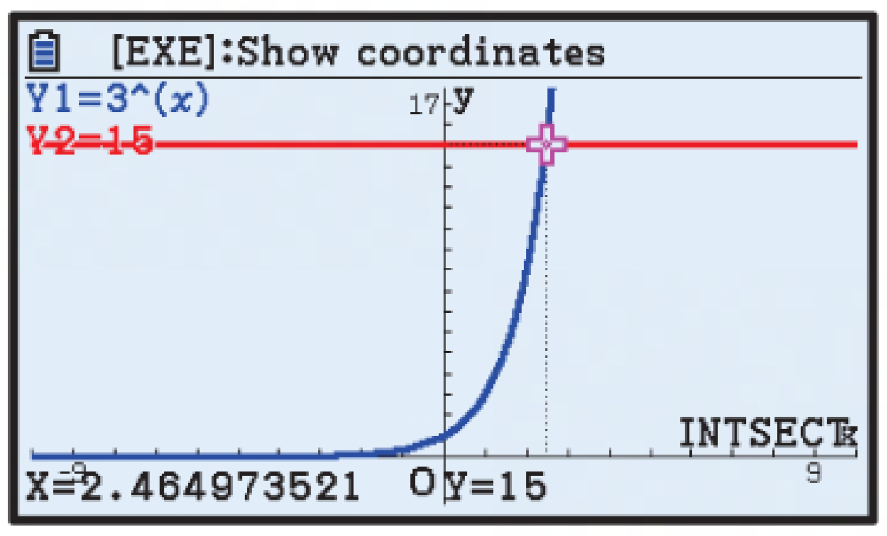
b We graph $y = \dfrac{2}{x}$ and $y = 3x + 4$ on the same set of axes.



The graphs intersect at $(-1.72,\ -1.16)$ and $(0.387,\ 5.16)$.

$\therefore$ the solutions are $x \approx -1.72$ or 0.387.

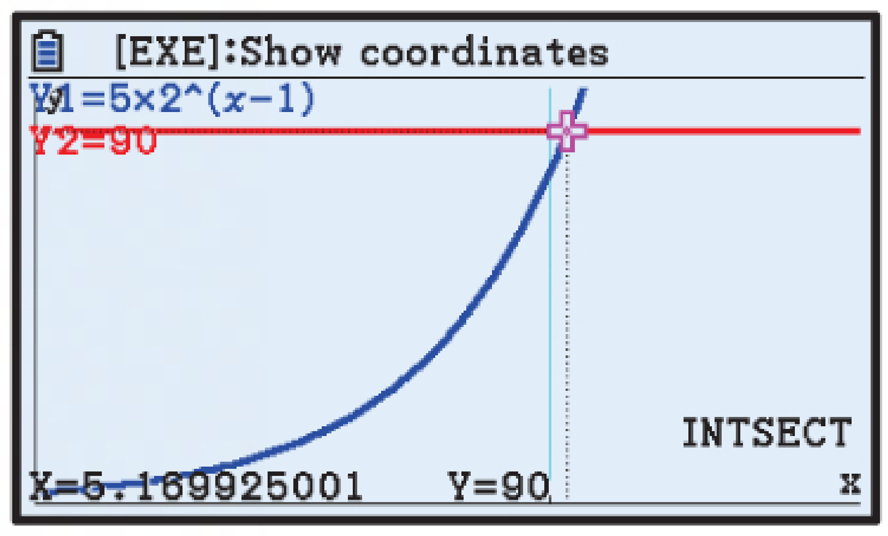
c We graph $y = 3^x$ and $y = 15$ on the same set of axes.



The graphs intersect at $(2.46, 15)$.

$\therefore$ the solution is $x \approx 2.46$.

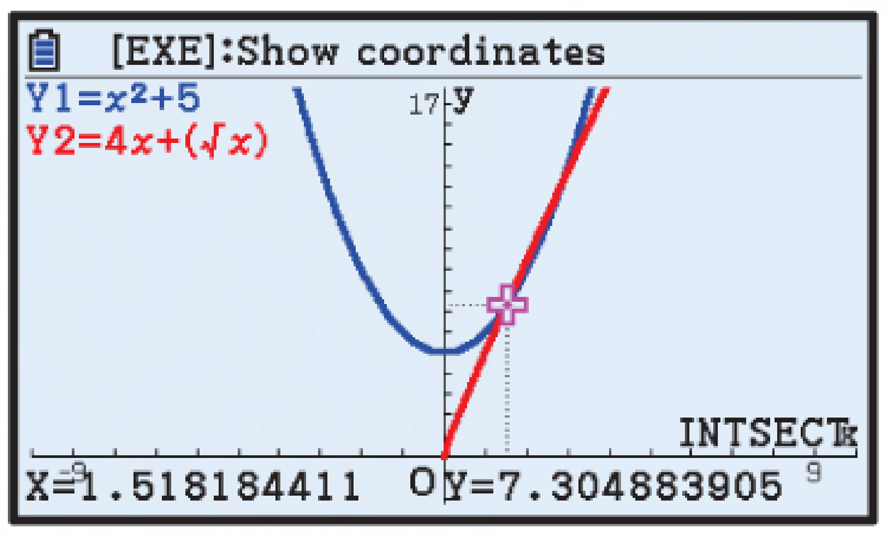
d We graph $y = 5 \times 2^{x-1}$ and $y = 90$ on the same set of axes.



The graphs intersect at $(5.17, 90)$.

$\therefore$ the solution is $x \approx 5.17$.

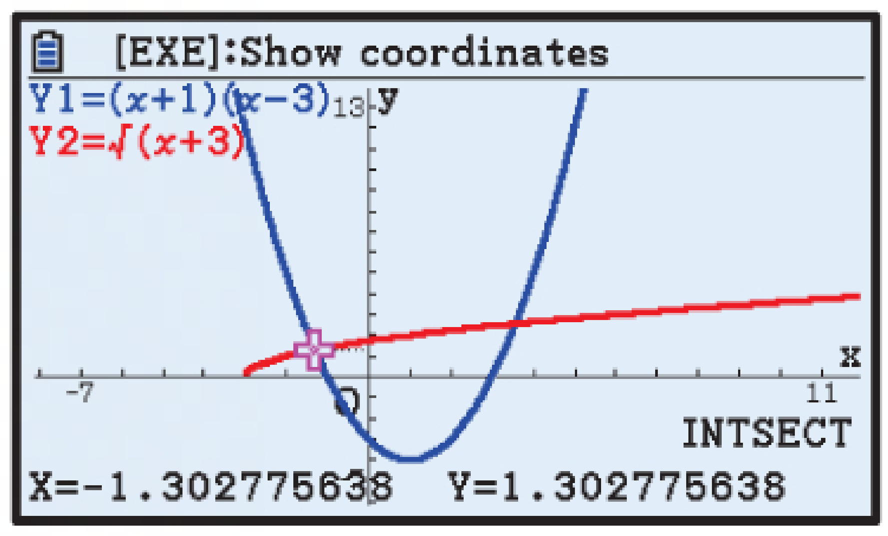
e We graph $y = x^2 + 5$ and $y = 4x + \sqrt{x}$ on the same set of axes.



The graphs intersect at $(1.52, 7.30)$ and $(2.83, 13.0)$.

$\therefore$ the solutions are $x \approx 1.52$ or 2.83.

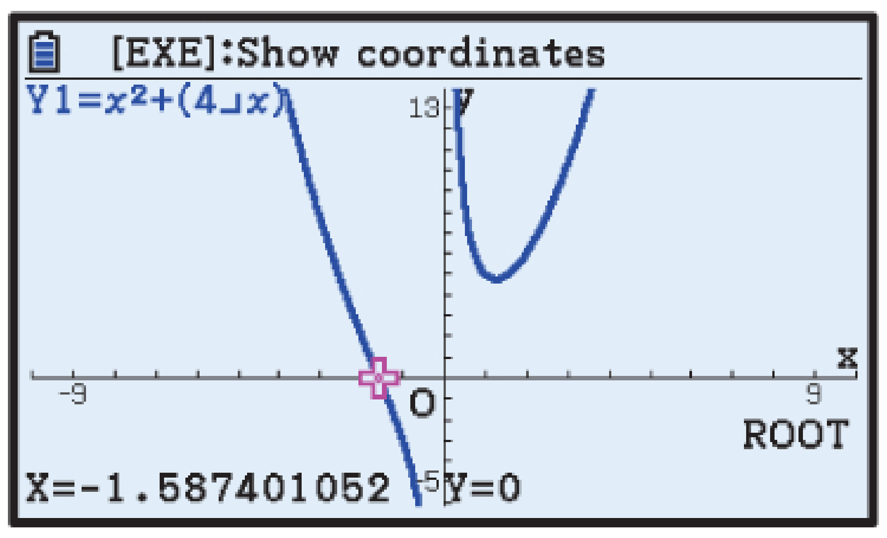
f We graph $y = (x+1)(x-3)$ and $y = \sqrt{x+3}$ on the same set of axes.



The graphs intersect at $(-1.30, 1.30)$ and $(3.56, 2.56)$.

$\therefore$ the solutions are $x \approx -1.30$ or 3.56.

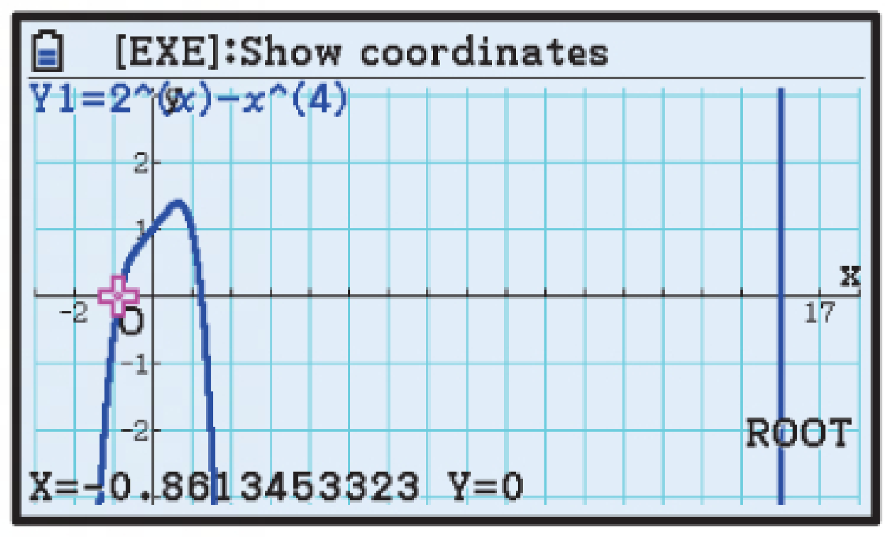
3 **a** We graph $y = x^2 + \dfrac{4}{x}$.



The x-intercept is ≈ -1.59.

$\therefore$ the solution is $x \approx -1.59$.

b We graph $y = 2^x - x^4$.

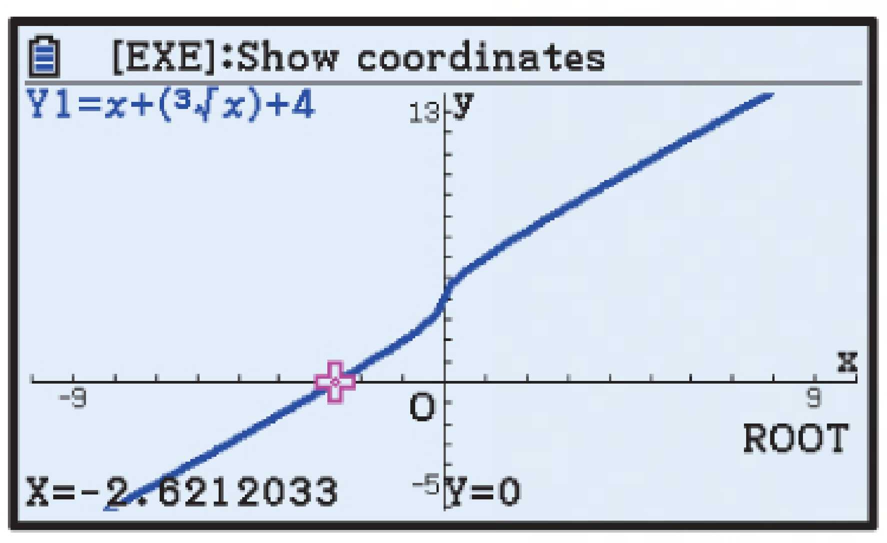


The x-intercepts are ≈ -0.861, 1.24, and 16.

$\therefore$ the solutions are

$x \approx -0.861$, 1.24, or 16.

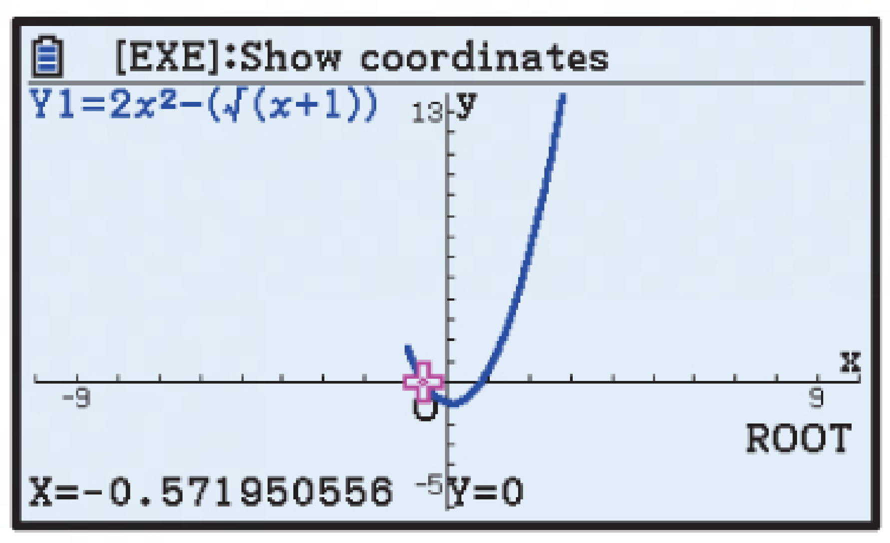
c We graph $y = x + \sqrt[3]{x} + 4$.



The x-intercept is ≈ -2.62.

$\therefore$ the solution is $x \approx -2.62$.

d We graph $y = 2x^2 - \sqrt{x+1}$.

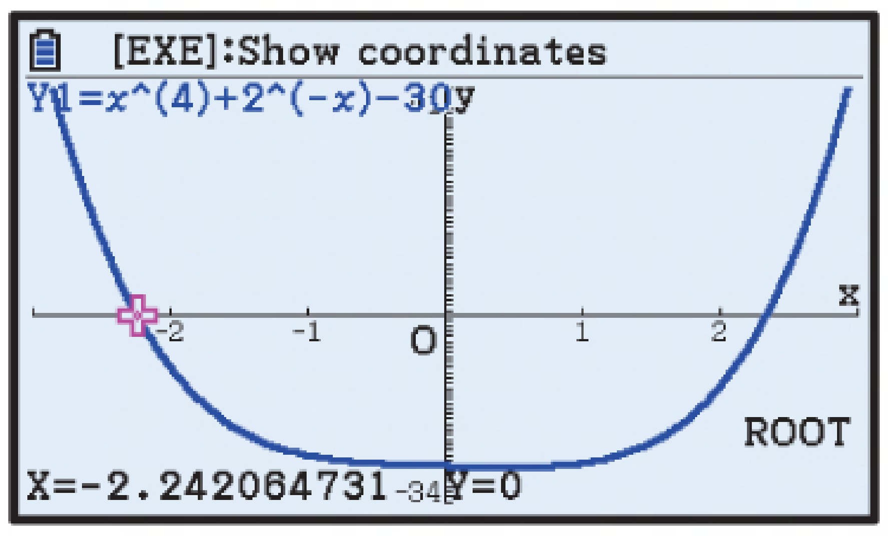


The x-intercepts are ≈ -0.572 and 0.821.

$\therefore$ the solutions are

$x \approx -0.572$ or 0.821.

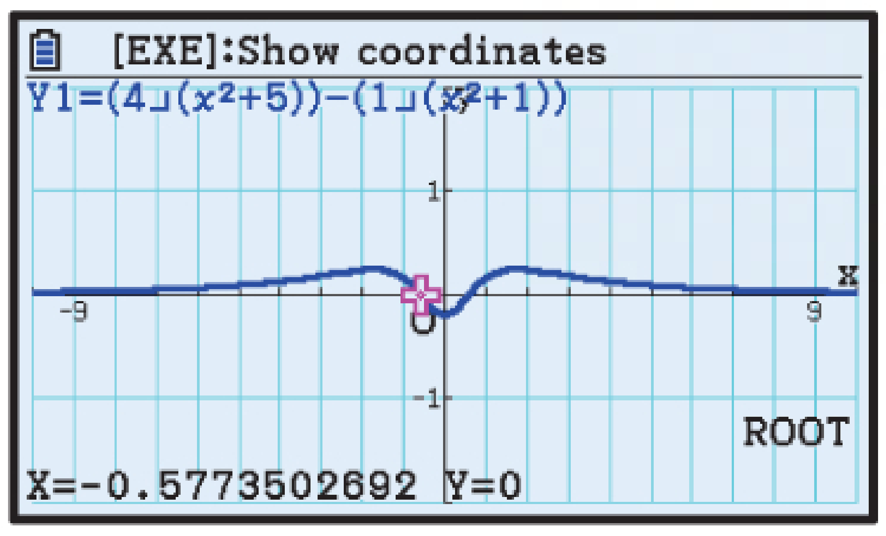
e We graph $y = x^4 + 2^{-x} - 30$.



The x-intercepts are ≈ -2.24 and 2.34.

$\therefore$ the solutions are $x \approx -2.24$ or 2.34.

f We graph $y = \dfrac{4}{x^2+5} - \dfrac{1}{x^2+1}$.

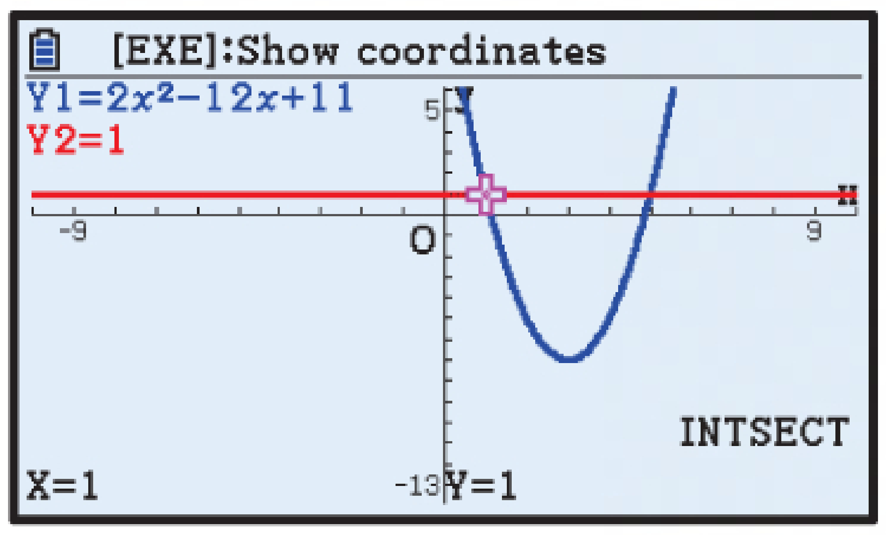


The x-intercepts are ≈ -0.577 and 0.577.

$\therefore$ the solutions are

$x \approx -0.577$ or 0.577.

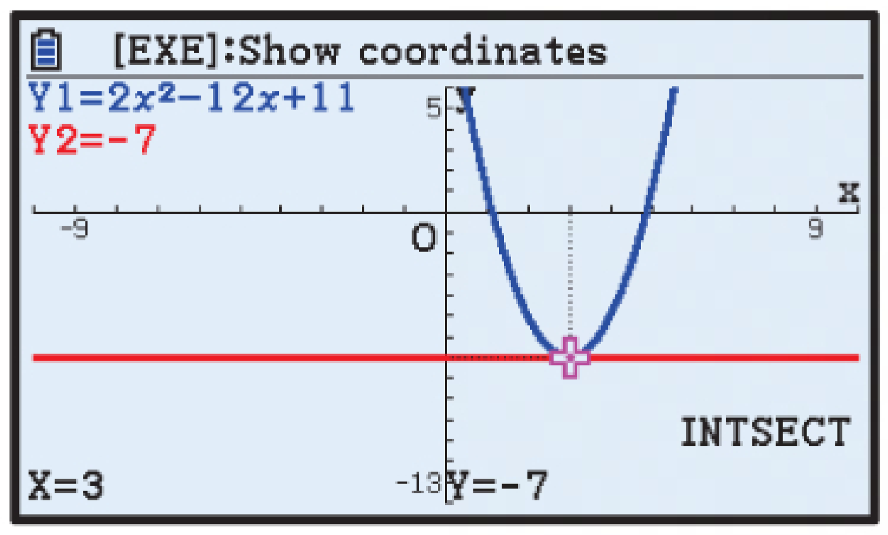
4 **a** **i** We graph $y = 2x^2 - 12x + 11$ and $y = 1$ on the same set of axes.



The graphs intersect at $(1, 1)$ and $(5, 1)$.

$\therefore$ the solutions are $x = 1$ or 5.

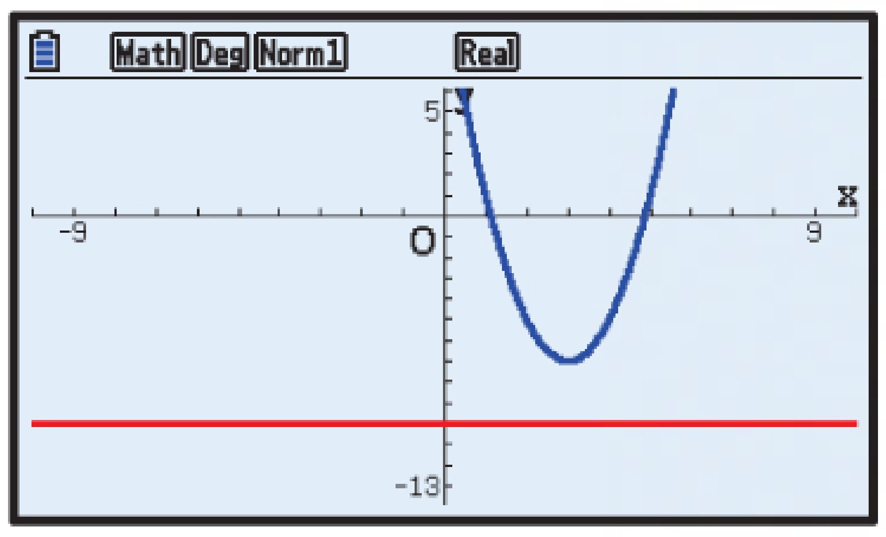
ii We graph $y = 2x^2 - 12x + 11$ and $y = -7$ on the same set of axes.



The graphs intersect (touch) at $(3, -7)$.

$\therefore$ the solution is $x = 3$.

iii We graph $y = 2x^2 - 12x + 11$ and $y = -10$ on the same set of axes.



The graphs do not intersect.

$\therefore$ there are no real solutions.

b $2x^2 - 12x + 11 = k$

$\therefore$ $2x^2 - 12x + (11-k) = 0$ has $a = 2$, $b = -12$, $c = 11 - k$

$$\begin{aligned}\therefore\ \Delta &= b^2 - 4ac \\ &= (-12)^2 - 4(2)(11-k) \\ &= 144 - 88 + 8k \\ &= 8k + 56\end{aligned}$$

i For two real solutions, $\Delta > 0$

$\therefore$ $8k + 56 > 0$

$\therefore$ $8k > -56$

$\therefore$ $k > -7$

ii For exactly one real solution, $\Delta = 0$

$\therefore$ $8k + 56 = 0$

$\therefore$ $8k = -56$

$\therefore$ $k = -7$

iii For no real solutions, $\Delta < 0$

$\therefore$ $8k + 56 < 0$

$\therefore$ $8k < -56$

$\therefore$ $k < -7$

which agrees with our results in **a**.

REVIEW SET 4A

1 **a** $2x^2 = 38$
$\therefore\ x^2 = 19$ {dividing both sides by 2}
$\therefore\ x = \pm\sqrt{19}$

b $(x-2)^2 = 25$
$\therefore\ x - 2 = \pm\sqrt{25}$
$\therefore\ x - 2 = \pm 5$
$\therefore\ x = 2 \pm 5$
$\therefore\ x = 7$ or -3

c $3(x-\sqrt{2})^2 = 6$
$\therefore\ (x-\sqrt{2})^2 = 2$ {dividing both sides by 3}
$\therefore\ x - \sqrt{2} = \pm\sqrt{2}$
$\therefore\ x = \sqrt{2} \pm \sqrt{2}$
$\therefore\ x = 2\sqrt{2}$ or 0

2 **a** $x^4 = -9$
has no real solutions as x^4 cannot be negative.

b $x^3 = \frac{1}{27}$
$\therefore\ x = \sqrt[3]{\frac{1}{27}}$
$\therefore\ x = \frac{1}{3}$

c $(x-1)^5 = 2$
$\therefore\ x - 1 = \sqrt[5]{2}$
$\therefore\ x = 1 + \sqrt[5]{2}$

3 **a** $x(x+2) = 0$
$\therefore\ x = 0$ or $x + 2 = 0$
$\therefore\ x = 0$ or -2

b $-(x+3)(2x-7) = 0$
$\therefore\ (x+3)(2x-7) = 0$
$\therefore\ x + 3 = 0$ or $2x - 7 = 0$
$\therefore\ x = -3$ or $\frac{7}{2}$

c $(x+5)(x+1)(x-6) = 0$
$\therefore\ x + 5 = 0$ or $x + 1 = 0$ or $x - 6 = 0$
$\therefore\ x = -5, -1,$ or 6

4 **a** $3x^2 - 5x = 0$
$\therefore\ x(3x-5) = 0$
$\therefore\ x = 0$ or $3x - 5 = 0$
$\therefore\ x = 0$ or $\frac{5}{3}$

b $x^2 - 4x - 5 = 0$
$\therefore\ (x+1)(x-5) = 0$
$\therefore\ x = -1$ or 5

c $x^2 + 6x + 9 = 0$
$\therefore\ (x+3)^2 = 0$
$\therefore\ x = -3$

d $4x - 3 = x^2$
$\therefore\ x^2 - 4x + 3 = 0$
$\therefore\ (x-1)(x-3) = 0$
$\therefore\ x = 1$ or 3

e $3x^2 = 2 - 5x$
$\therefore\ 3x^2 + 5x - 2 = 0$
$\therefore\ (3x-1)(x+2) = 0$
$\therefore\ x = \frac{1}{3}$ or -2

f $2x^2 - 108 = 6x$
$\therefore\ 2x^2 - 6x - 108 = 0$
$\therefore\ 2(x^2 - 3x - 54) = 0$
$\therefore\ 2(x+6)(x-9) = 0$
$\therefore\ (x+6)(x-9) = 0$
$\therefore\ x = -6$ or 9

5 **a**
$$(x+3)^2 = 5x+29$$
$$\therefore\ x^2+6x+9 = 5x+29$$
$$\therefore\ x^2+x-20=0$$
$$\therefore\ (x+5)(x-4)=0$$
$$\therefore\ x=-5 \text{ or } 4$$

b
$$x(x-4)-(x-6)=0$$
$$\therefore\ x^2-4x-x+6=0$$
$$\therefore\ x^2-5x+6=0$$
$$\therefore\ (x-2)(x-3)=0$$
$$\therefore\ x=2 \text{ or } 3$$

c
$$(1-2x)(4-x)=39$$
$$\therefore\ 4-x-8x+2x^2=39$$
$$\therefore\ 2x^2-9x-35=0$$
$$\therefore\ (2x+5)(x-7)=0$$
$$\therefore\ x=-\tfrac{5}{2} \text{ or } 7$$

6 **a** $x^2-7x+2=0$
has $a=1,\ b=-7,\ c=2$
$$\therefore\ x=\frac{-(-7)\pm\sqrt{(-7)^2-4(1)(2)}}{2(1)}$$
$$\therefore\ x=\frac{7\pm\sqrt{41}}{2}$$

b $-x^2+2x-4=0$
has $a=-1,\ b=2,\ c=-4$
$$\therefore\ x=\frac{-2\pm\sqrt{2^2-4(-1)(-4)}}{2(-1)}$$
$$\therefore\ x=\frac{-2\pm\sqrt{-12}}{-2}$$
which has no real solutions as $\sqrt{-12}$ is not real.

c $-3x^2-x+3=0$
has $a=-3,\ b=-1,\ c=3$
$$\therefore\ x=\frac{-(-1)\pm\sqrt{(-1)^2-4(-3)(3)}}{2(-3)}$$
$$\therefore\ x=\frac{1\pm\sqrt{37}}{-6}$$
$$\therefore\ x=\frac{-1\pm\sqrt{37}}{6}$$

7 $6x^2-x-2=0$ has $a=6,\ b=-1,\ c=-2$

a
$$\Delta=b^2-4ac$$
$$=(-1)^2-4(6)(-2)$$
$$=49$$
Since $\Delta>0$, and 49 is a square, there are 2 distinct rational roots.

b
$$x=\frac{-b\pm\sqrt{\Delta}}{2a}$$
$$\therefore\ x=\frac{-(-1)\pm\sqrt{49}}{2(6)}$$
$$\therefore\ x=\frac{1\pm7}{12}$$
$$\therefore\ x=\tfrac{8}{12} \text{ or } -\tfrac{6}{12}$$
$$\therefore\ x=\tfrac{2}{3} \text{ or } -\tfrac{1}{2}$$
So there are 2 distinct rational roots as expected.

8 $2x^2-5x+4=0$ has $a=2,\ b=-5,\ c=4$
$$\Delta=b^2-4ac$$
$$=(-5)^2-4(2)(4)$$
$$=-7$$
Since $\Delta<0$, there are no real roots.

9 $2x^2 - 3x + m = 0$ has $a = 2,\ b = -3,\ c = m$

$$\begin{aligned}\Delta &= b^2 - 4ac \\ &= (-3)^2 - 4(2)(m) \\ &= 9 - 8m\end{aligned}$$

a For a repeated root,
$$\Delta = 0$$
$$\therefore\ 9 - 8m = 0$$
$$\therefore\ -8m = -9$$
$$\therefore\ m = \tfrac{9}{8}$$

b For two distinct real roots,
$$\Delta > 0$$
$$\therefore\ 9 - 8m > 0$$
$$\therefore\ -8m > -9$$
$$\therefore\ m < \tfrac{9}{8}$$

c For no real roots,
$$\Delta < 0$$
$$\therefore\ 9 - 8m < 0$$
$$\therefore\ -8m < -9$$
$$\therefore\ m > \tfrac{9}{8}$$

10 For all quadratic equations with roots $\frac{1}{2}$ and -3,

the sum of the roots $= \frac{1}{2} + (-3) = -\frac{5}{2}$

and the product of the roots $= (\frac{1}{2})(-3) = -\frac{3}{2}$

So, we have $-\dfrac{b}{a} = -\dfrac{5}{2}$ and $\dfrac{c}{a} = -\dfrac{3}{2}$.

The simplest solution is $a = 2,\ b = 5,\ c = -3$.

$\therefore$ all quadratic equations with roots $\frac{1}{2}$ and -3 are of the form $k(2x^2 + 5x - 3) = 0,\quad k \in \mathbb{R},\quad k \neq 0$.

11 If α and β are the roots of $2x^2 - 3x = 4$, which can be rewritten as $2x^2 - 3x - 4 = 0$, then
$\alpha + \beta = -\dfrac{(-3)}{2} = \dfrac{3}{2}$ and $\alpha\beta = \dfrac{-4}{2} = -2$.

For the quadratic equation with roots $\dfrac{1}{\alpha}$ and $\dfrac{1}{\beta}$,

the sum of the roots
$$= \frac{1}{\alpha} + \frac{1}{\beta} = \frac{\beta + \alpha}{\alpha\beta} = \frac{\frac{3}{2}}{-2} = -\tfrac{3}{4}$$

and the product of the roots
$$= \left(\frac{1}{\alpha}\right)\left(\frac{1}{\beta}\right) = \frac{1}{\alpha\beta} = \frac{1}{-2} = -\tfrac{1}{2}$$

So, we have $-\dfrac{b}{a} = -\dfrac{3}{4}$ and $\dfrac{c}{a} = -\dfrac{1}{2} = -\dfrac{2}{4}$.

The simplest solution is $a = 4,\ b = 3,\ c = -2$.

$\therefore$ the quadratic equation is $4x^2 + 3x - 2 = 0$.

12 $kx^2 + (1 - 3k)x + (k - 6) = 0$ has $a = k,\ \ b = 1 - 3k,\ \ c = k - 6$

If α and $-\frac{1}{\alpha}$ are the roots, then $\alpha\left(-\frac{1}{\alpha}\right) = \frac{c}{a} = \frac{k-6}{k}$

$$\therefore\ -1 = \frac{k-6}{k}$$
$$\therefore\ -k = k - 6$$
$$\therefore\ -2k = -6$$
$$\therefore\ k = 3$$

For $k = 3$, the equation becomes $3x^2 + (1 - 3(3))x + (3 - 6) = 0$

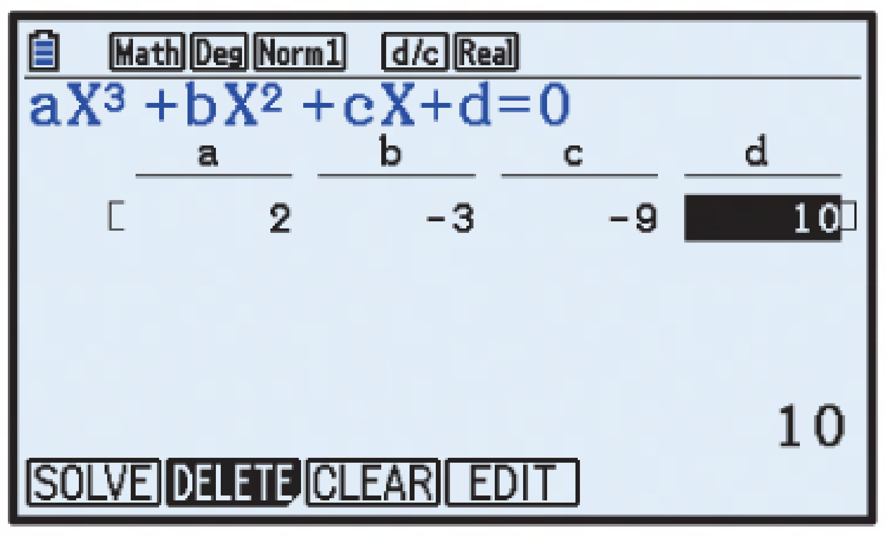
$$\therefore\ 3x^2 - 8x - 3 = 0$$
$$\therefore\ (3x + 1)(x - 3) = 0$$
$$\therefore\ x = -\tfrac{1}{3} \text{ or } 3$$

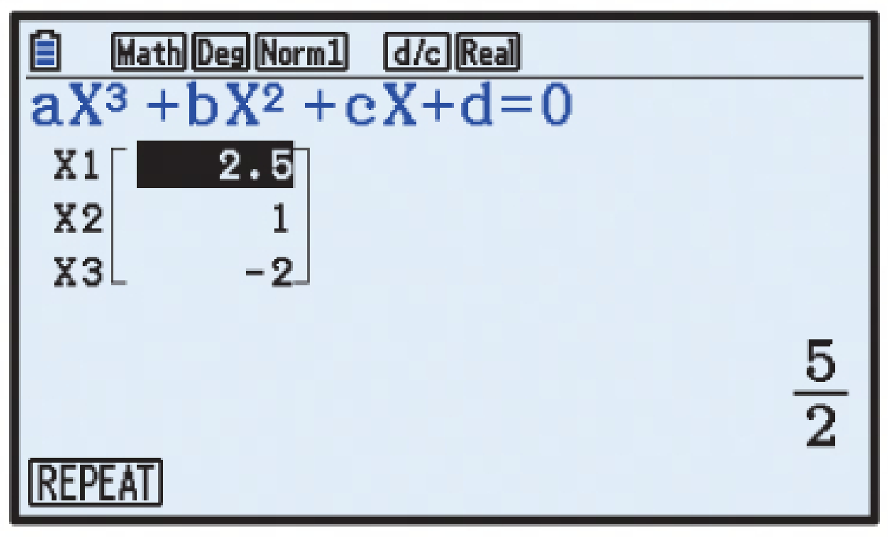
So, $k = 3$ and the two roots of the equation are $-\frac{1}{3}$ and 3.

13 **a** $2x^3 - 3x^2 - 9x + 10 = 0$

Using technology,

$x = \frac{5}{2}$, 1, or -2

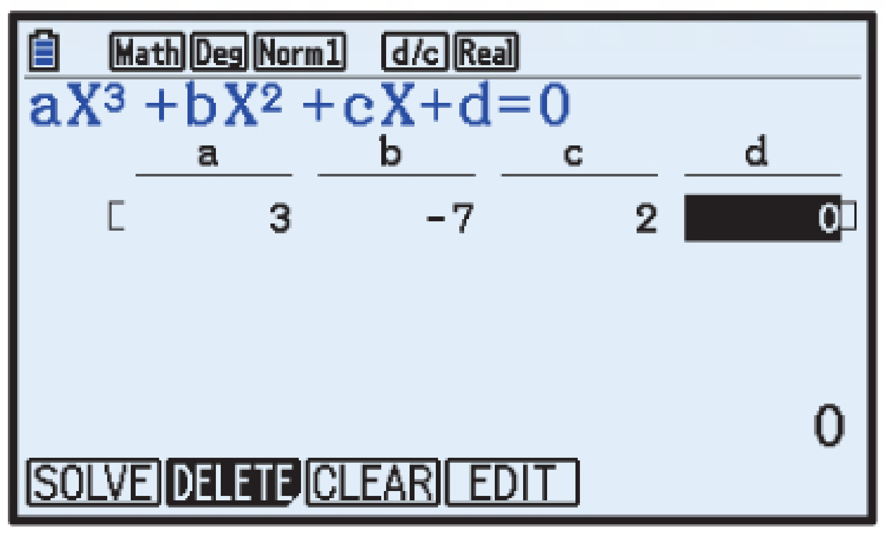


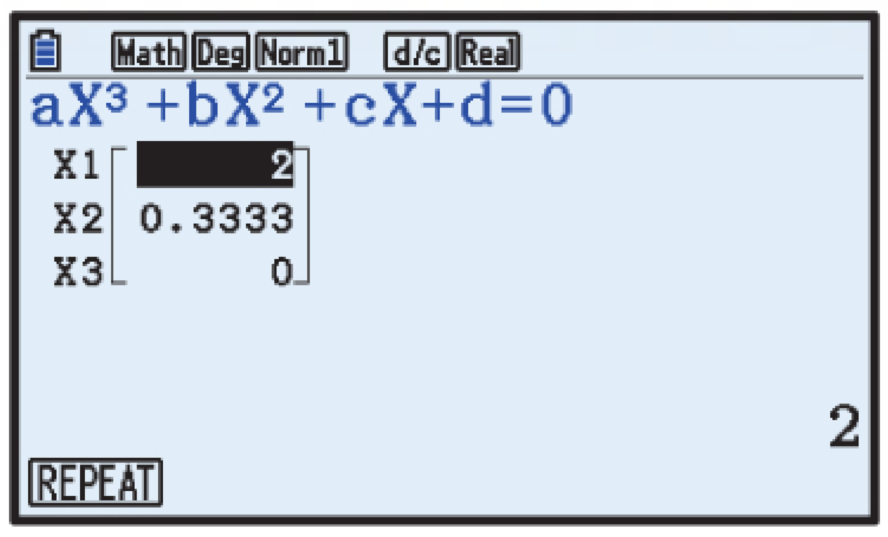


b $$3x^3 = x(7x - 2)$$
$$\therefore\ 3x^3 = 7x^2 - 2x$$
$$\therefore\ 3x^3 - 7x^2 + 2x = 0$$

Using technology,

$x = 2$, $\frac{1}{3}$, or 0

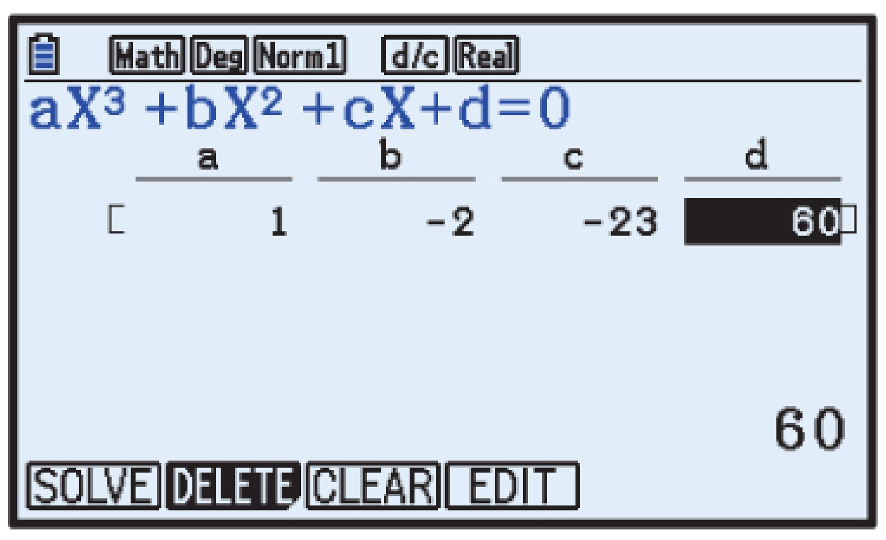


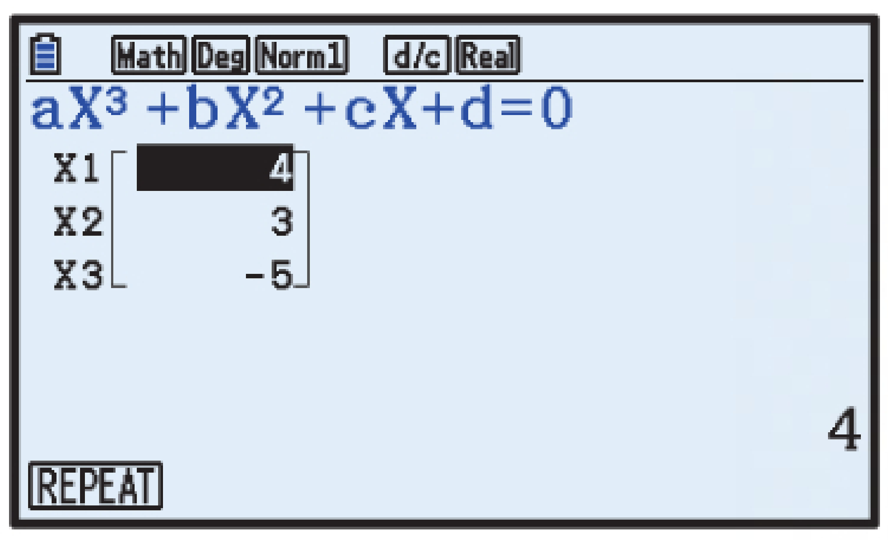


c $$x^3 + 60 = 23x + 2x^2$$
$$\therefore\ x^3 - 2x^2 - 23x + 60 = 0$$

Using technology,

$x = 4$, 3, or -5

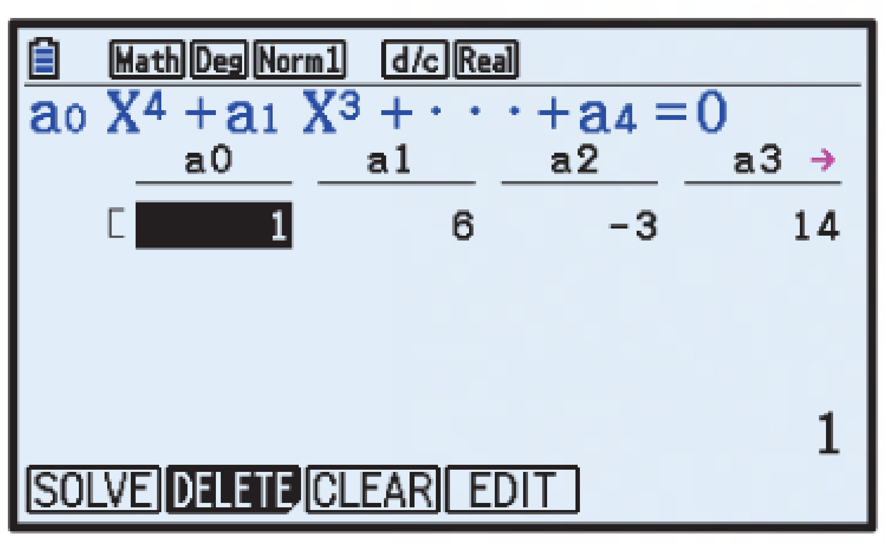


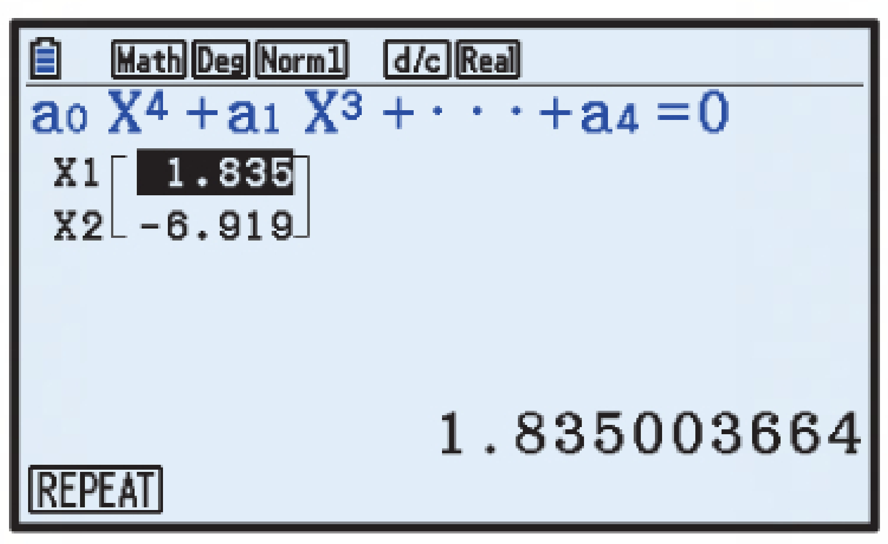


d $$x^2(x^2 - 3) = 64 - 6x^3 - 14x$$
$$\therefore\ x^4 - 3x^2 - 64 + 6x^3 + 14x = 0$$
$$\therefore\ x^4 + 6x^3 - 3x^2 + 14x - 64 = 0$$

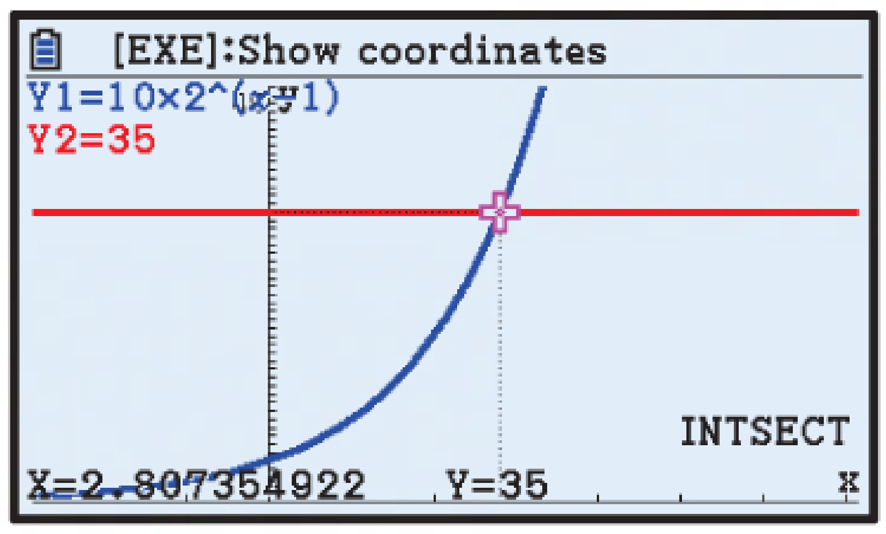
Using technology,

$x \approx 1.84$ or -6.92



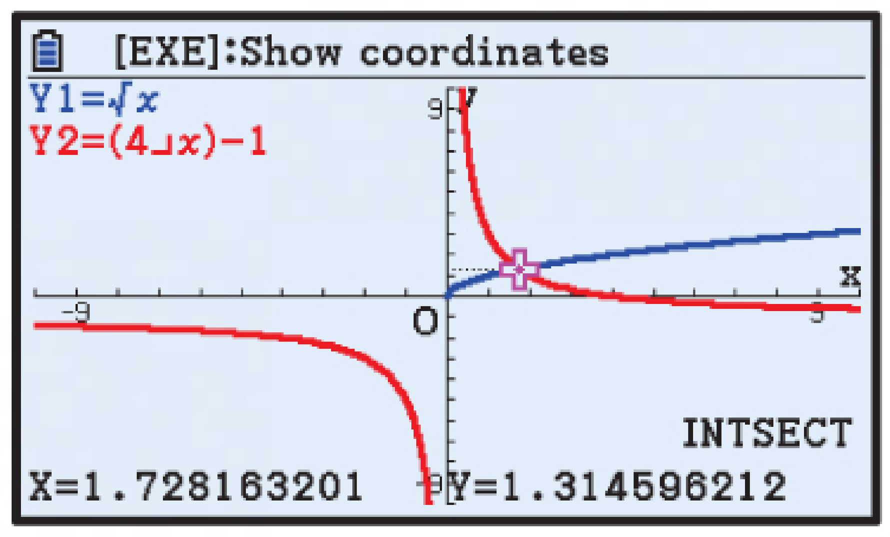


14 **a** We graph $y = 10 \times 2^{x-1}$ and $y = 35$ on the same set of axes.



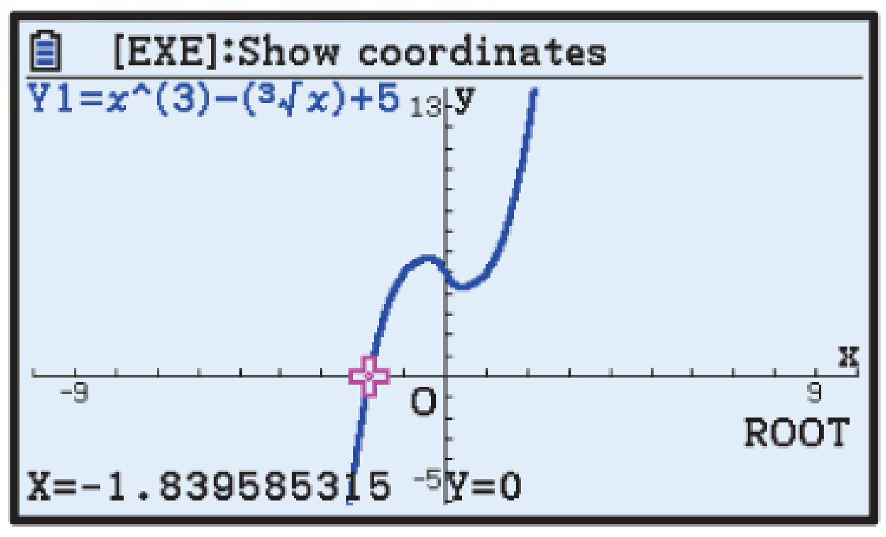
The graphs intersect at $(2.81, 35)$.
$\therefore$ the solution is $x \approx 2.81$.

b We graph $y = \sqrt{x}$ and $y = \dfrac{4}{x} - 1$ on the same set of axes.



The graphs intersect at $(1.73, 1.31)$.
$\therefore$ the solution is $x \approx 1.73$.

c We graph $y = x^3 - \sqrt[3]{x} + 5$.



The x-intercept is ≈ -1.84.
$\therefore$ the solution is $x \approx -1.84$.

REVIEW SET 4B

1 **a** $-7x^2 = 0$
$\therefore x^2 = 0$
$\therefore x = 0$

b $-4x^3 = \frac{125}{2}$
$\therefore x^3 = -\frac{125}{8}$
$\therefore x = \sqrt[3]{-\frac{125}{8}}$
$\therefore x = \dfrac{\sqrt[3]{-125}}{\sqrt[3]{8}}$
$\therefore x = -\frac{5}{2}$

c $(x - \sqrt{3})^2 = 16$
$\therefore x - \sqrt{3} = \pm\sqrt{16}$
$\therefore x - \sqrt{3} = \pm 4$
$\therefore x = \sqrt{3} \pm 4$

2 **a** $x^4 = \frac{81}{16}$
$\therefore x = \pm\sqrt[4]{\frac{81}{16}}$
$\therefore x = \pm\dfrac{\sqrt[4]{81}}{\sqrt[4]{16}}$
$\therefore x = \pm\frac{3}{2}$

b $x^5 = -18$
$\therefore x = \sqrt[5]{-18}$

c $(x-1)^{-2} = 4$
$\therefore (x-1)^2 = \frac{1}{4}$
$\therefore x - 1 = \pm\sqrt{\frac{1}{4}}$
$\therefore x - 1 = \pm\frac{1}{2}$
$\therefore x = 1 \pm \frac{1}{2}$
$\therefore x = \frac{3}{2}$ or $\frac{1}{2}$

3 **a** $\dfrac{p}{q} = 0$
$\therefore p = 0, \; q \neq 0$

b $\dfrac{2xz}{y} = 0$
$\therefore 2xz = 0, \; y \neq 0$
$\therefore xz = 0, \; y \neq 0$
$\therefore x = 0$ or $z = 0, \; y \neq 0$

c $-\dfrac{5}{ab} = 0$
$\therefore -5 = 0$
which is impossible
$\therefore$ there are no solutions.

4 **a**
$$2x^2 - 5x = 0$$
$$\therefore\ x(2x-5) = 0$$
$$\therefore\ x = 0 \text{ or } 2x - 5 = 0$$
$$\therefore\ x = 0 \text{ or } \tfrac{5}{2}$$

b
$$3x^2 - 12x = 0$$
$$\therefore\ 3x(x-4) = 0$$
$$\therefore\ 3x = 0 \text{ or } x - 4 = 0$$
$$\therefore\ x = 0 \text{ or } 4$$

c
$$x^2 - 7x + 6 = 0$$
$$\therefore\ (x-1)(x-6) = 0$$
$$\therefore\ x = 1 \text{ or } 6$$

d
$$x^2 + 4 = -4x$$
$$\therefore\ x^2 + 4x + 4 = 0$$
$$\therefore\ (x+2)^2 = 0$$
$$\therefore\ x = -2$$

e
$$x^2 - 12 = 4x$$
$$\therefore\ x^2 - 4x - 12 = 0$$
$$\therefore\ (x+2)(x-6) = 0$$
$$\therefore\ x = -2 \text{ or } 6$$

f
$$3x^2 - x - 10 = 0$$
$$\therefore\ (3x+5)(x-2) = 0$$
$$\therefore\ x = -\tfrac{5}{3} \text{ or } 2$$

5 **a**
$$x = \frac{9}{x}$$
$$\therefore\ x^2 = 9$$
$$\therefore\ x^2 - 9 = 0$$
$$\therefore\ (x+3)(x-3) = 0$$
$$\therefore\ x = \pm 3$$

b
$$\frac{1}{x} = \frac{3}{x^2} - 2$$
$$\therefore\ x = 3 - 2x^2$$
$$\therefore\ 2x^2 + x - 3 = 0$$
$$\therefore\ (2x+3)(x-1) = 0$$
$$\therefore\ x = -\tfrac{3}{2} \text{ or } 1$$

c
$$3x - 1 = \frac{2}{x}$$
$$\therefore\ 3x^2 - x = 2$$
$$\therefore\ 3x^2 - x - 2 = 0$$
$$\therefore\ (3x+2)(x-1) = 0$$
$$\therefore\ x = -\tfrac{2}{3} \text{ or } 1$$

6 **a** $x^2 + 5x + 3 = 0$
has $a = 1,\ b = 5,\ c = 3$
$$\therefore\ x = \frac{-5 \pm \sqrt{5^2 - 4(1)(3)}}{2(1)}$$
$$\therefore\ x = \frac{-5 \pm \sqrt{13}}{2}$$

b $3x^2 + 11x - 2 = 0$
has $a = 3,\ b = 11,\ c = -2$
$$\therefore\ x = \frac{-11 \pm \sqrt{11^2 - 4(3)(-2)}}{2(3)}$$
$$\therefore\ x = \frac{-11 \pm \sqrt{145}}{6}$$

7 **a**
$$x^2 - 6x + 4 = 0$$
$$\therefore\ x^2 - 6x = -4$$
$$\therefore\ x^2 - 6x + (-3)^2 = -4 + (-3)^2$$
$$\therefore\ (x-3)^2 = 5$$
$$\therefore\ x - 3 = \pm\sqrt{5}$$
$$\therefore\ x = 3 \pm \sqrt{5}$$

b
$$2x^2 + 8x = 1$$
$$\therefore\ x^2 + 4x = \tfrac{1}{2}$$
$$\therefore\ x^2 + 4x + 2^2 = \tfrac{1}{2} + 2^2$$
$$\therefore\ (x+2)^2 = \tfrac{9}{2}$$
$$\therefore\ x + 2 = \pm\sqrt{\tfrac{9}{2}}$$
$$\therefore\ x + 2 = \pm\frac{\sqrt{9}}{\sqrt{2}}$$
$$\therefore\ x + 2 = \pm\frac{3}{\sqrt{2}}$$
$$\therefore\ x = -2 \pm \frac{3}{\sqrt{2}}$$

c

$$
\begin{aligned}
4x^2 - 5x &= 6 \\
\therefore \quad x^2 - \tfrac{5}{4}x &= \tfrac{3}{2} \\
\therefore \quad x^2 - \tfrac{5}{4}x + (-\tfrac{5}{8})^2 &= \tfrac{3}{2} + (-\tfrac{5}{8})^2 \\
\therefore \quad (x - \tfrac{5}{8})^2 &= \tfrac{3}{2} + \tfrac{25}{64} \\
\therefore \quad (x - \tfrac{5}{8})^2 &= \tfrac{121}{64} \\
\therefore \quad x - \tfrac{5}{8} &= \pm\sqrt{\tfrac{121}{64}} \\
\therefore \quad x - \tfrac{5}{8} &= \pm\tfrac{\sqrt{121}}{\sqrt{64}} \\
\therefore \quad x - \tfrac{5}{8} &= \pm\tfrac{11}{8} \\
\therefore \quad x &= \tfrac{5}{8} \pm \tfrac{11}{8} \\
\therefore \quad x &= \tfrac{16}{8} \text{ or } -\tfrac{6}{8} \\
\therefore \quad x &= 2 \text{ or } -\tfrac{3}{4}
\end{aligned}
$$

8 **a** $x^2 - 8x + 16 = 0$
has $a = 1$, $b = -8$, $c = 16$

$$
\begin{aligned}
\Delta &= b^2 - 4ac \\
&= (-8)^2 - 4(1)(16) \\
&= 0
\end{aligned}
$$

Since $\Delta = 0$, there is one repeated root.

b $2x^2 - x - 5 = 0$
has $a = 2$, $b = -1$, $c = -5$

$$
\begin{aligned}
\Delta &= b^2 - 4ac \\
&= (-1)^2 - 4(2)(-5) \\
&= 41
\end{aligned}
$$

Since $\Delta > 0$, but 41 is not a square, there are 2 distinct irrational roots.

c $3x^2 + 5x + 3 = 0$
has $a = 3$, $b = 5$, $c = 3$

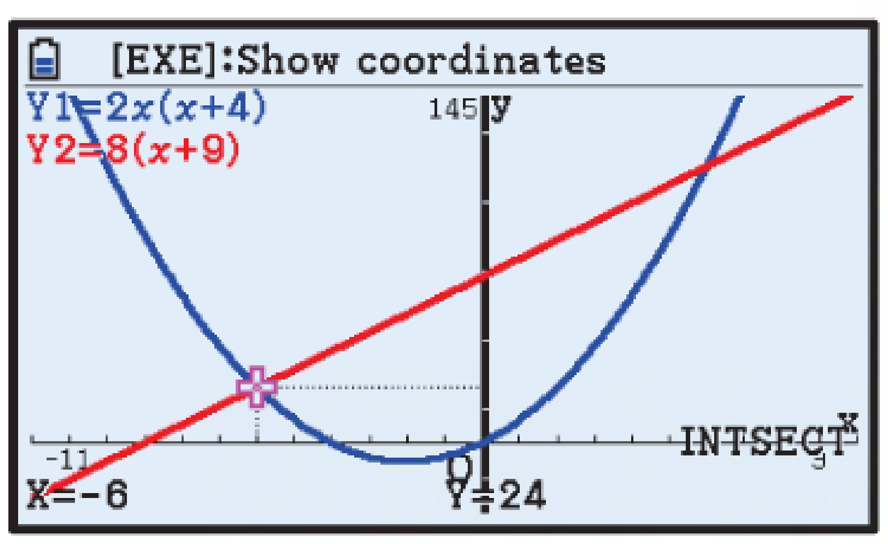
$$
\begin{aligned}
\Delta &= b^2 - 4ac \\
&= 5^2 - 4(3)(3) \\
&= -11
\end{aligned}
$$

Since $\Delta < 0$, there are no real roots.

9 **a** **i** $2x(x + 4) = 8(x + k)$
When $k = 9$,

$$
\begin{aligned}
2x(x + 4) &= 8(x + 9) \\
\therefore \quad 2x^2 + 8x &= 8x + 72 \\
\therefore \quad 2x^2 &= 72 \\
\therefore \quad x^2 &= 36 \\
\therefore \quad x &= \pm\sqrt{36} \\
\therefore \quad x &= \pm 6
\end{aligned}
$$

ii $2x(x + 4) = 8(x + k)$
When $k = 9$, $2x(x + 4) = 8(x + 9)$
We graph $y = 2x(x + 4)$ and $y = 8(x + 9)$ on the same set of axes.



The graphs intersect at $(-6,\ 24)$ and $(6,\ 120)$.
$\therefore$ the solutions are $x = -6$ or 6.

b $2x(x+4) = 8(x+k)$

$\therefore\ 2x^2 + 8x = 8x + 8k$

$\therefore\ 2x^2 = 8k$

$\therefore\ x^2 = 4k$

c $x^2 = 4k$

$\therefore\ x = \pm\sqrt{4k}$

$\therefore\ x = \pm 2\sqrt{k}$

i The equation $x^2 = 4k$ has two real solutions if $k > 0$.

ii The equation $x^2 = 4k$ has one real solution if $k = 0$.

iii The equation $x^2 = 4k$ has no real solutions if $k < 0$.

10 $ax^2 + bx + c = 0,\quad a \neq 0$

$$\therefore\ x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$\begin{aligned}\text{The sum of the solutions} &= \frac{-b + \sqrt{b^2-4ac}}{2a} + \frac{-b - \sqrt{b^2-4ac}}{2a}\\ &= \frac{-2b}{2a}\\ &= -\frac{b}{a},\quad a \neq 0\end{aligned}$$

11 $2x^2 + kx + 12 = 0$ has $a = 2,\ b = k,\ c = 12$

If α and 3α are the roots, then $\alpha + 3\alpha = -\dfrac{b}{a} = -\dfrac{k}{2}$

$$\therefore\ 4\alpha = -\frac{k}{2}$$

$$\therefore\ k = -8\alpha \quad \text{.... (1)}$$

and $\alpha(3\alpha) = \dfrac{c}{a} = \dfrac{12}{2}$

$$\therefore\ 3\alpha^2 = 6$$

$$\therefore\ \alpha^2 = 2 \quad \text{.... (2)}$$

Using (1), $\alpha = -\dfrac{k}{8}$

Substituting $\alpha = -\dfrac{k}{8}$ into (2) gives $\left(-\dfrac{k}{8}\right)^2 = 2$

$$\therefore\ \frac{k^2}{64} = 2$$

$$\therefore\ k^2 = 128$$

$$\therefore\ k = \pm 8\sqrt{2}$$

When $k = 8\sqrt{2}$, $\alpha = -\dfrac{8\sqrt{2}}{8} = -\sqrt{2}$

and $3\alpha = -3\sqrt{2}$

So, the two roots are $-\sqrt{2}$ and $-3\sqrt{2}$.

When $k = -8\sqrt{2}$, $\alpha = -\dfrac{(-8\sqrt{2})}{8} = \sqrt{2}$

and $3\alpha = 3\sqrt{2}$

So, the two roots are $\sqrt{2}$ and $3\sqrt{2}$.

12 If p and q are the roots of $4x^2 - 3x - 3 = 0$, then $p + q = -\frac{(-3)}{4} = \frac{3}{4}$ and $pq = \frac{-3}{4} = -\frac{3}{4}$.

For all quadratic equations with roots p^3 and q^3,

the sum of the roots $= p^3 + q^3$
$$= (p+q)^3 - 3p^2q - 3pq^2$$
$$= (p+q)^3 - 3(pq)(p+q)$$
$$= (\tfrac{3}{4})^3 - 3(-\tfrac{3}{4})(\tfrac{3}{4})$$
$$= \tfrac{27}{64} + \tfrac{27}{16}$$
$$= \tfrac{135}{64}$$

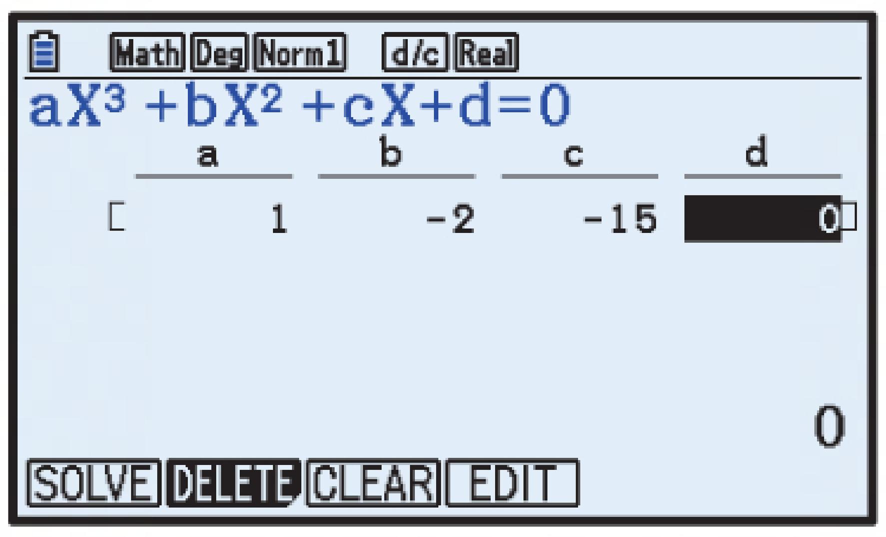
and the product of the roots $= p^3q^3$
$$= (pq)^3$$
$$= (-\tfrac{3}{4})^3$$
$$= -\tfrac{27}{64}$$

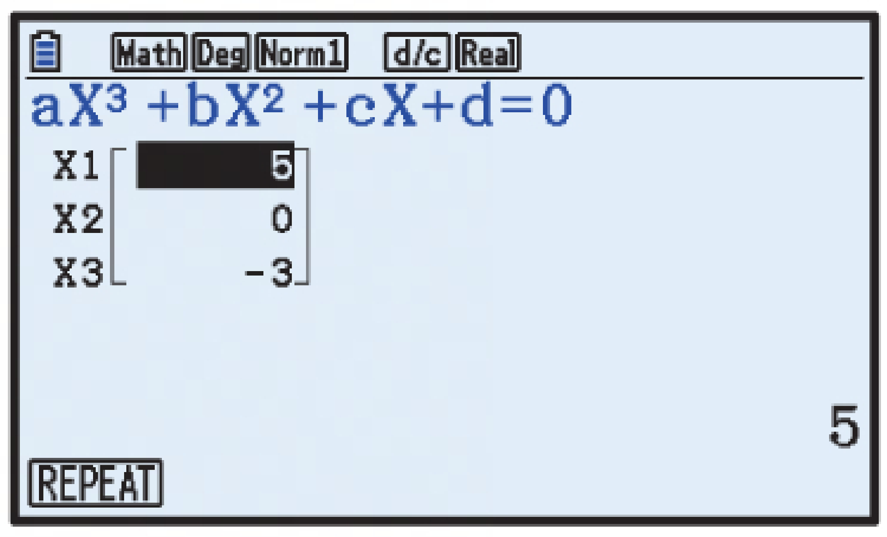
So, we have $-\frac{b}{a} = \frac{135}{64}$ and $\frac{c}{a} = -\frac{27}{64}$.

The simplest solution is $a = 64$, $b = -135$, $c = -27$.

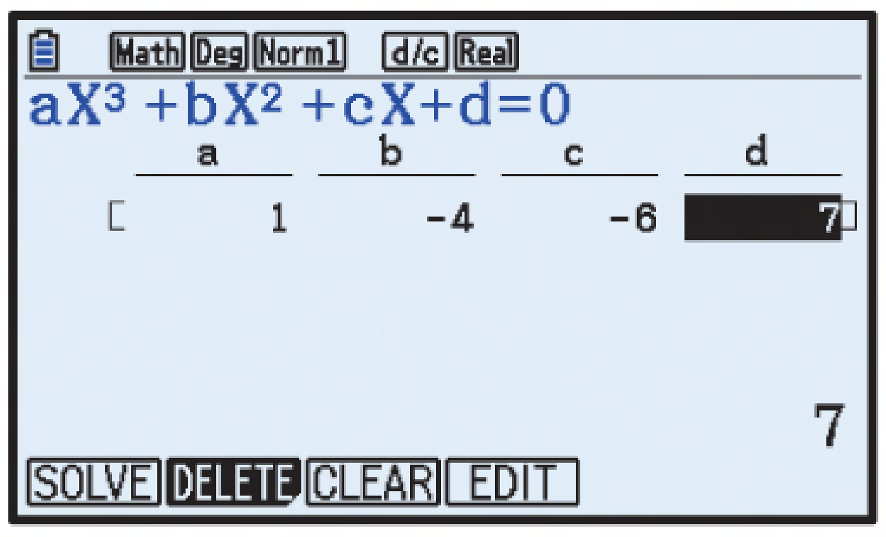
$\therefore$ all quadratic equations with roots p^3 and q^3 are of the form
$k(64x^2 - 135x - 27) = 0$, $k \in \mathbb{R}$, $k \neq 0$.

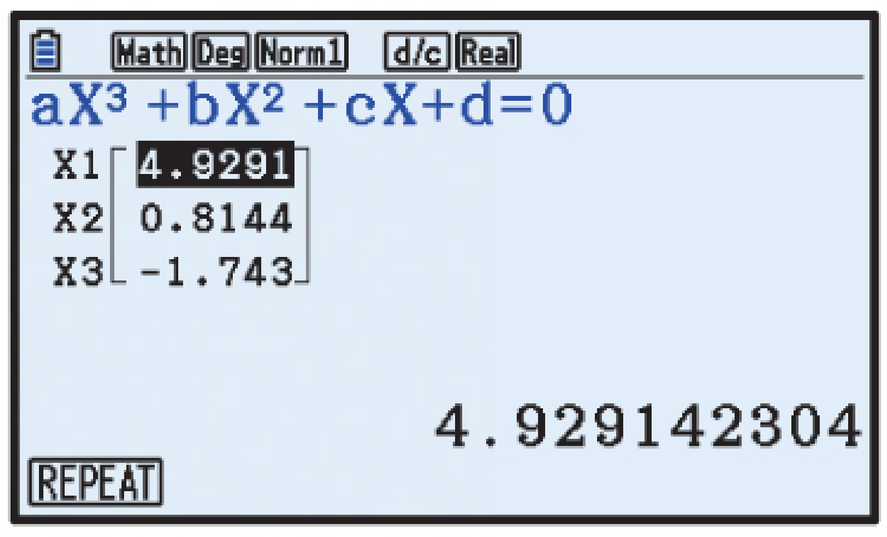
13 **a** $x^3 - 15x = 2x^2$
$\therefore$ $x^3 - 2x^2 - 15x = 0$
Using technology,
$x = 5$, 0, or -3



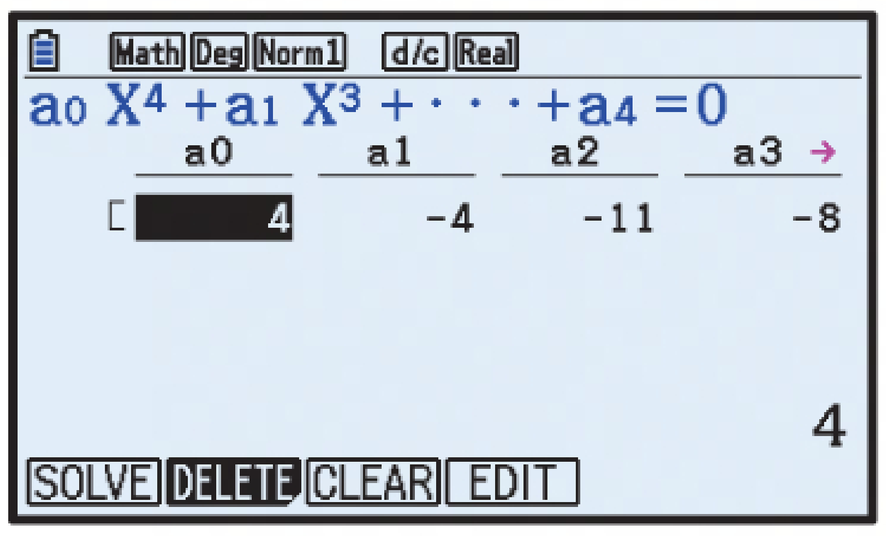


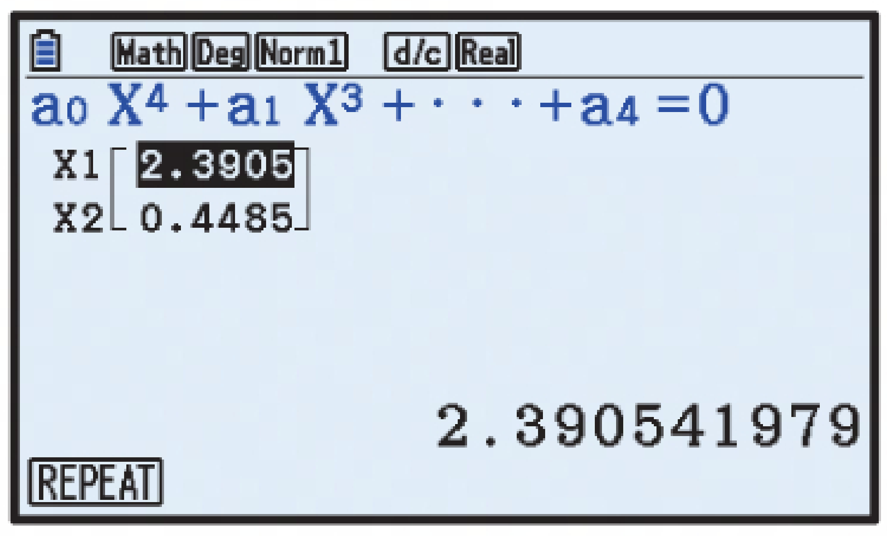
b $x^3 - 4x^2 - 6x + 7 = 0$
Using technology,
$x \approx 4.93$, 0.814, or -1.74



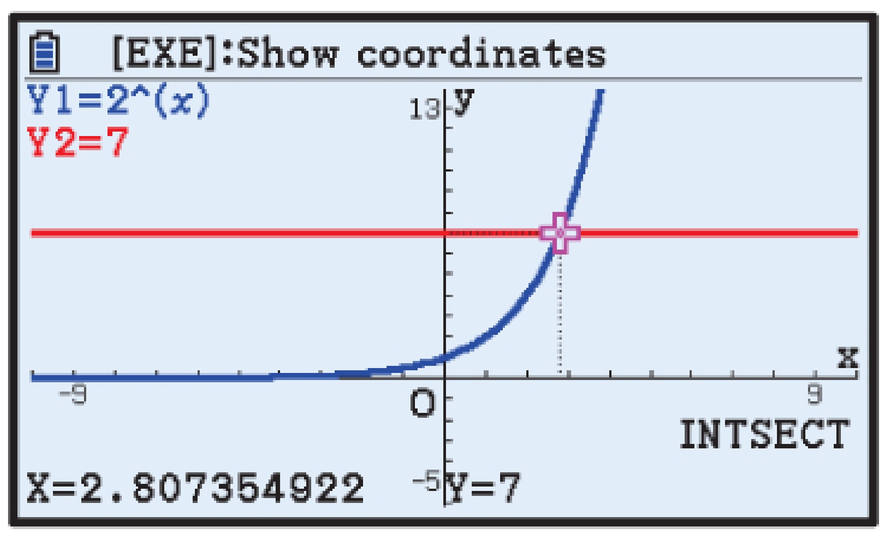


c $4x^4 - 4x^3 - 11x^2 - 8x + 6 = 0$
Using technology,
$x \approx 2.39$ or 0.449



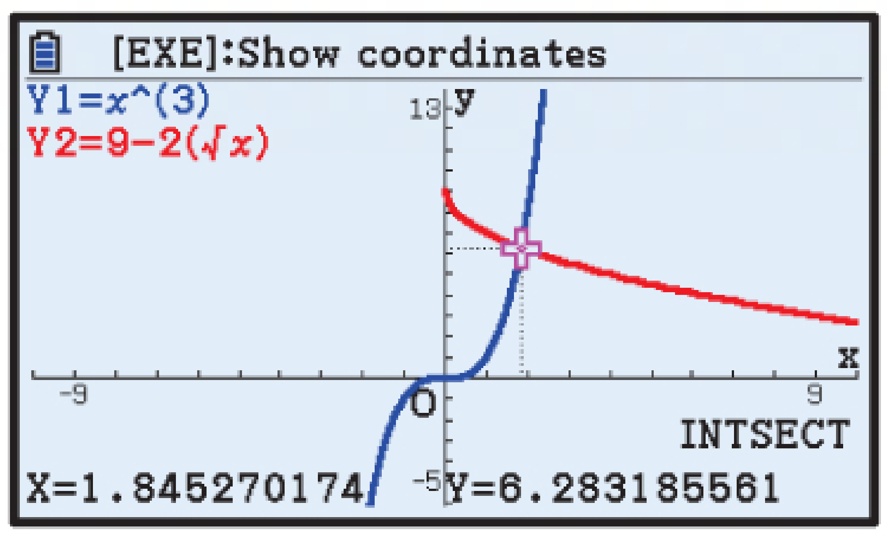


14 **a** We graph $y = 2^x$ and $y = 7$ on the same set of axes.



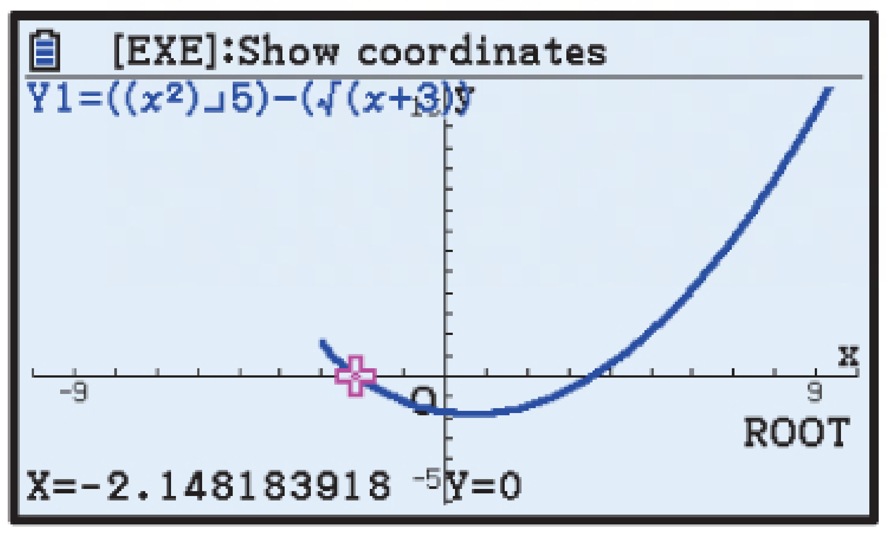
The graphs intersect at $(2.81, 7)$.
$\therefore$ the solution is $x \approx 2.81$.

b We graph $y = x^3$ and $y = 9 - 2\sqrt{x}$ on the same set of axes.



The graphs intersect at $(1.85, 6.28)$.
$\therefore$ the solution is $x \approx 1.85$.

c We graph $y = \dfrac{x^2}{5} - \sqrt{x+3}$.



The x-intercepts are ≈ -2.15 and 3.58.

$\therefore$ the solutions are $x \approx -2.15$ or 3.58.

Chapter 5

SEQUENCES AND SERIES

EXERCISE 5A

1 **a** 4, 13, 22, 31
$+9$ $+9$ $+9$

b 45, 39, 33, 27
-6 -6 -6

c 2, 6, 18, 54
$\times 3$ $\times 3$ $\times 3$

d 96, 48, 24, 12
$\div 2$ $\div 2$ $\div 2$

2 2, 3, 5, 7, 11, 13, 17, 19, ...

a $u_2 = 3$ **b** $u_5 = 11$ **c** $u_{10} = 29$ {the 10th prime number}

3 4, 7, 10, 13, 16, ...

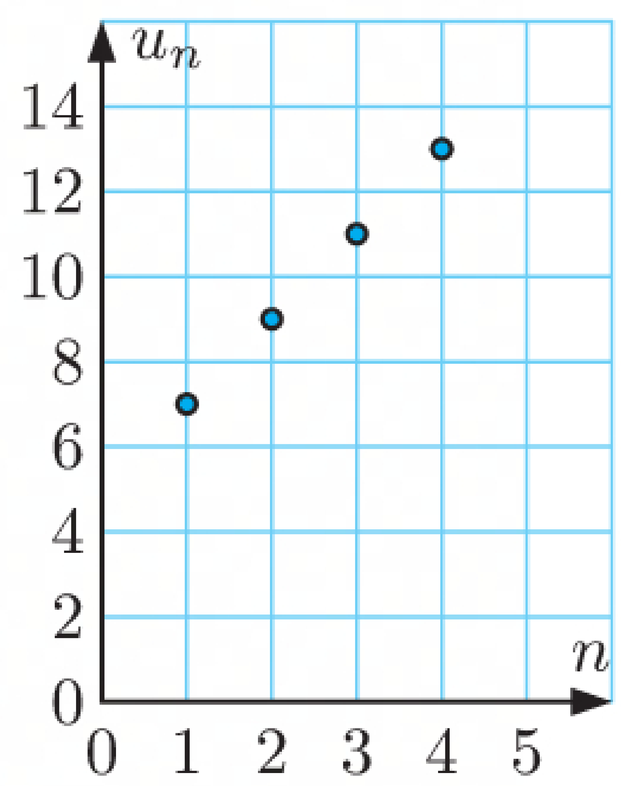
a We start with 4 and add 3 each time. **b** $u_1 = 4,\quad u_4 = 13$

c $$\begin{aligned} u_8 &= u_5 + 3 + 3 + 3 \\ &= 16 + 3 + 3 + 3 \\ &= 25 \end{aligned}$$

4 $u_n = 2n + 5$

a $$\begin{aligned} u_1 &= 2(1) + 5 \\ &= 7 \end{aligned} \qquad \begin{aligned} u_2 &= 2(2) + 5 \\ &= 9 \end{aligned} \qquad \begin{aligned} u_3 &= 2(3) + 5 \\ &= 11 \end{aligned} \qquad \begin{aligned} u_4 &= 2(4) + 5 \\ &= 13 \end{aligned}$$

b



5 $u_n = 3n - 2$

a $$\begin{aligned} u_1 &= 3(1) - 2 \\ &= 1 \end{aligned}$$ **b** $$\begin{aligned} u_5 &= 3(5) - 2 \\ &= 13 \end{aligned}$$ **c** $$\begin{aligned} u_{27} &= 3(27) - 2 \\ &= 79 \end{aligned}$$

6 $-9, -6, -1, 6, 15$

a Using **A**, $u_n = n - 10$

So, $$\begin{aligned} u_1 &= 1 - 10 \\ &= -9 \checkmark \end{aligned} \qquad \begin{aligned} u_2 &= 2 - 10 \\ &= -8 \times \end{aligned}$$

Using **B**, $u_n = n^2 - 10$

So, $$\begin{aligned} u_1 &= 1^2 - 10 \\ &= -9 \checkmark \end{aligned} \qquad \begin{aligned} u_2 &= 2^2 - 10 \\ &= -6 \checkmark \end{aligned} \qquad \begin{aligned} u_3 &= 3^2 - 10 \\ &= -1 \checkmark \end{aligned}$$

$$\begin{aligned} u_4 &= 4^2 - 10 \\ &= 6 \checkmark \end{aligned} \qquad \begin{aligned} u_5 &= 5^2 - 10 \\ &= 15 \checkmark \end{aligned}$$

Using **C**, $u_n = n^3 - 10$

So, $u_1 = 1^3 - 10$ $u_2 = 2^3 - 10$

$= -9$ ✓ $= -2$ ✗

So, **B** is the correct formula.

b $u_{20} = 20^2 - 10$

$= 390$

7 **a** 8, 16, 24, 32, ...

The sequence starts at 8 and each term is 8 more than the previous term.

The next two terms are 40 and 48.

b 2, 5, 8, 11, ...

The sequence starts at 2 and each term is 3 more than the previous term.

The next two terms are 14 and 17.

c 36, 31, 26, 21, ...

The sequence starts at 36 and each term is 5 less than the previous term.

The next two terms are 16 and 11.

d 96, 89, 82, 75, ...

The sequence starts at 96 and each term is 7 less than the previous term.

The next two terms are 68 and 61.

e 1, 4, 16, 64, ...

The sequence starts at 1 and each term is 4 times the previous term.

The next two terms are 256 and 1024.

f 2, 6, 18, 54, ...

The sequence starts at 2 and each term is 3 times the previous term.

The next two terms are 162 and 486.

g 480, 240, 120, 60, ...

The sequence starts at 480 and each term is half the previous term.

The next two terms are 30 and 15.

h 243, 81, 27, 9, ...

The sequence starts at 243 and each term is one third of the previous term.

The next two terms are 3 and 1.

i 50 000, 10 000, 2000, 400, ...

The sequence starts at 50 000 and each term is one fifth of the previous term.

The next two terms are 80 and 16.

8 **a** 1, 4, 9, 16, ...

Each term is the square of the term number. The next three terms are 25, 36, and 49.

b 1, 8, 27, 64, ...

Each term is the cube of the term number. The next three terms are 125, 216, and 343.

c 2, 6, 12, 20, ...

Each term is $n(n+1)$ where n is the term number. The next three terms are 30, 42, and 56.

9 **a** 95, 91, 87, 83, ...

Each term is 4 less than the previous term, so the next two terms are 79 and 75.

b 5, 20, 80, 320, ...
Each term is 4 times the previous term, so the next two terms are 1280 and 5120.

c 1, 16, 81, 256, ...
Each term is the fourth power of the term number, so the next two terms are $5^4 = 625$ and $6^4 = 1296$.

d 2, 3, 5, 7, 11, ...
This is the sequence of prime numbers, so the next two terms are 13 and 17.

e 2, 4, 7, 11, ...
$+2$ $+3$ $+4$
The difference between terms increases by 1 each time, so the next two terms are $11 + 5 = 16$ and $16 + 6 = 22$.

f 9, 8, 10, 7, 11, ...
Each odd numbered term is 1 more than the previous odd numbered term, and each even numbered term is 1 less than the previous even numbered term, so the next two terms are $7 - 1 = 6$ and $11 + 1 = 12$.

10 **a** The sequence $\{2n\}$ begins 2, 4, 6, 8, 10 (letting $n = 1, 2, 3, 4, 5, ...$).
b The sequence $\{2n - 3\}$ begins $-1, 1, 3, 5, 7$ (letting $n = 1, 2, 3, 4, 5, ...$).
c The sequence $\{2n + 11\}$ begins 13, 15, 17, 19, 21 (letting $n = 1, 2, 3, 4, 5, ...$).
d The sequence $\{4n - 3\}$ begins 1, 5, 9, 13, 17 (letting $n = 1, 2, 3, 4, 5, ...$).
e The sequence $\{2^n\}$ begins 2, 4, 8, 16, 32 (letting $n = 1, 2, 3, 4, 5, ...$).
f The sequence $\{6 \times (\frac{1}{2})^n\}$ begins $3, \frac{3}{2}, \frac{3}{4}, \frac{3}{8}, \frac{3}{16}$ (letting $n = 1, 2, 3, 4, 5, ...$).
g The sequence $\{(-2)^n\}$ begins $-2, 4, -8, 16, -32$ (letting $n = 1, 2, 3, 4, 5, ...$).
h The sequence $\{15 - (-2)^n\}$ begins $17, 11, 23, -1, 47$ (letting $n = 1, 2, 3, 4, 5, ...$).

ACTIVITY 1 RECURRENCE FORMULAE

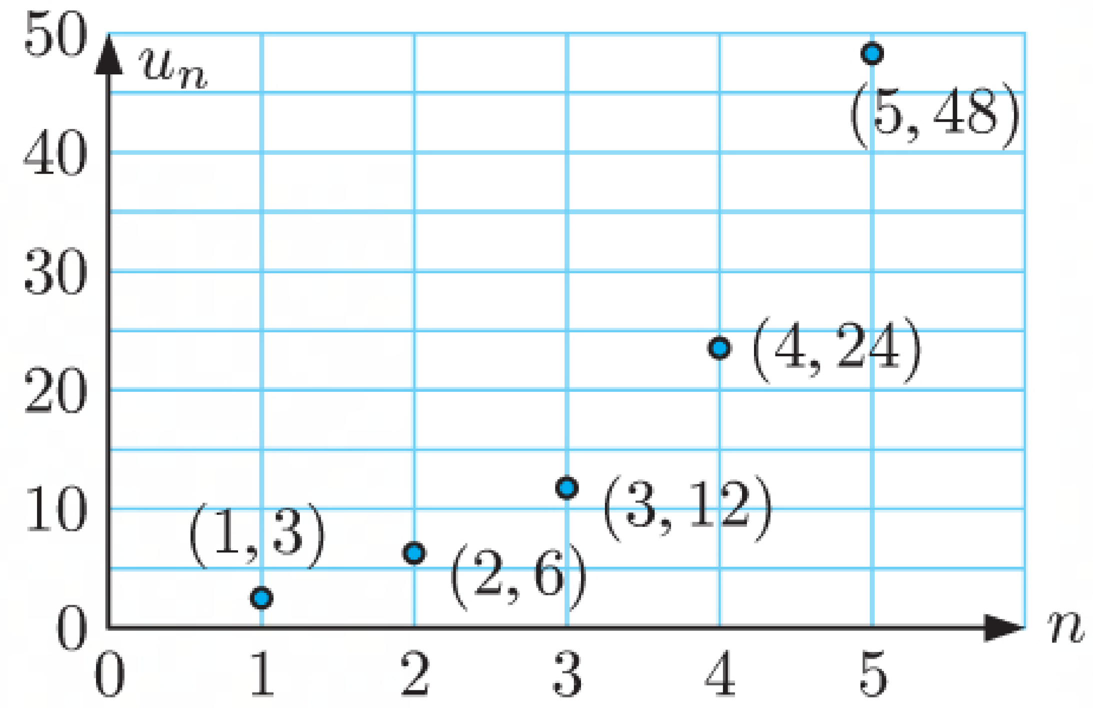
1 $u_1 = 3, \quad u_n = u_{n-1} - 4, \quad n > 1$

a $\begin{aligned} u_2 &= u_1 - 4 \\ &= 3 - 4 \\ &= -1 \end{aligned}$ **b** $\begin{aligned} u_3 &= u_2 - 4 \\ &= -1 - 4 \\ &= -5 \end{aligned}$ **c** $\begin{aligned} u_4 &= u_3 - 4 \\ &= -5 - 4 \\ &= -9 \end{aligned}$ **d** $\begin{aligned} u_5 &= u_4 - 4 \\ &= -9 - 4 \\ &= -13 \end{aligned}$

2 **a** The first term of the sequence is 3. So, $u_1 = 3$.
Each subsequent term is double the previous one. So, $u_n = 2u_{n-1}$.
$\therefore$ the sequence is defined by $u_1 = 3, \quad u_n = 2u_{n-1}, \quad n > 1$.

b $\begin{aligned} u_2 &= 2u_1 \\ &= 2 \times 3 \\ &= 6 \end{aligned}$ $\begin{aligned} u_3 &= 2u_2 \\ &= 2 \times 6 \\ &= 12 \end{aligned}$

$\begin{aligned} u_4 &= 2u_3 \\ &= 2 \times 12 \\ &= 24 \end{aligned}$ $\begin{aligned} u_5 &= 2u_4 \\ &= 2 \times 24 \\ &= 48 \end{aligned}$



3 $u_1 = 1, \quad u_n = \dfrac{1}{1+u_{n-1}}, \quad n > 1$

a

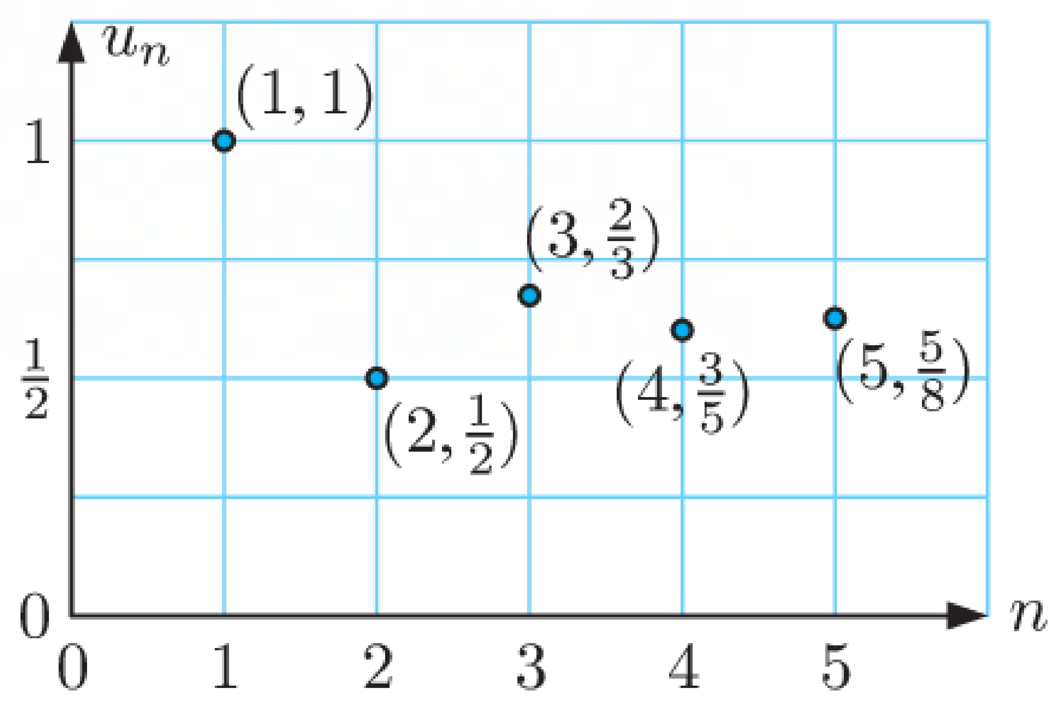
$$u_2 = \frac{1}{1+u_1} = \frac{1}{1+1} = \tfrac{1}{2}$$

$$u_3 = \frac{1}{1+u_2} = \frac{1}{1+\frac{1}{2}} = \frac{1}{\left(\frac{3}{2}\right)} = \tfrac{2}{3}$$

$$u_4 = \frac{1}{1+u_3} = \frac{1}{1+\frac{2}{3}} = \frac{1}{\left(\frac{5}{3}\right)} = \tfrac{3}{5}$$

$$u_5 = \frac{1}{1+u_4} = \frac{1}{1+\frac{3}{5}} = \frac{1}{\left(\frac{8}{5}\right)} = \tfrac{5}{8}$$

b



c As $n \to \infty$, $u_{n-1} \to u$ and $u_n \to u$.

So, as $n \to \infty$, $u_n = \dfrac{1}{1+u_{n-1}} \to \dfrac{1}{1+u}$ and $u_n \to u$

$$\therefore \quad u = \frac{1}{1+u}$$

$$\therefore \quad u(1+u) = 1$$

$$\therefore \quad u + u^2 = 1$$

$$\therefore \quad u^2 + u - 1 = 0$$

$$\therefore \quad u = \frac{-1 \pm \sqrt{1^2 - 4(1)(-1)}}{2(1)} = \frac{-1 \pm \sqrt{5}}{2}$$

But $u_n > 0$ for all n, so $u > 0$ and hence $u = \dfrac{-1+\sqrt{5}}{2}$

Now
$$u_n = \frac{1}{1+u_{n-1}} = \frac{1}{1+\frac{1}{1+u_{n-2}}} = \frac{1}{1+\frac{1}{1+\frac{1}{1+u_{n-3}}}} = \ldots.$$

$$\therefore \quad u = \frac{1}{1+\frac{1}{1+\frac{1}{1+\frac{1}{1+\ldots.}}}} = \frac{-1+\sqrt{5}}{2}$$

4 $u_1 = 1, \quad u_n = \sqrt{\dfrac{1}{1+u_{n-1}}}, \quad n > 1$

a
$$u_2 = \sqrt{\frac{1}{1+u_1}} = \sqrt{\frac{1}{1+1}} = \frac{1}{\sqrt{2}} \approx 0.707\,11$$

$$u_3 = \sqrt{\frac{1}{1+u_2}} = \sqrt{\frac{1}{1+\frac{1}{\sqrt{2}}}} \approx 0.765\,37$$

$$u_4 = \sqrt{\frac{1}{1+u_3}} \approx 0.752\,63$$

$$u_5 = \sqrt{\frac{1}{1+u_4}} \approx 0.755\,36$$

b As $n \to \infty, \quad u_{n-1} \to u$ and $u_n \to u$.

So, as $n \to \infty, \quad u_n = \sqrt{\dfrac{1}{1+u_{n-1}}} \to \sqrt{\dfrac{1}{1+u}}$ and $u_n \to u$

$$\therefore \quad u = \sqrt{\frac{1}{1+u}}$$

$$\therefore \quad u^2 = \frac{1}{1+u}$$

$$\therefore \quad u^2(1+u) = 1$$

$$\therefore \quad u^2 + u^3 = 1$$

$$\therefore \quad u^3 + u^2 - 1 = 0$$

c $u^3 + u^2 - 1 = 0$

$\therefore \quad u \approx 0.754\,88 \qquad$ {using technology}

and $\quad u = \sqrt{\dfrac{1}{1+\sqrt{\dfrac{1}{1+\sqrt{\dfrac{1}{1+\sqrt{\dots}}}}}}} \approx 0.754\,88$

5 Suppose a sequence is defined by $\quad u_1 = 1, \quad u_n = \dfrac{1}{1+u_{n-1}^{\,2}}, \quad n > 0.$

Assume that the sequence terms tend to the constant value u.
As $n \to \infty, \quad u_{n-1} \to u$ and $u_n \to u$.

So, as $n \to \infty, \quad u_n = \dfrac{1}{1+u_{n-1}^{\,2}} \to \dfrac{1}{1+u^2}$ and $u_n \to u$

$$\therefore \quad u = \frac{1}{1+u^2}$$

$$\therefore \quad u(1+u^2) = 1$$

$$\therefore \quad u + u^3 = 1$$

$$\therefore \quad u^3 + u - 1 = 0$$

$\therefore \quad u \approx 0.682\,33 \qquad$ {using technology}

and $\quad u = \dfrac{1}{1+\left(\dfrac{1}{1+\left(\dfrac{1}{1+\left(\frac{1}{1+\dots}\right)^2}\right)^2}\right)^2} \approx 0.682\,33$

6 **a** $u_1 = 1, \quad u_2 = 1, \quad u_n = u_{n-2} + u_{n-1}, \quad n > 2$

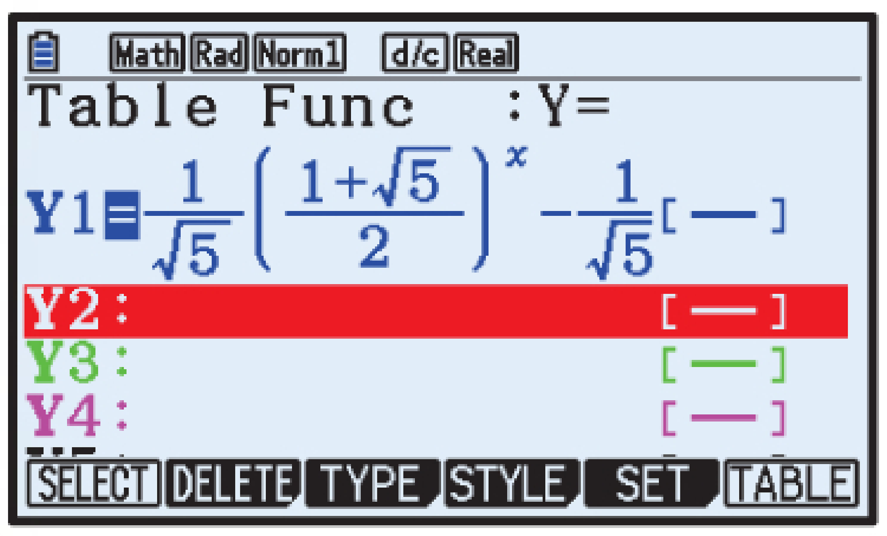
b The Fibonacci sequence is 1, 1, 2, 3, 5, 8, 13, 21, ...

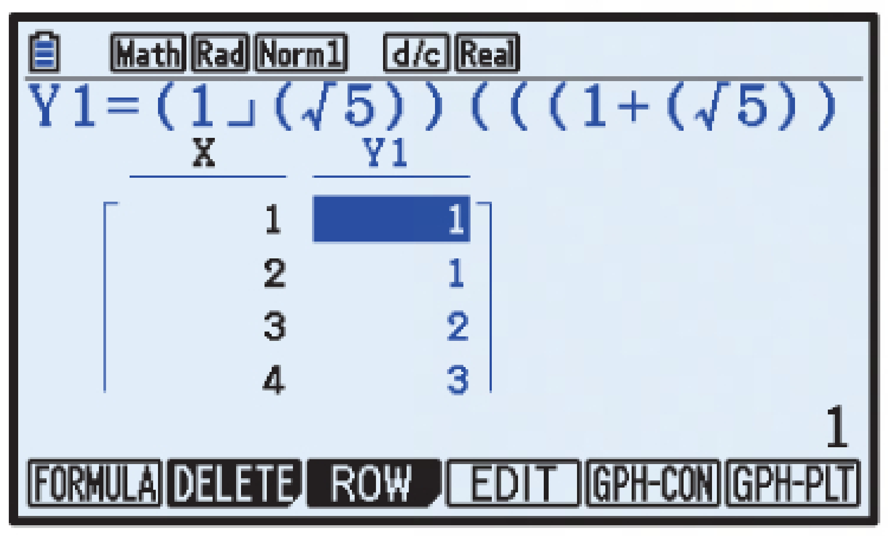
$$u_n = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^n, \quad n \geqslant 1$$

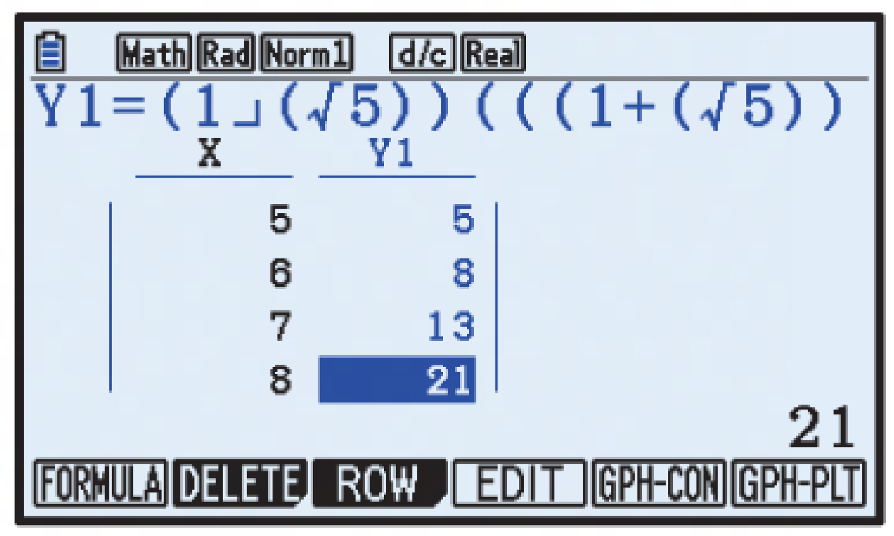
$$\begin{aligned} u_1 &= \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right) - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right) \\ &= \frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right) - \left(\frac{1-\sqrt{5}}{2}\right)\right) \\ &= \frac{1}{\sqrt{5}}\left(\frac{2\sqrt{5}}{2}\right) \\ &= 1 \quad \checkmark \end{aligned}$$

$$\begin{aligned} u_2 &= \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^2 - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^2 \\ &= \frac{1}{\sqrt{5}}\left(\frac{1+2\sqrt{5}+5}{4}\right) - \frac{1}{\sqrt{5}}\left(\frac{1-2\sqrt{5}+5}{4}\right) \\ &= \frac{1}{\sqrt{5}}\left(\frac{6+2\sqrt{5}}{4}\right) - \frac{1}{\sqrt{5}}\left(\frac{6-2\sqrt{5}}{4}\right) \\ &= \frac{1}{\sqrt{5}}\left(\frac{2(3+\sqrt{5})}{4}\right) - \frac{1}{\sqrt{5}}\left(\frac{2(3-\sqrt{5})}{4}\right) \\ &= \frac{1}{\sqrt{5}}\left(\frac{3+\sqrt{5}}{2}\right) - \frac{1}{\sqrt{5}}\left(\frac{3-\sqrt{5}}{2}\right) \\ &= \frac{1}{\sqrt{5}}\left(\left(\frac{3+\sqrt{5}}{2}\right) - \left(\frac{3-\sqrt{5}}{2}\right)\right) \\ &= \frac{1}{\sqrt{5}}\left(\frac{2\sqrt{5}}{2}\right) \\ &= 1 \quad \checkmark \end{aligned}$$

We can verify the remaining cases in a similar fashion, or use technology:







c There are many examples of the Fibonacci sequence in nature. For example, in the shell of a snail, and the arrangement of seeds on flower heads.

d **i** $\frac{u_2}{u_1} = \frac{1}{1} = 1, \quad \frac{u_3}{u_2} = \frac{2}{1} = 2, \quad \frac{u_4}{u_3} = \frac{3}{2} = 1.5,$

$\frac{u_5}{u_4} = \frac{5}{3} \approx 1.666\,67, \quad \frac{u_6}{u_5} = \frac{8}{5} = 1.6, \quad \frac{u_7}{u_6} = \frac{13}{8} = 1.625,$

$\frac{u_8}{u_7} = \frac{21}{13} \approx 1.615\,38, \quad \frac{u_9}{u_8} = \frac{34}{21} \approx 1.619\,05, \quad \frac{u_{10}}{u_9} = \frac{55}{34} \approx 1.617\,65,$

$\frac{u_{11}}{u_{10}} = \frac{89}{55} \approx 1.618\,18, \quad ....$

ii The ratio of consecutive terms of the Fibonacci sequence tends towards ≈ 1.62.

iii The golden ratio is $\frac{1+\sqrt{5}}{2} \approx 1.618$ which is very close to the ratio of consecutive terms of the Fibonacci sequence calculated in **ii**. In fact, the sequence terms of $\left\{\frac{u_{n+1}}{u_n}\right\}$ tend to the golden ratio.

EXERCISE 5B.1

1 **a** 19, 25, 31, 37, ...

i $25 - 19 = 6$
$31 - 25 = 6$
$37 - 31 = 6$
$u_1 = 19, \quad d = 6$

ii $u_n = u_1 + (n-1)d$
$\therefore\ u_n = 19 + 6(n-1)$
$\therefore\ u_n = 6n + 13$

iii $u_{15} = 6(15) + 13$
$= 103$

b 101, 97, 93, 89, ...

i $97 - 101 = -4$
$93 - 97 = -4$
$89 - 93 = -4$
$u_1 = 101, \quad d = -4$

ii $u_n = u_1 + (n-1)d$
$\therefore\ u_n = 101 - 4(n-1)$
$\therefore\ u_n = 105 - 4n$

iii $u_{15} = 105 - 4(15)$
$= 45$

c $8, 9\frac{1}{2}, 11, 12\frac{1}{2}$, ...

i $9\frac{1}{2} - 8 = 1\frac{1}{2}$
$11 - 9\frac{1}{2} = 1\frac{1}{2}$
$12\frac{1}{2} - 11 = 1\frac{1}{2}$
$u_1 = 8, \quad d = 1\frac{1}{2}$

ii $u_n = u_1 + (n-1)d$
$\therefore\ u_n = 8 + 1\frac{1}{2}(n-1)$
$\therefore\ u_n = 1\frac{1}{2}n + 6\frac{1}{2}$

iii $u_{15} = 1\frac{1}{2}(15) + 6\frac{1}{2}$
$= 29$

d 31, 36, 41, 46, ...

i $36 - 31 = 5$
$41 - 36 = 5$
$46 - 41 = 5$
$u_1 = 31, \quad d = 5$

ii $u_n = u_1 + (n-1)d$
$\therefore\ u_n = 31 + 5(n-1)$
$\therefore\ u_n = 5n + 26$

iii $u_{15} = 5(15) + 26$
$= 101$

e $5, -3, -11, -19$, ...

i $-3 - 5 = -8$
$-11 - (-3) = -8$
$-19 - (-11) = -8$
$u_1 = 5, \quad d = -8$

ii $u_n = u_1 + (n-1)d$
$\therefore\ u_n = 5 - 8(n-1)$
$\therefore\ u_n = 13 - 8n$

iii $u_{15} = 13 - 8(15)$
$= -107$

f $a, \ a+d, \ a+2d, \ a+3d$, ...

i $a + d - a = d$
$a + 2d - (a+d) = d$
$a + 3d - (a+2d) = d$
$u_1 = a, \quad d = d$

ii $u_n = u_1 + (n-1)d$
$\therefore\ u_n = a + (n-1)d$

iii $u_{15} = a + 14d$

2 6, 17, 28, 39, 50, ...

a $17 - 6 = 11$
$28 - 17 = 11$
$39 - 28 = 11$
$50 - 39 = 11$

The difference between successive terms is constant.
$\therefore$ the sequence is arithmetic with $u_1 = 6$ and $d = 11$.

b $u_n = u_1 + (n-1)d$
$= 6 + 11(n-1)$
$= 11n - 5$

c $u_{50} = 11(50) - 5$
$= 545$

d Let $u_n = 325 = 11n - 5$
$\therefore\ 330 = 11n$
$\therefore\ n = 30$
So, 325 is the 30th member of the sequence.

e Let $u_n = 761 = 11n - 5$
$\therefore\ 766 = 11n$
$\therefore\ n = 69\frac{7}{11}$
but n must be an integer, so 761 is not a member of the sequence.

3 87, 83, 79, 75, 71, ...

a $83 - 87 = -4$
$79 - 83 = -4$
$75 - 79 = -4$
$71 - 75 = -4$

The difference between successive terms is constant.
$\therefore$ the sequence is arithmetic with $u_1 = 87$ and $d = -4$.

b $u_n = u_1 + (n-1)d$
$= 87 - 4(n-1)$
$= 91 - 4n$

c $u_{40} = 91 - 4(40)$
$= 91 - 160$
$= -69$

d Let $u_n = -297 = 91 - 4n$
$\therefore\ 4n = 388$
$\therefore\ n = 97$
So, -297 is the 97th term of the sequence.

4 **a** $u_n = 3n - 2$
$u_{n+1} = 3(n+1) - 2$
$= 3n + 1$
$u_{n+1} - u_n = 3n + 1 - (3n - 2)$
$= 3$, a constant
Consecutive terms differ by 3.
$\therefore$ the sequence is arithmetic.

b $u_1 = 3(1) - 2 = 1$, $d = 3$

c $u_{57} = 3(57) - 2 = 169$

d Let $u_n = 450 = 3n - 2$
$\therefore\ 3n = 452$
$\therefore\ n = 150\frac{2}{3}$
We try the two values on either side of $n = 150\frac{2}{3}$, which are $n = 150$ and $n = 151$:
$u_{150} = 3(150) - 2 = 448$ and $u_{151} = 3(151) - 2 = 451$
So, $u_{150} = 448$ is the largest term which is smaller than 450.

5 **a** $u_n = \dfrac{71 - 7n}{2}$
$= 35\frac{1}{2} - \frac{7}{2}n$

$u_{n+1} = \dfrac{71 - 7(n+1)}{2}$
$= \dfrac{71 - 7n - 7}{2}$
$= \dfrac{64 - 7n}{2}$
$= 32 - \frac{7}{2}n$

$u_{n+1} - u_n = (32 - \frac{7}{2}n) - (35\frac{1}{2} - \frac{7}{2}n)$
$= -\frac{7}{2}$, a constant
So, consecutive terms differ by $-\frac{7}{2}$.
$\therefore$ the sequence is arithmetic.

b $u_1 = \dfrac{71-7(1)}{2} = 32, \quad d = -\frac{7}{2}$

c $u_{75} = \dfrac{71-7(75)}{2} = -227$

d Let $u_n = -200 = \dfrac{71-7n}{2}$

$$\therefore \quad -400 = 71 - 7n$$
$$\therefore \quad 7n = 471$$
$$\therefore \quad n = 67\tfrac{2}{7}$$

We try the two values on either side of $n = 67\frac{2}{7}$, which are $n = 67$ and $n = 68$:

$u_{67} = \dfrac{71-7(67)}{2} = -199$ and $u_{68} = \dfrac{71-7(68)}{2} = -202\frac{1}{2}$

So, the terms of the sequence are less than -200 for $n \geqslant 68$.

6 23, 36, 49, 62, ...

$36 - 23 = 13, \quad 49 - 36 = 13, \quad 62 - 49 = 13$

$u_1 = 23, \quad d = 13$

$$\begin{aligned} u_n &= u_1 + (n-1)d \\ &= 23 + 13(n-1) \\ &= 13n + 10 \end{aligned}$$

Let $u_n = 100\,000 = 13n + 10$

$$\therefore \quad 99\,990 = 13n$$
$$\therefore \quad n = 7691\tfrac{7}{13}$$

We try the two values on either side of $n = 7691\frac{7}{13}$, which are $n = 7691$ and $n = 7692$:

$u_{7691} = 13(7691) + 10 = 99\,993$ and $u_{7692} = 13(7692) + 10 = 100\,006$

So, the first term to exceed $100\,000$ is $u_{7692} = 100\,006$.

7 **a** $u_{n+1} = u_n + 7$

$\therefore \quad u_{n+1} - u_n = 7$

So, consecutive terms differ by 7.

$\therefore$ the sequence is arithmetic.

b $u_n = u_1 + (n-1)d$

$\therefore \quad u_{200} = -12 + 199(7)$

$\therefore \quad u_{200} = 1381$

c Let $u_n = 1000 = -12 + 7(n-1)$

$$\therefore \quad 7n - 7 = 1012$$
$$\therefore \quad 7n = 1019$$
$$\therefore \quad n = 145\tfrac{4}{7}$$

but n must be an integer, so 1000 is not a member of the sequence.

8 **a** 32, k, 3

Since the terms are consecutive, $k - 32 = 3 - k$ {equating differences}

$$\therefore \quad 2k = 35$$
$$\therefore \quad k = 17\tfrac{1}{2}$$

b k, 7, 10

Since the terms are consecutive, $7 - k = 10 - 7$ {equating differences}

$$\therefore \quad 7 - k = 3$$
$$\therefore \quad k = 4$$

c $k, \ 2k-1, \ 13$

Since the terms are consecutive, $2k-1-k = 13-(2k-1)$ {equating differences}

$$\therefore \ k-1 = 14-2k$$
$$\therefore \ 3k = 15$$
$$\therefore \ k = 5$$

d $k, \ 2k+1, \ 8-k$

Since the terms are consecutive, $2k+1-k = 8-k-(2k+1)$ {equating differences}

$$\therefore \ k+1 = 7-3k$$
$$\therefore \ 4k = 6$$
$$\therefore \ k = \tfrac{6}{4} = \tfrac{3}{2}$$

e $2k+7, \ 3k+5, \ 5k-4$

Since the terms are consecutive,

$$3k+5-(2k+7) = 5k-4-(3k+5) \quad \text{\{equating differences\}}$$
$$\therefore \ k-2 = 2k-9$$
$$\therefore \ k = 7$$

f $2k+18, \ -2-k, \ 2k+2$

Since the terms are consecutive,

$$-2-k-(2k+18) = 2k+2-(-2-k) \quad \text{\{equating differences\}}$$
$$\therefore \ -3k-20 = 3k+4$$
$$\therefore \ -6k = 24$$
$$\therefore \ k = -4$$

g k, k^2, k^2+6

Since the terms are consecutive, $k^2-k = k^2+6-k^2$ {equating differences}

$$\therefore \ k^2-k-6 = 0$$
$$\therefore \ (k+2)(k-3) = 0$$
$$\therefore \ k = -2 \text{ or } 3$$

h $5, k, k^2-8$

Since the terms are consecutive, $k-5 = k^2-8-k$ {equating differences}

$$\therefore \ k^2-2k-3 = 0$$
$$\therefore \ (k+1)(k-3) = 0$$
$$\therefore \ k = -1 \text{ or } 3$$

9 $10k+1, \ 2k, \ 4k^2-5$

a Since the terms are consecutive, $2k-(10k+1) = 4k^2-5-2k$ {equating differences}

$$\therefore \ 2k-10k-1 = 4k^2-5-2k$$
$$\therefore \ -8k-1 = 4k^2-5-2k$$
$$\therefore \ 4k^2+6k-4 = 0$$
$$\therefore \ 2(2k^2+3k-2) = 0$$
$$\therefore \ 2(2k-1)(k+2) = 0$$
$$\therefore \ k = \tfrac{1}{2} \text{ or } -2$$

b For $k = \frac{1}{2}$, the common difference $d = 2(\frac{1}{2}) - (10(\frac{1}{2}) + 1)$
$= 1 - 6$
$= -5$

For $k = -2$, the common difference $d = 2(-2) - (10(-2) + 1)$
$= -4 - (-19)$
$= 15$

10 **a** $u_7 = 41 \quad \therefore \; u_1 + 6d = 41 \quad .... (1) \qquad \{\text{using } u_n = u_1 + (n-1)d\}$
$u_{13} = 77 \quad \therefore \; u_1 + 12d = 77 \quad .... (2)$

We now solve (1) and (2) simultaneously:

$-u_1 - 6d = -41$ {multiplying both sides of (1) by -1}
$u_1 + 12d = 77$

$\therefore \; 6d = 36$ {adding the equations}
$\therefore \; d = 6$

So, in (1): $u_1 + 6(6) = 41$
$\therefore \; u_1 + 36 = 41$
$\therefore \; u_1 = 5$
Now $u_n = u_1 + (n-1)d$
$\therefore \; u_n = 5 + 6(n-1)$
$\therefore \; u_n = 6n - 1$

Check:
$u_7 = 6(7) - 1$
$= 42 - 1$
$= 41$ ✓
$u_{13} = 6(13) - 1$
$= 78 - 1$
$= 77$ ✓

b $u_5 = -2 \quad \therefore \; u_1 + 4d = -2 \quad .... (1) \qquad \{\text{using } u_n = u_1 + (n-1)d\}$
$u_{12} = -12\frac{1}{2} \quad \therefore \; u_1 + 11d = -12\frac{1}{2} \quad .... (2)$

We now solve (1) and (2) simultaneously:

$-u_1 - 4d = 2$ {multiplying both sides of (1) by -1}
$u_1 + 11d = -12\frac{1}{2}$

$\therefore \; 7d = -10\frac{1}{2}$ {adding the equations}
$\therefore \; d = -\frac{3}{2}$

So, in (1): $u_1 + 4(-\frac{3}{2}) = -2$
$\therefore \; u_1 - 6 = -2$
$\therefore \; u_1 = 4$
Now $u_n = u_1 + (n-1)d$
$\therefore \; u_n = 4 - \frac{3}{2}(n-1)$
$\therefore \; u_n = -\frac{3}{2}n + \frac{11}{2}$

Check:
$u_5 = -\frac{3}{2}(5) + \frac{11}{2}$
$= -\frac{15}{2} + \frac{11}{2}$
$= -\frac{4}{2} = -2$ ✓
$u_{12} = -\frac{3}{2}(12) + \frac{11}{2}$
$= -\frac{36}{2} + \frac{11}{2}$
$= -\frac{25}{2} = -12\frac{1}{2}$ ✓

c $u_7 = 1 \quad \therefore \ u_1 + 6d = 1 \quad .... \ (1) \qquad \{\text{using } u_n = u_1 + (n-1)d\}$

$u_{15} = -39 \quad \therefore \ u_1 + 14d = -39 \quad .... \ (2)$

We now solve (1) and (2) simultaneously:

$$\begin{aligned} -u_1 - \ \ 6d &= -1 \qquad \{\text{multiplying both sides of (1) by } -1\} \\ u_1 + 14d &= -39 \\ \hline \therefore \ 8d &= -40 \qquad \{\text{adding the equations}\} \\ \therefore \ d &= -5 \end{aligned}$$

So, in (1): $u_1 + 6(-5) = 1$

$\therefore \ u_1 - 30 = 1$

$\therefore \ u_1 = 31$

Now $u_n = u_1 + (n-1)d$

$\therefore \ u_n = 31 - 5(n-1)$

$\therefore \ u_n = 31 - 5n + 5$

$\therefore \ u_n = -5n + 36$

Check: $u_7 = -5(7) + 36$
$= -35 + 36$
$= 1$ ✓

$u_{15} = -5(15) + 36$
$= -75 + 36$
$= -39$ ✓

d $u_{11} = -16 \quad \therefore \ u_1 + 10d = -16 \quad .... \ (1) \qquad \{\text{using } u_n = u_1 + (n-1)d\}$

$u_8 = -11\frac{1}{2} \quad \therefore \ u_1 + 7d = -11\frac{1}{2} \quad .... \ (2)$

We now solve (1) and (2) simultaneously:

$$\begin{aligned} -u_1 - 10d &= 16 \qquad \{\text{multiplying both sides of (1) by } -1\} \\ u_1 + \ \ 7d &= -11\tfrac{1}{2} \\ \hline \therefore \ -3d &= 4\tfrac{1}{2} \qquad \{\text{adding the equations}\} \\ \therefore \ d &= -\tfrac{3}{2} \end{aligned}$$

So, in (1): $u_1 + 10(-\frac{3}{2}) = -16$

$\therefore \ u_1 - 15 = -16$

$\therefore \ u_1 = -1$

Now $u_n = u_1 + (n-1)d$

$\therefore \ u_n = -1 - \frac{3}{2}(n-1)$

$\therefore \ u_n = -\frac{3}{2}n + \frac{1}{2}$

Check: $u_{11} = -\frac{3}{2}(11) + \frac{1}{2}$
$= -\frac{33}{2} + \frac{1}{2}$
$= -\frac{32}{2} = -16$ ✓

$u_8 = -\frac{3}{2}(8) + \frac{1}{2}$
$= -\frac{24}{2} + \frac{1}{2}$
$= -\frac{23}{2} = -11\frac{1}{2}$ ✓

11 $u_1, u_2, u_3, u_4, u_5, u_6, ...$ is arithmetic, so $u_n = u_n + (n-1)d$.

a The sequence $u_1 + u_2, \ u_3 + u_4, \ u_5 + u_6, \ ....$ has nth term $u_{2n-1} + u_{2n}$

The difference between successive terms is

$$\begin{aligned} &(u_{2(n+1)-1} + u_{2(n+1)}) - (u_{2n-1} + u_{2n}) \\ &= (u_{2n+1} + u_{2n+2}) - (u_{2n-1} + u_{2n}) \\ &= (u_1 + 2nd + u_1 + (2n+1)d) - (u_1 + (2n-2)d + u_1 + (2n-1)d) \\ &\qquad\qquad \{\text{using } u_n = u_1 + (n-1)d\} \\ &= (2u_1 + (4n+1)d) - (2u_1 + (4n-3)d) \\ &= 4d \quad \text{which is constant} \end{aligned}$$

$\therefore \ u_1 + u_2, \ u_3 + u_4, \ u_5 + u_6, \ ....$ is also arithmetic.

b $u_2 = 5 \quad \therefore\ u_1 + d = 5 \quad .... (1) \quad \{\text{using } u_n = u_1 + (n-1)d\}$

$u_{10} = 61 \quad \therefore\ u_1 + 9d = 61 \quad .... (2)$

We now solve (1) and (2) simultaneously:

$$\begin{aligned} -u_1 - \ d &= -5 \quad \{\text{multiplying both sides of (1) by } -1\} \\ u_1 + 9d &= 61 \\ \hline \therefore\ 8d &= 56 \quad \{\text{adding the equations}\} \\ \therefore\ d &= 7 \end{aligned}$$

So, in (1): $u_1 + 7 = 5$

$\therefore\ u_1 = -2$

Now $u_n = u_1 + (n-1)d$

$\therefore\ u_n = -2 + 7(n-1)$

$\therefore\ u_n = -2 + 7n - 7$

$\therefore\ u_n = 7n - 9$

Check: $u_2 = 7(2) - 9 = 14 - 9 = 5$ ✓

$u_{10} = 7(10) - 9 = 70 - 9 = 61$ ✓

The nth term of the sequence in **a** is $u_{2n-1} + u_{2n}$.

$\therefore$ the 30th term of the sequence in **a** is $u_{59} + u_{60} = (7(59) - 9) + (7(60) - 9)$
$= 413 - 9 + 420 - 9$
$= 815$

12 Suppose the common difference is d.

$\therefore$ the numbers are $5,\ 5+d,\ 5+2d,\ 5+3d,$ and 10

$\therefore\ 5 + 4d = 10$

$\therefore\ 4d = 5$

$\therefore\ d = \frac{5}{4} = 1\frac{1}{4}$

So, the sequence is $5,\ 6\frac{1}{4},\ 7\frac{1}{2},\ 8\frac{3}{4},\ 10.$

13 Suppose the common difference is d.

$\therefore$ the numbers are $-1,\ -1+d,\ -1+2d,\ -1+3d,\ -1+4d,\ -1+5d,\ -1+6d,$ and 32.

$\therefore\ -1 + 7d = 32$

$\therefore\ 7d = 33$

$\therefore\ d = \frac{33}{7} = 4\frac{5}{7}$

So, the sequence is $-1,\ 3\frac{5}{7},\ 8\frac{3}{7},\ 13\frac{1}{7},\ 17\frac{6}{7},\ 22\frac{4}{7},\ 27\frac{2}{7},\ 32.$

14 **a** Suppose the common difference is d.

$\therefore$ the numbers are $50,\ 50+d,\ 50+2d,\ 50+3d,$ and 44.

$\therefore\ 50 + 4d = 44$

$\therefore\ 4d = -6$

$\therefore\ d = -\frac{6}{4} = -1\frac{1}{2}$

So, the sequence is $50,\ 48\frac{1}{2},\ 47,\ 45\frac{1}{2},\ 44.$

b $u_n = u_1 + (n-1)d$

$\therefore\ u_n = 50 - 1\frac{1}{2}(n-1)$

$\therefore\ u_n = 51\frac{1}{2} - 1\frac{1}{2}n$

Let $u_n = 0$

$\therefore\ 51\frac{1}{2} - 1\frac{1}{2}n = 0$

$\therefore\ 1\frac{1}{2}n = 51\frac{1}{2}$

$\therefore\ n = 34\frac{1}{3}$

We try the two values on either side of $n = 34\frac{1}{3}$, which are $n = 34$ and $n = 35$.

$u_{34} = 51\frac{1}{2} - 1\frac{1}{2}(34) = \frac{1}{2}$ and $u_{35} = 51\frac{1}{2} - 1\frac{1}{2}(35) = -1$

So, the first negative term of the sequence is $u_{35} = -1$.

15 **a** $u_3 = \dfrac{1}{k}$ $\therefore\ u_1 + 2d = \dfrac{1}{k}$ .... (1) {using $u_n = u_1 + (n-1)d$}

$u_4 = k$ $\therefore\ u_1 + 3d = k$ .... (2)

We now solve (1) and (2) simultaneously:

$-u_1 - 2d = -\dfrac{1}{k}$ {multiplying both sides of (1) by -1}

$u_1 + 3d = k$

$\therefore\ d = k - \dfrac{1}{k}$ {adding the equations}

So, in (1): $u_1 + 2\left(k - \dfrac{1}{k}\right) = \dfrac{1}{k}$

$\therefore\ u_1 = \dfrac{3}{k} - 2k$

Now, $u_6 = k^2 + 1$ $\therefore\ u_1 + 5d = k^2 + 1$

$\therefore\ \left(\dfrac{3}{k} - 2k\right) + 5\left(k - \dfrac{1}{k}\right) = k^2 + 1$

$\therefore\ 3k - \dfrac{2}{k} = k^2 + 1$

$\therefore\ 3k^2 - 2 = k^3 + k$ {multiplying both sides by k}

$\therefore\ k^3 - 3k^2 + k + 2 = 0$

$\therefore\ (k-2)(k^2 - k - 1) = 0$

$\therefore\ (k-2)[(k - \frac{1}{2})^2 - (\frac{1}{2})^2 - 1] = 0$

$\therefore\ (k-2)[(k - \frac{1}{2})^2 - \frac{5}{4}] = 0$

$\therefore\ k = 2$ or $\frac{1}{2} \pm \frac{\sqrt{5}}{2}$

but $k \in \mathbb{Q}$, $\therefore\ k = 2$

b Using **a**, when $k = 2$, $d = 2 - \frac{1}{2} = \frac{3}{2}$

and $u_1 = \frac{3}{2} - 2(2) = -\frac{5}{2}$

Now $u_n = u_1 + (n-1)d$

$\therefore\ u_n = -\frac{5}{2} + \frac{3}{2}(n-1)$

$\therefore\ u_n = -\frac{5}{2} + \frac{3}{2}n - \frac{3}{2}$

$\therefore\ u_n = \frac{3}{2}n - 4$

$\therefore\ u_n = \dfrac{3n-8}{2}$

Check: $u_3 = \dfrac{3(3)-8}{2} = \dfrac{1}{2}$ ✓

$u_4 = \dfrac{3(4)-8}{2} = 2$ ✓

$u_6 = \dfrac{3(6)-8}{2} = 5 = 2^2 + 1$ ✓

16 Since $x > y > z > 0$, then $\dfrac{1}{x} < \dfrac{1}{y} < \dfrac{1}{z}$

Now $\dfrac{1}{y} - \dfrac{1}{x} = \dfrac{1}{z} - \dfrac{1}{y}$ {equating differences}

$\therefore\ \dfrac{1}{x} + \dfrac{1}{z} = \dfrac{2}{y}$

$\therefore\ \dfrac{z+x}{xz} = \dfrac{2}{y}$

$\therefore\ yz + xy = 2xz$ .... (1)

For $x - z$, y, and $x - y + z$ to be the lengths of a right angled triangle, they must be positive and satisfy the Pythagorean identity.

Since $x > y > z > 0$, $x - z > 0$

and $x - y > 0$

$\therefore\ x - y + z > 0$

$\therefore$ $x - z$, y, and $x - y + z$ are positive.

Now $(x-y+z)^2 = (x-y+z)(x-y+z)$

$= x^2 - xy + xz - xy + y^2 - yz + xz - yz + z^2$

$= x^2 + y^2 + z^2 - 2xy - 2yz + 2xz$

$= x^2 + y^2 + z^2 - 2(xy + yz) + 2xz$

$= x^2 + y^2 + z^2 - 2(2xz) + 2xz$ {using (1)}

$= x^2 + y^2 + z^2 - 2xz$

$= x^2 - 2xz + z^2 + y^2$

$= (x-z)^2 + y^2$

$\therefore$ $x - z$, y, and $x - y + z$ satisfy the Pythagorean identity.

So, $x - z$, y, and $x - y + z$ could be the lengths of a right angled triangle.

17 No, it is not possible to construct an arithmetic sequence such that all of the terms are distinct primes. Such a sequence would require $u_1 =$ some prime number p, and $d \in \mathbb{Z}^+$.

But then $u_{p+1} = u_1 + ((p+1) - 1)d$

$= p + pd$

$= p(1+d)$ is a multiple of p, and so cannot be prime.

18 **a** *Month 1*: 5 cars

Month 2: $5 + 13 = 18$ cars

Month 3: $18 + 13 = 31$ cars

Month 4: $31 + 13 = 44$ cars

Month 5: $44 + 13 = 57$ cars

Month 6: $57 + 13 = 70$ cars

b Every month after the first, the factory assembles 13 cars, so the difference between successive months is always 13. Thus we have an arithmetic sequence with $u_1 = 5$ and $d = 13$.

c $u_n = u_1 + (n-1)d$
$= 5 + 13(n-1)$
$= 13n - 8$
$\therefore \; u_{12} = 13(12) - 8 \quad \{12 \text{ months} \equiv 1 \text{ year}\}$
$= 148$
So, 148 cars are made in the first year.

d Let $u_n = 250 = 13n - 8$
$\therefore \; 258 = 13n$
$\therefore \; n = \frac{258}{13} \approx 19.8$
So, the 250th car is made in the 20th month.

19 **a** *Week 1*: $3000 - 183 = 2817$ L
Week 2: $2817 - 183 = 2634$ L
Week 3: $2634 - 183 = 2451$ L
Week 4: $2451 - 183 = 2268$ L

b Every week Yafiah uses 183 L of water, so the difference between successive weeks is always -183. Thus we have an arithmetic sequence with $u_1 = 2817$ and $d = -183$.

c $u_n = u_1 + (n-1)d$
$= 2817 - 183(n-1)$
$= 3000 - 183n$
Let $u_n = 0$
$\therefore \; 3000 - 183n = 0$
$\therefore \; 183n = 3000$
$\therefore \; n \approx 16.4$
So, Yafiah's tank will run out of water in the 17th week.

EXERCISE 5B.2

1 **a** Average mass of oranges $= \dfrac{\text{total mass}}{\text{number of oranges}}$
$= \dfrac{1.126 \text{ kg}}{8}$
$= 0.140\,75$ kg
$= 140.75$ g

b $u_n = 140.75n$

2 Total mass of 12 eggs = mass of 12 eggs and carton − mass of carton
$= 743 - 32$
$= 711$ g

a Average mass of eggs $= \dfrac{\text{total mass}}{\text{number of eggs}}$
$= \dfrac{711 \text{ g}}{12}$
$= 59.25$ g

b $u_n = 32 + 59.25n$

c The carton can only hold a maximum of 12 eggs.
$\therefore$ the model is valid for $0 \leqslant n \leqslant 12$.

3 **a** *Day 1*: $580 - (8 \times 2) = 564$
Day 2: $564 - (8 \times 2) = 548$
Day 3: $548 - (8 \times 2) = 532$
Day 4: $532 - (8 \times 2) = 516$

$u_1 = 564$ and $d = -16$
$u_n = u_1 + (n-1)d$
$= 564 - 16(n-1)$
$\therefore \; u_n = 580 - 16n$

b Average mass of hay bales $= \dfrac{\text{total mass}}{\text{number of bales}}$

$= \dfrac{9850 \text{ kg}}{580}$

$= \frac{985}{58}$ kg

Since the farmer uses $8 \times 2 = 16$ bales of hay each day,

the average mass of hay used each day $= \frac{985}{58} \times 16$

$= \frac{15\,760}{58}$

$= \frac{7880}{29}$ kg

So, the mass of hay remaining after n days can be approximated by $u_n = 9850 - \frac{7880}{29}n$.

4 **a** $u_1 = 34, \quad u_9 = 80$

Average number of friends made each week

$= \dfrac{\text{total number of friends made from week 1 to week 9}}{\text{number of weeks}}$

$= \dfrac{80 - 34}{8}$

$= 5.75$ online friends

b $u_1 = 34, \quad d = 5.75$

$u_n = u_1 + (n-1)d$

$= 34 + 5.75(n-1)$

$\therefore \; u_n = 5.75n + 28.25$

c No, it is not a problem that the common difference is not an integer. The model is only intended to *estimate* the number of online friends. We can simply round to the nearest whole number.

d $u_{20} = 5.75(20) + 28.25$

$= 143.25$

Valéria will have approximately 143 online friends after 20 weeks.

5 **a** $u_{50} = 6950 \quad \therefore \; u_1 + 49d = 6950 \quad .... \; (1) \qquad \{\text{using } \; u_n = u_1(n-1)d\}$

$u_{100} = 11\,950 \quad \therefore \; u_1 + 99d = 11\,950 \quad .... \; (2)$

We now solve (1) and (2) simultaneously:

$-u_1 - 49d = -6950 \qquad \{\text{multiplying both sides of (1) by } -1\}$

$u_1 + 99d = 11\,950$

$\therefore \; 50d = 5000 \qquad \{\text{adding the equations}\}$

$\therefore \; d = 100$

So, in (1): $u_1 + 49(100) = 6950$

$\therefore \; u_1 + 4900 = 6950$

$\therefore \; u_1 = 2050$

Now $u_n = u_1 + (n-1)d$

$\therefore \; u_n = 2050 + 100(n-1)$

$\therefore \; u_n = 2050 + 100n - 100$

$\therefore \; u_n = 1950 + 100n$

Check:

$u_{50} = 100(50) + 1950$

$= 6950 \; \checkmark$

$u_{100} = 100(100) + 1950$

$= 11\,950 \; \checkmark$

b **i** The common difference is 100. This means that the catering cost is €100 per guest.

ii The constant term is 1950 (when $n = 0$). This means that the venue hire is €1950 (with 0 guests).

c $u_{85} = 100(85) + 1950$

$= 10\,450$

The cost of venue hire and catering for a reception with 85 guests would be €10 450.

EXERCISE 5C

1 **a** 3, 6, 12, 24, ...

i $\frac{6}{3} = 2$

$\therefore \quad r = 2, \quad u_1 = 3$

ii $u_n = u_1 r^{n-1}$

$\therefore \quad u_n = 3 \times 2^{n-1}$

iii $u_9 = 3 \times 2^8$

$= 768$

b 2, 10, 50, ...

i $\frac{10}{2} = 5$

$\therefore \quad r = 5, \quad u_1 = 2$

ii $u_n = u_1 r^{n-1}$

$\therefore \quad u_n = 2 \times 5^{n-1}$

iii $u_9 = 2 \times 5^8$

$= 781\,250$

c 512, 256, 128, ...

i $\frac{256}{512} = \frac{1}{2}$

$\therefore \quad r = \frac{1}{2}, \quad u_1 = 512$

ii $u_n = u_1 r^{n-1}$

$\therefore \quad u_n = 512 \times \left(\frac{1}{2}\right)^{n-1}$

$\therefore \quad u_n = 512 \times 2^{1-n}$

iii $u_9 = 512 \times 2^{-8}$

$= 2$

d 1, 3, 9, 27, ...

i $\frac{3}{1} = 3$

$\therefore \quad r = 3, \quad u_1 = 1$

ii $u_n = u_1 r^{n-1}$

$\therefore \quad u_n = 1 \times 3^{n-1}$

$\therefore \quad u_n = 3^{n-1}$

iii $u_9 = 3^8$

$= 6561$

e 12, 18, 27, ...

i $\frac{18}{12} = \frac{3}{2}$

$\therefore \quad r = \frac{3}{2}, \quad u_1 = 12$

ii $u_n = u_1 r^{n-1}$

$\therefore \quad u_n = 12 \times \left(\frac{3}{2}\right)^{n-1}$

iii $u_9 = 12 \times \left(\frac{3}{2}\right)^8$

$= \frac{2^2 \times 3 \times 3^8}{2^8}$

$= \frac{3^9}{2^6} = \frac{19\,683}{64}$

f $\frac{1}{16}, -\frac{1}{8}, \frac{1}{4}, -\frac{1}{2}, ...$

i $\frac{-\frac{1}{8}}{\frac{1}{16}} = -2$

$\therefore \quad r = -2, \quad u_1 = \frac{1}{16}$

ii $u_n = u_1 r^{n-1}$

$\therefore \quad u_n = \frac{1}{16} \times (-2)^{n-1}$

iii $u_9 = \frac{1}{16} \times (-2)^8$

$= 16$

2 **a** 5, 10, 20, 40, ...

$\frac{10}{5} = 2 \qquad \frac{20}{10} = 2 \qquad \frac{40}{20} = 2$

Consecutive terms have a common ratio of 2.

$\therefore$ the sequence is geometric with $u_1 = 5$ and $r = 2$.

b $u_n = u_1 r^{n-1}$

$\therefore \quad u_n = 5 \times 2^{n-1}$

$\therefore \quad u_{15} = 5 \times 2^{14}$

$= 81\,920$

3 **a** $12, -6, 3, -\frac{3}{2}, ...$

$\frac{-6}{12} = -\frac{1}{2}$ $\quad \frac{3}{-6} = -\frac{1}{2}$ $\quad \frac{-\frac{3}{2}}{3} = -\frac{1}{2}$

Consecutive terms have a common ratio of $-\frac{1}{2}$.

$\therefore$ the sequence is geometric with $u_1 = 12$ and $r = -\frac{1}{2}$.

b

$$u_n = u_1 r^{n-1}$$
$$\therefore\ u_n = 12 \times \left(-\tfrac{1}{2}\right)^{n-1}$$
$$\therefore\ u_{13} = 12 \times \left(-\tfrac{1}{2}\right)^{12} = \frac{2^2 \times 3}{2^{12}} = \frac{3}{2^{10}}$$
$$\therefore\ u_{13} = \frac{3}{1024} \approx 0.002\,93$$

4 **a** $8, -6, 4.5, -3.375, ...$

$\frac{-6}{8} = -\frac{3}{4}$ $\quad \frac{4.5}{-6} = -\frac{3}{4}$ $\quad \frac{-3.375}{4.5} = -\frac{3}{4}$

Consecutive terms have a common ratio of $-\frac{3}{4}$.

$\therefore$ the sequence is geometric with $u_1 = 8$ and $r = -\frac{3}{4}$.

b

$$u_n = u_1 r^{n-1}$$
$$\therefore\ u_n = 8 \times \left(-\tfrac{3}{4}\right)^{n-1}$$
$$\therefore\ u_{10} = 8 \times \left(-\tfrac{3}{4}\right)^{9}$$
$$\therefore\ u_{10} \approx -0.601$$

5 **a** $8, 4\sqrt{2}, 4, 2\sqrt{2}, ...$

$\frac{4\sqrt{2}}{8} = \frac{\sqrt{2}}{2} = \frac{1}{\sqrt{2}}$ $\quad \frac{4}{4\sqrt{2}} = \frac{1}{\sqrt{2}}$ $\quad \frac{2\sqrt{2}}{4} = \frac{\sqrt{2}}{2} = \frac{1}{\sqrt{2}}$

Consecutive terms have a common ratio of $\frac{1}{\sqrt{2}}$.

$\therefore$ the sequence is geometric with $u_1 = 8$ and $r = \frac{1}{\sqrt{2}}$.

b

$$u_n = u_1 r^{n-1}$$
$$\therefore\ u_n = 8 \times \left(\frac{1}{\sqrt{2}}\right)^{n-1}$$
$$\therefore\ u_n = 2^3 \times \left(2^{-\frac{1}{2}}\right)^{n-1}$$
$$\therefore\ u_n = 2^3 \times 2^{\frac{1}{2} - \frac{1}{2}n}$$
$$\therefore\ u_n = 2^{\frac{7}{2} - \frac{1}{2}n}$$

6 **a** $k,\ 3k,\ 54$

Since the terms are geometric, $\dfrac{3k}{k} = \dfrac{54}{3k}$ {equating the common ratio r}

$\therefore\ 9k^2 = 54k$

$\therefore\ 9k = 54$ {since $k \neq 0$}

$\therefore\ k = 6$

Check: If $k = 6$, the terms are: $6,\ 18,\ 54$. ✓ $\{r = 3\}$

b $1000,\ 4k,\ k$

Since the terms are geometric, $\dfrac{4k}{1000} = \dfrac{k}{4k}$ {equating the common ratio r}

$\therefore\ 16k^2 = 1000k$

$\therefore\ 16k = 1000$ {since $k \neq 0$}

$\therefore\ k = \frac{125}{2}$

Check: If $k = \frac{125}{2}$, the terms are: $1000,\ 250,\ \frac{125}{2}$. ✓ $\{r = \frac{1}{4}\}$

c $7,\ k,\ 28$

Since the terms are geometric, $\dfrac{k}{7} = \dfrac{28}{k}$ {equating the common ratio r}

$\therefore\ k^2 = 196$

$\therefore\ k = \pm\sqrt{196}$

$\therefore\ k = \pm 14$

Check: If $k = 14$, the terms are: $7,\ 14,\ 28$. ✓ $\{r = 2\}$

If $k = -14$, the terms are: $7,\ -14,\ 28$. ✓ $\{r = -2\}$

d $18,\ k,\ \frac{2}{9}$

Since the terms are geometric, $\dfrac{k}{18} = \dfrac{\frac{2}{9}}{k}$ {equating the common ratio r}

$\therefore\ k^2 = 4$

$\therefore\ k = \pm\sqrt{4}$

$\therefore\ k = \pm 2$

Check: If $k = 2$, the terms are: $18,\ 2,\ \frac{2}{9}$. ✓ $\{r = \frac{1}{9}\}$

If $k = -2$, the terms are: $18,\ -2,\ \frac{2}{9}$. ✓ $\{r = -\frac{1}{9}\}$

e $k,\ 12,\ \dfrac{k}{9}$

Since the terms are geometric, $\dfrac{12}{k} = \dfrac{\frac{k}{9}}{12}$ {equating the common ratio r}

$\therefore\ 144 = \dfrac{k^2}{9}$

$\therefore\ k^2 = 1296$

$\therefore\ k = \pm\sqrt{1296}$

$\therefore\ k = \pm 36$

Check: If $k = 36$, the terms are: $36,\ 12,\ 4$. ✓ $\{r = \frac{1}{3}\}$

If $k = -36$, the terms are: $-36,\ 12,\ -4$. ✓ $\{r = -\frac{1}{3}\}$

f $k,\ 20,\ \frac{25}{4}k$

Since the terms are geometric, $\dfrac{20}{k} = \dfrac{\frac{25}{4}k}{20}$ {equating the common ratio r}

$$\therefore\ 400 = \tfrac{25}{4}k^2$$
$$\therefore\ k^2 = 64$$
$$\therefore\ k = \pm\sqrt{64}$$
$$\therefore\ k = \pm 8$$

Check: If $k = 8$, the terms are: 8, 20, 50. ✓ $\{r = \frac{5}{2}\}$

If $k = -8$, the terms are: -8, 20, -50. ✓ $\{r = -\frac{5}{2}\}$

g $k,\ 3k,\ 20 - k$

Since the terms are geometric, $\dfrac{3k}{k} = \dfrac{20-k}{3k}$ {equating the common ratio r}

$$\therefore\ 9k^2 = k(20 - k)$$
$$\therefore\ 9k = 20 - k \qquad \{\text{since } k \neq 0\}$$
$$\therefore\ 10k = 20$$
$$\therefore\ k = 2$$

Check: If $k = 2$, the terms are: 2, 6, 18. ✓ $\{r = 3\}$

h $k,\ k+8,\ 9k$

Since the terms are geometric, $\dfrac{k+8}{k} = \dfrac{9k}{k+8}$ {equating the common ratio r}

$$\therefore\ (k+8)^2 = 9k^2$$
$$\therefore\ k^2 + 16k + 64 = 9k^2$$
$$\therefore\ 8k^2 - 16k - 64 = 0$$
$$\therefore\ 8(k^2 - 2k - 8) = 0$$
$$\therefore\ 8(k+2)(k-4) = 0$$
$$\therefore\ k = -2 \text{ or } 4$$

Check: If $k = -2$, the terms are: -2, 6, -18. ✓ $\{r = -3\}$

If $k = 4$, the terms are: 4, 12, 36. ✓ $\{r = 3\}$

7 $k - 1,\ 6,\ 3k$

a Since the terms are geometric, $\dfrac{6}{k-1} = \dfrac{3k}{6}$ {equating the common ratio r}

$$\therefore\ 36 = 3k(k-1)$$
$$\therefore\ 3k^2 - 3k - 36 = 0$$
$$\therefore\ 3(k^2 - k - 12) = 0$$
$$\therefore\ 3(k+3)(k-4) = 0$$
$$\therefore\ k = -3 \text{ or } 4$$

Check: If $k = -3$, the terms are: -4, 6, -9. ✓ $\{r = -\frac{3}{2}\}$

If $k = 4$, the terms are: 3, 6, 12. ✓ $\{r = 2\}$

b If $k = -3$, the next term is $-9 \times \left(-\frac{3}{2}\right) = \frac{27}{2}$.

If $k = 4$, the next term is $12 \times 2 = 24$.

8 **a**

$$u_4 = u_1 r^3 = 24 \quad .... (1)$$
$$\text{and} \quad u_7 = u_1 r^6 = 192 \quad .... (2)$$

$$\text{Now} \quad \frac{u_1 r^6}{u_1 r^3} = \frac{192}{24} \qquad \{(2) \div (1)\}$$
$$\therefore \; r^3 = 8$$
$$\therefore \; r = \sqrt[3]{8}$$
$$\therefore \; r = 2$$

$$\text{Using (1),} \quad u_1(2)^3 = 24$$
$$\therefore \; u_1 = 3$$
$$\text{Thus} \quad u_n = 3 \times 2^{n-1}$$

b

$$u_3 = u_1 r^2 = 8 \quad .... (1)$$
$$\text{and} \quad u_6 = u_1 r^5 = -1 \quad .... (2)$$

$$\text{Now} \quad \frac{u_1 r^5}{u_1 r^2} = \frac{-1}{8} \qquad \{(2) \div (1)\}$$
$$\therefore \; r^3 = -\tfrac{1}{8}$$
$$\therefore \; r = \sqrt[3]{-\tfrac{1}{8}}$$
$$\therefore \; r = -\tfrac{1}{2}$$

$$\text{Using (1),} \quad u_1\left(-\tfrac{1}{2}\right)^2 = 8$$
$$\therefore \; u_1 = 32$$
$$\text{Thus} \quad u_n = 32 \times \left(-\tfrac{1}{2}\right)^{n-1}$$

c

$$u_7 = u_1 r^6 = 24 \quad .... (1)$$
$$\text{and} \quad u_{15} = u_1 r^{14} = 384 \quad .... (2)$$

$$\text{Now} \quad \frac{u_1 r^{14}}{u_1 r^6} = \frac{384}{24} \qquad \{(2) \div (1)\}$$
$$\therefore \; r^8 = 16$$
$$\therefore \; r = \pm\sqrt[8]{16}$$
$$\therefore \; r = \pm\left(2^4\right)^{\frac{1}{8}}$$
$$\therefore \; r = \pm 2^{\frac{1}{2}}$$
$$\therefore \; r = \pm\sqrt{2}$$

$$\text{Using (1),} \quad u_1(\sqrt{2})^6 = 24$$
$$\therefore \; u_1 = 3$$

$$\text{Thus} \quad u_n = 3 \times (\sqrt{2})^{n-1}$$
$$\text{or} \quad u_n = 3 \times (-\sqrt{2})^{n-1}$$

d

$$u_3 = u_1 r^2 = 5 \quad .... (1)$$
$$\text{and} \quad u_7 = u_1 r^6 = \tfrac{5}{4} \quad .... (2)$$

$$\text{Now} \quad \frac{u_1 r^6}{u_1 r^2} = \frac{\frac{5}{4}}{5} \qquad \{(2) \div (1)\}$$
$$\therefore \; r^4 = \tfrac{1}{4}$$
$$\therefore \; r = \pm\sqrt[4]{\tfrac{1}{4}}$$
$$\therefore \; r = \pm\left(2^{-2}\right)^{\frac{1}{4}}$$
$$\therefore \; r = \pm 2^{-\frac{1}{2}}$$
$$\therefore \; r = \pm\tfrac{1}{\sqrt{2}}$$

$$\text{Using (1),} \quad u_1\left(\tfrac{1}{\sqrt{2}}\right)^2 = 5$$
$$\therefore \; u_1 = 10$$

$$\text{Thus} \quad u_n = 10 \times \left(\tfrac{1}{\sqrt{2}}\right)^{n-1} = 10 \times (\sqrt{2})^{1-n}$$
$$\text{or} \quad u_n = 10 \times \left(-\tfrac{1}{\sqrt{2}}\right)^{n-1} = 10 \times (-\sqrt{2})^{1-n}$$

9 **a**

$$u_3 = u_1 r^2 = 80 \quad .... (1)$$
$$\text{and} \quad u_6 = u_1 r^5 = 270 \quad .... (2)$$

$$\text{Now} \quad \frac{u_1 r^5}{u_1 r^2} = \frac{270}{80} \qquad \{(2) \div (1)\}$$
$$\therefore \; r^3 = \tfrac{27}{8}$$
$$\therefore \; r = \tfrac{3}{2}$$

$$\text{Using (1),} \quad u_1\left(\tfrac{3}{2}\right)^2 = 80$$
$$\therefore \; u_1 = \tfrac{320}{9} = 35\tfrac{5}{9}$$

b

$$u_1 = \tfrac{320}{9} \quad \text{and} \quad r = \tfrac{3}{2} \qquad \{\text{from } \textbf{a}\}$$
$$\text{Thus} \quad u_n = \tfrac{320}{9} \times \left(\tfrac{3}{2}\right)^{n-1}$$
$$\therefore \; u_{10} = \tfrac{320}{9} \times \left(\tfrac{3}{2}\right)^9$$
$$= \tfrac{320}{9} \times \tfrac{19\,683}{512}$$
$$= \tfrac{10\,935}{8}$$
$$= 1366\tfrac{7}{8}$$

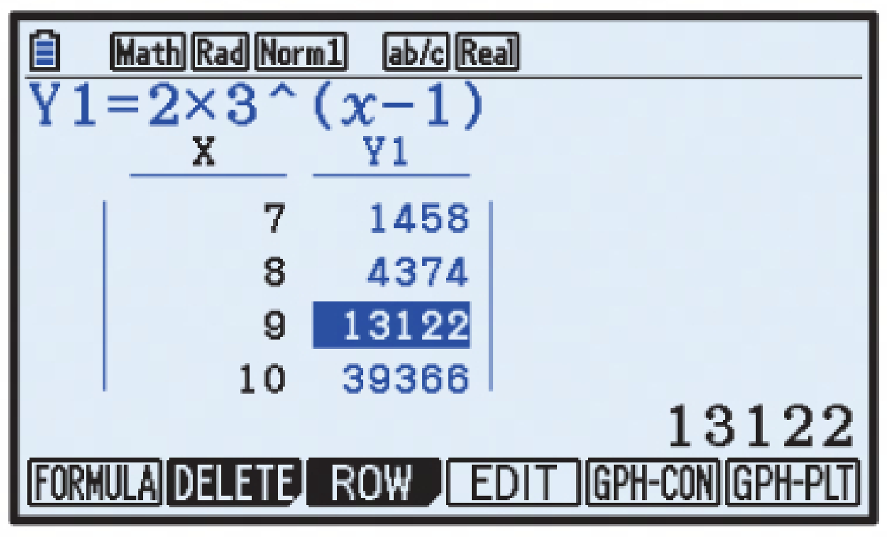
10 **a** 2, 6, 18, 54, ...

The sequence is geometric with $u_1 = 2$ and $r = 3$.

$\therefore \quad u_n = 2 \times 3^{n-1}$

We need to find n such that $u_n > 10\,000$.

Using a graphics calculator with $\text{Y}_1 = 2 \times 3\,^\wedge(\text{X} - 1)$, we view a table of values:



The first term to exceed 10 000 is $u_9 = 13\,122$.

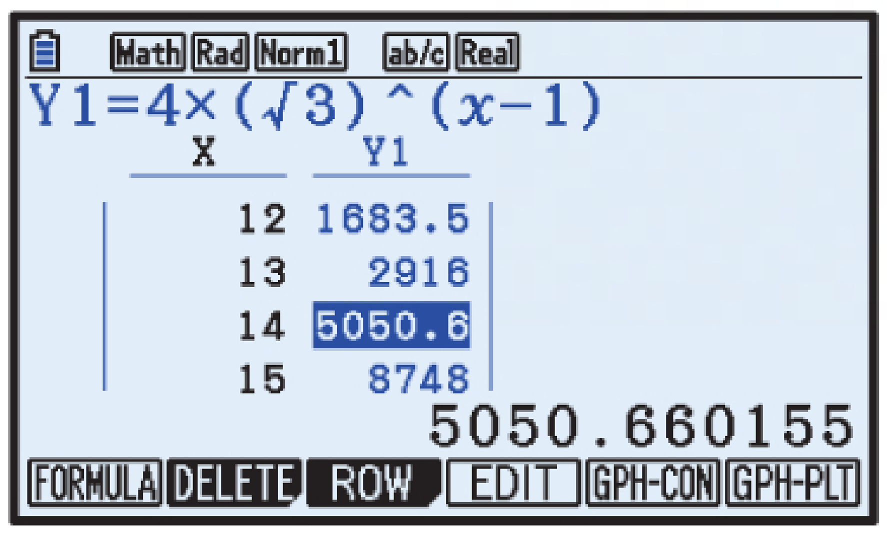
b $4,\ 4\sqrt{3},\ 12,\ 12\sqrt{3},\ ....$

The sequence is geometric with $u_1 = 4$ and $r = \sqrt{3}$.

$\therefore \quad u_n = 4 \times (\sqrt{3})^{n-1}$

We need to find n such that $u_n > 4800$.

Using a graphics calculator with $\text{Y}_1 = 4 \times (\sqrt{3})\,^\wedge(\text{X} - 1)$, we view a table of values:



The first term to exceed 4800 is $u_{14} = 2916\sqrt{3} \approx 5050$.

c 12, 6, 3, 1.5, ...

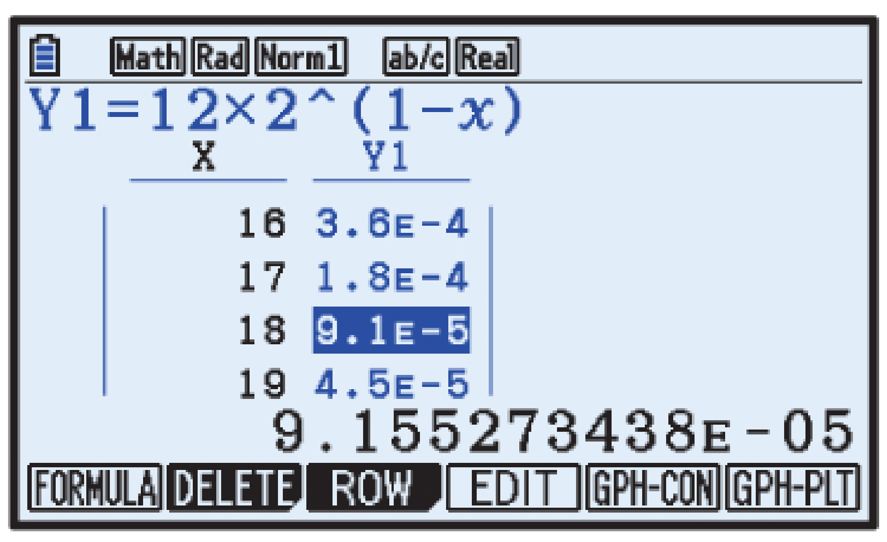
The sequence is geometric with $u_1 = 12$ and $r = \frac{1}{2}$.

$\therefore \quad u_n = 12 \times \left(\frac{1}{2}\right)^{n-1}$

$\therefore \quad u_n = 12 \times 2^{1-n}$

We need to find n such that $u_n < 0.0001$.

Using a graphics calculator with $\text{Y}_1 = 12 \times 2\,^\wedge(1 - \text{X})$, we view a table of values:



The first term which is less than 0.0001 is $u_{18} = \dfrac{3}{32\,768} \approx 0.000\,091\,6$.

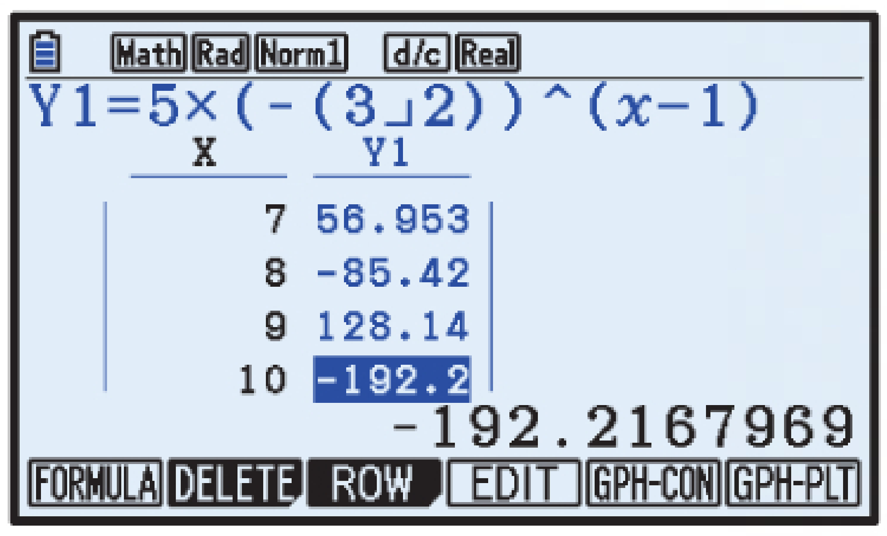
d $5, -\frac{15}{2}, \frac{45}{4}, -\frac{135}{8}, ...$

The sequence is geometric with $u_1 = 5$ and $r = -\frac{3}{2}$.

$\therefore \quad u_n = 5 \times \left(-\frac{3}{2}\right)^{n-1}$

We need to find n such that $u_n < -100$.

Using a graphics calculator with $Y_1 = 5 \times (-\frac{3}{2})^\wedge(X - 1)$, we view a table of values:



The first term which is less than -100 is $u_{10} = -\dfrac{98\,415}{512} \approx -192$.

11 Consider the general geometric sequence $u_n = u_1 \times r^{n-1}$ where u_1 is prime:

- If r is an integer, then all other terms are multiples of u_1 and hence there are no other primes.
- If r is rational, and not an integer, the most primes we can have in the sequence is 2, and all other terms will be non-integers.

  For example, $u_1 = 2$, $r = \frac{3}{2}$ has $u_2 = 2 \times \frac{3}{2} = 3$.
- If r is irrational, the most primes we can have in the sequence is 2, and all other terms will be non-integers.

  For example, $u_1 = 2$, $r = \dfrac{\sqrt{3}}{\sqrt{2}}$ has $u_2 = \dfrac{2\sqrt{3}}{\sqrt{2}} = \sqrt{6}$ and $u_3 = \dfrac{2 \times 3}{2} = 3$

$\therefore$ the maximum number of distinct prime numbers that can occur in a geometric sequence is 2.

12 **a** For the arithmetic sequence, $u_2 = u_1 + d$ and $u_3 = u_1 + 2d$

For the geometric sequence, $u_2 = u_1 \times r$ and $u_3 = u_1 \times r^2$

Since u_1 and u_2 are identical for each sequence, then $u_1 + d = u_1 \times r$

$$\therefore \quad d = u_1 r - u_1$$
$$\therefore \quad d = u_1(r - 1) \quad .... (1)$$

Also, the third terms of the geometric and arithmetic sequences are in the ratio $2:1$.

$$\therefore \quad u_1 \times r^2 = 2 \times (u_1 + 2d)$$
$$\therefore \quad u_1 r^2 = 2u_1 + 4d$$
$$\therefore \quad u_1 r^2 = 2u_1 + 4u_1(r - 1) \quad \{\text{using (1)}\}$$
$$\therefore \quad r^2 = 2 + 4r - 4$$
$$\therefore \quad r^2 - 4r = -2$$
$$\therefore \quad r^2 - 4r + (-2)^2 = -2 + (-2)^2 \quad \{\text{completing the square}\}$$
$$\therefore \quad (r - 2)^2 = 2$$
$$\therefore \quad r - 2 = \pm\sqrt{2}$$
$$\therefore \quad r = 2 \pm \sqrt{2}$$

b When $r = 2+\sqrt{2}$, $d = u_1(r-1)$ {using (1)}

$= u_1(1+\sqrt{2})$

For the arithmetic sequence, $u_4 = u_1 + 3d$

$= u_1 + 3u_1(1+\sqrt{2})$

$= u_1(1+3+3\sqrt{2})$

$= u_1(4+3\sqrt{2})$

For the geometric sequence, $u_4 = u_1 \times r^3$

$= u_1(2+\sqrt{2})^3$

$= u_1(2+\sqrt{2})(2+\sqrt{2})^2$

$= u_1(2+\sqrt{2})(4+4\sqrt{2}+2)$

$= u_1(2+\sqrt{2})(6+4\sqrt{2})$

$= 2u_1(2+\sqrt{2})(3+2\sqrt{2})$

$= 2u_1(6+4\sqrt{2}+3\sqrt{2}+4)$

$= 2u_1(10+7\sqrt{2})$

Now $\dfrac{2u_1(10+7\sqrt{2})}{u_1(4+3\sqrt{2})} = \left(\dfrac{2(10+7\sqrt{2})}{4+3\sqrt{2}}\right) \times \left(\dfrac{4-3\sqrt{2}}{4-3\sqrt{2}}\right)$

$= \dfrac{2(40-30\sqrt{2}+28\sqrt{2}-42)}{16-18}$

$= \dfrac{2(-2-2\sqrt{2})}{-2}$

$= 2+2\sqrt{2}$

So, the ratio of the fourth terms of the geometric to arithmetic sequences when $r = 2+\sqrt{2}$ is $2+2\sqrt{2} : 1$.

When $r = 2-\sqrt{2}$, $d = u_1(r-1)$ {using (1)}

$= u_1(1-\sqrt{2})$

For the arithmetic sequence, $u_4 = u_1 + 3d$

$= u_1 + 3u_1(1-\sqrt{2})$

$= u_1(1+3-3\sqrt{2})$

$= u_1(4-3\sqrt{2})$

For the geometric sequence, $u_4 = u_1 \times r^3$

$= u_1(2-\sqrt{2})^3$

$= u_1(2-\sqrt{2})(2-\sqrt{2})^2$

$= u_1(2-\sqrt{2})(4-4\sqrt{2}+2)$

$= u_1(2-\sqrt{2})(6-4\sqrt{2})$

$= 2u_1(2-\sqrt{2})(3-2\sqrt{2})$

$= 2u_1(6-4\sqrt{2}-3\sqrt{2}+4)$

$= 2u_1(10-7\sqrt{2})$

$$\text{Now}\quad \frac{2u_1(10-7\sqrt{2})}{u_1(4-3\sqrt{2})} = \left(\frac{2(10-7\sqrt{2})}{4-3\sqrt{2}}\right) \times \left(\frac{4+3\sqrt{2}}{4+3\sqrt{2}}\right)$$
$$= \frac{2(40+30\sqrt{2}-28\sqrt{2}-42)}{16-18}$$
$$= \frac{2(-2+2\sqrt{2})}{-2}$$
$$= 2-2\sqrt{2}$$

So, the ratio of the fourth terms of the geometric to arithmetic sequences when $r = 2-\sqrt{2}$ is $2-2\sqrt{2} : 1$.

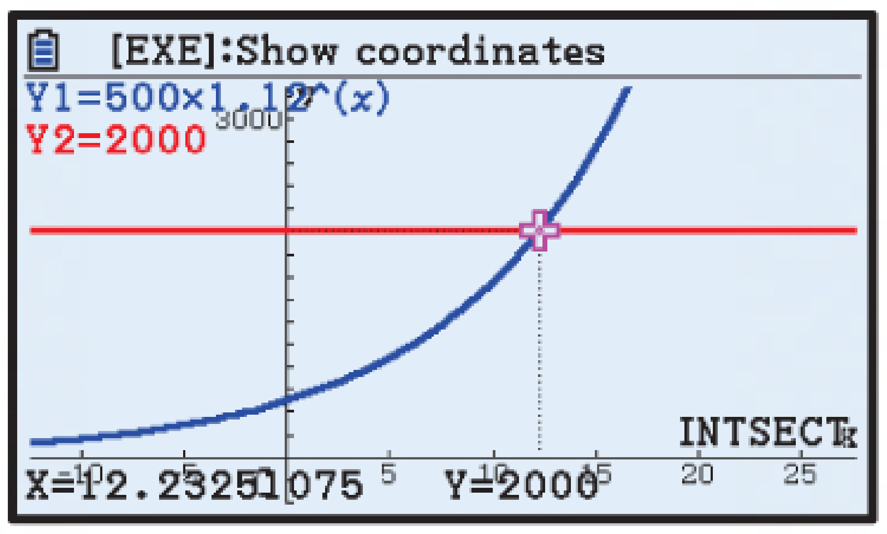
EXERCISE 5D

1 There is a fixed percentage increase each week, so the population forms a geometric sequence.
$u_0 = 500$ and $r = 1.12$
$\therefore$ the population after n weeks is $u_n = 500 \times (1.12)^n$.

a **i** $u_{10} = 500 \times (1.12)^{10}$
≈ 1552.92
There will be approximately 1550 ants after 10 weeks.

ii $u_{20} = 500 \times (1.12)^{20}$
≈ 4823.15
There will be approximately 4820 ants after 20 weeks.

b We need to find when $500 \times (1.12)^n = 2000$.

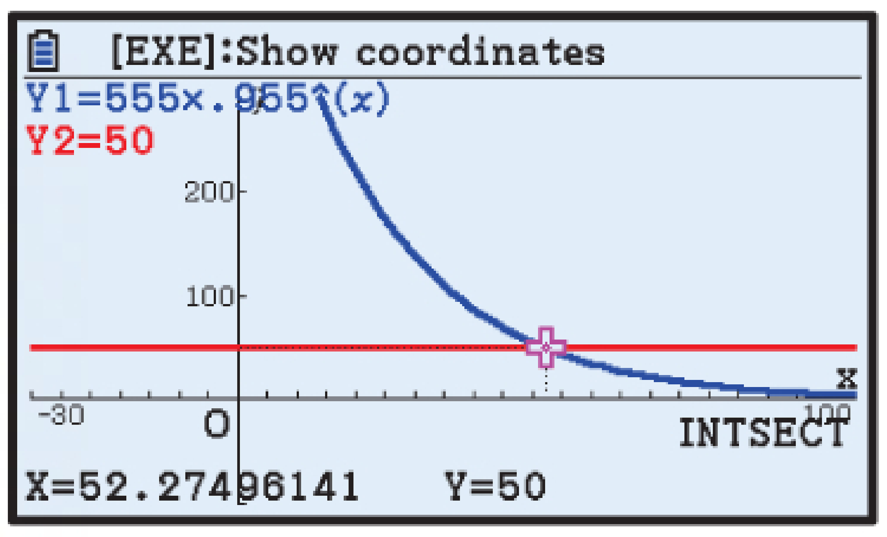


It will take approximately 12.2 weeks.

2 There is a fixed percentage decrease each year, so the population forms a geometric sequence.
$u_0 = 555$ and $r = 0.955$
$\therefore$ the population after n years is $u_n = 555 \times (0.955)^n$.

a 2020 is 15 years after 2005.
$u_{15} = 555 \times (0.955)^{15}$
≈ 278.19
The population is approximately 278 animals in the year 2020.

b For the population to have declined to 50, we need to find when $555 \times (0.955)^n = 50$.



So, in the 53rd year the population is 50. This is the year 2057.

3 There is a fixed percentage increase each year, so the herd size forms a geometric sequence.
$u_0 = 32$ and $r = 1.18$
$\therefore$ the herd size after n years is $u_n = 32 \times (1.18)^n$.

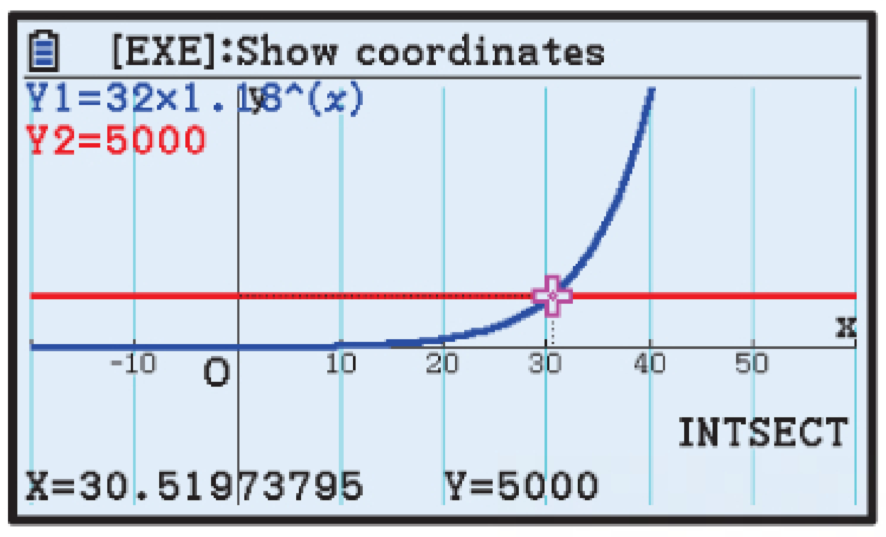
a **i** $u_5 = 32 \times (1.18)^5$
≈ 73.21

There will be approximately 73 deer after 5 years.

ii $u_{10} = 32 \times (1.18)^{10}$
≈ 167.48

There will be approximately 167 deer after 10 years.

b For the herd size to reach 5000, we need to find when $32 \times (1.18)^n = 5000$.



So, it will take approximately 30.5 years.

4 There is a fixed percentage increase each year, so the population forms a geometric sequence.
$u_0 = 178$ and $r = 1.32$
$\therefore$ the population after n years is $u_n = 178 \times (1.32)^n$.

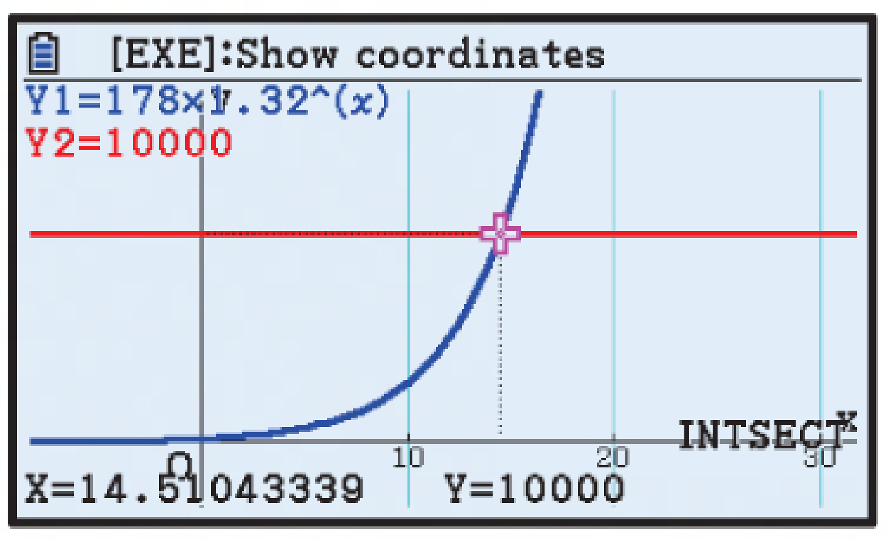
a **i** $u_{10} = 178 \times (1.32)^{10}$
≈ 2858.6

There will be approximately 2860 marsupials after 10 years.

ii $u_{25} = 178 \times (1.32)^{25}$
$\approx 183\,979.0$

There will be approximately 184 000 marsupials after 25 years.

b For the population to reach 10 000, we need to find when $178 \times (1.32)^n = 10\,000$.



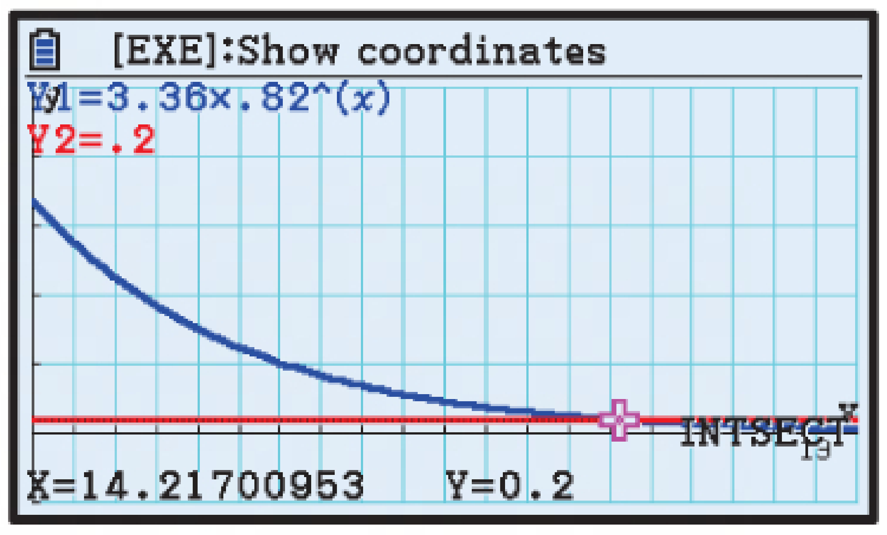
So, it will take approximately 14.5 years.

5 There is a fixed percentage decrease each year, so the amount of radioactive material forms a geometric sequence.
$u_4 = 1.52$ and $r = 0.82$

a
$$\begin{aligned} u_4 &= u_0 \times r^4 \\ \therefore\ 1.52 &= u_0 \times (0.82)^4 \\ \therefore\ u_0 &= \frac{1.52}{(0.82)^4} \\ &\approx 3.36 \end{aligned}$$

The initial quantity of radioactive material was approximately 3.36 g.

b The amount of radioactive material after n years is $u_n \approx 3.36 \times (0.82)^n$.
For the amount of radioactive material to reduce to 0.2 g, we need to find when $3.36 \times (0.82)^n = 0.2$.



So, it will take approximately 14.2 years, or 10.2 more years for the amount of radioactive material to reduce to 0.2 g.

6 There is a fixed percentage increase each year, so Maria's salary forms a geometric sequence.
$u_{10} = 49\,852$ and $r = 1.023$

a
$$\begin{aligned} u_{10} &= u_0 \times r^{10} \\ \therefore\ 49\,852 &= u_0 \times (1.023)^{10} \\ \therefore\ u_0 &= \frac{49\,852}{(1.023)^{10}} \\ &\approx 39\,712.41 \end{aligned}$$

So, Maria's salary was €39 712.41 p.a. when she joined the company.

b Maria's salary after n years is $u_n \approx 39\,712.41 \times (1.023)^n$.
$$\begin{aligned} u_{14} &\approx 39\,712.41 \times (1.023)^{14} \\ &\approx 54\,599.05 \end{aligned}$$
So, if Maria stays with the company for another 4 years, her salary will be €54 599.05 p.a.

EXERCISE 5E.1

1 The interest is calculated annually, so $n = 5$ time periods.
$$\begin{aligned} u_5 &= u_0 \times (1+i)^5 \\ &= 7000 \times (1.06)^5 \qquad \{6\% = 0.06\} \\ &\approx 9367.58 \end{aligned}$$
The investment will amount to £9367.58.

2 **a** The interest is calculated annually, so $n = 4$ time periods.
$$\begin{aligned} u_4 &= u_0 \times (1+i)^4 \\ &= 2000 \times (1.028)^4 \qquad \{2.8\% = 0.028\} \\ &\approx 2233.58 \end{aligned}$$
The investment will amount to €2233.58.

b The interest earned $=$ €2233.58 $-$ €2000
$=$ €233.58

3 The interest is calculated annually, so $n = 3$ time periods.

$$\begin{aligned} u_3 &= u_0 \times (1+i)^3 \\ &= 8000 \times (1.029)^3 \qquad \{2.9\% = 0.029\} \\ &\approx 8716.38 \end{aligned}$$

The investment will amount to \$8716.38.

$$\begin{aligned} \text{The interest earned} &= \$8716.38 - \$8000 \\ &= \$716.38 \end{aligned}$$

4 **a** The interest is calculated quarterly, so there are $n = 1 \times 4 = 4$ time periods.

Each time period the investment increases by $i = \dfrac{4.8\%}{4} = 1.2\%$.

$$\begin{aligned} \therefore \text{ the amount after 1 year is } \quad u_4 &= u_0 \times (1+i)^4 \\ &= 20\,000 \times (1.012)^4 \qquad \{1.2\% = 0.012\} \\ &\approx 20\,977.42 \end{aligned}$$

The investment will amount to \$20 977.42.

b The interest is calculated quarterly, so there are $n = 3 \times 4 = 12$ time periods.

$$\begin{aligned} \therefore \text{ the amount after 3 years is } \quad u_{12} &= u_0 \times (1+i)^{12} \\ &= 20\,000 \times (1.012)^{12} \\ &\approx 23\,077.89 \end{aligned}$$

The investment will amount to \$23 077.89.

5 **a** The interest is calculated annually, so $n = 4$ time periods.

$$\begin{aligned} u_4 &= u_0 \times (1+i)^4 \\ &= 30\,000 \times (1.056)^4 \qquad \{5.6\% = 0.056\} \\ &\approx 37\,305.85 \end{aligned}$$

The investment will amount to €37 305.85.

b The interest earned = €37 305.85 − €30 000
= €7305.85

6 The interest is calculated quarterly, so there are $n = 3 \times 4 = 12$ time periods.

Each time period the investment increases by $i = \dfrac{4.4\%}{4} = 1.1\%$.

$$\begin{aligned} \therefore \text{ the amount after 3 years is } \quad u_{12} &= u_0 \times (1+i)^{12} \\ &= 80\,000 \times (1.011)^{12} \qquad \{1.1\% = 0.011\} \\ &\approx 91\,222.90 \end{aligned}$$

The investment will amount to \$91 222.90.

$$\begin{aligned} \text{The interest earned} &= \$91\,222.90 - \$80\,000 \\ &= \$11\,222.90 \end{aligned}$$

7 *Bank A*:

The interest is calculated annually, so there are 10 time periods.

$$\begin{aligned} u_{10} &= u_0 \times (1+i)^{10} \\ &= 92\,000 \times (1.055)^{10} \qquad \{5\tfrac{1}{2}\% = 0.055\} \\ &\approx 157\,149.29 \end{aligned}$$

The investment will amount to $157 149.29 .

$$\begin{aligned} \text{The interest earned} &= \$157\,149.29 - \$92\,000 \\ &= \$65\,149.29 \end{aligned}$$

Bank B:

The interest is calculated quarterly, so there are $10 \times 4 = 40$ time periods.

Each time period the investment increases by $i = \dfrac{5.25\%}{4} = 1.3125\%$.

$\therefore$ the amount after 10 years is
$$\begin{aligned} u_{40} &= u_0 \times (1+i)^{40} \\ &= 92\,000 \times (1.013\,125)^{40} \qquad \{1.3125\% = 0.013\,125\} \\ &\approx 154\,991.94 \end{aligned}$$

The investment will amount to $154 991.94 .

$$\begin{aligned} \text{The interest earned} &= \$154\,991.94 - \$92\,000 \\ &= \$62\,991.94 \end{aligned}$$

Bank C:

The interest is calculated monthly, so there are $10 \times 12 = 120$ time periods.

Each time period the investment increases by $i = \dfrac{5\%}{12} = 0.41\overline{6}\%$.

$\therefore$ the amount after 10 years is
$$\begin{aligned} u_{120} &= u_0 \times (1+i)^{120} \\ &= 92\,000 \times (1.004\,1\overline{6})^{120} \qquad \{0.41\overline{6}\% = 0.004\,1\overline{6}\} \\ &\approx 151\,524.87 \end{aligned}$$

The investment will amount to $151 524.87 .

$$\begin{aligned} \text{The interest earned} &= \$151\,524.87 - \$92\,000 \\ &= \$59\,524.87 \end{aligned}$$

So, Bank A offers Jai the greatest interest on his inheritance.

8 The initial investment u_0 is unknown.

There are $n = 4$ time periods.

Now $u_4 = u_0 \times (1+i)^4$

$\therefore\ 20\,000 = u_0 \times (1.075)^4 \qquad \{7.5\% = 0.075\}$

$\therefore\ u_0 = \dfrac{20\,000}{(1.075)^4} \approx 14\,976.01$

Habib needs to invest £14 977 now. {rounded up to the next pound}

9 The initial investment u_0 is unknown.

There are $n = \frac{60}{12} = 5$ time periods.

Now $u_5 = u_0 \times (1+i)^5$

$\therefore\ 15\,000 = u_0 \times (1.055)^5 \qquad \{5.5\% = 0.055\}$

$\therefore\ u_0 = \dfrac{15\,000}{(1.055)^5} \approx 11\,477.02$

An initial investment of \$11 478 is required. {rounded up to the next dollar}

10 The initial investment u_0 is unknown.

There are $n = 3 \times 4 = 12$ time periods.

Each time period the investment increases by $i = \dfrac{4.2\%}{4} = 1.05\%$.

Now $u_{12} = u_0 \times (1+i)^{12}$

$\therefore\ 25\,000 = u_0 \times (1.0105)^{12} \qquad \{1.05\% = 0.0105\}$

$\therefore\ u_0 = \dfrac{25\,000}{(1.0105)^{12}} \approx 22\,054.85$

An investment of \$22 054.85 is required now.

11 The initial investment u_0 is unknown.

There are $n = 8 \times 12 = 96$ time periods.

Each time period the investment increases by $i = \dfrac{3.6\%}{12} = 0.3\%$.

Now $u_{96} = u_0 \times (1+i)^{96}$

$\therefore\ 4\,000\,000 = u_0 \times (1.003)^{96} \qquad \{0.3\% = 0.003\}$

$\therefore\ u_0 = \dfrac{4\,000\,000}{(1.003)^{96}} \approx 3\,000\,340$

An initial investment of ¥3 000 340 is required.

EXERCISE 5E.2

1 **a** To index the amount of money for inflation, we increase it by 3% each year for 2 years.

$\therefore$ indexed value $= \$8000 \times (1.03)^2$
$= \$8487.20$

b To index the amount of money for inflation, we increase it by 3% each year for 5 years.

$\therefore$ indexed value $= \$14\,000 \times (1.03)^5$
$= \$16\,229.84$

c To index the amount of money for inflation, we increase it by 3% each year for 7 years.

$\therefore$ indexed value $= \$22\,500 \times (1.03)^7$
$= \$27\,672.16$

2 **a** To index the amount of money for inflation, we increase it by 2% each year for 10 years.

$\therefore$ indexed value $= \$1000 \times (1.02)^{10}$
$= \$1218.99$

In 10 years' time Hoang will require \$1218.99 per week to maintain his current lifestyle.

b To index the amount of money for inflation, we increase it by 2% each year for 20 years.

$\therefore$ indexed value $= \$1000 \times (1.02)^{20}$
$= \$1485.95$

In 20 years' time Hoang will require \$1485.95 per week to maintain his current lifestyle.

c To index the amount of money for inflation, we increase it by 2% each year for 30 years.

$\therefore$ indexed value $= \$1000 \times (1.02)^{30}$
$= \$1811.36$

In 30 years' time Hoang will require \$1811.36 per week to maintain his current lifestyle.

3 To index the value of the holiday package for inflation, we increase it by 2% each year for 4 years.

$\therefore$ indexed value $= \$15\,000 \times (1.02)^4$
$= \$16\,236.48$

EXERCISE 5E.3

1 **a** There are $n = 3 \times 4 = 12$ time periods.

Each period, the investment increases by $i = \frac{3.6\%}{4} = 0.9\%$.

$\therefore$ the amount after 3 years is $u_{12} = u_0 \times (1+i)^{12}$
$= 5000 \times (1.009)^{12}$
≈ 5567.55

The investment will amount to \$5567.55 .

b real value $\times (1.02)^3 = \$5567.55$

$\therefore$ real value $= \frac{\$5567.55}{(1.02)^3}$
$= \$5246.43$

2 **a** There are $n = 4 \times 12 = 48$ time periods.

Each period, the investment increases by $i = \frac{4.2\%}{12} = 0.35\%$.

$\therefore$ the amount after 4 years is $u_{48} = u_0 \times (1+i)^{48}$
$= 20\,000 \times (1.0035)^{48}$
$\approx 23\,651.79$

The investment will amount to €23 651.79 .

b real value $\times (1.034)^4 =$ €23 651.79

$\therefore$ real value $= \frac{€23\,651.79}{(1.034)^4}$
$=$ €20 691.02

3 **a** There are $n = 6 \times 2 = 12$ time periods.
Each period, the investment increases by $i = \frac{3\%}{2} = 1.5\%$.

$$\therefore \text{ the amount after 6 years is } \quad \begin{aligned} u_{12} &= u_0 \times (1+i)^{12} \\ &= 4000 \times (1.015)^{12} \\ &\approx 4782.47 \end{aligned}$$

The final value of the investment is \$4782.47.

b The interest earned $= \$4782.47 - \$4000$
$= \$782.47$

c real value $\times (1.032)^6 = \$4782.47$

$$\therefore \text{ real value} = \frac{\$4782.47}{(1.032)^6} = \$3958.90$$

d The investment has not been effective. The real value of the investment after 6 years is less than what was originally invested.

4 **a** The account pays 0.5% interest per month and inflation is 0.1% per month, so the real interest rate $= \frac{1.005}{1.001} \approx 1.003\,996 \approx 0.4\%$ per month.

b There are $n = 2 \times 12 = 24$ time periods.
Each period, the investment increases by $i \approx 0.3996\%$. {from **a**}

$$\therefore \text{ the real value after 2 years is } \quad \begin{aligned} u_{24} &= u_0 \times (1+i)^{24} \\ &\approx 6000 \times (1.003\,996)^{24} \\ &\approx 6602.66 \end{aligned}$$

The investment will amount to \$6602.66.

5 The account pays $i\%$ interest per quarter and inflation is $r\%$ per quarter,

so the real interest rate $= \dfrac{1 + \frac{i}{100}}{1 + \frac{r}{100}} = \dfrac{100+i}{100+r}$ per quarter.

There are $n = y \times 4 = 4y$ time periods, and $\$u_0$ is initially invested.

$\therefore$ the real value of the investment after y years is $u_{4y} = u_0 \left(\dfrac{100+i}{100+r}\right)^{4y}$.

EXERCISE 5E.4

1 $u_3 = u_0 \times (1-d)^3$
$= 2500 \times (0.8)^3 \qquad \{20\% = 0.2\}$
$= 1280$

So, after 3 years the value of the lathe is €1280.

2 **a** $u_5 = u_0 \times (1-d)^5$
$= 110\,000 \times (0.75)^5 \quad \{25\% = 0.25\}$
$\approx 26\,103.52$

So, after 5 years the value of the tractor is €26 103.52 .

b The depreciation $=$ €110 000 $-$ €26 103.52
$=$ €83 896.48

3 **a** $u_3 = u_0 \times (1-d)^3$
$= 87\,500 \times (0.7)^3 \quad \{30\% = 0.3\}$
$= 30\,012.50$

So, after 3 years the value of the laptop is ¥30 013.

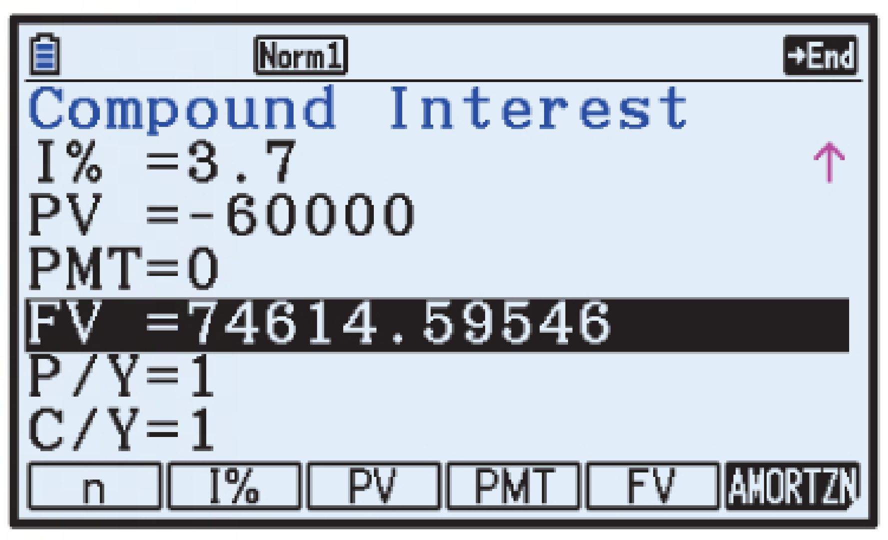
b The depreciation $=$ ¥87 500 $-$ ¥30 013
$=$ ¥57 487

4
$$
\begin{aligned}
u_4 &= u_0 \times (1-d)^4 \\
\therefore \quad 80\,000 &= 250\,000 \times (1-d)^4 \\
\therefore \quad (1-d)^4 &= \frac{80\,000}{250\,000} \\
&= 0.32 \\
\therefore \quad 1-d &= \pm\sqrt[4]{0.32} \\
\therefore \quad d &= 1 \pm \sqrt[4]{0.32} \\
\therefore \quad d &\approx 1.75 \text{ or } 0.248 \\
\therefore \quad d &\approx 0.248 \qquad \{\text{as } 0 < d < 1\}
\end{aligned}
$$

So, the printing press depreciated in value by 24.8% per year.

EXERCISE 5E.5

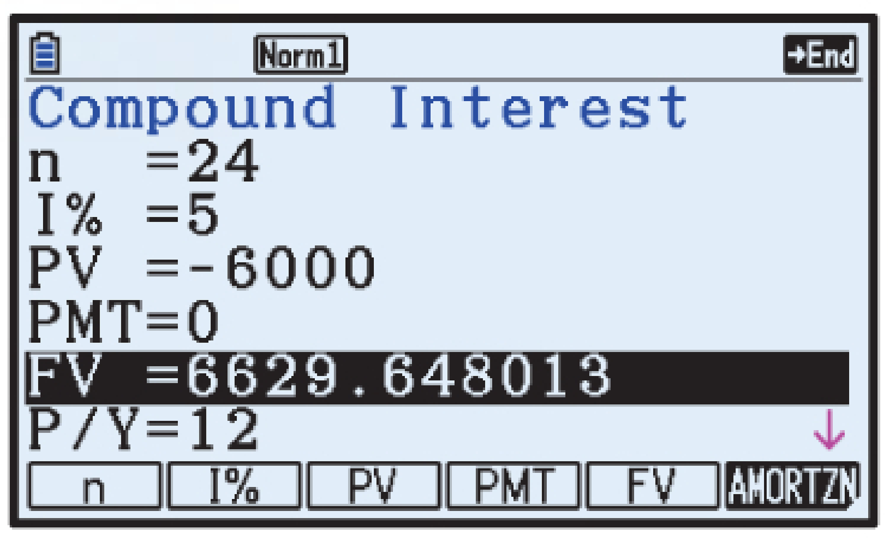
1 $N = 6, \quad I\% = 3.7, \quad PV = -60\,000, \quad PMT = 0, \quad P/Y = 1, \quad C/Y = 1$



$\therefore \quad FV \approx 74\,614.60$

Enrique's investment is worth 74 614.60 pesos after 6 years.

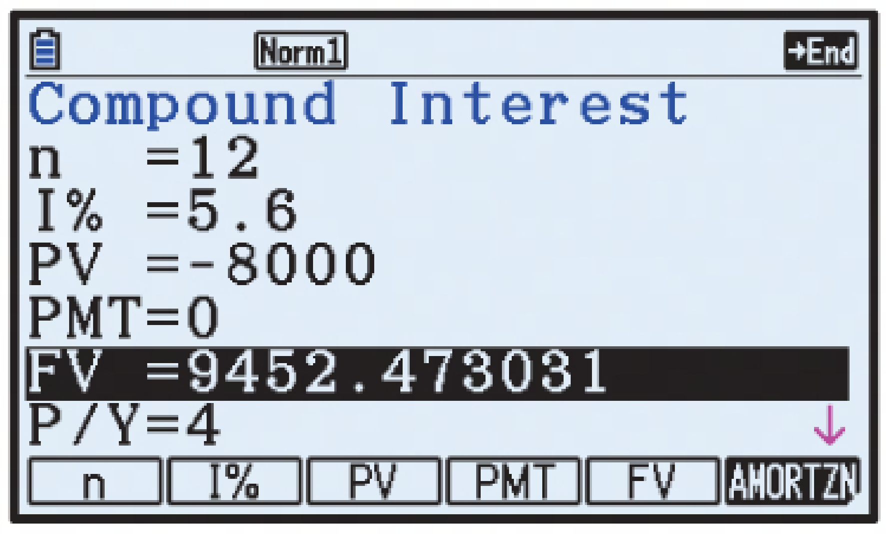
2 $N = 2 \times 12 = 24, \quad I\% = 5, \quad PV = -6000, \quad PMT = 0, \quad P/Y = 12, \quad C/Y = 12$



$\therefore \quad FV \approx 6629.65$

I will have \$6629.65 in my account after 2 years.

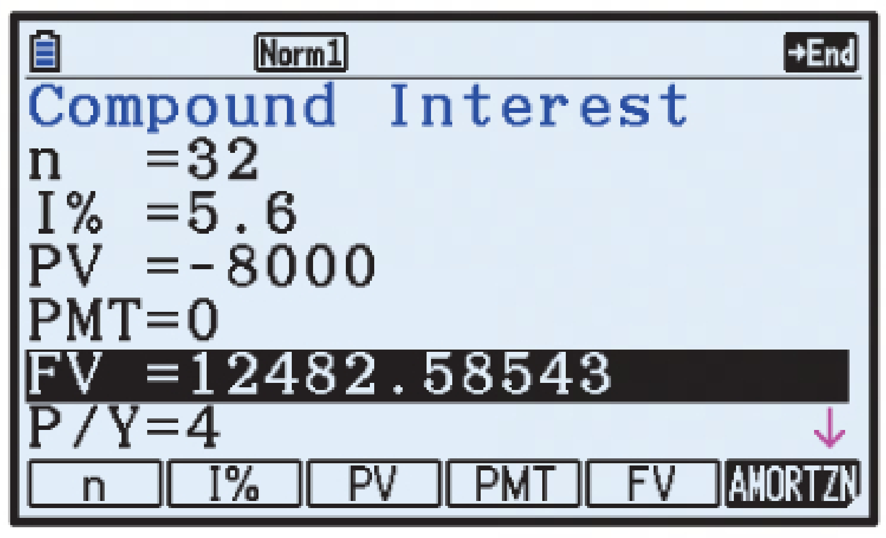
3 **a** $N = 3 \times 4 = 12, \quad I\% = 5.6, \quad PV = -8000, \quad PMT = 0, \quad P/Y = 4, \quad C/Y = 4$



$\therefore \quad FV \approx 9452.47$

Kenneth will have \$9452.47 in his account after 3 years.

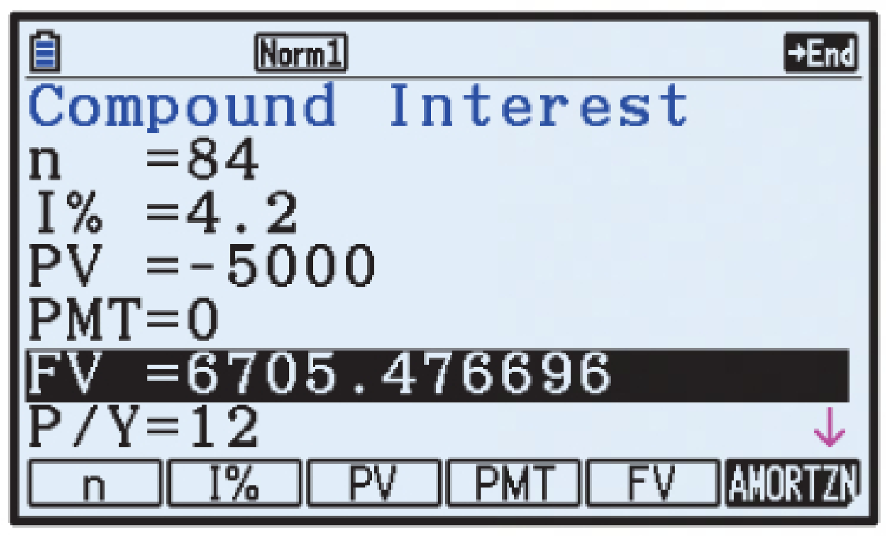
b $N = 8 \times 4 = 32, \quad I\% = 5.6, \quad PV = -8000, \quad PMT = 0, \quad P/Y = 4, \quad C/Y = 4$



$\therefore \quad FV \approx 12\,482.59$

Kenneth will have \$12 482.59 in his account after 8 years.

4 **a** $N = 7 \times 12 = 84, \quad I\% = 4.2, \quad PV = -5000, \quad PMT = 0, \quad P/Y = 12, \quad C/Y = 12$



$\therefore \quad FV \approx 6705.48$

There will be €6705.48 in the account after 7 years.

b Interest earned $= €6705.48 - €5000$
$= €1705.48$

5 **a** The account pays 1.2% interest per quarter and inflation is 0.5% per quarter.

So the real interest rate $= \dfrac{1.012}{1.005} \approx 1.006\,965 \approx 0.6965\%$ per quarter.

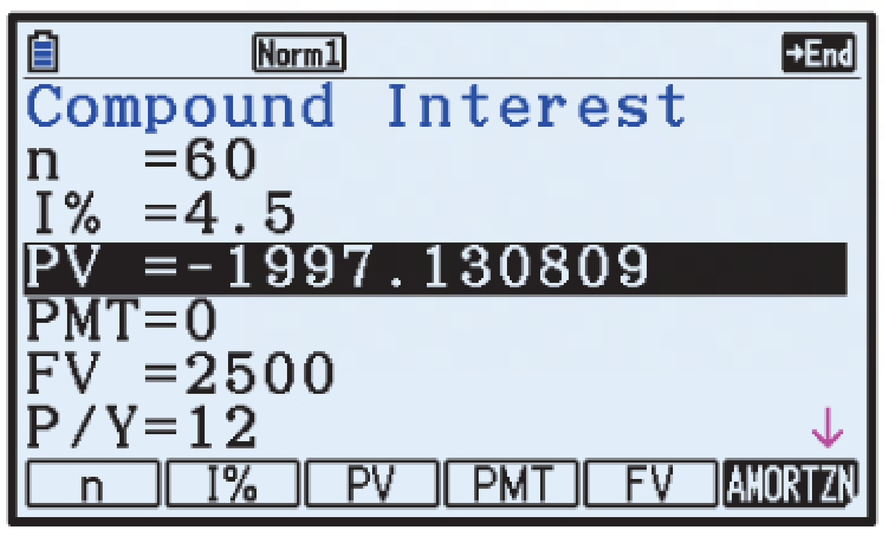
$\therefore$ the real interest rate per year $\approx (1.006\,965)^4 \approx 1.028\,153 \approx 2.82\%$ p.a.

b There are $n = 5$ time periods.

Each period, the investment increases by $i \approx 0.028\,153\%$. {from **a**}

$\therefore$ the real value of the investment after 5 years is $u_5 \approx 4000(1.028\,153)^5$
$\approx €4595.67$

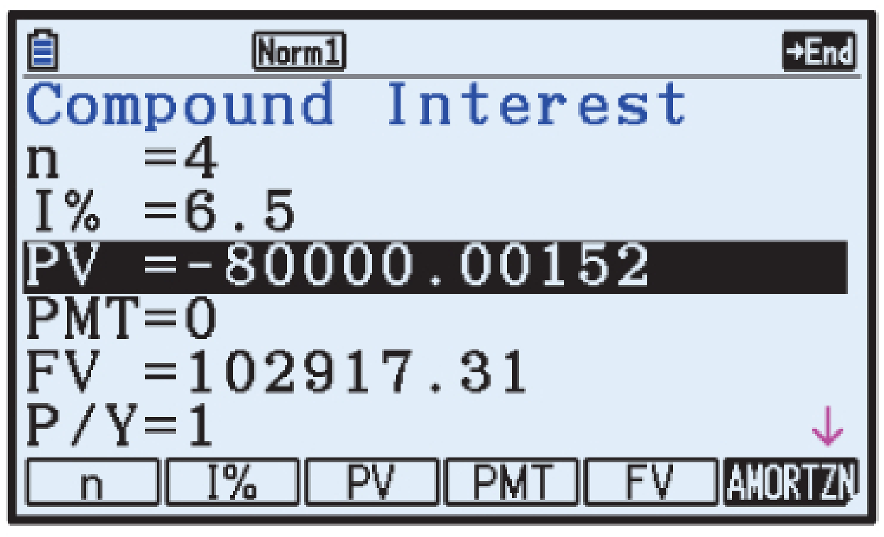
6 $N = 5 \times 12 = 60, \quad I\% = 4.5, \quad PMT = 0, \quad FV = 2500, \quad P/Y = 12, \quad C/Y = 12$



$\therefore \quad PV \approx -1997.13$

\$1997.13 needs to be invested now.

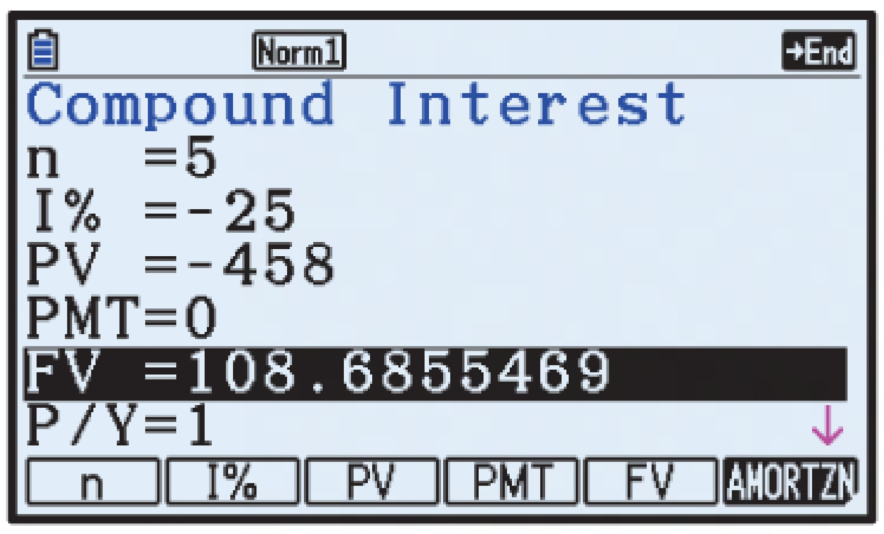
7 $N = 4, \quad I\% = 6.5, \quad PMT = 0, \quad FV = 102\,917.31, \quad P/Y = 1, \quad C/Y = 1$



$\therefore \quad PV \approx -80\,000.00$

You won \$80 000 in the lottery.

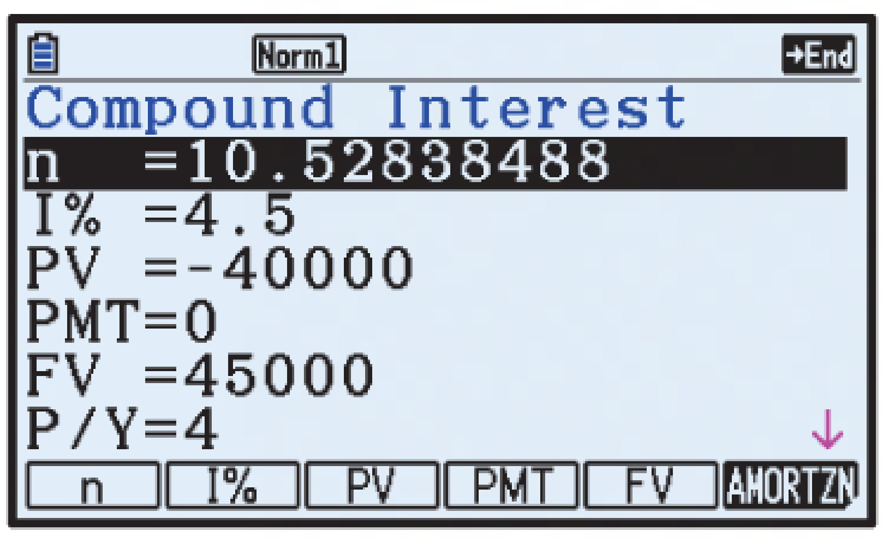
8 $N = 5, \quad I\% = -25, \quad PV = -458, \quad PMT = 0, \quad P/Y = 1, \quad C/Y = 1$



$\therefore \quad FV \approx 108.69$

The value of the stereo after 5 years is \$108.69 .

9 $I\% = 4.5, \quad PV = -40\,000, \quad PMT = 0, \quad FV = 45\,000, \quad P/Y = 4, \quad C/Y = 4$

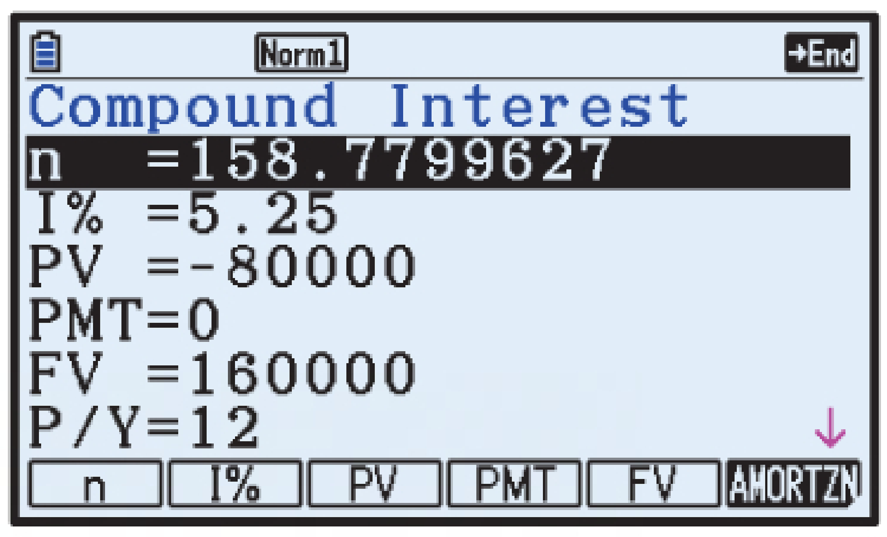


$\therefore \quad N \approx 10.5$

They kept the money in the account for 11 quarters, or 2 years 9 months.

{rounded up to the next quarter}

10 $I\% = 5.25, \quad PV = -80\,000, \quad PMT = 0, \quad FV = 160\,000, \quad P/Y = 12, \quad C/Y = 12$

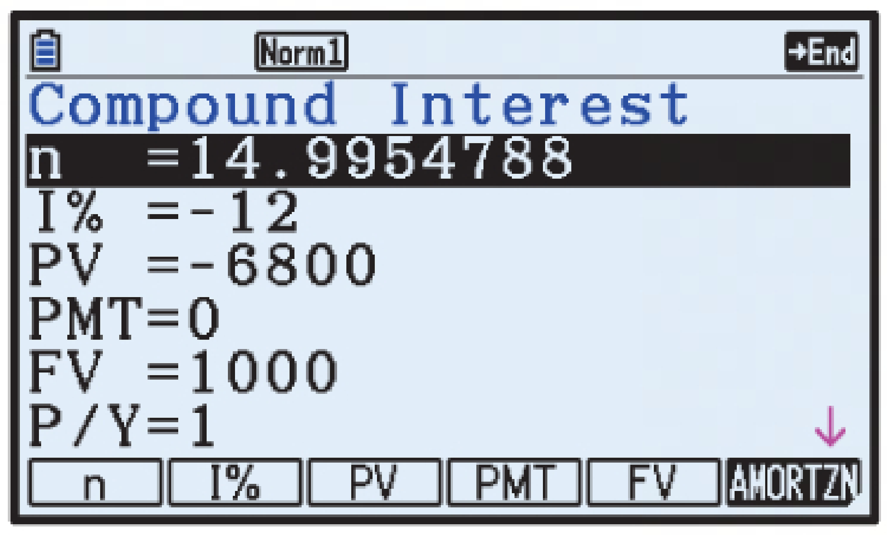


$\therefore \quad N \approx 158.8$

The money is doubled after 159 months, or 13 years 3 months.

{rounded up to the next month}

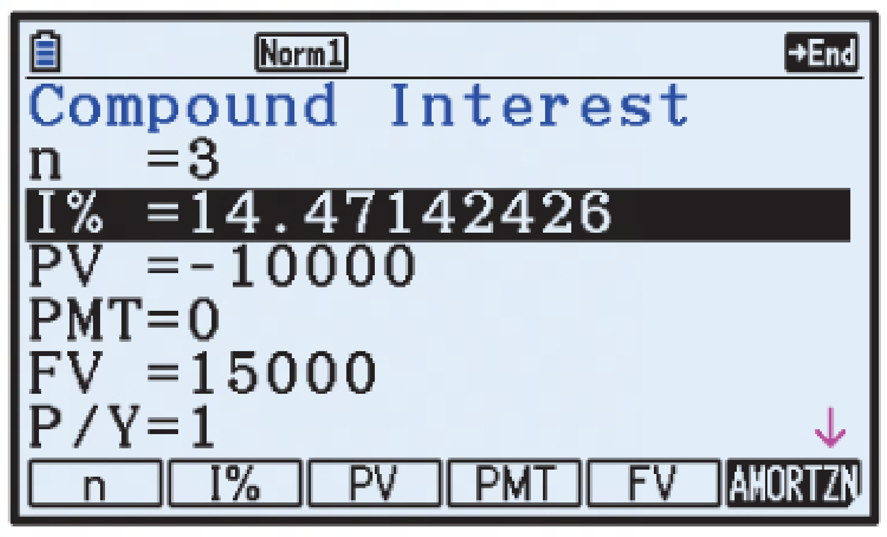
11 $I\% = -12, \quad PV = -6800, \quad PMT = 0, \quad FV = 1000, \quad P/Y = 1, \quad C/Y = 1$



$\therefore \quad N \approx 15.0$

It will take 15 years for the value to reduce to \$1000.

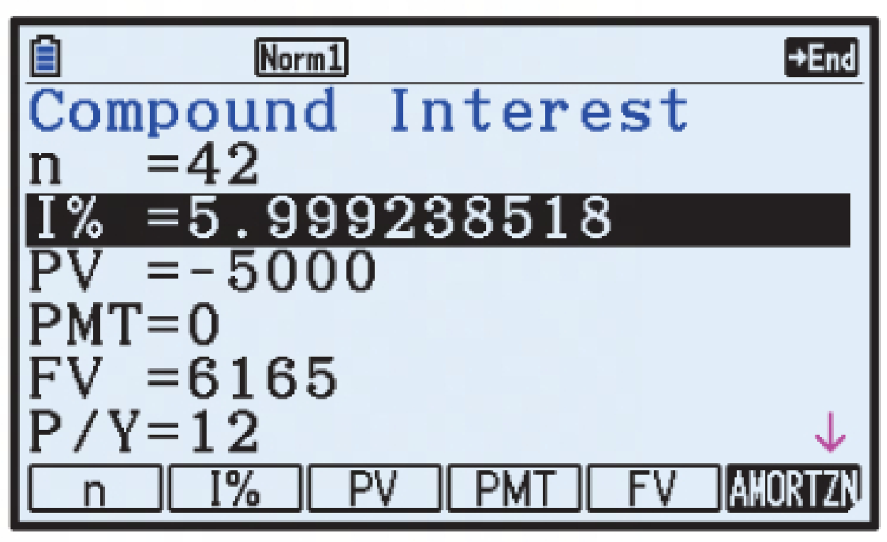
12 $N = 3, \quad PV = -10\,000, \quad PMT = 0, \quad FV = 15\,000, \quad P/Y = 1, \quad C/Y = 1$



$\therefore \quad I\% \approx 14.5$

An annual increase of 14.5% is required.

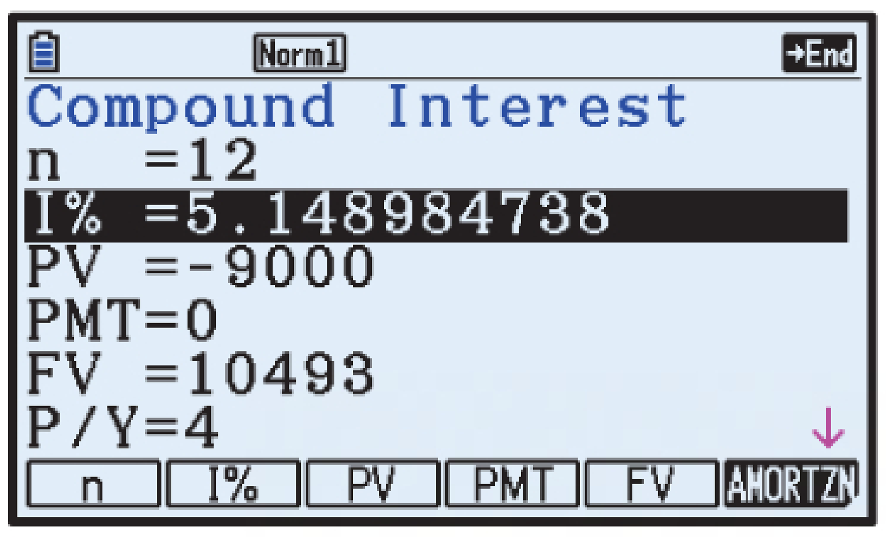
13 $N = 3.5 \times 12 = 42, \quad PV = -5000, \quad PMT = 0, \quad FV = 6165, \quad P/Y = 12, \quad C/Y = 12$



$\therefore \quad I\% \approx 6.00$

The account paid 6.00% p.a. interest.

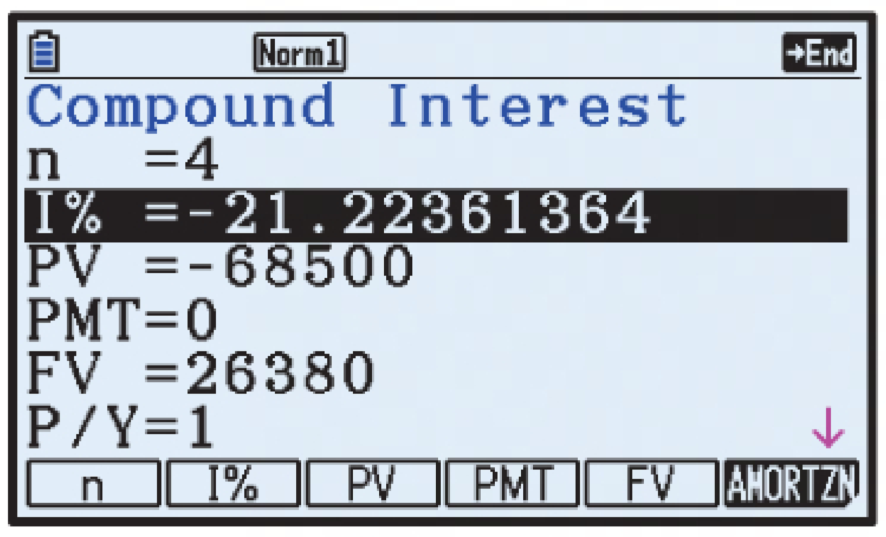
14 $N = 3 \times 4 = 12, \quad PV = -9000, \quad PMT = 0, \quad FV = 10\,493, \quad P/Y = 4, \quad C/Y = 4$



$\therefore \quad I\% \approx 5.15$

The interest rate paid was 5.15% p.a.

15 $N = 4, \quad PV = -68\,500, \quad PMT = 0, \quad FV = 26\,380, \quad P/Y = 1, \quad C/Y = 1$



$\therefore \quad I\% \approx -21.2$

The annual rate of depreciation was 21.2%.

EXERCISE 5F

1 4, 6, 8, 9, 10, 12, 14, 15, 16, ...

a $S_3 = 4 + 6 + 8$
$\quad = 18$

b $S_5 = 4 + 6 + 8 + 9 + 10$
$\quad = 37$

c $S_{12} = 4 + 6 + 8 + 9 + 10 + 12 + 14 + 15 + 16 + 18 + 20 + 21$
$\quad = 153$

2 $S_5 = S_4 + u_5$

$\therefore \quad 20 = 13 + u_5$

$\therefore \quad u_5 = 7$

3 **a** 3, 11, 19, 27, ...

i $11 - 3 = 8 \qquad 19 - 11 = 8 \qquad 27 - 19 = 8$

The difference between successive terms is constant.

$\therefore$ the sequence is arithmetic with $u_1 = 3$ and $d = 8$.

$u_n = u_1 + (n-1)d$

$\therefore \quad u_n = 3 + 8(n-1)$

$\therefore \quad u_n = 8n - 5$

Now $S_n = 3 + 11 + 19 + 27 + ... + (8n - 5)$

$$\therefore \quad S_n = \sum_{k=1}^{n} (8k - 5)$$

ii $S_5 = \sum_{k=1}^{5}(8k-5)$

$= 3 + 11 + 19 + 27 + 35$

$= 95$

b 42, 37, 32, 27, ...

i $37 - 42 = -5 \qquad 32 - 37 = -5 \qquad 27 - 32 = -5$

The difference between successive terms is constant.

$\therefore$ the sequence is arithmetic with $u_1 = 42$ and $d = -5$.

$u_n = u_1 + (n-1)d$

$\therefore \; u_n = 42 - 5(n-1)$

$\therefore \; u_n = 47 - 5n$

Now $S_n = 42 + 37 + 32 + 27 + ... + (47 - 5n)$

$\therefore \; S_n = \sum_{k=1}^{n}(47 - 5k)$

ii $S_5 = \sum_{k=1}^{5}(47-5k)$

$= 42 + 37 + 32 + 27 + 22$

$= 160$

c 12, 6, 3, $1\frac{1}{2}$, ...

i $\frac{6}{12} = \frac{1}{2} \qquad \frac{3}{6} = \frac{1}{2} \qquad \frac{1\frac{1}{2}}{3} = \frac{1}{2}$

Consecutive terms have a common ratio of $\frac{1}{2}$.

$\therefore$ the sequence is geometric with $u_1 = 12$ and $r = \frac{1}{2}$.

$u_n = u_1 r^{n-1}$

$\therefore \; u_n = 12 \times \left(\frac{1}{2}\right)^{n-1}$

$\therefore \; u_n = 12 \times 2^{1-n}$

Now $S_n = 12 + 6 + 3 + 1\frac{1}{2} + ... + (12 \times 2^{1-n})$

$\therefore \; S_n = \sum_{k=1}^{n}(12 \times 2^{1-k})$ or $\sum_{k=1}^{n} 12\left(\frac{1}{2}\right)^{k-1}$

ii $S_5 = \sum_{k=1}^{5}(12 \times 2^{1-k})$

$= 12 + 6 + 3 + 1\frac{1}{2} + \frac{3}{4}$

$= 23\frac{1}{4}$

d 2, 3, $4\frac{1}{2}$, $6\frac{3}{4}$, ...

i $\dfrac{3}{2} = \dfrac{3}{2}$ $\quad \dfrac{4\frac{1}{2}}{3} = \dfrac{3}{2}$ $\quad \dfrac{6\frac{3}{4}}{4\frac{1}{2}} = \dfrac{3}{2}$

Consecutive terms have a common ratio of $\frac{3}{2}$.

$\therefore$ the sequence is geometric with $u_1 = 2$ and $r = \frac{3}{2}$.

$$u_n = u_1 r^{n-1}$$

$$\therefore\ u_n = 2 \times \left(\tfrac{3}{2}\right)^{n-1}$$

Now $S_n = 2 + 3 + 4\frac{1}{2} + 6\frac{3}{4} + ... + \left(2 \times \left(\frac{3}{2}\right)^{n-1}\right)$

$$\therefore\ S_n = \sum_{k=1}^{n} \left(2 \times \left(\tfrac{3}{2}\right)^{k-1}\right) \quad \text{or} \quad \sum_{k=1}^{n} 2\left(\tfrac{3}{2}\right)^{k-1}$$

ii $S_5 = \sum_{k=1}^{5} \left(2 \times \left(\frac{3}{2}\right)^{k-1}\right)$

$= 2 + 3 + 4\frac{1}{2} + 6\frac{3}{4} + 10\frac{1}{8}$

$= 26\frac{3}{8}$

e 1, $\frac{1}{2}$, $\frac{1}{4}$, $\frac{1}{8}$, ...

i $\dfrac{\frac{1}{2}}{1} = \dfrac{1}{2}$ $\quad \dfrac{\frac{1}{4}}{\frac{1}{2}} = \dfrac{1}{2}$ $\quad \dfrac{\frac{1}{8}}{\frac{1}{4}} = \dfrac{1}{2}$

Consecutive terms have a common ratio of $\frac{1}{2}$.

$\therefore$ the sequence is geometric with $u_1 = 1$ and $r = \frac{1}{2}$.

$$u_n = u_1 r^{n-1}$$

$$\therefore\ u_n = 1 \times \left(\tfrac{1}{2}\right)^{n-1}$$

$$\therefore\ u_n = 2^{1-n}$$

Now $S_n = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + ... + 2^{1-n}$

$$\therefore\ S_n = \sum_{k=1}^{n} 2^{1-k} \quad \text{or} \quad \sum_{k=1}^{n} \frac{1}{2^{k-1}}$$

ii $S_5 = \sum_{k=1}^{5} 2^{1-k}$

$= 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16}$

$= 1\frac{15}{16}$

f 1, 8, 27, 64, ...

i $S_n = 1^3 + 2^3 + 3^3 + 4^3 + ... + n^3$ {all terms are cubes}

$$\therefore\ S_n = \sum_{k=1}^{n} k^3$$

ii $S_5 = 1 + 8 + 27 + 64 + 125$

$= 225$

4 **a** $\sum_{k=1}^{3} 4k = 4 + 8 + 12$

$= 24$

b $\sum_{k=1}^{6} (k+1) = 2 + 3 + 4 + 5 + 6 + 7$

$= 27$

c $\sum_{k=1}^{4} (3k-5) = -2 + 1 + 4 + 7$

$= 10$

d $\sum_{k=1}^{5} (11-2k) = 9 + 7 + 5 + 3 + 1$

$= 25$

e $\sum_{k=1}^{7} k(k+1) = 2 + 6 + 12 + 20 + 30 + 42 + 56$

$= 168$

f $\sum_{k=1}^{5} 10 \times 2^{k-1} = 10 + 20 + 40 + 80 + 160$

$= 310$

5 $u_n = 3n - 1$

$$\therefore\ u_1 + u_2 + u_3 + ... + u_{20} = \sum_{k=1}^{20} (3k - 1)$$
$$= 2 + 5 + 8 + 11 + 14 + 17 + 20 + 23 + 26 + 29 + 32 + 35 + 38 + 41 + 44 + 47 + 50 + 53 + 56 + 59$$
$$= 610$$

6 **a** $$\sum_{k=1}^{n} c = \underbrace{c + c + c + ... + c}_{n \text{ times}} = cn$$

b $$\sum_{k=1}^{n} ca_k = ca_1 + ca_2 + ... + ca_n$$
$$= c(a_1 + a_2 + ... + a_n)$$
$$= c \sum_{k=1}^{n} a_k$$

c $$\sum_{k=1}^{n} (a_k + b_k) = (a_1 + b_1) + (a_2 + b_2) + ... + (a_n + b_n)$$
$$= (a_1 + a_2 + ... + a_n) + (b_1 + b_2 + ... + b_n)$$
$$= \sum_{k=1}^{n} a_k + \sum_{k=1}^{n} b_k$$

7 **a** $$\sum_{k=1}^{n} k = 1 + 2 + 3 + ... + (n-1) + n$$
or $$n + (n-1) + (n-2) + ... + 2 + 1$$

b $$2\sum_{k=1}^{n} k = (n+1) + (n+1) + (n+1) + ... + (n+1) + (n+1)$$
$$= n(n+1)$$
$$\therefore\ \sum_{k=1}^{n} k = \frac{n(n+1)}{2}$$
$$\therefore\ S_n = \frac{n(n+1)}{2}$$

c $$\sum_{k=1}^{n} (ak + b) = \sum_{k=1}^{n} ak + \sum_{k=1}^{n} b$$
$$= a\sum_{k=1}^{n} k + nb$$
$$= \frac{an(n+1)}{2} + nb \qquad \{\text{using } \mathbf{b}\}$$

But $$\sum_{k=1}^{n} (ak + b) = 8n^2 + 11n$$
$$\therefore\ \frac{an(n+1)}{2} + nb = 8n^2 + 11n$$
$$\therefore\ \frac{an^2 + an}{2} + nb = 8n^2 + 11n$$
$$\therefore\ \frac{a}{2}n^2 + \frac{a}{2}n + nb = 8n^2 + 11n$$
$$\therefore\ \frac{a}{2}n^2 + \left(\frac{a}{2} + b\right)n = 8n^2 + 11n$$

Comparing coefficients, we get $\frac{a}{2} = 8$ and $\frac{a}{2} + b = 11$

$\therefore\ a = 16$ $\qquad$ $\therefore\ \frac{16}{2} + b = 11$

$\therefore\ b = 3$

8 The sequence of positive odd integers is $1, 3, 5, 7, 9, ...$

This is an arithmetic sequence with $u_1 = 1$ and $d = 2$.

$$u_n = u_1 + (n-1)d$$
$$\therefore \quad u_n = 1 + 2(n-1)$$
$$\therefore \quad u_n = 2n - 1$$

So, the sum of the first n positive odd integers can be represented by

$$\begin{aligned} S_n = \sum_{k=1}^{n}(2k-1) &= 1 + 3 + 5 + ... + (2n-3) + (2n-1) \\ \text{or} \quad & (2n-1) + (2n-3) + (2n-5) + ... + 3 + 1 \\ \therefore \quad 2\sum_{k=1}^{n}(2k-1) &= 2n + 2n + 2n + ... + 2n + 2n \\ &= n \times 2n \\ &= 2n^2 \\ \therefore \quad \sum_{k=1}^{n}(2k-1) &= n^2 \end{aligned}$$

So, $S_n = \sum_{k=1}^{n}(2k-1) = n^2$

9 **a**
$$\begin{aligned}(x+1)^3 &= (x+1)(x+1)^2 \\ &= (x+1)(x^2+2x+1) \\ &= x^3 + 2x^2 + x + x^2 + 2x + 1 \\ &= x^3 + 3x^2 + 3x + 1\end{aligned}$$

b
$$\begin{aligned}(0+1)^3 &= 0^3 + 3(0)^2 + 3(0) + 1 \\ (1+1)^3 &= 1^3 + 3(1)^2 + 3(1) + 1 \\ (2+1)^3 &= 2^3 + 3(2)^2 + 3(2) + 1 \\ (3+1)^3 &= 3^3 + 3(3)^2 + 3(3) + 1 \\ &\vdots \\ (n+1)^3 &= n^3 + 3n^2 + 3n + 1\end{aligned}$$

c Adding the terms in **b** vertically, gives

$$\begin{aligned}1^3 + 2^3 + 3^3 + ... + n^3 + (n+1)^3 &= 0^3 + 1^3 + 2^3 + 3^3 + ... + n^3 \\ &\quad + 3(0^2 + 1^2 + 2^2 + 3^2 + ... + n^2) \\ &\quad + 3(0 + 1 + 2 + 3 + ... + n) \\ &\quad + (n+1)(1) \\ \therefore \quad (n+1)^3 &= 3\sum_{k=1}^{n}k^2 + 3\sum_{k=1}^{n}k + (n+1)\end{aligned}$$

d $\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$

$$\text{Now} \quad (n+1)^3 = 3\sum_{k=1}^{n} k^2 + 3\sum_{k=1}^{n} k + (n+1) \qquad \{\text{from } \mathbf{c}\}$$

$$\therefore \quad n^3 + 3n^2 + 3n + 1 = 3\sum_{k=1}^{n} k^2 + 3\left(\frac{n(n+1)}{2}\right) + n + 1$$

$$\therefore \quad n^3 + 3n^2 + 2n = 3\sum_{k=1}^{n} k^2 + \frac{3n(n+1)}{2}$$

$$= 3\sum_{k=1}^{n} k^2 + \frac{3n^2}{2} + \frac{3n}{2}$$

$$\therefore \quad n^3 + \frac{3n^2}{2} + \frac{n}{2} = 3\sum_{k=1}^{n} k^2$$

$$\therefore \quad \sum_{k=1}^{n} k^2 = \frac{n^3}{3} + \frac{n^2}{2} + \frac{n}{6}$$

$$= \frac{2n^3 + 3n^2 + n}{6}$$

$$= \frac{n(2n^2 + 3n + 1)}{6}$$

$$\therefore \quad \sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$$

10 $\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$ and $\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$

$$\text{Now} \quad \sum_{k=1}^{n} (k+1)(k+2) = \sum_{k=1}^{n} (k^2 + 3k + 2)$$

$$= \sum_{k=1}^{n} k^2 + \sum_{k=1}^{n} 3k + \sum_{k=1}^{n} 2$$

$$= \frac{n(n+1)(2n+1)}{6} + 3\sum_{k=1}^{n} k + 2n$$

$$= \frac{n(n+1)(2n+1)}{6} + 3 \times \frac{n(n+1)}{2} + 2n$$

$$= \frac{n(n+1)(2n+1)}{6} + \frac{9n(n+1)}{6} + \frac{12n}{6}$$

$$= \frac{n(n+1)(2n+1) + 9n(n+1) + 12n}{6}$$

$$= \frac{n(2n^2 + 3n + 1) + n(9n + 9) + 12n}{6}$$

$$= \frac{n(2n^2 + 3n + 1 + 9n + 9 + 12)}{6}$$

$$= \frac{n(2n^2 + 12n + 22)}{6}$$

$$= \frac{2n(n^2 + 6n + 11)}{6}$$

$$= \frac{n(n^2 + 6n + 11)}{3}$$

When $n = 10$, $\displaystyle\sum_{k=1}^{10}(k+1)(k+2) = 6 + 12 + 20 + 30 + 42 + 56 + 72 + 90 + 110 + 132$
$= 570$

but $\displaystyle\sum_{k=1}^{n}(k+1)(k+2) = \frac{n(n^2+6n+11)}{3}$ {from above}

$\therefore$ when $n = 10$, $\displaystyle\sum_{k=1}^{10}(k+1)(k+2) = \frac{10(10^2+6(10)+11)}{3}$
$= \frac{10(171)}{3}$
$= 570$ ✓

EXERCISE 5G

1 **a** The series is arithmetic with $u_1 = 7$, $d = 2$, and $n = 10$.

Now $S_n = \frac{n}{2}(2u_1 + (n-1)d)$

$\therefore S_{10} = \frac{10}{2}(2 \times 7 + 9 \times 2)$
$= 5(14 + 18)$
$= 160$

b The series is arithmetic with $u_1 = 3$, $d = 4$, and $n = 20$.

Now $S_n = \frac{n}{2}(2u_1 + (n-1)d)$

$\therefore S_{20} = \frac{20}{2}(2 \times 3 + 19 \times 4)$
$= 10(6 + 76)$
$= 820$

c The series is arithmetic with $u_1 = \frac{1}{2}$, $d = \frac{5}{2}$, and $n = 50$.

Now $S_n = \frac{n}{2}(2u_1 + (n-1)d)$

$\therefore S_{50} = \frac{50}{2}\left(2 \times \frac{1}{2} + 49 \times \frac{5}{2}\right)$
$= 25(1 + 122\frac{1}{2})$
$= 3087\frac{1}{2}$

d The series is arithmetic with $u_1 = 100$, $d = -7$, and $n = 40$.

Now $S_n = \frac{n}{2}(2u_1 + (n-1)d)$

$\therefore S_{40} = \frac{40}{2}(2 \times 100 + 39 \times (-7))$
$= 20(200 - 273)$
$= -1460$

e The series is arithmetic with $u_1 = -31$, $d = 3$, and $n = 15$.

Now $S_n = \frac{n}{2}(2u_1 + (n-1)d)$

$\therefore S_{15} = \frac{15}{2}(2 \times (-31) + 14 \times 3)$
$= \frac{15}{2}(-62 + 42)$
$= -150$

f The series is arithmetic with $u_1 = 50$, $d = -\frac{3}{2}$, and $n = 80$.

Now $S_n = \frac{n}{2}(2u_1 + (n-1)d)$

$\therefore S_{80} = \frac{80}{2}\left(2 \times 50 + 79 \times (-\frac{3}{2})\right)$
$= 40\left(100 - \frac{237}{2}\right)$
$= -740$

2 **a** $5 + 8 + 11 + 14 + ... + 101$

The series is arithmetic with
$u_1 = 5$, $d = 3$, and $u_n = 101$.
First we need to find n.

$$\begin{aligned} \text{Now} \quad u_n &= 101 \\ \therefore \quad u_1 + (n-1)d &= 101 \\ \therefore \quad 5 + 3(n-1) &= 101 \\ \therefore \quad 3(n-1) &= 96 \\ \therefore \quad n - 1 &= 32 \\ \therefore \quad n &= 33 \end{aligned}$$

$$\begin{aligned} \text{Using} \quad S_n &= \frac{n}{2}(u_1 + u_n), \\ S_{33} &= \tfrac{33}{2}(5 + 101) \\ &= \tfrac{33}{2} \times 106 \\ &= 1749 \end{aligned}$$

b $37 + 33 + 29 + 25 + ... + 9$

The series is arithmetic with
$u_1 = 37$, $d = -4$, and $u_n = 9$.
First we need to find n.

$$\begin{aligned} \text{Now} \quad u_n &= 9 \\ \therefore \quad u_1 + (n-1)d &= 9 \\ \therefore \quad 37 - 4(n-1) &= 9 \\ \therefore \quad -4(n-1) &= -28 \\ \therefore \quad n - 1 &= 7 \\ \therefore \quad n &= 8 \end{aligned}$$

$$\begin{aligned} \text{Using} \quad S_n &= \frac{n}{2}(u_1 + u_n), \\ S_8 &= \tfrac{8}{2}(37 + 9) \\ &= 4 \times 46 \\ &= 184 \end{aligned}$$

c $50 + 49\frac{1}{2} + 49 + 48\frac{1}{2} + ... + (-20)$

The series is arithmetic with
$u_1 = 50$, $d = -\frac{1}{2}$, and $u_n = -20$,
First we need to find n.

$$\begin{aligned} \text{Now} \quad u_n &= -20 \\ \therefore \quad u_1 + (n-1)d &= -20 \\ \therefore \quad 50 - \tfrac{1}{2}(n-1) &= -20 \\ \therefore \quad -\tfrac{1}{2}(n-1) &= -70 \\ \therefore \quad n - 1 &= 140 \\ \therefore \quad n &= 141 \end{aligned}$$

$$\begin{aligned} \text{Using} \quad S_n &= \frac{n}{2}(u_1 + u_n), \\ S_{141} &= \tfrac{141}{2}(50 + (-20)) \\ &= \tfrac{141}{2} \times 30 \\ &= 2115 \end{aligned}$$

d $8 + 10\frac{1}{2} + 13 + 15\frac{1}{2} + ... + 83$

The series is arithmetic with
$u_1 = 8$, $d = \frac{5}{2}$, and $u_n = 83$.
First we need to find n.

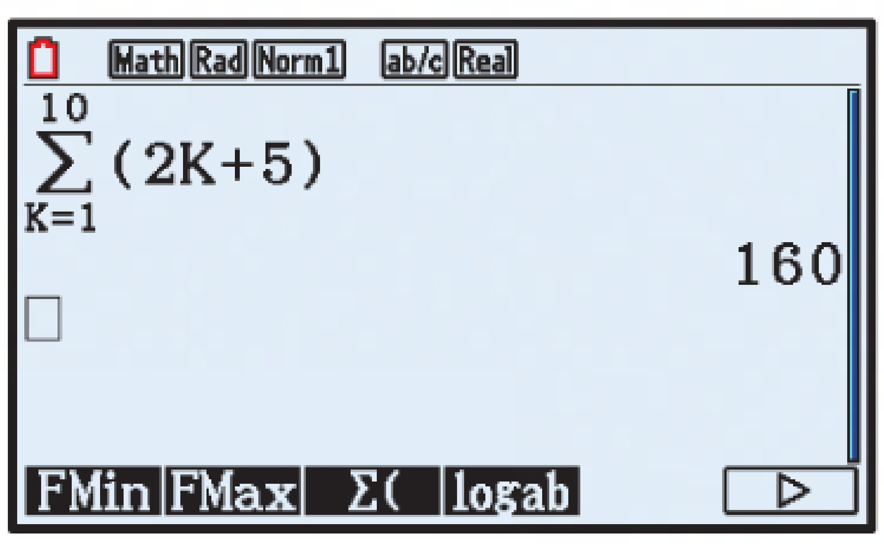
$$\begin{aligned} \text{Now} \quad u_n &= 83 \\ \therefore \quad u_1 + (n-1)d &= 83 \\ \therefore \quad 8 + \tfrac{5}{2}(n-1) &= 83 \\ \therefore \quad \tfrac{5}{2}(n-1) &= 75 \\ \therefore \quad n - 1 &= 30 \\ \therefore \quad n &= 31 \end{aligned}$$

$$\begin{aligned} \text{Using} \quad S_n &= \frac{n}{2}(u_1 + u_n), \\ S_{31} &= \tfrac{31}{2}(8 + 83) \\ &= \tfrac{31}{2} \times 91 \\ &= 1410\tfrac{1}{2} \end{aligned}$$

3 **a** $\displaystyle\sum_{k=1}^{10}(2k+5) = 7 + 9 + 11 + ... + 25$

This series is arithmetic with $u_1 = 7$, $d = 2$, and $n = 10$.

$$\begin{aligned} \text{Using} \quad S_n &= \frac{n}{2}(u_1 + u_n), \\ S_{10} &= \tfrac{10}{2}(7 + 25) \\ &= 5 \times 32 \\ &= 160 \end{aligned}$$



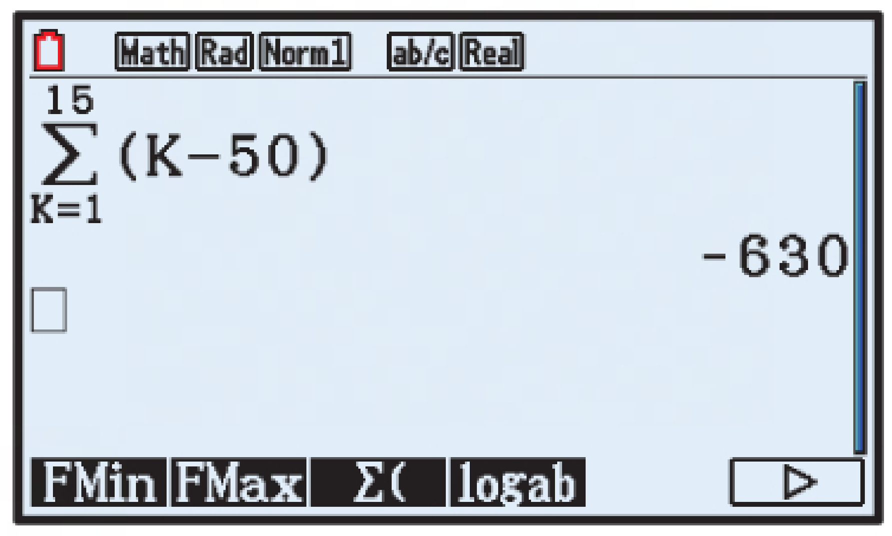
$\displaystyle\sum_{k=1}^{10}(2k+5) = 160$ ✓

b $\sum_{k=1}^{15}(k-50) = (-49)+(-48)+(-47)+\ldots.+(-35)$

This series is arithmetic with $u_1 = -49$, $d = 1$, and $n = 15$.

Using $S_n = \frac{n}{2}(u_1 + u_n)$,

$$S_{15} = \tfrac{15}{2}(-49+(-35))$$
$$= \tfrac{15}{2} \times (-84)$$
$$= -630$$



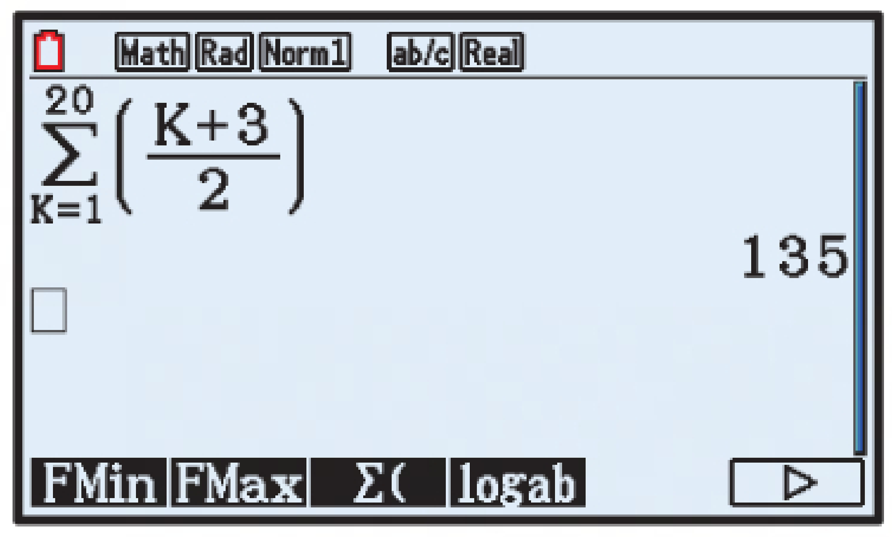
$\sum_{k=1}^{15}(k-50) = -630$ ✓

c $\sum_{k=1}^{20}\left(\frac{k+3}{2}\right) = 2 + \frac{5}{2} + 3 + \ldots. + \frac{23}{2}$

This series is arithmetic with $u_1 = 2$, $d = \frac{1}{2}$, and $n = 20$.

Using $S_n = \frac{n}{2}(u_1 + u_n)$,

$$S_{20} = \tfrac{20}{2}\left(2 + \tfrac{23}{2}\right)$$
$$= 10 \times \tfrac{27}{2}$$
$$= 135$$

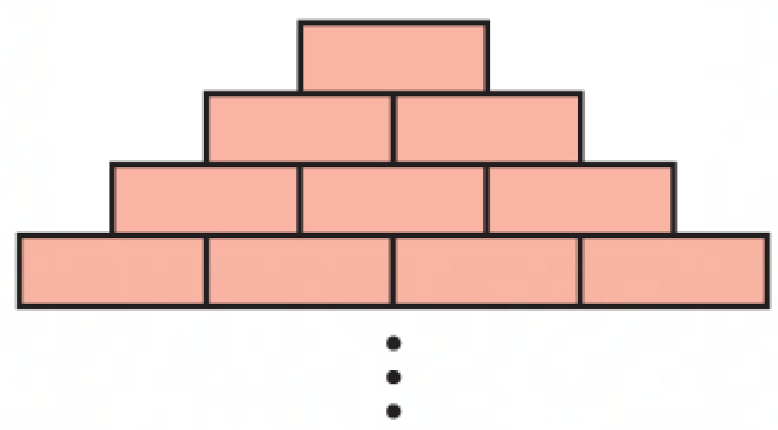


$\sum_{k=1}^{20}\left(\frac{k+3}{2}\right) = 135$ ✓

4 $u_1 = 6$, $n = 11$, $u_n = -27$

$$S_n = \frac{n}{2}(u_1 + u_n)$$
$$\therefore \quad S_{11} = \tfrac{11}{2}(6+(-27))$$
$$= \tfrac{11}{2} \times (-21)$$
$$= -115\tfrac{1}{2}$$

5 The total number of bricks can be expressed as an arithmetic series: $1+2+3+4+\ldots.+n$
We know that the total number of bricks is 171, so $S_n = 171$.
Also, $u_1 = 1$, $d = 1$ and we need to find n, the number of terms (layers) of the series.



$$S_n = \frac{n}{2}(2u_1 + (n-1)d) = 171$$
$$\therefore \quad \frac{n}{2}(2 \times 1 + (n-1) \times 1) = 171$$
$$\therefore \quad n(2+n-1) = 342$$
$$\therefore \quad n(n+1) = 342$$
$$\therefore \quad n^2 + n - 342 = 0$$
$$\therefore \quad (n+19)(n-18) = 0$$
$$\therefore \quad n = -19 \text{ or } 18$$

But $n > 0$, so $n = 18$

So, the bricklayer built 18 layers.

6 **a** The number of laps Vicki swims each day can be expressed as an arithmetic sequence 20, 22, 24, 26, ...

So, $u_1 = 20$ and $d = 2$.

$$u_n = u_1 + (n-1)d$$
$$\therefore\ u_n = 20 + 2(n-1)$$
$$\therefore\ u_n = 2n + 18$$

i $u_{10} = 2(10) + 18$
$= 38$

Vicki swims 38 laps on the tenth day.

ii $u_{30} = 2(30) + 18$
$= 78$

Vicki swims 78 laps on the final day.

b $S_n = \frac{n}{2}(u_1 + u_n)$

$$\therefore\ S_{30} = \tfrac{30}{2}(20 + 78)$$
$$= 15 \times 98$$
$$= 1470$$

Vicki swims 1470 laps in total.

7 **a** The amount of money the woman deposits each birthday can be expressed as an arithmetic sequence 100, 125, 150, ...

So, $u_1 = 100$ and $d = 25$.

$$u_n = u_1 + (n-1)d$$
$$\therefore\ u_n = 100 + 25(n-1)$$
$$\therefore\ u_n = 25n + 75$$
$$u_{15} = 25(15) + 75$$
$$= 450$$

The woman will deposit \$450 into her son's account on his 15th birthday.

b $S_n = \frac{n}{2}(u_1 + u_n)$

$$\therefore\ S_{15} = \tfrac{15}{2}(100 + 450)$$
$$= \tfrac{15}{2} \times 550$$
$$= 4125$$

The woman will have deposited \$4125 over the 15 years.

8 The total number of seats in n rows can be expressed as an arithmetic series: $22 + 23 + 24 + ... + u_n$

Row 1 has 22 seats, so $u_1 = 22$. Row 2 has 23 seats, so $d = 1$.

$$S_n = \frac{n}{2}\left(2u_1 + (n-1)d\right)$$
$$= \frac{n}{2}\left(2 \times 22 + 1(n-1)\right)$$
$$= \frac{n}{2}(44 + n - 1)$$
$$\therefore\ S_n = \frac{n}{2}(n + 43)$$ which is the total number of seats in n rows.

a Number of seats in row 44 of one section

$= $ total number of seats in every row $-$ number of seats in the first 43 rows

$$= S_{44} - S_{43}$$
$$= \tfrac{44}{2}(44 + 43) - \tfrac{43}{2}(43 + 43)$$
$$= 1914 - 1849$$
$$= 65 \text{ seats}$$

b Number of seats in each section $= S_{44} = 1914$ seats {from **a**}

c Number of seats in the whole stadium (25 sections) $= S_{44} \times 25$

$$= 1914 \times 25$$
$$= 47\,850 \text{ seats}$$

9 **a** The sum of the first 50 multiples of 11 can be expressed as an arithmetic series:

$11 + 22 + 33 + ... + u_{50}$ where $u_1 = 11$, $d = 11$, $n = 50$.

$$S_n = \frac{n}{2}(2u_1 + (n-1)d)$$
$$\therefore \quad S_{50} = \tfrac{50}{2}(2 \times 11 + 49 \times 11)$$
$$= 25(22 + 539)$$
$$= 14\,025$$

b The sum of the multiples of 7 between 0 and 1000 can be expressed as an arithmetic series:

$7 + 14 + 21 + 28 + ... + u_n$ where $u_1 = 7$, $d = 7$.

To find u_n, we need to find the largest multiple of 7 less than 1000.

Now $\dfrac{1000}{7} \approx 142.9$, so $u_n = 142 \times 7 = 994$

Also, $n = 142$ {since $u_n = 994$ is the 142nd multiple of 7}

$$S_n = \frac{n}{2}(u_1 + u_n)$$
$$\therefore \quad S_{142} = \tfrac{142}{2}(7 + 994)$$
$$= 71\,071$$

c The integers from 1 to 100 which are not divisible by 3 can be expressed as:

1, 2, 4, 5, 7, 8, ..., 100 where $u_1 = 1$, $u_n = 100$.

Alternatively, these integers can be expressed as two separate arithmetic series A and B:

$S_A = 1 + 4 + 7 + ... + 97 + 100$ where $u_1 = 1$, $d = 3$, $u_n = 100$

and $S_B = 2 + 5 + 8 + ... + 95 + 98$ where $u_1 = 2$, $d = 3$, $u_n = 98$

Now for S_A, $u_n = u_1 + (n-1)d$

$\therefore \ 100 = 1 + 3(n-1)$

$\therefore \ 99 = 3(n-1)$

$\therefore \ 33 = n - 1$

$\therefore \ n = 34$

and for S_B, $u_n = u_1 + (n-1)d$

$\therefore \ 98 = 2 + 3(n-1)$

$\therefore \ 96 = 3(n-1)$

$\therefore \ 32 = n - 1$

$\therefore \ n = 33$

Using $S_n = \frac{n}{2}(u_1 + u_n)$, $S_A = \frac{34}{2}(1 + 100) = 1717$ and $S_B = \frac{33}{2}(2 + 98) = 1650$

The total sum $= S_A + S_B$

$$= 1717 + 1650$$
$$= 3367$$

d The three-digit numbers which start or end with a "4" can be expressed as:

104, 114, 124, 134, ..., 394, 400, 401, 402, ..., 499, 504, 514, 524, ..., 994 where $u_1 = 104$, $u_n = 994$.

Alternatively, these integers can be expressed as two separate arithmetic series A and B:

$S_A = 104 + 114 + 124 + ... + 984 + 994$ where $u_1 = 104$, $d = 10$, $u_n = 994$

and $S_B = 400 + 401 + 402 + ... + 498 + 499$ where $u_1 = 400$, $d = 1$, $u_n = 499$

The numbers 404, 414, 424, ..., 494 appear in both series. We name this arithmetic series C:
$S_C = 404 + 414 + 424 + ... + 494$ where $u_1 = 404$, $d = 10$, $u_n = 494$.

Now for S_A, $u_n = u_1 + (n-1)d$
$\therefore\ 994 = 104 + 10(n-1)$
$\therefore\ 890 = 10(n-1)$
$\therefore\ 89 = n - 1$
$\therefore\ n = 90$

for S_B, $u_n = u_1 + (n-1)d$
$\therefore\ 499 = 400 + 1(n-1)$
$\therefore\ 99 = n - 1$
$\therefore\ n = 100$

and for S_C, $u_n = u_1 + (n-1)d$
$\therefore\ 494 = 404 + 10(n-1)$
$\therefore\ 90 = 10(n-1)$
$\therefore\ 9 = n - 1$
$\therefore\ n = 10$

Using $S_n = \frac{n}{2}(u_1 + u_n)$, $S_A = \frac{90}{2}(104 + 994) = 49\,410$, $S_B = \frac{100}{2}(400 + 499) = 44\,950$,

and $S_C = \frac{10}{2}(404 + 494) = 4490$.

The total sum $= S_A + S_B - S_C$
$= 49\,410 + 44\,950 - 4490$
$= 89\,870$

10 $k - 1,\ 2k + 3,\ 27 - k$

a Since the terms are consecutive,
$(2k+3) - (k-1) = (27-k) - (2k+3)$ {equating differences}
$\therefore\ k + 4 = 24 - 3k$
$\therefore\ 4k = 20$
$\therefore\ k = 5$

b Since $k = 5$, the first three terms are $5 - 1 = 4$, $2(5) + 3 = 13$, and $27 - 5 = 22$.
The series is arithmetic with $u_1 = 4$, $d = 9$, and $n = 25$.

Now $S_n = \frac{n}{2}(2u_1 + (n-1)d)$

$\therefore\ S_{25} = \frac{25}{2}(2 \times 4\ +\ 24 \times 9)$
$= 2800$

11 $u_6 = 21$, $S_{17} = 0$

We need to find u_1 and u_2.

$S_n = \frac{n}{2}(2u_1 + (n-1)d)$ $\therefore\ S_{17} = \frac{17}{2}(2u_1 + 16d) = 0$
$\therefore\ u_1 + 8d = 0$
$\therefore\ u_1 = -8d$ .... (1)

Also, $u_n = u_1 + (n-1)d$
$\therefore\ u_6 = u_1 + 5d$
$\therefore\ 21 = -8d + 5d$ {using (1)}
$\therefore\ 21 = -3d$
$\therefore\ d = -7$

So, $u_1 = -8(-7) = 56$ and $u_2 = 56 - 7 = 49$

The first two terms are 56 and 49.

12 $S_3 = 9$ and $S_6 = 90$

$$S_n = \frac{n}{2}(2u_1 + (n-1)d)$$
$$\therefore\ S_3 = \tfrac{3}{2}(2u_1 + 2d)$$
$$\therefore\ 9 = 3u_1 + 3d$$
$$\therefore\ u_1 + d = 3$$
$$\therefore\ d = 3 - u_1 \quad .... (1)$$

Also, $S_6 = \tfrac{6}{2}(2u_1 + 5d)$
$$\therefore\ 90 = 3(2u_1 + 5d)$$
$$\therefore\ 30 = 2u_1 + 5d$$
$$\therefore\ 5d = 30 - 2u_1$$
$$\therefore\ d = 6 - \tfrac{2}{5}u_1 \quad .... (2)$$

Equating (1) and (2) gives $3 - u_1 = 6 - \tfrac{2}{5}u_1$
$$\therefore\ -\tfrac{3}{5}u_1 = 3$$
$$\therefore\ u_1 = -5$$

Substituting $u_1 = -5$ into (1) gives $d = 3 - (-5)$
$$\therefore\ d = 8$$

Now $S_{10} = \tfrac{10}{2}(2 \times (-5)\ +\ 9 \times 8)$
$$= 310$$

13 The sequence is arithmetic with $u_1 = 4$ and $d = 6$.

Now $S_n = 200$, so $\dfrac{n}{2}(2u_1 + (n-1)d) = 200$
$$\therefore\ \frac{n}{2}(2 \times 4 + 6(n-1)) = 200$$
$$\therefore\ \frac{n}{2}(8 + 6(n-1)) = 200$$
$$\therefore\ 4n + 3n(n-1) = 200$$
$$\therefore\ 3n^2 + n - 200 = 0$$
$$\therefore\ (3n + 25)(n - 8) = 0$$
$$\therefore\ n = -\tfrac{25}{3} \text{ or } 8$$
$$\therefore\ n = 8 \quad \{\text{as } n > 0\}$$

$\therefore$ there are 8 terms in the sequence.

14 **a** The sequence is arithmetic with $u_1 = 7$ and $S_2 = 17$.

Now $S_2 = 17$, so $\dfrac{n}{2}(2u_1 + (n-1)d) = 17$
$$\therefore\ \tfrac{2}{2}(2 \times 7 + (2-1)d) = 17$$
$$\therefore\ 14 + d = 17$$
$$\therefore\ d = 3$$

b $S_n = 242$, so $\dfrac{n}{2}(2u_1 + (n-1)d) = 242$
$$\therefore\ \frac{n}{2}(2 \times 7 + 3(n-1)) = 242$$
$$\therefore\ \frac{n}{2}(14 + 3n - 3) = 242$$
$$\therefore\ n(3n + 11) = 484$$
$$\therefore\ 3n^2 + 11n - 484 = 0$$
$$\therefore\ (3n + 44)(n - 11) = 0$$
$$\therefore\ n = -\tfrac{44}{3} \text{ or } 11$$
$$\therefore\ n = 11 \quad \{\text{as } n > 0\}$$

$\therefore$ there are 11 terms in the sequence.

15 13, 21, 29, 37, ... is arithmetic with $u_1 = 13$ and $d = 8$.

Now $S_n = 1000$, so $\frac{n}{2}(2u_1 + (n-1)d) = 1000$

$$\therefore \quad \frac{n}{2}(2 \times 13 + 8(n-1)) = 1000$$
$$\therefore \quad \frac{n}{2}(26 + 8n - 8) = 1000$$
$$\therefore \quad \frac{n}{2}(8n + 18) = 1000$$
$$4n^2 + 9n = 1000$$
$$\therefore \quad 4n^2 + 9n - 1000 = 0$$

Using technology, $n \approx -16.98$ or 14.73

$\therefore \quad n \approx 14.73 \quad \{\text{as } n > 0\}$

But n must be an integer, and the sequence must exceed 1000, so we round up to 15.

$\therefore \quad n = 15$ which means 15 terms are required.

16 Using the arithmetic sequence formula $u_n = u_1 + (n-1)d$, the terms are

$u_1,\ u_1 + d,\ u_1 + 2d,\ ....,\ u_n - 2d,\ u_n - d,\ u_n,$ or, in reverse order, the terms are

$u_n,\ u_n - d,\ u_n - 2d,\ ....,\ u_1 + 2d,\ u_1 + d,\ u_1$

Adding the terms together vertically, we obtain

$$2 \times S_n = \underbrace{(u_1 + u_n) + (u_1 + u_n) + (u_1 + u_n) + ... + (u_1 + u_n) + (u_1 + u_n) + (u_1 + u_n)}_{n \text{ of these}}$$

$$\therefore \quad 2S_n = n(u_1 + u_n)$$
$$\therefore \quad S_n = \frac{n}{2}(u_1 + u_n)$$
$$= \frac{n}{2}(1 + n) \qquad \{\text{since } u_1 = 1 \text{ and } u_n = n \text{ for the sequence of the first } n \text{ integers}\}$$
$$= \frac{n(n+1)}{2} \quad \text{as required}$$

17 **a** The series of odd integers can be expressed as an arithmetic series:

$1 + 3 + 5 + 7 + ...$ where $u_1 = 1$ and $d = 2$.

Now $u_n = u_1 + (n-1)d$

$$= 1 + 2(n-1)$$
$$\therefore \quad u_n = 2n - 1$$

b We need to show that S_n is n^2.

The sum of the first n odd integers can be expressed as an arithmetic series:

$1 + 3 + 5 + 7 + ... + (2n - 1) \qquad \{\text{using } u_n = 2n - 1 \text{ from } \mathbf{a}\}$

So, $S_n = \frac{n}{2}(u_1 + u_n)$

$$= \frac{n}{2}(1 + (2n - 1))$$
$$= \frac{n}{2}(2n)$$
$$\therefore \quad S_n = n^2 \quad \text{as required}$$

18 Let the three consecutive terms be $x - d$, x, and $x + d$.

Now, sum of terms $= 12$

$\therefore$ $(x - d) + x + (x + d) = 12$

$\therefore$ $3x = 12$

$\therefore$ $x = 4$

So, the terms are $4 - d$, 4, $4 + d$

Also, product of terms $= -80$

$\therefore$ $(4 - d)(4)(4 + d) = -80$

$\therefore$ $4(4^2 - d^2) = -80$

$\therefore$ $16 - d^2 = -20$

$\therefore$ $d^2 = 36$

$\therefore$ $d = \pm 6$

So, the three terms could be $4 - 6$, 4, $4 + 6$, which are -2, 4, 10
or $4 - (-6)$, 4, $4 + (-6)$, which are 10, 4, -2.

19

$$S_{15} = 480$$

$$\therefore \frac{n}{2}(u_1 + u_n) = 480$$

$$\therefore \tfrac{15}{2}(u_1 + u_{15}) = 480$$

$$\therefore u_1 + u_{15} = 64$$

$$\therefore u_1 + (u_1 + 14d) = 64 \qquad \{\text{since } u_n = u_1 + (n-1)d\}$$

$$\therefore 2u_1 + 14d = 64$$

$$\therefore u_1 + 7d = 32$$

$$\therefore u_8 = 32$$

20 Let the five consecutive terms be $x - 2d$, $x - d$, x, $x + d$, and $x + 2d$.
The sum of the terms is 40.

$$\therefore (x - 2d) + (x - d) + x + (x + d) + (x + 2d) = 40$$

$$\therefore 5x = 40$$

$$\therefore x = 8$$

So, the terms are $8 - 2d$, $8 - d$, 8, $8 + d$, $8 + 2d$.

The product of the first, middle, and last terms is 224.

$$\therefore (8 - 2d)(8)(8 + 2d) = 224$$

$$\therefore 8(8^2 - (2d)^2) = 224$$

$$\therefore 64 - 4d^2 = 28$$

$$\therefore 4d^2 = 36$$

$$\therefore d^2 = 9$$

$$\therefore d = \pm 3$$

So, the five terms could be $8 - 2(3)$, $8 - 3$, 8, $8 + 3$, $8 + 2(3)$, which are $2, 5, 8, 11, 14$
or $8 - 2(-3)$, $8 - (-3)$, 8, $8 + (-3)$, $8 + 2(-3)$, which are $14, 11, 8, 5, 2$.

21 **a** $S_n = \dfrac{n(3n + 11)}{2}$

When $n = 1$, $S_1 = \dfrac{1(3(1) + 11)}{2} = 7$

So, $u_1 = 7$.

When $n = 2$, $S_2 = \dfrac{2(3(2) + 11)}{2} = 17$

So, $u_2 = S_2 - S_1$
$= 17 - 7$
$= 10$

$\therefore$ $u_1 = 7$ and $u_2 = 10$

b The sequence is arithmetic with $u_1 = 7$ and $d = 10 - 7 = 3$.

$$u_n = u_1 + (n - 1)d$$

$$\therefore u_n = 7 + 3(n - 1)$$

$$\therefore u_n = 3n + 4$$

$$\therefore u_{20} = 3(20) + 4$$

$$\therefore u_{20} = 64$$

22 $3-5+7-9+11-13+15-\ldots$ can be expressed as two separate arithmetic series:

$3+7+11+15+\ldots$ where $u_1=3,\ d=4,\ n=40$

and $-5-9-13-\ldots$ where $u_1=-5,\ d=-4,\ n=40$

Using $S_n=\frac{n}{2}(2u_1+(n-1)d)$,

$$\begin{aligned}\text{the sum of the first series} &= \tfrac{40}{2}(2(3)+39(4))\\ &= 20(6+156)\\ &= 3240\\ \text{and the sum of the second series} &= \tfrac{40}{2}(2(-5)+39(-4))\\ &= 20(-10-156)\\ &= -3320\\ \therefore\ \text{the sum of both series} &= 3240+(-3320)\\ &= -80\end{aligned}$$

So, $3-5+7-9+11-13+15-\ldots$ to 80 terms is -80.

23 $S_n=n^2-\frac{9}{2}n$

Now $u_n=S_n-S_{n-1}$

$$\begin{aligned}&=n^2-\tfrac{9}{2}n-\left((n-1)^2-\tfrac{9}{2}(n-1)\right)\\ &=n^2-\tfrac{9}{2}n-\left(n^2-2n+1-\tfrac{9}{2}n+\tfrac{9}{2}\right)\\ &=n^2-\tfrac{9}{2}n-n^2+2n-1+\tfrac{9}{2}n-\tfrac{9}{2}\\ &=2n-\tfrac{11}{2}\end{aligned}$$

When $n=1$, $u_1=2(1)-\frac{11}{2}=-\frac{7}{2}$

Now $u_n=u_1+(n-1)d$

$\therefore\ 2n-\frac{11}{2}=-\frac{7}{2}+(n-1)d$

$\therefore\ 2n-2=(n-1)d$

$\therefore\ (n-1)d=2(n-1)$

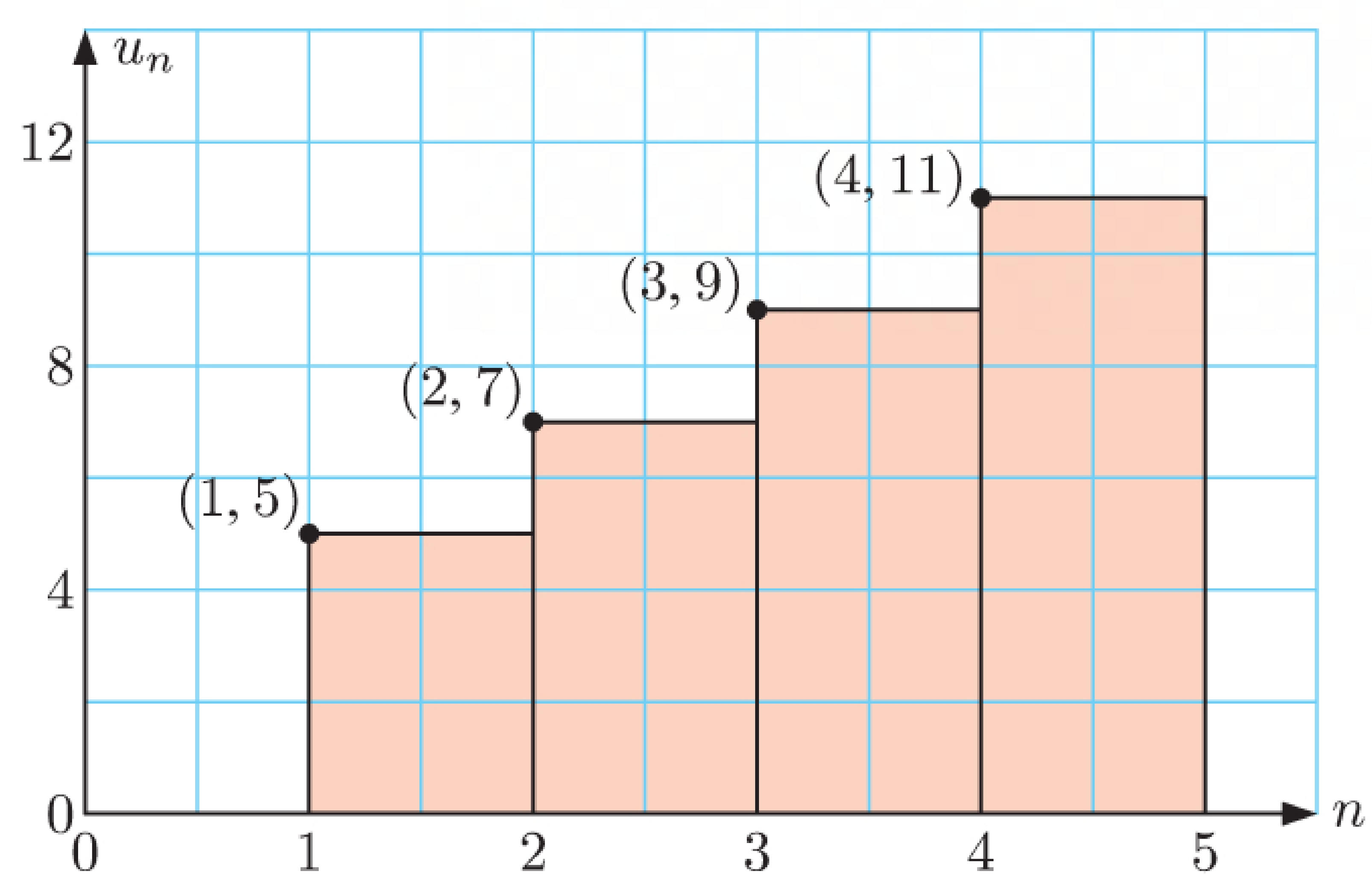
$\therefore\ d=2$

So, the sequence is arithmetic with $u_1=-\frac{7}{2}$ and $d=2$.

24 **a** $u_n=3+2n$

$u_1=3+2(1)=5,\quad u_2=3+2(2)=7,\quad u_3=3+2(3)=9,\quad u_4=3+2(4)=11$

So, the graph is:



b $S_n = 5 + 7 + 9 + 11 + \ldots$ is an arithmetic series with $u_1 = 5$ and $d = 2$.
S_n is the sum of the areas of the first n rectangles.

c **i** The height of each rectangle increases by 2 units from the previous rectangle, so $u_{n+1} = u_n + 2$.

ii The area of the $(n+1)$ th rectangle is u_{n+1}.
S_{n+1} is the sum of the areas of the first n rectangles and the $(n+1)$ th rectangle, so $S_{n+1} = S_n + u_{n+1}$.

25 $S_n = \frac{n}{2}(2u_1 + (n-1)d)$

So, $S_2 = \frac{2}{2}(2u_1 + d) = 2u_1 + d$ and $S_5 = \frac{5}{2}(2u_1 + 4d) = 5u_1 + 10d$ and $S_7 = \frac{7}{2}(2u_1 + 6d) = 7u_1 + 21d$

Now, S_2, S_5, S_7 form an arithmetic sequence with common difference $S_5 - S_2$ or $S_7 - S_5$.

$\therefore \; S_5 - S_2 = S_7 - S_5$ {equating differences}

$\therefore \; (5u_1 + 10d) - (2u_1 + d) = (7u_1 + 21d) - (5u_1 + 10d)$

$\therefore \; 3u_1 + 9d = 2u_1 + 11d$

$\therefore \; u_1 = 2d$

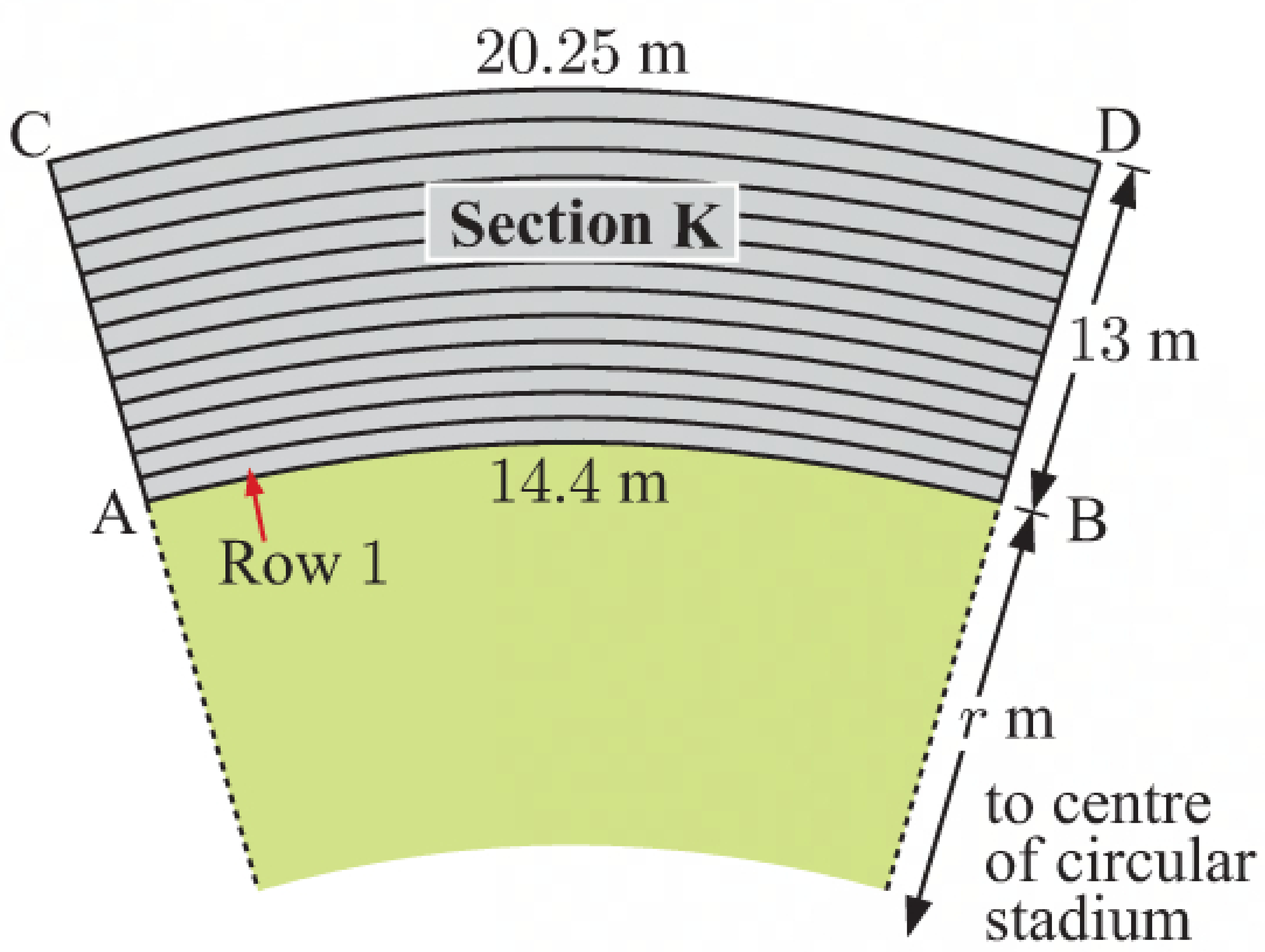
So the common difference for the sequence S_2, S_5, S_7 is

$$\begin{aligned} S_5 - S_2 &= (5(2d) + 10d) - (2(2d) + d) \\ &= 10d + 10d - 4d - d \\ &= 15d \end{aligned}$$

ACTIVITY 2 STADIUM SEATING

1 There are 13 tiers of concrete steps with total width 13 m.

$\therefore$ each concrete step is $\dfrac{13 \text{ m}}{13} = 1$ m wide.



2 The spacing between rows is constant, so the arc lengths of the rows form an arithmetic sequence.

$\therefore$ the length of the arc at the back of each row increases by $\dfrac{20.25 - 14.4 \text{ m}}{13} = 0.45$ m for each row.

We can summarise this in a table:

Row number	1	2	3	4	5	6	7
Arc length at back of row (m)	14.85	15.3	15.75	16.2	16.65	17.1	17.55

Row number	8	9	10	11	12	13
Arc length at back of row (m)	18.0	18.45	18.9	19.35	19.8	20.25

3 Number of seats in Row 1 $= \dfrac{\text{arc length at front of Row 1}}{\text{width of each seat}}$

$$= \frac{14.4 \text{ m}}{0.45 \text{ m}}$$

$$= 32 \text{ seats}$$

Since each row is 0.45 m longer than the previous row and each seat is 0.45 m wide, then there is one more seat in each row than the previous row.
We can summarise this in a table:

Row number	1	2	3	4	5	6	7	8	9	10	11	12	13
Number of seats in row	32	33	34	35	36	37	38	39	40	41	42	43	44

4 arc length $= \dfrac{\theta}{360} \times 2\pi r$

$$\therefore\ 14.4 = \frac{\theta}{360} \times 2\pi r$$

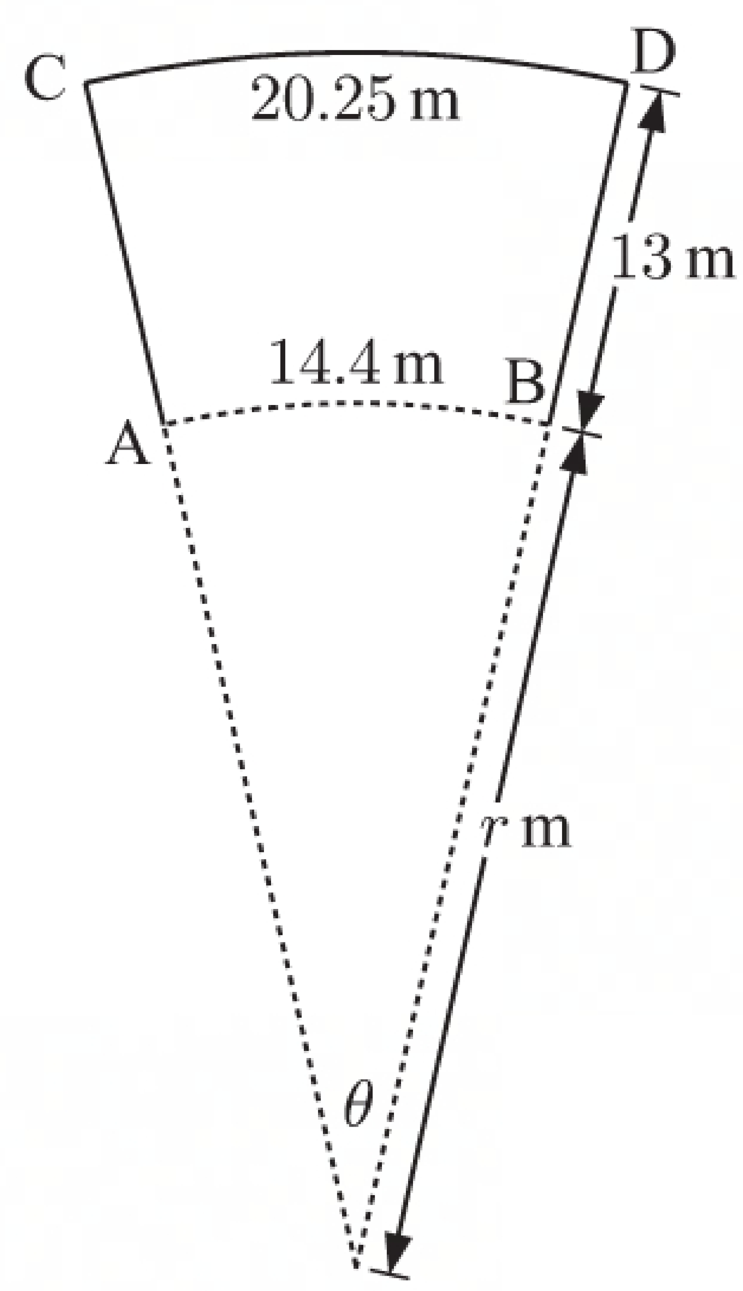
$$\therefore\ \theta = \frac{14.4 \times 360}{2\pi r}$$

$$\therefore\ \theta = \frac{2592}{\pi r} \quad .... (1)$$

and $\quad 20.25 = \dfrac{\theta}{360} \times 2\pi \times (r + 13)$

$$\therefore\ \theta = \frac{20.25 \times 360}{2\pi(r + 13)}$$

$$\therefore\ \theta = \frac{3645}{\pi(r + 13)} \quad .... (2)$$



Equating (1) and (2): $\quad \dfrac{2592}{\pi r} = \dfrac{3645}{\pi(r + 13)}$

$$\therefore\ \frac{2592}{r} = \frac{3645}{r + 13}$$

$$\therefore\ 2592(r + 13) = 3645r$$

$$\therefore\ 2592r + 33\,696 = 3645r$$

$$\therefore\ 1053r = 33\,696$$

$$\therefore\ r = 32$$

Substituting $r = 32$ into (1) gives: $\quad \theta = \dfrac{2592}{\pi \times 32}$

$$\therefore\ \theta = \frac{81}{\pi}$$

So, the angle of each sector of the circle is $\dfrac{81}{\pi} \approx 25.8°$.

Number of sectors in the circle $= \dfrac{360°}{\theta°}$

$$= \frac{360°}{\left(\frac{81}{\pi}\right)^{\circ}}$$

$$\approx 13.96$$

So, the stadium has 13 sections as there is insufficient space for 14.

5 Total seating capacity = number of seats per section × number of sections

$$= (32 + 33 + 34 + ... + 43 + 44) \times 13$$
$$= \tfrac{13}{2}(32 + 44) \times 13$$
$$= 494 \times 13$$
$$= 6422 \text{ seats}$$

6 From **4**, the radius r is 32 m.

EXERCISE 5H

1 **a** The series is geometric with $u_1 = 2$, $r = 3$, $n = 8$.

$$S_n = \frac{u_1(r^n - 1)}{r - 1}$$
$$\therefore \quad S_8 = \frac{2(3^8 - 1)}{3 - 1} = \frac{2(6561 - 1)}{2} = 6560$$

b The series is geometric with $u_1 = 5$, $r = 2$, $n = 10$.

$$S_n = \frac{u_1(r^n - 1)}{r - 1}$$
$$\therefore \quad S_{10} = \frac{5(2^{10} - 1)}{2 - 1} = \frac{5(1024 - 1)}{1} = 5 \times 1023 = 5115$$

c The series is geometric with $u_1 = 12$, $r = \frac{1}{2}$, $n = 10$.

$$S_n = \frac{u_1(1 - r^n)}{1 - r}$$
$$\therefore \quad S_{10} = \frac{12\left(1 - (\frac{1}{2})^{10}\right)}{1 - \frac{1}{2}} = \frac{3069}{128} \approx 24.0$$

d The series is geometric with $u_1 = \sqrt{7}$, $r = \sqrt{7}$, $n = 12$.

$$S_n = \frac{u_1(r^n - 1)}{r - 1}$$
$$\therefore \quad S_{12} = \frac{\sqrt{7}\left((\sqrt{7})^{12} - 1\right)}{\sqrt{7} - 1} \approx 189\,000$$

e The series is geometric with $u_1 = 6$, $r = -\frac{1}{2}$, $n = 15$.

$$S_n = \frac{u_1(1 - r^n)}{1 - r}$$
$$\therefore \quad S_{15} = \frac{6\left(1 - (-\frac{1}{2})^{15}\right)}{1 - (-\frac{1}{2})} = \frac{32\,769}{8192} \approx 4.00$$

f The series is geometric with $u_1 = 1$, $r = -\frac{1}{\sqrt{2}}$, $n = 20$.

$$S_n = \frac{u_1(1 - r^n)}{1 - r}$$
$$\therefore \quad S_{20} = \frac{1\left(1 - (-\frac{1}{\sqrt{2}})^{20}\right)}{1 - (-\frac{1}{\sqrt{2}})} \approx 0.585$$

2 **a** $\sqrt{3}+3+3\sqrt{3}+9+....$

The series is geometric with

$u_1=\sqrt{3}$ and $r=\sqrt{3}$.

$$\begin{aligned} S_n &= \frac{u_1(r^n-1)}{r-1} \\ &= \frac{\sqrt{3}\left((\sqrt{3})^n-1\right)}{\sqrt{3}-1}\times\left(\frac{\sqrt{3}+1}{\sqrt{3}+1}\right) \\ &= \frac{\left(3+\sqrt{3}\right)\left((\sqrt{3})^n-1\right)}{3-1} \\ &= \frac{3+\sqrt{3}}{2}\left((\sqrt{3})^n-1\right) \end{aligned}$$

b $12+6+3+1\frac{1}{2}+....$

The series is geometric with

$u_1=12$ and $r=\frac{1}{2}$.

$$\begin{aligned} S_n &= \frac{u_1(1-r^n)}{1-r} \\ &= \frac{12\left(1-(\frac{1}{2})^n\right)}{1-\frac{1}{2}} \\ &= 24\left(1-(\tfrac{1}{2})^n\right) \end{aligned}$$

c $0.9+0.09+0.009+0.0009+....$

The series is geometric with

$u_1=0.9$ and $r=0.1$.

$$\begin{aligned} S_n &= \frac{u_1(1-r^n)}{1-r} \\ &= \frac{0.9\,(1-(0.1)^n)}{1-0.1} \\ &= 1-(0.1)^n \end{aligned}$$

d $20-10+5-2\frac{1}{2}+....$

The series is geometric with

$u_1=20$ and $r=-\frac{1}{2}$.

$$\begin{aligned} S_n &= \frac{u_1(1-r^n)}{1-r} \\ &= \frac{20\left(1-(-\frac{1}{2})^n\right)}{1-(-\frac{1}{2})} \\ &= \frac{20\left(1-(-\frac{1}{2})^n\right)}{(\frac{3}{2})} \\ &= \tfrac{40}{3}\left(1-(-\tfrac{1}{2})^n\right) \end{aligned}$$

3 **a** $\sum_{k=1}^{10} 3\times 2^{k-1}=3+6+12+....+384+768+1536$

This series is geometric with $u_1=3$, $r=2$, and $n=10$.

$$\begin{aligned} S_n &= \frac{u_1(r^n-1)}{r-1} \\ \therefore\quad S_{10} &= \frac{3(2^{10}-1)}{2-1} \\ &= 3069 \end{aligned}$$

b $\displaystyle\sum_{k=1}^{12} \left(\tfrac{1}{2}\right)^{k-2} = 2 + 1 + \tfrac{1}{2} + ... + \tfrac{1}{256} + \tfrac{1}{512} + \tfrac{1}{1024}$

This series is geometric with $u_1 = 2$, $r = \frac{1}{2}$, and $n = 12$.

$$S_n = \frac{u_1(1-r^n)}{1-r}$$

$$\therefore \quad S_{12} = \frac{2\left(1-(\frac{1}{2})^{12}\right)}{1-\frac{1}{2}}$$

$$= 4\left(1-(\tfrac{1}{2})^{12}\right)$$

$$= 4\left(1-\frac{1}{2^{12}}\right)$$

$$= \frac{2^{12}-1}{2^{10}}$$

$$= \frac{4095}{1024} \approx 4.00$$

c $\displaystyle\sum_{k=1}^{25} 6 \times (-2)^k = -12 + 24 + (-48) + ... + 100\,663\,296 + (-201\,326\,592)$

This series is geometric with $u_1 = -12$, $r = -2$, and $n = 25$.

$$S_n = \frac{u_1(1-r^n)}{1-r}$$

$$\therefore \quad S_{25} = \frac{-12(1-(-2)^{25})}{1+2}$$

$$= -4(1-(-2)^{25})$$

$$= -134\,217\,732$$

4 **a** $A_3 = A_2 \times 1.06 + 2000$

$= (A_1 \times 1.06 + 2000) \times 1.06 + 2000$

$= (2000 \times 1.06 + 2000) \times 1.06 + 2000$

$\therefore \; A_3 = 2000 + 2000 \times 1.06 + 2000 \times (1.06)^2$ as required

b $A_4 = A_3 \times 1.06 + 2000$

$= \left[2000 + 2000 \times 1.06 + 2000 \times (1.06)^2\right] \times 1.06 + 2000$ {from **a**}

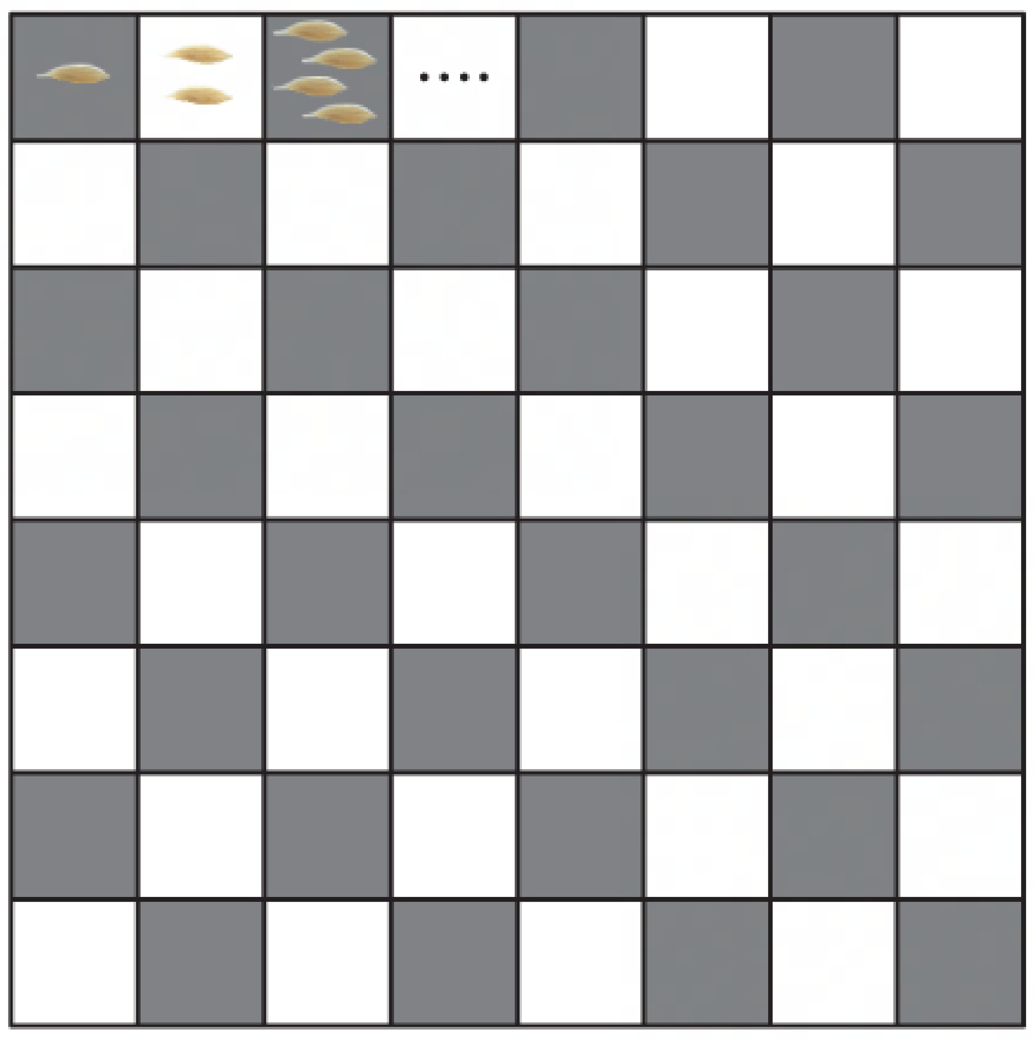
$\therefore \; A_4 = 2000\left[1 + 1.06 + (1.06)^2 + (1.06)^3\right]$ as required

c $A_{10} = 2000[1 + 1.06 + (1.06)^2 + (1.06)^3 + (1.06)^4 + (1.06)^5 + (1.06)^6 + (1.06)^7$
$+ (1.06)^8 + (1.06)^9]$

$\therefore \; A_{10} \approx 26\,361.59$

$\therefore$ the total bank balance after 10 years is \$26 361.59 .

5 **a** The number of grains of wheat starts at 1, and each square has double the number of grains of the previous square.



b The number of grains of wheat on each square can be expressed as a geometric sequence 1, 2, 4, 8, ...

So, $u_1 = 1$ and $r = 2$.

$$u_n = u_1 r^{n-1}$$
$$\therefore \quad u_n = 1 \times 2^{n-1}$$
$$\therefore \quad u_n = 2^{n-1}$$

c There are 64 squares in total, so $n = 64$.

$$S_n = \frac{u_1(r^n - 1)}{r - 1}$$
$$\therefore \quad S_{64} = \frac{1(2^{64} - 1)}{2 - 1}$$
$$= 2^{64} - 1$$
$$\approx 1.84 \times 10^{19}$$

So, the king owed $2^{64} - 1 \approx 1.84 \times 10^{19}$ grains of wheat.

6 There is a fixed percentage increase each year, so Paula's annual rent forms a geometric sequence.
$u_1 = 5000$ and $r = 1.05$

$\therefore$ Paula's annual rent after n years is $u_n = 5000 \times (1.05)^{n-1}$.

a $u_4 = 5000 \times (1.05)^3$
$= 5788.125$

So, in the 4th year, Paula paid approximately \$5790.

b
$$S_n = \frac{u_1(r^n - 1)}{r - 1}$$
$$\therefore \quad S_n = \frac{5000((1.05)^n - 1)}{1.05 - 1}$$
$$\therefore \quad S_n = \frac{5000((1.05)^n - 1)}{0.05}$$
$$\therefore \quad S_n = 100\,000((1.05)^n - 1)$$

c $S_7 = 100\,000((1.05)^7 - 1)$
$\approx 40\,710.04$

So, Paula paid approximately \$40 710 in rent during the first 7 years.

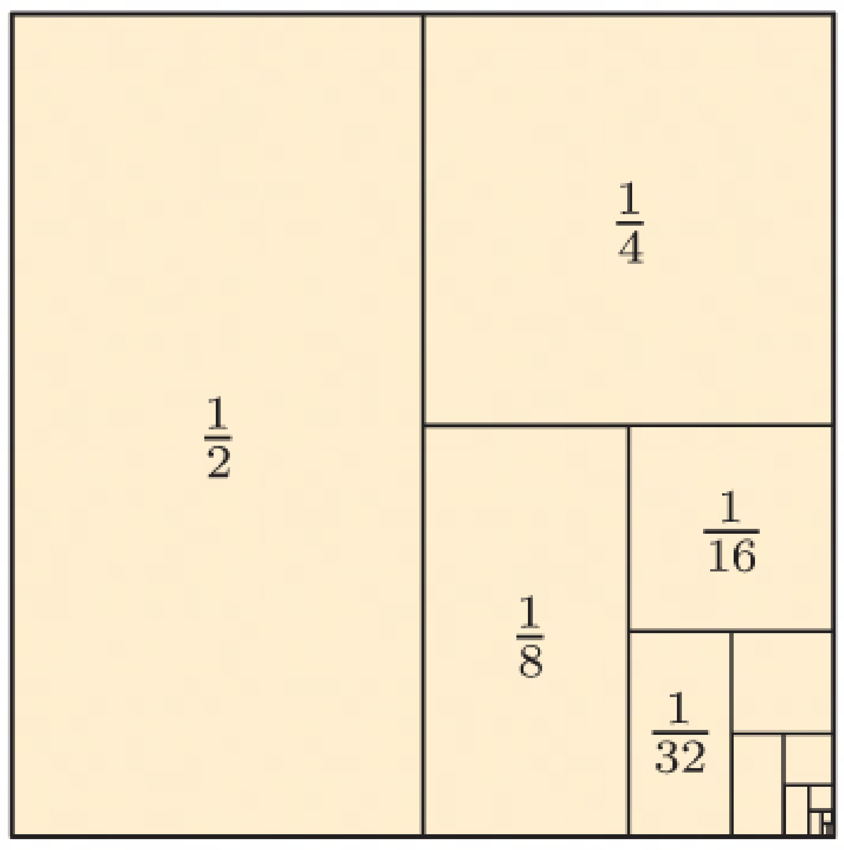
7 The amount of money in Jim's account at the end of each year will be:

$$\begin{aligned}
A_0 &= 6000\\
A_1 &= A_0 \times 1.05 + 1000\\
&= 6000 \times 1.05 + 1000\\
A_2 &= A_1 \times 1.05 + 1000\\
&= (6000 \times 1.05 + 1000) \times 1.05 + 1000\\
&= 6000 \times (1.05)^2 + 1000 \times 1.05 + 1000\\
A_3 &= A_2 \times 1.05 + 1000\\
&= (6000 \times (1.05)^2 + 1000 \times 1.05 + 1000) \times 1.05 + 1000\\
&= 6000 \times (1.05)^3 + 1000 \times (1.05)^2 + 1000 \times 1.05 + 1000 \quad \text{and so on}
\end{aligned}$$

So,
$$\begin{aligned}
A_8 &= 6000 \times (1.05)^8 + 1000 \times (1.05)^7 + 1000 \times (1.05)^6 + 1000 \times (1.05)^5\\
&\quad + 1000 \times (1.05)^4 + 1000 \times (1.05)^3 + 1000 \times (1.05)^2 + 1000 \times 1.05 + 1000\\
&\approx 18\,413.84
\end{aligned}$$

The value of the account after 8 years is £18 413.84 .

8 $S_n = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \dots + \frac{1}{2^n}$



a $S_1 = \frac{1}{2}, \quad S_2 = \frac{1}{2} + \frac{1}{4} = \frac{3}{4}, \quad S_3 = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} = \frac{7}{8},$

$S_4 = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} = \frac{15}{16},$

$S_5 = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} = \frac{31}{32}$

b $S_n = \dfrac{2^n - 1}{2^n}$

c
$$\begin{aligned}
S_n &= \frac{u_1(1 - r^n)}{1 - r}, \quad \text{where } u_1 = \tfrac{1}{2} \text{ and } r = \tfrac{1}{2}\\
&= \frac{\frac{1}{2}\left(1 - (\frac{1}{2})^n\right)}{1 - \frac{1}{2}}\\
\therefore\ S_n &= 1 - (\tfrac{1}{2})^n\\
&= 1 - \frac{1}{2^n}\\
&= \frac{2^n - 1}{2^n}
\end{aligned}$$

d As $n \to \infty$, $\left(\frac{1}{2}\right)^n \to 0$, and so $S_n \to 1$ (from below)

e The diagram represents one whole unit divided into smaller and smaller fractions. As $n \to \infty$, the area which the fraction represents becomes smaller and smaller, and the total area approaches the area of a 1×1 unit square.

9 The series is geometric with $u_2 = 6$ and $S_3 = -14$.

$$u_2 = u_1 \times r \quad \text{and} \quad u_3 = u_2 \times r$$
$$\therefore\ 6 = u_1 r \qquad\qquad = 6r$$
$$\therefore\ u_1 = \frac{6}{r}$$

Now $u_1 + u_2 + u_3 = \dfrac{6}{r} + 6 + 6r = -14$

$$\therefore\ \frac{6}{r} + 6r = -20$$
$$\therefore\ 6 + 6r^2 = -20r$$
$$\therefore\ 6r^2 + 20r + 6 = 0$$
$$\therefore\ 2(3r^2 + 10r + 3) = 0$$
$$\therefore\ 2(3r+1)(r+3) = 0$$
$$\therefore\ r = -\tfrac{1}{3} \text{ or } -3$$

$$\begin{aligned} u_4 &= u_3 \times r \\ &= 6r \times r \\ &= 6r^2 \end{aligned}$$

If $r = -\frac{1}{3}$, $u_4 = 6\left(-\frac{1}{3}\right)^2 = \frac{2}{3}$

If $r = -3$, $u_4 = 6(-3)^2 = 54$

10 $u_1 = 1$ for both sequences.

For the arithmetic sequence, $u_2 = u_1 + d = 1 + d$

and for the geometric sequence, $u_2 = u_1 \times r = 1 \times r = r$

Since the second terms are equal, $1 + d = r$

$$\therefore\ d = r - 1 \quad .... \ (*)$$

For the arithmetic sequence, $u_{14} = u_1 + 13d = 1 + 13(r-1) = 13r - 12$

For the geometric sequence, $u_3 = u_1 \times r^2 = 1 \times r^2 = r^2$

Since the 14th term of the arithmetic sequence is 3 times the third term of the geometric sequence,

$$13r - 12 = 3r^2$$
$$\therefore\ 3r^2 - 13r + 12 = 0$$
$$\therefore\ (3r-4)(r-3) = 0$$
$$\therefore\ r = \tfrac{4}{3} \text{ or } 3$$

When $r = \frac{4}{3}$, $d = \frac{4}{3} - 1 = \frac{1}{3}$ {using $(*)$}

So for the arithmetic sequence,

$$\begin{aligned} u_{20} &= u_1 + 19d \\ &= 1 + 19(\tfrac{1}{3}) \\ &= \tfrac{22}{3} = 7\tfrac{1}{3} \end{aligned}$$

and for the geometric sequence,

$$\begin{aligned} u_{20} &= u_1 \times r^{19} \\ &= 1 \times (\tfrac{4}{3})^{19} \\ &= (\tfrac{4}{3})^{19} \end{aligned}$$

When $r = 3$, $d = 3 - 1 = 2$ {using $(*)$}

So for the arithmetic sequence,

$$\begin{aligned} u_{20} &= u_1 + 19d \\ &= 1 + 19(2) \\ &= 39 \end{aligned}$$

and for the geometric sequence,

$$\begin{aligned} u_{20} &= u_1 \times r^{19} \\ &= 1 \times 3^{19} \\ &= 3^{19} \end{aligned}$$

11 $u_1, u_2, ..., u_n$ is a geometric sequence, so $u_n = u_1 r^{n-1}$.

The general term of the sequence $(u_1+u_2)^2, \ (u_2+u_3)^2, \ ...., \ (u_{n-1}+u_n)^2$ is:

$$\begin{aligned}(u_n+u_{n+1})^2 &= u_n^2 + 2u_n u_{n+1} + u_{n+1}^2\\ &= (u_1\times r^{n-1})^2 + 2(u_1\times r^{n-1})(u_1\times r^n) + (u_1\times r^n)^2 \qquad \{\text{using } u_n = u_1\times r^n\}\\ &= u_1^2\, r^{2(n-1)} + 2u_1^2\, r^{2n-1} + u_1^2\, r^{2n}\\ &= u_1^2 (r^2)^{n-1} + 2u_1^2\, r(r^2)^{n-1} + u_1^2\, r^2(r^2)^{n-1}\\ &= u_1^2(1+2r+r^2)(r^2)^{n-1}\\ &= u_1^2(1+r)^2(r^2)^{n-1}\end{aligned}$$

So it is a geometric sequence with first term $u_1^2(1+r)^2$ and common ratio r^2.

$\therefore \quad (u_1+u_2)^2 + (u_2+u_3)^2 + (u_3+u_4)^2 + ... + (u_{n-1}+u_n)^2$

$$= \frac{u_1^2(1+r)^2((r^2)^{n-1}-1)}{r^2-1} \qquad \{\text{sum of the first } (n-1) \text{ terms of the geometric sequence } \{u_n+u_{n+1}\}\}$$

$$= \frac{u_1^2(1+r)^2(r^{2n-2}-1)}{(r+1)(r-1)}$$

$$= \frac{u_1^2(1+r)(r^{2n-2}-1)}{r-1}$$

$$= \frac{u_1^2(r^{2n-2}-1+r^{2n-1}-r)}{r-1}$$

Now

$$\begin{aligned}\frac{2u_1^2(r^{2n-1}-1)}{r-1} - (u_1^2+u_n^2) &= \frac{2u_1^2(r^{2n-1}-1)}{r-1} - \frac{(u_1^2+u_1^2\, r^{2n-2})(r-1)}{r-1}\\ &= \frac{2u_1^2(r^{2n-1}-1) - u_1^2(1+r^{2n-2})(r-1)}{r-1}\\ &= \frac{u_1^2(2r^{2n-1}-2-(r-1+r^{2n-1}-r^{2n-2}))}{r-1}\\ &= \frac{u_1^2(r^{2n-2}-1+r^{2n-1}-r)}{r-1}\end{aligned}$$

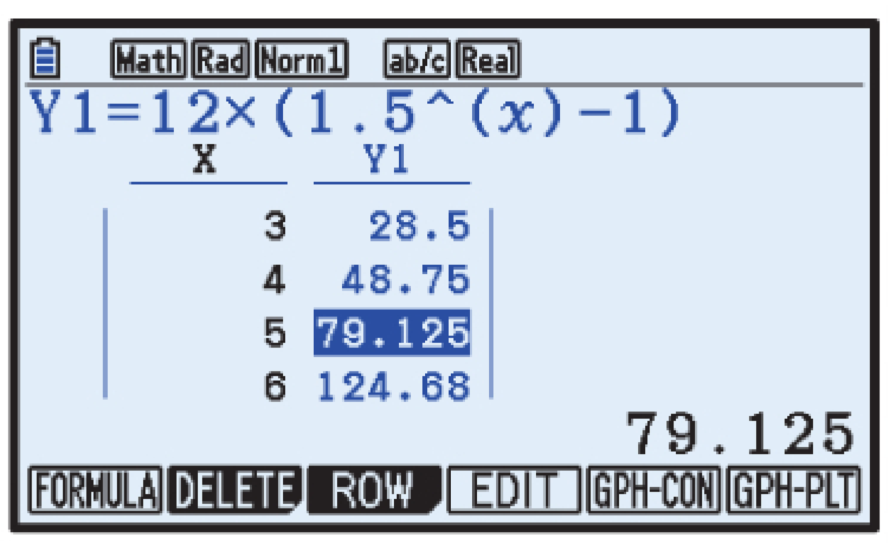
$\therefore \quad (u_1+u_2)^2 + (u_2+u_3)^2 + (u_3+u_4)^2 + ... + (u_{n-1}+u_n)^2$

$$= \frac{2u_1^2(r^{2n-1}-1)}{r-1} - (u_1^2+u_n^2) \quad \text{as required}$$

12 The sequence is geometric with $u_1 = 6$ and $r = 1.5$.

$$\begin{aligned}S_n &= \frac{u_1(r^n-1)}{r-1}\\ &= \frac{6((1.5)^n-1)}{1.5-1}\\ &= 12((1.5)^n-1)\end{aligned}$$

To find n such that $S_n = 79.125$, we use a table of values with $Y_1 = 12\times(1.5\text{^}X{-}1)$.



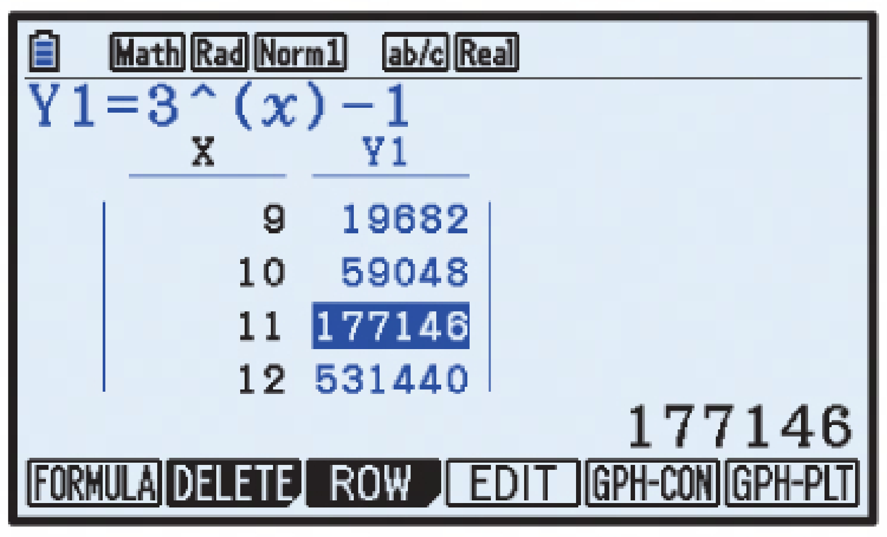
$S_5 = 79.125$, so $n = 5$.

13 $\displaystyle\sum_{k=1}^{n} 2 \times 3^{k-1} = 2 + 6 + 18 + \dots$

This series is geometric with $u_1 = 2$ and $r = 3$.

$$\begin{aligned} S_n &= \frac{u_1(r^n - 1)}{r - 1} \\ &= \frac{2(3^n - 1)}{3 - 1} \\ &= 3^n - 1 \end{aligned}$$

To find n such that $S_n = 177\,146$, we use a table of values with $\mathrm{Y}_1 = 3\hat{}\mathrm{X}{-}1$.



$S_{11} = 177\,146$, so $n = 11$.

14 **a** *Option A*:

$$\begin{aligned} \text{First year salary} &= \$40\,000 \\ \text{Second year salary} &= \$40\,000 + 5\% \times \$40\,000 = \$42\,000 \\ \text{Third year salary} &= \$42\,000 + 5\% \times \$42\,000 = \$44\,100 \\ \text{Total earned in three years} &= \$40\,000 + \$42\,000 + \$44\,100 = \$126\,100 \end{aligned}$$

Option B:

$$\begin{aligned} \text{First year salary} &= \$60\,000 \\ \text{Second year salary} &= \$60\,000 + \$1000 = \$61\,000 \\ \text{Third year salary} &= \$61\,000 + \$1000 = \$62\,000 \\ \text{Total earned in three years} &= \$60\,000 + \$61\,000 + \$62\,000 = \$183\,000 \end{aligned}$$

So, over three years Felicity would earn more under *Option B*.

b **i** Let A_n be the amount of money earned under *Option A* in the nth year.

A_n forms a geometric sequence with $A_1 = 40\,000$ and $r = 1.05$.

$\therefore\ A_n = 40\,000 \times (1.05)^{n-1}$

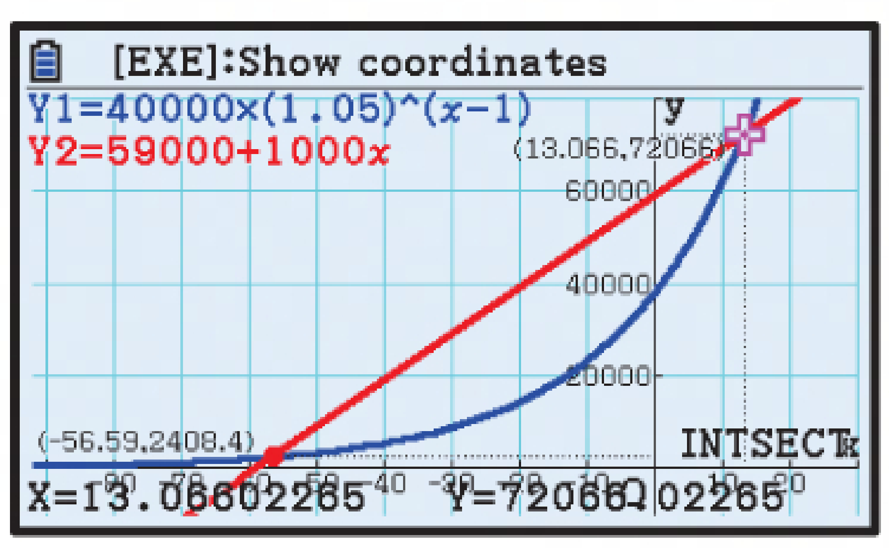
ii Let B_n be the amount of money earned under *Option B* in the nth year.

B_n forms an arithmetic sequence with $B_1 = 60\,000$ and $d = 1000$.

$$\begin{aligned} \therefore\ B_n &= 60\,000 + 1000(n - 1) \\ &= 59\,000 + 1000n \end{aligned}$$

c If $A_n = B_n$, $40\,000 \times (1.05)^{n-1} = 59\,000 + 1000n$

We graph $A_n = 40\,000 \times (1.05)^{n-1}$ and $B_n = 59\,000 + 1000n$ on the same set of axes and find their points of intersection.



Since $n \geqslant 0$, $n \approx 13.07$.

$\therefore$ the money earned under *Option A* will exceed that of *Option B* after approximately 13.1 years.

d **i** $S_n = \dfrac{u_1(r^n - 1)}{r - 1}$ for a geometric series

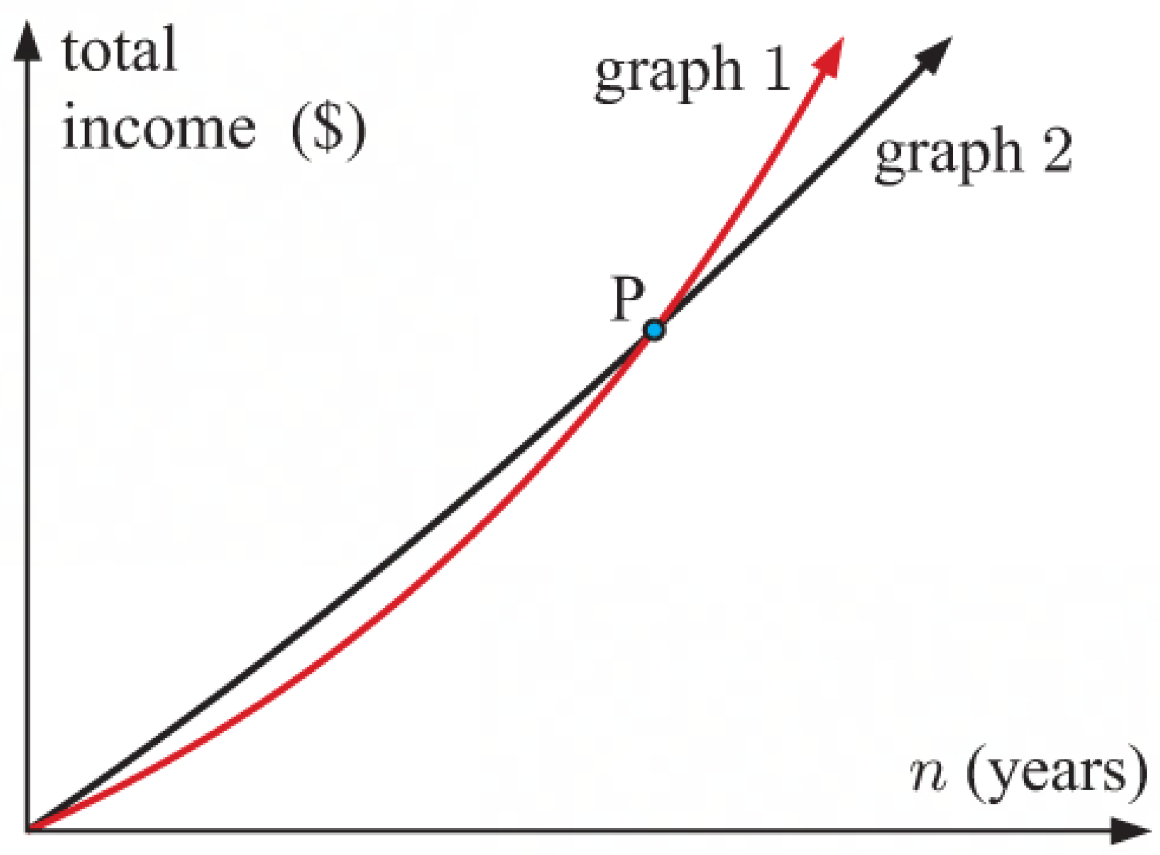
$$T_A = \frac{40\,000 \times ((1.05)^n - 1)}{1.05 - 1}$$
$$= \frac{40\,000 \times ((1.05)^n - 1)}{0.05}$$
$$= 800\,000((1.05)^n - 1) \text{ dollars}$$

ii $S_n = \dfrac{n}{2}(2u_1 + (n-1)d)$ for an arithmetic series

$$T_B = \frac{n}{2}(2(60\,000) + 1000(n-1))$$
$$= 60\,000n + 500n(n-1)$$
$$= 60\,000n + 500n^2 - 500n$$
$$= 500n^2 + 59\,500n \text{ dollars}$$

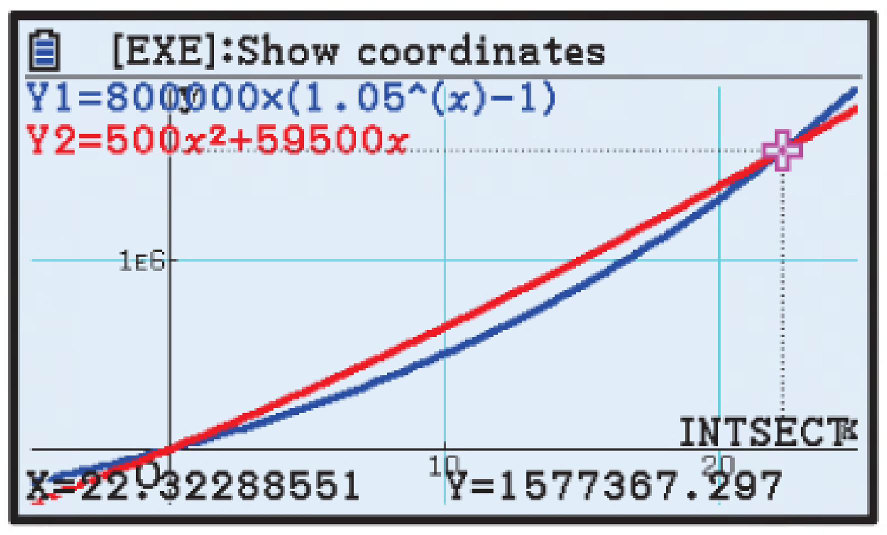
e **i** Initially *Option B* is better than *Option A*, so $T_B > T_A$ for small values of n.

$\therefore$ graph 1 represents T_A, graph 2 represents T_B.



ii The point P is where T_A meets T_B, which is when $800\,000((1.05)^n - 1) = 500n^2 + 59\,500n$.

We graph $T_A = 800\,000(1.05^n - 1)$ and $T_B = 500n^2 + 59\,500n$ on the same set of axes and find their points of intersection.



Since $n > 0$, $\text{P} \approx (22.3, 1\,580\,000)$.

iii *Option B* provides the greater total income for $0 \leqslant n \leqslant 22$ years.

15 **a** $A_3 = A_2 \times 1.03 - R$

$= (\$8000 \times (1.03)^2 - 1.03R - R) \times 1.03 - R$

$= \$8000 \times (1.03)^3 - (1.03)^2R - 1.03R - R$

b $A_8 = \$8000 \times (1.03)^8 - (1.03)^7R - (1.03)^6R - (1.03)^5R - (1.03)^4R - (1.03)^3R - (1.03)^2R - 1.03R - R$

c $A_8 = 0$

$\therefore \$8000 \times (1.03)^8 = R(1 + 1.03 + (1.03)^2 + (1.03)^3 + (1.03)^4 + ... + (1.03)^7)$

$\therefore \$8000 \times (1.03)^8 = R\left(\dfrac{1 \times ((1.03)^8 - 1)}{1.03 - 1}\right)$ {using $S_n = \dfrac{u_1(r^n - 1)}{r - 1}$}

$$\therefore R = \frac{\$8000 \times (1.03)^8 \times 0.03}{(1.03)^8 - 1}$$
$$= \$1139.65$$

d Notice in **c** that $P = 8000$ and $(1.03)^8 = (1 + 0.03)^8 = (1 + r)^m$

$\therefore r = 0.03$ and $m = 8$

So, each repayment is $R = \dfrac{P(1+r)^m \times r}{(1+r)^m - 1}$ dollars.

EXERCISE 5I

1 **a** $0.\overline{3} = \frac{3}{10} + \frac{3}{100} + \frac{3}{1000} + ...$

$$\frac{u_2}{u_1} = \frac{\left(\frac{3}{100}\right)}{\left(\frac{3}{10}\right)} = \frac{\left(\frac{3}{1000}\right)}{\left(\frac{3}{100}\right)} = \frac{1}{10}$$

$\therefore$ the series is geometric with $u_1 = \frac{3}{10}$ and $r = \frac{1}{10}$.

Since we are adding all the terms, it is an infinite geometric series.

b We need to show that $0.\overline{3} = \frac{1}{3}$.

$$\text{Now} \quad 0.\overline{3} = \frac{3}{10} + \frac{3}{100} + \frac{3}{1000} + ...$$

$$\text{So, let} \quad S_n = \frac{3}{10} + \frac{3}{100} + \frac{3}{1000} + ...$$

$$\begin{aligned}\text{Since } n \to \infty, \text{ then } \quad S &= \frac{u_1}{1-r} \\ &= \frac{\frac{3}{10}}{1 - \left(\frac{1}{10}\right)} \\ &= \frac{1}{3}\end{aligned}$$

$\therefore \quad 0.\overline{3} = \frac{1}{3}$ as required

2 **a** $0.\overline{4} = 0.444\,444\,....$

$$= \frac{4}{10} + \frac{4}{100} + \frac{4}{1000} + ...$$

is an infinite geometric series with $u_1 = \frac{4}{10}$ and $r = \frac{1}{10}$.

$$\begin{aligned}\therefore \quad S &= \frac{u_1}{1-r} \\ &= \frac{\frac{4}{10}}{1 - \frac{1}{10}} \\ &= \frac{4}{9}\end{aligned}$$

$\therefore \quad 0.\overline{4} = \frac{4}{9}$

b $0.\overline{16} = 0.161\,616\,....$

$$= \frac{16}{10^2} + \frac{16}{10^4} + \frac{16}{10^6} + ...$$

is an infinite geometric series with $u_1 = \frac{16}{100}$ and $r = \frac{1}{100}$.

$$\begin{aligned}\therefore \quad S &= \frac{u_1}{1-r} \\ &= \frac{\frac{16}{100}}{1 - \frac{1}{100}} \\ &= \frac{16}{99}\end{aligned}$$

$\therefore \quad 0.\overline{16} = \frac{16}{99}$

c $0.\overline{312} = 0.312\,312\,312\,....$

$$= \frac{312}{10^3} + \frac{312}{10^6} + \frac{312}{10^9} + ...$$

is an infinite geometric series with $u_1 = \frac{312}{1000}$ and $r = \frac{1}{1000}$.

$$\begin{aligned}\therefore \quad S &= \frac{u_1}{1-r} \\ &= \frac{\frac{312}{1000}}{1 - \frac{1}{1000}} \\ &= \frac{312}{999} \\ &= \frac{104}{333}\end{aligned}$$

$\therefore \quad 0.\overline{312} = \frac{104}{333}$

3 Checking **Exercise 5H** question **8 d**: $S = \dfrac{u_1}{1-r} = \dfrac{\frac{1}{2}}{1-\frac{1}{2}} = 1$ ✓

4 **a** $18 + 12 + 8 + \frac{16}{3} + \ldots$ is an infinite geometric series with $u_1 = 18$ and $r = \frac{2}{3}$.

$$\therefore \quad S = \frac{u_1}{1-r} = \frac{18}{1-\frac{2}{3}} = 54$$

b $18.9 - 6.3 + 2.1 - 0.7 + \ldots$ is an infinite geometric series with $u_1 = 18.9$ and $r = -\frac{1}{3}$.

$$\therefore \quad S = \frac{u_1}{1-r} = \frac{18.9}{1-(-\frac{1}{3})} = 14.175$$

5 **a** $\displaystyle\sum_{k=1}^{\infty} \frac{3}{4^k} = \frac{3}{4} + \frac{3}{16} + \frac{3}{64} + \ldots$ is an infinite geometric series with $u_1 = \frac{3}{4}$ and $r = \frac{1}{4}$.

$$\therefore \quad S = \frac{u_1}{1-r} = \frac{\frac{3}{4}}{1-\frac{1}{4}} = 1$$

b $\displaystyle\sum_{k=0}^{\infty} 6\left(-\tfrac{2}{5}\right)^k = 6 - 6 \times (\tfrac{2}{5}) + 6 \times (\tfrac{2}{5})^2 - \ldots$ is an infinite geometric series with $u_1 = 6$ and $r = -\frac{2}{5}$.

$$\therefore \quad S = \frac{u_1}{1-r} = \frac{6}{1-(-\frac{2}{5})} = \frac{30}{7} \quad (= 4\tfrac{2}{7})$$

6 Let the terms of the geometric series be $u_1,\ u_1 r,\ u_1 r^2,\ \ldots$

$$u_1 + u_1 r + u_1 r^2 = 19$$
$$\therefore \quad u_1(1 + r + r^2) = 19$$
$$\therefore \quad u_1 = \frac{19}{1+r+r^2} \quad \ldots (1)$$

and $S = \dfrac{u_1}{1-r} = 27$

$$\therefore \quad u_1 = 27(1-r) \quad \ldots (2)$$

Equating (1) and (2), $\dfrac{19}{1+r+r^2} = 27(1-r)$

$$\therefore \quad \tfrac{19}{27} = (1-r)(1+r+r^2)$$
$$\therefore \quad \tfrac{19}{27} = 1 + r + r^2 - r - r^2 - r^3$$
$$\therefore \quad \tfrac{19}{27} = 1 - r^3$$
$$\therefore \quad r^3 = \tfrac{8}{27}$$
$$\therefore \quad r = \tfrac{2}{3}$$

Substituting $r = \frac{2}{3}$ into (2) gives $u_1 = 27(1 - \frac{2}{3}) = 9$

$\therefore$ the first term is 9 and the common ratio is $\frac{2}{3}$.

7 Let the terms of the geometric series be $u_1,\ u_1r,\ u_1r^2,\ \ldots.$

$u_1r = \frac{8}{5}$ and $S = \dfrac{u_1}{1-r} = 10$

$\therefore\ u_1 = \dfrac{8}{5r}$ .... (1) $\therefore\ u_1 = 10 - 10r$ .... (2)

Equating (1) and (2), $\dfrac{8}{5r} = 10 - 10r$

$$\therefore\ 8 = 50r - 50r^2$$
$$\therefore\ 50r^2 - 50r + 8 = 0$$
$$\therefore\ 2(25r^2 - 25r + 4) = 0$$
$$\therefore\ 2(5r-1)(5r-4) = 0$$
$$\therefore\ r = \tfrac{1}{5} \text{ or } \tfrac{4}{5}$$

Using (2), if $r = \frac{1}{5}$, $u_1 = 10 - 10(\frac{1}{5}) = 8$

if $r = \frac{4}{5}$, $u_1 = 10 - 10(\frac{4}{5}) = 2$

$\therefore$ either $u_1 = 8,\ r = \frac{1}{5}$ or $u_1 = 2,\ r = \frac{4}{5}$.

8 Let the terms of the geometric series be $u_1,\ u_1r,\ u_1r^2,\ \ldots.$

$S_3 = u_1 + u_1r + u_1r^2 = 21$ and $S = \dfrac{u_1}{1-r} = \frac{64}{3}$

$\therefore\ u_1(1 + r + r^2) = 21$ $\therefore\ u_1 = \frac{64}{3}(1-r)$ .... (2)

$\therefore\ u_1 = \dfrac{21}{1+r+r^2}$ .... (1)

Equating (1) and (2), $\dfrac{21}{1+r+r^2} = \frac{64}{3}(1-r)$

$$\therefore\ \tfrac{21\times 3}{64} = (1-r)(1+r+r^2)$$
$$\therefore\ \tfrac{63}{64} = 1 + r + r^2 - r - r^2 - r^3$$
$$\therefore\ \tfrac{63}{64} = 1 - r^3$$
$$\therefore\ r^3 = \tfrac{1}{64}$$
$$\therefore\ r = \tfrac{1}{4}$$

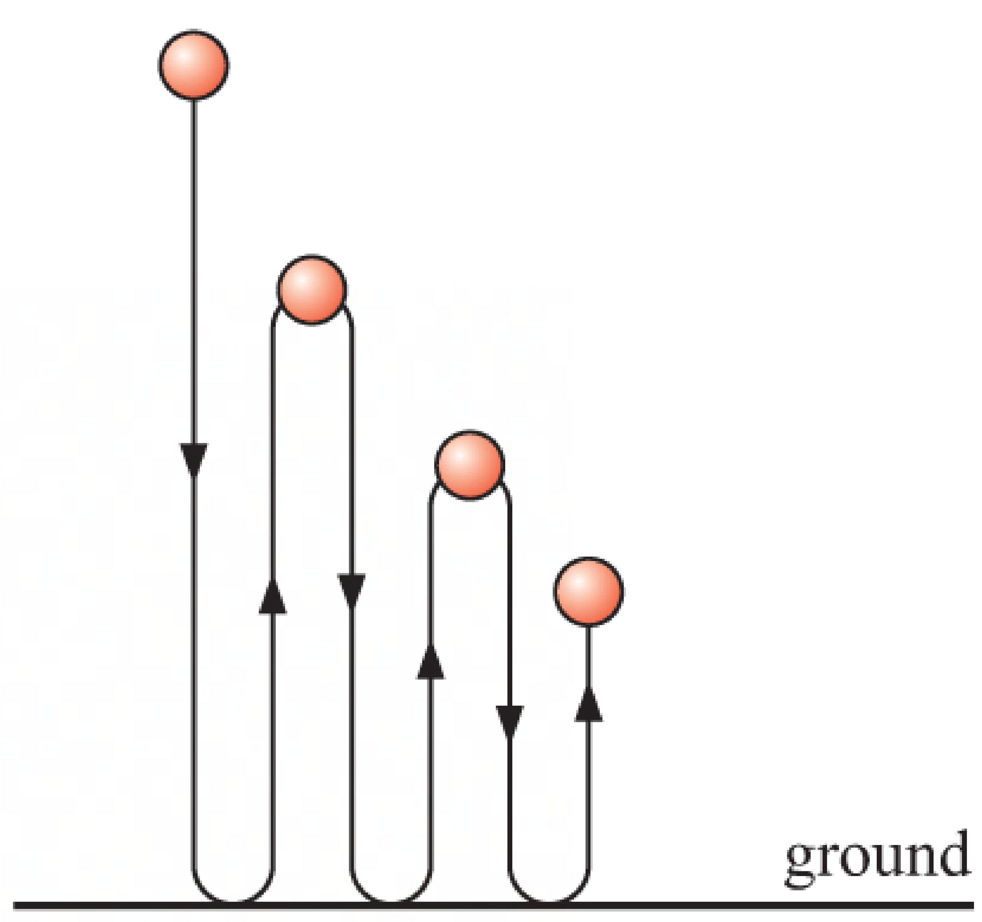
Substituting $r = \frac{1}{4}$ into (2) gives $u_1 = \frac{64}{3}(1 - \frac{1}{4}) = \frac{64}{3} \times \frac{3}{4} = 16.$

So, $S_5 = u_1 + u_1r + u_1r^2 + u_1r^3 + u_1r^4$

$$= u_1(1 + r + r^2 + r^3 + r^4)$$
$$= 16(1 + \tfrac{1}{4} + (\tfrac{1}{4})^2 + (\tfrac{1}{4})^3 + (\tfrac{1}{4})^4)$$
$$= \tfrac{341}{16}$$

9 **a** Total time of motion

$$= 1 + (90\% \times 1) + (90\% \times 1) + (90\% \times 90\% \times 1)$$
$$+ (90\% \times 90\% \times 1) + (90\% \times 90\% \times 90\% \times 1) + \ldots.$$
$$= 1 + 0.9 + 0.9 + (0.9)^2 + (0.9)^2 + (0.9)^3 + \ldots.$$
$$= 1 + 2(0.9) + 2(0.9)^2 + 2(0.9)^3 + \ldots. \quad \text{as required}$$



b The total time of motion can be written as $[2 + 2(0.9) + 2(0.9)^2 + 2(0.9)^3 + ...] - 1$

So, $S_n = \dfrac{u_1(1-r^n)}{1-r} - 1$, where $u_1 = 2$, $r = 0.9$

$\therefore \quad S_n = \dfrac{2(1-0.9^n)}{1-0.9} - 1$

$\therefore \quad S_n = \dfrac{2(1-0.9^n)}{0.1} - 1$

$\therefore \quad S_n = 20(1-0.9^n) - 1$

$\therefore \quad S_n = 20 - 20 \times 0.9^n - 1$

$\therefore \quad S_n = 19 - 20(0.9)^n$

c For the ball to come to rest, n must approach infinity.

As $n \to \infty$, $0.9^n \to 0$ and so $20 \times 0.9^n \to 0$ also.

$\therefore \quad S_n \to 19$ (from below)

So, it takes 19 seconds for the ball to come to rest.

10 Total distance travelled

$= h + 2\left(\frac{3}{4}\right)h + 2\left(\frac{3}{4}\right)\left(\frac{3}{4}\right)h + ...$

$= h + 2\left(\frac{3}{4}\right)h\left[1 + \frac{3}{4} + \left(\frac{3}{4}\right)^2 + \left(\frac{3}{4}\right)^3 + ...\right]$

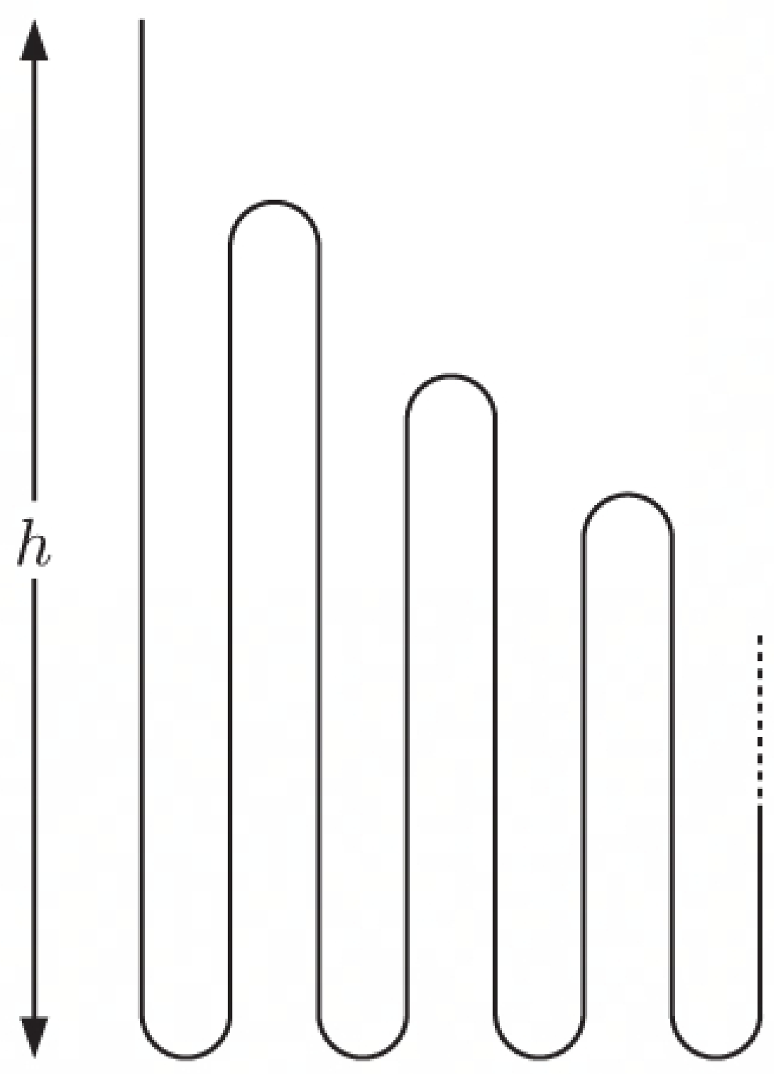
$= h + \frac{3}{2}h\left(\dfrac{1}{1-\frac{3}{4}}\right) \quad \left\{\text{as } |r| = \left|\frac{3}{4}\right| < 1 \text{ and } S = \dfrac{u_1}{1-r}\right\}$

$= h + \frac{3}{2}h(4)$

$= 7h$

But $7h = 490$, so $h = 70$.

The ball was dropped from a height of 70 cm.



11 **a** $0.\overline{9} = \frac{9}{10} + \frac{9}{100} + \frac{9}{1000} + ...$ which is geometric with $u_1 = \frac{9}{10}$ and $r = \frac{1}{10}$

$\therefore \quad 0.\overline{9} = S = \dfrac{\frac{9}{10}}{1-\frac{1}{10}} = 1$

b $u_n = \dfrac{9}{10^n}$

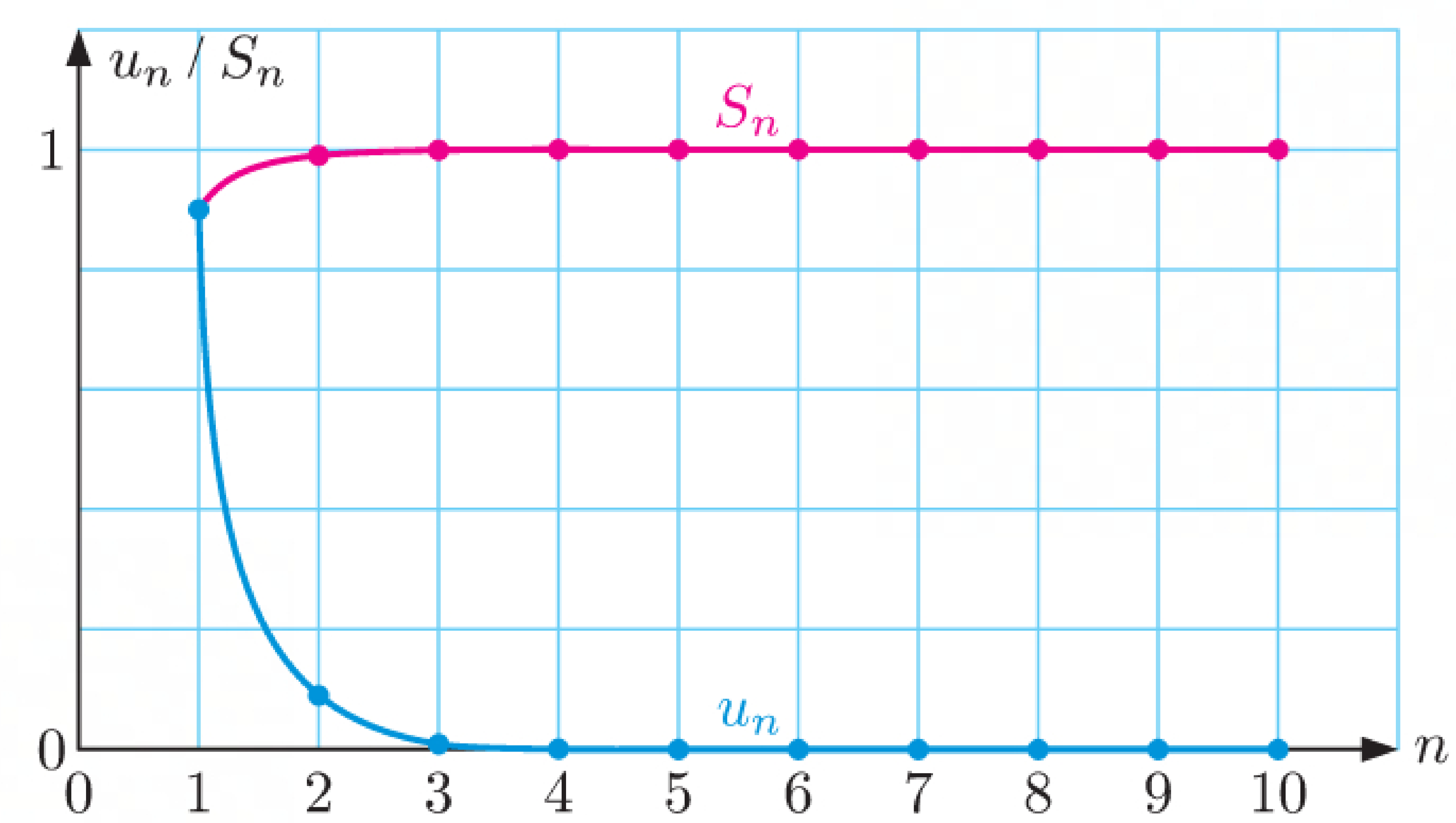
$S_n = \dfrac{u_1(1-r^n)}{1-r}$

$= \dfrac{\frac{9}{10^1}\left(1-(\frac{1}{10})^n\right)}{1-\frac{1}{10}}$

$= \dfrac{\frac{9}{10}\left(1-\frac{1^n}{10^n}\right)}{\frac{9}{10}}$

$\therefore \quad S_n = 1 - \dfrac{1}{10^n}$

c



12
$$\sum_{k=1}^{\infty}\left(\frac{3x}{2}\right)^{k-1} = \left(\frac{3x}{2}\right)^0 + \left(\frac{3x}{2}\right)^1 + \left(\frac{3x}{2}\right)^2 + \left(\frac{3x}{2}\right)^3 + \ldots.$$
$$= 1 + \frac{3x}{2} + \left(\frac{3x}{2}\right)^2 + \left(\frac{3x}{2}\right)^3 + \ldots.$$
$$= \frac{u_1}{1-r} \qquad \{\text{as it converges to 4 and is geometric}\}$$
$$= \frac{1}{1-\frac{3x}{2}} = \frac{2}{2-3x}$$

$$\therefore\ \frac{2}{2-3x} = 4 \quad \text{and so} \quad 2-3x = \tfrac{1}{2}$$
$$\therefore\ 3x = 1\tfrac{1}{2}$$
$$\therefore\ x = \tfrac{1}{2}$$

13 $u_1 + u_2 + u_3 + u_4 + \ldots.$ has common ratio r, $0 < r < 1$ as $u_1, u_2, u_3, u_4, \ldots. > 0$

a $u_1 - u_2 + u_3 - u_4 + \ldots.$ has common ratio $-r$

$\therefore\ -1 < -r < 1$ and so the series is convergent.

$\sqrt{u_1} + \sqrt{u_2} + \sqrt{u_3} + \sqrt{u_4} + \ldots.$ has common ratio $\sqrt{r}$

$\therefore\ -1 < \sqrt{r} < 1$ and so the series is convergent.

b $u_1 - u_2 + u_3 - u_4 + \ldots. = \frac{81}{10}$ and $\sqrt{u_1} + \sqrt{u_2} + \sqrt{u_3} + \sqrt{u_4} + \ldots. = \frac{9}{2}$

$$u_1 - u_2 + u_3 - u_4 + \ldots. = \frac{u_1}{1-(-r)} = \frac{81}{10}$$
$$\therefore\ \frac{u_1}{1+r} = \frac{81}{10}$$
$$\therefore\ u_1 = \tfrac{81}{10}(1+r) \quad \ldots.\ (1)$$

$$\sqrt{u_1} + \sqrt{u_2} + \sqrt{u_3} + \sqrt{u_4} + \ldots. = \frac{\sqrt{u_1}}{1-\sqrt{r}} = \frac{9}{2} \quad \ldots.\ (2)$$

Substituting (1) into (2) gives $\dfrac{\sqrt{\frac{81}{10}(1+r)}}{1-\sqrt{r}} = \dfrac{9}{2}$

$$\therefore\ \sqrt{\tfrac{81}{10}(1+r)} = \tfrac{9}{2} - \tfrac{9}{2}\sqrt{r}$$
$$\therefore\ \tfrac{81}{10}(1+r) = \left(\tfrac{9}{2} - \tfrac{9}{2}\sqrt{r}\right)^2$$
$$\therefore\ \tfrac{81}{10}(1+r) = \tfrac{81}{4} - \tfrac{81}{2}\sqrt{r} + \tfrac{81}{4}r$$
$$\therefore\ \tfrac{81}{10} + \tfrac{81}{10}r = \tfrac{81}{4} - \tfrac{81}{2}\sqrt{r} + \tfrac{81}{4}r$$
$$\therefore\ \tfrac{243}{20}r - \tfrac{81}{2}\sqrt{r} + \tfrac{243}{20} = 0$$
$$\therefore\ \tfrac{243}{20}X^2 - \tfrac{81}{2}X + \tfrac{243}{20} = 0 \qquad \{\text{letting } r = X^2\}$$
$$\therefore\ 243X^2 - 810X + 243 = 0$$
$$\therefore\ 3X^2 - 10X + 3 = 0$$
$$\therefore\ (3X-1)(X-3) = 0$$
$$\therefore\ X = \tfrac{1}{3} \text{ or } 3$$
$$\therefore\ \sqrt{r} = \tfrac{1}{3} \text{ or } 3$$
$$\therefore\ r = \tfrac{1}{9} \text{ or } 9$$

But $-1 < r < 1$, so $r = \frac{1}{9}$

Substituting $r = \frac{1}{9}$ into (1) gives
$$u_1 = \tfrac{81}{10}(1 + \tfrac{1}{9}) = \tfrac{81}{10} \times \tfrac{10}{9}$$
$$\therefore \quad u_1 = 9$$

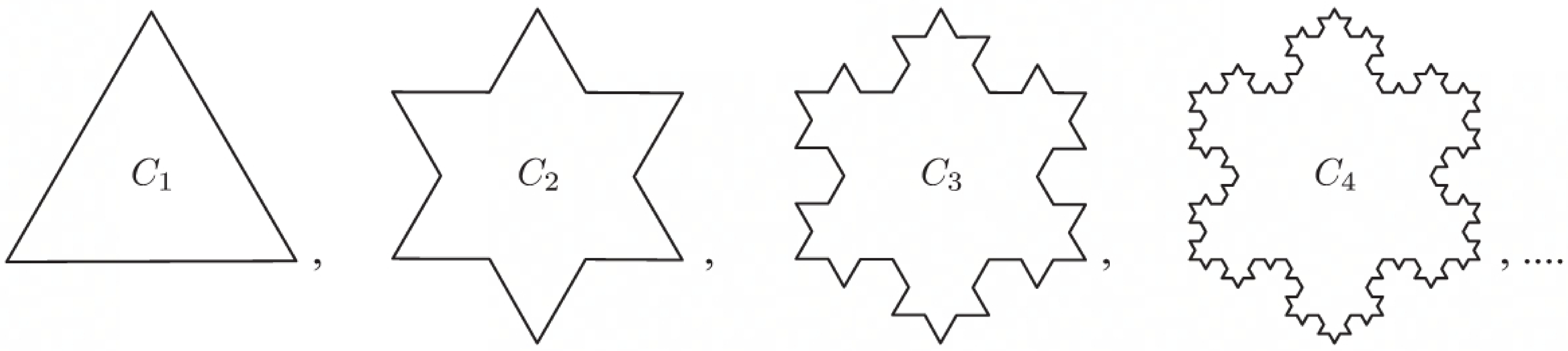
$$\therefore \quad u_1 + u_2 + u_3 + u_4 + \dots = \frac{u_1}{1-r} = \frac{9}{1-\frac{1}{9}} = \frac{81}{8}$$

14 Let $S = \frac{1}{4} + \frac{2}{8} + \frac{3}{16} + \frac{4}{32} + \frac{5}{64} + \dots$

$$\therefore \quad \frac{S}{2} = \tfrac{1}{8} + \tfrac{2}{16} + \tfrac{3}{32} + \tfrac{4}{64} + \dots$$

Now
$$S - \frac{S}{2} = \tfrac{1}{4} + \tfrac{2}{8} + \tfrac{3}{16} + \tfrac{4}{32} + \tfrac{5}{64} + \dots - \tfrac{1}{8} - \tfrac{2}{16} - \tfrac{3}{32} - \tfrac{4}{64} - \dots$$
$$\therefore \quad \frac{S}{2} = \tfrac{1}{4} + \tfrac{1}{8} + \tfrac{1}{16} + \tfrac{1}{32} + \tfrac{1}{64} + \dots$$
$$\therefore \quad \frac{S}{2} = \frac{\frac{1}{4}}{1-\frac{1}{2}} \qquad \{u_1 = \tfrac{1}{4}, \quad r = \tfrac{1}{2}\}$$
$$\therefore \quad \frac{S}{2} = \tfrac{1}{2}$$
$$\therefore \quad S = 1$$
$$\therefore \quad \tfrac{1}{4} + \tfrac{2}{8} + \tfrac{3}{16} + \tfrac{4}{32} + \tfrac{5}{64} + \dots = 1$$

ACTIVITY 4 VON KOCH'S SNOWFLAKE CURVE



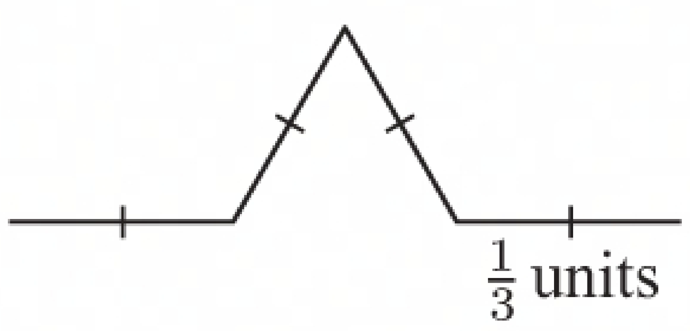
1 **a** C_1 has perimeter 3 units.

$\therefore$ each side of triangle C_1 has length 1 unit.

Now, C_2 has been formed by dividing each side of C_1 into thirds, then making another equilateral triangle along the middle third of each side.

So each side of C_2 has length $\frac{1}{3}$ units.

$\therefore$ the perimeter of $C_2 = \frac{1}{3} \times 4 \times 3$
$= 4$ units



Similarly, C_3 has been formed by dividing each side of C_2 into thirds, then making another equilateral triangle along the middle third of each side.

So each side of C_3 has length $\frac{1}{3} \times \frac{1}{3} = \frac{1}{9}$ units.

$$\begin{aligned}\therefore\quad \text{the perimeter of } C_3 &= \tfrac{1}{9} \times 4 \times 4 \times 3\\ &= \tfrac{16}{3} \text{ units}\end{aligned}$$

Also, each side of C_4 has length $\frac{1}{3} \times \frac{1}{9} = \frac{1}{27}$ units.

$$\begin{aligned}\therefore\quad \text{the perimeter of } C_4 &= \tfrac{1}{27} \times 4 \times 4 \times 4 \times 3\\ &= \tfrac{64}{9} \text{ units}\end{aligned}$$

Also, each side of C_5 has length $\frac{1}{3} \times \frac{1}{27} = \frac{1}{81}$ units.

$$\begin{aligned}\therefore\quad \text{the perimeter of } C_5 &= \tfrac{1}{81} \times 4 \times 4 \times 4 \times 4 \times 3\\ &= \tfrac{256}{27} \text{ units}\end{aligned}$$

b $\dfrac{C_5}{C_4} = \dfrac{\frac{256}{27}}{\frac{64}{9}} = \dfrac{4}{3}$ $\qquad \dfrac{C_4}{C_3} = \dfrac{\frac{64}{9}}{\frac{16}{3}} = \dfrac{4}{3}$ $\qquad \dfrac{C_3}{C_2} = \dfrac{\frac{16}{3}}{4} = \dfrac{4}{3}$ $\qquad \dfrac{C_2}{C_1} = \dfrac{4}{3}$

Consecutive terms have a common ratio of $\frac{4}{3}$.

$\therefore$ the sequence is geometric with $u_1 = 3$ and $r = \frac{4}{3}$.

$$\begin{aligned}u_n &= u_1 r^{n-1}\\ \therefore\quad u_n &= 3 \times \left(\tfrac{4}{3}\right)^{n-1}\end{aligned}$$

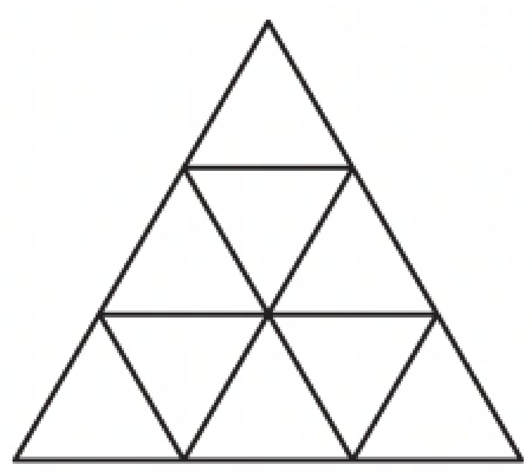
$\therefore$ the perimeter of $C_n = 3 \times \left(\frac{4}{3}\right)^{n-1}$ units

As $n \to \infty$, $\left(\frac{4}{3}\right)^{n-1} \to \infty$

So, von Koch's curve has infinite perimeter.

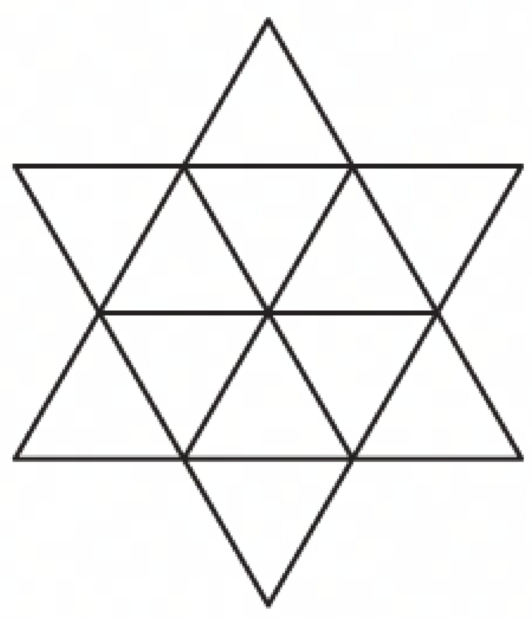
2 C_1 has area 1 unit^2.

a C_1 can be divided into 9 equilateral triangles as shown, each with area $\frac{1}{9}$ units^2.



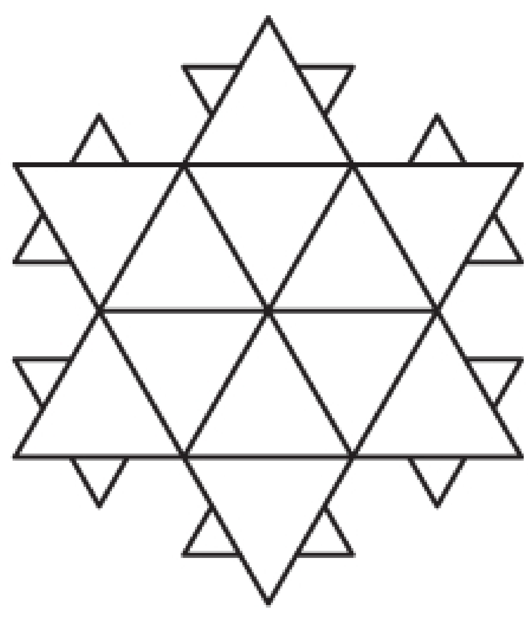
C_2 is formed by adding 3 equilateral triangles to C_1, each with area $\frac{1}{9}$ units^2.

$$\begin{aligned}\text{So, } C_2 \text{ has area } A_2 &= 1 + 3 \times \tfrac{1}{9}\\ &= 1 + \tfrac{1}{3} \text{ units}^2\end{aligned}$$

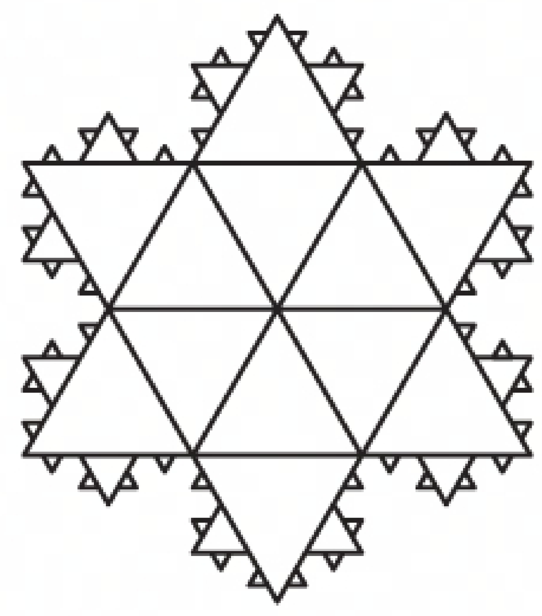


C_3 is formed by adding $4 \times 3 = 12$ equilateral triangles to C_2, each with area $\frac{1}{9} \times \frac{1}{9} = \frac{1}{81}$ units^2.

$$\begin{aligned}\text{So, } C_3 \text{ has area } A_3 &= 1 + \tfrac{1}{3} + 12 \times \tfrac{1}{81}\\ &= 1 + \tfrac{1}{3}[1 + \tfrac{4}{9}] \text{ units}^2\end{aligned}$$



C_4 is formed by adding $4 \times 12 = 48$ equilateral triangles to C_3, each with area $\frac{1}{9} \times \frac{1}{81} = \frac{1}{729}$ units^2.



So, C_4 has area $A_4 = 1 + \frac{1}{3}[1 + \frac{4}{9}] + 48 \times \frac{1}{729}$
$= 1 + \frac{1}{3}[1 + \frac{4}{9} + \frac{16}{81}]$
$= 1 + \frac{1}{3}[1 + \frac{4}{9} + (\frac{4}{9})^2]$ units^2

C_5 is formed by adding $4 \times 48 = 192$ equilateral triangles to C_4, each with area $\frac{1}{9} \times \frac{1}{729} = \frac{1}{6561}$ units^2.

So, C_5 has area $A_5 = 1 + \frac{1}{3}[1 + \frac{4}{9} + (\frac{4}{9})^2] + 192 \times \frac{1}{6561}$
$= 1 + \frac{1}{3}[1 + \frac{4}{9} + (\frac{4}{9})^2 + \frac{64}{729}]$
$= 1 + \frac{1}{3}[1 + \frac{4}{9} + (\frac{4}{9})^2 + (\frac{4}{9})^3]$ units^2

b $A_1 = 1$ unit^2

$A_2 = 1 + \frac{1}{3} = 1.333\,333\,333 ....$ units^2

$A_3 = 1 + \frac{1}{3}[1 + \frac{4}{9}] = 1.481\,481\,481 ....$ units^2

$A_4 = 1 + \frac{1}{3}[1 + \frac{4}{9} + (\frac{4}{9})^2] \approx 1.547\,325\,103$ units^2

$A_5 = 1 + \frac{1}{3}[1 + \frac{4}{9} + (\frac{4}{9})^2 + (\frac{4}{9})^3] \approx 1.576\,588\,935$ units^2

$A_6 = 1 + \frac{1}{3}[1 + \frac{4}{9} + (\frac{4}{9})^2 + (\frac{4}{9})^3 + (\frac{4}{9})^4] \approx 1.589\,595\,082$ units^2

$A_7 = 1 + \frac{1}{3}[1 + \frac{4}{9} + (\frac{4}{9})^2 + (\frac{4}{9})^3 + (\frac{4}{9})^4 + (\frac{4}{9})^5] \approx 1.595\,375\,592$ units^2

c Area within von Koch's snowflake curve

$= 1 + \frac{1}{3}[1 + \frac{4}{9} + (\frac{4}{9})^2 + (\frac{4}{9})^3 + ...]$

$= 1 + \frac{1}{3} \times \dfrac{1}{1 - \frac{4}{9}}$ {as $r = \frac{4}{9}$, $|r| < 1$ so converges}

$= 1 + \frac{1}{3} \times \frac{9}{5}$

$= \frac{8}{5} = 1.6$ units^2

3 Yes, the perimeter of von Koch's curve is infinite whereas the area of von Koch's curve is finite.

REVIEW SET 5A

1 5, 9, 11, 12, 15, 19

a $u_2 = 9$ **b** $u_6 = 19$ **c** $S_4 = 5 + 9 + 11 + 12 = 37$

2 Since the terms are consecutive, $(k-2) - 3k = k + 7 - (k-2)$ {equating differences}

$\therefore\ k - 2 - 3k = k + 7 - k + 2$
$\therefore\ -2 - 2k = 9$
$\therefore\ 2k = -11$
$\therefore\ k = -\frac{11}{2}$

3 $u_n = 6\left(\frac{1}{2}\right)^{n-1}$

a $\dfrac{u_{n+1}}{u_n} = \dfrac{6(\frac{1}{2})^{n+1-1}}{6(\frac{1}{2})^{n-1}} = \frac{1}{2}$ for all n

$\therefore$ $\{u_n\}$ is a geometric sequence.

b $u_1 = 6, \quad r = \frac{1}{2}$

c $u_{16} = 6\left(\frac{1}{2}\right)^{15}$

$\approx 0.000\,183$

4 $u_6 = \frac{16}{3} \qquad \therefore\ u_1 \times r^5 = \frac{16}{3} \quad .... (1)$

$u_{10} = \frac{256}{3} \qquad \therefore\ u_1 \times r^9 = \frac{256}{3} \quad .... (2)$

Now $\dfrac{u_1 r^9}{u_1 r^5} = \dfrac{(\frac{256}{3})}{(\frac{16}{3})}$ $\quad \{(2) \div (1)\}$

$\therefore\ r^4 = \frac{256}{16} = 16$

$\therefore\ r = \pm\sqrt[4]{16}$

$\therefore\ r = \pm 2$

Substituting $r = 2$ into (1) gives

$u_1 \times 2^5 = \frac{16}{3}$

$\therefore\ u_1 \times 32 = \frac{16}{3}$

$\therefore\ u_1 = \frac{1}{6}$

Substituting $r = -2$ into (1) gives

$u_1 \times (-2)^5 = \frac{16}{3}$

$\therefore\ u_1 \times (-32) = \frac{16}{3}$

$\therefore\ u_1 = -\frac{1}{6}$

Now $u_n = u_1 r^{n-1}$

$\therefore\ u_n = \frac{1}{6} \times 2^{n-1}$ or $-\frac{1}{6} \times (-2)^{n-1}$

5 Let the numbers be $23,\ 23+d,\ 23+2d,\ 23+3d,\ 23+4d,\ 23+5d,\ 23+6d,\ 9$

Then $23 + 7d = 9$

$\therefore\ 7d = -14$

$\therefore\ d = -2$

So, the numbers are $23, 21, 19, 17, 15, 13, 11, 9$.

6 **a** Average amount of juice collected $= \dfrac{\text{total amount of juice collected}}{\text{number of lemons}}$

$= \dfrac{274.3 \text{ mL}}{6}$

≈ 45.7 mL

b $u_n \approx 45.7n$

c $u_{13} \approx 45.7 \times 13$

≈ 594

So, approximately 594 mL of juice would be collected from squeezing 13 lemons.

7 **a** $18 - 12 + 8 - ....$
is an infinite geometric series with
$u_1 = 18$ and $r = -\frac{2}{3}$.

$$\therefore \quad S = \frac{u_1}{1-r} = \frac{18}{1-(-\frac{2}{3})} = \frac{54}{5} \text{ or } 10\tfrac{4}{5}$$

b $8 + 4\sqrt{2} + 4 + ...$
is an infinite geometric series with
$u_1 = 8$ and $r = \frac{1}{\sqrt{2}}$.

$$\therefore \quad S = \frac{u_1}{1-r} = \frac{8}{(1-\frac{1}{\sqrt{2}})} \times \frac{(1+\frac{1}{\sqrt{2}})}{(1+\frac{1}{\sqrt{2}})} = \frac{8+\frac{8}{\sqrt{2}}}{1-\frac{1}{2}} = \frac{8+4\sqrt{2}}{\frac{1}{2}} = 16 + 8\sqrt{2}$$

8 **a** $7 + 11 + 15 + 19 + ... + 99$
The series is arithmetic with
$u_1 = 7$, $d = 4$, and $u_n = 99$.
First we need to find n.

$$\begin{aligned} \text{Now} \quad u_n &= 99 \\ \therefore \quad u_1 + (n-1)d &= 99 \\ \therefore \quad 7 + 4(n-1) &= 99 \\ \therefore \quad 4(n-1) &= 92 \\ \therefore \quad n - 1 &= 23 \\ \therefore \quad n &= 24 \end{aligned}$$

Using $S_n = \frac{n}{2}(u_1 + u_n)$,

$$\begin{aligned} S_{24} &= \tfrac{24}{2}(7+99) \\ &= 12 \times 106 \\ &= 1272 \end{aligned}$$

b $35 + 33\frac{1}{2} + 32 + 30\frac{1}{2} + ... + 20$
The series is arithmetic with
$u_1 = 35$, $d = -\frac{3}{2}$, and $u_n = 20$.
First we need to find n.

$$\begin{aligned} \text{Now} \quad u_n &= 20 \\ \therefore \quad u_1 + (n-1)d &= 20 \\ \therefore \quad 35 - \tfrac{3}{2}(n-1) &= 20 \\ \therefore \quad \tfrac{3}{2}(n-1) &= 15 \\ \therefore \quad n - 1 &= 10 \\ \therefore \quad n &= 11 \end{aligned}$$

Using $S_n = \frac{n}{2}(u_1 + u_n)$,

$$\begin{aligned} S_{11} &= \tfrac{11}{2}(35+20) \\ &= \tfrac{11}{2} \times 55 \\ &= 302\tfrac{1}{2} \end{aligned}$$

9 **a** Year 2011: $700\,000 \times 0.9 = 630\,000$ sheets of paper
Year 2012: $630\,000 \times 0.9 = 567\,000$ sheets of paper

b There is a fixed percentage decrease each year, so the amount of paper used each year forms a geometric sequence.
In 2008, the school used $700\,000 \div 0.9 \div 0.9 \approx 864\,198$ sheets of paper.
$\therefore \quad u_1 \approx 864\,198$ and $r = 0.9$
For the decade from 2008 to 2017, $n = 10$.

$$S_n = \frac{u_1(1-r^n)}{1-r}$$

$$\therefore \quad S_{10} \approx \frac{864\,198(1-(0.9)^{10})}{1-0.9} \approx 5\,628\,705$$

The school used approximately 5 630 000 sheets of paper in total in the decade from 2008 to 2017.

10 **a** $\displaystyle\sum_{k=1}^{7} k^2 = 1^2 + 2^2 + 3^2 + 4^2 + 5^2 + 6^2 + 7^2$

$= 1 + 4 + 9 + 16 + 25 + 36 + 49$

$= 140$

b $\displaystyle\sum_{k=1}^{4} \frac{k+3}{k+2} = \tfrac{4}{3} + \tfrac{5}{4} + \tfrac{6}{5} + \tfrac{7}{6}$

$= \tfrac{80}{60} + \tfrac{75}{60} + \tfrac{72}{60} + \tfrac{70}{60}$

$= \tfrac{297}{60}$

$= \tfrac{99}{20}$

11 **a** $S_n = \dfrac{3n^2 + 5n}{2}$

$\therefore\ u_n = S_n - S_{n-1}$

$= \dfrac{3n^2 + 5n}{2} - \dfrac{3(n-1)^2 + 5(n-1)}{2}$

$= \dfrac{3n^2 + 5n - 3(n^2 - 2n + 1) - 5(n-1)}{2}$

$= \dfrac{3n^2 + 5n - 3n^2 + 6n - 3 - 5n + 5}{2}$

$= \dfrac{6n+2}{2}$

$\therefore\ u_n = 3n + 1$

b Using part **a**, $u_n - u_{n-1} = (3n+1) - (3(n-1)+1)$

$= 3n + 1 - 3n + 3 - 1$

$= 3$

The difference between consecutive terms is constant for all n, so the sequence is arithmetic.

12 **a** The interest is calculated half-yearly, so there are $n = 5 \times 2 = 10$ time periods.

Each time period the investment increases by $i = \dfrac{4.25\%}{2} = 2.125\%$.

$\therefore$ the amount after 5 years is $u_{10} = u_0 \times (1+i)^{10}$

$= 12\,500 \times (1.021\,25)^{10}$ $\quad\{2.125\% = 0.021\,25\}$

$\approx 15\,425.20$

The investment will amount to £15 425.20 .

b The interest is calculated monthly, so there are $n = 5 \times 12 = 60$ time periods.

Each time period the investment increases by $i = \dfrac{4.25\%}{12} \approx 0.354\%$.

$\therefore$ the amount after 5 years is $u_{60} = u_0 \times (1+i)^{60}$

$\approx 12\,500 \times (1.003\,54)^{60}$ $\quad\{0.354\% = 0.003\,54\}$

$\approx 15\,453.77$

The investment will amount to £15 453.77 .

13 The initial investment u_0 is unknown.

There are $n = 4 \times 4 = 16$ time periods.

Each time period the investment increases by $i = \frac{6.5\%}{4} = 1.625\%$.

$$\text{Now,} \quad u_{16} = u_0 \times (1+i)^{16}$$
$$\therefore \quad 6212.27 = u_0 \times (1.016\,25)^{16} \qquad \{1.625\% = 0.016\,25\}$$
$$\therefore \quad u_0 = \frac{6212.27}{(1.016\,25)^{16}} \approx 4800.00$$

Chelsea originally invested \$4800.

14 **a** To index the amount of money for inflation, we increase it by 2.5% each year for 4 years.

$$\therefore \text{ indexed value} = €6000 \times (1.025)^4$$
$$= €6622.88$$

b To index the amount of money for inflation, we increase it by 2.5% each year for 7 years.

$$\therefore \text{ indexed value} = €11\,200 \times (1.025)^7$$
$$= €13\,313.28$$

15 **a** There are $n = 3 \times 12 = 36$ time periods.

Each period, the investment increases by $i = \frac{6.2\%}{12} \approx 0.517\%$.

$$\therefore \text{ the amount after 3 years is } \quad u_{36} = u_0 \times (1+i)^{36}$$
$$= 20\,000 \times (1.005\,17)^{36} \qquad \{0.517\% = 0.005\,17\}$$
$$\approx 24\,076.91$$

The future value of the investment is \$24 076.91 .

b real value $\times (1.018)^3 = \$24\,076.91$

$$\therefore \text{ real value} = \frac{\$24\,076.91}{(1.018)^3}$$
$$= \$22\,822.20$$

16 28, 23, 18, 13, ...

$23 - 28 = -5$ The difference between successive terms is constant.

$18 - 23 = -5$ $\therefore$ the sequence is arithmetic with $u_1 = 28$ and $d = -5$.

$13 - 18 = -5$

$$u_n = u_1 + (n-1)d$$
$$\therefore \quad u_n = 28 - 5(n-1)$$
$$\therefore \quad u_n = 33 - 5n$$

$$S_n = \frac{n}{2}(2u_1 + (n-1)d)$$
$$= \frac{n}{2}(2 \times 28 - 5(n-1))$$
$$= \frac{n}{2}(56 - 5n + 5)$$
$$= \frac{n}{2}(61 - 5n)$$

17 **a** 128, 64, 32, 16, ..., $\frac{1}{512}$

The sequence is geometric with

$u_1 = 128, \quad r = \frac{1}{2}, \quad u_n = \frac{1}{512}$

$$\begin{aligned} u_n &= u_1 r^{n-1} \\ &= 128\left(\tfrac{1}{2}\right)^{n-1} \\ &= 2^7 \times 2^{1-n} \end{aligned}$$

$$\begin{aligned} \therefore \quad \tfrac{1}{512} &= 2^7 \times 2^{1-n} \\ \therefore \quad 2^{-9} &= 2^{8-n} \\ \therefore \quad -9 &= 8-n \\ \therefore \quad n &= 17 \end{aligned}$$

So, there are 17 terms in the sequence.

b
$$\begin{aligned} S_n &= \frac{u_1(1-r^n)}{1-r} \\ \therefore \quad S_{17} &= \frac{128\left(1-(\frac{1}{2})^{17}\right)}{1-\frac{1}{2}} \\ &= \frac{131\,071}{512} \\ &\approx 256 \end{aligned}$$

18 Let the terms of the geometric series be $u_1, u_1r, u_1r^2, ...$

$$u_1 + u_1 r = 90 \qquad \text{and} \qquad u_1 r^2 = 24$$

$$\therefore \quad u_1(1+r) = 90 \qquad\qquad \therefore \quad u_1 = \frac{24}{r^2} \quad .... (2)$$

$$\therefore \quad u_1 = \frac{90}{1+r} \quad .... (1)$$

Equating (1) and (2) gives $\dfrac{90}{1+r} = \dfrac{24}{r^2}$

$$\begin{aligned} \therefore \quad 90r^2 &= 24 + 24r \\ \therefore \quad 90r^2 - 24r - 24 &= 0 \\ \therefore \quad 6(15r^2 - 4r - 4) &= 0 \\ \therefore \quad 6(5r+2)(3r-2) &= 0 \\ \therefore \quad r &= -\tfrac{2}{5} \text{ or } \tfrac{2}{3} \end{aligned}$$

Using (2), if $r = -\frac{2}{5}$, $\quad u_1 = \dfrac{24}{(-\frac{2}{5})^2} = \dfrac{24}{\frac{4}{25}} = 150$

if $r = \frac{2}{3}$, $\quad u_1 = \dfrac{24}{(\frac{2}{3})^2} = \dfrac{24}{\frac{4}{9}} = 54$

$\therefore$ either $u_1 = 150, \ r = -\frac{2}{5}$ or $u_1 = 54, \ r = \frac{2}{3}$.

Since $|\,r\,| < 1$ in each case, both series converge.

19 **a** Every week after the first, Tim smokes 5 less cigarettes, so the difference between successive weeks is always -5. Thus we have an arithmetic sequence with $u_1 = 120 - 5 = 115$ and $d = -5$.

b
$$\begin{aligned} u_n &= u_1 + (n-1)d \\ &= 115 - 5(n-1) \\ &= 120 - 5n \end{aligned}$$

Let $u_n = 0$

$$\begin{aligned} \therefore \quad 120 - 5n &= 0 \\ \therefore \quad 5n &= 120 \\ \therefore \quad n &= 24 \end{aligned}$$

So, it will take 24 weeks before Tim has smoked his last cigarette.

c
$$\begin{aligned} S_n &= \frac{n}{2}(u_1 + u_n) \\ \therefore \quad S_{24} &= \tfrac{24}{2}(115 + 0) \\ &= 12 \times 115 \\ &= 1380 \end{aligned}$$

Tim will smoke 1380 cigarettes before he successfully quits.

20 **a** $160,\ 80\sqrt{2},\ 80,\ 40\sqrt{2},\ ....$

The sequence is geometric with $u_1 = 160$ and $r = \frac{1}{\sqrt{2}}$.

$$\begin{aligned} u_n &= u_1 r^{n-1} \\ &= 160 \times \left(\tfrac{1}{\sqrt{2}}\right)^{n-1} \\ \therefore\ u_{12} &= 160 \times \left(\tfrac{1}{\sqrt{2}}\right)^{11} \\ &= 160 \times \frac{1}{(\sqrt{2})^{11}} \\ &= \frac{160}{32\sqrt{2}} \\ &= \frac{5}{\sqrt{2}} \\ &= \tfrac{5}{2}\sqrt{2} \end{aligned}$$

b **i**

$$\begin{aligned} S_n &= \frac{u_1(1-r^n)}{1-r} \\ &= \frac{160\left(1-\left(\frac{1}{\sqrt{2}}\right)^n\right)}{1-\frac{1}{\sqrt{2}}} \\ \therefore\ S_{10} &= \frac{160\left(1-\left(\frac{1}{\sqrt{2}}\right)^{10}\right)}{1-\frac{1}{\sqrt{2}}} \\ &= \frac{160\left(1-\frac{1}{(\sqrt{2})^{10}}\right)}{1-\frac{1}{\sqrt{2}}} \\ &= \frac{160\left(1-\frac{1}{32}\right)}{1-\frac{1}{\sqrt{2}}} \\ &= \frac{160 \times \frac{31}{32}}{1-\frac{1}{\sqrt{2}}} \\ &= \frac{155}{1-\frac{1}{\sqrt{2}}} \times \frac{\sqrt{2}}{\sqrt{2}} \\ &= \frac{155\sqrt{2}}{\sqrt{2}-1} \times \frac{\sqrt{2}+1}{\sqrt{2}+1} \\ &= \frac{155\sqrt{2}(\sqrt{2}+1)}{2-1} \\ &= 310 + 155\sqrt{2} \end{aligned}$$

ii

$$\begin{aligned} S &= \frac{u_1}{1-r} \\ &= \frac{160}{1-\frac{1}{\sqrt{2}}} \times \frac{\sqrt{2}}{\sqrt{2}} \\ &= \frac{160\sqrt{2}}{\sqrt{2}-1} \times \frac{\sqrt{2}+1}{\sqrt{2}+1} \\ &= \frac{160\sqrt{2}(\sqrt{2}+1)}{2-1} \\ &= 320 + 160\sqrt{2} \end{aligned}$$

21 Let the first three terms of the geometric series be $u_1,\ u_1r,\ u_1r^2$.

$$S_3 = u_1 + u_1r + u_1r^2 = 39$$
$$\therefore\ u_1(1+r+r^2) = 39$$
$$\therefore\ u_1 = \frac{39}{1+r+r^2} \quad .... (1)$$

The middle term u_1r is increased by $66\frac{2}{3}\%$, so it can now be expressed as $\frac{5}{3}u_1r$.

The three terms thus become $u_1,\ \frac{5}{3}u_1r,\ u_1r^2$ and these form an arithmetic sequence.

Since the terms are consecutive, $\frac{5}{3}u_1r - u_1 = u_1r^2 - \frac{5}{3}u_1r$ {equating differences}

$$\therefore\ u_1r^2 - \tfrac{10}{3}u_1r + u_1 = 0$$
$$\therefore\ u_1(r^2 - \tfrac{10}{3}r + 1) = 0$$
$$\therefore\ r^2 - \tfrac{10}{3}r + 1 = 0 \qquad \{u_1 \text{ cannot be equal to zero}\}$$
$$\therefore\ 3r^2 - 10r + 3 = 0$$
$$\therefore\ (3r-1)(r-3) = 0$$
$$\therefore\ r = \tfrac{1}{3} \text{ or } 3$$

Using (1), $u_1 = \dfrac{39}{1+r+r^2}$

When $r = \frac{1}{3}$, $u_1 = \dfrac{39}{1+\frac{1}{3}+(\frac{1}{3})^2} = 27$

When $r = 3$, $u_1 = \dfrac{39}{1+3+3^2} = 3$

$\therefore$ the smallest possible value of the first term is 3.

22 $S_n = \frac{n}{2}(u_1 + u_n)$ and $u_n = u_1 + (n-1)d$

So, $S_1 = \frac{1}{2}(u_1+u_1) = u_1$ and $S_2 = \frac{2}{2}(u_1+u_2) = u_1 + u_1 + d = 2u_1 + d$ and $S_4 = \frac{4}{2}(u_1+u_4) = 2(u_1+u_1+3d) = 4u_1 + 6d$

Now S_1, S_2, S_4 form a geometric sequence with common ratio $\dfrac{S_2}{S_1}$ or $\dfrac{S_4}{S_2}$.

$$\frac{S_2}{S_1} = \frac{S_4}{S_2} \qquad \{\text{equating ratios}\}$$
$$\therefore\ \frac{2u_1+d}{u_1} = \frac{4u_1+6d}{2u_1+d}$$
$$\therefore\ (2u_1+d)^2 = u_1(4u_1+6d)$$
$$\therefore\ 4u_1^2 + 4u_1d + d^2 = 4u_1^2 + 6u_1d$$
$$\therefore\ 2u_1d - d^2 = 0$$
$$\therefore\ d(2u_1 - d) = 0$$
$$\therefore\ d = 0 \text{ or } 2u_1 - d = 0$$
$$\therefore\ u_1 = \frac{d}{2} \qquad \{d \text{ cannot be equal to zero}\}$$

So the common ratio for the sequence S_1, S_2, S_3 is $\dfrac{S_2}{S_1} = \dfrac{2(\frac{d}{2})+d}{\frac{d}{2}} = \dfrac{d+d}{\frac{d}{2}} = 4$

23 $x+y+z=\frac{7}{3}$

$\therefore\ x+z=\frac{7}{3}-y$ .... (1)

$x^2+y^2+z^2=\frac{91}{9}$ .... (2)

x, y, and z are consecutive terms of a geometric sequence.

$\therefore\ \frac{y}{x}=\frac{z}{y}$ {equating common ratios}

$\therefore\ y^2=xz$ .... (3)

Substituting (3) into (2) gives $x^2+xz+z^2=\frac{91}{9}$

$\therefore\ x^2+2xz+z^2=\frac{91}{9}+xz$ {adding xz to both sides}

$\therefore\ (x+z)^2=\frac{91}{9}+y^2$ {using (3)}

$\therefore\ (\frac{7}{3}-y)^2=\frac{91}{9}+y^2$ {using (1)}

$\therefore\ \frac{49}{9}-\frac{14}{3}y+y^2=\frac{91}{9}+y^2$

$\therefore\ -\frac{14}{3}y=\frac{42}{9}$

$\therefore\ y=-1$

Substituting $y=-1$ into (3) gives $(-1)^2=xz$

$\therefore\ z=\frac{1}{x}$ .... (4)

Substituting $y=-1$ into (1) gives $x+z=\frac{7}{3}-(-1)$

$\therefore\ x+\frac{1}{x}=\frac{10}{3}$ {using (4)}

$\therefore\ x^2+1=\frac{10}{3}x$

$\therefore\ 3x^2+3=10x$

$\therefore\ 3x^2-10x+3=0$

$\therefore\ (3x-1)(x-3)=0$

$\therefore\ x=\frac{1}{3}$ or 3

When $x=\frac{1}{3}$, $z=\frac{1}{(\frac{1}{3})}=3$ {using (4)}

When $x=3$, $z=\frac{1}{3}$ {using (4)}

So, the two solutions are $x=\frac{1}{3}$, $y=-1$, $z=3$

or $x=3$, $y=-1$, $z=\frac{1}{3}$

24 **a** Total prize value for *Option 1* $=\$8000\times 24$

$=\$192\,000$

b **i** Amount won in each of the first three months:

Month 1: \$1000

Month 2: $\$1000+\$600=\$1600$

Month 3: $\$1600+\$600=\$2200$

ii The amount (in dollars) won in month n forms an arithmetic sequence with $u_1 = 1000$ and $d = 600$.

$$S_n = \frac{n}{2}(2u_1 + (n-1)d)$$

$$\therefore \quad S_{24} = \tfrac{24}{2}(2 \times 1000 + 23 \times 600)$$
$$= 12(2000 + 13\,800)$$
$$= 189\,600$$

So, the total amount won over the 24 month period was \$189 600.

c **i** Amount won in each of the first three months:

Month 1: \$500
Month 2: \$500 × 1.2 = \$600
Month 3: \$600 × 1.2 = \$720

ii The amount (in dollars) won in month n forms a geometric sequence with $u_1 = 500$ and $r = 1.2$.

$$S_n = \frac{u_1(r^n - 1)}{r - 1}$$

$$\therefore \quad S_{24} = \frac{500((1.2)^{24} - 1)}{1.2 - 1}$$
$$\approx 196\,242.12$$

So, the total amount won over the 24 month period was \$196 242.12 .

d *Option 3* is worth the greatest amount of money overall.

e $S_{24} = 250\,000$, $r = 1.2$ but u_1 is unknown

$$\therefore \quad 250\,000 = \frac{u_1((1.2)^{24} - 1)}{1.2 - 1}$$

$$\therefore \quad u_1 = \frac{250\,000 \times 0.2}{(1.2)^{24} - 1}$$
$$\approx 636.97$$

So, the new initial amount is \$636.97 .

25 $2x$ and $x-2$ are the first two terms of a convergent geometric series.

$\therefore\ u_1 = 2x$ and $r = \dfrac{x-2}{2x}$

$$S = \frac{u_1}{1-r} = \tfrac{18}{7}$$

$$\therefore\ \frac{2x}{1-\left(\frac{x-2}{2x}\right)} = \tfrac{18}{7}$$

$$\therefore\ \frac{2x}{\left(\frac{2x-(x-2)}{2x}\right)} = \tfrac{18}{7}$$

$$\therefore\ \frac{2x}{\left(\frac{x+2}{2x}\right)} = \tfrac{18}{7}$$

$$\therefore\ \frac{(2x)^2}{x+2} = \tfrac{18}{7}$$

$$\therefore\ \frac{4x^2}{x+2} = \tfrac{18}{7}$$

$$\therefore\ 4x^2 = \tfrac{18}{7}(x+2)$$

$$\therefore\ 4x^2 = \tfrac{18}{7}x + \tfrac{36}{7}$$

$$\therefore\ 4x^2 - \tfrac{18}{7}x - \tfrac{36}{7} = 0$$

$$\therefore\ 28x^2 - 18x - 36 = 0$$

$$\therefore\ 2(14x^2 - 9x - 18) = 0$$

$$\therefore\ 2(7x+6)(2x-3) = 0$$

$$\therefore\ x = -\tfrac{6}{7} \text{ or } \tfrac{3}{2}$$

When $x = -\frac{6}{7}$, $r = \dfrac{-\frac{6}{7}-2}{2(-\frac{6}{7})} = \dfrac{-\frac{20}{7}}{-\frac{12}{7}} = \frac{20}{12} = \frac{5}{3}$

When $x = \frac{3}{2}$, $r = \dfrac{\frac{3}{2}-2}{2(\frac{3}{2})} = \dfrac{-\frac{1}{2}}{3} = -\frac{1}{6}$

$|\,r\,| < 1$ only when $x = \frac{3}{2}$, so $x = \frac{3}{2}$ is the only possible solution.

26 a, b, and c are consecutive terms of an arithmetic sequence.

$\therefore\ b - a = c - b = d$ where d is the common difference.

Also, $b = a + d$ and $c = a + 2d$ $\{$using $u_n = u_1 + (n-1)d\}$

$\therefore\ c - a = 2d$ .... (1)

a $(c+a) - (b+c) = a - b = -(b-a) = -d$

$(a+b) - (c+a) = b - c = -(c-b) = -d$

$\therefore$ the difference between successive terms is constant.

$\therefore$ $b+c$, $c+a$, and $a+b$ are also consecutive terms of an arithmetic sequence.

b $$\frac{1}{\sqrt{c}+\sqrt{a}} - \frac{1}{\sqrt{b}+\sqrt{c}} = \left(\frac{1}{\sqrt{c}+\sqrt{a}}\right)\left(\frac{\sqrt{c}-\sqrt{a}}{\sqrt{c}-\sqrt{a}}\right) - \left(\frac{1}{\sqrt{b}+\sqrt{c}}\right)\left(\frac{\sqrt{b}-\sqrt{c}}{\sqrt{b}-\sqrt{c}}\right)$$

$$= \frac{\sqrt{c}-\sqrt{a}}{c-a} - \left(\frac{\sqrt{b}-\sqrt{c}}{b-c}\right)$$

$$= \frac{\sqrt{c}-\sqrt{a}}{c-a} - \left(\frac{\sqrt{b}-\sqrt{c}}{-(c-b)}\right)$$

$$= \frac{\sqrt{c}-\sqrt{a}}{2d} - \left(\frac{\sqrt{b}-\sqrt{c}}{-d}\right) \qquad \{\text{using (1)}\}$$

$$= \frac{\sqrt{c}-\sqrt{a}}{2d} + \frac{\sqrt{b}-\sqrt{c}}{d}$$

$$= \frac{\sqrt{c}-\sqrt{a}+2(\sqrt{b}-\sqrt{c})}{2d}$$

$$= \frac{-\sqrt{a}+2\sqrt{b}-\sqrt{c}}{2d}$$

$$\frac{1}{\sqrt{a}+\sqrt{b}} - \frac{1}{\sqrt{c}+\sqrt{a}} = \left(\frac{1}{\sqrt{a}+\sqrt{b}}\right)\left(\frac{\sqrt{a}-\sqrt{b}}{\sqrt{a}-\sqrt{b}}\right) - \left(\frac{1}{\sqrt{c}+\sqrt{a}}\right)\left(\frac{\sqrt{c}-\sqrt{a}}{\sqrt{c}-\sqrt{a}}\right)$$

$$= \frac{\sqrt{a}-\sqrt{b}}{a-b} - \left(\frac{\sqrt{c}-\sqrt{a}}{c-a}\right)$$

$$= \frac{\sqrt{a}-\sqrt{b}}{-(b-a)} - \left(\frac{\sqrt{c}-\sqrt{a}}{c-a}\right)$$

$$= \frac{\sqrt{a}-\sqrt{b}}{-d} - \left(\frac{\sqrt{c}-\sqrt{a}}{2d}\right) \qquad \{\text{using (1)}\}$$

$$= \frac{\sqrt{b}-\sqrt{a}}{d} - \left(\frac{\sqrt{c}-\sqrt{a}}{2d}\right)$$

$$= \frac{2(\sqrt{b}-\sqrt{a})-(\sqrt{c}-\sqrt{a})}{2d}$$

$$= \frac{-\sqrt{a}+2\sqrt{b}-\sqrt{c}}{2d}$$

$\therefore$ the difference between successive terms is constant.

$\therefore$ $\dfrac{1}{\sqrt{b}+\sqrt{c}}$, $\dfrac{1}{\sqrt{c}+\sqrt{a}}$, and $\dfrac{1}{\sqrt{a}+\sqrt{b}}$ are also consecutive terms of an arithmetic sequence.

REVIEW SET 5B

1 **a** The sequence $\{(\frac{1}{3})^n\}$ begins $\frac{1}{3}$, $\frac{1}{9}$, $\frac{1}{27}$, $\frac{1}{81}$, $\frac{1}{243}$ (letting $n = 1, 2, 3, 4, 5, ...$).

b The sequence $\{12+5n\}$ begins 17, 22, 27, 32, 37 (letting $n = 1, 2, 3, 4, 5, ...$).

c The sequence $\left\{\dfrac{4}{n+2}\right\}$ begins $\frac{4}{3}$, 1, $\frac{4}{5}$, $\frac{2}{3}$, $\frac{4}{7}$ (letting $n = 1, 2, 3, 4, 5, ...$).

2 **a** $u_n = 68 - 5n,$ $\quad u_{n+1} = 68 - 5(n+1)$

$$= 63 - 5n$$

$$u_{n+1} - u_n = (63 - 5n) - (68 - 5n)$$
$$= -5, \quad \text{a constant}$$

Consecutive terms differ by -5.

$\therefore$ the sequence is arithmetic.

b $u_1 = 68 - 5(1)$ $\quad d = -5$
$= 63$

c $u_{37} = 68 - 5(37)$
$= -117$

d Let $u_n = -200 = 68 - 5n$

$$\therefore \; 5n = 268$$
$$\therefore \; n = 53\tfrac{3}{5}$$

We try the two values on either side of $n = 53\frac{3}{5}$, which are $n = 53$ and $n = 54$:

$u_{53} = 68 - 5(53)$ and $u_{54} = 68 - 5(54)$
$= -197$ $\qquad\qquad = -202$

So, $u_{54} = -202$ is the first term which is less than -200.

3 **a** $u_7 = 31 \quad \therefore \; u_1 + 6d = 31 \quad .... (1) \quad$ {using $u_n = u_1 + (n-1)d$}

$u_{15} = -17 \quad \therefore \; u_1 + 14d = -17 \quad .... (2)$

We now solve (1) and (2) simultaneously:

$$-u_1 - 6d = -31 \quad \text{\{multiplying both sides of (1) by } -1\text{\}}$$
$$u_1 + 14d = -17$$
$$\therefore \; 8d = -48 \quad \text{\{adding the equations\}}$$
$$\therefore \; d = -6$$

So in (1), $u_1 + 6(-6) = 31$

$$\therefore \; u_1 - 36 = 31$$
$$\therefore \; u_1 = 67$$

Now $u_n = u_1 + (n-1)d$

$$\therefore \; u_n = 67 - 6(n-1)$$
$$\therefore \; u_n = 73 - 6n$$

Check:

$u_7 = 73 - 6(7)$
$= 31$ ✓

$u_{15} = 73 - 6(15)$
$= -17$ ✓

b $u_{34} = 73 - 6(34)$
$= -131$

4 **a** The series is arithmetic with $u_1 = 3$, $d = 6$, $n = 12$.

Now $S_n = \frac{n}{2}(2u_1 + (n-1)d)$

$$\therefore \; S_{12} = \tfrac{12}{2}(2 \times 3 + 11 \times 6)$$
$$\therefore \; S_{12} = 6(6 + 66)$$
$$= 432$$

b The series is geometric with $u_1 = 24$, $r = \frac{1}{2}$, $n = 12$.

$$S_n = \frac{u_1(1 - r^n)}{1 - r}$$
$$\therefore \; S_{12} = \frac{24\left(1 - (\frac{1}{2})^{12}\right)}{1 - \frac{1}{2}}$$
$$= 48\left(1 - (\tfrac{1}{2})^{12}\right)$$
$$= \frac{12\,285}{256}$$
$$\approx 48.0$$

5 **a** $u_{25} = 60 \quad \therefore \quad u_1 + 24d = 60 \quad .... (1) \qquad \{\text{using } u_n = u_1 + (n-1)d\}$

$u_{43} = 135 \quad \therefore \quad u_1 + 42d = 135 \quad .... (2)$

We now solve (1) and (2) simultaneously:

$$\begin{aligned} -u_1 - 24d &= -60 \qquad \{\text{multiplying both sides of (1) by } -1\} \\ u_1 + 42d &= 135 \\ \hline \therefore \quad 18d &= 75 \\ \therefore \quad d &= \tfrac{75}{18} = \tfrac{25}{6} \end{aligned}$$

So in (1),

$$\begin{aligned} u_1 + 24\left(\tfrac{25}{6}\right) &= 60 \\ \therefore \quad u_1 + 100 &= 60 \\ \therefore \quad u_1 &= -40 \end{aligned}$$

Now

$$\begin{aligned} u_n &= u_1 + (n-1)d \\ \therefore \quad u_n &= -40 + \tfrac{25}{6}(n-1) \\ \therefore \quad u_n &= -40 + \tfrac{25}{6}n - \tfrac{25}{6} \\ \therefore \quad u_n &= \tfrac{25}{6}n - \tfrac{265}{6} \end{aligned}$$

Check:

$$\begin{aligned} u_{25} &= \tfrac{25}{6}(25) - \tfrac{265}{6} \\ &= 60 \quad \checkmark \\ u_{43} &= \tfrac{25}{6}(43) - \tfrac{265}{6} \\ &= 135 \quad \checkmark \end{aligned}$$

b The common difference is $\frac{25}{6} \approx 4.17$. This means that Stacy makes approximately £4.17 profit per customer.

The constant term is $-\frac{265}{6} \approx -44.17$. This means that the setup fee for the hot dog stand is approximately £44.17.

c

$$\begin{aligned} u_n &= \tfrac{25}{6}n - \tfrac{265}{6} \\ \therefore \quad u_{36} &= \tfrac{25}{6}(36) - \tfrac{265}{6} \\ &\approx 105.83 \end{aligned}$$

So, Stacy made an estimated profit of £105.83 on the third day.

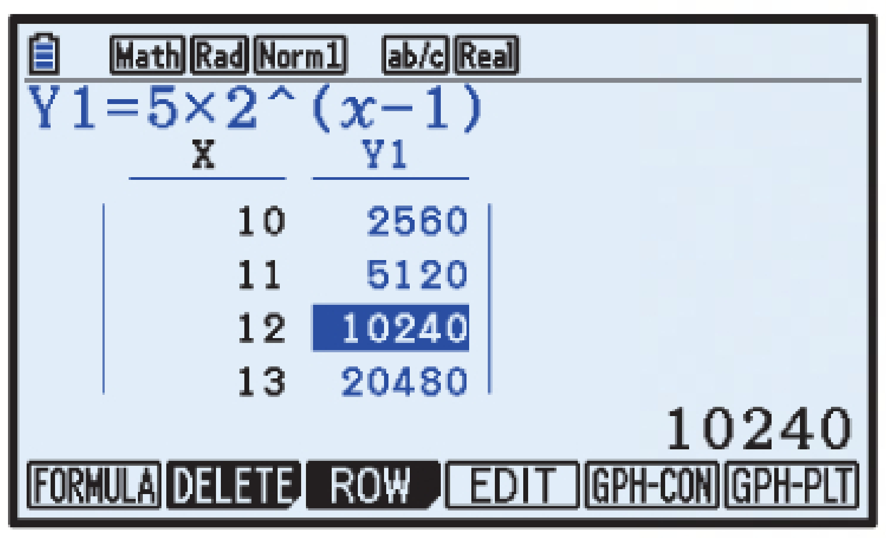
6 5, 10, 20, 40, ...

The sequence is geometric with $u_1 = 5$ and $r = 2$.

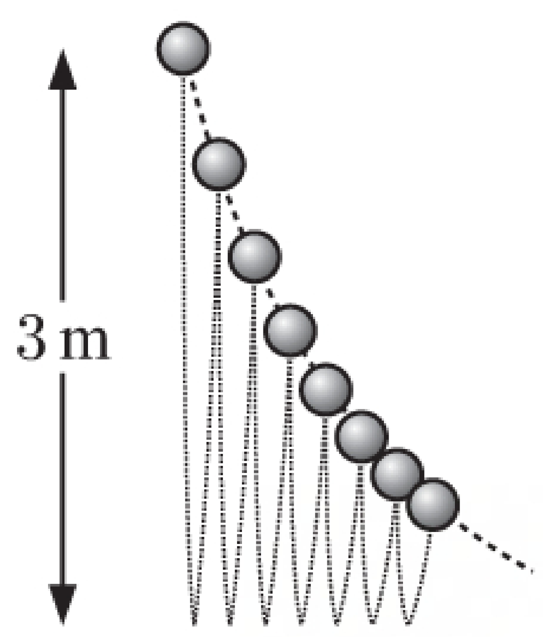
$\therefore \quad u_n = 5 \times 2^{n-1}$

We need to find n such that $u_n > 10\,000$.

Using a graphics calculator with $\text{Y}_1 = 5 \times 2\text{^}(\text{X}-1)$, we view a table of values:



The first term to exceed 10 000 is $u_{12} = 10\,240$.

7 

Total distance travelled

$= 3 + 2 \times 3 \times 0.8 + 2 \times 3 \times (0.8)^2 + 2 \times 3 \times (0.8)^3 + ...$

$= 3 + 2 \times 3 \times 0.8\left[1 + 0.8 + (0.8)^2 + (0.8)^3 + ...\right]$

$= 3 + 4.8 \times \dfrac{1}{1-0.8}$ $\left\{\text{as } r = 0.8, \ |r| < 1 \text{ so converges to } \dfrac{u_1}{1-r}\right\}$

$= 3 + \dfrac{4.8}{0.2}$

$= 3 + 24$

$= 27$ metres

8 **a** The interest is calculated annually, so $n = 3$ time periods.

$u_3 = u_0 \times (1+i)^3$

$= 7000 \times (1.06)^3$ $\{6\% = 0.06\}$

≈ 8337.11

The investment will amount to \$8337.11 .

b The interest is calculated quarterly, so there are $n = 3 \times 4 = 12$ time periods.

Each time period the investment increases by $i = \dfrac{6\%}{4} = 1.5\%$.

$\therefore$ the amount after 3 years is $u_{12} = u_0 \times (1+i)^{12}$

$= 7000 \times (1.015)^{12}$ $\{1.5\% = 0.015\}$

≈ 8369.33

The investment will amount to \$8369.33 .

c The interest is calculated monthly, so there are $n = 3 \times 12 = 36$ time periods.

Each time period the investment increases by $i = \dfrac{6\%}{12} = 0.5\%$.

$\therefore$ the amount after 3 years is $u_{36} = u_0 \times (1+i)^{36}$

$= 7000 \times (1.005)^{36}$ $\{0.5\% = 0.005\}$

≈ 8376.76

The investment will amount to \$8376.76 .

9 **a** Since the terms are geometric,

$\dfrac{k}{4} = \dfrac{k^2-1}{k}$ {equating the common ratio r}

$\therefore \ k^2 = 4(k^2 - 1)$

$\therefore \ k^2 = 4k^2 - 4$

$\therefore \ 3k^2 = 4$

$\therefore \ k^2 = \frac{4}{3}$

$\therefore \ k = \pm\frac{2}{\sqrt{3}} = \pm\frac{2\sqrt{3}}{3}$

b When $k = \frac{2\sqrt{3}}{3}$, $r = \dfrac{k}{4} = \dfrac{\frac{2\sqrt{3}}{3}}{4}$

$\therefore \ r = \frac{\sqrt{3}}{6}$

When $k = -\frac{2\sqrt{3}}{3}$, $r = \dfrac{k}{4} = \dfrac{-\frac{2\sqrt{3}}{3}}{4}$

$\therefore \ r = -\frac{\sqrt{3}}{6}$

10 Since Seve walks an additional $500 \text{ m} = 0.5 \text{ km}$ each week, we have an arithmetic sequence with $u_1 = 10$ and common difference $d = 0.5$.

$$u_n = u_1 + (n-1)d$$
$$\therefore \quad u_n = 10 + 0.5(n-1)$$

a $u_{52} = 10 + 0.5(52-1)$ {52 weeks in a year}
$= 35.5$

$\therefore$ Seve walks 35.5 km in the last week.

b In total, Seve walks $10 + 10.5 + 11 + ... + 35.5$, which is an arithmetic series with $u_1 = 10$, $u_n = 35.5$, $n = 52$.

$$S_n = \frac{n}{2}(u_1 + u_n)$$
$$\therefore \quad S_{52} = \tfrac{52}{2}(10 + 35.5) = 1183$$

$\therefore$ Seve walks 1183 km in total.

11 **a** $1.21 - 1.1 + 1 - ....$ is an infinite geometric series with $u_1 = 1.21$ and $r = -\frac{10}{11}$.

$$\therefore \quad S = \frac{u_1}{1-r} = \frac{1.21}{1-(-\frac{10}{11})} = \frac{1331}{2100} \approx 0.634$$

b $\frac{14}{3} + \frac{4}{3} + \frac{8}{21} + ...$ is an infinite geometric series with $u_1 = \frac{14}{3}$ and $r = \frac{2}{7}$.

$$\therefore \quad S = \frac{u_1}{1-r} = \frac{\frac{14}{3}}{1-\frac{2}{7}} = \tfrac{98}{15} \approx 6.53$$

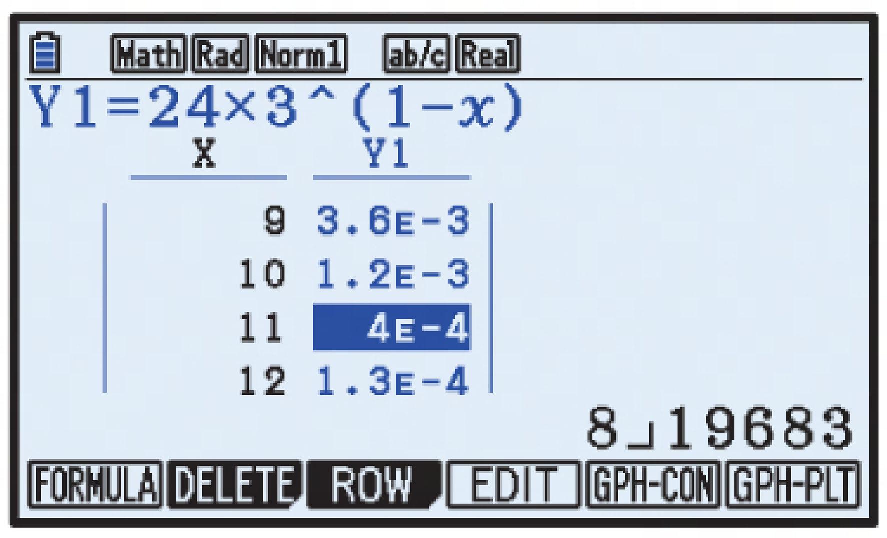
12 $24, 8, \frac{8}{3}, \frac{8}{9}, ...$

The sequence is geometric with $u_1 = 24$ and $r = \frac{1}{3}$.

$$\therefore \quad u_n = 24 \times \left(\tfrac{1}{3}\right)^{n-1}$$
$$\therefore \quad u_n = 24 \times 3^{1-n}$$

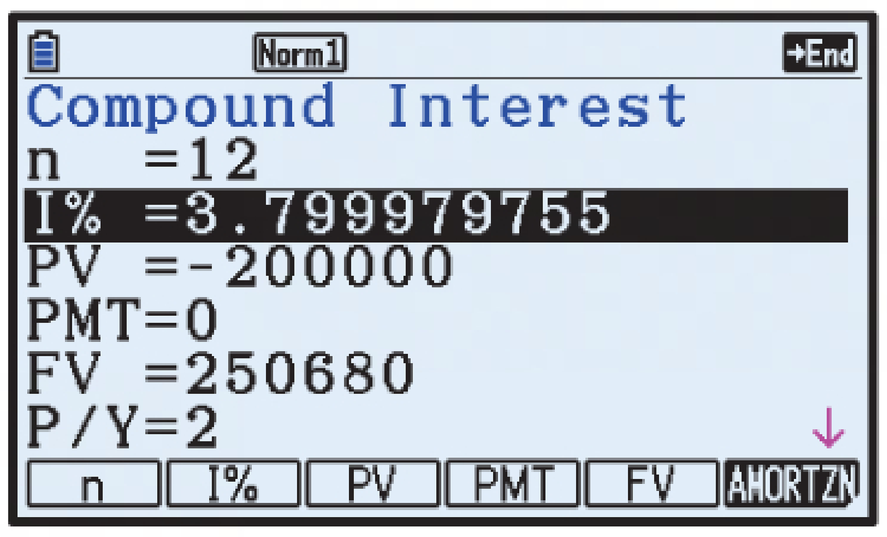
We need to find n such that $u_n < 0.001$.

Using a graphics calculator with $\text{Y}_1 = 24 \times 3\text{\^{}}(1-\text{X})$, we view a table of values:



The first term which is less than 0.001 is $u_{11} = \dfrac{8}{19\,683} \approx 0.000\,406$.

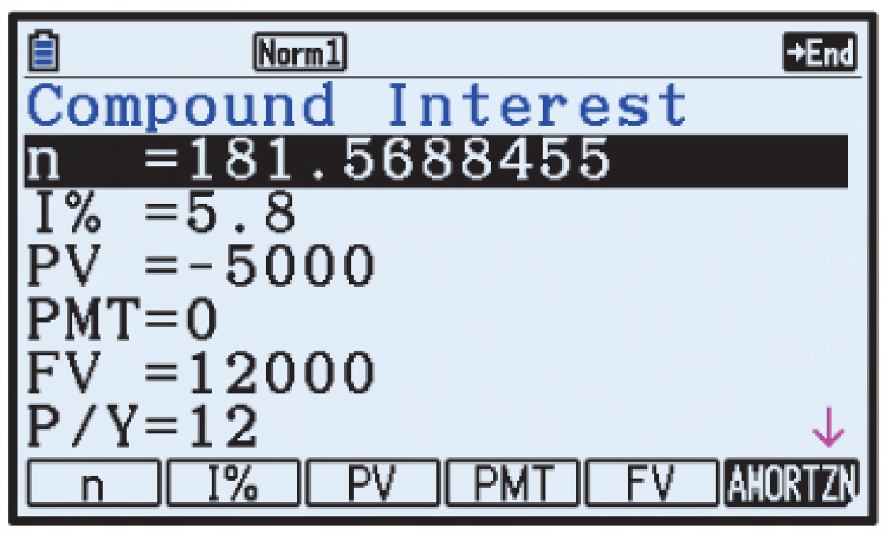
13 $N = 6 \times 2 = 12$, $PV = -200\,000$, $PMT = 0$, $FV = 250\,680$, $P/Y = 2$, $C/Y = 2$



$\therefore \quad I\% \approx 3.80$

The interest rate is 3.80% p.a.

14 $I\% = 5.8$, $PV = -5000$, $PMT = 0$, $FV = 12\,000$, $P/Y = 12$, $C/Y = 12$



$\therefore \quad N \approx 181.6$

It will take 182 months, or 15 years 2 months for the investment to amount to €12 000.

15 **a** There are $n = 8 \times 4 = 32$ time periods.

Each period, the investment increases by $i = \dfrac{3.7\%}{4} = 0.925\%$.

$\therefore$ the amount after 8 years is
$$\begin{aligned} u_{32} &= u_0 \times (1+i)^{32} \\ &= 7500 \times (1.009\,25)^{32} \quad \{0.925\% = 0.009\,25\} \\ &\approx 10\,069.82 \end{aligned}$$

The future value of Richard's investment is \$10 069.82.

b real value $\times (1.031)^8 = \$10\,069.82$

$$\begin{aligned} \therefore \quad \text{real value} &= \frac{\$10\,069.82}{(1.031)^8} \\ &= \$7887.74 \end{aligned}$$

16
$$\begin{aligned} u_5 &= u_0 \times (1-d)^5 \\ &= 9800 \times (0.74)^5 \quad \{26\% = 0.26\} \\ &= 2174.63 \end{aligned}$$

So, after 5 years the value of the photocopier is \$2174.63.

17 **a** $\displaystyle\sum_{k=1}^{8} \left(\frac{31-3k}{2}\right) = 14 + 12\tfrac{1}{2} + 11 + 9\tfrac{1}{2} + 8 + 6\tfrac{1}{2} + 5 + 3\tfrac{1}{2}$

This series is arithmetic with $u_1 = 14$, $d = -\frac{3}{2}$, and $n = 8$.

Using $S_n = \frac{n}{2}(u_1 + u_n)$,
$$\begin{aligned} S_8 &= \tfrac{8}{2}(14 + 3\tfrac{1}{2}) \\ &= 4 \times 17\tfrac{1}{2} \\ &= 70 \end{aligned}$$

b $\displaystyle\sum_{k=1}^{15} 50(0.8)^{k-1} \approx 50 + 40 + 32 + ... + 3.436 + 2.749 + 2.199$

This series is geometric with $u_1 = 50$, $r = 0.8$, and $n = 15$.

$$S_n = \frac{u_1(1-r^n)}{1-r}$$

$$\therefore \quad S_{15} = \frac{50\left(1-(0.8)^{15}\right)}{1-0.8}$$
$$= 250\left(1-(0.8)^{15}\right)$$
$$\approx 241$$

c $\displaystyle\sum_{k=7}^{\infty} 5\left(\tfrac{2}{5}\right)^{k-1} = 5\left(\tfrac{2}{5}\right)^6 + 5\left(\tfrac{2}{5}\right)^7 + 5\left(\tfrac{2}{5}\right)^8 + ...$

This is an infinite geometric series with $u_1 = 5\left(\frac{2}{5}\right)^6$ and $r = \frac{2}{5}$.

$$S = \frac{u_1}{1-r}$$
$$= \frac{5(\frac{2}{5})^6}{1-\frac{2}{5}}$$
$$= \frac{25}{3} \times \frac{2^6}{5^6}$$
$$= \frac{64}{1875}$$
$$\approx 0.0341$$

18 **a** $u_6 = 24 \quad \therefore \quad u_1 \times r^5 = 24 \quad .... (1)$

$u_{11} = 768 \quad \therefore \quad u_1 \times r^{10} = 768 \quad .... (2)$

Now $\dfrac{u_1 r^{10}}{u_1 r^5} = \dfrac{768}{24} \quad \{(2) \div (1)\}$

$\therefore \quad r^5 = 32$

$\therefore \quad r = 2$

Using (1), $u_1 \times 2^5 = 24$

$\therefore \quad u_1 = \frac{24}{32} = \frac{3}{4}$

$u_n = u_1 r^{n-1}$

$\therefore \quad u_n = \left(\frac{3}{4}\right)2^{n-1}$

b $$S_n = \frac{u_1(r^n-1)}{r-1}$$
$$= \frac{\frac{3}{4}(2^n-1)}{2-1}$$
$$= \tfrac{3}{4}(2^n-1)$$

$\therefore \quad S_{15} = \frac{3}{4}(2^{15}-1) = 24\,575\frac{1}{4}$

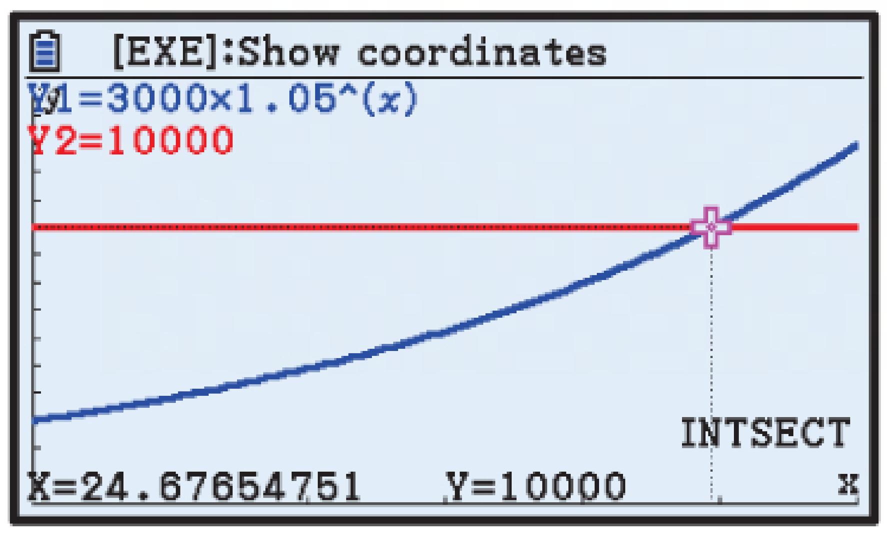
19 There is a fixed percentage increase each year, so the population forms a geometric sequence.
$u_0 = 3000$ and $r = 1.05$
$\therefore$ the population after n years is $u_n = 3000 \times (1.05)^n$.

a 2007 is 3 years after 2004.

$u_3 = 3000 \times (1.05)^3$
$= 3472.875$

There were approximately 3470 iguanas on the island in 2007.

b We need to find when $3000 \times (1.05)^n = 10\,000$.



So, it will take approximately 24.7 years for the population to reach 10 000.
$\therefore$ the population will exceed 10 000 in the 25th year, which is 2029.

20 **a** $\sum_{k=1}^{\infty} 50(2x-1)^{k-1}$ is a geometric series with $r = 2x-1$ and converges if $-1 < r < 1$

$\therefore -1 < 2x - 1 < 1$
$\therefore 0 < 2x < 2$
$\therefore 0 < x < 1$

b When $x = 0.3$, $2x - 1 = 0.6 - 1 = -0.4$

$\therefore \sum_{k=1}^{\infty} 50(2x-1)^{k-1} = 50(-0.4)^0 + 50(-0.4)^1 + 50(-0.4)^2 + ...$

which is geometric with $u_1 = 50$ and $r = -0.4$.

Now as $0 < 0.3 < 1$, the series converges and
$$S = \frac{u_1}{1-r} = \frac{50}{1+0.4} = \frac{50}{\frac{7}{5}} = 35\tfrac{5}{7}$$

21 $u_1, u_2, u_3, ...$ is an arithmetic sequence, so $u_n = u_1 + (n-1)d$

a The sequence $2^{u_1}, 2^{u_2}, 2^{u_3}, ...$ has general term
$$2^{u_n} = 2^{u_1+(n-1)d} = 2^{u_1} \times (2^d)^{n-1}$$

which is a geometric sequence with first term 2^{u_1} and common ratio 2^d.

b $u_3 = u_1 + 2d = -4$
$\therefore u_1 = -4 - 2d$ .... (1)

$u_5 = u_1 + 4d = -10$
$\therefore u_1 = -10 - 4d$ .... (2)

Equating (1) and (2) gives $-4 - 2d = -10 - 4d$
$\therefore 2d = -6$
$\therefore d = -3$

Substituting $d = -3$ into (1) gives $u_1 = -4 - 2(-3)$
$= 2$

Now $S = \dfrac{u_1}{1-r}$

$= \dfrac{2^{u_1}}{1-2^d}$ $\{r = 2^d$ from **a**$\}$

$= \dfrac{2^2}{1-2^{-3}}$

$= \dfrac{4}{1-\frac{1}{8}}$

$= \dfrac{4}{\frac{7}{8}}$

$= \frac{32}{7}$

22 **a** The initial investment u_0 is unknown.

There are $n = 3 \times 2 = 6$ time periods.

Each time period the investment increases by $i = \dfrac{6.5\%}{2} = 3.25\%$.

Now $u_6 = u_0 \times (1+i)^6$

$\therefore$ $100\,000 = u_0 \times (1.0325)^6$ $\{3.25\% = 0.0325\}$

$\therefore$ $u_0 = \dfrac{100\,000}{(1.0325)^6} \approx 82\,539.08$

Michael invested \$82 539.08 three years ago.

b

n (years)	0	1	2	3	4
V_n (\$)	100 000	106 000	112 360	$112\,360 \times 1.06$ $= 119\,101.60$	$119\,101.60 \times 1.06$ $\approx 126\,247.70$

c $V_n = 100\,000 \times (1.06)^n$ dollars

d Each year, the amount of money in the safe increases by a constant amount of \$6000, with an initial amount of \$6000.

So, we have an arithmetic sequence with $u_1 = 6000$ and $d = 6000$.

$S_n = 6000 + (n-1)6000$

$\therefore$ $S_n = 6000n$ dollars

e $T_n = V_n + S_n$

$\therefore$ $T_3 = V_3 + S_3$ and $T_4 = V_4 + S_4$

$= 119\,101.60 + 6000(3)$ $\approx 126\,247.70 + 6000(4)$

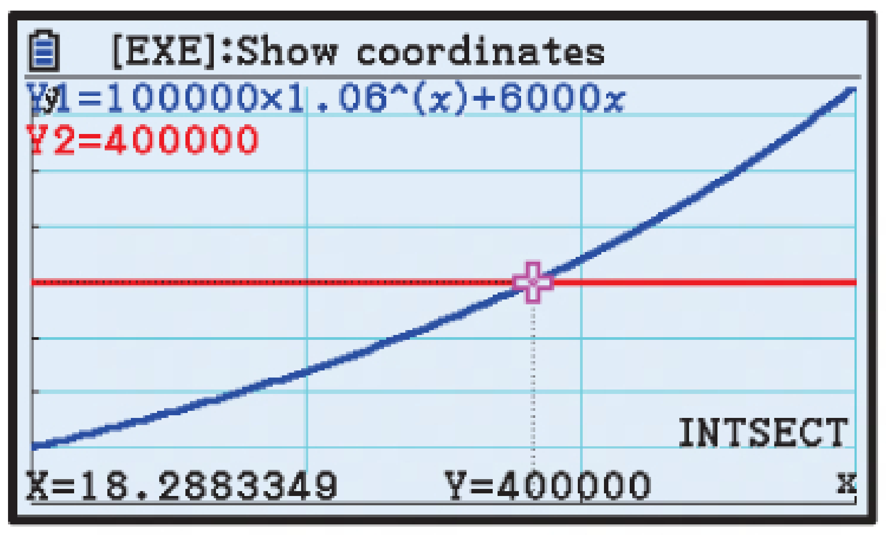
$= 137\,101.60$ dollars $\approx 150\,247.70$ dollars

So, the table is:

n (years)	0	1	2	3	4
T_n (\$)	100 000	112 000	124 360	137 101.60	150 247.70

f We need to find when $T_n = V_n + S_n = 400\,000$

$$\therefore \quad 100\,000 \times (1.06)^n + 6000n = 400\,000$$



So, it will take approximately 18.3 years, or 19 whole years, for Michael to have the $400 000 needed to buy his house.

23 Let the terms of the geometric series be $u_1, \; u_1 r, \; u_1 r^2, \; ....$

$u_1 r = 10$ and $S = \dfrac{u_1}{1-r} = 49$

$$\therefore \quad u_1 = \frac{10}{r} \quad .... \ (1)$$

$$\therefore \quad \frac{\frac{10}{r}}{1-r} = 49 \qquad \{\text{using (1)}\}$$

$$\therefore \quad \frac{10}{r} = 49(1-r)$$

$$\therefore \quad \frac{10}{r} = 49 - 49r$$

$$\therefore \quad 49r^2 - 49r + 10 = 0$$

$$\therefore \quad (7r-2)(7r-5) = 0$$

$$\therefore \quad r = \tfrac{2}{7} \text{ or } \tfrac{5}{7}$$

When $r = \frac{2}{7}$, $u_1 = \dfrac{10}{\frac{2}{7}} = 35$, so the first three terms are $35, \; 10, \; \frac{20}{7}$
with sum $35 + 10 + \frac{20}{7} = 47\frac{6}{7}$

When $r = \frac{5}{7}$, $u_1 = \dfrac{10}{\frac{5}{7}} = 14$, so the first three terms are $14, \; 10, \; \frac{50}{7}$
with sum $14 + 10 + \frac{50}{7} = 31\frac{1}{7}$

24 Let the terms of the geometric sequence be $1, \underbrace{r, r^2, r^3, ..., r^{n-1}, r^n}_{n \text{ terms}}, 2.$

$$\therefore \quad r^{n+1} = 2$$

$$\therefore \quad \left(r^{n+1}\right)^{\frac{1}{n+1}} = 2^{\frac{1}{n+1}}$$

$$\therefore \quad r = 2^{\frac{1}{n+1}}$$

The required sum is $r + r^2 + r^3 + ... + r^{n-1} + r^n$ which is a finite geometric series of n terms with first term $u_1 = r$ and common ratio r.

$$S_n = \frac{u_1(r^n - 1)}{r-1}$$

$$= \frac{r(r^n - 1)}{r-1}$$

$$= \frac{r^{n+1} - r}{r-1}$$

$$\therefore \quad S_n = \frac{2 - 2^{\frac{1}{n+1}}}{2^{\frac{1}{n+1}} - 1}$$

25 **a** For the arithmetic sequence, $u_3 = u_1 + 2d$, $u_4 = u_1 + 3d$, $u_8 = u_1 + 7d$.

$\therefore$ the first three terms of the geometric sequence are $u_1 + 2d$, $u_1 + 3d$, $u_1 + 7d$.

The first three terms of the geometric sequence can also be expressed as $u_1 + 2d$, $(u_1 + 2d)r$, $(u_1 + 2d)r^2$, where r is the common ratio.

$$\begin{aligned}\therefore\ u_1 + 3d &= (u_1 + 2d)r\\ \therefore\ u_1 + 3d &= u_1 r + 2dr\\ \therefore\ u_1 - u_1 r &= 2dr - 3d\\ \therefore\ u_1(1 - r) &= d(2r - 3)\\ \therefore\ u_1 &= \frac{d(2r-3)}{1-r} \quad \text{.... (1)}\end{aligned}$$

and

$$\begin{aligned}u_1 + 7d &= (u_1 + 2d)r^2\\ \therefore\ u_1 + 7d &= u_1 r^2 + 2dr^2\\ \therefore\ u_1 - u_1 r^2 &= 2dr^2 - 7d\\ \therefore\ u_1(1 - r^2) &= d(2r^2 - 7)\\ \therefore\ u_1 &= \frac{d(2r^2-7)}{1-r^2} \quad \text{.... (2)}\end{aligned}$$

Equating (1) and (2) gives $\dfrac{d(2r-3)}{1-r} = \dfrac{d(2r^2-7)}{1-r^2}$

$$\begin{aligned}\therefore\ \frac{2r-3}{1-r} &= \frac{2r^2-7}{(1+r)(1-r)}\\ \therefore\ 2r - 3 &= \frac{2r^2-7}{1+r} \qquad \{\text{since } r \neq 1\}\\ \therefore\ (2r-3)(1+r) &= 2r^2 - 7\\ \therefore\ 2r + 2r^2 - 3 - 3r &= 2r^2 - 7\\ \therefore\ -r &= -4\\ \therefore\ r &= 4\end{aligned}$$

So, the common ratio r for the geometric sequence is 4.

b Substituting $r = 4$ into (1) gives
$$\begin{aligned}u_1 &= \frac{d(2(4)-3)}{1-4}\\ &= \frac{5d}{-3}\\ \therefore\ u_1 &= -\tfrac{5}{3}d \quad \text{.... (3)}\end{aligned}$$

For the arithmetic sequence,
$$\begin{aligned}u_{24} &= u_1 + 23d\\ &= -\tfrac{5}{3}d + 23d \qquad \{\text{using (3)}\}\\ &= 21\tfrac{1}{3}d\end{aligned}$$

For the geometric sequence,
$$\begin{aligned}u_4 &= (u_1 + 2d)r^3\\ &= (-\tfrac{5}{3}d + 2d) \times 4^3\\ &= (\tfrac{1}{3}d) \times 64\\ &= 21\tfrac{1}{3}d\end{aligned}$$

$\therefore$ the 24th term of the arithmetic sequence is equal to the 4th term of the geometric sequence.

26 The series $1 + 10 + 10^2 + ... + 10^{n-1}$ is geometric with $u_1 = 1$ and $r = 10$.

$$\therefore \underbrace{111111....1}_{2n \text{ lots of } 1} = 1 + 10 + 10^2 + ... + 10^{2n-1}$$

$$= \frac{1 \times (10^{2n} - 1)}{10 - 1} \quad \{\text{using } S_n = \frac{u_1(r^n - 1)}{r - 1}\}$$

$$= \frac{10^{2n} - 1}{9}$$

Also, $\underbrace{22222....2}_{n \text{ lots of } 2} = 2 \times \underbrace{111111....1}_{n \text{ lots of } 1}$

$$= 2 \times (1 + 10 + 10^2 + ... + 10^{n-1})$$

$$= 2 \times \left(\frac{1 \times (10^n - 1)}{10 - 1}\right) \quad \{\text{using } S_n = \frac{u_1(r^n - 1)}{r - 1}\}$$

$$= \frac{2 \times 10^n - 2}{9}$$

$$\therefore \underbrace{(111111....1)}_{2n \text{ lots of } 1} - \underbrace{(22222....2)}_{n \text{ lots of } 2} = \frac{10^{2n} - 1}{9} - \frac{2 \times 10^n - 2}{9}$$

$$= \frac{10^{2n} - 1 - 2 \times 10^n + 2}{9}$$

$$= \frac{(10^n)^2 - 2 \times 10^n + 1}{9}$$

$$= \frac{(10^n - 1)^2}{3^2}$$

$$= \left(\frac{10^n - 1}{3}\right)^2$$

$10^n - 1 = \underbrace{99999....9}_{n \text{ lots of } 9}$ is divisible by 3, so $\frac{10^n - 1}{3}$ is an integer.

$\therefore \underbrace{(111111....1)}_{2n \text{ lots of } 1} - \underbrace{(22222....2)}_{n \text{ lots of } 2}$ is a perfect square.

Chapter 6

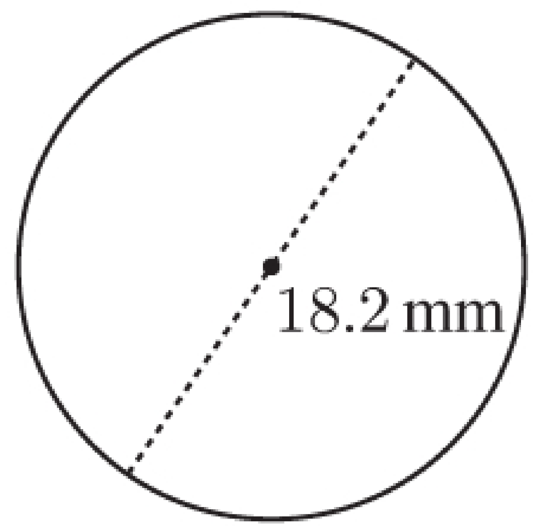
MEASUREMENT

EXERCISE 6A

1 **a** Perimeter $= 2\pi r$

$= 2 \times \pi \times \frac{18.2}{2}$ mm

≈ 57.2 mm



b Arc length $= \frac{\theta}{360} \times 2\pi r$

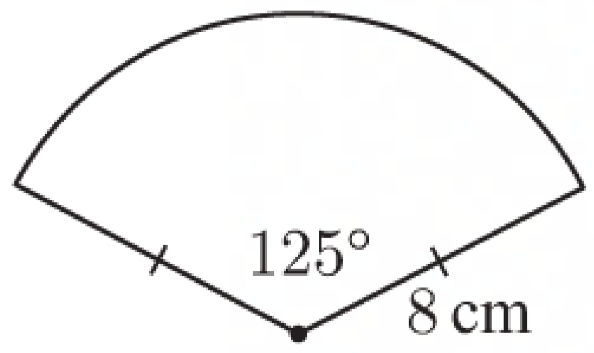
$= \frac{125}{360} \times 2\pi \times 8$

≈ 17.5 cm

$\therefore$ perimeter $= 2r +$ arc length

$\approx 2 \times 8 + 17.5$ cm

≈ 33.5 cm



c Arc length $= \frac{\theta}{360} \times 2\pi r$

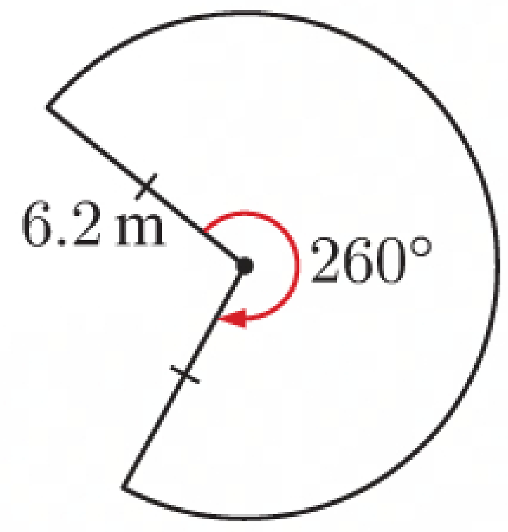
$= \frac{260}{360} \times 2\pi \times 6.2$

≈ 28.1 m

$\therefore$ perimeter $= 2r +$ arc length

$\approx 2 \times 6.2 + 28.1$ m

≈ 40.5 m



d Length of shorter arc $= \frac{\theta}{360} \times 2\pi r$

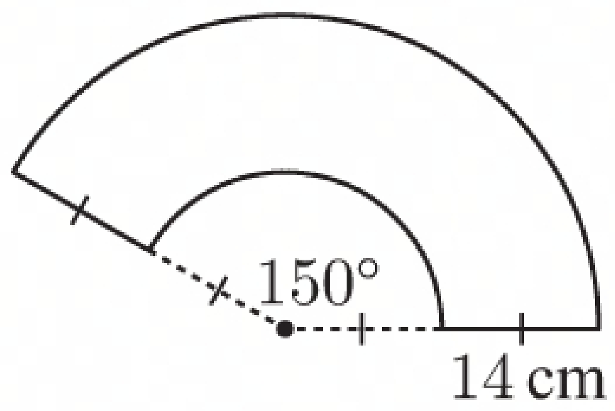
$= \frac{150}{360} \times 2\pi \times 14$ cm

≈ 36.7 cm

Length of longer arc $= \frac{\theta}{360} \times 2\pi r$

$= \frac{150}{360} \times 2\pi \times (14 + 14)$ cm

≈ 73.3 cm



$\therefore$ perimeter = length of shorter arc + length of longer arc + length of two ends

$\approx 36.7 + 73.3 + 2 \times 14$ cm

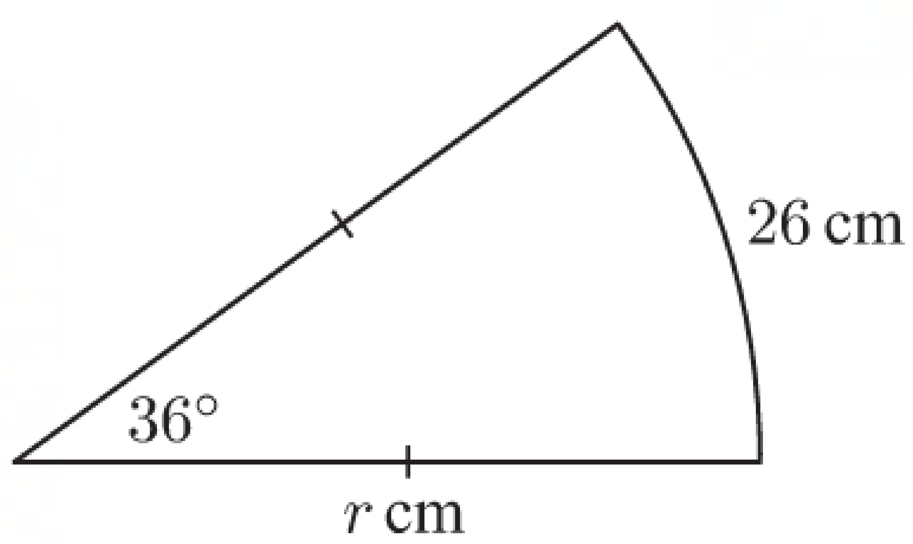
≈ 138 cm

2 Arc length $= \frac{\theta}{360} \times 2\pi r$

$\therefore \; 26 = \frac{36}{360} \times 2\pi \times r$

$\therefore \; r = \frac{26 \times 360}{36 \times 2\pi}$

≈ 41.4



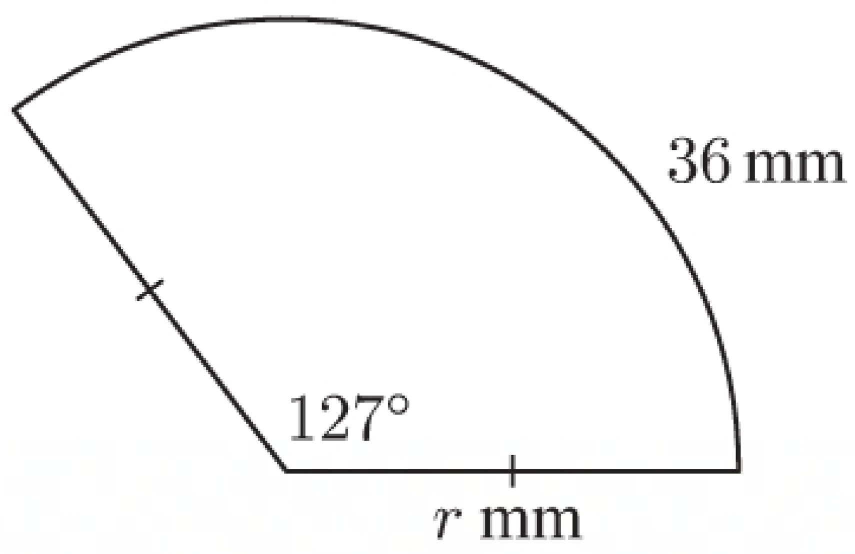
The radius of the circle is approximately 41.4 cm.

3 Arc length $= \frac{\theta}{360} \times 2\pi r$

$\therefore\ 36 = \frac{127}{360} \times 2\pi \times r$

$\therefore\ r = \frac{36 \times 360}{127 \times 2\pi}$

≈ 16.2



Perimeter $= 2r +$ arc length

$\approx 2 \times 16.2 + 36$ mm

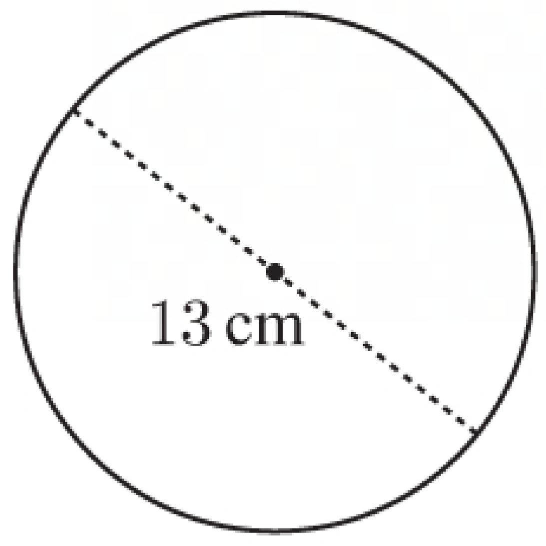
≈ 68.5 mm

4 **a** Area

$= \pi r^2$

$= \pi \times \left(\frac{13}{2}\right)^2$

$\approx 133\ \text{cm}^2$

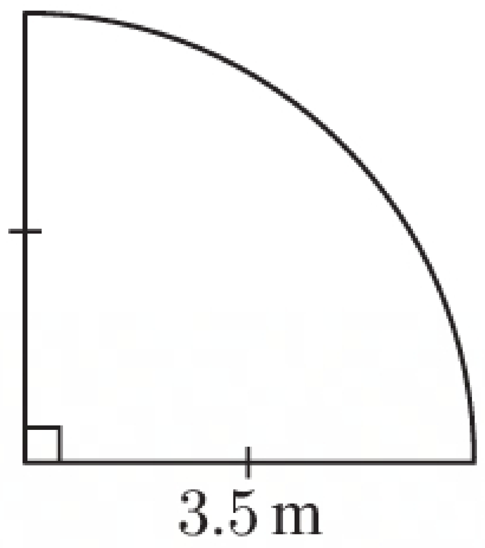


b Area

$= \frac{1}{4} \times \pi r^2$

$= \frac{1}{4} \times \pi \times (3.5)^2$

$\approx 9.62\ \text{m}^2$

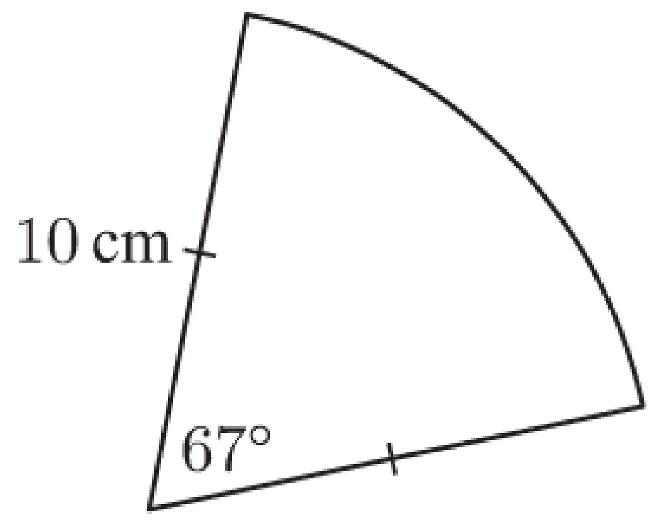


c Area

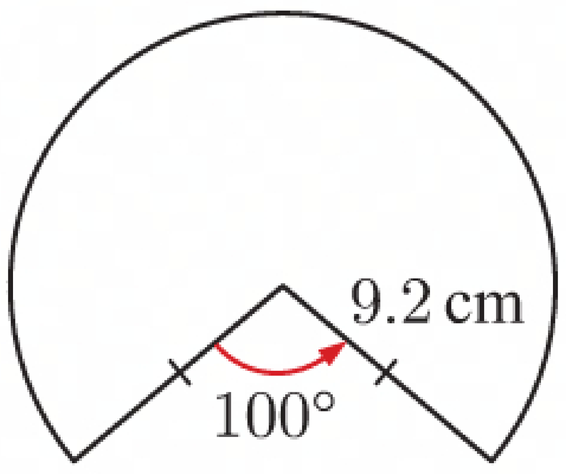
$= \frac{\theta}{360} \times \pi r^2$

$= \frac{67}{360} \times \pi \times 10^2$

$\approx 58.5\ \text{cm}^2$



d



Area $= \frac{\theta}{360} \times \pi r^2$

$= \frac{360 - 100}{360} \times \pi \times (9.2)^2$

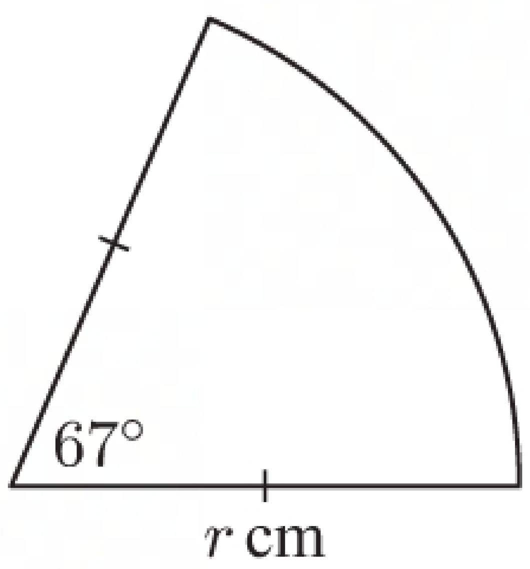
$\approx 192\ \text{cm}^2$

5 Area $= \frac{\theta}{360} \times \pi r^2$

$\therefore\ 16.2 = \frac{67}{360} \times \pi \times r^2$

$\therefore\ r^2 = \frac{16.2 \times 360}{67 \times \pi}$

$\therefore\ r \approx 5.26 \qquad \{\text{as } r > 0\}$



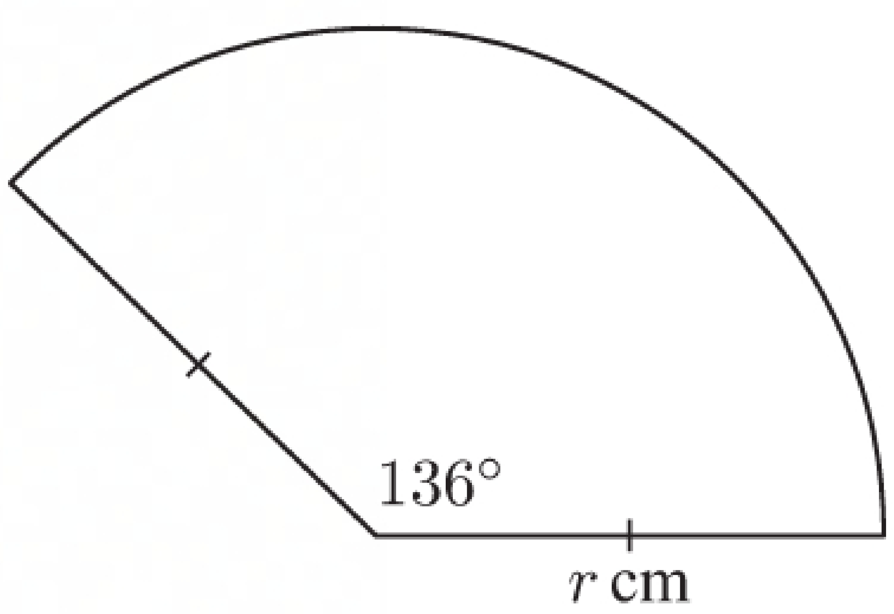
The radius of the sector is approximately 5.26 cm.

6 Area $= \frac{\theta}{360} \times \pi r^2$

$\therefore\ 28.8 = \frac{136}{360} \times \pi \times r^2$

$\therefore\ r^2 = \frac{28.8 \times 360}{136 \times \pi}$

$\therefore\ r \approx 4.93 \qquad \{\text{as } r > 0\}$



Now, arc length $= \frac{\theta}{360} \times 2\pi r$

$\approx \frac{136}{360} \times 2\pi \times 4.93$ cm

≈ 11.7 cm

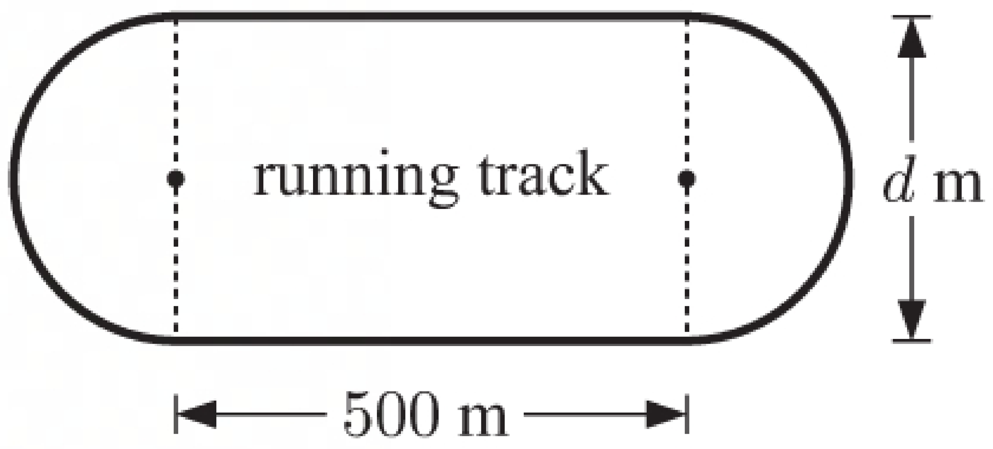
Perimeter $= 2r +$ arc length

$\approx 2 \times 4.93 + 11.7$ cm

≈ 21.5 cm

7 **a** Total perimeter of track

$= \text{length of two straight segments}$
$\quad + \text{ length of two semi-circular ends}$
$= 2 \times 500 + 2\pi r$
$= 1000 + \pi d$



$\therefore \quad 1000 + \pi d = 1600$
$\therefore \quad \pi d = 600$
$\therefore \quad d = \dfrac{600}{\pi}$
≈ 191

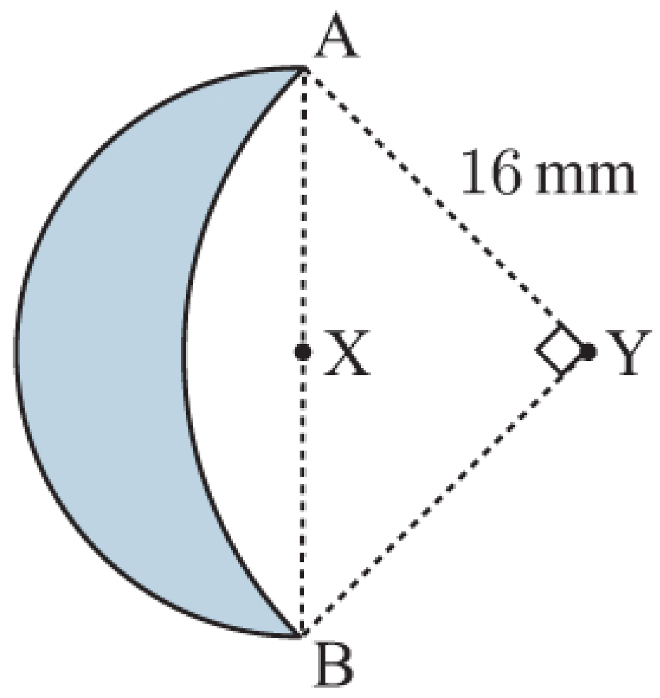
The diameter of the semi-circular ends is approximately 191 m.

b 4 minutes 25 seconds
$= 4 \times 60 + 25$ seconds
$= 265$ seconds

$\text{speed} = \dfrac{\text{distance}}{\text{time}}$
$= \dfrac{1600 \text{ m}}{265 \text{ s}}$
$\approx 6.04 \text{ m s}^{-1}$

Jason's average speed is approximately 6.04 m s^{-1}.

8 **a** In $\triangle$ABY, $(\text{AB})^2 = 16^2 + 16^2$ {Pythagoras}
$\therefore \quad \text{AB} = \sqrt{16^2 + 16^2}$ {as AB > 0}
$\therefore \quad \text{AB} = 16\sqrt{2}$ mm



The circle with centre X has diameter AB, and radius $r_\text{X} = \text{AX}$.

$\therefore \quad \text{AX} = \frac{1}{2} \times \text{AB}$
$\therefore \quad \text{AX} = \frac{1}{2} \times 16\sqrt{2}$ mm
$\therefore \quad \text{AX} = 8\sqrt{2} \approx 11.3$ mm

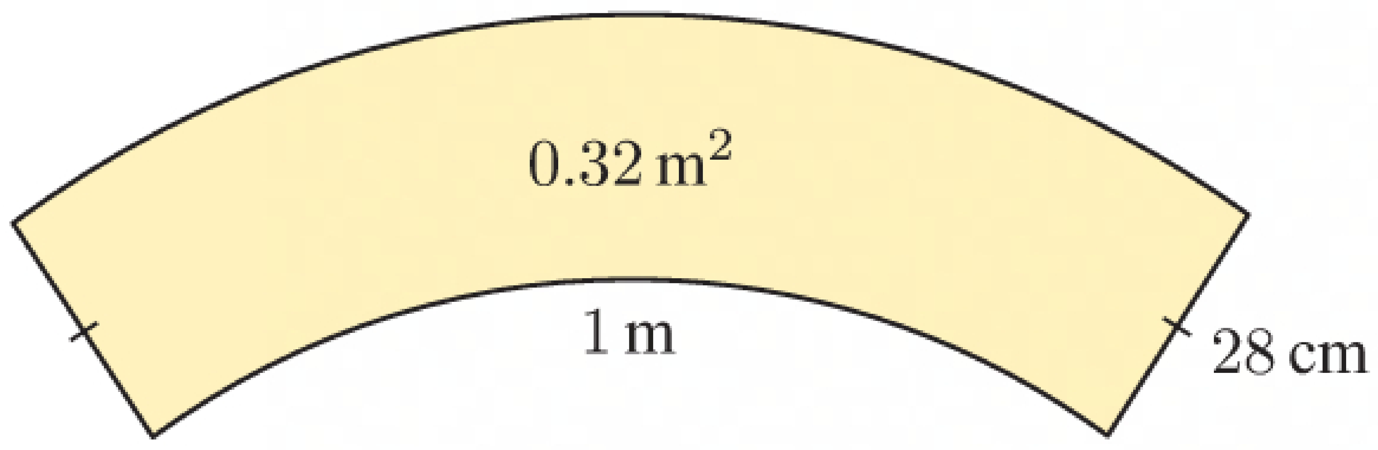
b Perimeter of shaded crescent

$= \text{arc length of circle with centre X} + \text{arc length of circle with centre Y}$
$= \pi r_\text{X} + \frac{1}{2} \times \pi r_\text{Y}$
$= \pi \times 8\sqrt{2} + \frac{1}{2} \times \pi \times 16$
$= 8\pi(1 + \sqrt{2})$
≈ 60.7 mm

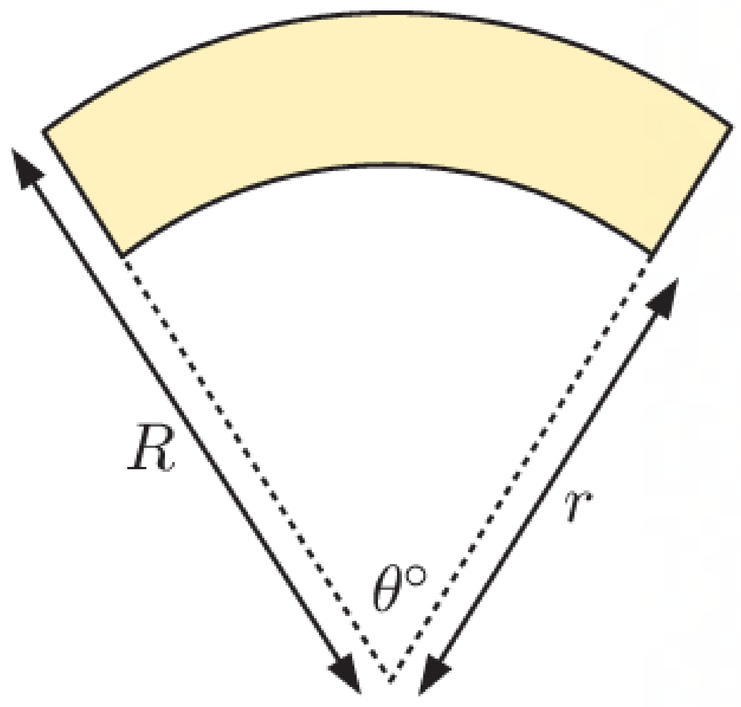
c Area of shaded crescent

$= \text{area of semi-circle with centre X} + \text{area of } \triangle\text{ABY} - \text{area of quarter-circle with centre Y}$
$= \frac{1}{2} \times \pi r_\text{X}^2 + \frac{1}{2} \times \text{base} \times \text{height} - \frac{1}{4} \times \pi r_\text{Y}^2$
$= \frac{1}{2} \times \pi \times (8\sqrt{2})^2 + \frac{1}{2} \times 16 \times 16 - \frac{1}{4} \times \pi \times 16^2$
$= 128 \text{ mm}^2$

9

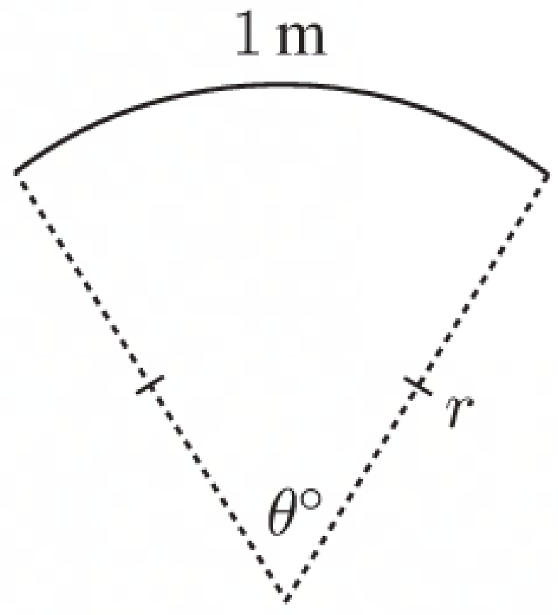


a Area of lampshade



$$\begin{aligned}
&= \text{area of sector with radius } R - \text{area of sector with radius } r \\
&= \frac{\theta}{360} \times \pi R^2 - \frac{\theta}{360} \times \pi r^2 \\
&= \frac{\theta}{360}\, \pi\left(R^2 - r^2\right) \\
&= \frac{\theta}{360}\, \pi\left((r+0.28)^2 - r^2\right) \\
&= \frac{\theta}{360}\, \pi\left(r^2 + 0.56r + 0.0784 - r^2\right) \\
&= \frac{\theta}{360}\, \pi(0.56r + 0.0784) \\
&= \frac{0.28\theta}{360}\, \pi(2r + 0.28) \text{ m}^2
\end{aligned}$$

b Arc length $= \dfrac{\theta}{360} \times 2\pi r$



$$\therefore \quad 1 = \frac{\theta}{360} \times 2\pi r$$

$$\therefore \quad \theta = \frac{360}{2\pi r}$$

$$\therefore \quad \theta = \frac{180}{\pi r}$$

c Area of lampshade $= \dfrac{0.28\theta}{360}\, \pi(2r + 0.28)$ {from **a**}

$$\therefore \quad 0.32 = \frac{0.28}{360} \times \frac{180}{\pi r} \times \pi(2r + 0.28) \qquad \{\text{using } \textbf{b}\}$$

$$= \frac{0.28}{2r}(2r + 0.28)$$

$$= 0.28 + \frac{(0.28)^2}{2r}$$

$$\therefore \quad 0.04 = \frac{(0.28)^2}{2r}$$

$$\therefore \quad 2r = \frac{(0.28)^2}{0.04}$$

$$\therefore \quad r = 0.98 \text{ m}$$

Now $\quad \theta = \dfrac{180}{\pi r}$

$$\therefore \quad \theta = \frac{180}{\pi \times 0.98}$$

$$\therefore \quad \theta \approx 58.5$$

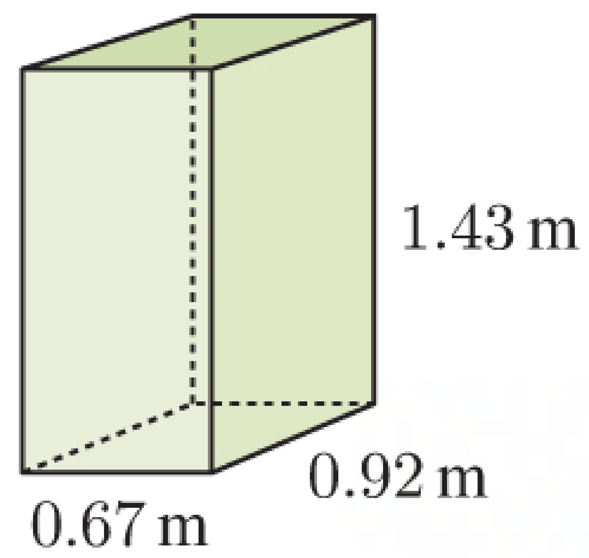
d Arc length $= \dfrac{\theta}{360} \times 2\pi R$

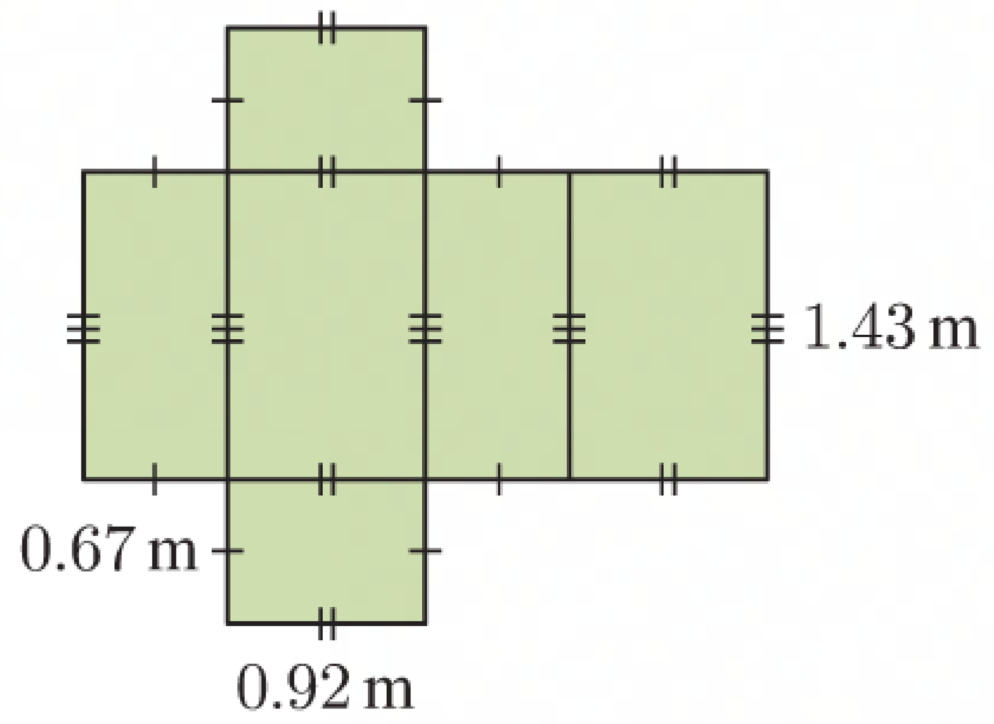
$\approx \dfrac{58.5}{360} \times 2\pi \times (0.98 + 0.28)$ m

≈ 1.29 m

EXERCISE 6B.1

1 **a**



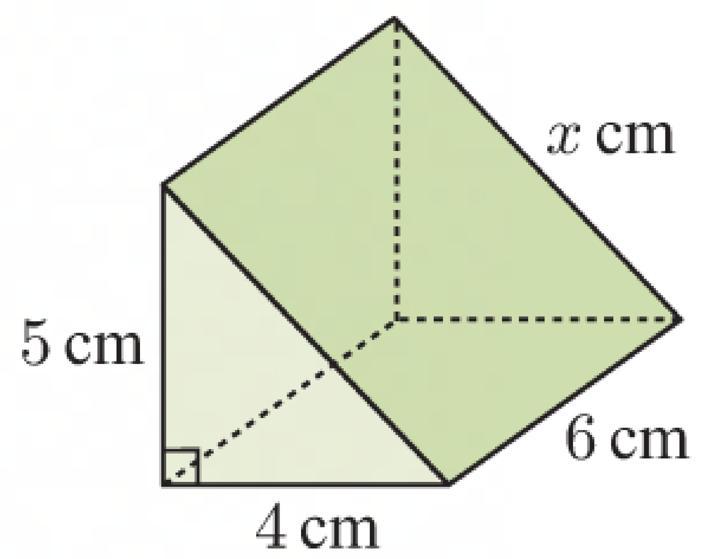


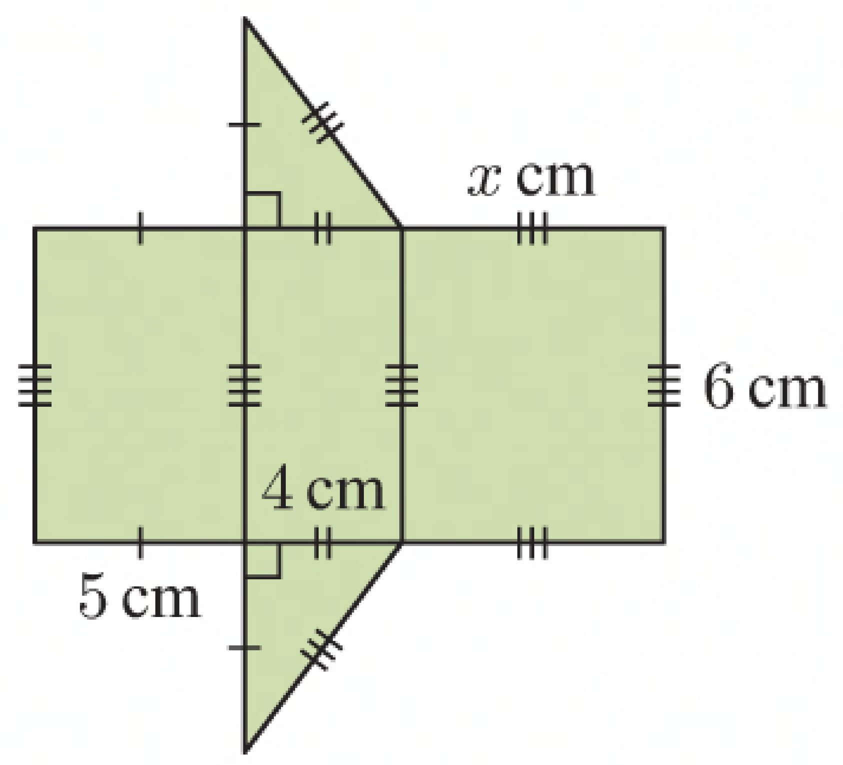
The net of the rectangular prism includes six rectangles: two with length 1.43 m and width 0.92 m, two with length 1.43 m and width 0.67 m, and two with length 0.92 m and width 0.67 m.

$\therefore$ the surface area $= 2 \times (1.43 \times 0.92) + 2 \times (1.43 \times 0.67) + 2 \times (0.92 \times 0.67)$ m^2

$= 5.7802$ m^2

b





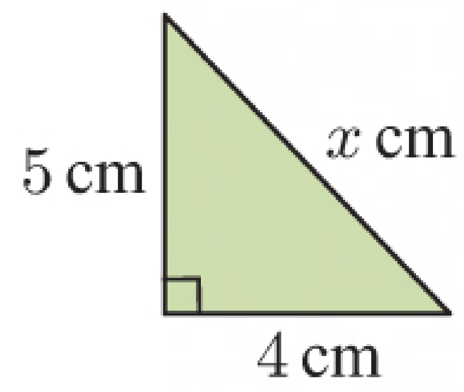
The net of the triangular prism includes two triangles with base 4 cm and height 5 cm, a rectangle with length 6 cm and width 4 cm, a rectangle with length 6 cm and width 5 cm, and a rectangle with side lengths 6 cm and x cm.

Let the hypotenuse of the triangular end be x cm.

$x^2 = 5^2 + 4^2$ {Pythagoras}

$\therefore\ x = \sqrt{5^2 + 4^2}$ {as $x > 0$}

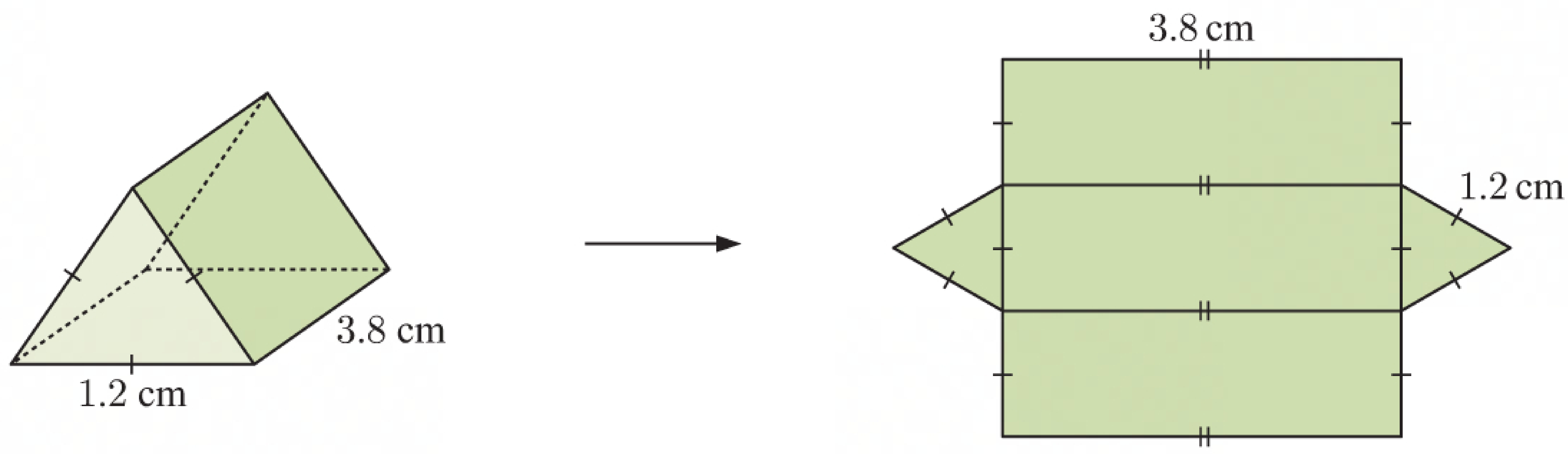
$= \sqrt{41}$



$\therefore$ the surface area $= 2 \times (\frac{1}{2} \times 4 \times 5) + (6 \times 4) + (6 \times 5) + (\sqrt{41} \times 6)$ cm^2

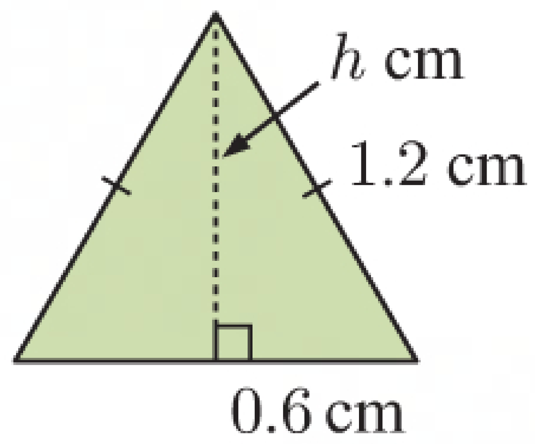
≈ 112 cm^2

c



The net of the triangular prism includes two equilateral triangles with side lengths 1.2 cm, and three rectangles with length 3.8 cm and width 1.2 cm.

Let the height of the triangular end be h cm.



$h^2 + (0.6)^2 = (1.2)^2$ {Pythagoras}

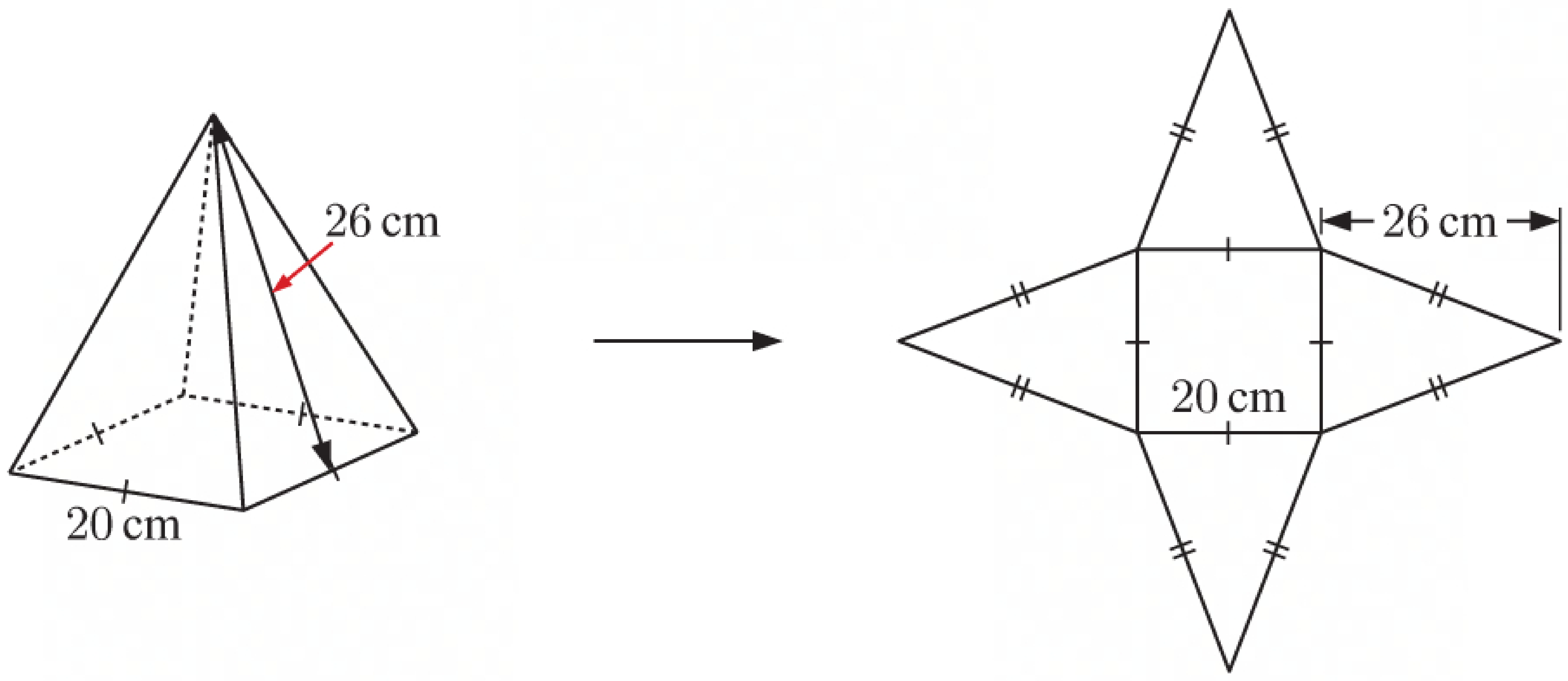
$\therefore \quad h = \sqrt{(1.2)^2 - (0.6)^2}$ {as $h > 0$}

$= \sqrt{1.08}$

$\therefore$ the surface area $= 2 \times (\frac{1}{2} \times 1.2 \times \sqrt{1.08}) + 3 \times (3.8 \times 1.2)$ cm^2

≈ 14.9 cm^2

2 **a**

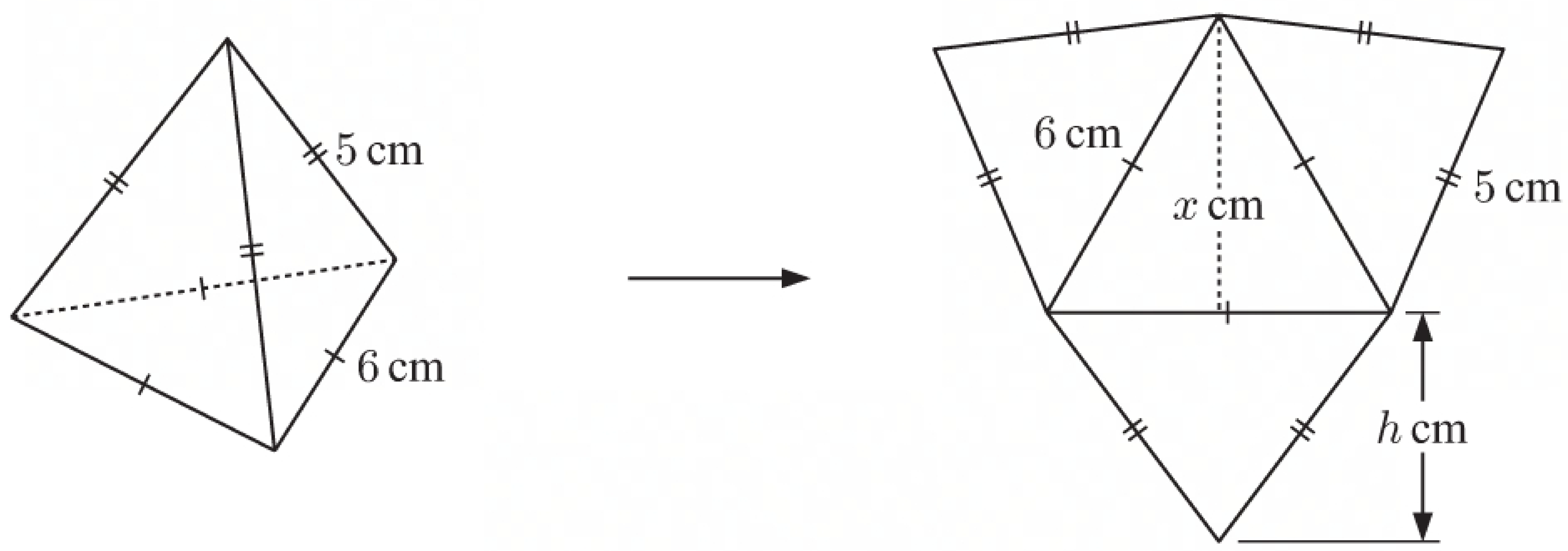


The net of the pyramid includes one square with side length 20 cm, and four isosceles triangles with base 20 cm and height 26 cm.

$\therefore$ the surface area $= 20^2 + 4 \times (\frac{1}{2} \times 20 \times 26)$ cm^2

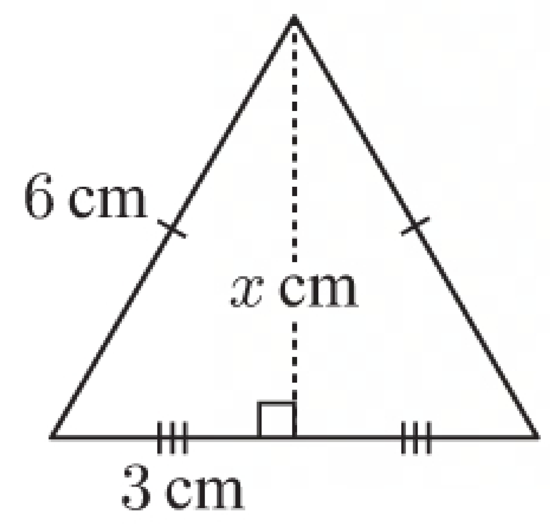
$= 1440$ cm^2

b



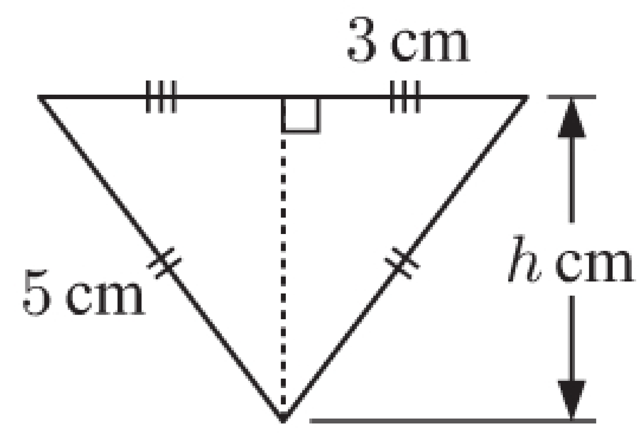
The net of the pyramid includes one equilateral triangle with side length 6 cm, and three isosceles triangles with base 6 cm and slant height 5 cm.

Let the height of the triangular base be x cm.



$$\begin{aligned} x^2+3^2&=6^2 && \{\text{Pythagoras}\}\\ \therefore\ x&=\sqrt{6^2-3^2} && \{\text{as } x>0\}\\ &=\sqrt{27}\\ &=3\sqrt{3} \end{aligned}$$

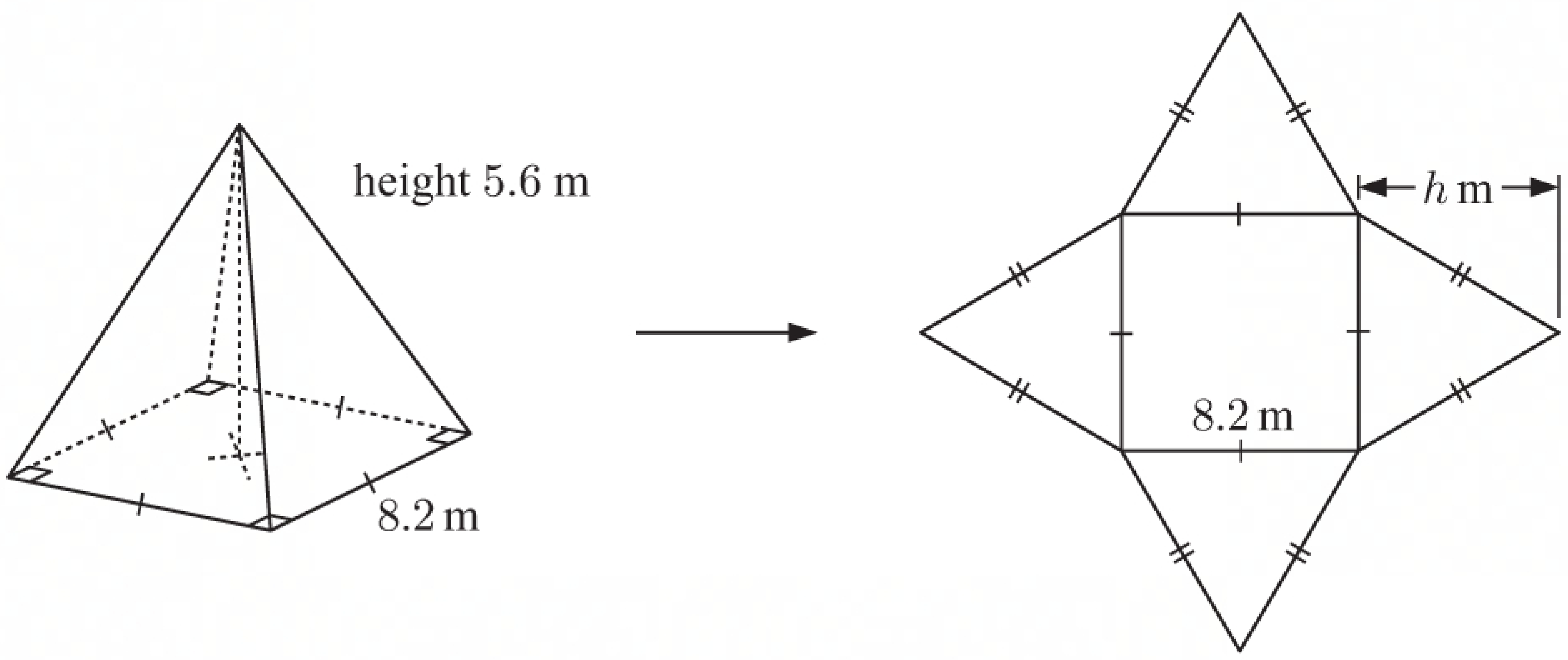
Let the height of the triangular sides be h cm.



$$\begin{aligned} h^2+3^2&=5^2 && \{\text{Pythagoras}\}\\ \therefore\ h&=\sqrt{5^2-3^2} && \{\text{as } h>0\}\\ &=\sqrt{16}\\ &=4 \end{aligned}$$

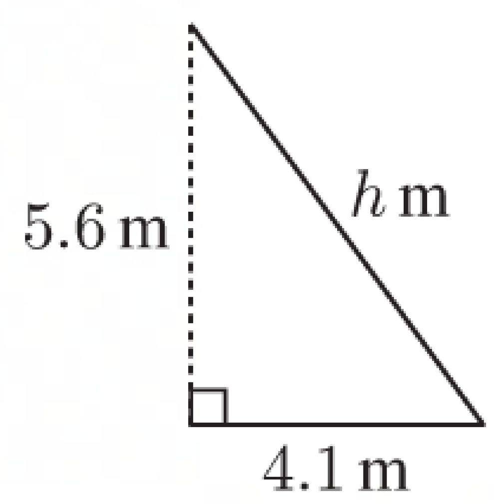
$$\begin{aligned} \therefore\ \text{the surface area} &= \left(\tfrac{1}{2}\times 6\times 3\sqrt{3}\right)+3\times\left(\tfrac{1}{2}\times 6\times 4\right)\ \text{cm}^2\\ &\approx 51.6\ \text{cm}^2 \end{aligned}$$

c



The net of the pyramid includes one square with side length 8.2 m, and four isosceles triangles with base 8.2 m.

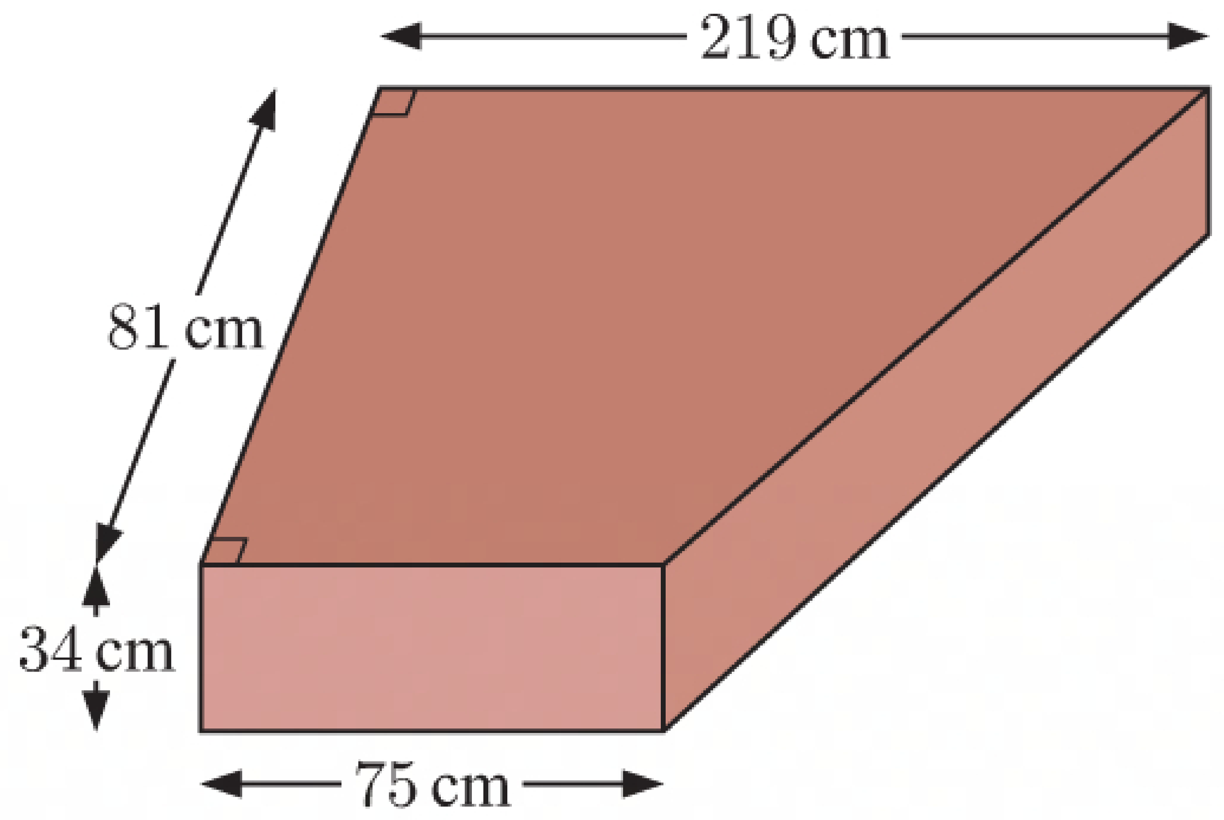
Let the height of the triangles be h m.



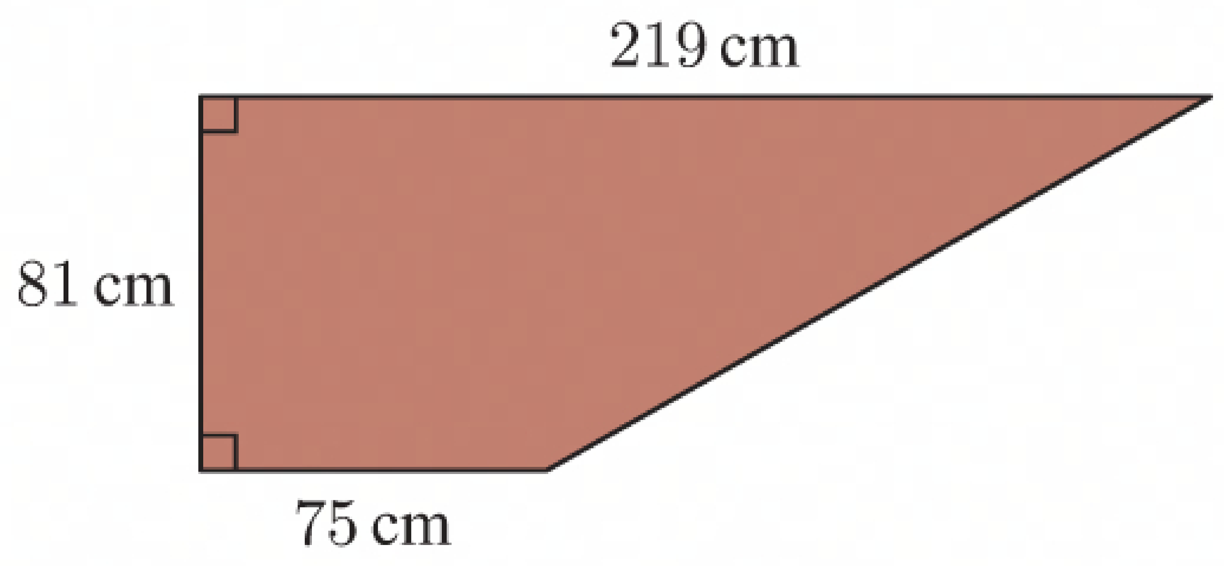
$$\begin{aligned} h^2&=(5.6)^2+(4.1)^2 && \{\text{Pythagoras}\}\\ \therefore\ h&=\sqrt{(5.6)^2+(4.1)^2} && \{\text{as } h>0\}\\ &=\sqrt{48.17} \end{aligned}$$

$$\begin{aligned} \text{Surface area} &= (8.2)^2+4\times\left(\tfrac{1}{2}\times 8.2\times\sqrt{48.17}\right)\ \text{m}^2\\ &\approx 181\ \text{m}^2 \end{aligned}$$

3



a



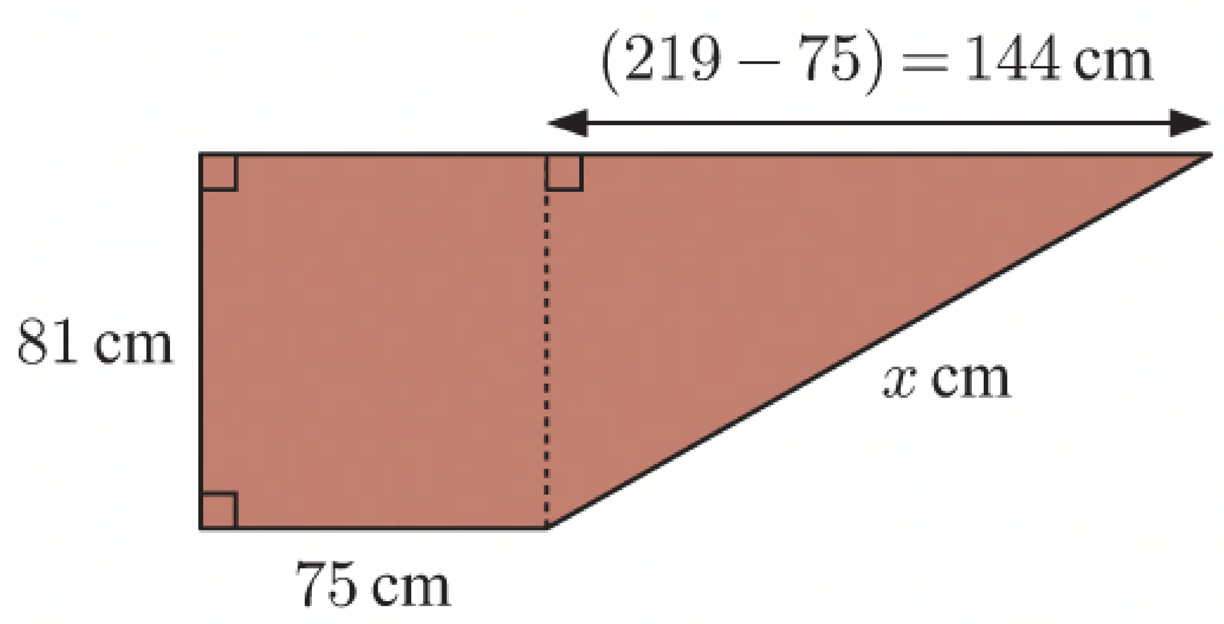
The area of the top surface is the same as the bottom surface. This area is a trapezium, so

total area of top and bottom surfaces

$= 2 \times \left(\frac{75 + 219}{2}\right) \times 81 \text{ cm}^2$

$= 23\,814 \text{ cm}^2$

b

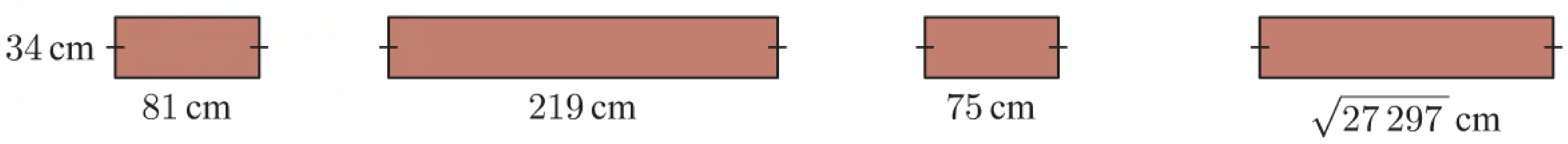


$x^2 = 81^2 + 144^2$ {Pythagoras}

$\therefore\ x^2 = 27\,297$

$\therefore\ x = \sqrt{27\,297}$ {as $x > 0$}

So, the four sides are:



area	area	area	area
$= 81 \times 34 \text{ cm}^2$	$= 219 \times 34 \text{ cm}^2$	$= 75 \times 34 \text{ cm}^2$	$= \sqrt{27\,297} \times 34 \text{ cm}^2$
$= 2754 \text{ cm}^2$	$= 7446 \text{ cm}^2$	$= 2550 \text{ cm}^2$	$\approx 5617 \text{ cm}^2$

c The surface area = area of top and bottom surfaces + area of four sides

$\approx 23\,814 + 2550 + 2754 + 7446 + 5617 \text{ cm}^2$ {using **a** and **b**}

$\approx 42\,181 \text{ cm}^2$

$\approx 42\,181 \div 10\,000 \text{ m}^2$

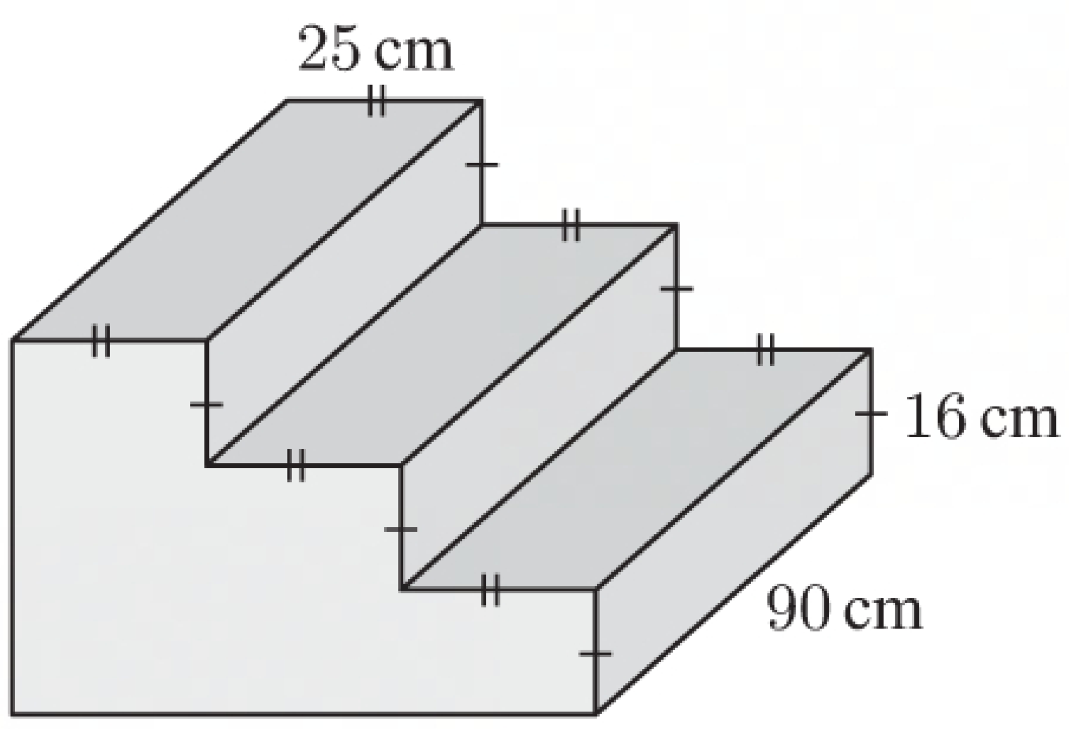
$\approx 4.2181 \text{ m}^2$

$\therefore$ timber cost $\approx 4.2181 \text{ m}^2 \times €128/\text{m}^2$

$\approx €539.92$

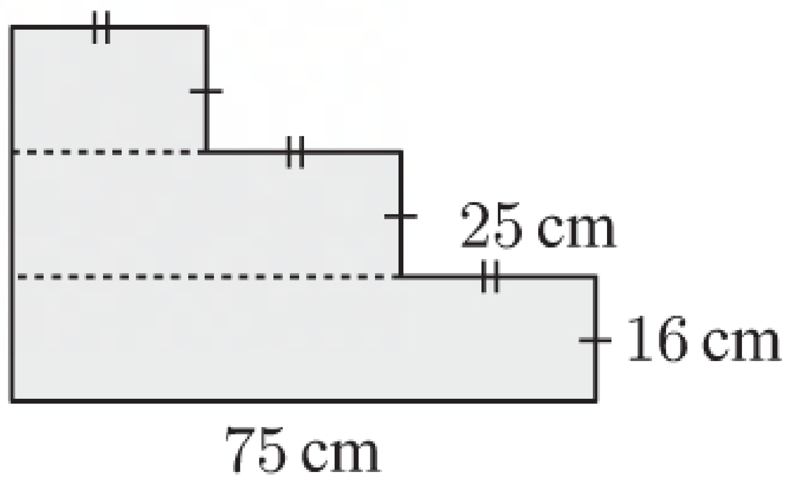
$\approx €540$

4 **a**



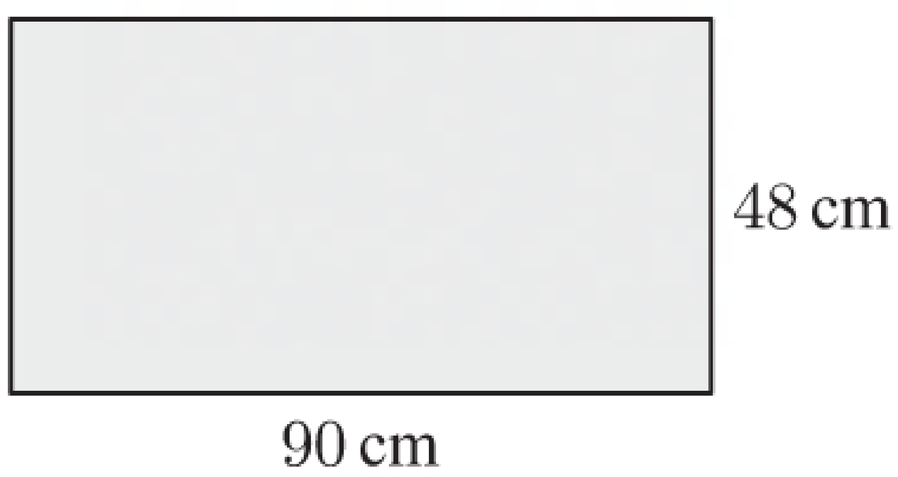
The surfaces of the steps are:

side



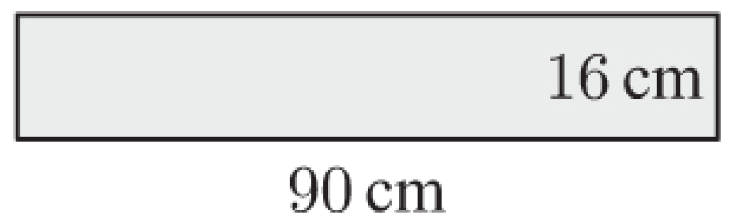
area
$= 25 \times 16 + 50 \times 16$
$+ 75 \times 16 \text{ cm}^2$
$= 2400 \text{ cm}^2$

back



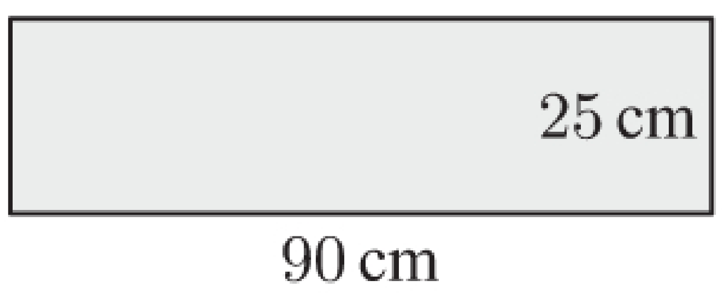
area
$= 90 \times 48 \text{ cm}^2$
$= 4320 \text{ cm}^2$

front of step



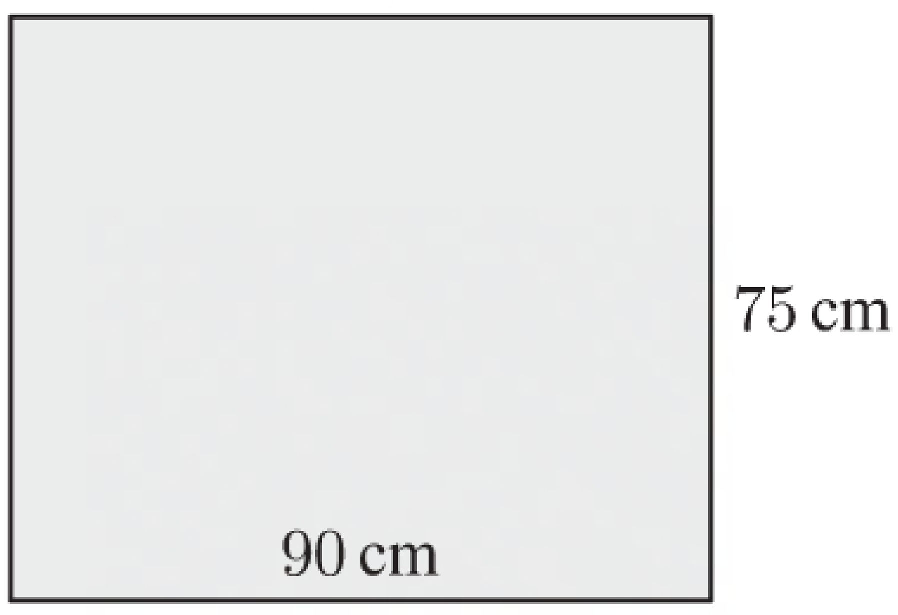
area
$= 90 \times 16 \text{ cm}^2$
$= 1440 \text{ cm}^2$

top of step



area
$= 90 \times 25 \text{ cm}^2$
$= 2250 \text{ cm}^2$

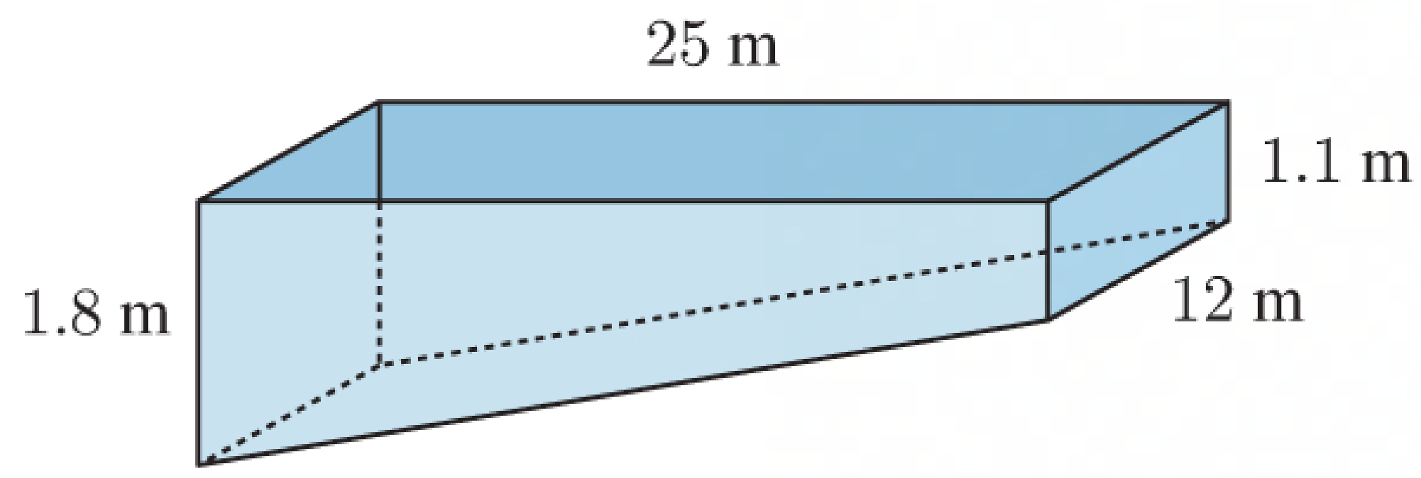
base



area
$= 90 \times 75 \text{ cm}^2$
$= 6750 \text{ cm}^2$

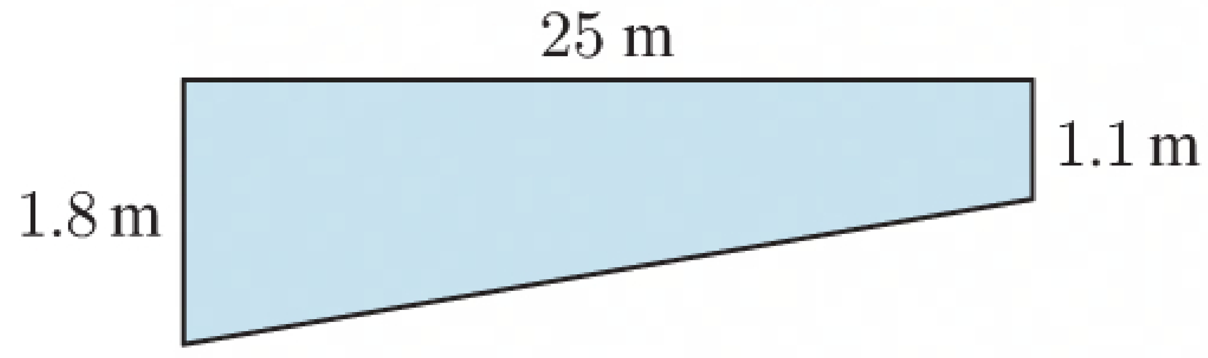
$\therefore$ the surface area = area of two sides + area of back + area of three front of steps
+ area of three top of steps + area of base
$= 2 \times 2400 + 4320 + 3 \times 1440 + 3 \times 2250 + 6750 \text{ cm}^2$
$= 26\,940 \text{ cm}^2$

b



The sides of the pool are:

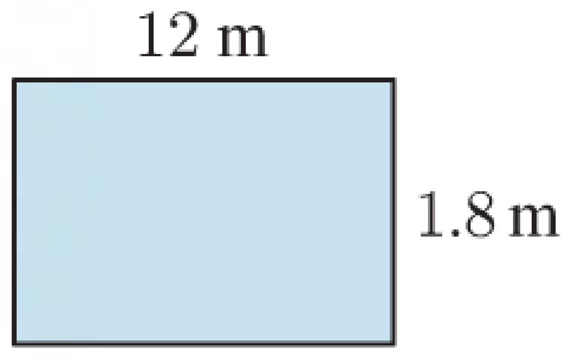
side



area

$$= \left(\frac{1.8 + 1.1}{2}\right) \times 25 \text{ m}^2$$
$$= 36.25 \text{ m}^2$$

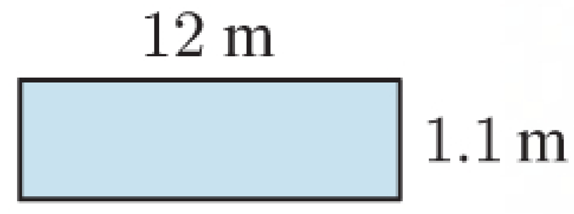
deep end side



area

$$= 12 \times 1.8 \text{ m}^2$$
$$= 21.6 \text{ m}^2$$

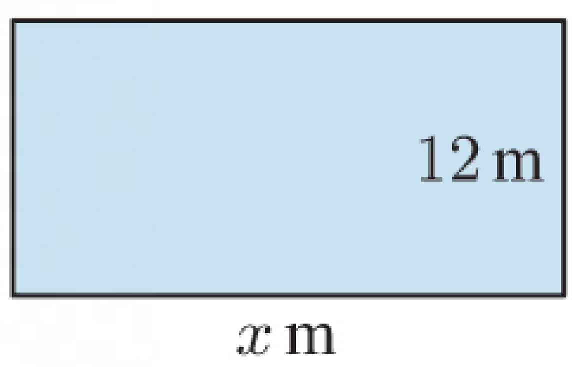
shallow end side

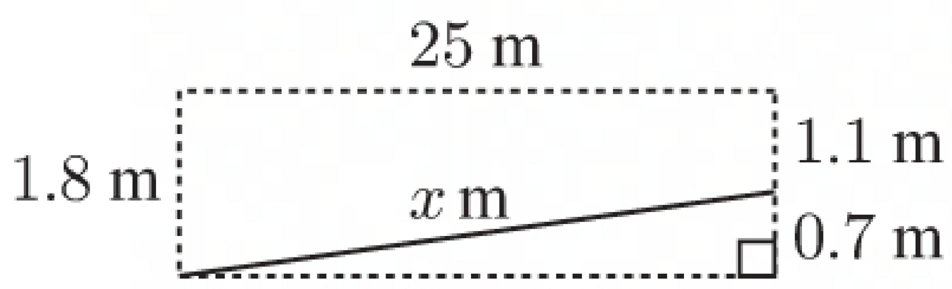


area

$$= 12 \times 1.1 \text{ m}^2$$
$$= 13.2 \text{ m}^2$$

The base of the pool is:





$x^2 = 25^2 + (0.7)^2$ {Pythagoras}

$\therefore\ x = \sqrt{25^2 + (0.7)^2}$ {as $x > 0$}

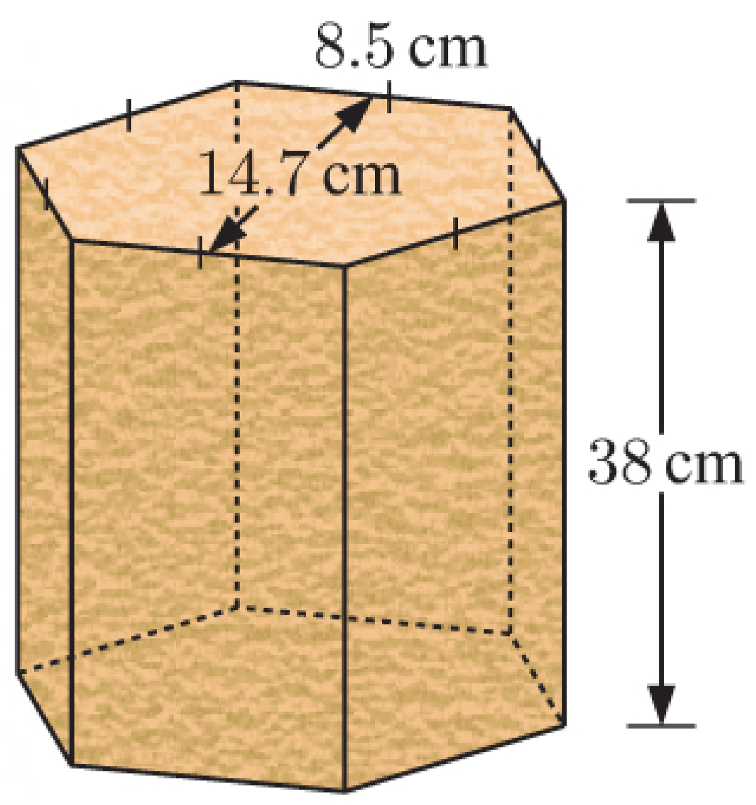
$= \sqrt{625.49}$

$$\text{area} = \sqrt{625.49} \times 12 \text{ m}^2$$
$$\approx 300 \text{ m}^2$$

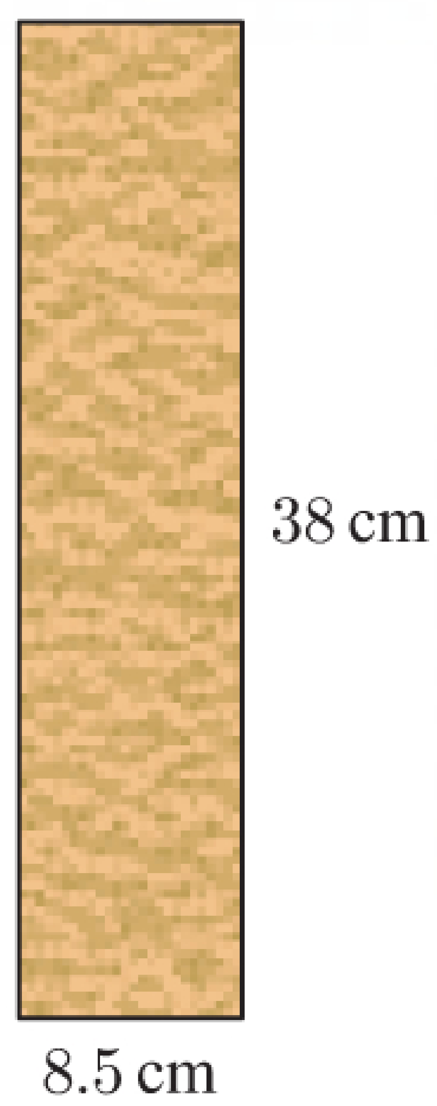
$\therefore$ the surface area = area of two sides + area of deep end side + area of shallow end side + area of base

$$\approx 2 \times 36.25 + 21.6 + 13.2 + 300 \text{ m}^2$$
$$\approx 407 \text{ m}^2$$

5

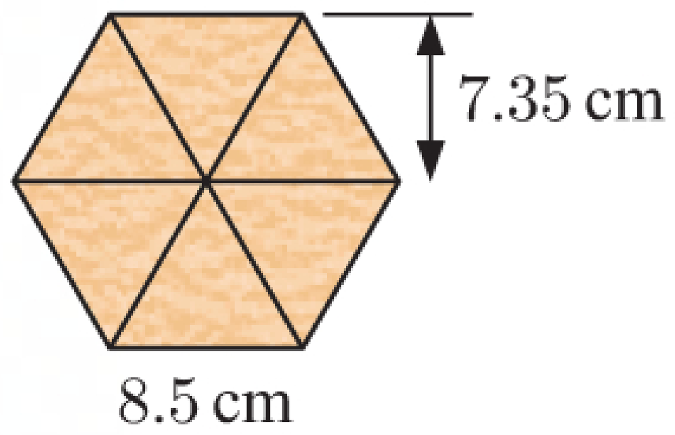


The sides are:



$$\text{area} = 8.5 \times 38 \text{ cm}^2$$
$$= 323 \text{ cm}^2$$

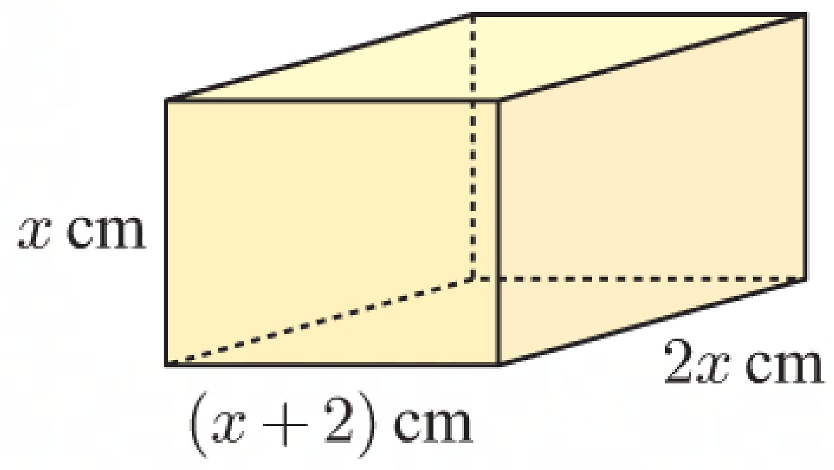
The ends are:



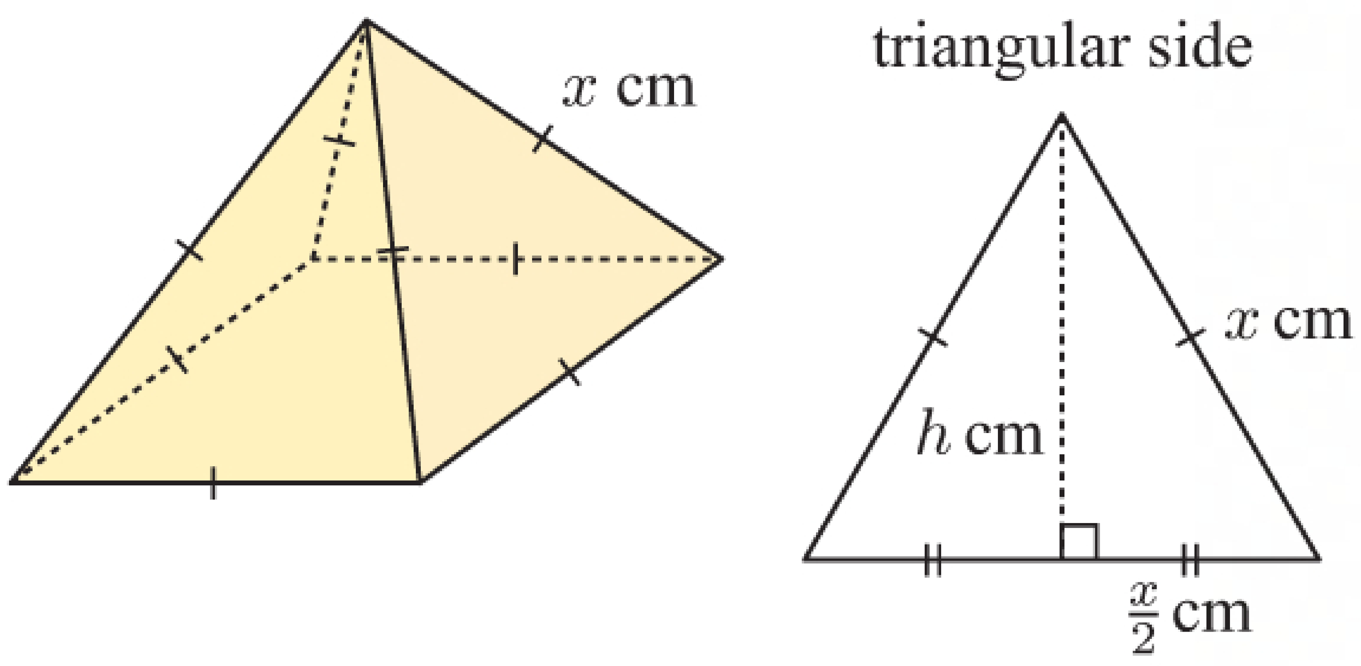
$$\begin{aligned}\text{area of triangle} &= \tfrac{1}{2} \times 8.5 \times 7.35 \text{ cm}^2\\ &= 31.2375 \text{ cm}^2\\ \therefore\ \text{area of end} &= 6 \times 31.2375 \text{ cm}^2\\ &= 187.425 \text{ cm}^2\\ \therefore\ \text{total surface area} &= 6 \times \text{sides} + 2 \times \text{ends}\\ &= 6 \times 323 + 2 \times 187.425 \text{ cm}^2\\ &= 2312.85 \text{ cm}^2\\ &\approx 2310 \text{ cm}^2\end{aligned}$$

6 **a** Surface area of prism

$$\begin{aligned}&= 2 \times x(x+2) + 2 \times x(2x) + 2 \times 2x(x+2) \text{ cm}^2\\ &= 2x(x+2) + 2x(2x) + 4x(x+2) \text{ cm}^2\\ &= 2x^2 + 4x + 4x^2 + 4x^2 + 8x \text{ cm}^2\\ &= (10x^2 + 12x) \text{ cm}^2\end{aligned}$$



b



Let the height of the triangular sides be h cm.

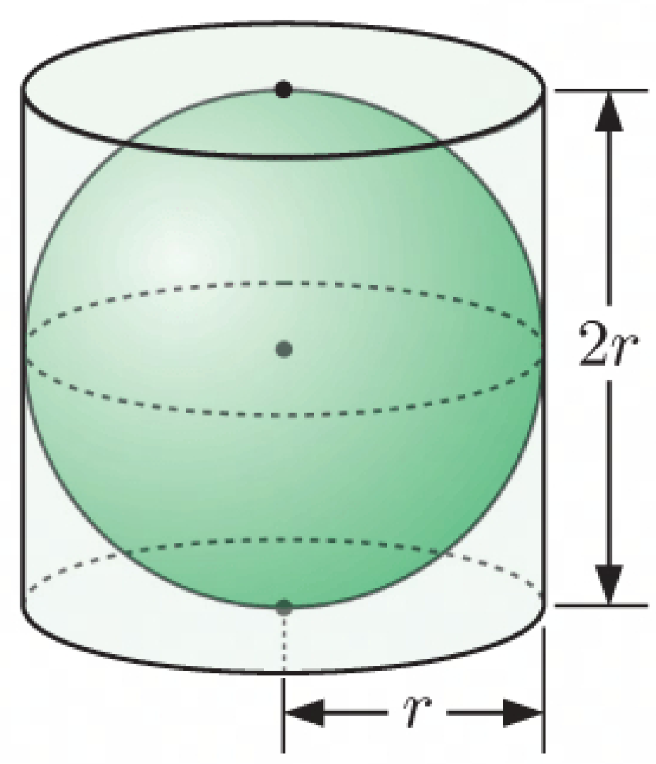
$$h^2 + \left(\frac{x}{2}\right)^2 = x^2 \qquad \{\text{Pythagoras}\}$$

$$\begin{aligned}\therefore\ h &= \sqrt{x^2 - \frac{x^2}{4}} \qquad \{\text{as } h > 0\}\\ &= \sqrt{\frac{3x^2}{4}}\\ &= \frac{\sqrt{3}x}{2}\end{aligned}$$

$$\begin{aligned}\text{Surface area of pyramid} &= \text{area of base} + \text{area of four triangular sides}\\ &= x \times x + 4 \times \left(\frac{1}{2} \times x \times \frac{\sqrt{3}x}{2}\right) \text{ cm}^2\\ &= x^2 + \frac{2x(\sqrt{3}x)}{2} \text{ cm}^2\\ &= x^2 + \sqrt{3}x^2 \text{ cm}^2\\ &= (1+\sqrt{3})x^2 \text{ cm}^2\end{aligned}$$

INVESTIGATION 1 ARCHIMEDES AND THE SPHERE

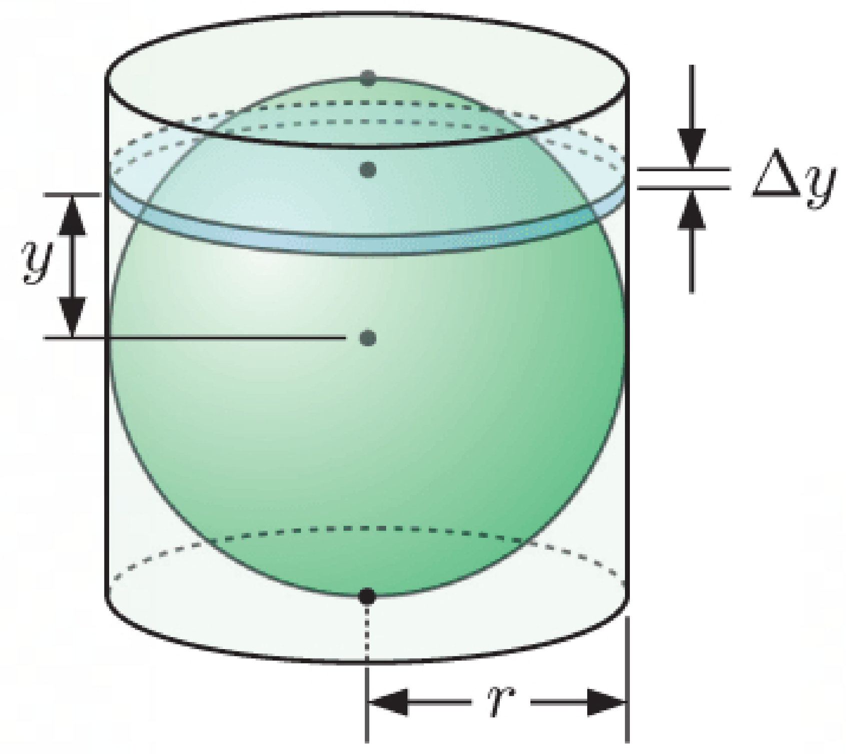
1
$$\begin{aligned}\text{Curved surface area of the cylinder} &= 2 \times \pi \times \text{radius} \times \text{height}\\ &= 2 \times \pi \times r \times 2r\\ &= 4\pi r^2\end{aligned}$$



2 **a** Curved surface area of the slice $= 2 \times \pi \times \text{radius} \times \text{height}$

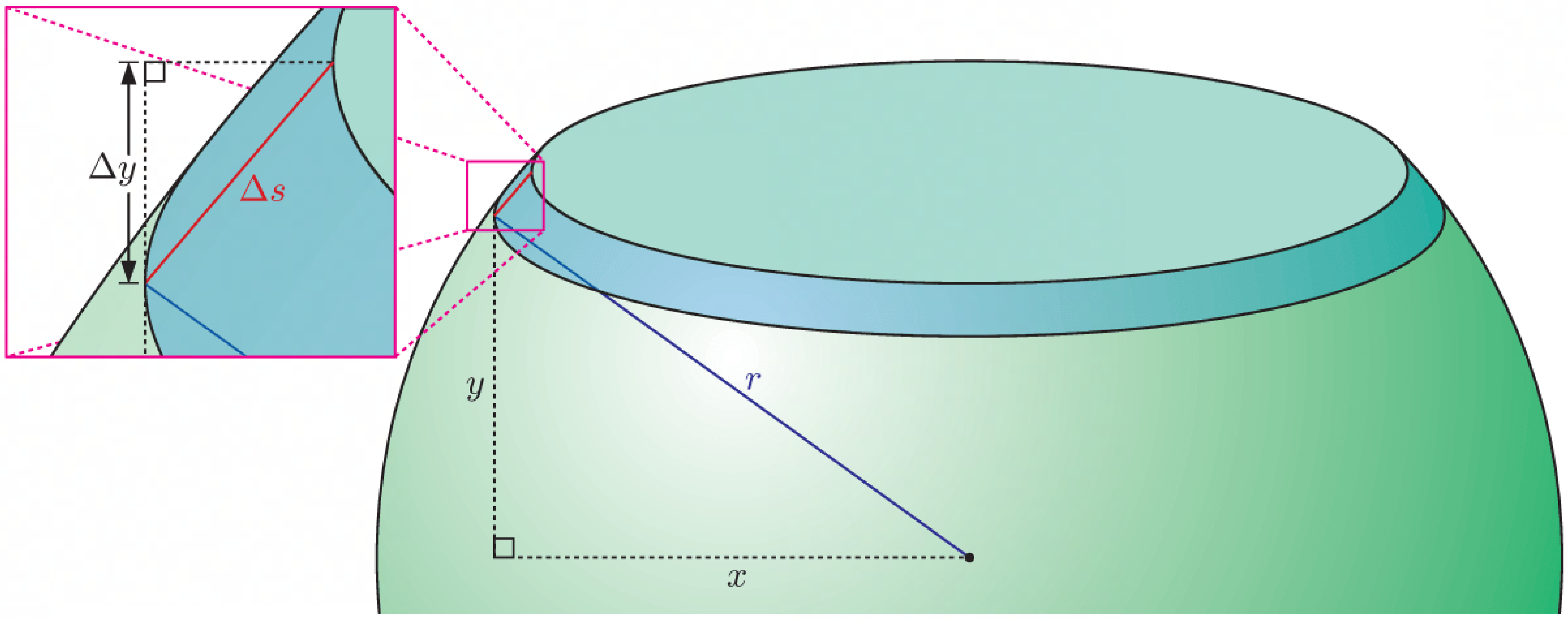
$= 2 \times \pi \times r \times \Delta y$

$= 2\pi r \Delta y$

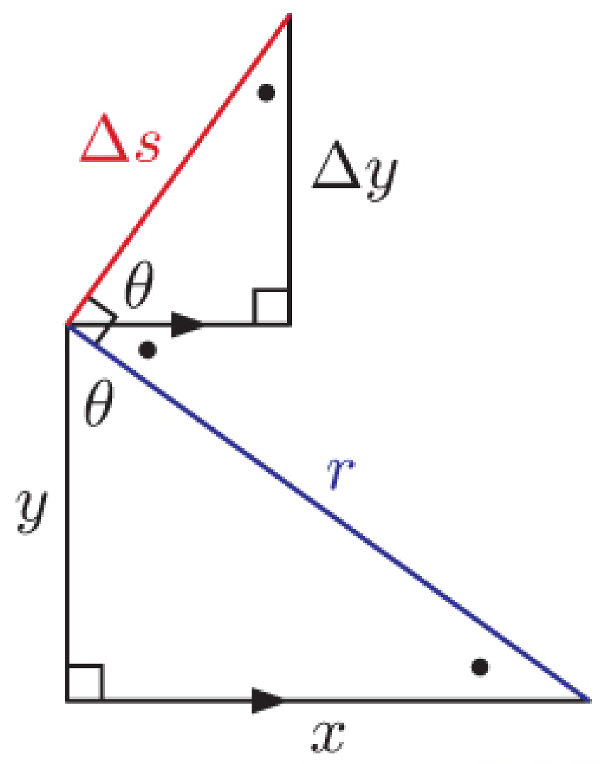


b In the diagram below, $x^2 + y^2 = r^2$ {Pythagoras}

$\therefore\ x^2 = r^2 - y^2$



c



Using similar triangles, $\dfrac{x}{\Delta y} = \dfrac{r}{\Delta s}$

$\therefore\ x\Delta s = r\Delta y$

d The curved surface area of the slice is a rectangle with length $2\pi x$ and width Δs.

$\therefore$ the surface area of the sphere from this slice $= 2\pi x \times \Delta s$

$= 2\pi r \Delta y$ {using **c**}

3 Each thin slice of thickness Δy contributes $2\pi r\Delta y$ to the surface area of the sphere *and* the surface area of the curved surface of the cylinder which just contains it. So, the surface areas are therefore equal as we sum these contributions over the height $2r$.

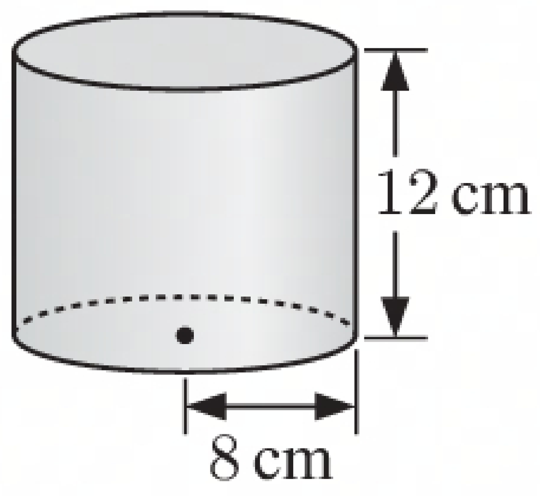
$\therefore$ the surface area of the sphere = the curved surface area of the cylinder

$= 2\pi r \times 2r$

$= 4\pi r^2$

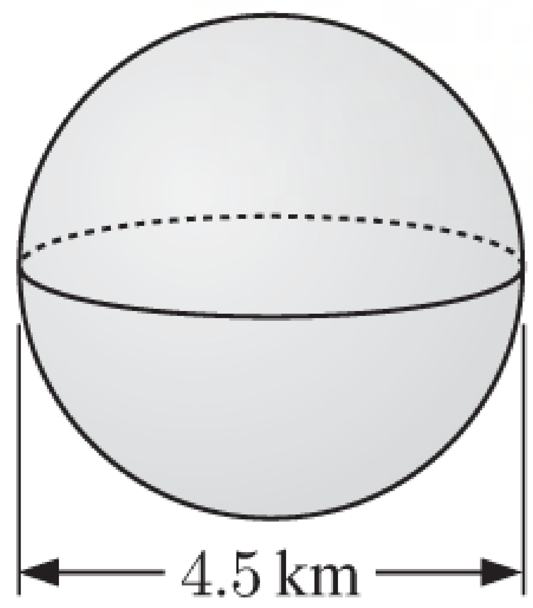
EXERCISE 6B.2

1 **a**



$A = 2\pi rh + 2\pi r^2$
$= 2 \times \pi \times 8 \times 12 + 2 \times \pi \times 8^2$
$\approx 1005.3\ \text{cm}^2$

b

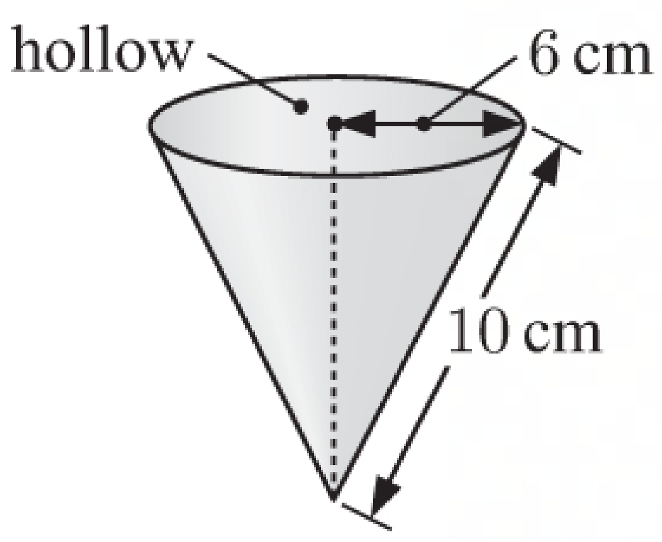


The diameter $d = 4.5$ km,

so the radius $r = \frac{4.5}{2} = 2.25$ km.

$A = 4\pi r^2$
$= 4 \times \pi \times (2.25)^2$
$\approx 63.6\ \text{km}^2$

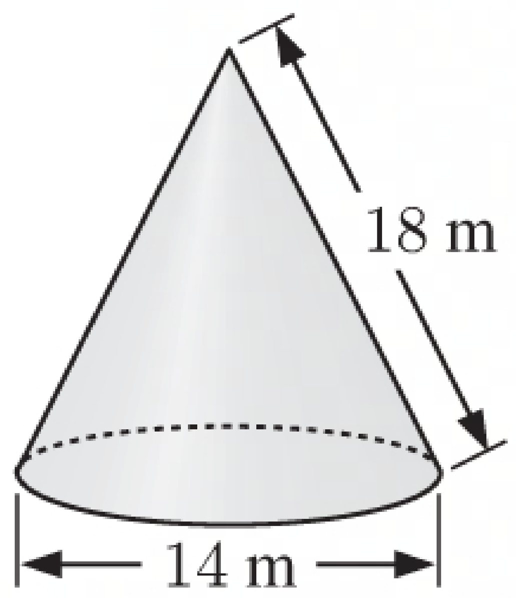
c



The cone is hollow at the top, so we only have the curved surface.

$A = \pi rs$
$= \pi \times 6 \times 10$
$\approx 188.5\ \text{cm}^2$

d

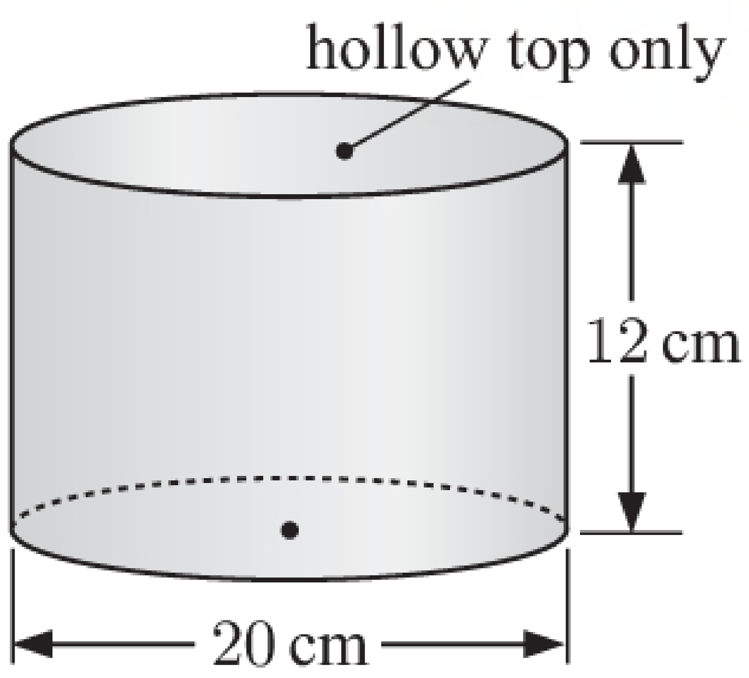


The diameter $d = 14$ m,

so the radius $r = \frac{14}{2} = 7$ m.

$A = \pi rs + \pi r^2$
$= \pi \times 7 \times 18 + \pi \times 7^2$
$\approx 549.8\ \text{m}^2$

e



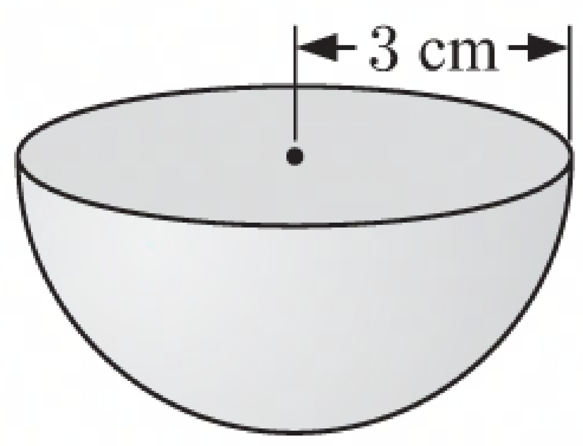
The diameter $d = 20$ cm,

so the radius $r = \frac{20}{2} = 10$ cm.

The cylinder is hollow at the top, so we only have the curved surface and one circular end.

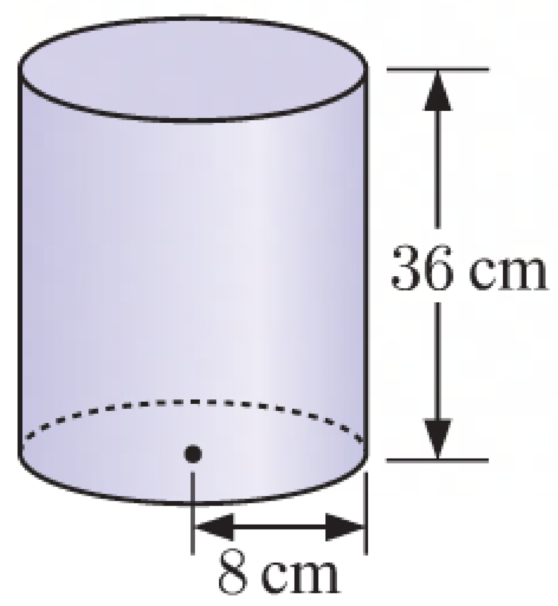
$A = 2\pi rh + \pi r^2$
$= 2 \times \pi \times 10 \times 12 + \pi \times 10^2$
$\approx 1068.1\ \text{cm}^2$

f



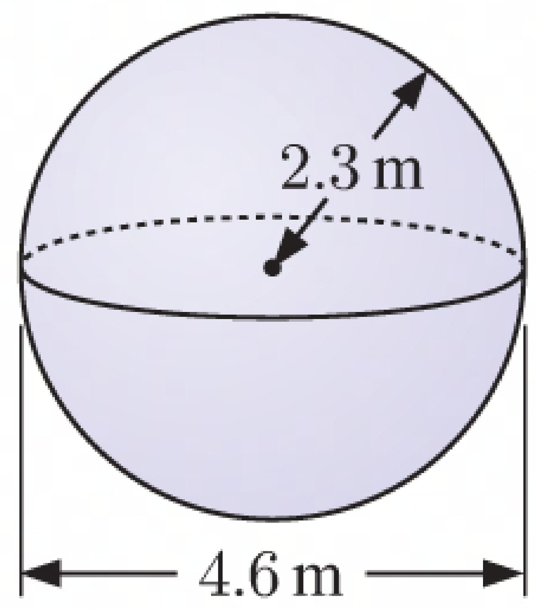
$A =$ area of curved surface
$+$ area of flat end
$= \frac{1}{2} \times 4\pi r^2 + \pi r^2$
$= 2 \times \pi \times 3^2 + \pi \times 3^2$
$\approx 84.8\ \text{cm}^2$

2 **a**



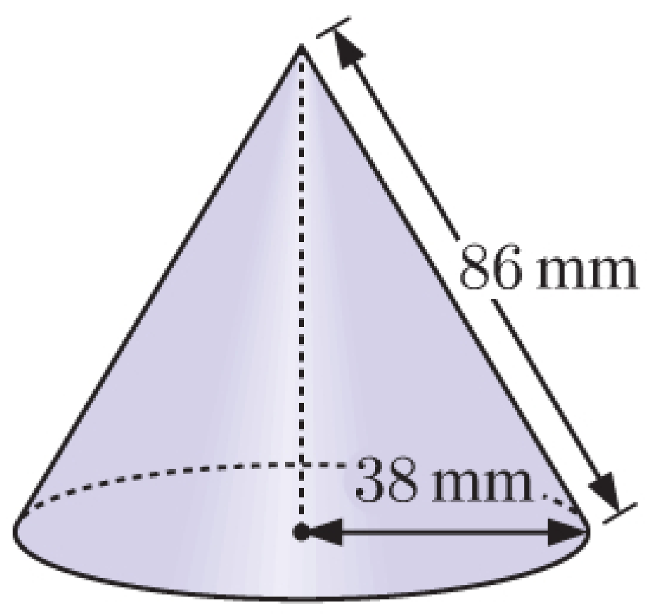
$$\begin{aligned} A &= 2\pi r h + 2\pi r^2 \\ &= 2 \times \pi \times 8 \times 36 + 2 \times \pi \times 8^2 \\ &\approx 2210 \text{ cm}^2 \end{aligned}$$

b



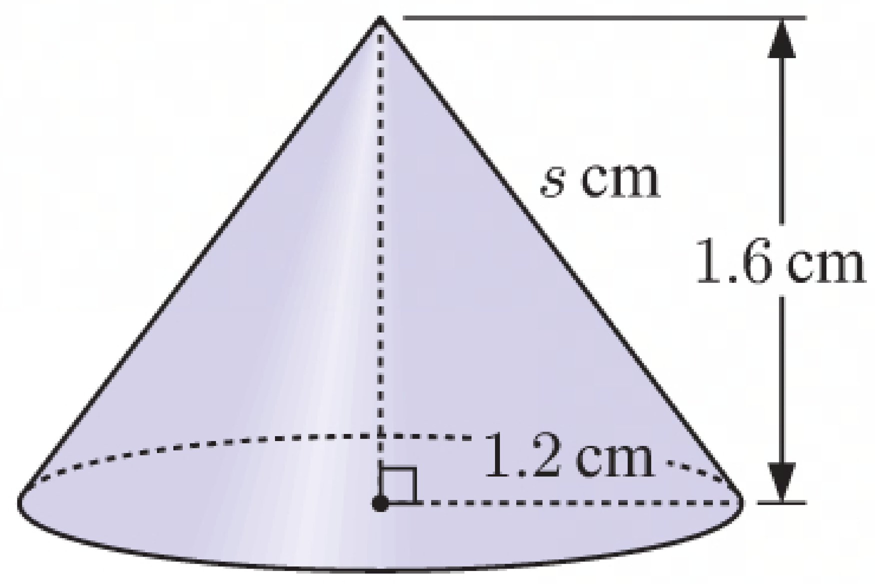
$$\begin{aligned} A &= 4\pi r^2 \\ &= 4 \times \pi \times (2.3)^2 \\ &\approx 66.5 \text{ m}^2 \end{aligned}$$

c



$$\begin{aligned} A &= \pi r s + \pi r^2 \\ &= \pi \times 38 \times 86 + \pi \times 38^2 \\ &\approx 14\,800 \text{ mm}^2 \end{aligned}$$

d



Let the slant height of the cone be s cm.

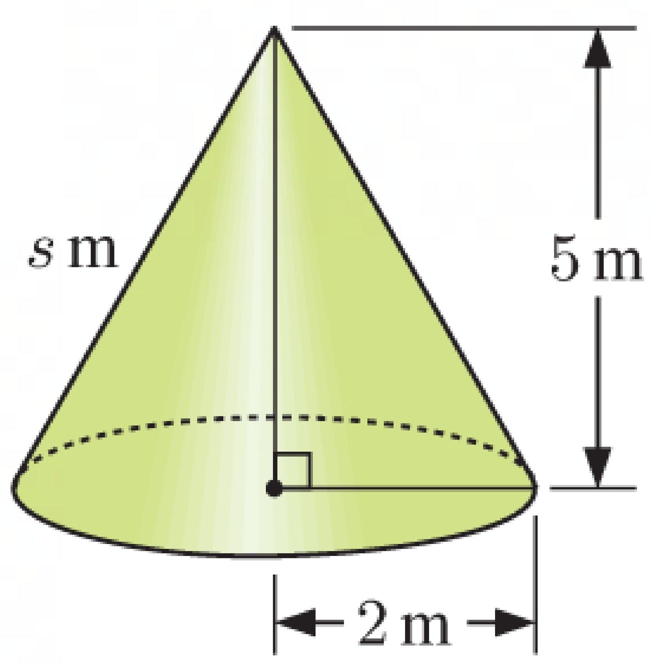
Now $s^2 = (1.6)^2 + (1.2)^2$ {Pythagoras}

$$\begin{aligned} \therefore\ s &= \sqrt{(1.6)^2 + (1.2)^2} \quad \{\text{as } s > 0\} \\ &= \sqrt{4} \\ &= 2 \end{aligned}$$

$$\begin{aligned} A &= \pi r s + \pi r^2 \\ &= \pi \times 1.2 \times 2 + \pi \times (1.2)^2 \\ &\approx 12.1 \text{ cm}^2 \end{aligned}$$

3 **a**

$$\begin{aligned} s^2 &= 2^2 + 5^2 \quad \{\text{Pythagoras}\} \\ \therefore\ s &= \sqrt{2^2 + 5^2} \quad \{\text{as } s > 0\} \\ &= \sqrt{29} \\ &\approx 5.39 \end{aligned}$$



b Area of canvas $= \pi r s + \pi r^2$

$$\begin{aligned} &= \pi \times 2 \times \sqrt{29} + \pi \times 2^2 \\ &\approx 46.4 \text{ m}^2 \end{aligned}$$

$\therefore$ approximately 46.4 m^2 of canvas is required to make the tent.

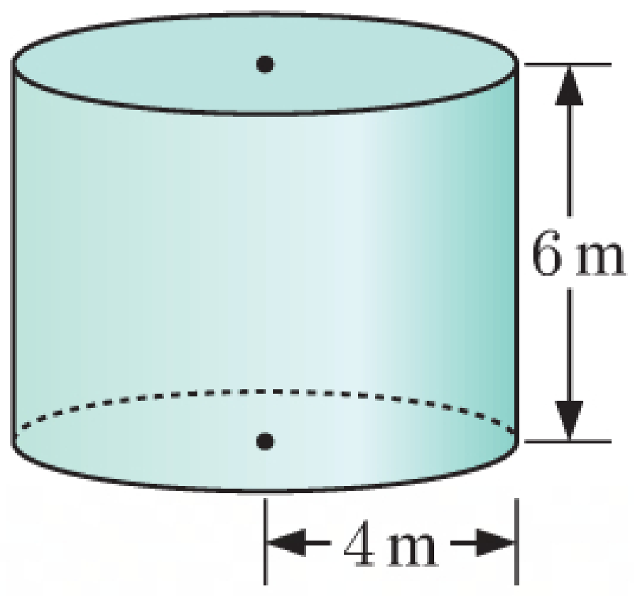
c Cost of canvas = area of canvas $\times$ cost per m^2

$$\begin{aligned} &\approx 46.4 \text{ m}^2 \times \$18/\text{m}^2 \\ &\approx \$835.24 \end{aligned}$$

4 **a** The diameter $d = 8$ m,

so the radius $r = \frac{8}{2} = 4$ m.

$$\begin{aligned}\text{Area of base} &= \pi r^2 \\ &= \pi \times 4^2 \\ &\approx 50.3 \text{ m}^2\end{aligned}$$



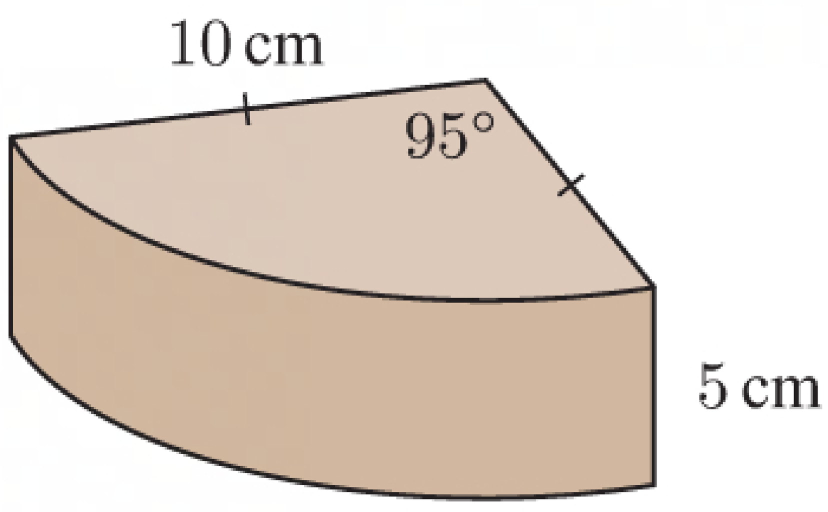
b $$\begin{aligned}\text{Cost of lining the base} &= \text{area of base} \times \text{cost per m}^2 \\ &\approx 50.3 \text{ m}^2 \times \$23.20/\text{m}^2 \\ &\approx \$1166.16\end{aligned}$$

c $$\begin{aligned}\text{Area of curved wall} &= 2\pi rh \\ &= 2 \times \pi \times 4 \times 6 \\ &\approx 150.8 \text{ m}^2\end{aligned}$$

d $$\begin{aligned}\text{Cost of lining the curved wall} &= \text{area of wall} \times \text{cost per m}^2 \\ &\approx 150.8 \text{ m}^2 \times \$18.50/\text{m}^2 \\ &\approx \$2789.73\end{aligned}$$

e $$\begin{aligned}\text{Total cost of the lining} &= \text{cost of lining the base} + \text{cost of lining the curved wall} \\ &\approx \$1166.16 + \$2789.73 \qquad \{\text{using } \mathbf{b} \text{ and } \mathbf{d}\} \\ &\approx \$3960\end{aligned}$$

5 $$\begin{aligned}\text{Area of top} &= \frac{\theta}{360} \times \pi r^2 \\ &= \tfrac{95}{360} \times \pi \times 10^2 \\ &\approx 82.9 \text{ cm}^2\end{aligned}$$



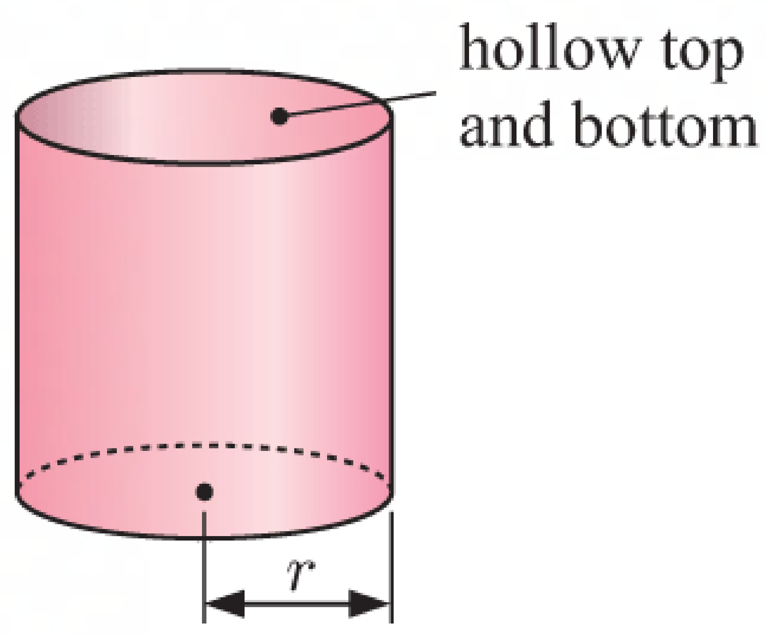
$$\begin{aligned}\text{Area of curved surface} &= \frac{\theta}{360} \times 2\pi rh \\ &= \tfrac{95}{360} \times 2 \times \pi \times 10 \times 5 \\ &\approx 82.9 \text{ cm}^2\end{aligned}$$

$$\begin{aligned}\text{Area of two flat surfaces} &= 2 \times \text{length} \times \text{width} \\ &= 2 \times 10 \times 5 \\ &= 100 \text{ cm}^2\end{aligned}$$

$$\begin{aligned}\text{Total surface area to be iced} &= \text{area of top} + \text{area of curved surface} + \text{area of two flat surfaces} \\ &\approx 82.9 + 82.9 + 100 \text{ cm}^2 \\ &\approx 266 \text{ cm}^2\end{aligned}$$

6 **a** The radius is r, so the diameter is $2r$.
Since the height is the same as the diameter, the height is also $2r$.

$$\begin{aligned}\text{Surface area} &= 2\pi rh \\ &= 2 \times \pi \times r \times 2r \\ &= 4\pi r^2\end{aligned}$$



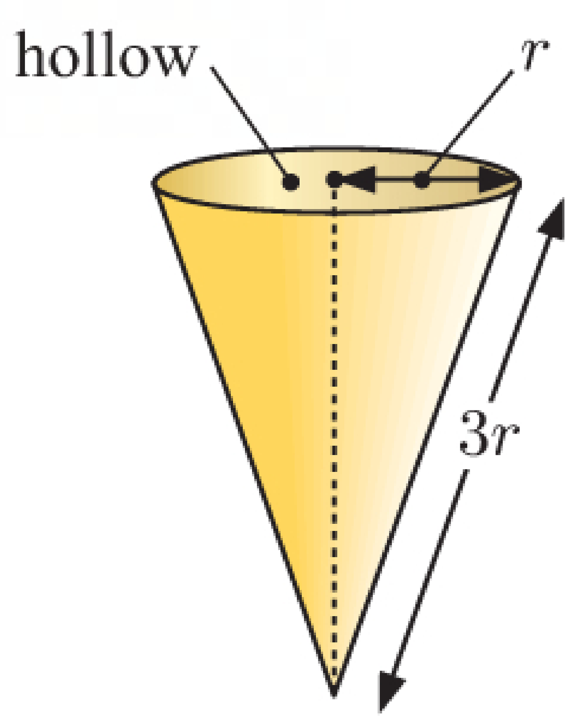
b The surface area is 91.6 m^2

$$\therefore\ 4\pi r^2 = 91.6$$
$$\therefore\ r^2 = \frac{91.6}{4\pi}$$
$$\therefore\ r = \sqrt{\frac{91.6}{4\pi}} \qquad \{\text{as } r > 0\}$$
$$\approx 2.70$$
$$\therefore\ h \approx 2 \times 2.70 \qquad \{h = 2r\}$$
$$\approx 5.40$$

So, the height of the cylinder is approximately 5.40 m.

7 **a** Surface area $= \pi r s$

$$= \pi \times r \times 3r$$
$$= 3\pi r^2$$



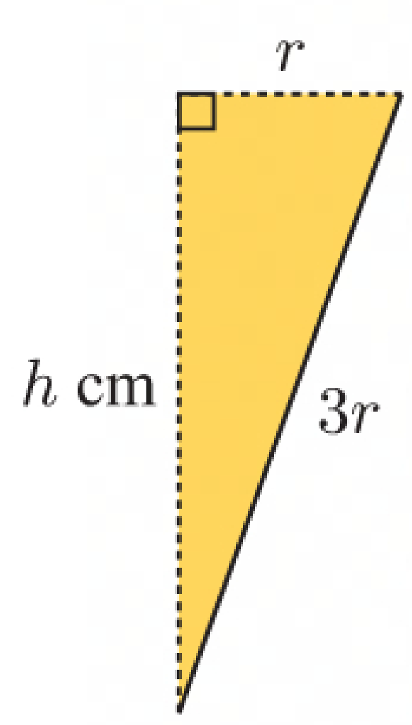
b **i** The surface area is 21.2 cm^2

$$\therefore\ 3\pi r^2 = 21.2$$
$$\therefore\ r^2 = \frac{21.2}{3\pi}$$
$$\therefore\ r = \sqrt{\frac{21.2}{3\pi}} \qquad \{\text{as } r > 0\}$$
$$\therefore\ r \approx 1.50$$
$$\therefore\ 3r \approx 3 \times 1.50$$
$$\therefore\ s \approx 4.50$$

So, the slant height of the cone is approximately 4.50 cm.

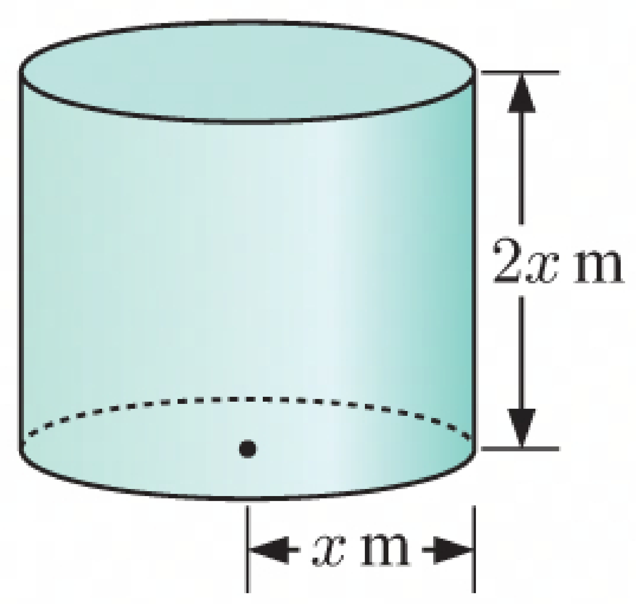
ii Let the height of the cone be h cm.

$$h^2 + r^2 = (3r)^2 \qquad \{\text{Pythagoras}\}$$
$$\therefore\ h^2 + r^2 = 9r^2$$
$$\therefore\ h^2 = 8r^2$$
$$\therefore\ h = \sqrt{8r^2} \qquad \{\text{as } h > 0\}$$
$$\approx \sqrt{8 \times (1.50)^2} \qquad \{\text{from } \mathbf{b\ i}\}$$
$$\approx 4.24$$

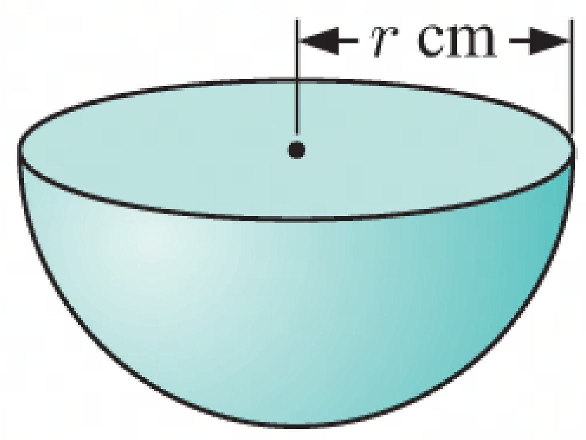


So, the height of the cone is approximately 4.24 cm.

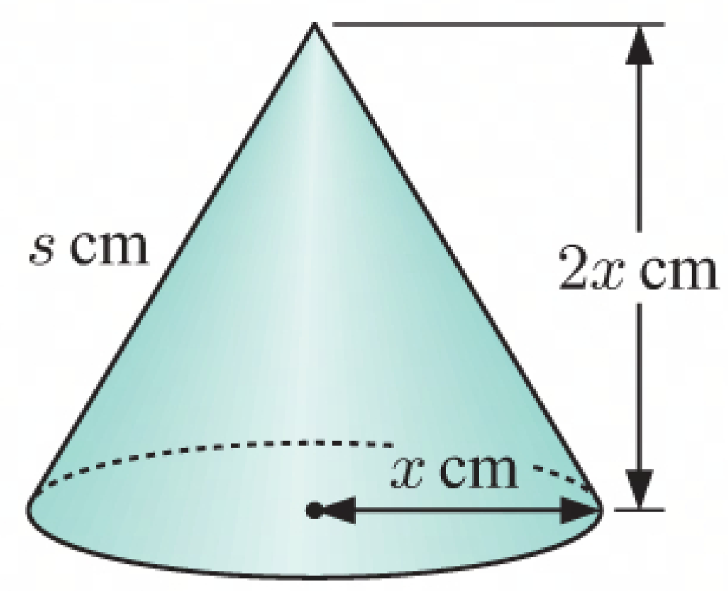
8 **a** Surface area $= 2\pi rh + 2\pi r^2$
$$= 2 \times \pi \times x \times 2x + 2 \times \pi \times x^2$$
$$= 4\pi x^2 + 2\pi x^2$$
$$= 6\pi x^2 \text{ cm}^2$$



b Surface area $= \frac{1}{2} \times 4\pi r^2 + \pi r^2$
$$= 2\pi r^2 + \pi r^2$$
$$= 3\pi r^2 \text{ cm}^2$$



c Let the slant height of the cone be s cm.
$$s^2 = x^2 + (2x)^2 \quad \{\text{Pythagoras}\}$$
$$= 5x^2$$
$$\therefore \quad s = \sqrt{5}x \quad \{\text{as } s > 0\}$$
Surface area $= \pi rs + \pi r^2$
$$= \pi \times x \times \sqrt{5}x + \pi \times x^2$$
$$= \sqrt{5}\pi x^2 + \pi x^2$$
$$= \pi x^2(\sqrt{5} + 1) \text{ cm}^2$$

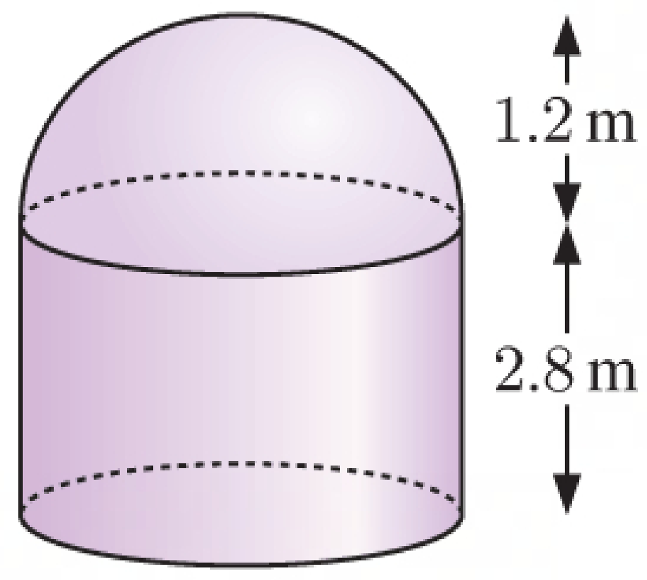


9 **a** Surface area of a sphere $= 4\pi r^2$
The surface area is 64π cm^2
$$\therefore \quad 4\pi r^2 = 64\pi$$
$$\therefore \quad r^2 = 16$$
$$\therefore \quad r = 4 \quad \{\text{as } r > 0\}$$
The radius of the sphere is 4 cm.

b Surface area of a solid cylinder $= 2\pi rh + 2\pi r^2$
The radius is 6.3 cm and the surface area is 1243 cm^2
$$\therefore \quad 2 \times \pi \times 6.3 \times h + 2 \times \pi \times 6.3^2 = 1243$$
$$\therefore \quad 12.6\pi h + 79.38\pi = 1243$$
$$\therefore \quad 12.6\pi h = 1243 - 79.38\pi$$
$$\therefore \quad h = \frac{1243 - 79.38\pi}{12.6\pi}$$
$$\therefore \quad h \approx 25.1$$
The height of the cylinder is approximately 25.1 cm.

c Surface area of a cone $= \pi rs + \pi r^2$
The slant height is 143 mm and the surface area is 60 000 mm^2
$$\therefore \quad \pi \times r \times 143 + \pi \times r^2 = 60\,000$$
$$\therefore \quad 143\pi r + \pi r^2 = 60\,000$$
$$\therefore \quad \pi r^2 + 143\pi r - 60\,000 = 0$$
Using technology, $r \approx 84.1$ or -227 but $r > 0$ $\therefore$ $r \approx 84.1$.
The radius of the cone is approximately 84.1 mm.

10 **a**



Surface area

$= \text{surface area of hemispherical top}$

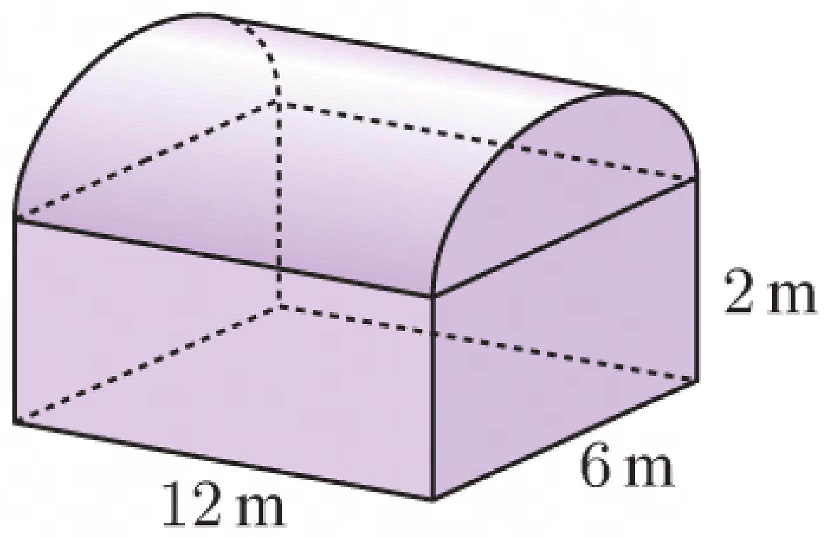
$\quad + \text{ surface area of cylindrical base}$

$= \frac{1}{2} \times 4\pi r^2 \ + \ (2\pi rh + \pi r^2)$

$= 2 \times \pi \times (1.2)^2 \ + \ 2 \times \pi \times 1.2 \times 2.8 + \pi \times (1.2)^2$

$\approx 34.7 \text{ m}^2$

b



Surface area

$= \text{surface area of half cylindrical top}$

$\quad + \text{ surface area of rectangular prism base}$

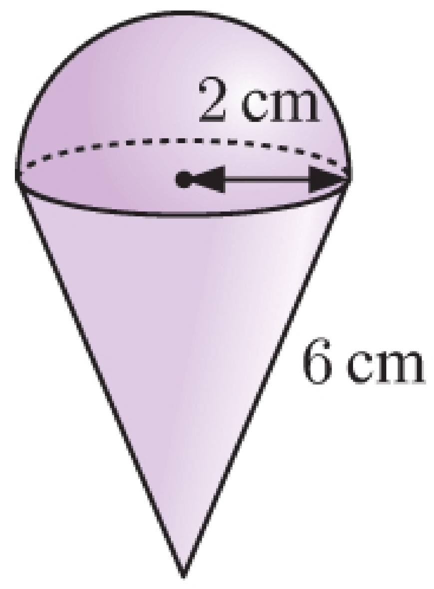
$= \frac{1}{2} \times (2\pi rh + 2\pi r^2)$

$\quad + \ (2 \times 6 \times 2 + 2 \times 12 \times 2 + 12 \times 6)$

$= \pi \times 3 \times 12 + \pi \times 3^2 \ + \ (24 + 48 + 72)$

$\approx 285.4 \text{ m}^2$

c



Surface area

$= \text{surface area of hemispherical top}$

$\quad + \text{ surface area of conical base}$

$= \frac{1}{2} \times 4\pi r^2 \ + \ \pi rs$

$= 2 \times \pi \times 2^2 \ + \ \pi \times 2 \times 6$

$\approx 62.8 \text{ cm}^2$

11 Surface area of a sphere $= 4\pi r^2$

The surface area is $\approx 7.618 \times 10^9 \text{ km}^2$

$\therefore \ 4\pi r^2 \approx 7.618 \times 10^9$

$\therefore \ r^2 \approx \dfrac{7.618 \times 10^9}{4\pi}$

$\therefore \ r \approx \sqrt{\dfrac{7.618 \times 10^9}{4\pi}} \qquad \{\text{as } r > 0\}$

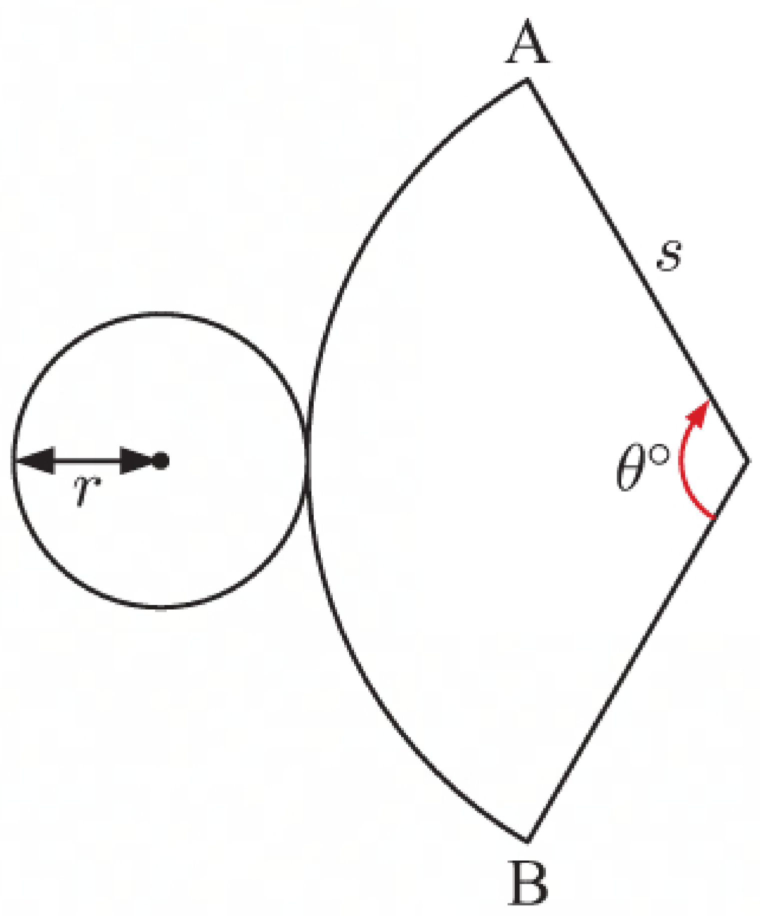
$\approx 24\,600$

The radius of Neptune is approximately 24 600 km.

12 **a** Arc length AB $= \dfrac{\theta}{360} \times 2 \times \pi \times \text{radius of sector}$

$= \dfrac{\theta}{360} \times 2\pi s$

$= \dfrac{\theta \pi s}{180}$



b Arc length AB = circumference of base circle

$\therefore \ \dfrac{\theta \pi s}{180} = 2\pi r$

$\therefore \ \theta = \dfrac{360r}{s}$

c Surface area of cone = area of sector + area of base circle

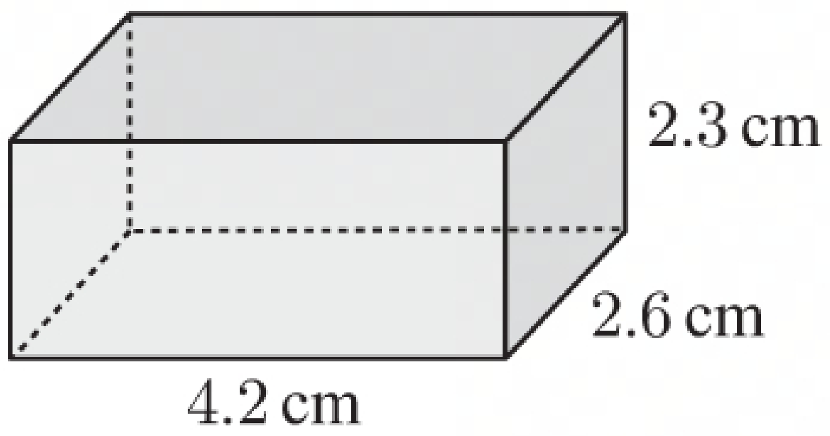
$$= \frac{\theta}{360} \times \pi s^2 + \pi r^2 \qquad \{\text{using } \mathbf{a} \text{ and } \mathbf{b}\}$$

$$= \frac{\frac{360r}{s}}{360} \times \pi s^2 + \pi r^2$$

$$= \pi r s + \pi r^2$$

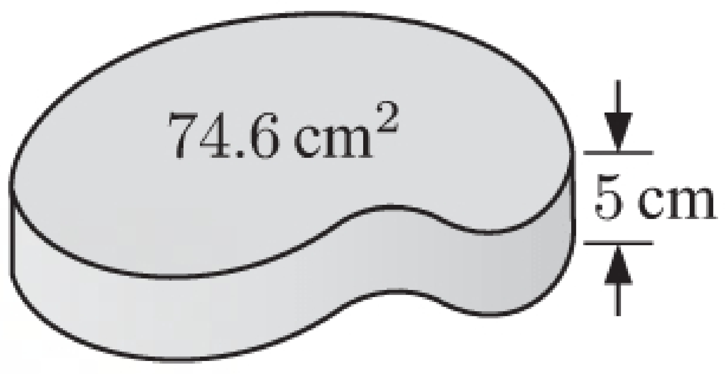
EXERCISE 6C.1

1 **a**



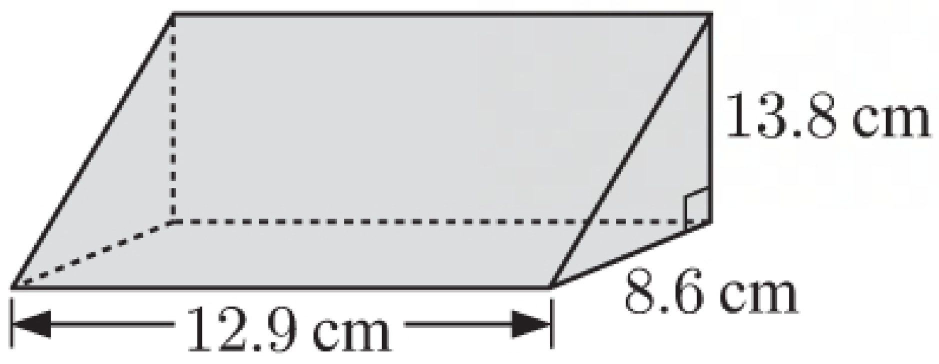
V = length × width × height
$= 4.2 \times 2.6 \times 2.3 \text{ cm}^3$
$= 25.116 \text{ cm}^3$

b



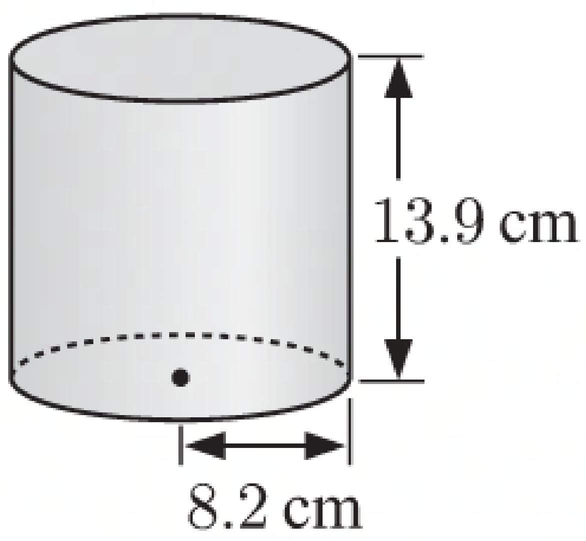
V = area of cross-section × length
$= 74.6 \times 5 \text{ cm}^3$
$= 373 \text{ cm}^3$

c



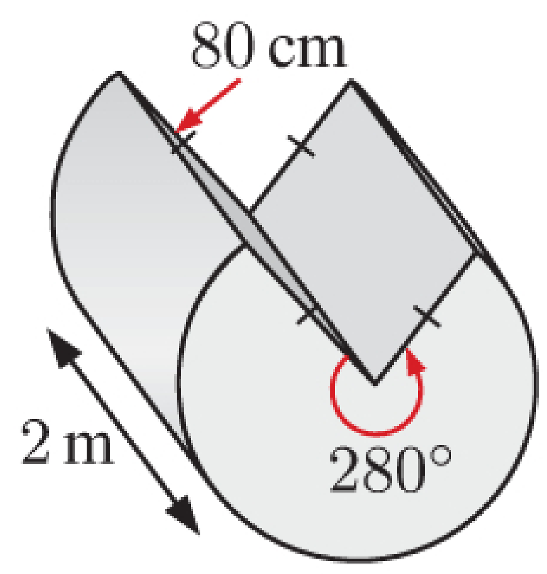
V = area of cross-section × length
$= (\frac{1}{2} \times \text{base} \times \text{height}) \times \text{length}$
$= (\frac{1}{2} \times 8.6 \times 13.8) \times 12.9 \text{ cm}^3$
$= 765.486 \text{ cm}^3$

d



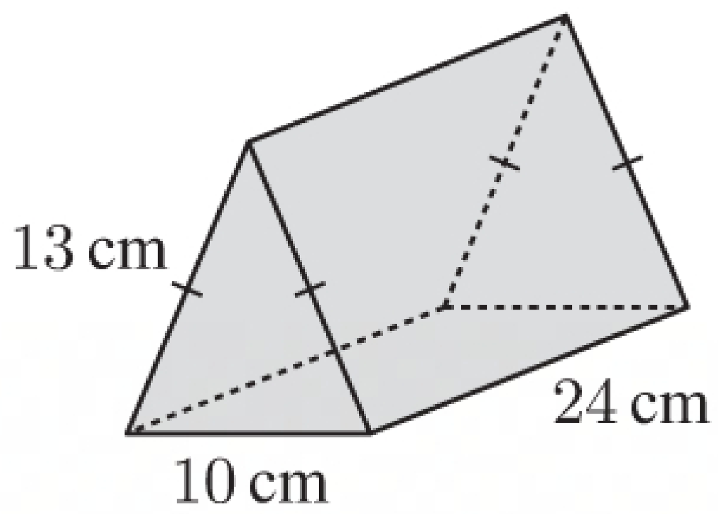
$V = \pi r^2 h$
$= \pi \times (8.2)^2 \times 13.9 \text{ cm}^3$
$\approx 2940 \text{ cm}^3$

e



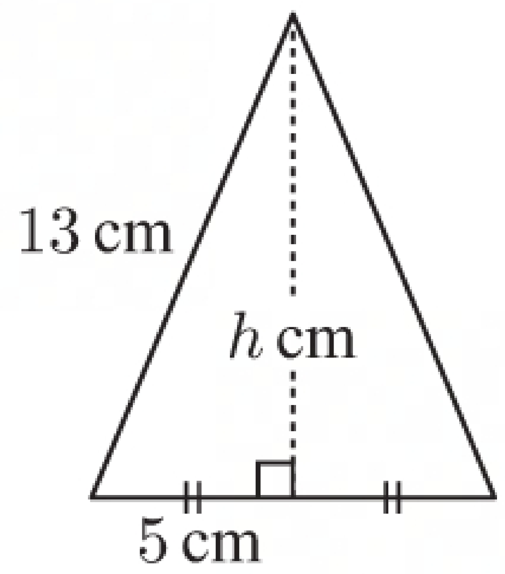
V = area of cross-section × length
$= \left(\frac{280}{360} \times \pi \times (0.8)^2\right) \times 2 \text{ m}^3 \quad \{80 \text{ cm} \equiv 0.8 \text{ m}\}$
$\approx 3.13 \text{ m}^3$

f



Let the prism have height h cm.

$h^2 + 5^2 = 13^2 \qquad \{\text{Pythagoras}\}$
$\therefore \; h^2 + 25 = 169$
$\therefore \; h^2 = 144$
$\therefore \; h = 12 \qquad \{\text{as } h > 0\}$



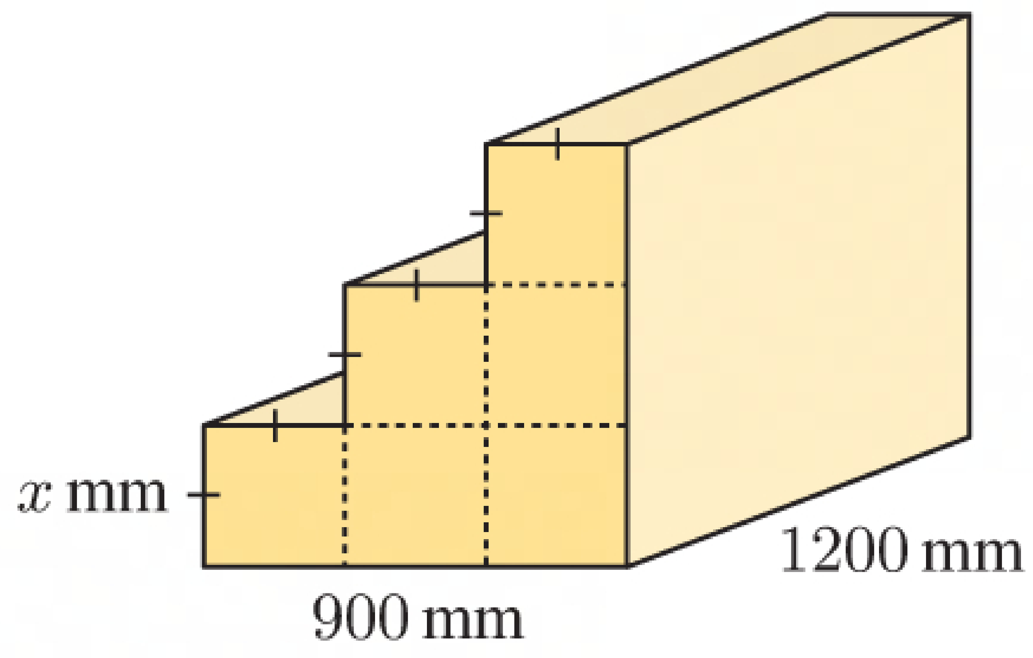
Volume = area of cross-section × length
$= \left(\frac{1}{2} \times 10 \times 12\right) \times 24 \text{ cm}^3$
$= 1440 \text{ cm}^3$

2 **a** Let the equal length sides be x mm.

$\therefore\ 3x = 900$

$\therefore\ x = 300$

So, the end is made up of six 300 mm by 300 mm squares.

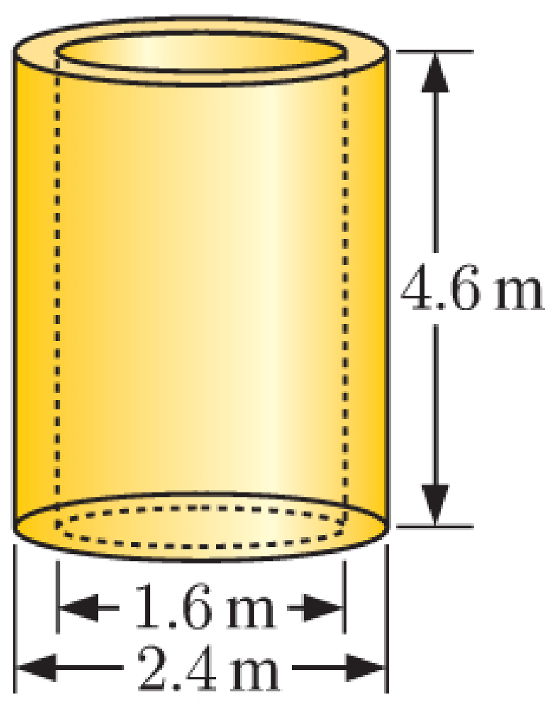


V = area of end × length

$= (6 \times 300 \times 300) \times 1200\ \text{mm}^3$

$= 648\,000\,000\ \text{mm}^3$

b



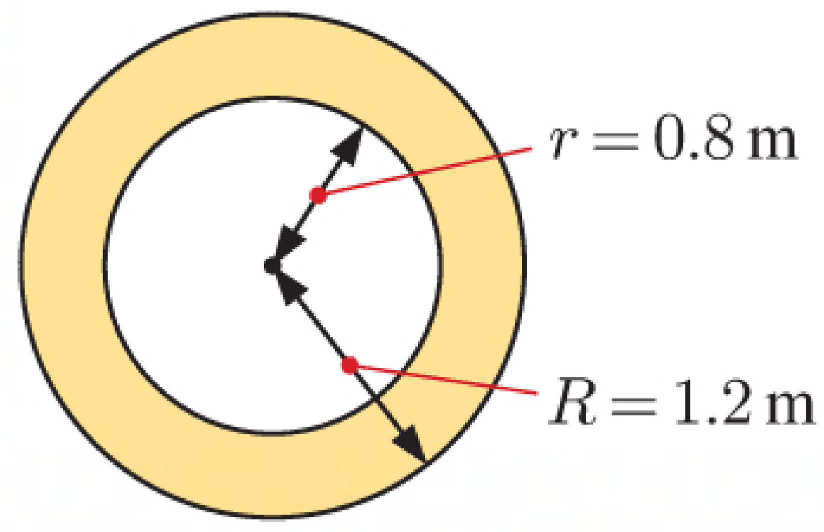
V = area of end × length

= (area of large circle − area of small circle) × length

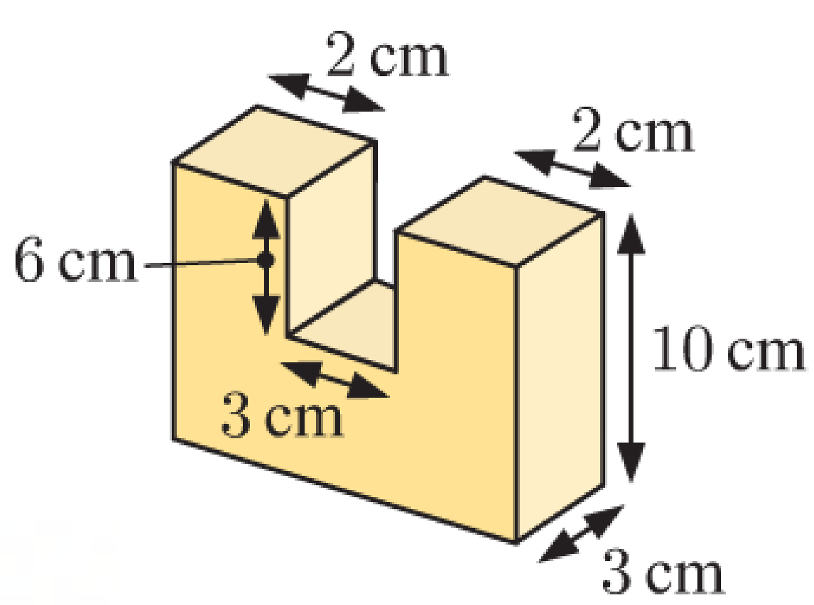
$= (\pi R^2 - \pi r^2) \times \text{length}$

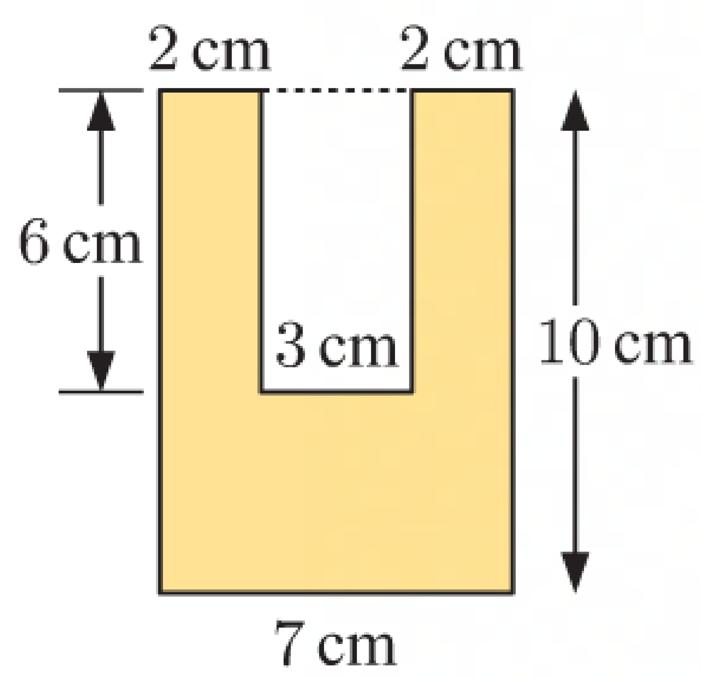
$= (\pi \times 1.2^2 - \pi \times 0.8^2) \times 4.6\ \text{m}^3$

$\approx 11.6\ \text{m}^3$



c





V = area of end × length

= (area of large rectangle − area of small rectangle) × length

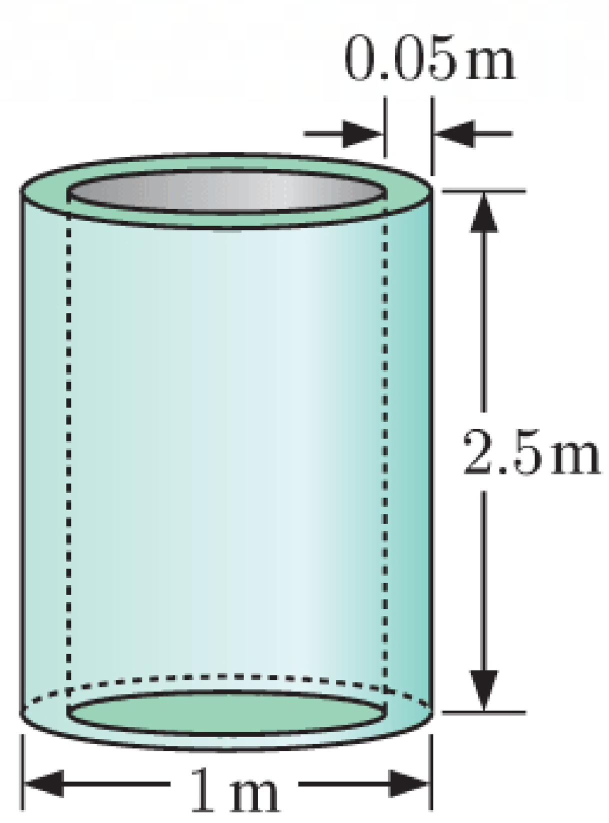
$= (7 \times 10 - 3 \times 6) \times 3\ \text{cm}^3$

$= 156\ \text{cm}^3$

3 **a** The diameter $d = 1$ m

so the radius $r = \frac{1}{2} = 0.5$ m.

So, the external radius of a pipe is 0.5 m.



b internal radius = external radius − width of concrete

$\therefore\ r = 0.5 - 0.05$ m

$= 0.45$ m

So, the internal radius of a pipe is 0.45 m.

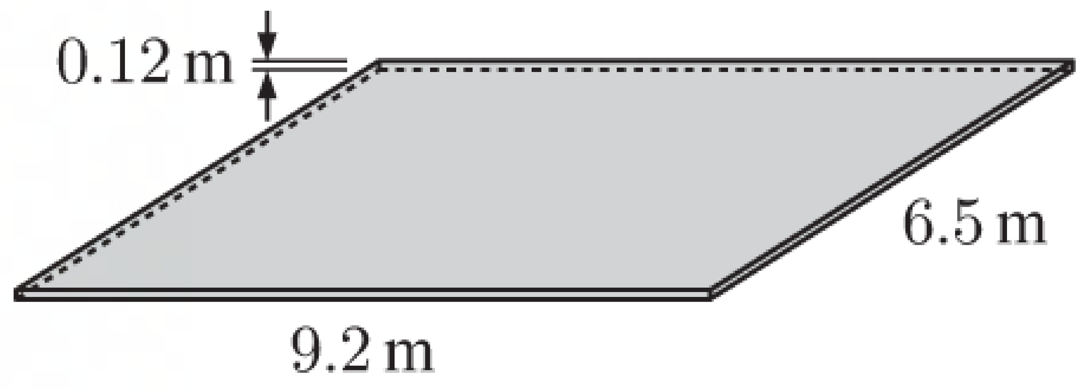
c Volume of concrete necessary to make one pipe

= volume of whole cylinder − volume of hollow section

$= \pi \times (0.5)^2 \times 2.5 - \pi \times (0.45)^2 \times 2.5\ \text{m}^3$

$\approx 0.373\ \text{m}^3$

4 **a** Depth of floor $= 120$ mm
$= 120 \div 10 \div 100$ m
$= 0.12$ m



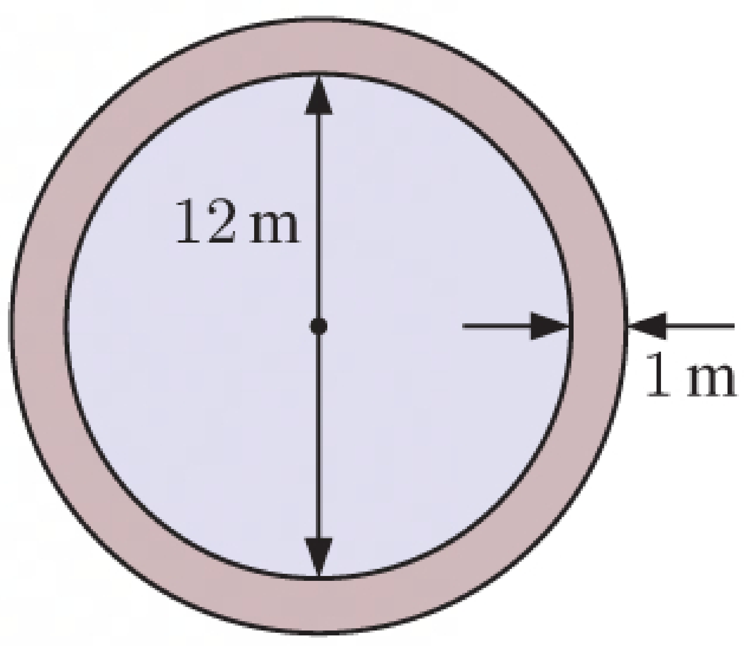
Volume of concrete $=$ length $\times$ width $\times$ depth
$= 9.2 \times 6.5 \times 0.12 \text{ m}^3$
$= 7.176 \text{ m}^3$

b 7.176 m^3 of concrete is needed. Since concrete is only supplied in multiples of 0.2 m^3, 7.2 m^3 of concrete will need to be ordered.

Cost of concrete $=$ volume of concrete $\times$ cost per m^3
$= 7.2 \text{ m}^3 \times \$135/\text{m}^3$
$= \$972$

So, it will cost \$972 to concrete the floor.

5 **a**



b The diameter of the small circle $d = 12$, so the radius $r = \frac{12}{2} = 6$ m.

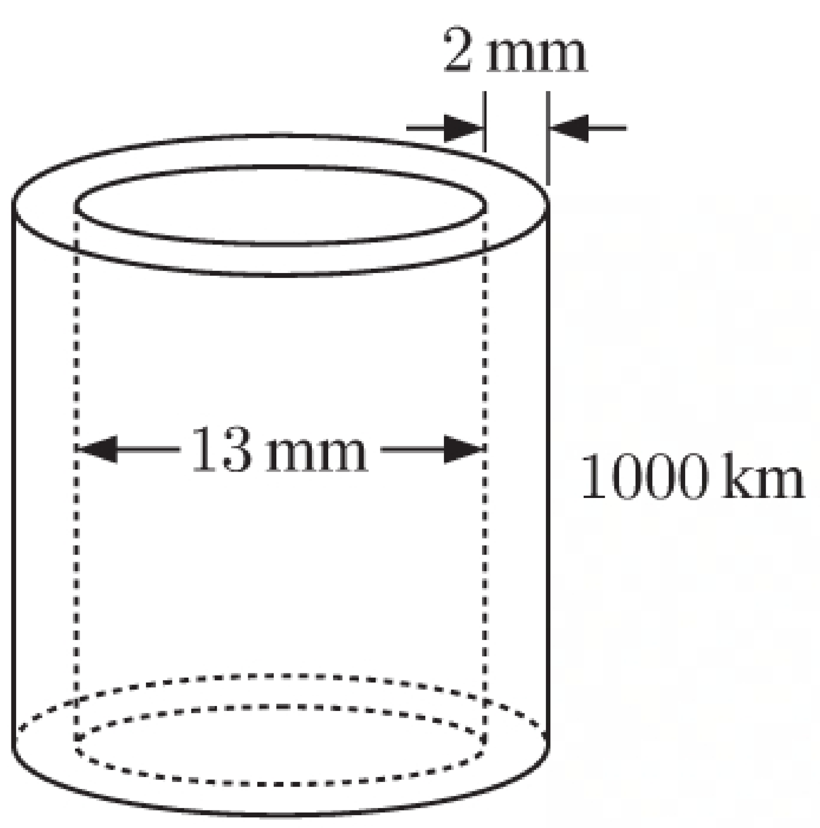
So the radius of the large circle $R = r + 1 = 7$ m.

Surface area of concrete
$=$ area of large circle $-$ area of small circle
$= \pi R^2 - \pi r^2$
$= \pi \times 7^2 - \pi \times 6^2$
$\approx 40.8 \text{ m}^2$

c Volume $=$ surface area of concrete $\times$ depth
$\approx 40.8 \times 0.1 \text{ m}^3$ {as 10 cm $= 0.1$ m}
$\approx 4.08 \text{ m}^3$

$\therefore$ approximately 4.08 m^3 of concrete is needed for the path.

6



Length of piping required $= 1000$ km
$= 1000 \times 1000$ m
$= 1\,000\,000$ m

Internal diameter of piping $= 13$ mm
$= (13 \div 10 \div 100)$ m
$= 0.013$ m

$\therefore$ internal radius $= 0.013 \div 2$ m
$= 0.0065$ m

External radius of piping $=$ internal radius $+$ wall
$= 0.0065 \text{ m} + 2 \text{ mm}$
$= 0.0065 \text{ m} + (2 \div 10 \div 100) \text{ m}$
$= 0.0065 \text{ m} + 0.002 \text{ m}$
$= 0.0085$ m

Total volume of piping $=$ volume of whole cylinder $-$ volume of hollow section
$= \pi \times (0.0085)^2 \times 1\,000\,000 - \pi \times (0.0065)^2 \times 1\,000\,000 \text{ m}^3$
$\approx 94.248 \text{ m}^3$

Weight of plastic required = volume of plastic required × weight of plastic per cubic metre
$\approx 94.248\ \text{m}^3 \times 0.86\ \text{tonnes/m}^3$
≈ 81.1 tonnes

7 **a** Volume of garden = length × width × depth
$= 8.6 \times 2.4 \times 0.15\ \text{m}^3$ $\{15\ \text{cm} \equiv 0.15\ \text{m}\}$
$= 3.096\ \text{m}^3$

Volume of trailer = length × width × height
$= 2.2 \times 1.8 \times (0.6 - 0.2)\ \text{m}^3$ $\{60\ \text{cm} \equiv 0.6\ \text{m},\ \ 20\ \text{cm} \equiv 0.2\ \text{m}\}$
$= 1.584\ \text{m}^3$

Number of trailer loads required $= \dfrac{\text{volume of soil required}}{\text{volume of soil per trailer load}}$
$= \dfrac{3.096\ \text{m}^3}{1.584\ \text{m}^3}$
≈ 1.95

So, I will need 2 trailer loads of soil.

b Cost of soil = number of loads × cost per load
$= 2 \times \$87.30$
$= \$174.60$

c **i** Area of garden = length × width
$= 8.6 \times 2.4\ \text{m}^2$
$= 20.64\ \text{m}^2$

Number of loads of bark needed
$= \dfrac{\text{area of garden}}{\text{area covered by one load}}$
$= \dfrac{20.64\ \text{m}^2}{11\ \text{m}^2}$
≈ 1.88

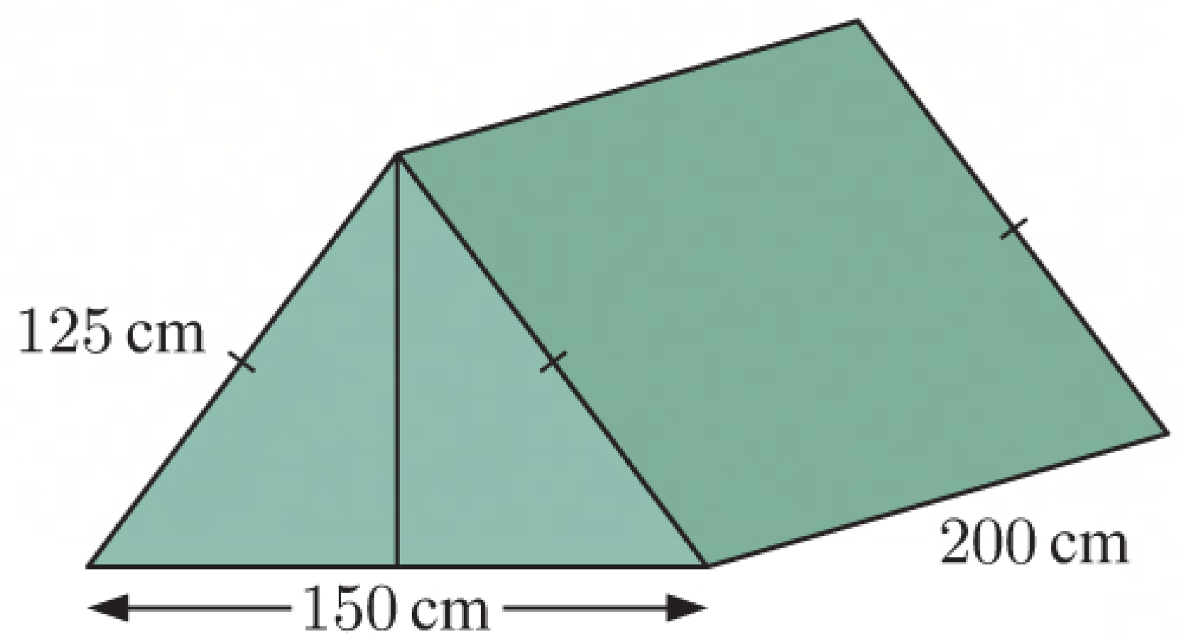
So, I will need 2 trailer loads of bark.

ii Cost of bark = number of loads × cost per load
$= 2 \times \$47.95$
$= \$95.90$

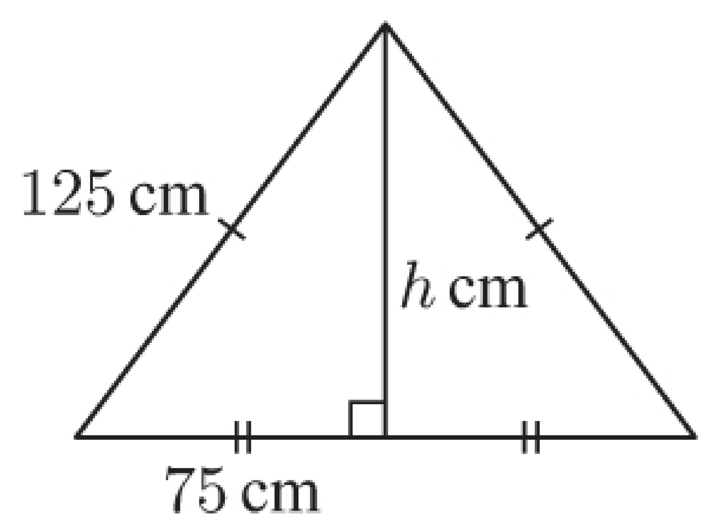
d Total cost of establishing garden = cost of soil + cost of bark
$= \$174.60 + \$95.90$
$= \$270.50$

8 **a** Let the triangular prism have height h cm.
$h^2 + 75^2 = 125^2$ {Pythagoras}
$\therefore\ h^2 + 5625 = 15\,625$
$\therefore\ h^2 = 10\,000$
$\therefore\ h = 100$ {as $h > 0$}

Each vertical support post is 100 cm in height.



b Volume of tent = area of triangular end × length
$= \frac{1}{2} \times 150 \times 100 \times 200\ \text{cm}^3$
$= 1\,500\,000\ \text{cm}^3$
$= 1\,500\,000 \div 100^3\ \text{m}^3$
$= 1.5\ \text{m}^3$



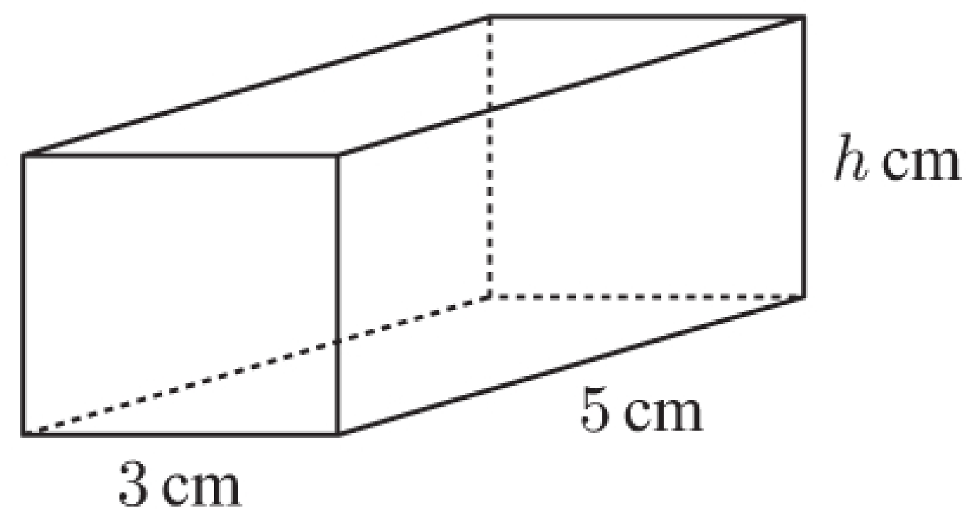
c Total area of canvas

$= \text{area of two triangular ends} + \text{area of two rectangular sides} + \text{area of rectangular base}$

$= 2 \times (\frac{1}{2} \times 150 \times 100) + 2 \times (200 \times 125) + (200 \times 150)\ \text{cm}^2$

$= 95\,000\ \text{cm}^2$

9 **a**



Let the height of the rectangular prism be h cm.

$$V = 40\ \text{cm}^3$$

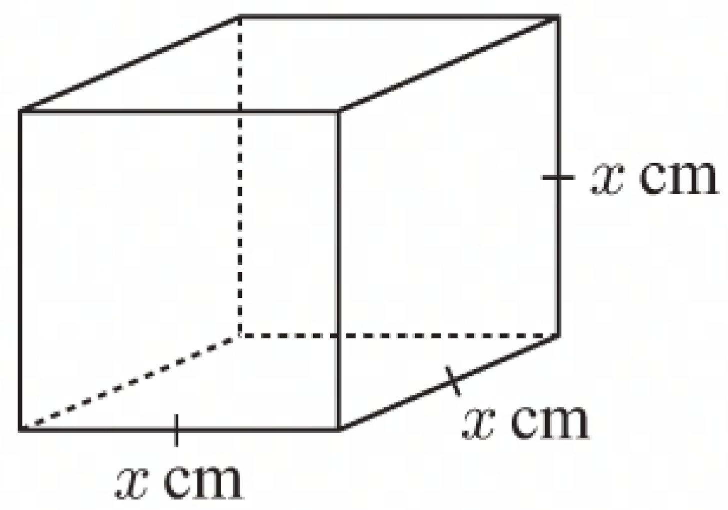
$$\therefore\ 5 \times 3 \times h = 40$$

$$\therefore\ h = \frac{40}{15}$$

$$\therefore\ h = \tfrac{8}{3} \approx 2.67$$

The height is approximately 2.67 cm.

b



Let the sides of the cube be x cm.

$$V = 34.01\ \text{cm}^3$$

$$\therefore\ x \times x \times x = 34.01$$

$$\therefore\ x^3 = 34.01$$

$$\therefore\ x = \sqrt[3]{34.01}$$

$$\approx 3.24$$

The side length is approximately 3.24 cm.

c Let the radius be r cm.

$$V = 43.75\ \text{cm}^3$$

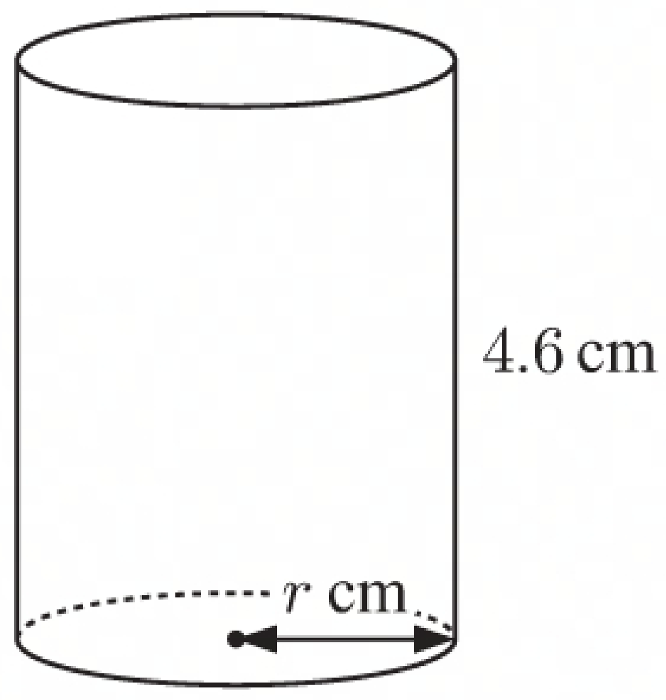
$$\therefore\ \pi \times r^2 \times 4.6 = 43.75$$

$$\therefore\ r^2 = \frac{43.75}{\pi \times 4.6}$$

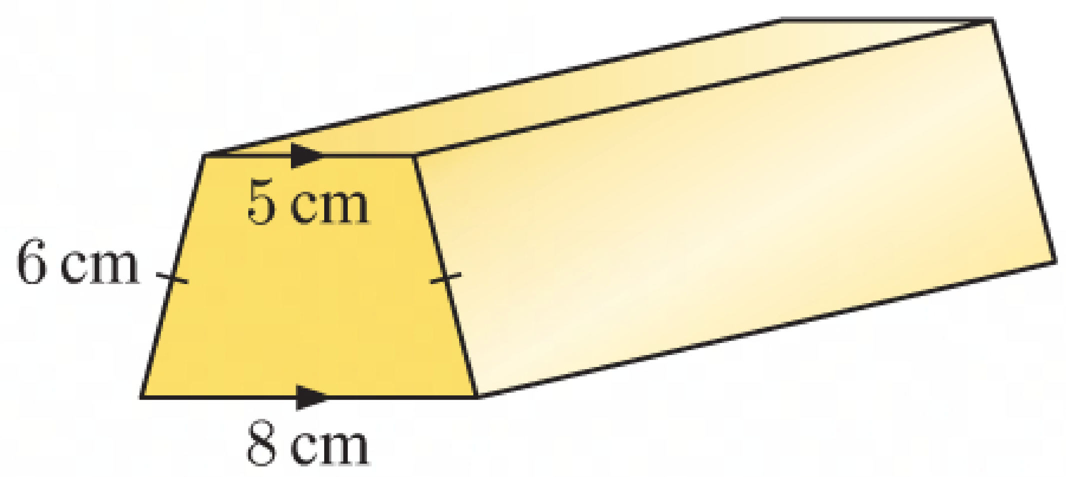
$$\therefore\ r = \sqrt{\frac{43.75}{\pi \times 4.6}} \quad \{\text{as } r > 0\}$$

$$\approx 1.74$$

The radius is approximately 1.74 cm.



10



Let the height of the trapezoidal cross-section be h cm.

$$h^2 + 1.5^2 = 6^2 \qquad \{\text{Pythagoras}\}$$

$$\therefore\ h = \sqrt{6^2 - 1.5^2}$$

$$= \sqrt{33.75} \qquad \{\text{as } h > 0\}$$

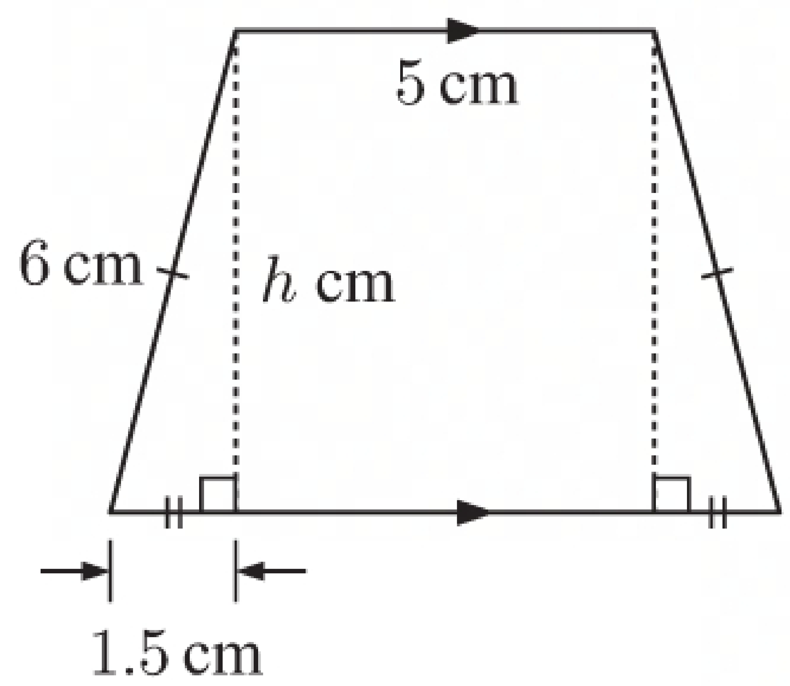
Volume of gold bar = area of cross-section × length

$$\therefore\ 480 = \left(\frac{5+8}{2}\right) \times \sqrt{33.75} \times \text{length}$$

$$\therefore\ \text{length} = \frac{480}{\frac{13}{2}\sqrt{33.75}}$$

$$\approx 12.7$$

The length of the gold bar is approximately 12.7 cm.



INVESTIGATION 2 VOLUME FORMULAE

1 **a** **i** $$\frac{\sum_{k=1}^{10} k}{10^2} = \frac{\frac{10(10+1)}{2}}{100} = \frac{110}{200} = \frac{11}{20} = 0.55$$

ii $$\frac{\sum_{k=1}^{100} k}{100^2} = \frac{\frac{100(100+1)}{2}}{10\,000} = \frac{10\,100}{20\,000} = \frac{101}{200} = 0.505$$

iii $$\frac{\sum_{k=1}^{1000} k}{1000^2} = \frac{\frac{1000(1000+1)}{2}}{1\,000\,000} = \frac{1\,001\,000}{2\,000\,000} = \frac{1001}{2000} = 0.5005$$

iv $$\frac{\sum_{k=1}^{10\,000} k}{10\,000^2} = \frac{\frac{10\,000(10\,000+1)}{2}}{100\,000\,000} = \frac{100\,010\,000}{200\,000\,000} = \frac{10\,001}{20\,000} = 0.500\,05$$

b As $n \to \infty$, $\dfrac{\sum_{k=1}^{n} k}{n^2} \to \frac{1}{2}$.

2 **a** **i** $$\frac{\sum_{k=1}^{10} k^2}{10^3} = \frac{\frac{10(10+1)(2(10)+1)}{6}}{1000} = \frac{2310}{6000} = \frac{231}{600} = 0.385$$

ii $$\frac{\sum_{k=1}^{100} k^2}{100^3} = \frac{\frac{100(100+1)(2(100)+1)}{6}}{1\,000\,000} = \frac{2\,030\,100}{6\,000\,000} = \frac{20\,301}{60\,000} = 0.338\,35$$

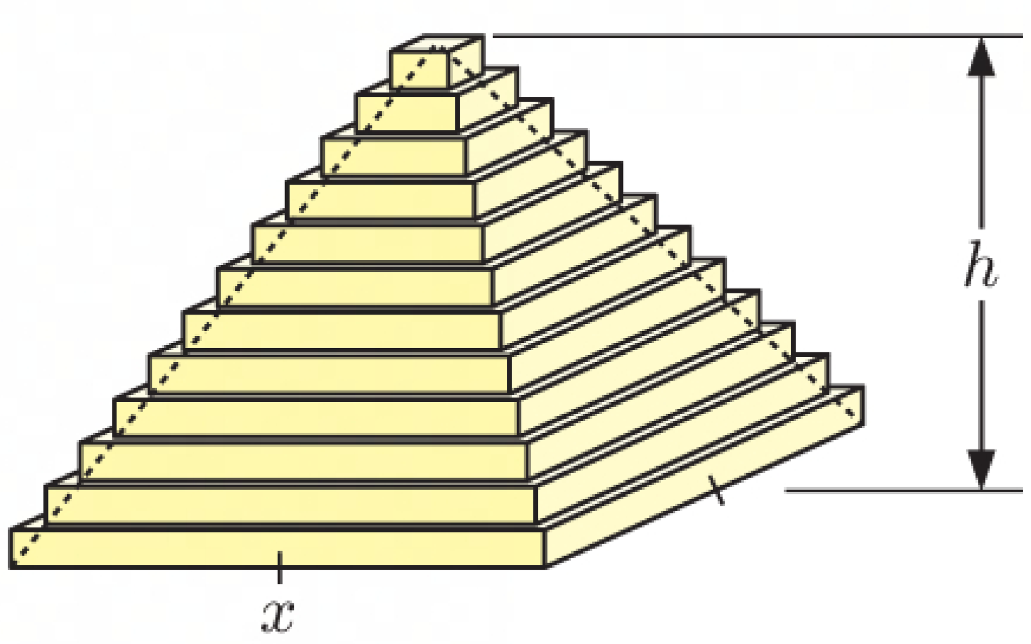
iii $$\frac{\sum_{k=1}^{1000} k^2}{1000^3} = \frac{\frac{1000(1000+1)(2(1000)+1)}{6}}{1\,000\,000\,000} = \frac{2\,003\,001\,000}{6\,000\,000\,000} = \frac{2\,003\,001}{6\,000\,000} \approx 0.333\,83$$

iv $$\frac{\sum_{k=1}^{10\,000} k^2}{10\,000^3} = \frac{\frac{10\,000(10\,000+1)(2(10\,000)+1)}{6}}{1\,000\,000\,000\,000} = \frac{2\,000\,300\,010\,000}{6\,000\,000\,000\,000} = \frac{200\,030\,001}{600\,000\,000} \approx 0.333\,38$$

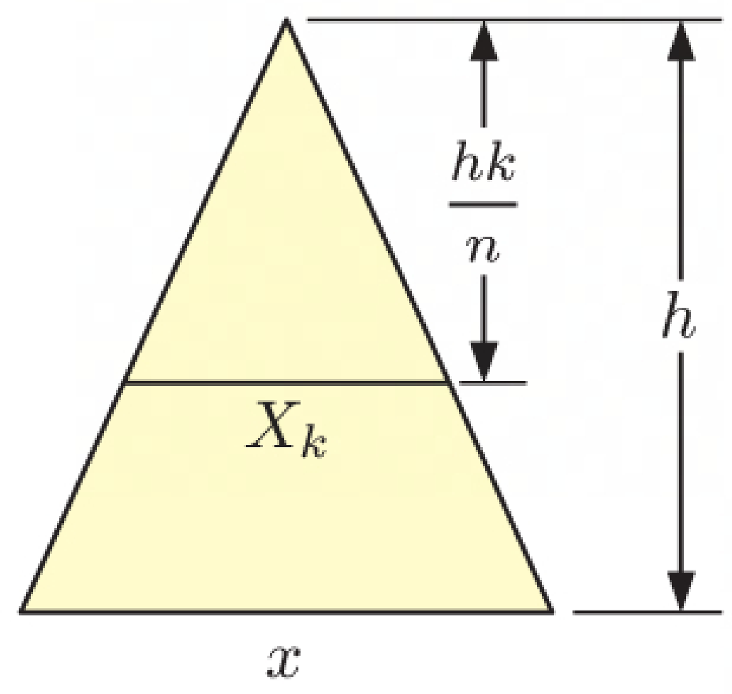
b As $n \to \infty$, $\dfrac{\sum_{k=1}^{n} k^2}{n^3} \to \frac{2}{6} = \frac{1}{3}$.

3 **a** There are n prisms with equal thickness and total height h.

$\therefore$ the thickness of each prism is $\dfrac{h}{n}$.



b From the diagram alongside, the distance from the apex to the base of the kth prism is $\dfrac{hk}{n}$.



Using similar triangles, $\dfrac{X_k}{x} = \dfrac{\frac{hk}{n}}{h}$

$\therefore \dfrac{X_k}{x} = \dfrac{k}{n}$

$\therefore X_k = \dfrac{xk}{n}$

c The kth prism will have length = width = $\dfrac{xk}{n}$, and height $\dfrac{h}{n}$.

$\therefore$ the volume of the kth prism = length × width × height

$$= \frac{xk}{n} \times \frac{xk}{n} \times \frac{h}{n}$$

$$= \frac{h}{n}\left(\frac{xk}{n}\right)^2$$

The volume of the pyramid can be approximated by the sum of the volume of all n prisms.

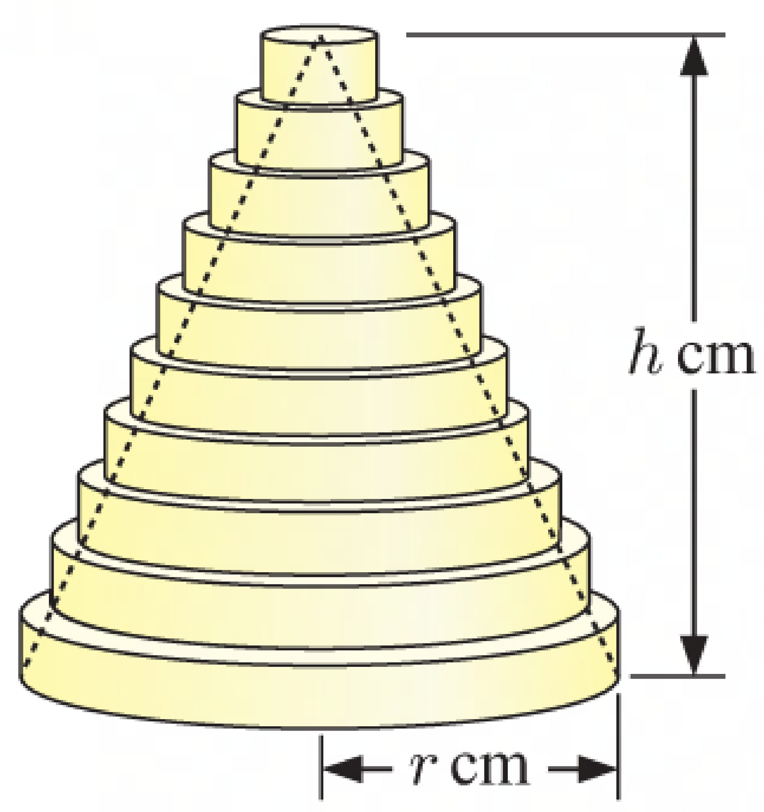
So, $V \approx \displaystyle\sum_{k=1}^{n} \frac{h}{n}\left(\frac{xk}{n}\right)^2 = \sum_{k=1}^{n} x^2h\,\frac{k^2}{n^3} = x^2h\,\frac{\sum_{k=1}^{n} k^2}{n^3}$

d As $n \to \infty$, $\dfrac{\sum_{k=1}^{n} k^2}{n^3} \to \frac{1}{3}$ {from **2 b**}

$\therefore$ as $n \to \infty$, $x^2h\,\dfrac{\sum_{k=1}^{n} k^2}{n^3} \to x^2h \times \frac{1}{3} = \frac{1}{3} \times$ base area × height

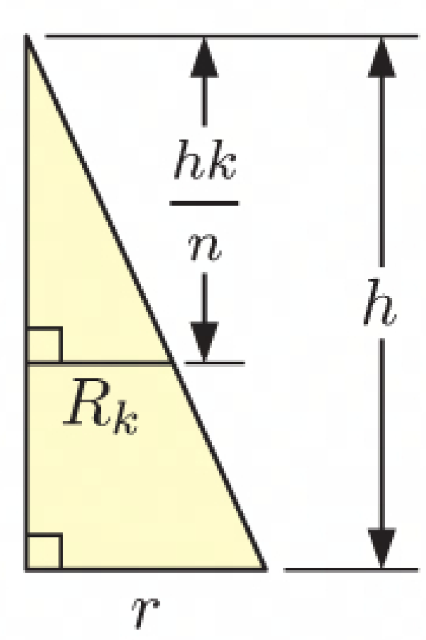
As the number of prisms in our approximation approaches infinity, the volume is given by $\frac{1}{3}x^2h = \frac{1}{3} \times$ base area × height.

4 **a** There are n cylinders of equal thickness and total height h.



$\therefore$ the height of each cylinder is $\dfrac{h}{n}$.

Let R_k be the radius of the kth cylinder. From the diagram alongside, the distance from the apex to the base of the kth cylinder is $\dfrac{hk}{n}$.



Using similar triangles, $\dfrac{R_k}{r} = \dfrac{\frac{hk}{n}}{h}$

$\therefore \dfrac{R_k}{r} = \dfrac{k}{n}$

$\therefore R_k = \dfrac{rk}{n}$

b The volume of the kth cylinder $=$ base area $\times$ height

$$= \pi \times R_k^2 \times \frac{h}{n}$$

$$= \frac{h}{n}\pi\left(\frac{rk}{n}\right)^2$$

The volume of the cone can be approximated by the sum of the volumes of all n cones.

So, $V \approx \sum_{k=1}^{n} \frac{h}{n}\pi\left(\frac{rk}{n}\right)^2 = \sum_{k=1}^{n} \pi r^2 h\,\frac{k^2}{n^3} = \pi r^2 h\,\frac{\sum_{k=1}^{n} k^2}{n^3}$

c As $n \to \infty$, $\dfrac{\sum_{k=1}^{n} k^2}{n^3} \to \frac{1}{3}$ {from **2 b**}

$\therefore$ as $n \to \infty$, $\pi r^2 h\,\dfrac{\sum_{k=1}^{n} k^2}{n^3} \to \pi r^2 h \times \frac{1}{3} = \frac{1}{3} \times$ base area $\times$ height

The volume of the cone is given by $\frac{1}{3}\pi r^2 h = \frac{1}{3} \times$ base area $\times$ height.

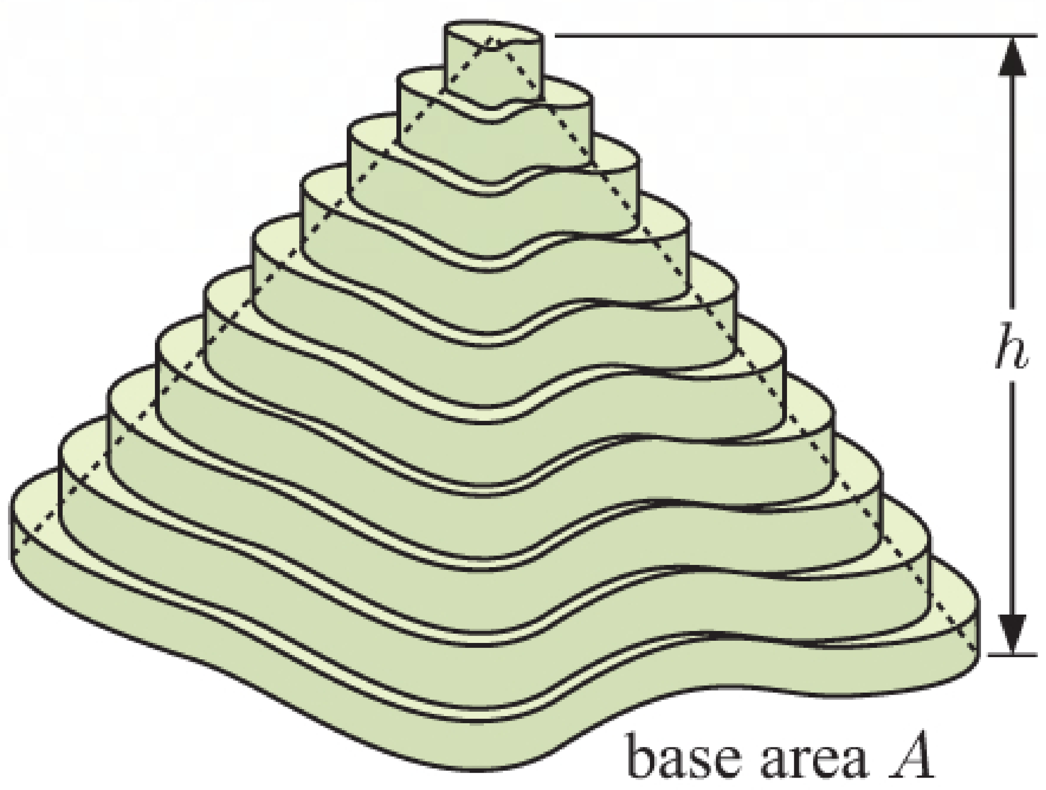
5 **a** There are n solids of uniform cross-section with equal thickness and total height h.

$\therefore$ the height of each solid of uniform cross-section is $\dfrac{h}{n}$.

Let A_k be the base area of the kth solid of uniform cross-section.

The base of the kth solid of uniform cross-section is a reduction of the base of the tapered solid with scale factor $\dfrac{k}{n}$.

$\therefore$ $A_k = A\left(\frac{k}{n}\right)^2$



b The volume of the kth solid of uniform cross-section $=$ base area $\times$ height

$$= A\left(\frac{k}{n}\right)^2 \times \frac{h}{n}$$

$$= \frac{h}{n} A\left(\frac{k}{n}\right)^2$$

The volume of the tapered solid can be approximated by the sum of the volumes of all n solids of uniform cross-section.

So, $V \approx \sum_{k=1}^{n} \frac{h}{n} A\left(\frac{k}{n}\right)^2$.

c $\displaystyle\sum_{k=1}^{n} \frac{h}{n} A\left(\frac{k}{n}\right)^2 = \sum_{k=1}^{n} Ah\,\frac{k^2}{n^3}$

$$= Ah\,\frac{\sum_{k=1}^{n} k^2}{n^3}$$

As $n \to \infty$, $\dfrac{\sum_{k=1}^{n} k^2}{n^3} \to \frac{1}{3}$

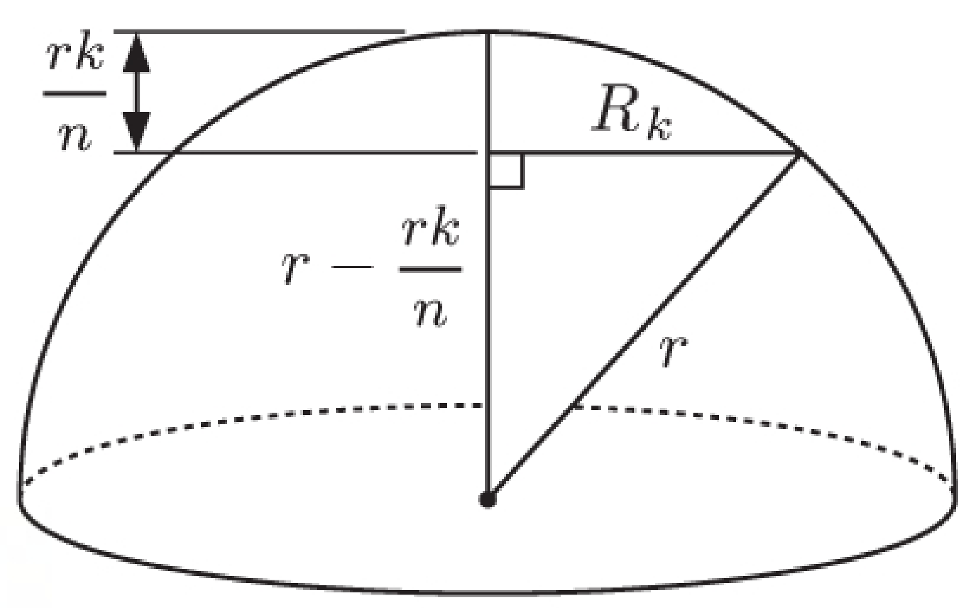
$\therefore$ as $n \to \infty$, $Ah\,\dfrac{\sum_{k=1}^{n} k^2}{n^3} \to Ah \times \frac{1}{3} = \frac{1}{3} \times \text{base area} \times \text{height}$

The volume of the tapered solid is given by $\frac{1}{3} \times$ base area $\times$ height.

6 **a** There are n cylinders of equal thickness and the total height is h.

$\therefore$ the height of the kth cylinder is $\dfrac{r}{n}$.

b $\left(r - \dfrac{rk}{n}\right)^2 + R_k^2 = r^2$ {Pythagoras}



$$\therefore\ R_k^2 = r^2 - \left(r - \frac{rk}{n}\right)^2$$

$$= r^2 - \left(r^2 - \frac{2r^2k}{n} + \frac{r^2k^2}{n^2}\right)$$

$$= r^2 - r^2 + \frac{2r^2k}{n} - \frac{r^2k^2}{n^2}$$

$$= \frac{r^2}{n}\left(2k - \frac{k^2}{n}\right)$$

c The volume of the kth cylinder $=$ base area $\times$ height

$$= \pi R_k^2 \times \frac{r}{n}$$

$$= \frac{r}{n}\,\pi\left(\frac{r^2}{n}\left(2k - \frac{k^2}{n}\right)\right)$$

The volume of the hemisphere can be approximated by the sum of the volumes of all n cylinders.

So, $\displaystyle V \approx \sum_{k=1}^{n} \frac{r}{n}\,\pi\left(\frac{r^2}{n}\left(2k - \frac{k^2}{n}\right)\right) = \sum_{k=1}^{n} \pi r^3\left(\frac{1}{n^2}\left(2k - \frac{k^2}{n}\right)\right)$

$$= \pi r^3 \sum_{k=1}^{n}\left(\frac{2k}{n^2} - \frac{k^2}{n^3}\right)$$

$$= \pi r^3\left(\frac{2\sum_{k=1}^{n} k}{n^2} - \frac{\sum_{k=1}^{n} k^2}{n^3}\right)$$

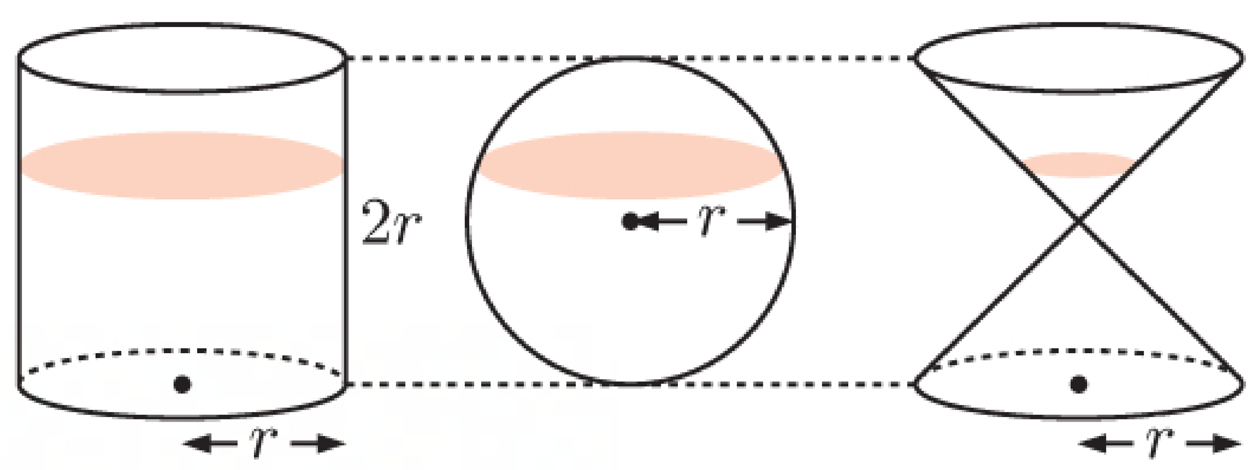
d As $n \to \infty$, $\dfrac{\sum_{k=1}^{n} k}{n^2} \to \frac{1}{2}$, and $\dfrac{\sum_{k=1}^{n} k^2}{n^3} \to \frac{1}{3}$ {from **1** and **2**}

$$\therefore \text{ as } n \to \infty, \quad \pi r^3 \left(\frac{2\sum_{k=1}^{n} k}{n^2} - \frac{\sum_{k=1}^{n} k^2}{n^3} \right) \to \pi r^3 \left(2 \times \tfrac{1}{2} - \tfrac{1}{3}\right) = \tfrac{2}{3}\pi r^3$$

As the number of cylinders in our approximation approaches infinity, the volume of the hemisphere approaches $\frac{2}{3}\pi r^3$.

$\therefore$ the volume of a sphere with radius r is $2 \times \frac{2}{3}\pi r^3 = \frac{4}{3}\pi r^3$.

7 **a** Consider a horizontal slice made at height h above (or below) the centre of the sphere.



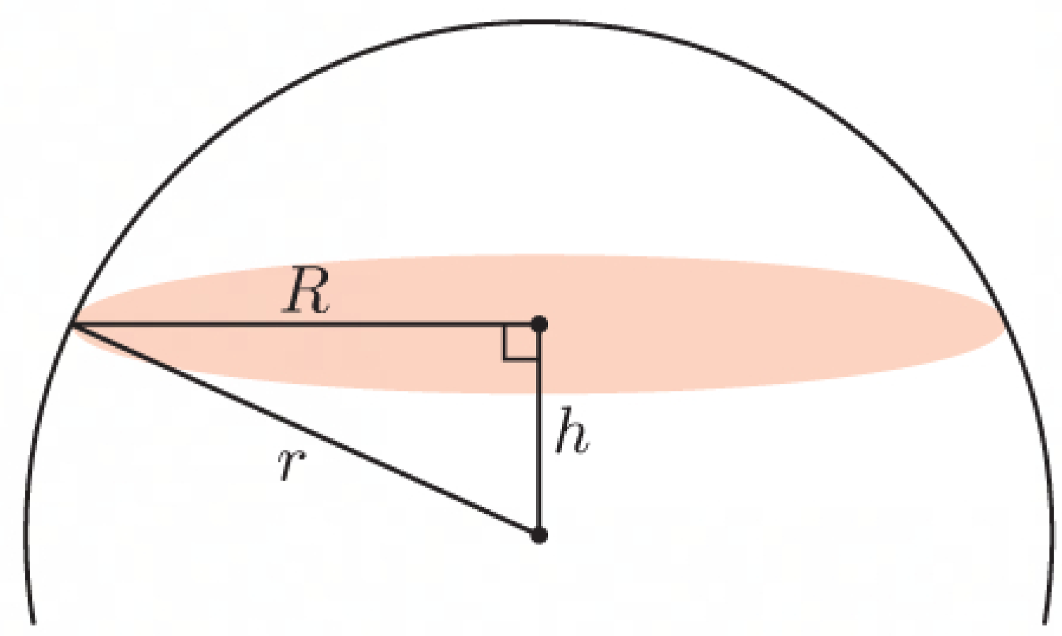
At this height:

- the cross-sectional area of the cylinder is πr^2
- the cross-sectional area of the sphere is πR^2 where R is the radius of the circle.

  Using Pythagoras, $r^2 = R^2 + h^2$

  $\therefore \quad R^2 = r^2 - h^2$

  $\therefore$ the cross-sectional area of the sphere is $\pi(r^2 - h^2)$

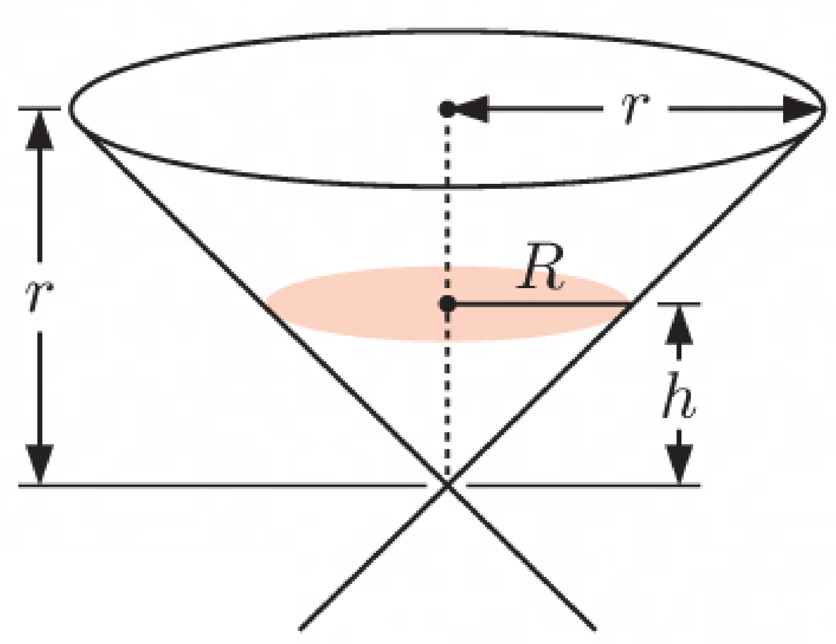


- the cross-sectional area of the double-cone is πR^2 where R is the radius of the circle.

  Using similar triangles, $\dfrac{R}{h} = \dfrac{r}{r} = 1$

  $\therefore \quad R = h$

  $\therefore$ the cross-sectional area of the double-cone is πh^2



So, the sum of the areas from the sphere and the double-cone is $\pi(r^2 - h^2) + \pi h^2 = \pi r^2$, which is the area from the cylinder.

b Using the property in **a**, if we take a thin slice across the object, at a height h above (or below) the centre of the sphere, the volume of the slice of the sphere plus the volume of the slice of the double-cone will equal the volume of the slice of the cylinder.

Since this is true at *any* height above (or below) the centre of the sphere, the sum of the volumes of the sphere and the double-cone equals the volume of the cylinder.

c
$$V_{\text{sphere}} + V_{\text{double-cone}} = V_{\text{cylinder}}$$

$$\therefore \quad V_{\text{sphere}} + 2\left(\tfrac{1}{3} \times \text{base of cone} \times \text{height of cone}\right) = \text{base of cylinder} \times \text{height of cylinder}$$

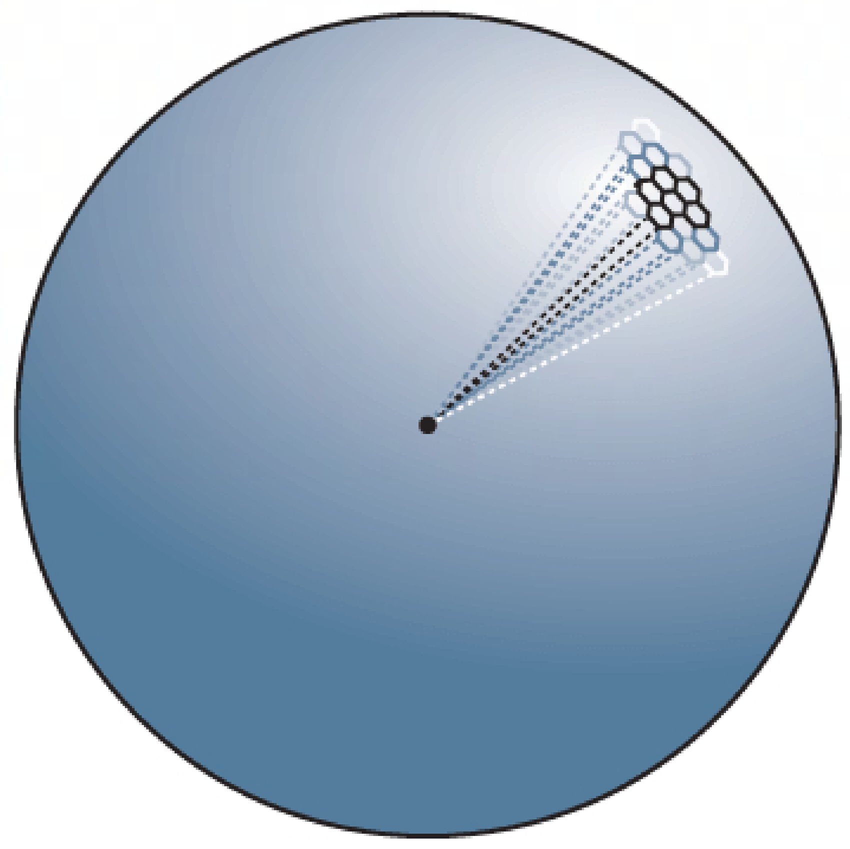
{using known formulae}

$$\therefore \quad V_{\text{sphere}} + \tfrac{2}{3} \times \pi r^2 \times r = \pi r^2 \times 2r$$

$$\therefore \quad V_{\text{sphere}} + \tfrac{2}{3}\pi r^3 = 2\pi r^3$$

$$\therefore \quad V_{\text{sphere}} = \tfrac{4}{3}\pi r^3$$

8 **a** Each tapered solid has volume $\frac{1}{3} \times \text{base area} \times \text{height}$
$= \frac{1}{3}A_k r$



We know that the volume of a sphere with radius r is $\frac{4}{3}\pi r^3$.

The volume of the sphere can be approximated by the sum of the volumes of all n tapered solids, $\sum\limits_{k=1}^{n} \frac{1}{3}A_k r \approx \frac{4}{3}\pi r^3$.

b $\sum\limits_{k=1}^{n} \frac{1}{3}A_k r \approx \frac{4}{3}\pi r^3$

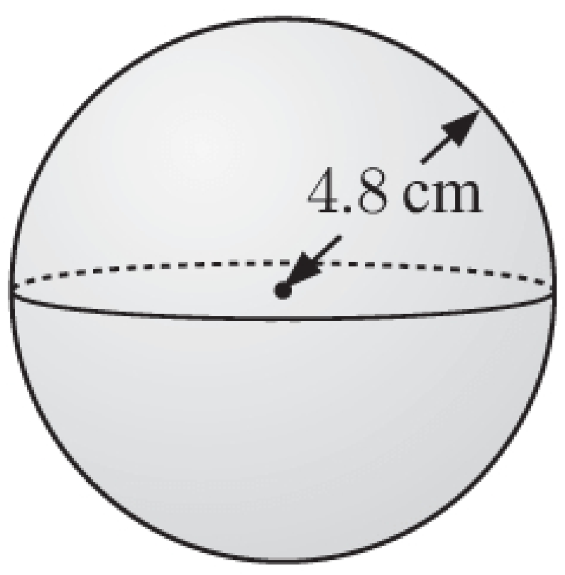
$\therefore \frac{1}{3}r \sum\limits_{k=1}^{n} A_k \approx \frac{4}{3}\pi r^3$

$\therefore \sum\limits_{k=1}^{n} A_k \approx 4\pi r^2$

The sum of the base areas of all n tapered solids is approximately $4\pi r^2$. The base of each tapered solid lies on the surface of the sphere. As the number of tapered solids in our approximation approaches infinity, the surface area of a sphere approaches $A = 4\pi r^2$.

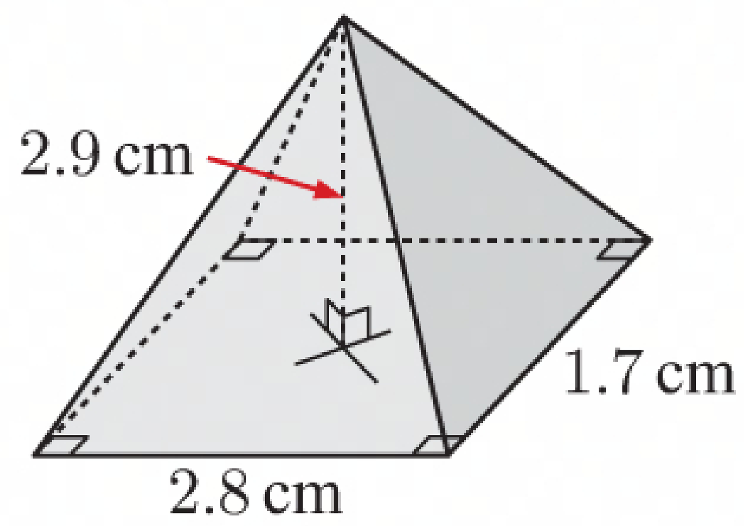
EXERCISE 6C.2

1 **a**



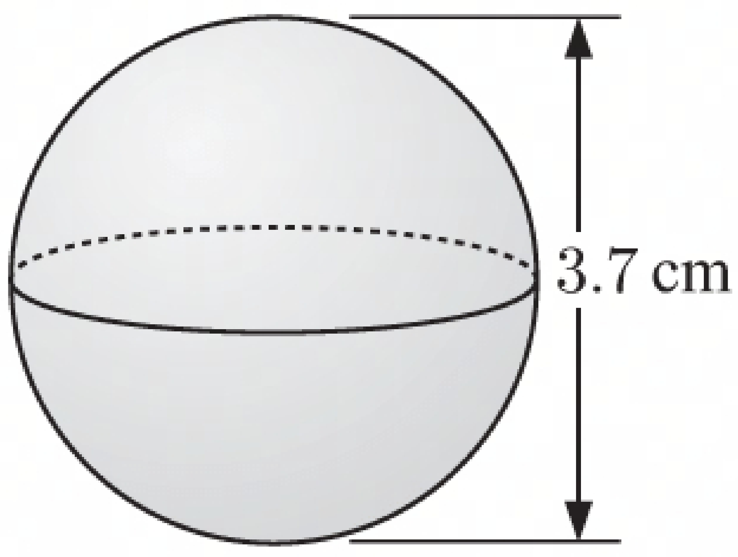
$V = \frac{4}{3}\pi r^3$
$= \frac{4}{3} \times \pi \times (4.8)^3 \text{ cm}^3$
$\approx 463 \text{ cm}^3$

b



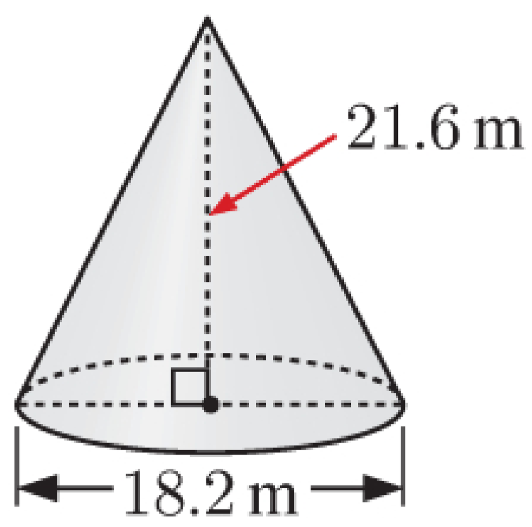
$V = \frac{1}{3}(\text{area of base} \times \text{height})$
$= \frac{1}{3}(\text{length} \times \text{width} \times \text{height})$
$= \frac{1}{3}(2.8 \times 1.7 \times 2.9) \text{ cm}^3$
$\approx 4.60 \text{ cm}^3$

c



$V = \frac{4}{3}\pi r^3$
$= \frac{4}{3} \times \pi \times \left(\frac{3.7}{2}\right)^3 \text{ cm}^3$
$\approx 26.5 \text{ cm}^3$

d

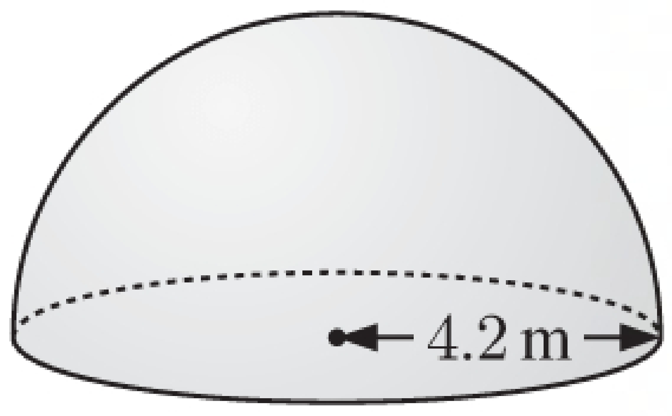


$V = \frac{1}{3}(\text{area of base} \times \text{height})$
$= \frac{1}{3}(\pi r^2 h)$
$= \frac{1}{3}\left(\pi \times \left(\frac{18.2}{2}\right)^2 \times 21.6\right) \text{ m}^3$
$\approx 1870 \text{ m}^3$

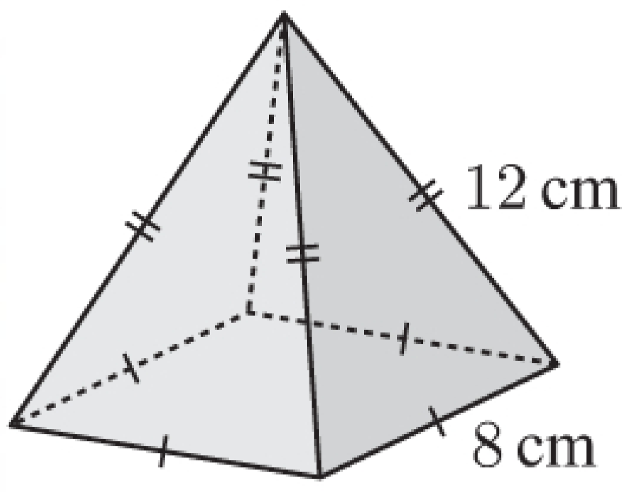
e $V = \frac{1}{2}(\frac{4}{3}\pi r^3)$

$= \frac{2}{3} \times \pi \times (4.2)^3 \text{ m}^3$

$\approx 155 \text{ m}^3$



f



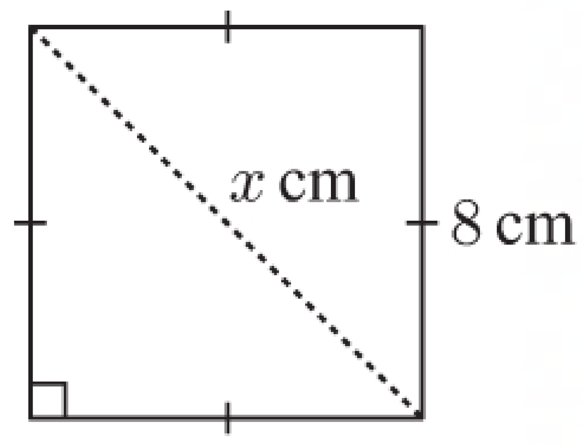
Let the diagonal of the square base be x cm.

$x^2 = 8^2 + 8^2$ {Pythagoras}

$\therefore\ x^2 = 128$

$\therefore\ x = \sqrt{128}$ {as $x > 0$}

$= 8\sqrt{2}$



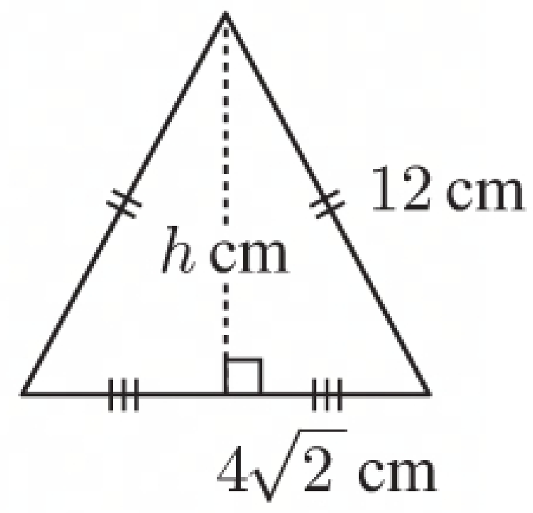
Let the height of the pyramid be h cm.

$h^2 + (4\sqrt{2})^2 = 12^2$ {Pythagoras}

$\therefore\ h^2 + 32 = 144$

$\therefore\ h^2 = 112$

$\therefore\ h = \sqrt{112}$ {as $h > 0$}



Now $V = \frac{1}{3}(\text{area of base} \times \text{height})$

$= \frac{1}{3}(8 \times 8 \times \sqrt{112}) \text{ cm}^3$

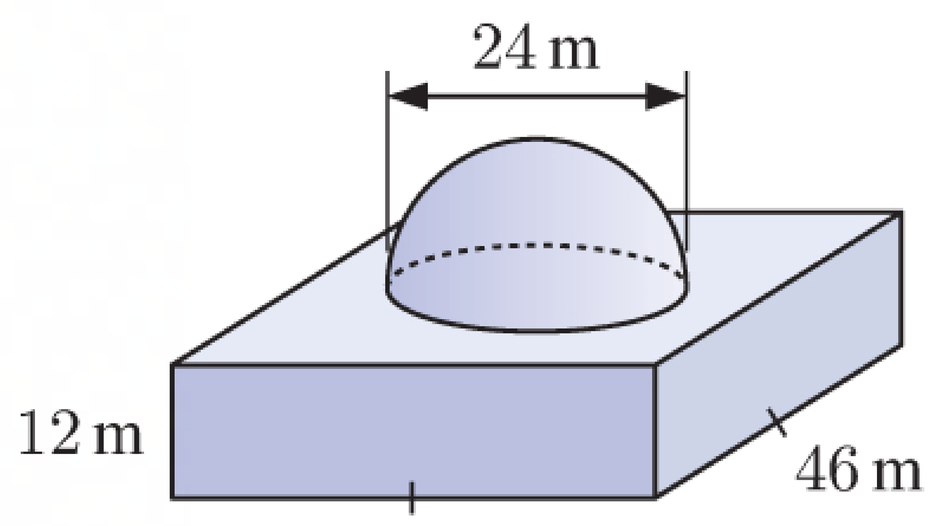
$\approx 226 \text{ cm}^3$

2 **a** V = volume of rectangular prism + volume of hemisphere

$= \text{length} \times \text{width} \times \text{height} + \frac{1}{2} \times \frac{4}{3}\pi r^3$

$= 46 \times 46 \times 12 + \frac{2}{3} \times \pi \times \left(\frac{24}{2}\right)^3$

$\approx 29\,000 \text{ m}^3$

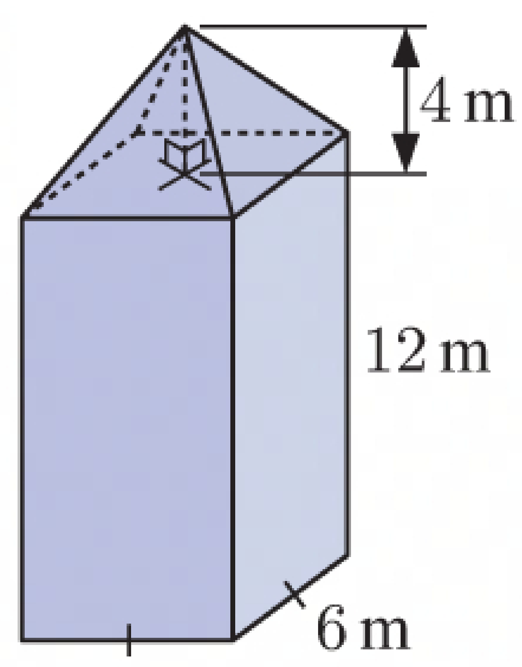


b V = volume of rectangular prism + volume of pyramid

$= \text{length} \times \text{width} \times \text{height} + \frac{1}{3}(\text{area of base} \times \text{height})$

$= 6 \times 6 \times 12 + \frac{1}{3}(6 \times 6 \times 4) \text{ m}^3$

$= 480 \text{ m}^3$

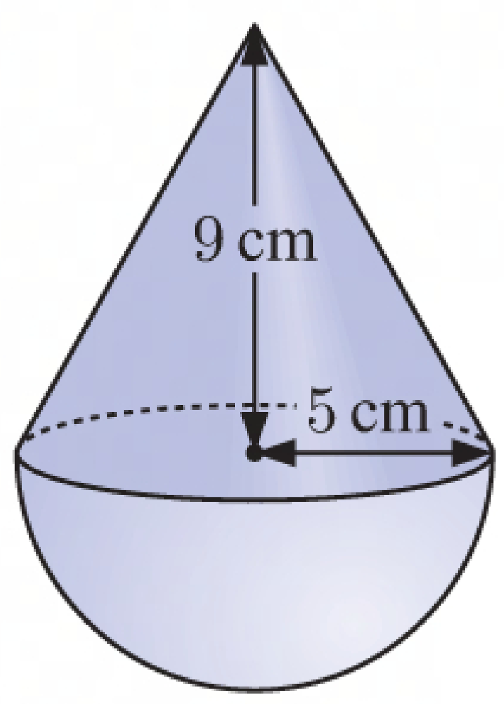


c V = volume of hemisphere + volume of cone

$= \frac{1}{2} \times \frac{4}{3}\pi r^3 + \frac{1}{3}(\text{area of base} \times \text{height})$

$= \frac{2}{3} \times \pi \times 5^3 + \frac{1}{3} \times \pi \times 5^2 \times 9 \text{ cm}^3$

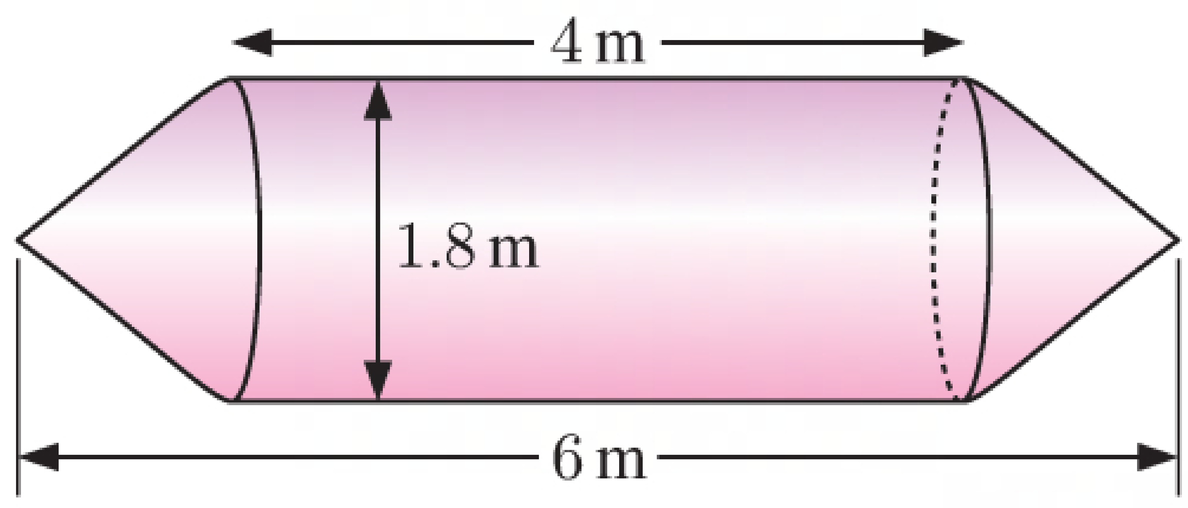
$\approx 497 \text{ cm}^3$



3 **a** Volume of cylinder $= \pi r^2 h$

$$= \pi \times \left(\frac{1.8}{2}\right)^2 \times 4 \text{ m}^3$$

$$\approx 10.179 \text{ m}^3$$



Volume of each conical end $= \frac{1}{3}$(area of base $\times$ height)

$$= \tfrac{1}{3}(\pi r^2 h)$$

$$= \tfrac{1}{3} \times \pi \times \left(\frac{1.8}{2}\right)^2 \times 1 \text{ m}^3$$

$$\approx 0.848 \text{ m}^3$$

Total volume of tanker = volume of cylinder + volume of 2 conical ends

$$\approx 10.179 + 2 \times 0.848 \text{ m}^3$$

$$\approx 11.875 \text{ m}^3$$

$$\approx 11.9 \text{ m}^3$$

So, about 11.9 m^3 of concrete can be held in the tanker.

b If the ends were hemispheres, the end sections would be as long as the radius of the hemisphere.

$\therefore$ total length $= 4 \text{ m} + 2 \times 0.9 \text{ m}$

$$= 5.8 \text{ m}$$

c The two hemispherical ends combine to make one sphere with radius 0.9 m.

Volume of sphere $= \frac{4}{3}\pi r^3$

$$= \tfrac{4}{3} \times \pi \times (0.9)^3 \text{ m}^3$$

$$\approx 3.054 \text{ m}^3$$

Volume of 2 conical ends $\approx 2 \times 0.848 \text{ m}^3$

$$\approx 1.696 \text{ m}^3$$

Difference in volume of ends = volume of sphere − volume of 2 conical ends

$$\approx 3.054 - 1.696 \text{ m}^3$$

$$\approx 1.36 \text{ m}^3$$

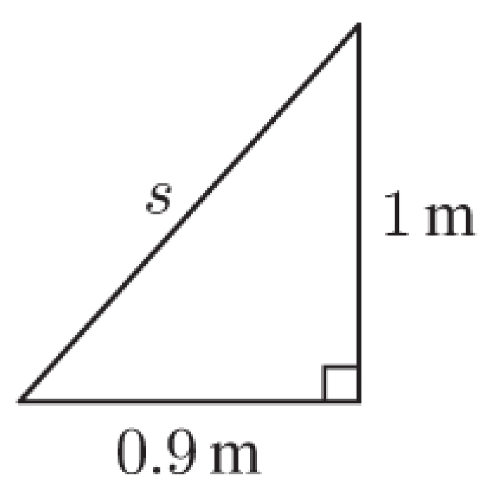
So, the tanker could fit about 1.36 m^3 more concrete if the ends were hemispheres instead of cones.

d Surface area of cylindrical part of tanker $= 2\pi r h$

$$= 2 \times \pi \times 0.9 \times 4 \text{ m}^2$$

$$\approx 22.62 \text{ m}^2$$

i



Let the slant height of the cone be s m.

$s^2 = 1^2 + (0.9)^2$ {Pythagoras}

$\therefore$ $s = \sqrt{1^2 + 0.9^2}$ {as $s > 0$}

$= \sqrt{1.81}$ m

Surface area of 2 conical ends $= 2\pi r s$

$$= 2 \times \pi \times 0.9 \times \sqrt{1.81} \text{ m}^2$$

$$\approx 7.61 \text{ m}^2$$

Total surface area of tanker
$= \text{surface area of cylindrical part} + \text{surface area of 2 conical ends}$
$\approx 22.62 + 7.61 \text{ m}^2$
$\approx 30.2 \text{ m}^2$
So, the surface area of the tanker with conical ends is about 30 m^2.

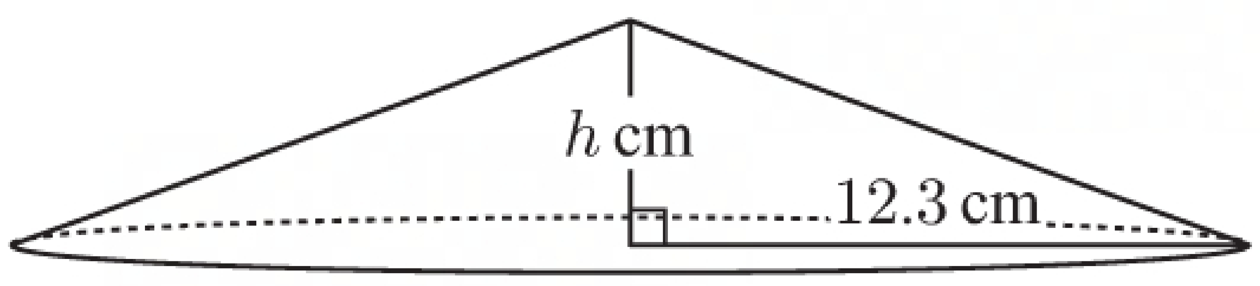
ii The two hemispherical ends combine to make one sphere with radius 0.9 m.

$$\text{Surface area of sphere} = 4\pi r^2 = 4 \times \pi \times (0.9)^2 \text{ m}^2 \approx 10.18 \text{ m}^2$$

Total surface area of tanker
$= \text{surface area of cylindrical part} + \text{surface area of sphere}$
$\approx 22.62 + 10.18 \text{ m}^2$
$\approx 32.8 \text{ m}^2$
So, the surface area of the tanker with hemispherical ends is about 33 m^2.

e The hemispherical ends allow a greater volume to be carried by the tanker. They also allow the length of the vehicle to be shorter. However they have a greater surface area which means they require more steel to manufacture, so they would cost more to produce. This would be a one-off cost however, so for the permanent advantages, the hemispherical design is better.

4 **a**

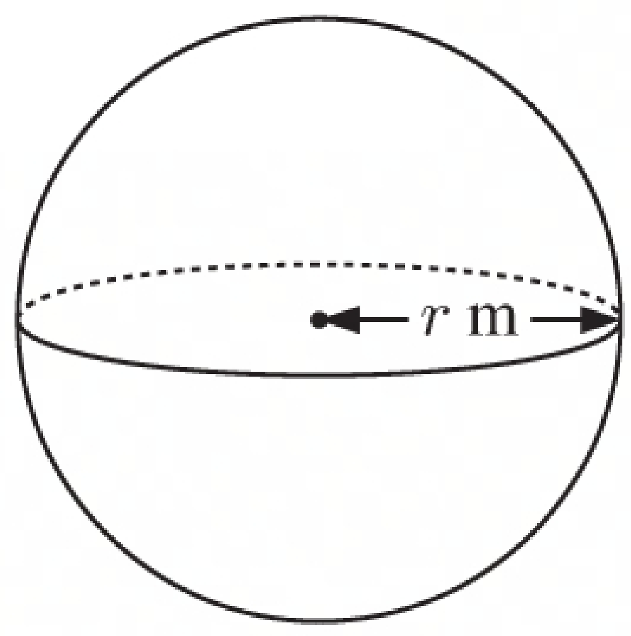


Let the height of the cone be h cm.

$$V = 706 \text{ cm}^3$$
$$\therefore \tfrac{1}{3} \times \pi \times (12.3)^2 \times h = 706$$
$$\therefore 50.43 \times \pi \times h = 706$$
$$\therefore h = \frac{706}{50.43 \times \pi} \approx 4.46$$

The height is approximately 4.46 cm.

b



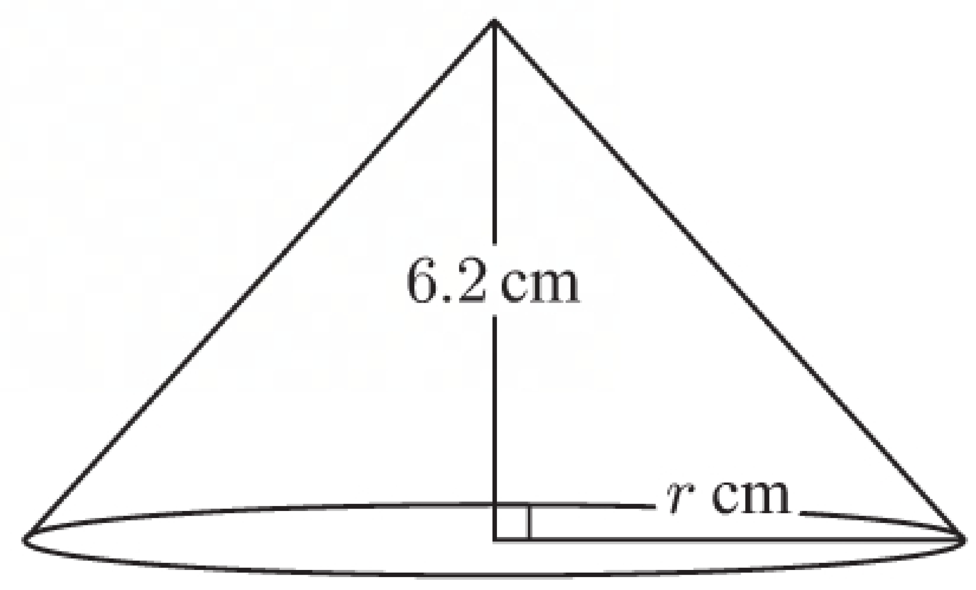
Let the radius be r m.

$$V = 73.62 \text{ m}^3$$
$$\therefore \tfrac{4}{3} \times \pi \times r^3 = 73.62$$
$$\therefore r^3 = \frac{73.62}{\frac{4}{3} \times \pi}$$
$$\therefore r = \sqrt[3]{\frac{73.62}{\frac{4}{3} \times \pi}} \approx 2.60$$

The radius is approximately 2.60 m.

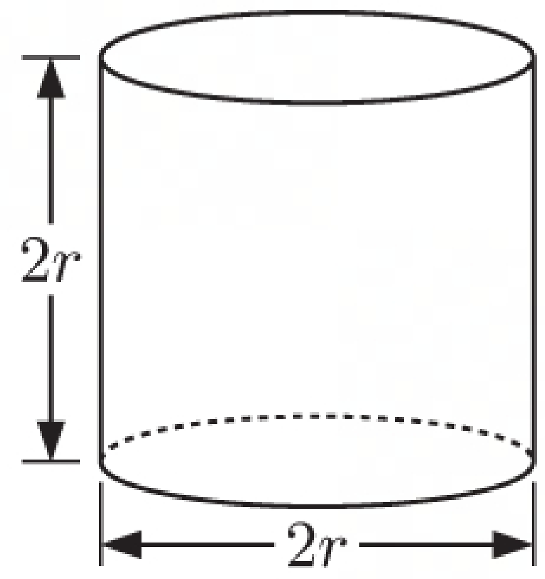
c Let the radius be r cm.

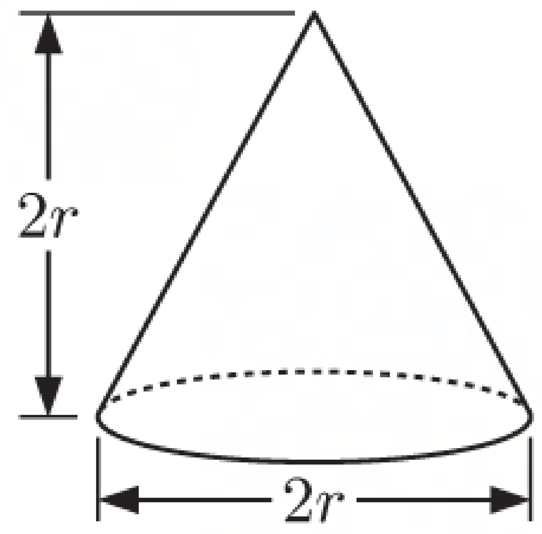
$$V = 203.9 \text{ cm}^3$$
$$\therefore \ \tfrac{1}{3} \times \pi \times r^2 \times 6.2 = 203.9$$
$$\therefore \ r^2 = \frac{203.9}{\frac{1}{3} \times \pi \times 6.2}$$
$$\therefore \ r = \sqrt{\frac{203.9}{\frac{1}{3} \times \pi \times 6.2}} \qquad \{\text{as } r > 0\}$$
$$\approx 5.60$$



The radius is approximately 5.60 cm.

5





Let the height and diameter of the cylinder be $2r$.

$$V = \pi r^2 h$$
$$= \pi \times r^2 \times 2r$$
$$= 2\pi r^3$$

$$V = \tfrac{1}{3}(\text{area of base} \times \text{height})$$
$$= \tfrac{1}{3}(\pi \times r^2 \times 2r)$$
$$= \tfrac{2}{3}\pi r^3$$

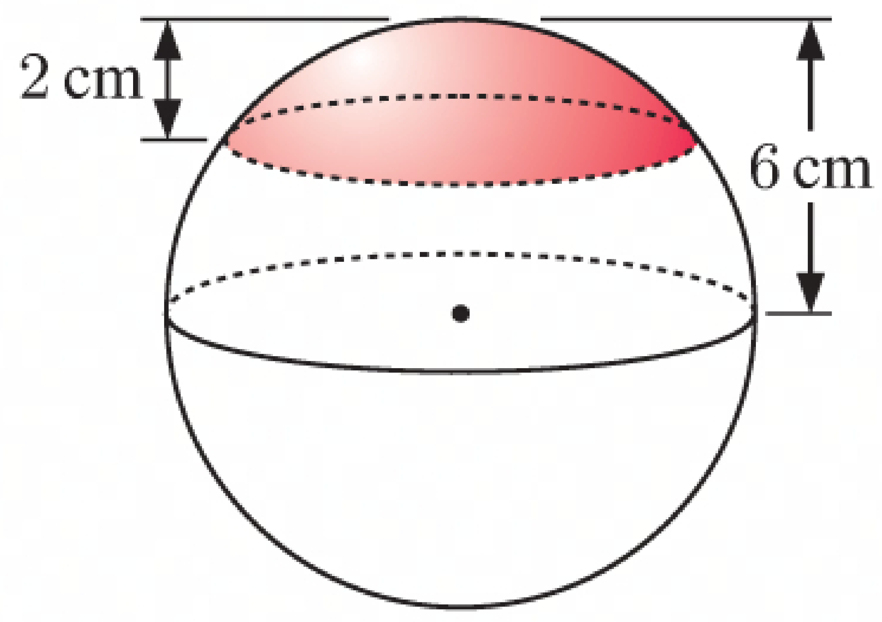
Remaining volume = volume of cylinder − volume of cone

$$= 2\pi r^3 - \tfrac{2}{3}\pi r^3$$
$$= \tfrac{4}{3}\pi r^3$$

which is the volume of a sphere with the same diameter, $2r$.

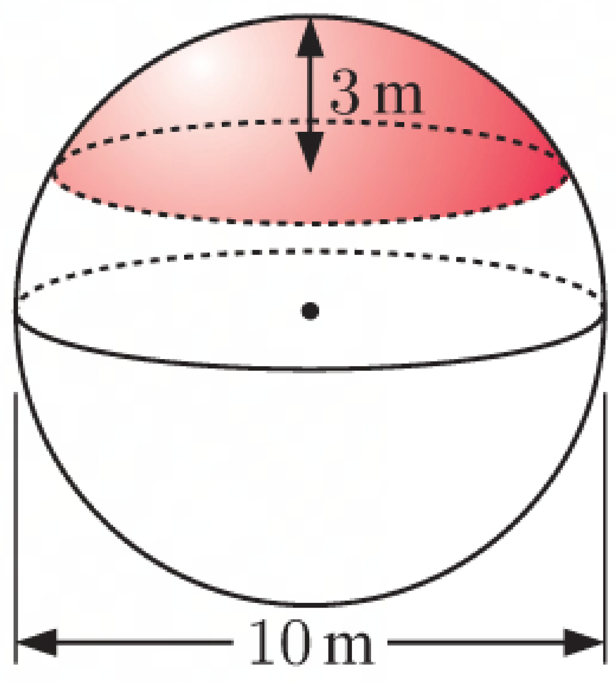
6 a i $V = \dfrac{\pi h^2}{3}(3r - h)$

$$= \frac{\pi \times 2^2}{3}(3 \times 6 - 2) \text{ cm}^3$$
$$= \tfrac{4\pi}{3} \times 16 \text{ cm}^3$$
$$\approx 67.0 \text{ cm}^3$$



ii $V = \dfrac{\pi h^2}{3}(3r - h)$

$$= \frac{\pi \times 3^2}{3}(3 \times 5 - 3) \text{ m}^3$$
$$= 3\pi \times 12 \text{ m}^3$$
$$\approx 113 \text{ m}^3$$



b $$V = \frac{\pi h^2}{3}(3r - h)$$

When $h = r$,
$$\begin{aligned} V &= \frac{\pi r^2}{3}(3r - r) \\ &= \frac{\pi r^2}{3}(2r) \\ &= \tfrac{2}{3}\pi r^3 \\ &= \tfrac{1}{2} \times \tfrac{4}{3}\pi r^3 \\ &= \tfrac{1}{2} \times \text{volume of sphere} \end{aligned}$$

This volume is half the volume of a sphere because when $h = r$, the cap is a hemisphere.

ACTIVITY 1 DENSITY

1 **a**
$$\begin{aligned} \text{Density} &= \frac{\text{mass}}{\text{volume}} \\ &= \frac{10 \text{ g}}{2 \text{ cm}^3} \\ &= 5 \text{ g cm}^{-3} \end{aligned}$$

b
$$\begin{aligned} \text{Volume} &= \text{length} \times \text{width} \times \text{height} \\ &= 2 \times 2 \times 2 \text{ cm}^3 \\ &= 8 \text{ cm}^3 \end{aligned}$$

$$\begin{aligned} \text{Density} &= \frac{\text{mass}}{\text{volume}} \\ &= \frac{10.6 \text{ g}}{8 \text{ cm}^3} \\ &= 1.325 \text{ g cm}^{-3} \end{aligned}$$

c
$$\begin{aligned} \text{Volume} &= \tfrac{4}{3}\pi r^3 \\ &= \tfrac{4}{3} \times \pi \times (4.5)^3 \text{ mm}^3 \\ &\approx 382 \text{ mm}^3 \end{aligned}$$

$$\begin{aligned} \text{Density} &= \frac{\text{mass}}{\text{volume}} \\ &\approx \frac{1.03 \text{ g}}{382 \text{ mm}^3} \\ &\approx 0.002\,70 \text{ g mm}^{-3} \end{aligned}$$

2 $$\text{Density} = \frac{\text{mass}}{\text{volume}}$$

a Mass = density × volume

b $$\text{Volume} = \frac{\text{mass}}{\text{density}}$$

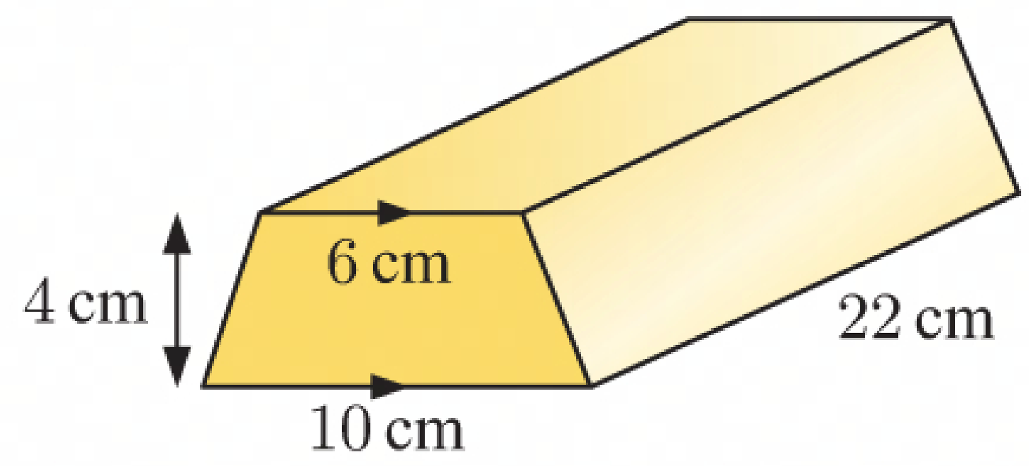
3
$$\begin{aligned} \text{Volume of salt} &= \frac{\text{mass of salt}}{\text{density of salt}} \\ &= \frac{80 \text{ g}}{2.16 \text{ g cm}^{-3}} \\ &\approx 37.0 \text{ cm}^3 \end{aligned}$$

4 $1 \text{ mm} \equiv 0.1 \text{ cm}, \quad 250 \text{ m} \equiv 25\,000 \text{ cm}$

$$\begin{aligned} \therefore \quad \text{volume of copper wire} &= \pi r^2 h \\ &= \pi \times 0.1^2 \times 25\,000 \text{ cm}^3 \\ &\approx 785 \text{ cm}^3 \end{aligned}$$

$$\begin{aligned} \therefore \quad \text{mass of copper wire} &= \text{density of copper} \times \text{volume of copper wire} \\ &\approx 8.96 \text{ g cm}^{-3} \times 785 \text{ cm}^3 \\ &\approx 7040 \text{ g} \end{aligned}$$

5
$$\begin{aligned} \text{Volume of gold bar} &= \text{area of cross-section} \times \text{length} \\ &= \left(\frac{6+10}{2}\right) \times 4 \times 22 \text{ cm}^3 \\ &= 704 \text{ cm}^3 \end{aligned}$$



$$\begin{aligned} \therefore \quad \text{density of gold} &= \frac{\text{mass of gold}}{\text{volume of gold}} \\ &= \frac{13.60 \text{ kg}}{704 \text{ cm}^3} \\ &\approx 0.0193 \text{ kg cm}^{-3} \quad \text{or} \quad 19.3 \text{ g cm}^{-3} \end{aligned}$$

6
$$\begin{aligned} \text{Volume of steel ball} &= \tfrac{4}{3}\pi r^3 \\ &= \tfrac{4}{3}\pi(1.4)^3 \\ &\approx 11.5 \text{ cm}^3 \end{aligned}$$

$$\begin{aligned} \therefore \quad \text{mass of steel ball} &= \text{density of steel} \times \text{volume of steel ball} \\ &\approx 8.05 \text{ g cm}^{-3} \times 11.5 \text{ cm}^3 \\ &\approx 92.5 \text{ g} \end{aligned}$$

$$\begin{aligned} \text{Volume of lead sphere} &= \tfrac{4}{3}\pi r^3 \\ &= \tfrac{4}{3}\pi(1.2)^3 \\ &\approx 7.24 \text{ cm}^3 \end{aligned}$$

$$\begin{aligned} \therefore \quad \text{mass of lead sphere} &= \text{density of lead} \times \text{volume of lead sphere} \\ &\approx 11.34 \text{ g cm}^{-3} \times 7.24 \text{ cm}^3 \\ &\approx 82.1 \text{ g} \end{aligned}$$

$$\frac{\text{mass of steel ball}}{\text{mass of lead sphere}} \approx \frac{92.5 \text{ g}}{82.1 \text{ g}} \approx 1.127$$

$\therefore$ the steel ball weighs $\approx 12.7\%$ more than the lead sphere.

7 Density of water $= 1 \text{ g cm}^{-3}$

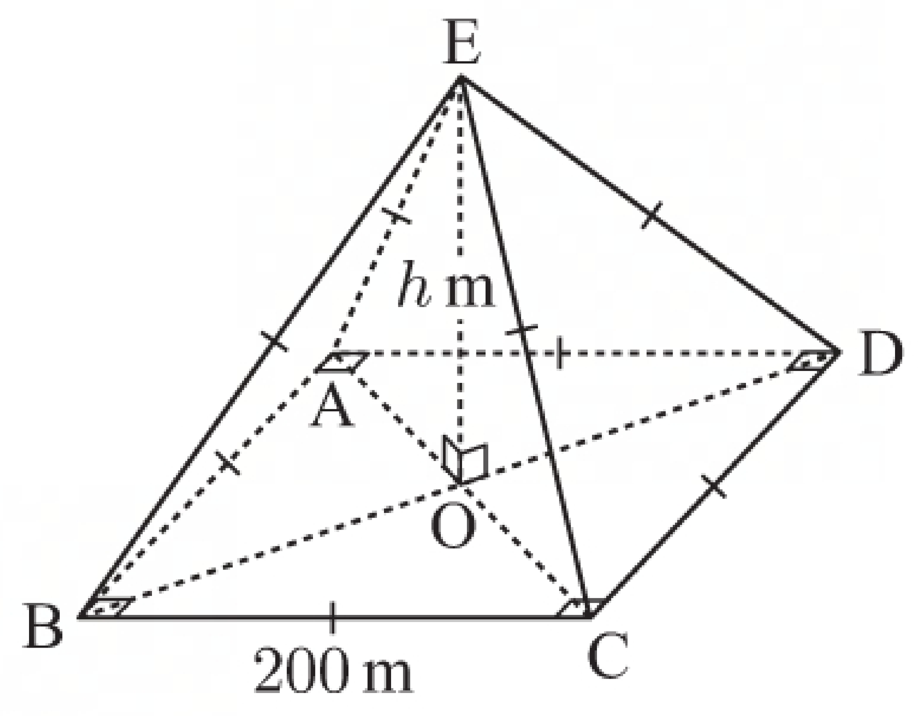
Density of oil $= 0.92 \text{ g cm}^{-3}$

Oil has a lower density than water, so oil will float on water.

8 **a** Density $= \frac{\text{mass}}{\text{volume}}$, so if a heated substance expands, its volume will *increase*, resulting in a *decrease* in density.

b Water in its solid state is ice, which floats in water.

9 Let the height of the pyramid be h m.



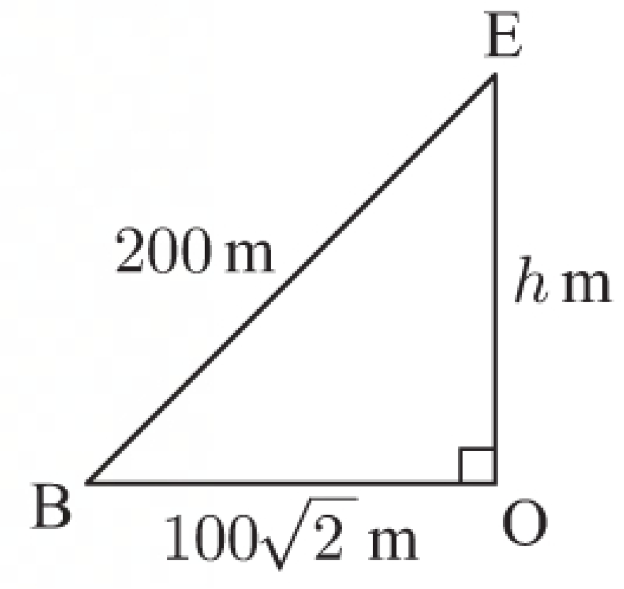
In ΔABD, $\text{BD}^2 = 200^2 + 200^2$ {Pythagoras}

$\therefore \text{BD}^2 = 80\,000$

$\therefore \text{BD} = \sqrt{80\,000}$ {as BD > 0}

$= 200\sqrt{2}$ m

$\therefore \text{BO} = \frac{1}{2}\text{BD} = 100\sqrt{2}$ m



In ΔEBO, $h^2 + (100\sqrt{2})^2 = 200^2$ {Pythagoras}

$\therefore h^2 + 20\,000 = 40\,000$

$\therefore h^2 = 20\,000$

$\therefore h = \sqrt{20\,000}$ {as $h > 0$}

$= 100\sqrt{2}$

$\therefore$ volume of pyramid $= \frac{1}{3}$(area of base $\times$ height)

$= \frac{1}{3}$(length $\times$ width $\times$ height)

$= \frac{1}{3}(200 \times 200 \times 100\sqrt{2})$ m^3

$= \dfrac{4\,000\,000\sqrt{2}}{3}$ m^3

$\therefore$ mass of pyramid = density of stone $\times$ volume of pyramid

$= 2.25 \text{ t m}^{-3} \times \dfrac{4\,000\,000\sqrt{2}}{3}$ m^3

$\approx 4\,240\,000$ t

10 **a** Volume of Uranus $\approx \frac{4}{3}\pi r^3$

$\approx \frac{4}{3} \times \pi \times (2.536 \times 10^7)^3$ m^3

$\approx 6.83 \times 10^{22}$ m^3

b Density of Uranus $= \dfrac{\text{mass of Uranus}}{\text{volume of Uranus}}$

$\approx \dfrac{8.681 \times 10^{25} \text{ kg}}{6.83 \times 10^{22} \text{ m}^3}$

$\approx 1270 \text{ kg m}^{-3}$

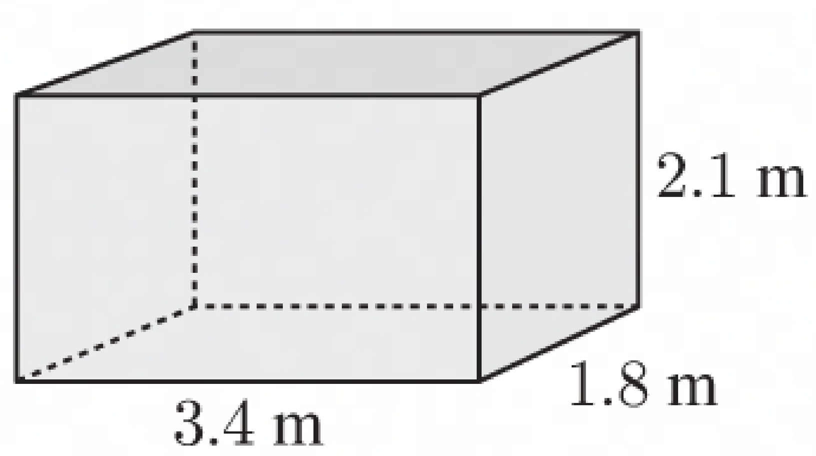
EXERCISE 6D

1 **a** V = length $\times$ width $\times$ height

$= 3.4 \times 1.8 \times 2.1$ m^3

$= 12.852$ m^3

The tank's capacity is 12.852 kL.

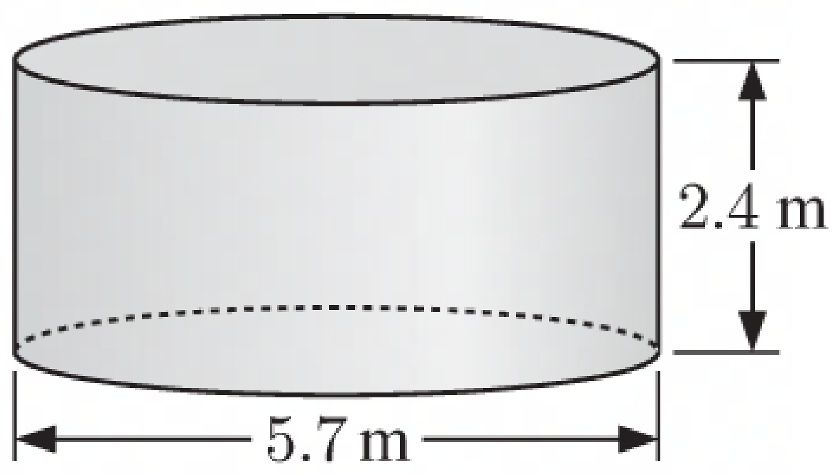


b $V = \pi r^2 h$

$= \pi \times \left(\frac{5.7}{2}\right)^2 \times 2.4 \text{ m}^3$

$\approx 61.2 \text{ m}^3$

The tank's capacity is approximately 61.2 kL.



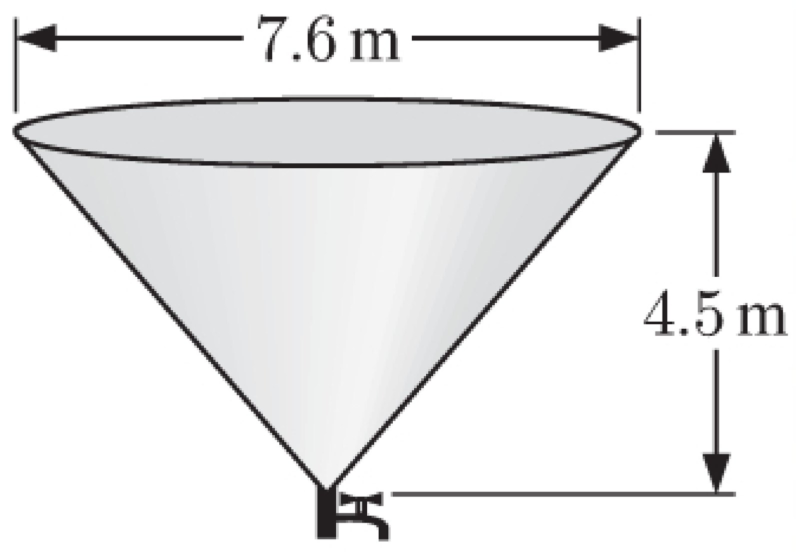
c $V = \frac{1}{3}(\text{area of base} \times \text{height})$

$= \frac{1}{3}(\pi r^2 \times h)$

$= \frac{1}{3} \times \pi \times \left(\frac{7.6}{2}\right)^2 \times 4.5 \text{ m}^3$

$\approx 68.0 \text{ m}^3$

The tank's capacity is approximately 68.0 kL.



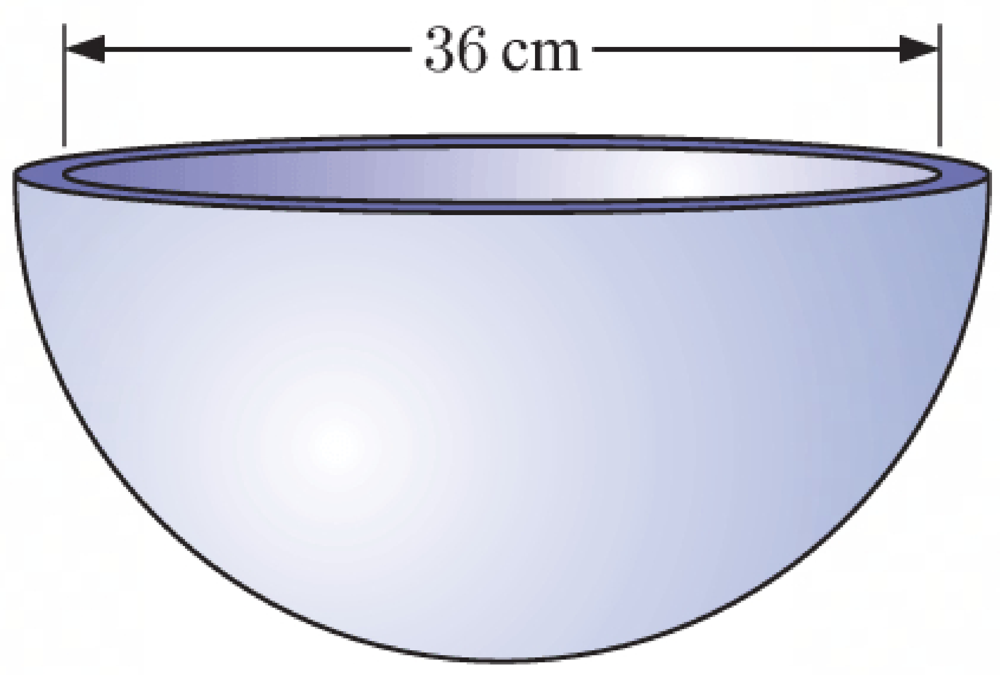
2 **a** $V = \frac{1}{2} \times \text{volume of sphere}$

$= \frac{1}{2} \times \frac{4}{3}\pi r^3$

$= \frac{2}{3} \times \pi \times \left(\frac{36}{2}\right)^3 \text{ cm}^3$

$\approx 12\,200 \text{ cm}^3$

Approximately $12\,200 \text{ cm}^3$ of soup fits in the pot.



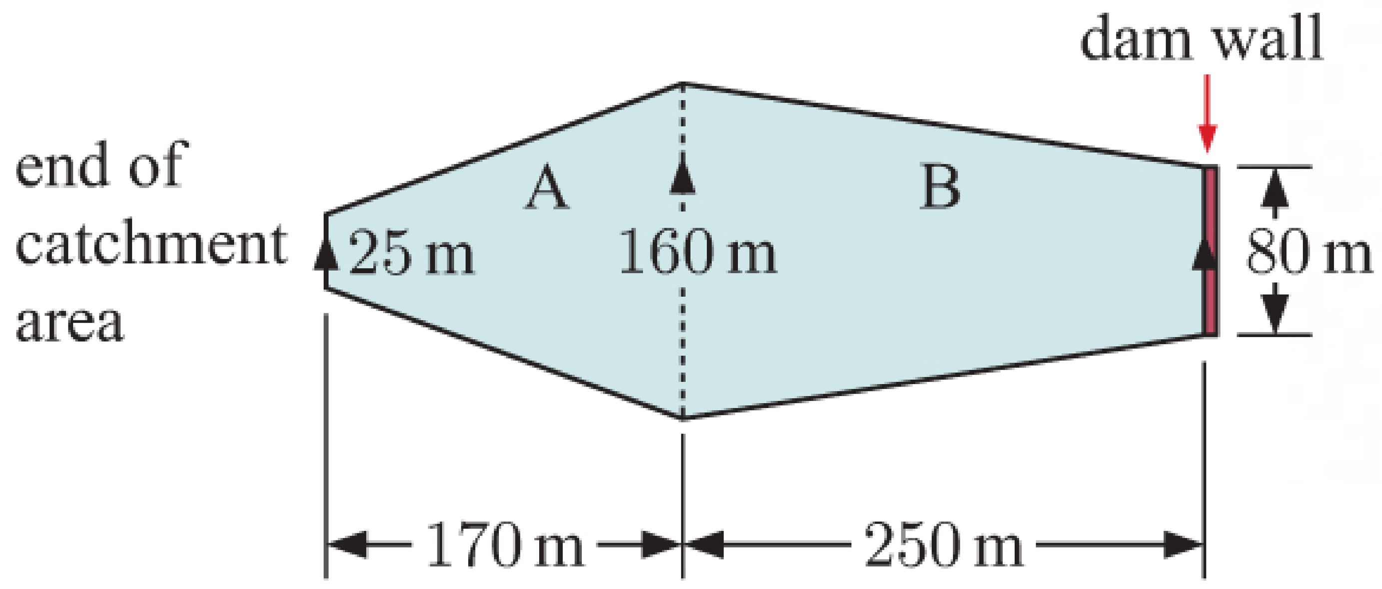
b Capacity $\approx 12\,200$ mL

$\approx (12\,200 \div 1000)$ L

≈ 12.2 L

Approximately 12.2 L of soup fits in the pot.

3



area of trapezium A $= \left(\frac{a+b}{2}\right) \times h$

$= \left(\frac{25+160}{2}\right) \times 170 \text{ m}^2$

$= 15\,725 \text{ m}^2$

area of trapezium B $= \left(\frac{a+b}{2}\right) \times h$

$= \left(\frac{80+160}{2}\right) \times 250 \text{ m}^2$

$= 30\,000 \text{ m}^2$

Total surface area of the reservoir $= 15\,725 + 30\,000 \text{ m}^2$

$= 45\,725 \text{ m}^2$

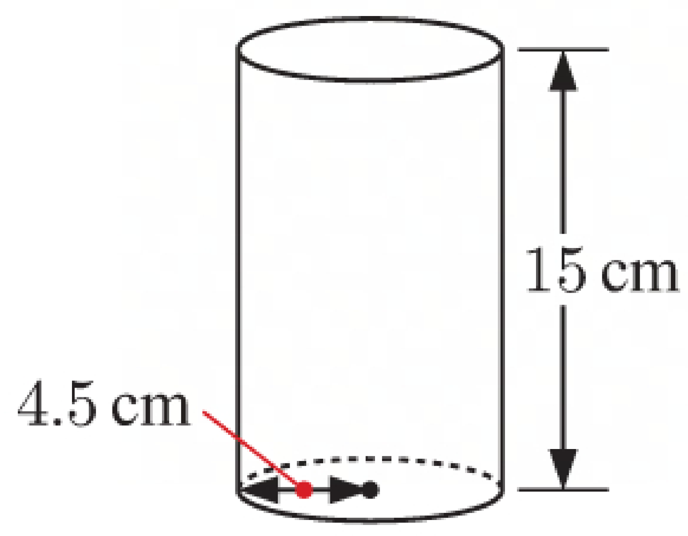
$V = \text{area of cross-section} \times \text{depth}$

$= 45\,725 \times 13 \text{ m}^3$

$= 594\,425 \text{ m}^3$

The capacity of the reservoir is 594 425 kL.

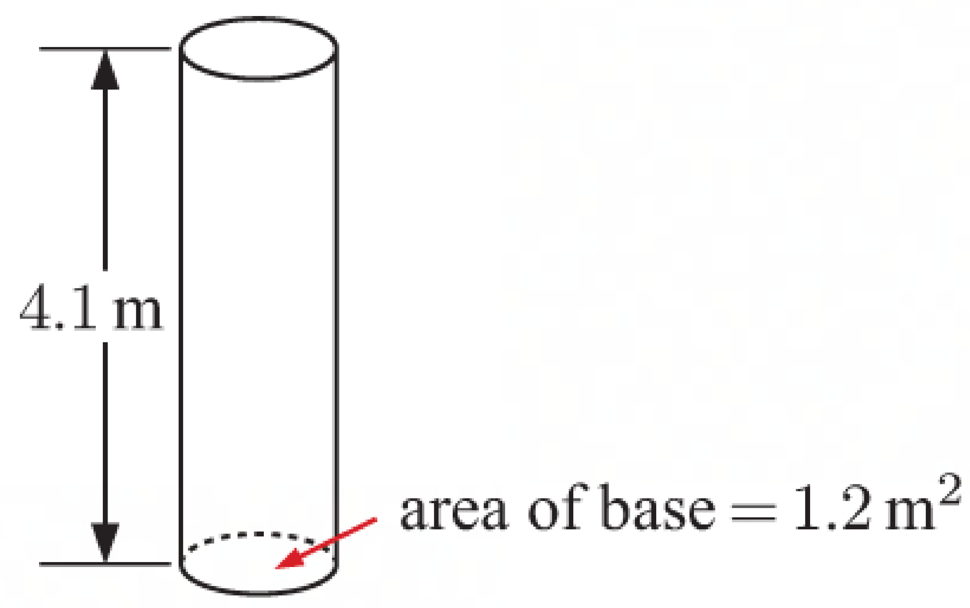
4 **a**



$$V = \pi r^2 h$$
$$= \pi \times (4.5)^2 \times 15 \text{ cm}^3$$
$$\approx 954 \text{ cm}^3$$

The capacity of each tin is approximately 954 mL.

b



$$V = \text{area of base} \times \text{height}$$
$$= 1.2 \times 4.1 \text{ m}^3$$
$$= 4.92 \text{ m}^3$$

The capacity of the mixing vat is 4.92 kL.

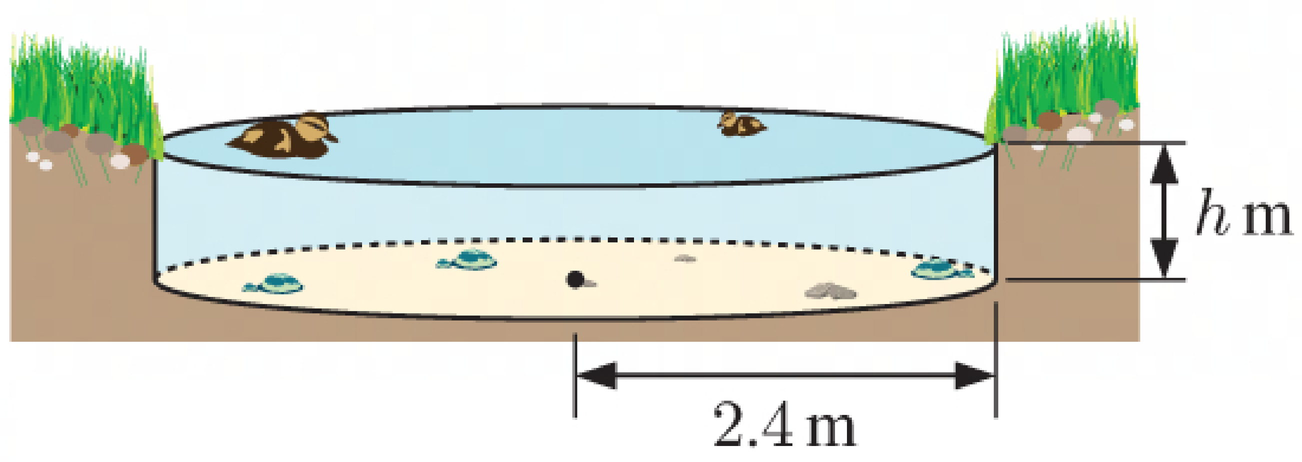
c Number of tins to be filled from one vat $= \dfrac{\text{capacity of vat}}{\text{capacity of one tin}}$

$$\approx \frac{4.92 \text{ kL}}{954 \text{ mL}}$$
$$\approx \frac{(4.92 \times 1000 \times 1000) \text{ mL}}{954 \text{ mL}}$$
$$\approx \frac{4\,920\,000}{954}$$
$$\approx 5155.8$$

So, 5155 tins could be filled from one vat.

d Value of one vat of jam = number of tins × cost per tin
$= 5155 \times \$3.50$
$= \$18\,042.50$

5 $10 \text{ kL} \equiv 10 \text{ m}^3$



Volume of pond = area of base × depth
$= \pi r^2 h$

$\therefore\ 10 = \pi \times (2.4)^2 \times h$

$\therefore\ h = \dfrac{10}{\pi \times (2.4)^2}$

≈ 0.553

The pond is approximately 0.553 m (or $\approx$ 55.3 cm) deep.

6 **a** The area of the roof is in m², so we convert 12 mm to metres.
$12 \text{ mm} = (12 \div 1000) \text{ m} = 0.012 \text{ m}$

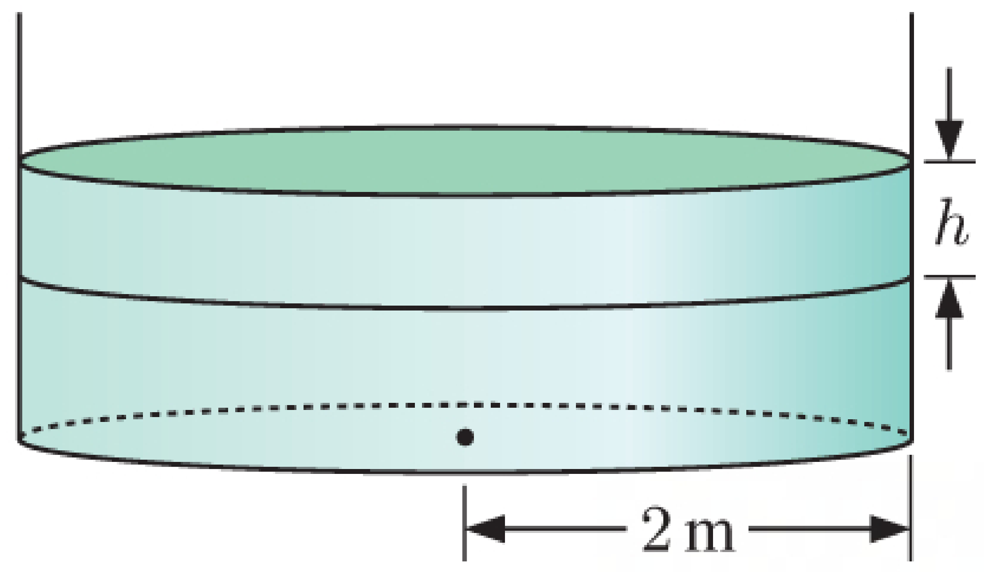
The volume of water which fell on the roof = area of roof × depth
$= 110 \times 0.012 \text{ m}^3$
$= 1.32 \text{ m}^3$

b $1.32 \text{ m}^3 \equiv 1.32 \text{ kL}$, so 1.32 kL of water entered the tank.

c The volume added to the tank = area of base × height

$= \pi \times 2^2 \times h$ m^3

$= 4\pi \times h$ m^3



The volume added to the tank must equal the volume which falls on the roof, so

$4\pi \times h = 1.32$

$\therefore \quad h = \dfrac{1.32}{4\pi}$ {dividing both sides by 4π}

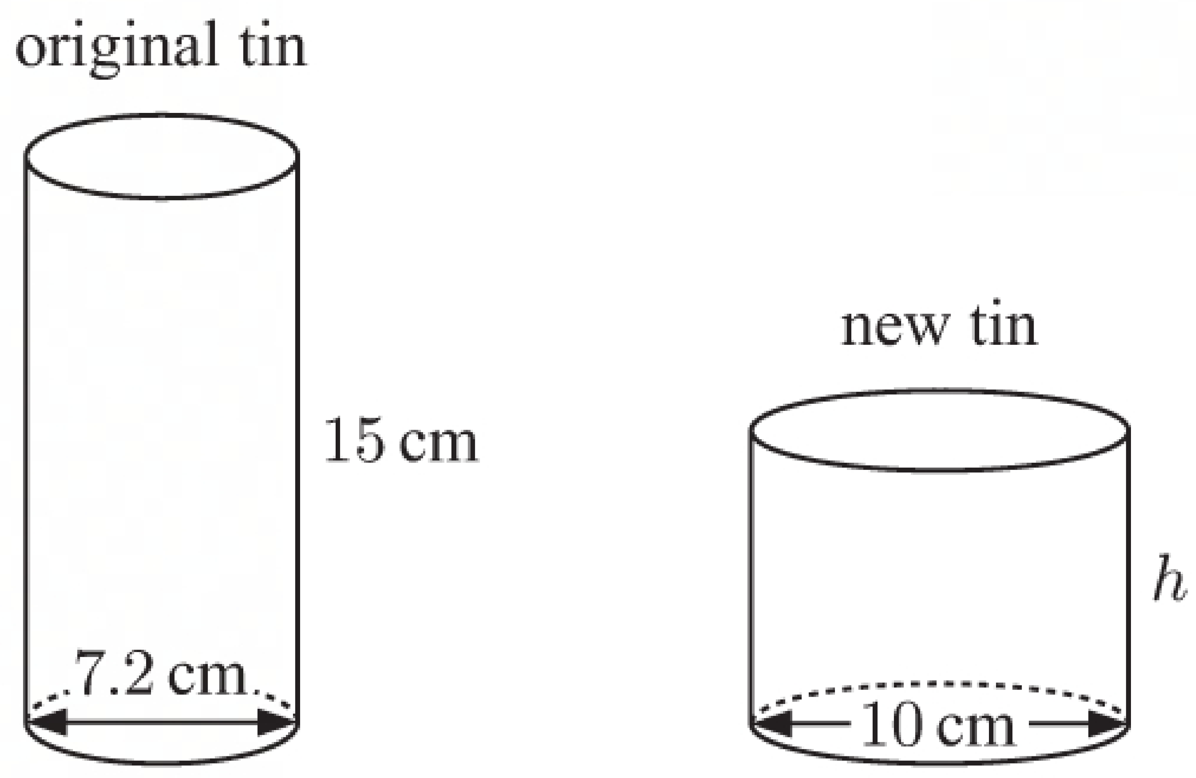
$\therefore \quad h \approx 0.105$ m

The water level rises by about 10.5 cm.

7 Original tin: $V = \pi r^2 h$

$= \pi \times \left(\dfrac{7.2}{2}\right)^2 \times 15$ cm^3

$= \dfrac{972\pi}{5}$ cm^3



New tin: $V = \dfrac{972\pi}{5}$ cm^3

$\therefore \quad \pi \times \left(\dfrac{10}{2}\right)^2 \times h = \dfrac{972\pi}{5}$

$\therefore \quad h = \dfrac{972}{5 \times 5^2}$

$= 7.776$ cm

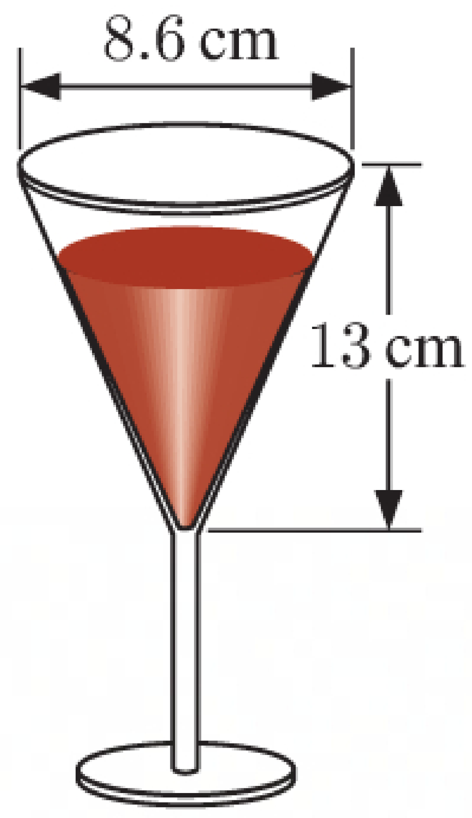
The height of the new tin must be about 7.8 cm.

8 **a** $V = \frac{1}{3}$(area of base × height)

$= \frac{1}{3} \times \pi \times \left(\dfrac{8.6}{2}\right)^2 \times 13$ cm^3

$= \dfrac{24\,037\pi}{300}$ cm^3

≈ 252 cm^3



The capacity of the glass is about 252 mL.

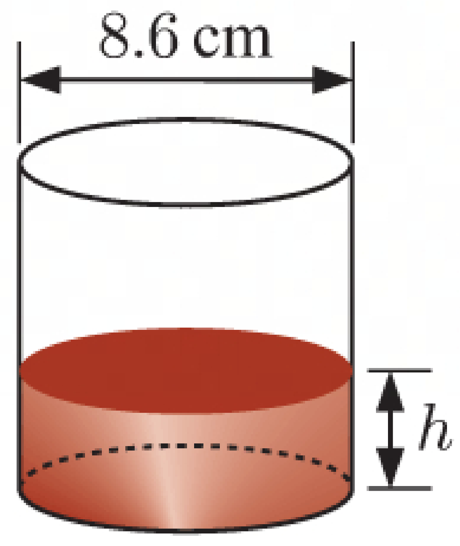
b **i** When 75% full, the glass holds approximately $252 \times 0.75 \approx 189$ mL of wine.

ii $V = \dfrac{24\,037}{300}\pi \times 0.75$ cm^3

$\therefore \quad \pi \times \left(\dfrac{8.6}{2}\right)^2 \times h = \dfrac{24\,037}{300}\pi \times 0.75$

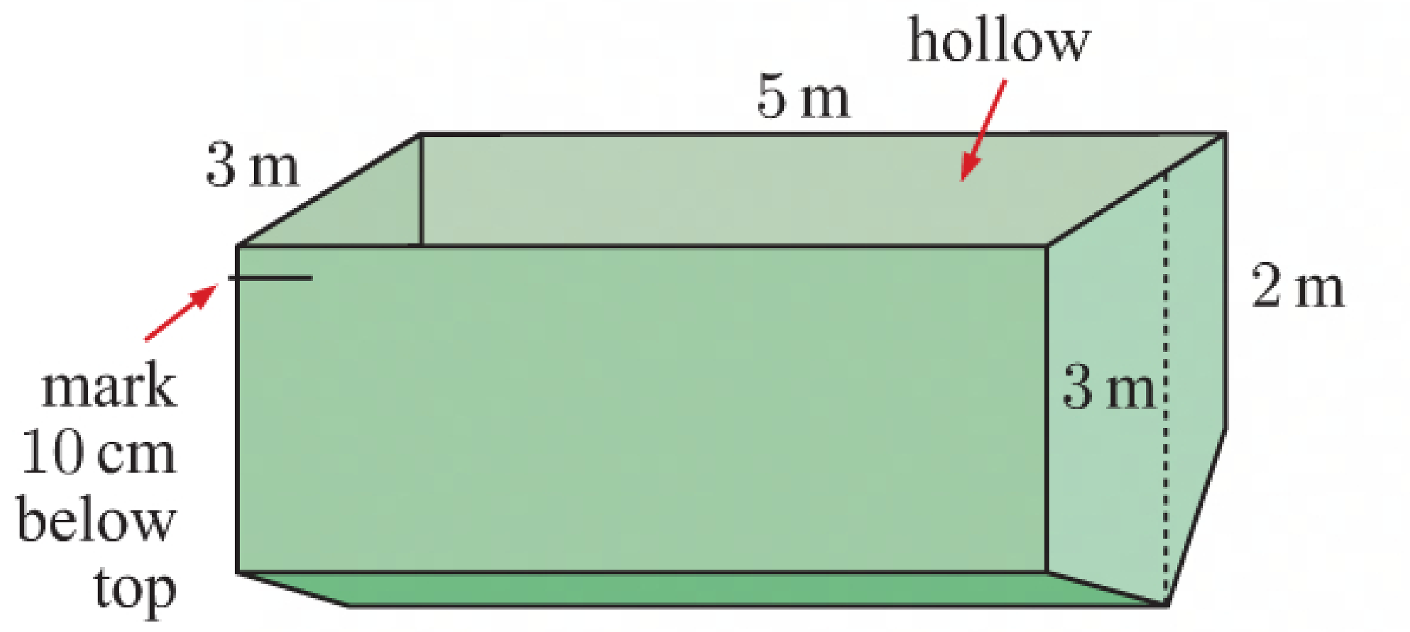
$\therefore \quad h \approx \dfrac{24\,037 \times 0.75}{300 \times (4.3)^2}$

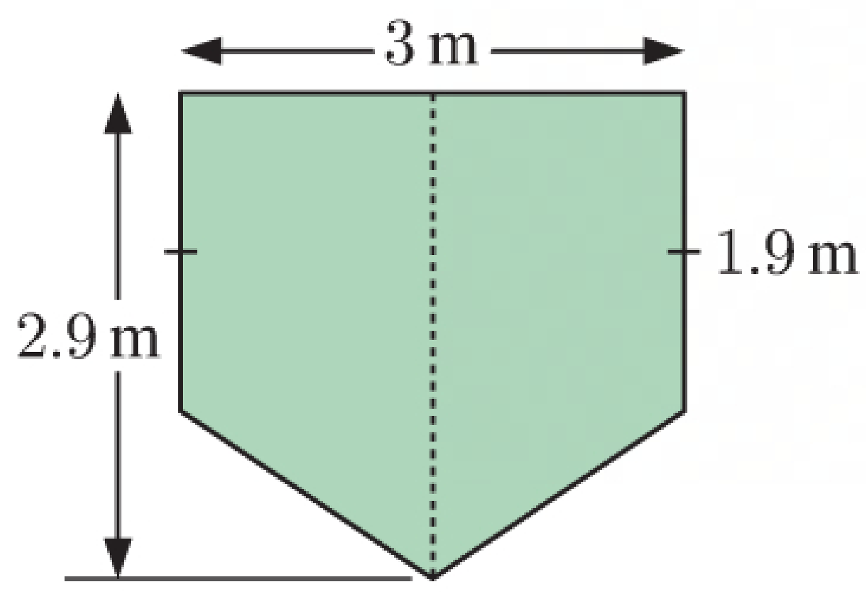
$= 3.25$ cm



The wine will rise 3.25 cm.

9





Subtracting the 10 cm line from the top, the end of the container up to the level which can be filled looks like the diagram shown.

Area of end $= 2 \times$ area of trapezium

$$= 2 \times \left(\left(\frac{1.9 + 2.9}{2} \right) \times 1.5 \right) \text{ m}^2$$

$$= 7.2 \text{ m}^2$$

Volume of wheat in a container $=$ area of end $\times$ length

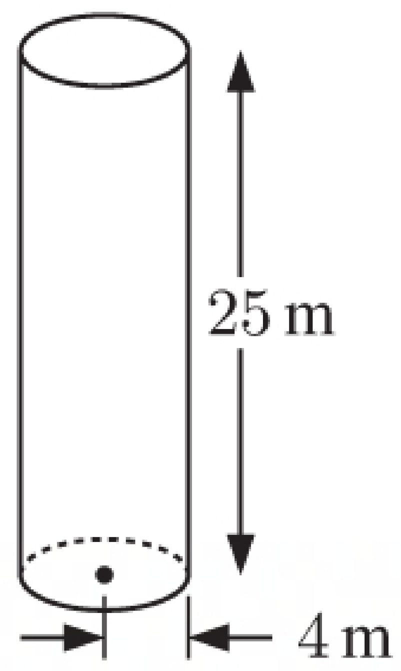
$$= 7.2 \times 5 \text{ m}^3$$

$$= 36 \text{ m}^3$$

Volume of cylindrical silo $= \pi r^2 h$

$$= \pi \times 4^2 \times 25 \text{ m}^3$$

$$\approx 1256.64 \text{ m}^3$$



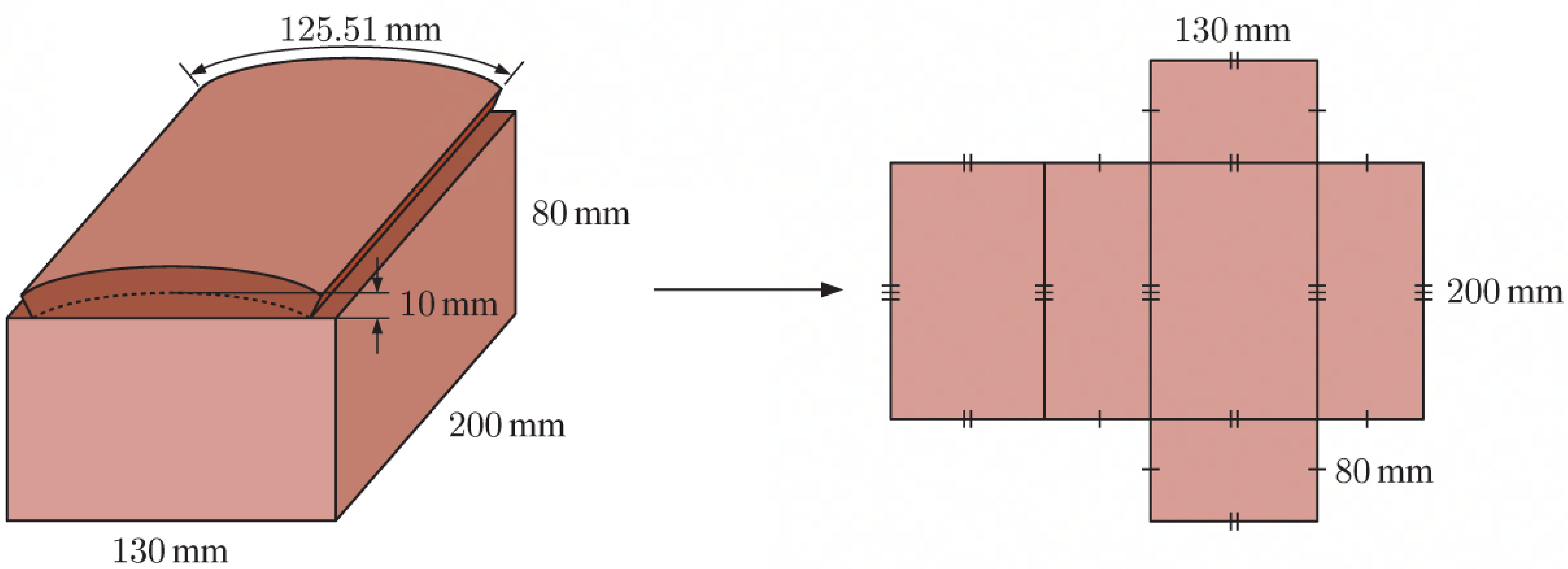
Number of truck loads $= \dfrac{\text{volume of silo}}{\text{volume of container}}$

$$\approx \frac{1256.64}{36}$$

$$\approx 34.9$$

So, 35 truck loads are needed to fill the silo.

10 **a**



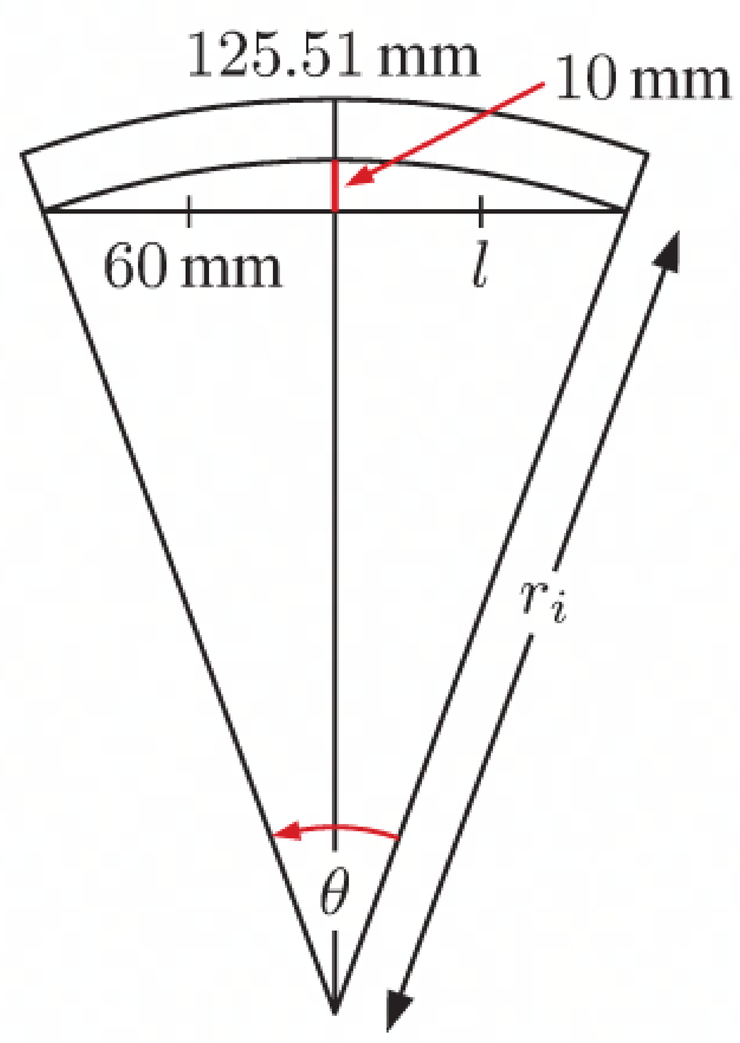
To find the external surface area S_{total} of the jewellery box, we need to calculate:

- the external surface area S_{base} of the rectangular prism base
- the external surface area S_{lid} of the lid.

For the rectangular prism base:

$$S_{\text{base}} = \text{area of top not covered by lid} + \text{area of sides} + \text{area of base}$$
$$= 2 \times 5 \times 200 + 2 \times 80 \times 130 + 2 \times 80 \times 200 + 130 \times 200 \text{ mm}^2$$
$$= 80\,800 \text{ mm}^2$$

For the lid, consider the diagram alongside.
We find the angle θ subtended by the curved edge.



We know that $l = \dfrac{130 - 2 \times 5}{2} = 60$ mm.

Let r_i be the radius for the *internal* arc, and r_e be the radius for the *external* arc.

$$\therefore \quad {r_i}^2 = l^2 + (r_i - 10)^2 \quad \{\text{Pythagoras}\}$$
$$\therefore \quad {r_i}^2 = 60^2 + {r_i}^2 - 20r_i + 100$$
$$\therefore \quad 20r_i = 3700$$
$$\therefore \quad r_i = 185 \text{ mm}$$

So, the radius for the *external* arc is $r_e = r_i + 5$ mm {the wood is 5 mm thick}

$$= 190 \text{ mm}$$

Using the arc length formula, $125.51 = \theta \times 190$

$$\therefore \quad \theta = \left(\frac{125.51}{190}\right)^c$$

So, area of side = area of larger sector − 2 × area of right angled triangle

$$= \frac{190^2 \times \frac{125.51}{190}}{2} - 2 \times \tfrac{1}{2} \times 60 \times (185 - 10)$$
$$= 1423.45 \text{ mm}^2$$

$$\therefore \quad S_{\text{lid}} = \text{area of curved top surface} + 2 \times \text{area of side} + 2 \times \text{area of rectangular sides}$$
$$= 125.51 \times 200 + 2 \times 1423.45 + 2 \times 5 \times 200$$
$$= 29\,948.9 \text{ mm}^2$$

$$\therefore \quad S_{\text{total}} = S_{\text{base}} + S_{\text{lid}}$$
$$= 80\,800 + 29\,948.9 \text{ mm}^2$$
$$= 110\,748.9 \text{ mm}^2$$
$$\approx 111\,000 \text{ mm}^2$$

b It is useful to specify the "external" surface area when talking about a container as the external surface area may be different from the internal surface area.

c **i** To find the total volume V_{total} of the jewellery box, we need to calculate:

- the volume V_{base} enclosed by the rectangular prism base
- the volume V_{lid} enclosed by the lid.

The wood is 5 mm thick, so $V_{\text{base}} = (130 - 2 \times 5) \times (200 - 2 \times 5) \times (80 - 5)$

$$= 1\,710\,000 \text{ mm}^3$$

and $V_{\text{lid}} = (200 - 2 \times 5) \times \left(\dfrac{{r_i}^2\theta}{2} - l(r_i - 10) \right)$

$= (200 - 2 \times 5) \times \left(\dfrac{185^2 \times \frac{125.51}{190}}{2} - 2 \times \frac{1}{2} \times 60 \times (185 - 10) \right)$

$= 152\,789.875 \text{ mm}^3$

$\therefore\ V_{\text{total}} = V_{\text{base}} + V_{\text{lid}}$

$= 1\,710\,000 + 152\,789.875 \text{ mm}^3$

$= 1\,862\,789.875 \text{ mm}^3$

$\approx 1\,860\,000 \text{ mm}^3$

The box can hold approximately $1\,860\,000$ mm^3 of jewellery.

ii Capacity of box $\approx (1\,860\,000 \div 1000)$ mL $\quad \{1 \text{ cm}^3 = 1000 \text{ mm}^3\}$

≈ 1860 mL

≈ 1.86 L

iii To find the total volume W_{total} of wood used to make the box, we need to calculate:

- the volume W_{base} of wood to make the rectangular prism base
- the volume W_{lid} of wood to make the lid.

Using the volumes calculated in **c i**:

$W_{\text{base}} = 130 \times 200 \times 80 - V_{\text{base}}$

$= 2\,080\,000 - 1\,710\,000 \text{ mm}^3$

$= 370\,000 \text{ mm}^3$

$W_{\text{lid}} = 200 \times \left(\dfrac{{r_e}^2\theta}{2} - l(r_i - 10) \right) - V_{\text{lid}}$

$= 200 \times \left(\dfrac{190^2 \times \frac{125.51}{190}}{2} - 2 \times \frac{1}{2} \times 60 \times (185 - 10) \right) - 152\,789.875 \text{ mm}^3$

$= 131\,900.125 \text{ mm}^3$

$\therefore\ W_{\text{total}} = W_{\text{base}} + W_{\text{lid}}$

$= 370\,000 + 131\,900.125 \text{ mm}^3$

$= 501\,900.125 \text{ mm}^3$

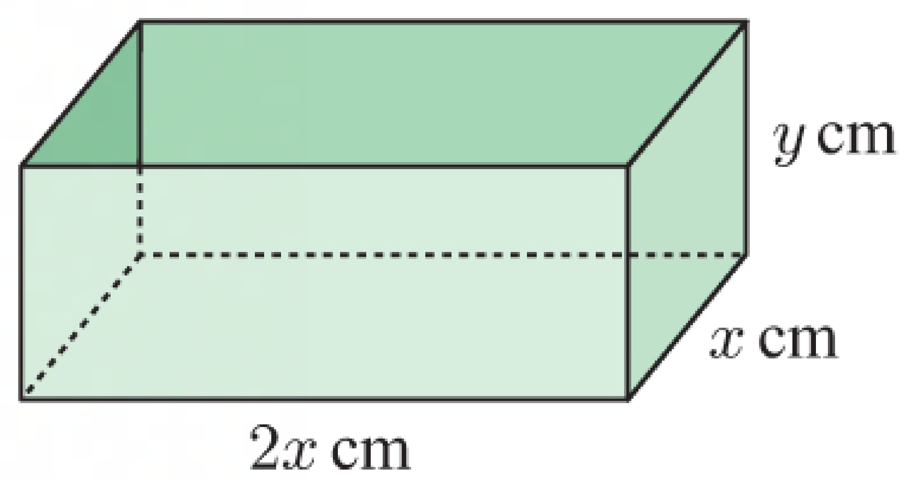
$\approx 502\,000 \text{ mm}^3$

ACTIVITY 2 MINIMISING MATERIAL

1 **a** Volume = length × width × height

$= 2x \times x \times y$

$\therefore\ V = 2x^2y$



b The container must hold exactly 1 litre of fluid.

$1 \text{ L} \equiv 1000 \text{ cm}^3$

$\therefore\ 2x^2y = 1000$

c $2x^2y = 1000$

$\therefore \quad x^2y = 500$

$\therefore \quad y = \dfrac{500}{x^2}$

2 Surface area $= 2 \times$ (area of longer rectangular ends) $+ 2 \times$ (area of shorter rectangular ends) $+$ area of bottom

$= 2 \times (2x \times y) + 2 \times (x \times y) + 2x \times x$

$= 4xy + 2xy + 2x^2$

$\therefore \quad A = 2x^2 + 6xy$

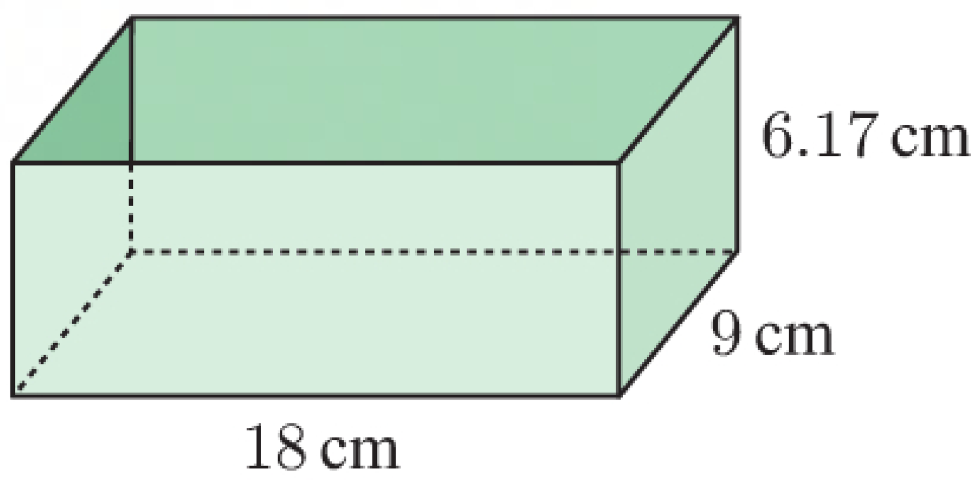
3

	A	B	C
1	x values	y values	A values
2	1	500	3002
3	2	125	1508
4	3	55.555556	1018
5	4	31.25	782
6	5	20	650
7	6	13.888889	572
8	7	10.204082	526.5714286
9	8	7.8125	503
10	9	6.1728395	495.3333333
11	10	5	500

4 The smallest value of A is ≈ 495.33, when $x = 9$.

When $x = 9$, $y = \dfrac{500}{9^2} \approx 6.17$

The dimensions of the box that your boss desires are shown alongside.

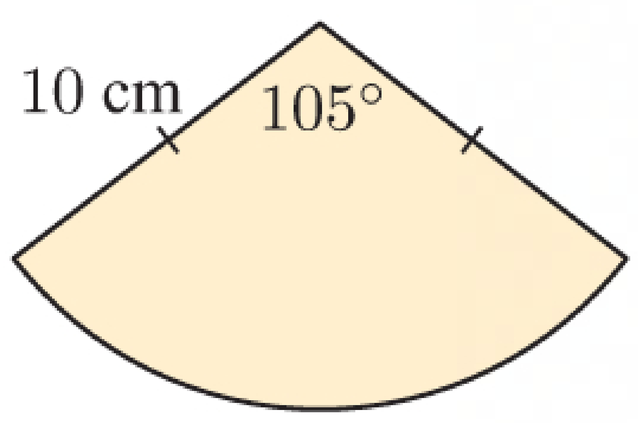


REVIEW SET 6A

1 **a** Arc length $= \dfrac{\theta}{360} \times 2\pi r$

$= \frac{105}{360} \times 2\pi \times 10$ cm

≈ 18.3 cm



b Perimeter $= 2r +$ arc length

$\approx 2 \times 10 + 18.3$ cm

≈ 38.3 cm

c Area $= \dfrac{\theta}{360} \times \pi r^2$

$= \frac{105}{360} \times \pi \times 10^2$

$\approx 91.6\ \text{cm}^2$

2

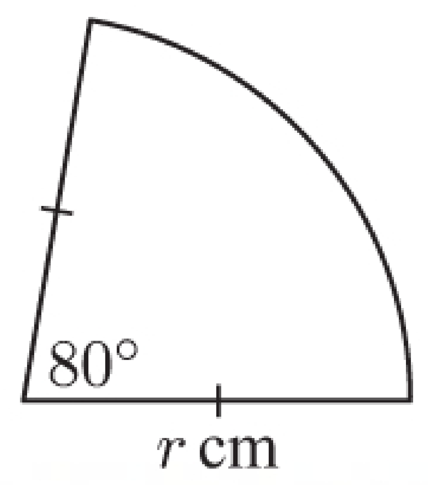
$$\text{Area} = \frac{\theta}{360} \times \pi r^2$$

$$\therefore\ 24\pi = \frac{80}{360} \times \pi \times r^2$$

$$\therefore\ r^2 = \frac{24\pi}{\frac{80}{360} \times \pi}$$

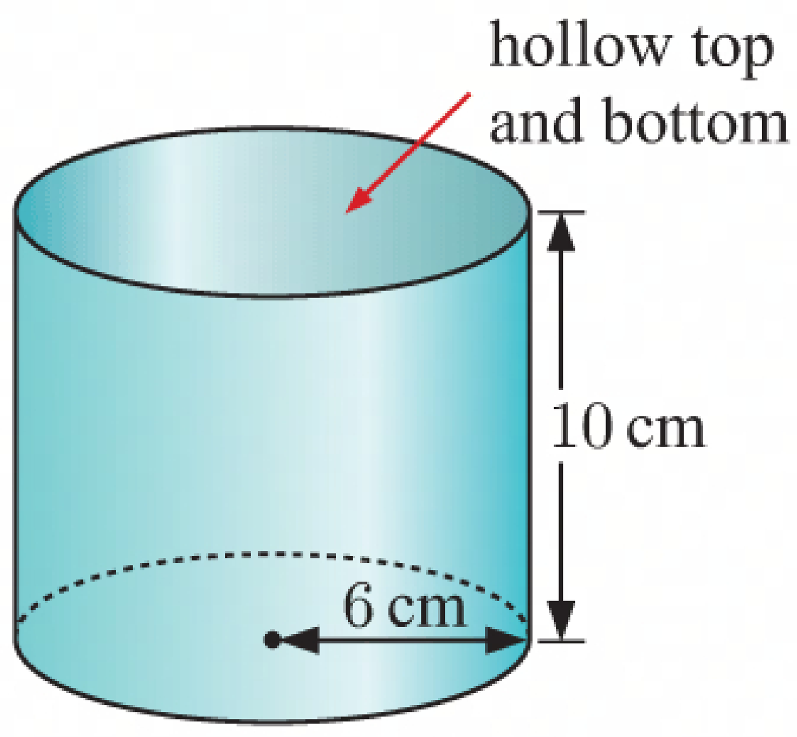
$$\therefore\ r = \sqrt{108} \quad \{\text{as } r > 0\}$$

$$\approx 10.4$$



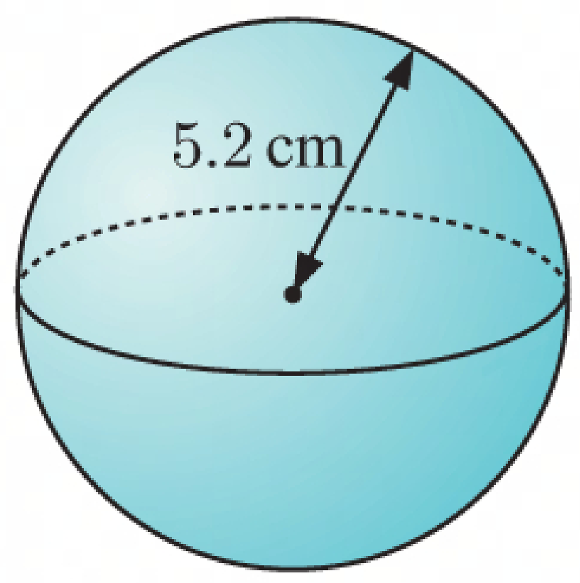
The radius of the sector is approximately 10.4 cm.

3 **a**



$$\text{Surface area} = 2\pi r h$$
$$= 2 \times \pi \times 6 \times 10 \text{ cm}^2$$
$$\approx 377.0 \text{ cm}^2$$

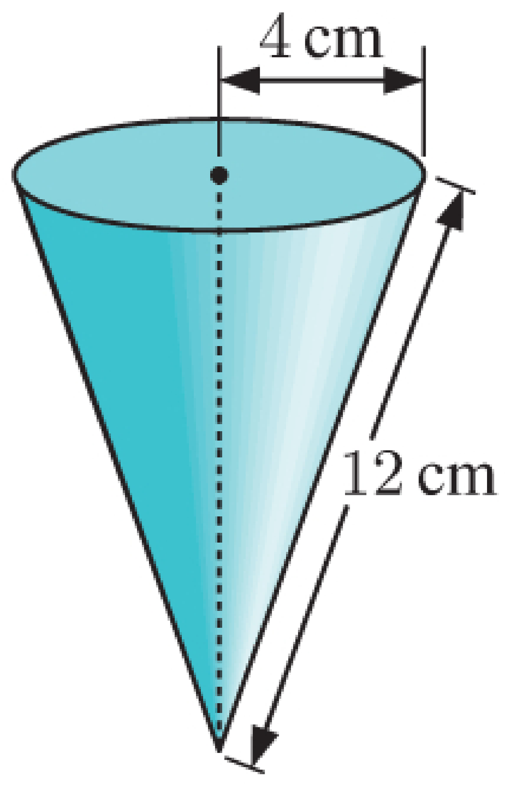
b



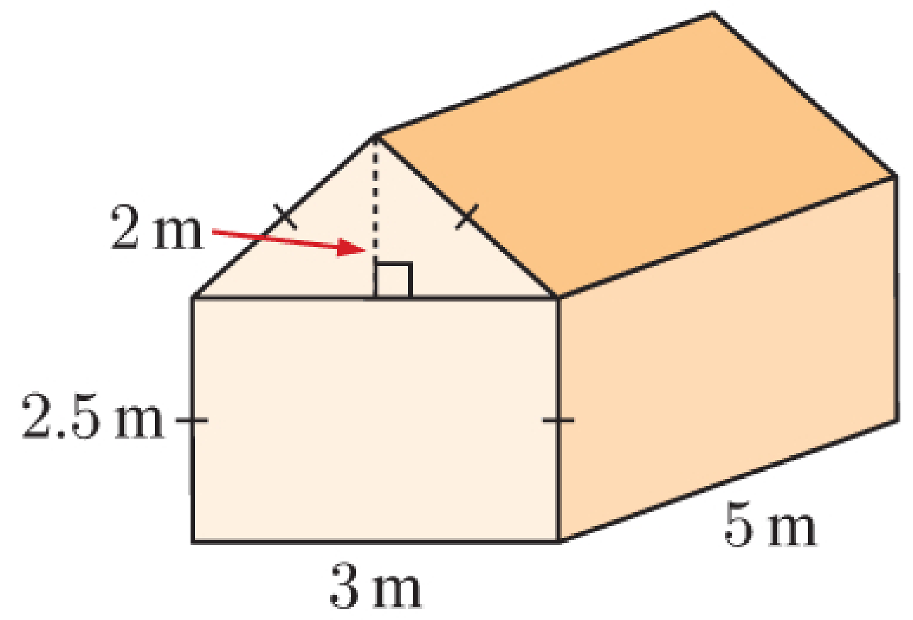
$$\text{Surface area} = 4\pi r^2$$
$$= 4 \times \pi \times (5.2)^2 \text{ cm}^2$$
$$\approx 339.8 \text{ cm}^2$$

c

$$\text{Surface area} = \pi r s + \pi r^2$$
$$= \pi \times 4 \times 12 + \pi \times 4^2 \text{ cm}^2$$
$$\approx 201.1 \text{ cm}^2$$



4 **a**



Surface area of shed

= area of 2 rectangular ends + area of 4 rectangular sides + area of 2 triangular ends

$= 2 \times (3 \times 2.5) + 4 \times (5 \times 2.5) + 2 \times (\frac{1}{2} \times 3 \times 2)$ m^2

$= 71$ m^2

b Since the shed is to be painted with two coats of zinc-alum, we need enough zinc-alum to cover an area of $2 \times 71 = 142\ \text{m}^2$.

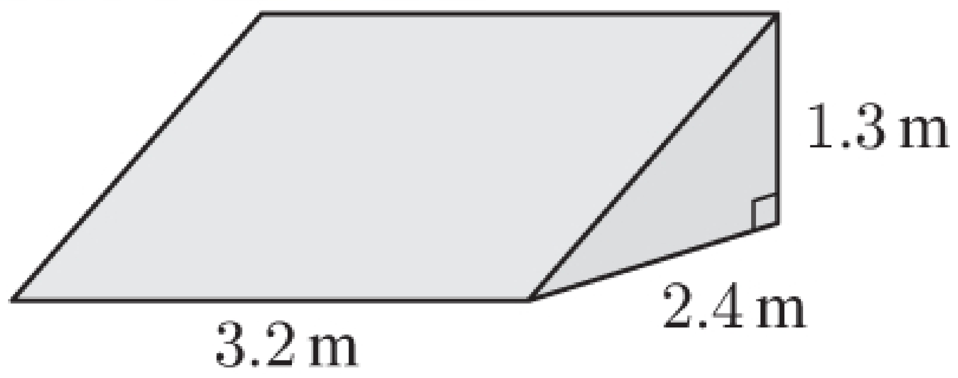
The zinc-alum covers $5\ \text{m}^2$ per litre.

$$\begin{aligned}\text{Number of litres of zinc-alum needed} &= \frac{\text{total area to be painted}}{\text{area covered per litre}}\\ &= \frac{142\ \text{m}^2}{5\ \text{m}^2\ \text{L}^{-1}}\\ &= 28.4\ \text{L}\end{aligned}$$

Since the zinc-alum must be purchased in whole litres, we need to purchase 29 L of zinc-alum.

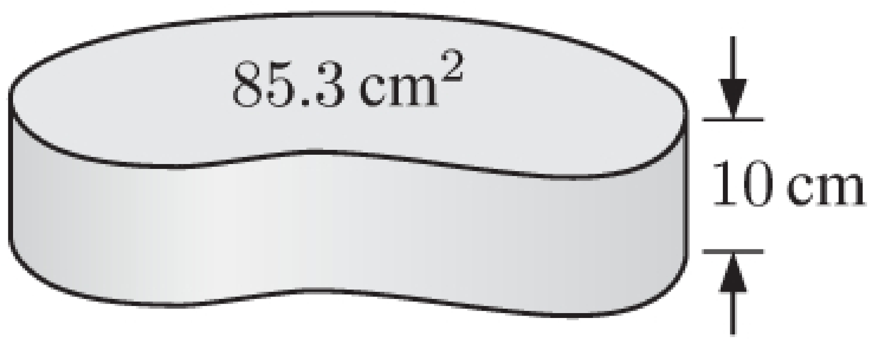
$$\begin{aligned}\text{Total cost of the zinc-alum} &= \text{number of litres to be purchased} \times \text{cost per litre}\\ &= 29\ \text{L} \times \$8.25/\text{L}\\ &= \$239.25\end{aligned}$$

5 **a**



$$\begin{aligned}V &= \text{area of end} \times \text{length}\\ &= \tfrac{1}{2} \times 2.4 \times 1.3 \times 3.2\ \text{m}^3\\ &= 4.992\ \text{m}^3\\ &\approx 4.99\ \text{m}^3\end{aligned}$$

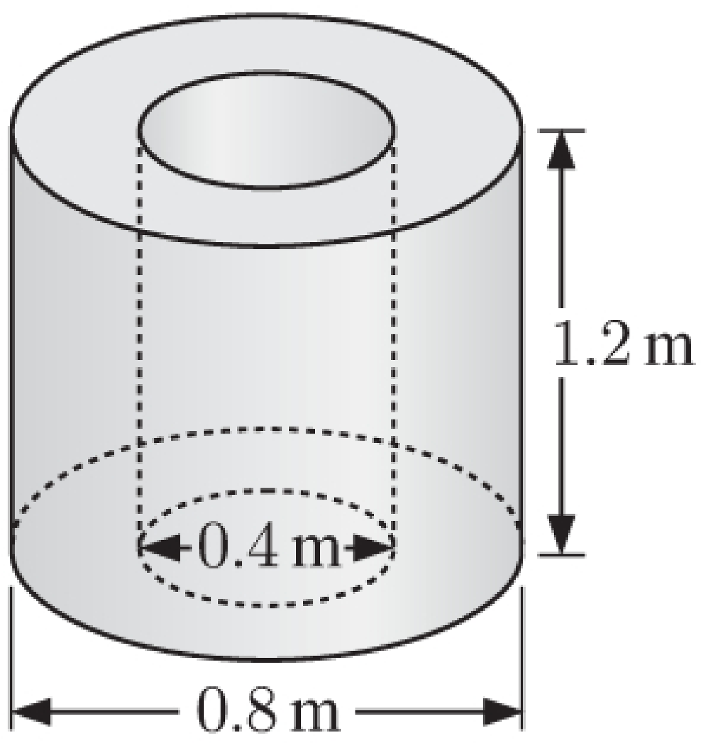
b



$$\begin{aligned}V &= \text{area of end} \times \text{height}\\ &= 85.3 \times 10\ \text{cm}^3\\ &= 853\ \text{cm}^3\end{aligned}$$

c

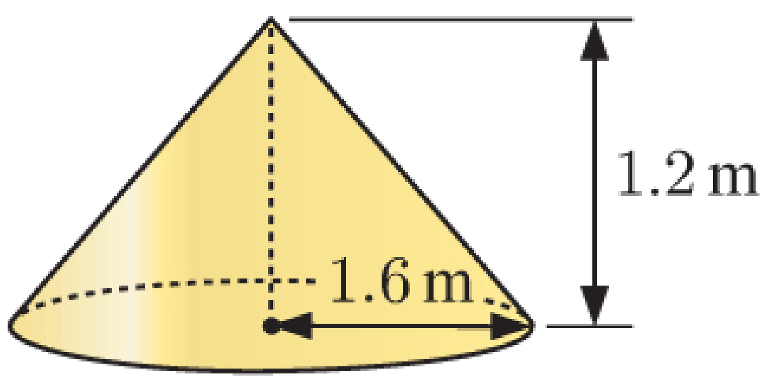
$$\begin{aligned}V &= \text{volume of external cylinder} - \text{volume of internal cylinder}\\ &= \pi R^2 h - \pi r^2 h\\ &= \pi \times \left(\frac{0.8}{2}\right)^2 \times 1.2 - \pi \times \left(\frac{0.4}{2}\right)^2 \times 1.2\ \text{m}^3\\ &\approx 0.452\ \text{m}^3\end{aligned}$$



6

$$\begin{aligned}\text{Volume of cone} &= \tfrac{1}{3}(\text{area of base} \times \text{height})\\ &= \tfrac{1}{3} \times \pi \times (1.6)^2 \times 1.2\ \text{m}^3\\ &\approx 3.22\ \text{m}^3\end{aligned}$$

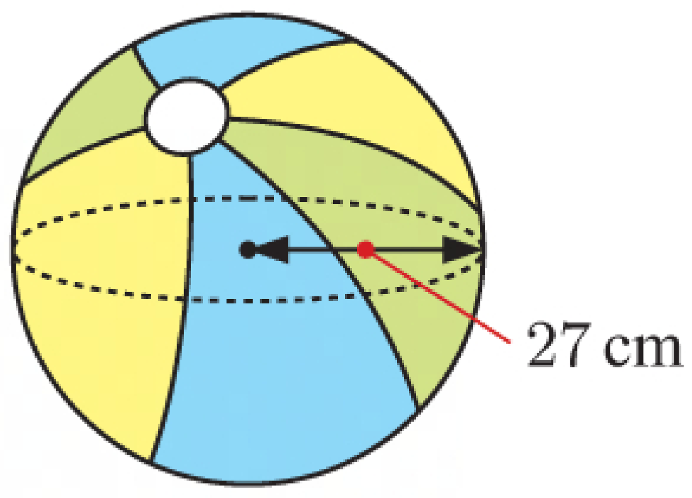
Tom has had approximately $3.22\ \text{m}^3$ of sand delivered.



7

$$\begin{aligned}\text{Volume of sphere} &= \tfrac{4}{3}\pi r^3\\ &= \tfrac{4}{3} \times \pi \times 27^3\ \text{cm}^3\\ &\approx 82\,400\ \text{cm}^3\end{aligned}$$

So, the beach ball has volume of approximately $82\,400\ \text{cm}^3$.



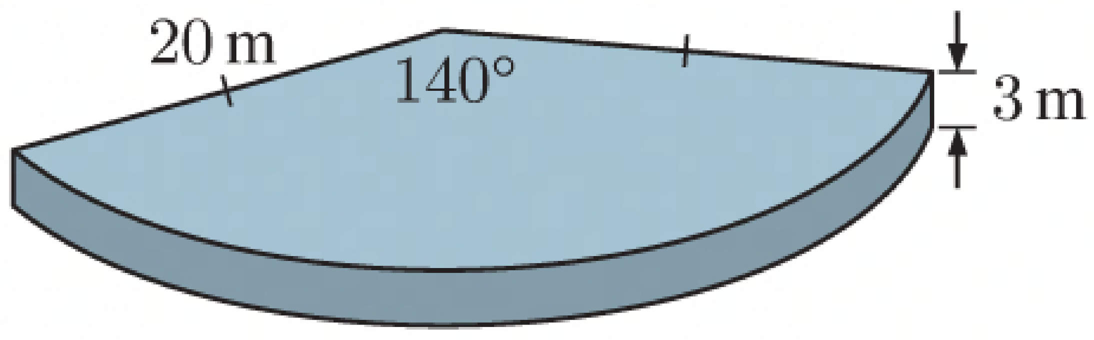
8 $V = \text{area of end} \times \text{height}$

$= \text{area of sector} \times \text{height}$

$= \left(\frac{\theta}{360} \times \pi r^2\right) \times \text{height}$

$= \frac{140}{360} \times \pi \times 20^2 \times 3$

$\approx 1470 \text{ m}^3$



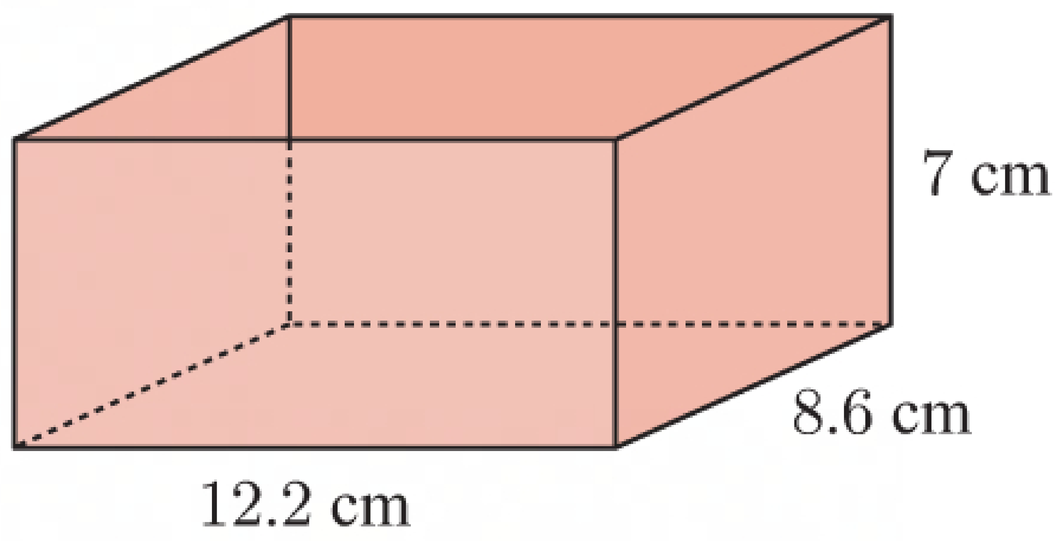
So, the volume of material required is approximately 1470 m^3.

9 **a** $V = \text{length} \times \text{width} \times \text{height}$

$= 12.2 \times 8.6 \times 7 \text{ cm}^3$

$= 734.44 \text{ cm}^3$

The capacity is 734.44 mL.

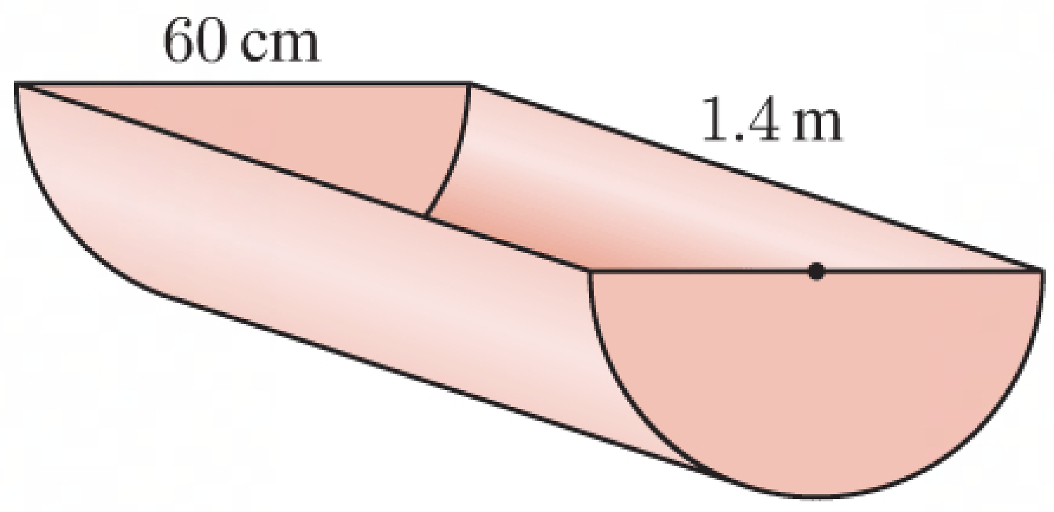


b $V = \text{area of end} \times \text{length}$

$= \frac{1}{2} \times \pi r^2 \times \text{length}$

$= \frac{1}{2} \times \pi \times \left(\frac{60}{2}\right)^2 \times 140 \text{ cm}^3 \quad \{1.4 \text{ m} \equiv 140 \text{ cm}\}$

$\approx 198\,000 \text{ cm}^3$



The capacity is approximately 198 000 mL or 198 L.

10 The dimensions of the roof are in m, so we convert 15.4 mm to metres.

$15.4 \text{ mm} = (15.4 \div 1000) \text{ m} = 0.0154 \text{ m}$

The volume of water collected by the roof $= \text{area of roof} \times \text{depth}$

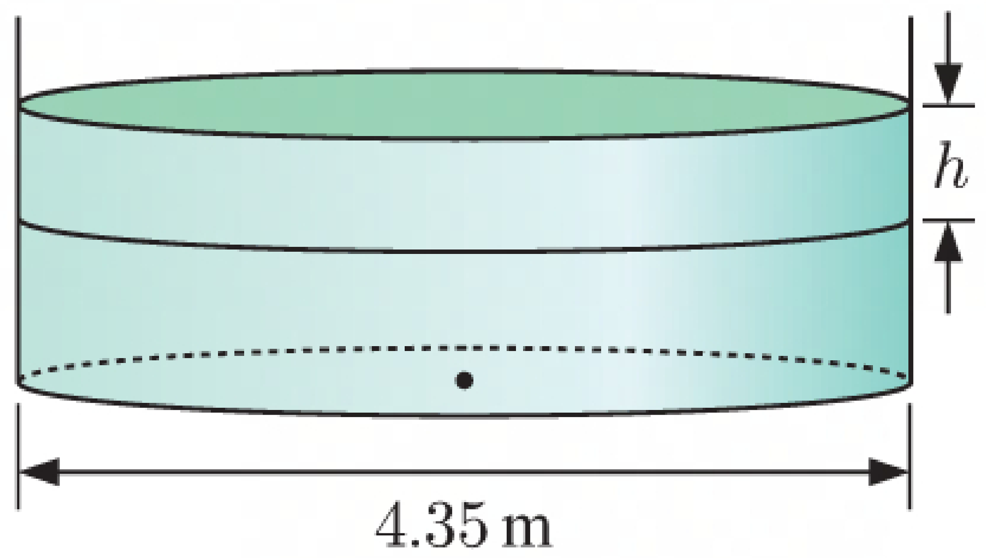
$= 12 \times 5.5 \times 0.0154 \text{ m}^3$

$= 1.0164 \text{ m}^3$

The volume added to the tank $= \text{area of base} \times \text{height}$

$= \pi \times \left(\frac{4.35}{2}\right)^2 \times h \text{ m}^3$

$= 4.6225\pi \times h \text{ m}^3$



The volume added to the tank must equal the volume which falls on the roof, so

$4.6225\pi \times h = 1.0164$

$\therefore \quad h = \frac{1.0164}{4.6225\pi}$

$\therefore \quad h \approx 0.0684 \text{ m}$

So, the level in the tank rises by about 68.4 mm.

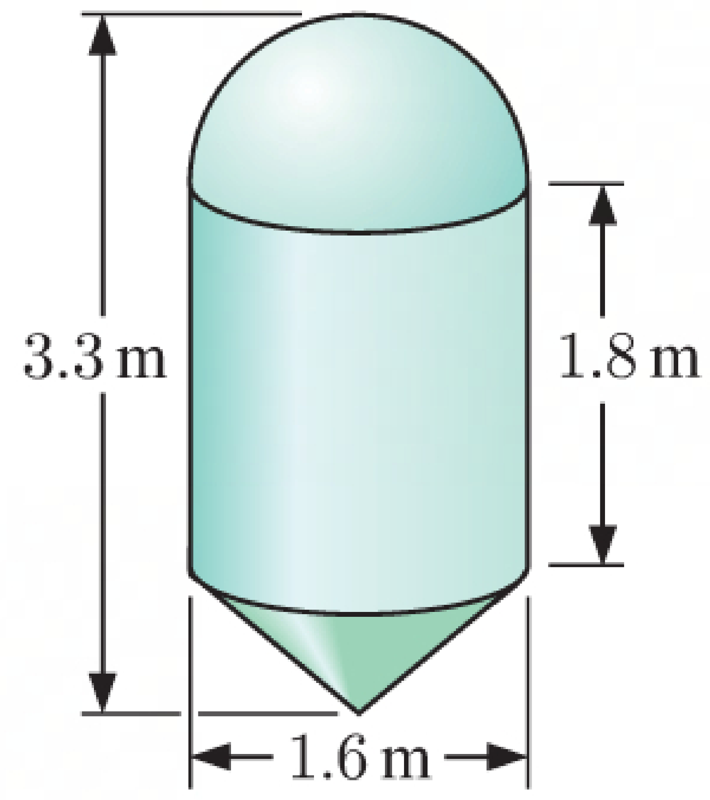
11 **a** Height of cone

$= $ total height of silo $-$ height of cylinder $-$ height of hemisphere

$= 3.3 - 1.8 - \frac{1.6}{2}$ m

$= 0.7$ m

$= 70$ cm

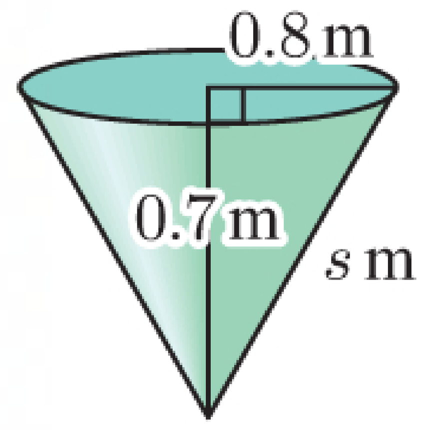


b Let the slant height be s m.

$s^2 = (0.7)^2 + (0.8)^2$ {Pythagoras}

$\therefore \quad s = \sqrt{(0.7)^2 + (0.8)^2}$

≈ 1.06 {as $s > 0$}



So, the slant height of the cone is approximately 1.06 m.

c Surface area of hemisphere $= \frac{1}{2}$(surface area of sphere)

$= \frac{1}{2}(4\pi r^2)$

$= \frac{1}{2}(4 \times \pi \times (0.8)^2)$ m^2

≈ 4.02 m^2

Surface area of cylinder $= 2\pi rh$

$= 2 \times \pi \times 0.8 \times 1.8$ m^2

≈ 9.05 m^2

Surface area of cone $= \pi rs$

$\approx \pi \times 0.8 \times 1.06$

≈ 2.67 m^2

$\therefore$ total surface area $\approx 4.02 + 9.05 + 2.67$ m^2

≈ 15.7 m^2

So, approximately 15.7 m^2 of steel is used to make the feed silo.

d Volume of hemisphere $= \frac{1}{2}\left(\frac{4}{3}\pi r^3\right)$

$= \frac{1}{2} \times \frac{4}{3} \times \pi \times (0.8)^3$ m^3

≈ 1.07 m^3

Volume of cylinder $= \pi r^2 h$

$= \pi \times (0.8)^2 \times 1.8$ m^3

≈ 3.62 m^3

Volume of cone $= \frac{1}{3}(\pi r^2 h)$

$= \frac{1}{3} \times \pi \times (0.8)^2 \times 0.7$ m^3

≈ 0.469 m^3

$\therefore$ volume of silo $=$ volume of hemisphere $+$ volume of cylinder $+$ volume of cone

$\approx 1.07 + 3.62 + 0.47$ m^3

≈ 5.16 m^3

≈ 5.2 m^3

So, the silo can hold about 5.2 m^3 of grain.

e $5.2\text{ m}^3 \equiv 5.2\text{ kL}$

So, the capacity of the silo is about 5.2 kL.

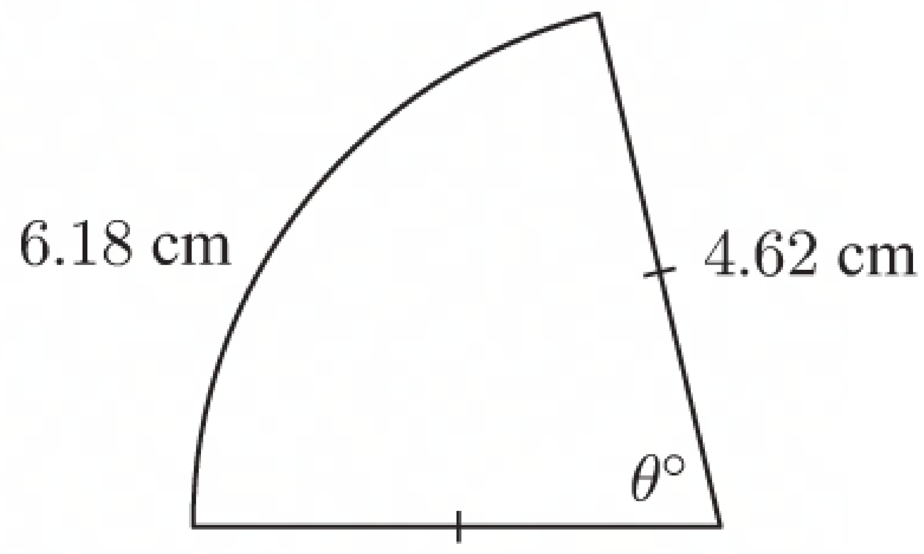
REVIEW SET 6B

1 **a** Arc length $= \dfrac{\theta}{360} \times 2\pi r$

$\therefore\ 6.18 = \dfrac{\theta}{360} \times 2\pi \times 4.62$

$\therefore\ \theta = \dfrac{6.18 \times 360}{2\pi \times 4.62}$

$\therefore\ \theta° \approx 76.6°$



b Area $= \dfrac{\theta}{360} \times \pi r^2$

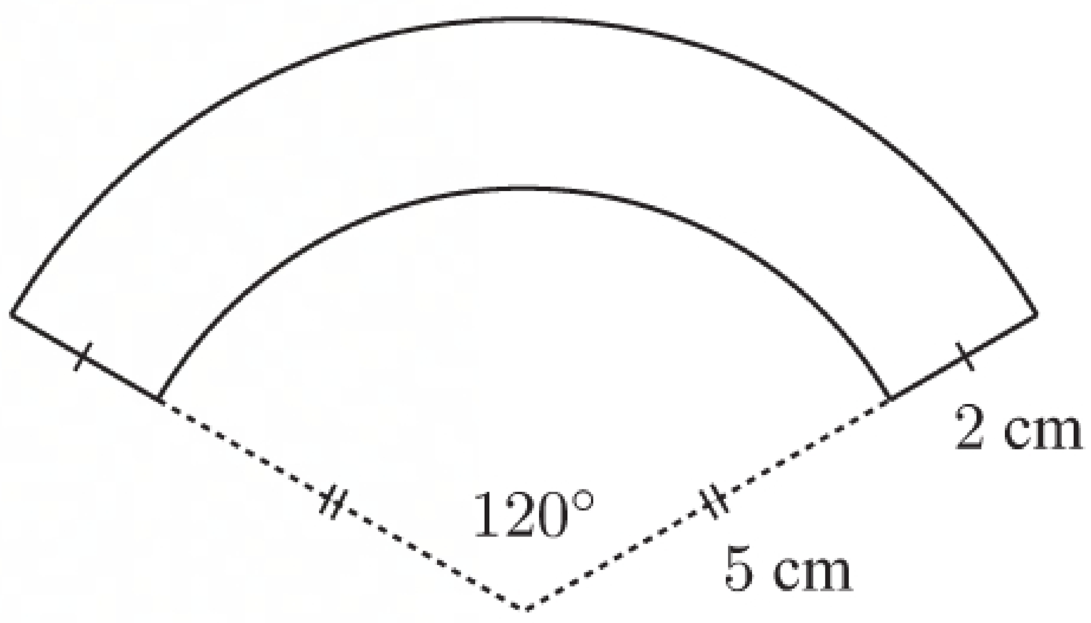
$\approx \dfrac{76.6}{360} \times \pi \times (4.62)^2$

$\approx 14.3\text{ cm}^2$

2 **a** Length of shorter arc $= \dfrac{\theta}{360} \times 2\pi r$

$= \frac{120}{360} \times 2\pi \times 5\text{ cm}$

$\approx 10.47\text{ cm}$



Length of longer arc $= \dfrac{\theta}{360} \times 2\pi r$

$= \frac{120}{360} \times 2\pi \times (5+2)\text{ cm}$

$\approx 14.66\text{ cm}$

Perimeter = length of shorter arc + length of longer arc + length of two ends

$\approx 10.47 + 14.66 + 2 \times 2\text{ cm}$

$\approx 29.1\text{ cm}$

b Area of larger sector $= \dfrac{\theta}{360} \times \pi r^2$

$= \frac{120}{360} \times \pi \times (5+2)^2\text{ cm}^2$

$\approx 51.31\text{ cm}^2$

Area of smaller sector $= \dfrac{\theta}{360} \times \pi r^2$

$= \frac{120}{360} \times \pi \times 5^2\text{ cm}^2$

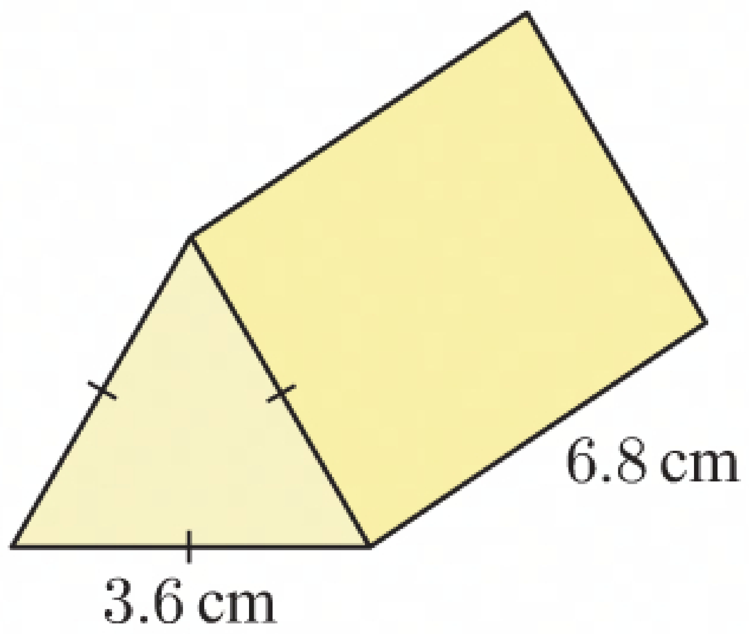
$\approx 26.18\text{ cm}^2$

Area of figure = area of larger sector − area of smaller sector

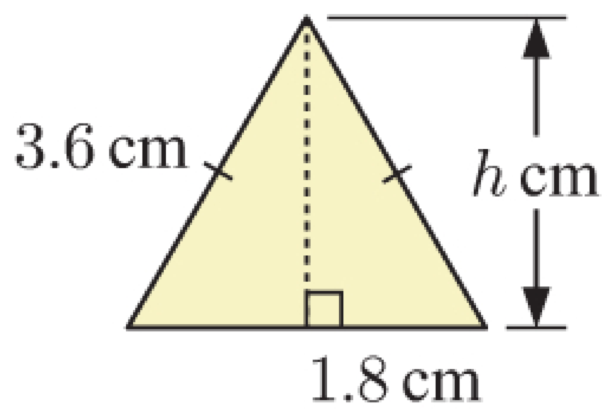
$\approx 51.31 - 26.18\text{ cm}^2$

$\approx 25.1\text{ cm}^2$

3 **a**



Let the height of the triangular end be h cm.



$h^2 + 1.8^2 = 3.6^2$ {Pythagoras}

$\therefore\ h = \sqrt{3.6^2 - 1.8^2}$ {as $h > 0$}

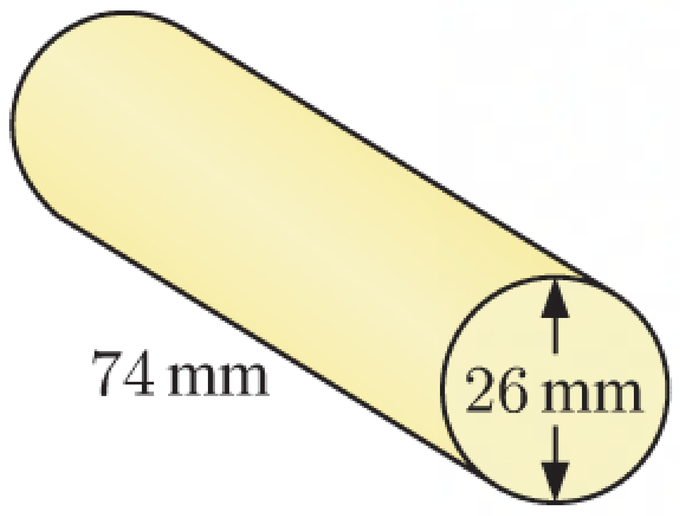
≈ 3.12

Surface area = area of two triangular ends + area of three rectangular sides

$\approx 2 \times \frac{1}{2} \times 3.6 \times 3.12 \ + \ 3 \times 6.8 \times 3.6 \text{ cm}^2$

$\approx 84.7 \text{ cm}^2$

b

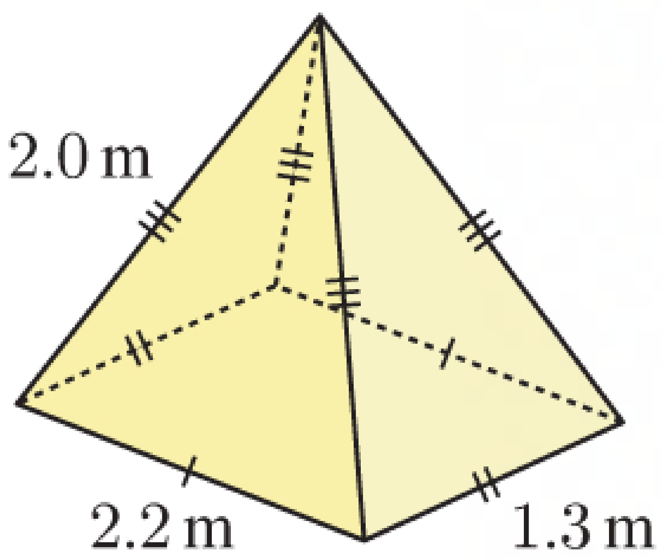


Surface area $= 2\pi r h + 2\pi r^2$

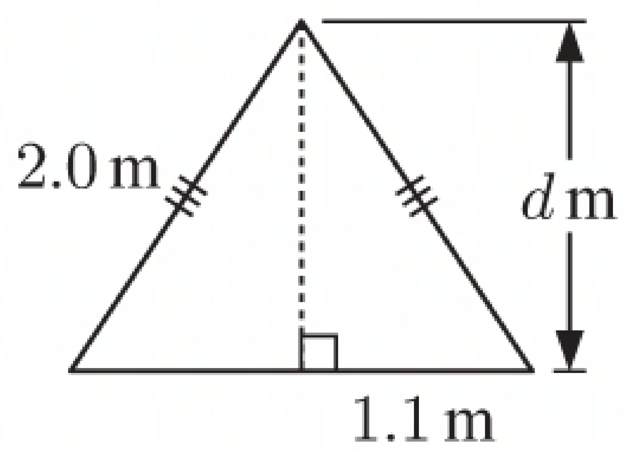
$= 2\pi \times \left(\frac{26}{2}\right) \times 74 \ + \ 2\pi \times \left(\frac{26}{2}\right)^2 \text{ mm}^2$

$\approx 7110 \text{ mm}^2$

c



Let the height of the triangular side with base 2.2 m be d m.

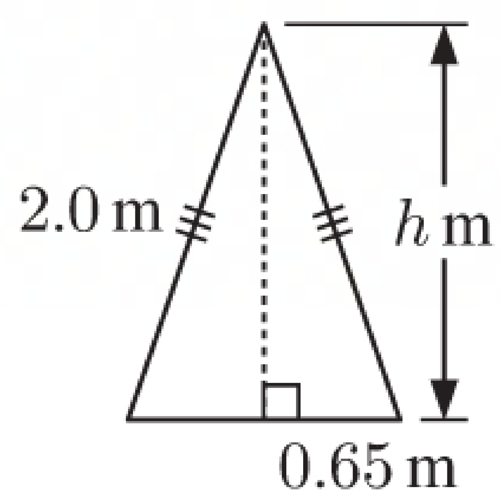


$d^2 + 1.1^2 = 2.0^2$ {Pythagoras}

$\therefore\ d = \sqrt{2.0^2 - 1.1^2}$ {as $d > 0$}

≈ 1.67

Let the height of the triangular side with base 1.3 m be h m.



$h^2 + 0.65^2 = 2.0^2$ {Pythagoras}

$\therefore\ h = \sqrt{2.0^2 - 0.65^2}$ {as $h > 0$}

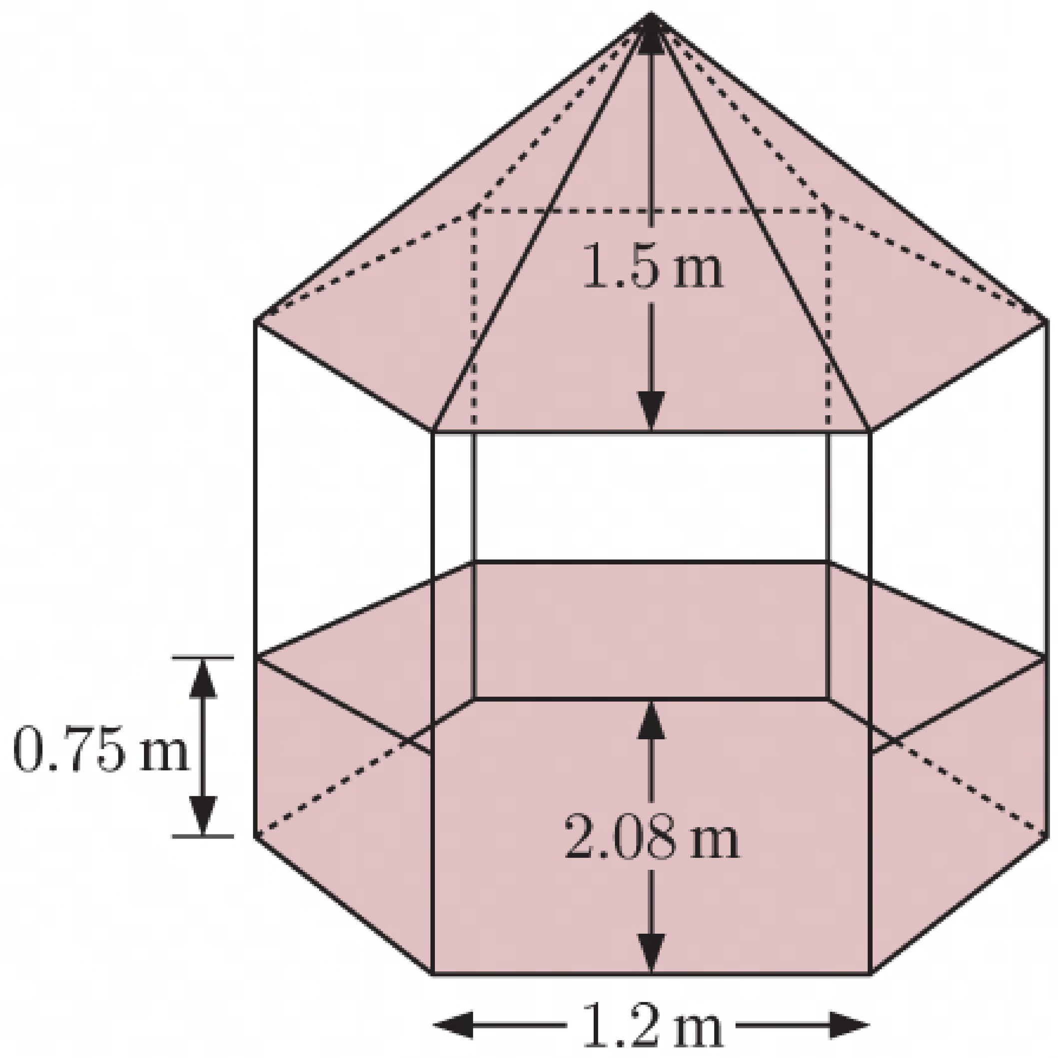
≈ 1.89

Surface area of pyramid = area of base + area of two triangular sides with base 2.2 m
+ area of two triangular sides with base 1.3 m

$\approx 2.2 \times 1.3 \ + \ 2 \times \frac{1}{2} \times 2.2 \times 1.67 \ + \ 2 \times \frac{1}{2} \times 1.3 \times 1.89 \text{ m}^2$

$\approx 8.99 \text{ m}^2$

4 The panelling for the gazebo includes 6 interior and 6 exterior triangular panels for the roof, 5 interior and 5 exterior rectangular panels for the walls, and one hexagonal panel for the floor.



Total surface area

$= \text{roof area} + \text{wall area} + \text{floor area}$

$= 2 \times 6 \times (\frac{1}{2} \times 1.2 \times 1.5) + 2 \times 5 \times (1.2 \times 0.75)$

$\quad + 6 \times (\frac{1}{2} \times 1.2 \times 1.04)\ \text{m}^2$

$= 23.544\ \text{m}^2$

$\approx 23.5\ \text{m}^2$

5 Length of cylinder is 325 mm = 32.5 cm,

and radius of cylinder is $\frac{40}{2} = 20$ mm = 2 cm.

Surface area of cylinder

$= 2\pi rh + 2\pi r^2$

$= (2 \times \pi \times 2 \times 32.5) + (2 \times \pi \times 2^2)\ \text{cm}^2$

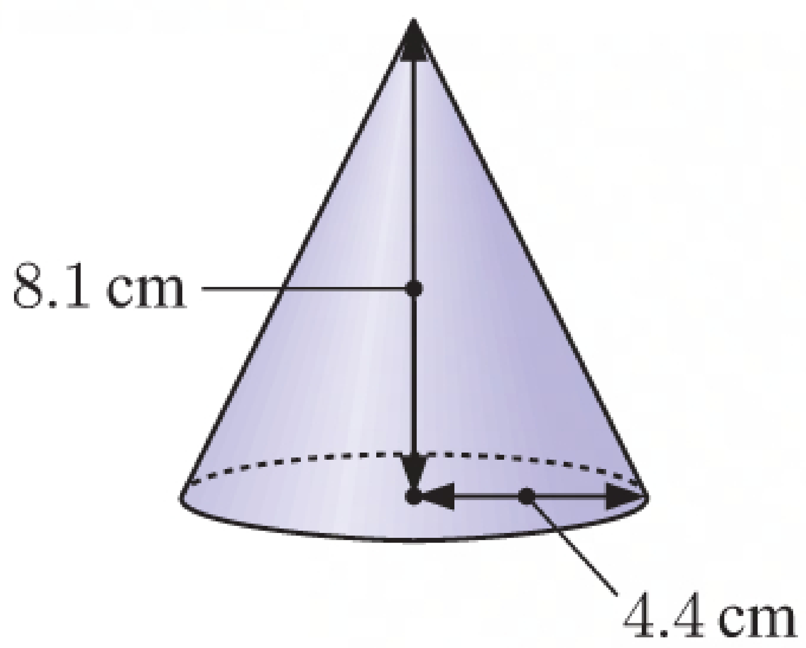
$\approx 434\ \text{cm}^2$

So, approximately 434 cm^2 of bubble wrap is needed to line the cylinder walls.

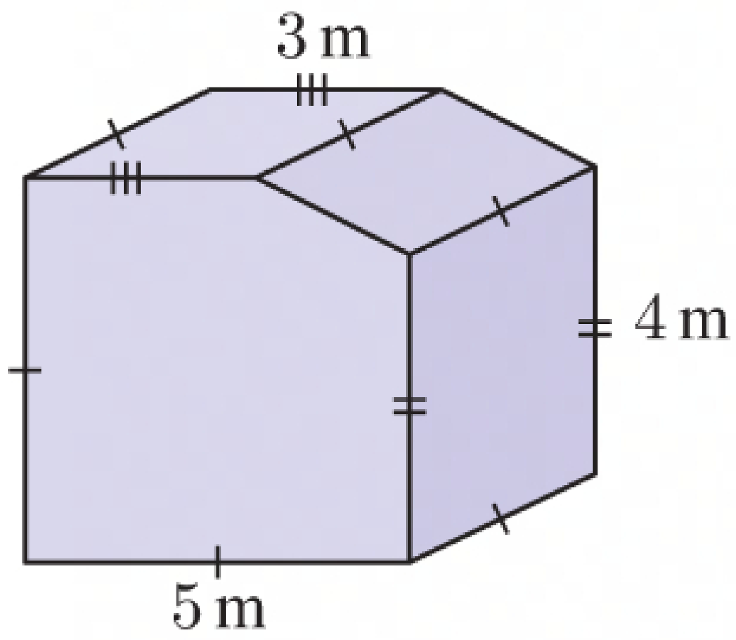
6 **a** $V = \frac{1}{3}(\text{area of base} \times \text{height})$

$= \frac{1}{3} \times \pi \times (4.4)^2 \times 8.1\ \text{cm}^3$

$\approx 164\ \text{cm}^3$



b

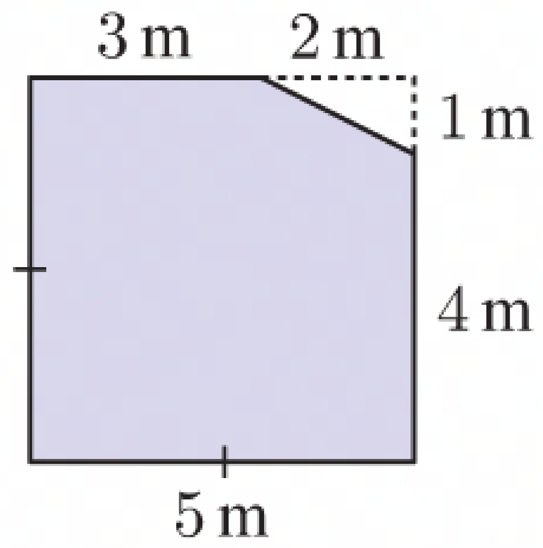


$V = \text{area of end} \times \text{length}$

$= (5 \times 5 - \frac{1}{2} \times 2 \times 1) \times 5\ \text{m}^3$

$= 24 \times 5\ \text{m}^3$

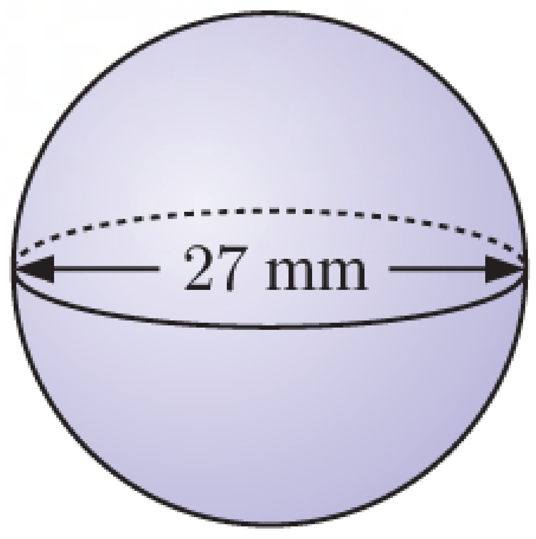
$= 120\ \text{m}^3$



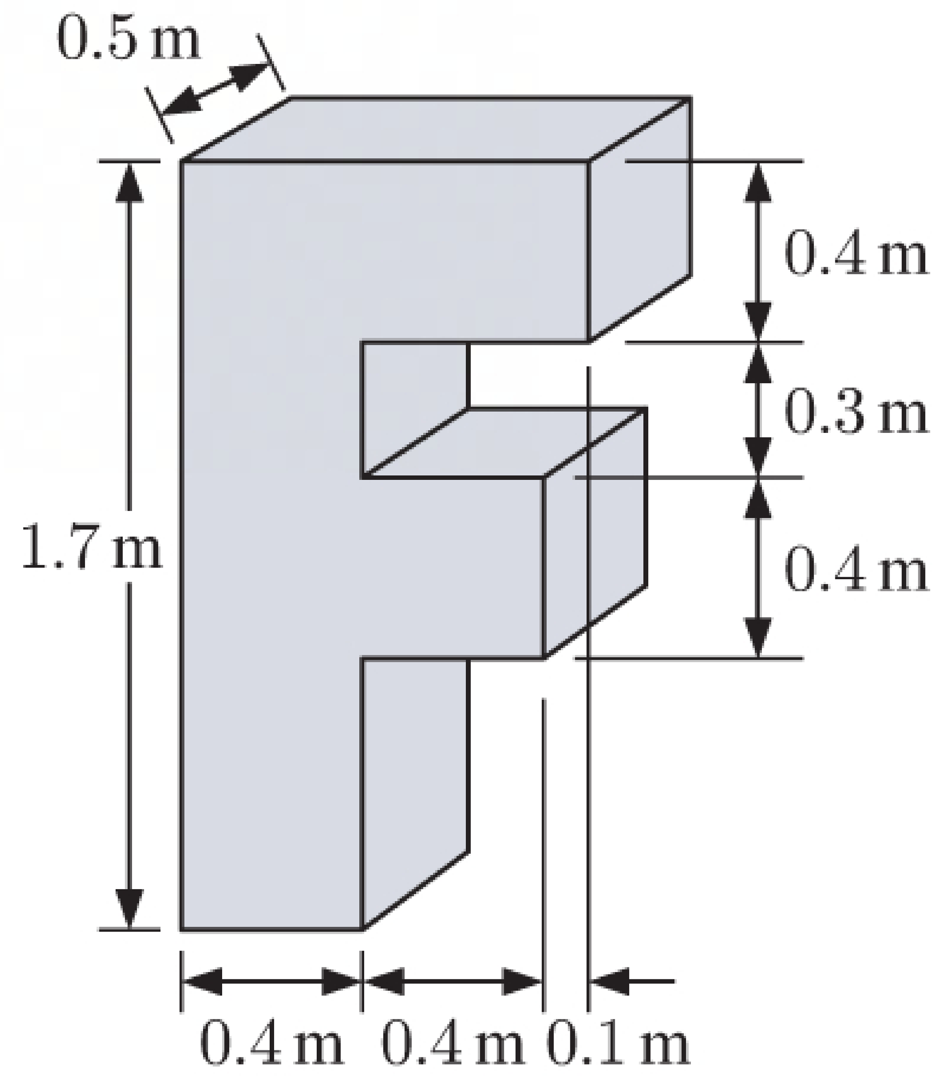
c $V = \frac{4}{3}\pi r^3$

$= \frac{4}{3} \times \pi \times (\frac{27}{2})^3\ \text{mm}^3$

$\approx 10\,300\ \text{mm}^3$



7 **a** The front face of the letter F is made up of 3 rectangles.



Volume of letter F

$= \text{area of front face} \times \text{length}$

$= (1.7 \times 0.4 + 0.4 \times 0.4 + 0.5 \times 0.4) \times 0.5\ \text{m}^3$

$= 1.04 \times 0.5\ \text{m}^3$

$= 0.52\ \text{m}^3$

Frank will need $0.52\ \text{m}^3$ of plastic.

b Surface area of letter F

$= 2 \times \text{area of front face} + 2 \times \text{area of side face}$
$\quad + \text{area of top horizontal faces}$
$\quad + \text{area of bottom horizontal faces}$

$= (2 \times 1.04) + (2 \times 1.7 \times 0.5) + [(0.9 \times 0.5) + (0.5 \times 0.4)]$
$\quad + [(0.5 \times 0.5) + (0.5 \times 0.4) + (0.5 \times 0.4)]\ \text{m}^2$

$= 2.08 + 1.7 + 0.45 + 0.2 + 0.25 + 0.2 + 0.2\ \text{m}^2$

$= 5.08\ \text{m}^2$

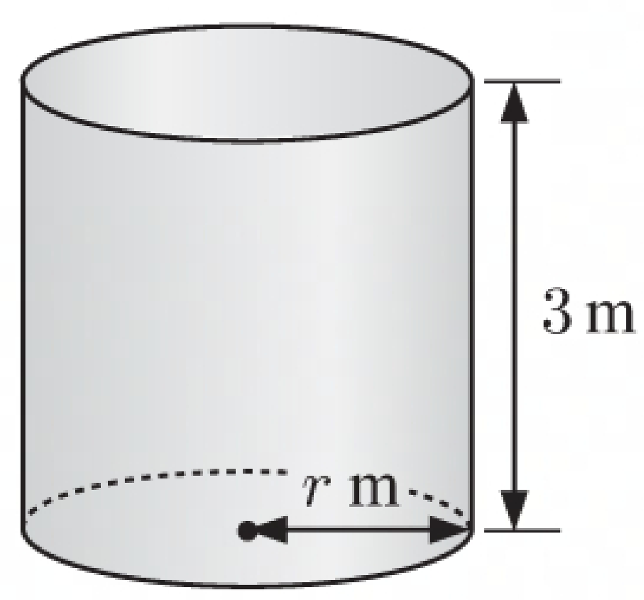
Frank will need $5.08\ \text{m}^2$ of fibreglass.

8 Available volume of bench = total volume of bench − volume of sink

$$\begin{aligned}
\text{Available volume of bench} &= \text{total volume of bench} - \text{volume of sink}\\
&= 3845 \times 1260 \times 1190 - 750 \times 550 \times 195\ \text{mm}^3\\
&= 5\,684\,755\,500\ \text{mm}^3\\
&= (5\,684\,755\,500 \div 10^3)\ \text{cm}^3\\
&= 5\,684\,755.5\ \text{cm}^3\\
\therefore\ \text{capacity} &= (5\,684\,755.5 \div 1000)\ \text{L}\\
&= 5\,684.7555\ \text{L}\\
&\approx 5680\ \text{L}
\end{aligned}$$

The bench has a storage capacity of approximately 5680 L.

9



$$\begin{aligned}
10\ \text{kL} &\equiv 10\ \text{m}^3\\
\text{Volume of cylindrical drum} &= 10\ \text{m}^3\\
\therefore\ \pi r^2 h &= 10\\
\therefore\ \pi \times r^2 \times 3 &= 10\\
\therefore\ r^2 &= \frac{10}{3\pi}\\
\therefore\ r &= \sqrt{\frac{10}{3\pi}} \qquad \{\text{as } r > 0\}\\
&\approx 1.03
\end{aligned}$$

So, the radius of the drum is approximately 1.03 m.

10 **a** Surface area of the Sun $\approx 4\pi r^2$

$\approx 4\pi \times (6.955 \times 10^8)^2\ \text{m}^2$

$\approx 6.08 \times 10^{18}\ \text{m}^2$

The Sun's surface area is approximately $6.08 \times 10^{18}\ \text{m}^2$.

b Volume of the Sun $\approx \frac{4}{3}\pi r^3$

$\approx \frac{4}{3}\pi \times (6.955 \times 10^8)^3 \text{ m}^3$

$\approx 1.41 \times 10^{27} \text{ m}^3$

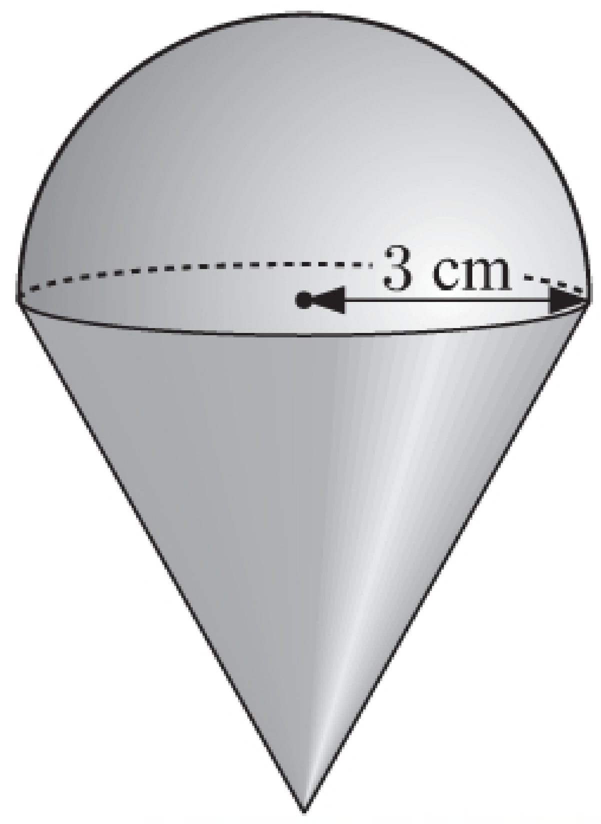
The Sun's volume is approximately $1.41 \times 10^{27} \text{ m}^3$.

11 **a** Volume of hemispherical top $= \frac{1}{2} \times \frac{4}{3}\pi r^3$

$= \frac{2}{3}\pi \times 3^3 \text{ cm}^3$

$= 18\pi \text{ cm}^3$

$\approx 56.5 \text{ cm}^3$



b Volume of cone-shaped base $= \frac{1}{3} \times \text{area of base} \times \text{height}$

$= \frac{1}{3} \times \pi r^2 h$

$= \frac{1}{3}\pi \times 3^2 \times h \text{ cm}^3$

$= 3\pi h \text{ cm}^3$

Now, volume of cone $= \frac{1}{2} \times$ volume of hemisphere

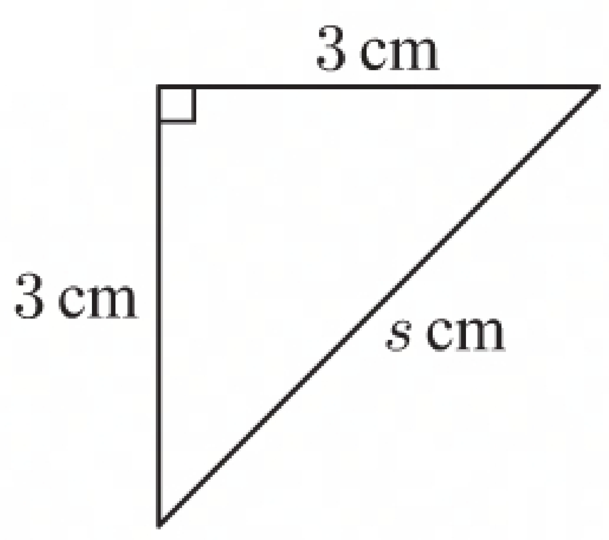
$\therefore\ 3\pi h = \frac{1}{2} \times 18\pi$

$\therefore\ 3h = 9$

$\therefore\ h = 3$

So, the cone-shaped base has height 3 cm.

c



Let the slant height of the cone-shaped base be s cm.

$s^2 = 3^2 + 3^2$ {Pythagoras}

$\therefore\ s = \sqrt{18}$ {as $s > 0$}

$= 3\sqrt{2}$

Outer surface area of spinning top

$=$ surface area of hemispherical top $+$ surface area of cone-shaped base

$= \frac{1}{2} \times 4\pi r^2 + \pi r s$

$= 2\pi \times 3^2 + \pi \times 3 \times 3\sqrt{2}$

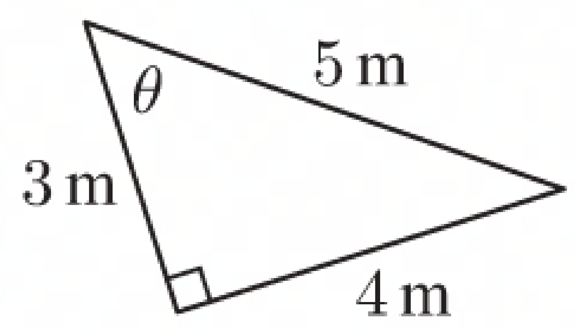
$\approx 96.5 \text{ cm}^2$

Chapter 7

RIGHT ANGLED TRIANGLE TRIGONOMETRY

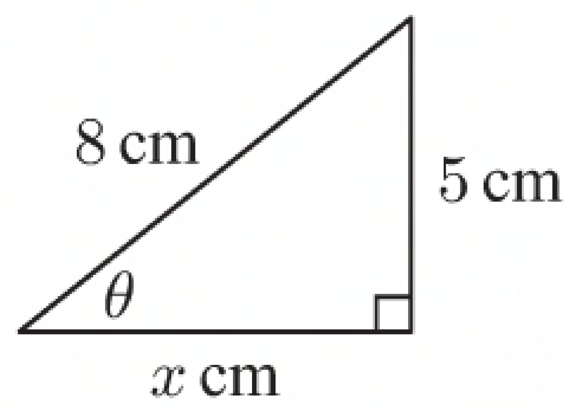
EXERCISE 7A

1 **a**



i $\sin\theta = \frac{\text{OPP}}{\text{HYP}} = \frac{4}{5}$ **ii** $\cos\theta = \frac{\text{ADJ}}{\text{HYP}} = \frac{3}{5}$ **iii** $\tan\theta = \frac{\text{OPP}}{\text{ADJ}} = \frac{4}{3}$

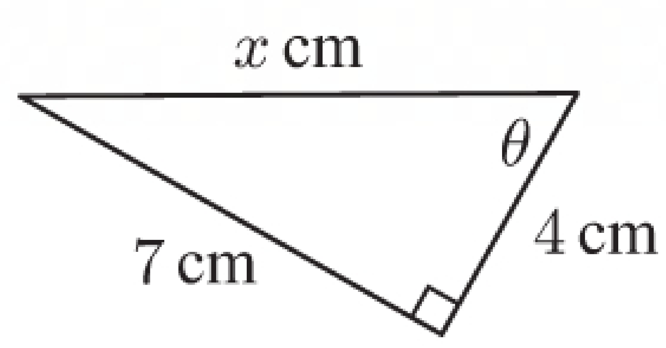
b Let the unknown side be x cm.



$$x^2 + 5^2 = 8^2 \quad \{\text{Pythagoras}\}$$
$$\therefore\ x^2 + 25 = 64$$
$$\therefore\ x^2 = 39$$
$$\therefore\ x = \sqrt{39} \quad \{\text{as } x > 0\}$$

i $\sin\theta = \frac{\text{OPP}}{\text{HYP}} = \frac{5}{8}$ **ii** $\cos\theta = \frac{\text{ADJ}}{\text{HYP}} = \frac{\sqrt{39}}{8}$ **iii** $\tan\theta = \frac{\text{OPP}}{\text{ADJ}} = \frac{5}{\sqrt{39}}$

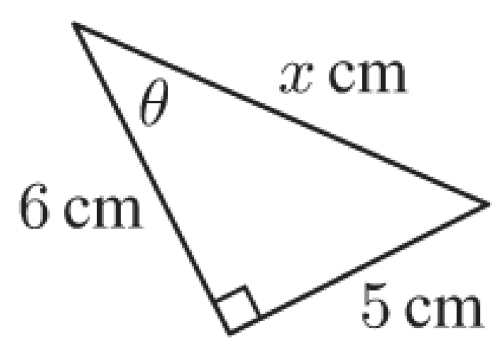
c Let the unknown side be x cm.



$$x^2 = 7^2 + 4^2 \quad \{\text{Pythagoras}\}$$
$$\therefore\ x^2 = 49 + 16$$
$$\therefore\ x^2 = 65$$
$$\therefore\ x = \sqrt{65} \quad \{\text{as } x > 0\}$$

i $\sin\theta = \frac{\text{OPP}}{\text{HYP}} = \frac{7}{\sqrt{65}}$ **ii** $\cos\theta = \frac{\text{ADJ}}{\text{HYP}} = \frac{4}{\sqrt{65}}$ **iii** $\tan\theta = \frac{\text{OPP}}{\text{ADJ}} = \frac{7}{4}$

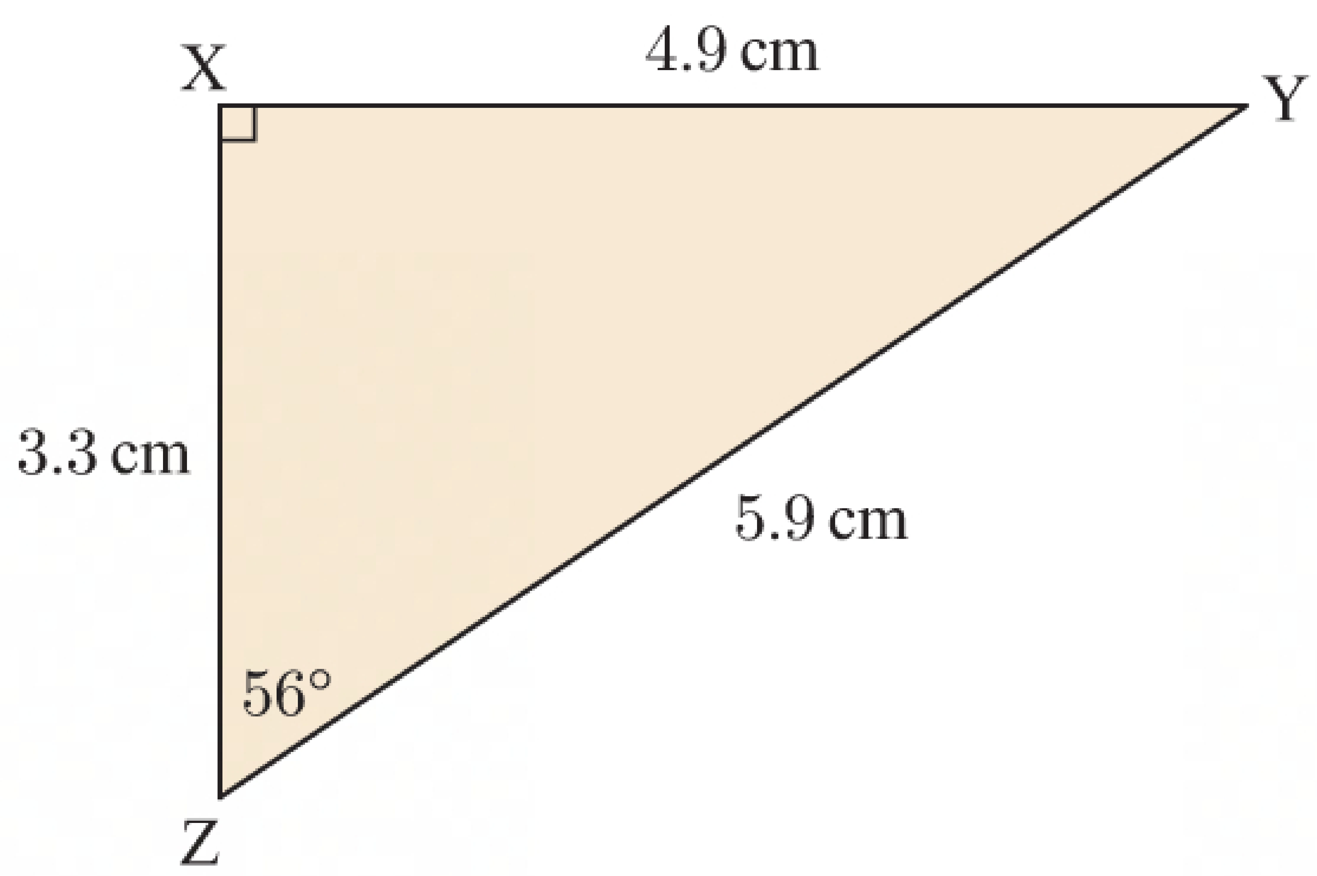
d Let the unknown side be x cm.



$$x^2 = 6^2 + 5^2 \quad \{\text{Pythagoras}\}$$
$$\therefore\ x^2 = 36 + 25$$
$$\therefore\ x^2 = 61$$
$$\therefore\ x = \sqrt{61} \quad \{\text{as } x > 0\}$$

i $\sin\theta = \frac{\text{OPP}}{\text{HYP}} = \frac{5}{\sqrt{61}}$ **ii** $\cos\theta = \frac{\text{ADJ}}{\text{HYP}} = \frac{6}{\sqrt{61}}$ **iii** $\tan\theta = \frac{\text{OPP}}{\text{ADJ}} = \frac{5}{6}$

2 **a**



b **i** $\sin 56° = \frac{\text{OPP}}{\text{HYP}} \approx \frac{4.9}{5.9} \approx 0.83$

ii $\cos 56° = \frac{\text{ADJ}}{\text{HYP}} \approx \frac{3.3}{5.9} \approx 0.56$

iii $\tan 56° = \frac{\text{OPP}}{\text{ADJ}} \approx \frac{4.9}{3.3} \approx 1.48$

c **i** $\sin 56° \approx 0.83$

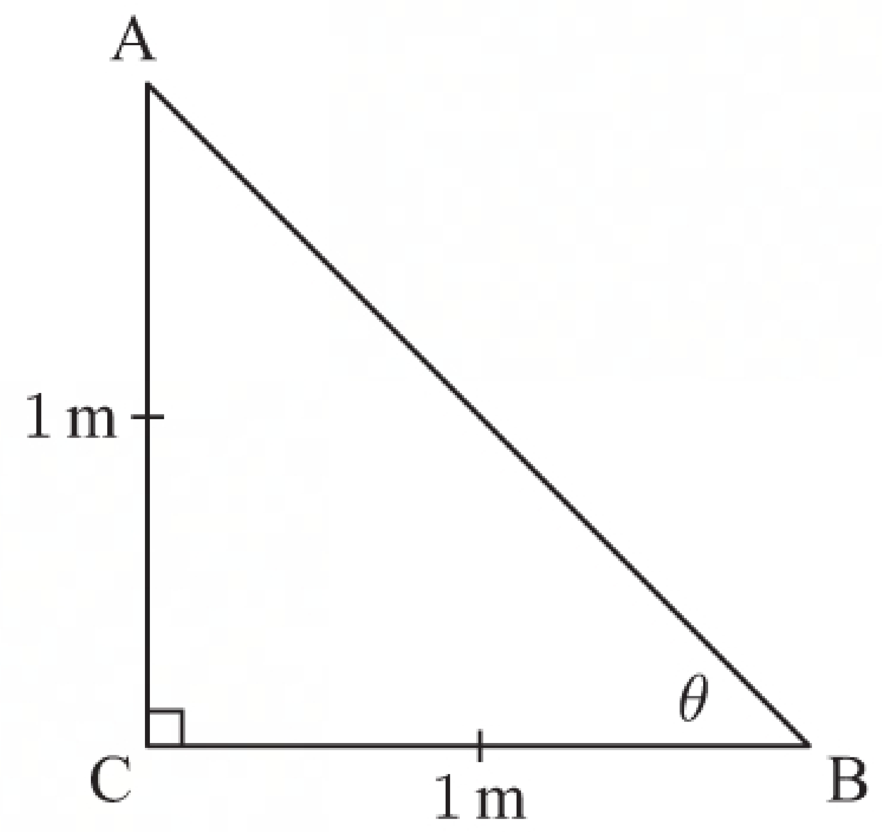
ii $\cos 56° \approx 0.56$

iii $\tan 56° \approx 1.48$

3 **a** Let $\widehat{ABC} = \theta$.

Base angles of an isosceles triangle are equal.

$\therefore \quad \widehat{BAC} = \theta$



$$\widehat{ABC} + \widehat{BAC} + 90^\circ = 180^\circ \quad \{\text{angles in a triangle}\}$$
$$\therefore \quad \theta + \theta = 90^\circ$$
$$\therefore \quad 2\theta = 90^\circ$$
$$\therefore \quad \theta = 45^\circ$$

So, $\widehat{ABC} = 45^\circ$

b
$$\begin{aligned} AB^2 &= AC^2 + BC^2 \quad \{\text{Pythagoras}\} \\ &= 1^2 + 1^2 \\ &= 2 \end{aligned}$$
$$\begin{aligned} \therefore \quad AB &= \sqrt{2} \quad \{\text{as } AB > 0\} \\ &\approx 1.41 \text{ m} \end{aligned}$$

c **i** $\sin 45^\circ = \dfrac{\text{OPP}}{\text{HYP}} = \dfrac{1}{\sqrt{2}} \approx 0.707$ **ii** $\cos 45^\circ = \dfrac{\text{ADJ}}{\text{HYP}} = \dfrac{1}{\sqrt{2}} \approx 0.707$

iii $\tan 45^\circ = \dfrac{\text{OPP}}{\text{ADJ}} = \dfrac{1}{1} = 1$

d $\sin 45^\circ \approx 0.707$, $\cos 45^\circ \approx 0.707$, $\tan 45^\circ = 1$

4 The hypotenuse of a right angled triangle is always the longest side of the triangle.

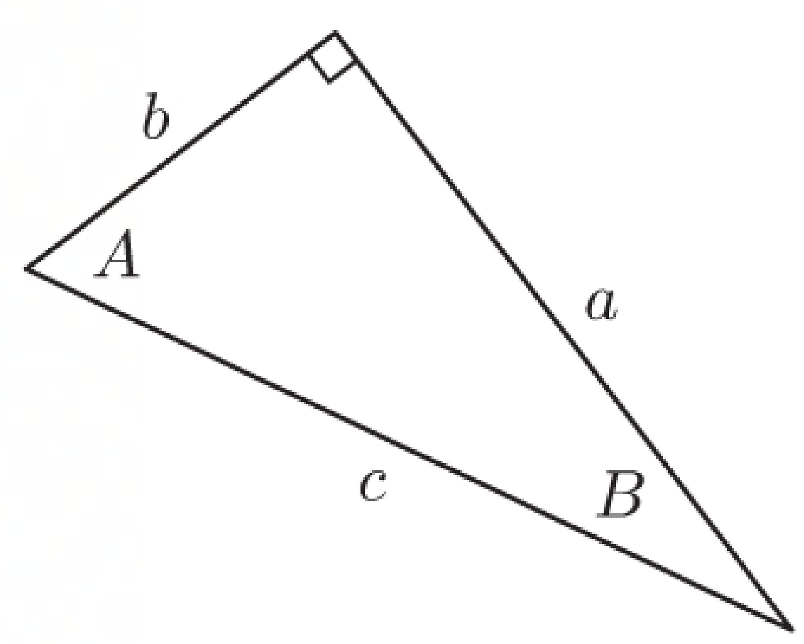
$\therefore$ the opposite and adjacent sides will always be shorter than the hypotenuse.

So, $\sin\theta = \dfrac{\text{OPP}}{\text{HYP}}$ and $\cos\theta = \dfrac{\text{ADJ}}{\text{HYP}}$ will always be less than or equal to 1.

5 **a** **i** $\sin A = \dfrac{\text{OPP}}{\text{HYP}} = \dfrac{a}{c}$ **ii** $\cos A = \dfrac{\text{ADJ}}{\text{HYP}} = \dfrac{b}{c}$

iii $\tan A = \dfrac{\text{OPP}}{\text{ADJ}} = \dfrac{a}{b}$ **iv** $\sin B = \dfrac{\text{OPP}}{\text{HYP}} = \dfrac{b}{c}$

v $\cos B = \dfrac{\text{ADJ}}{\text{HYP}} = \dfrac{a}{c}$ **vi** $\tan B = \dfrac{\text{OPP}}{\text{ADJ}} = \dfrac{b}{a}$



b
$$A + B + 90^\circ = 180^\circ \quad \{\text{angles in a triangle}\}$$
$$\therefore \quad A + B = 90^\circ$$
$$\therefore \quad A = 90^\circ - B$$

c Let $\theta = B$.

i $\sin B = \dfrac{b}{c} = \cos A$ {using **a**}

$\therefore \quad \sin B = \cos(90^\circ - B)$ {using **b**}

$\therefore \quad \sin\theta = \cos(90^\circ - \theta)$

ii $\cos B = \dfrac{a}{c} = \sin A$ {using **a**}

$\therefore \quad \cos B = \sin(90^\circ - B)$ {using **b**}

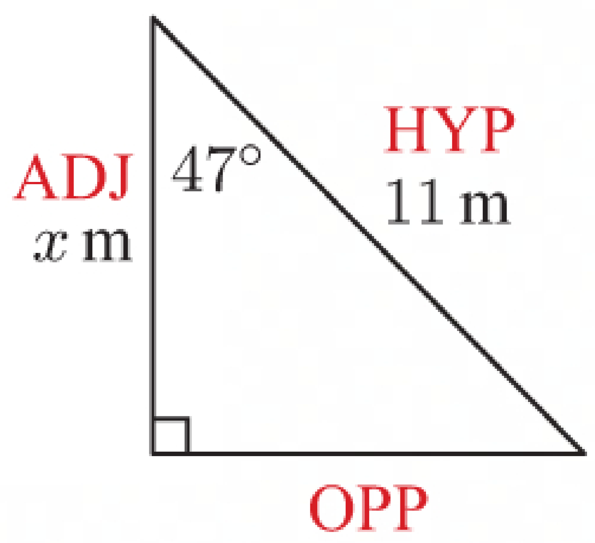
$\therefore \quad \cos\theta = \sin(90^\circ - \theta)$

iii $\tan B = \dfrac{b}{a} = \dfrac{1}{\left(\frac{a}{b}\right)} = \dfrac{1}{\tan A}$ {using **a**}

$\therefore \quad \tan B = \dfrac{1}{\tan(90^\circ - B)}$ {using **b**}

$\therefore \quad \tan\theta = \dfrac{1}{\tan(90^\circ - \theta)}$

6 **a**



The relevant sides are ADJ and HYP, so we use the *cosine* ratio.

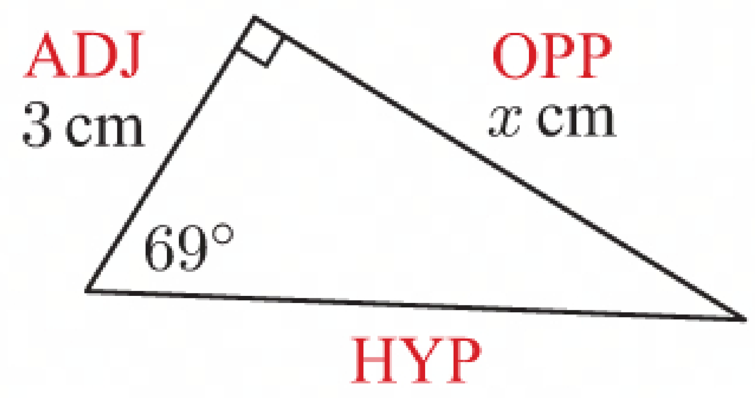
$$\cos 47° = \frac{x}{11} \qquad \{\cos\theta = \frac{\text{ADJ}}{\text{HYP}}\}$$

$$\therefore\ 11 \times \cos 47° = x$$

$$\therefore\ x \approx 7.50$$

So, the side is about 7.50 m long.

b



The relevant sides are ADJ and OPP, so we use the *tangent* ratio.

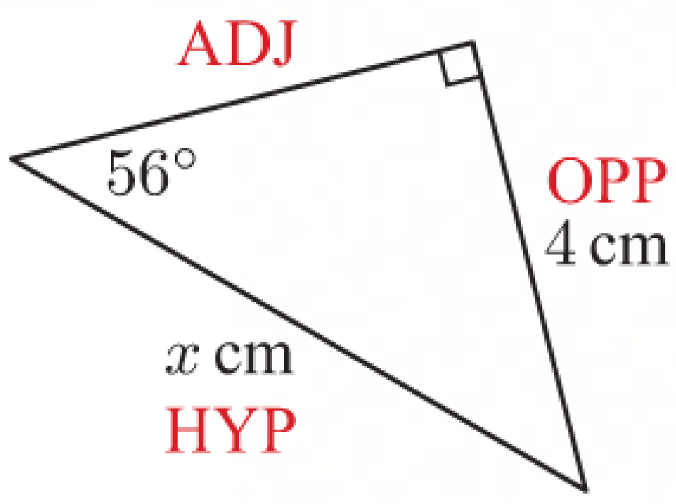
$$\tan 69° = \frac{x}{3} \qquad \{\tan\theta = \frac{\text{OPP}}{\text{ADJ}}\}$$

$$\therefore\ 3 \times \tan 69° = x$$

$$\therefore\ x \approx 7.82$$

So, the side is about 7.82 cm long.

c



The relevant sides are HYP and OPP, so we use the *sine* ratio.

$$\sin 56° = \frac{4}{x} \qquad \{\sin\theta = \frac{\text{OPP}}{\text{HYP}}\}$$

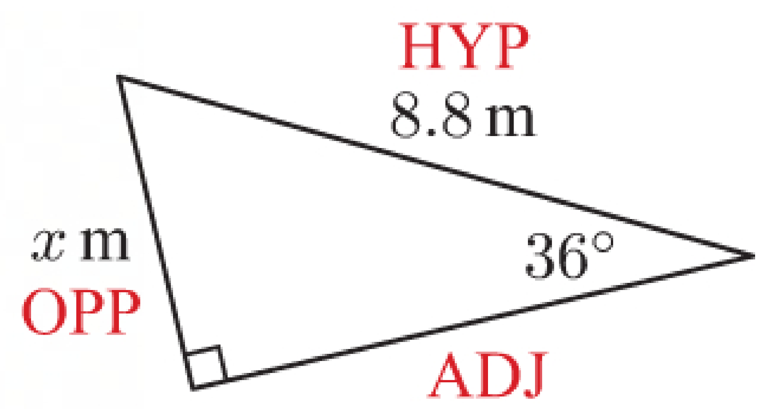
$$\therefore\ x \times \sin 56° = 4$$

$$\therefore\ x = \frac{4}{\sin 56°}$$

$$\therefore\ x \approx 4.82$$

So, the side is about 4.82 cm long.

d



The relevant sides are HYP and OPP, so we use the *sine* ratio.

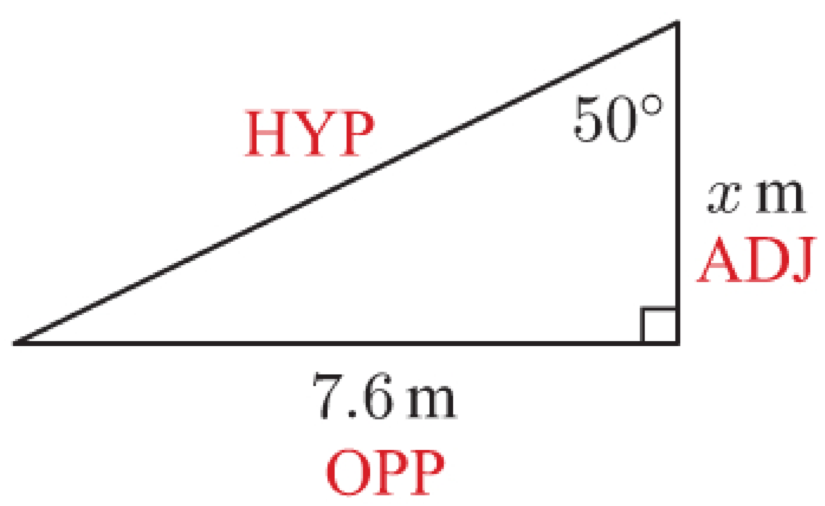
$$\sin 36° = \frac{x}{8.8} \qquad \{\sin\theta = \frac{\text{OPP}}{\text{HYP}}\}$$

$$\therefore\ 8.8 \times \sin 36° = x$$

$$\therefore\ x \approx 5.17$$

So, the side is about 5.17 m long.

e



The relevant sides are ADJ and OPP, so we use the *tangent* ratio.

$$\tan 50° = \frac{7.6}{x} \qquad \{\tan\theta = \frac{\text{OPP}}{\text{ADJ}}\}$$

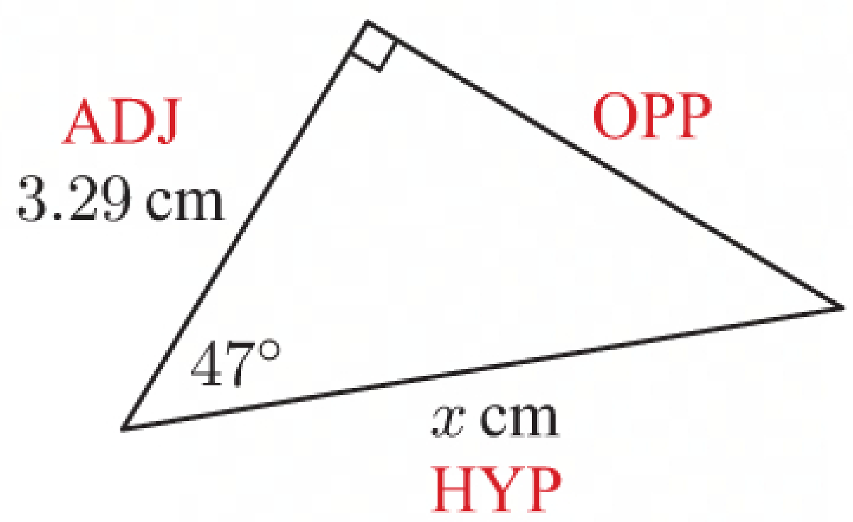
$$\therefore\ x \times \tan 50° = 7.6$$

$$\therefore\ x = \frac{7.6}{\tan 50°}$$

$$\therefore\ x \approx 6.38$$

So, the side is about 6.38 m long.

f



The relevant sides are ADJ and HYP, so we use the *cosine* ratio.

$$\cos 47° = \frac{3.29}{x} \qquad \{\cos\theta = \frac{\text{ADJ}}{\text{HYP}}\}$$

$$\therefore\ x \times \cos 47° = 3.29$$

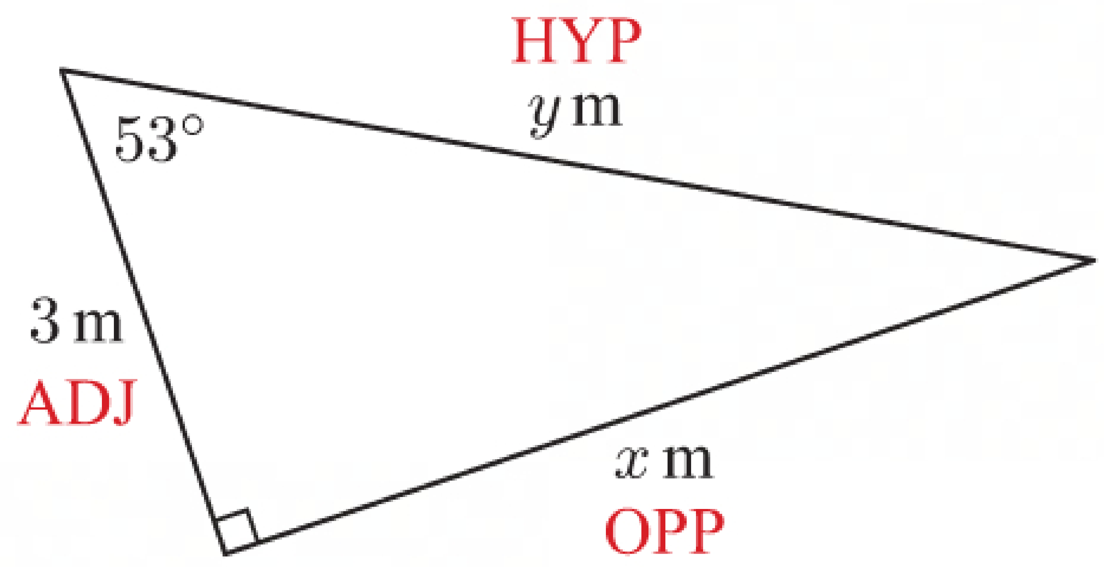
$$\therefore\ x = \frac{3.29}{\cos 47°}$$

$$\therefore\ x \approx 4.82$$

So, the side is about 4.82 cm long.

7 **a**

$$\tan 53^\circ = \frac{x}{3} \qquad \{\tan\theta = \frac{\text{OPP}}{\text{ADJ}}\}$$
$$\therefore\ 3 \times \tan 53^\circ = x$$
$$\therefore\ x \approx 3.98$$



b **i**

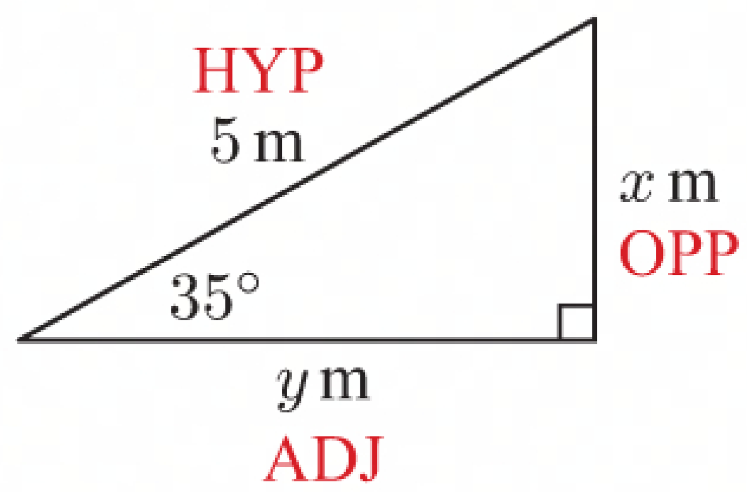
$$y^2 = 3^2 + x^2 \qquad \{\text{Pythagoras}\}$$
$$\therefore\ y^2 \approx 9 + 3.98^2$$
$$\therefore\ y \approx \sqrt{24.85} \qquad \{\text{as } y > 0\}$$
$$\therefore\ y \approx 4.98$$

ii

$$\cos 53^\circ = \frac{3}{y} \qquad \{\cos\theta = \frac{\text{ADJ}}{\text{HYP}}\}$$
$$\therefore\ y \times \cos 53^\circ = 3$$
$$\therefore\ y = \frac{3}{\cos 53^\circ}$$
$$\therefore\ y \approx 4.98$$

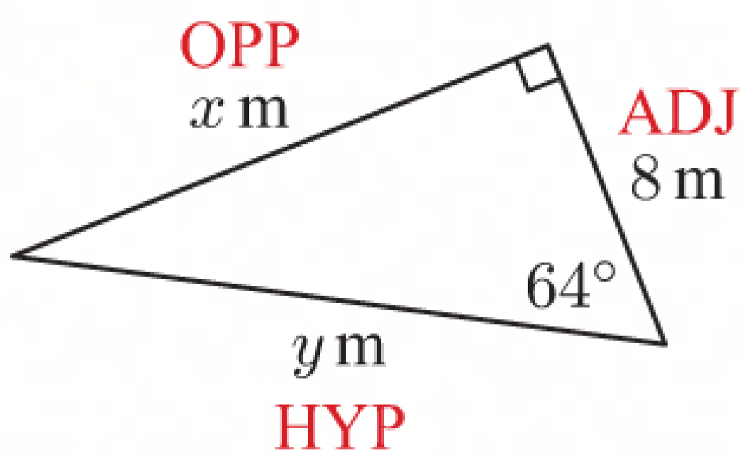
8 **a**

$$\sin 35^\circ = \frac{x}{5} \qquad \{\sin\theta = \frac{\text{OPP}}{\text{HYP}}\}$$
$$\therefore\ 5 \times \sin 35^\circ = x$$
$$\therefore\ x \approx 2.87$$
$$\cos 35^\circ = \frac{y}{5} \qquad \{\cos\theta = \frac{\text{ADJ}}{\text{HYP}}\}$$
$$\therefore\ 5 \times \cos 35^\circ = y$$
$$\therefore\ y \approx 4.10$$



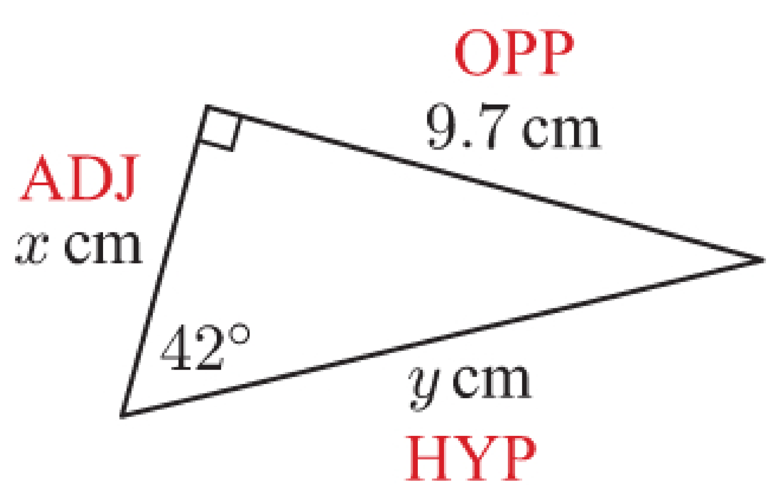
b

$$\tan 64^\circ = \frac{x}{8} \qquad \{\tan\theta = \frac{\text{OPP}}{\text{ADJ}}\}$$
$$\therefore\ 8 \times \tan 64^\circ = x$$
$$\therefore\ x \approx 16.40$$
$$\cos 64^\circ = \frac{8}{y} \qquad \{\cos\theta = \frac{\text{ADJ}}{\text{HYP}}\}$$
$$\therefore\ y \times \cos 64^\circ = 8$$
$$\therefore\ y = \frac{8}{\cos 64^\circ}$$
$$\therefore\ y \approx 18.25$$

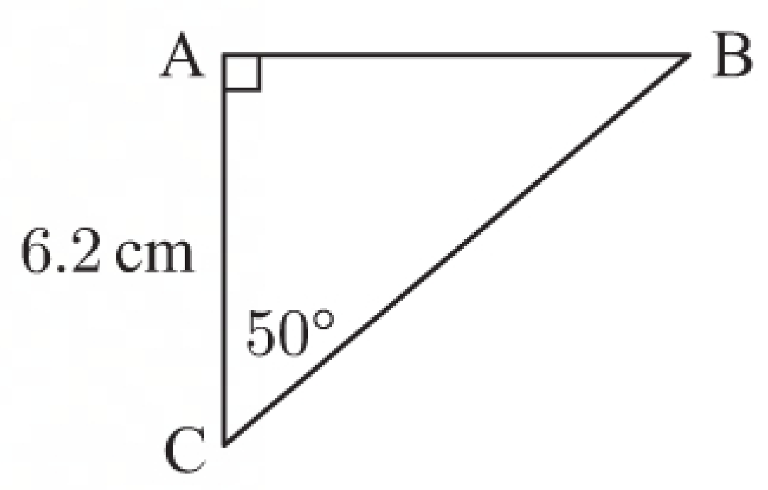


c

$$\tan 42^\circ = \frac{9.7}{x} \qquad \{\tan\theta = \frac{\text{OPP}}{\text{ADJ}}\}$$
$$\therefore\ x \times \tan 42^\circ = 9.7$$
$$\therefore\ x = \frac{9.7}{\tan 42^\circ}$$
$$\therefore\ x \approx 10.77$$
$$\sin 42^\circ = \frac{9.7}{y} \qquad \{\sin\theta = \frac{\text{OPP}}{\text{HYP}}\}$$
$$\therefore\ y \times \sin 42^\circ = 9.7$$
$$\therefore\ y = \frac{9.7}{\sin 42^\circ}$$
$$\therefore\ y \approx 14.50$$



9 **a**



$$\tan 50^\circ = \frac{AB}{6.2} \qquad \{\tan\theta = \frac{OPP}{ADJ}\}$$

$$\therefore\ 6.2 \times \tan 50^\circ = AB$$

$$\therefore\ AB \approx 7.39 \text{ cm}$$

$$\cos 50^\circ = \frac{6.2}{BC} \qquad \{\cos\theta = \frac{ADJ}{HYP}\}$$

$$\therefore\ BC \times \cos 50^\circ = 6.2$$

$$\therefore\ BC = \frac{6.2}{\cos 50^\circ}$$

$$\therefore\ BC \approx 9.65 \text{ cm}$$

Perimeter of triangle ABC $= AB + AC + BC$

$\approx 7.39 + 6.2 + 9.65$ cm

≈ 23.2 cm

Area of triangle ABC $= \frac{1}{2} \times \text{base} \times \text{height}$

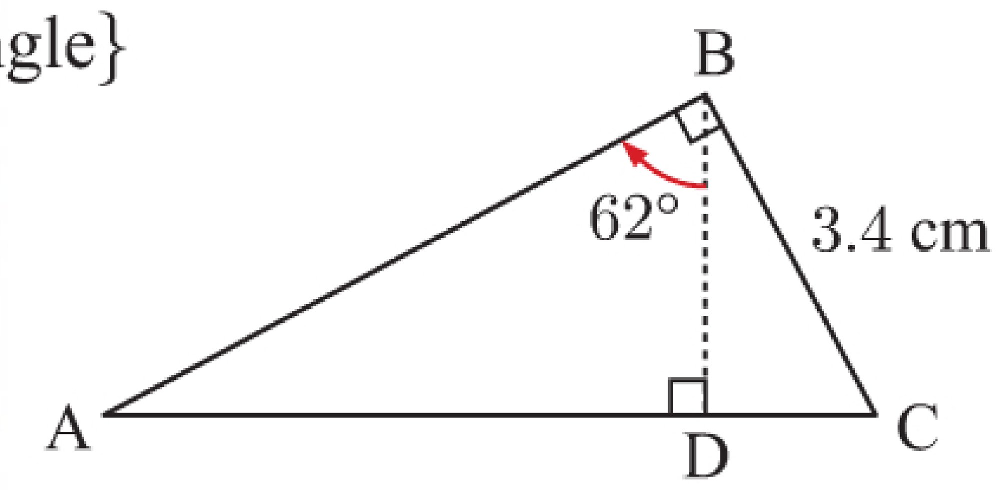
$= \frac{1}{2} \times AC \times AB$

$\approx \frac{1}{2} \times 6.2 \times 7.39 \text{ cm}^2$

$\approx 22.9 \text{ cm}^2$

b In $\triangle ABD$, $B\widehat{A}D = 180^\circ - 90^\circ - 62^\circ$ {angles in a triangle}

$\therefore\ B\widehat{A}D = 28^\circ$



In $\triangle ABC$, $\tan B\widehat{A}C = \frac{3.4}{AB} \qquad \{\tan\theta = \frac{OPP}{ADJ}\}$

$$\therefore\ \tan 28^\circ = \frac{3.4}{AB} \qquad \{B\widehat{A}C = B\widehat{A}D\}$$

$$\therefore\ AB \times \tan 28^\circ = 3.4$$

$$\therefore\ AB = \frac{3.4}{\tan 28^\circ}$$

$$\therefore\ AB \approx 6.39 \text{ cm}$$

Also, $\sin 28^\circ = \frac{3.4}{AC} \qquad \{\sin\theta = \frac{OPP}{HYP}\}$

$$\therefore\ AC \times \sin 28^\circ = 3.4$$

$$\therefore\ AC = \frac{3.4}{\sin 28^\circ}$$

$$\therefore\ AC \approx 7.24 \text{ cm}$$

Perimeter of triangle ABC $= AB + AC + BC$

$\approx 6.39 + 7.24 + 3.4$ cm

≈ 17.0 cm

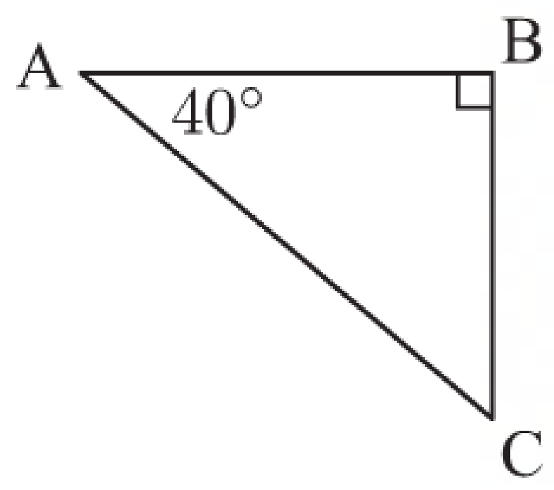
Area of triangle ABC $= \frac{1}{2} \times \text{base} \times \text{height}$

$= \frac{1}{2} \times AB \times BC$

$\approx \frac{1}{2} \times 6.39 \times 3.4 \text{ cm}^2$

$\approx 10.9 \text{ cm}^2$

10 Area of triangle ABC $= \frac{1}{2} \times \text{base} \times \text{height}$



$$\therefore\ 20 = \tfrac{1}{2} \times AB \times BC$$
$$\therefore\ 40 = AB \times BC$$
$$\therefore\ BC = \frac{40}{AB} \quad .... (1)$$

Also, $\tan 40^\circ = \dfrac{BC}{AB}$ $\quad \{\tan\theta = \dfrac{OPP}{ADJ}\}$

$$= \frac{\left(\frac{40}{AB}\right)}{AB} \quad \{\text{using } (1)\}$$
$$= \frac{40}{AB^2}$$
$$\therefore\ AB^2 = \frac{40}{\tan 40^\circ}$$
$$\therefore\ AB = \sqrt{\frac{40}{\tan 40^\circ}} \quad \{\text{as } AB > 0\}$$
$$\therefore\ AB \approx 6.904 \text{ cm}$$

Substituting $AB = \sqrt{\dfrac{40}{\tan 40^\circ}}$ into (1) gives $BC = \dfrac{40}{\sqrt{\frac{40}{\tan 40^\circ}}}$

$$\therefore\ BC \approx 5.793 \text{ cm}$$

Also, $AC^2 = AB^2 + BC^2$ $\quad$ {Pythagoras}

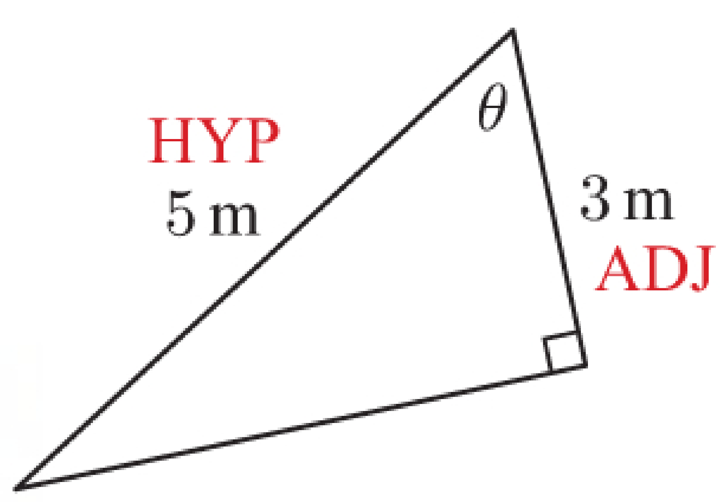
$$\therefore\ AC = \sqrt{AB^2 + BC^2} \quad \{\text{as } AC > 0\}$$
$$= \sqrt{\frac{40}{\tan 40^\circ} + \left(\frac{40}{\sqrt{\frac{40}{\tan 40^\circ}}}\right)^2}$$
$$\therefore\ AC \approx 9.013 \text{ cm}$$

Perimeter of triangle ABC $= AB + BC + AC$

$$\approx 6.904 + 5.793 + 9.013 \text{ cm}$$
$$\approx 21.7 \text{ cm}$$

EXERCISE 7B

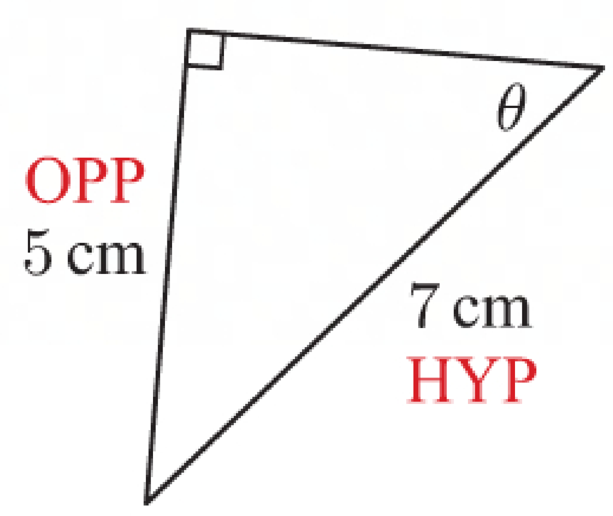
1 **a**



$\cos\theta = \dfrac{3}{5}$ $\quad \{\cos\theta = \dfrac{ADJ}{HYP}\}$

$$\therefore\ \theta = \cos^{-1}\left(\tfrac{3}{5}\right)$$
$$\therefore\ \theta \approx 53.1^\circ$$

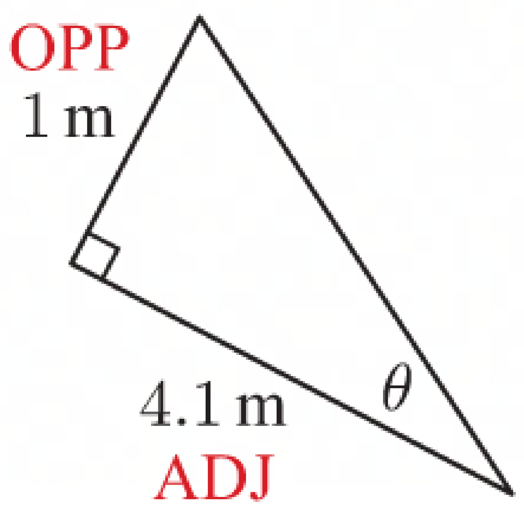
b



$\sin\theta = \dfrac{5}{7}$ $\quad \{\sin\theta = \dfrac{OPP}{HYP}\}$

$$\therefore\ \theta = \sin^{-1}\left(\tfrac{5}{7}\right)$$
$$\therefore\ \theta \approx 45.6^\circ$$

c

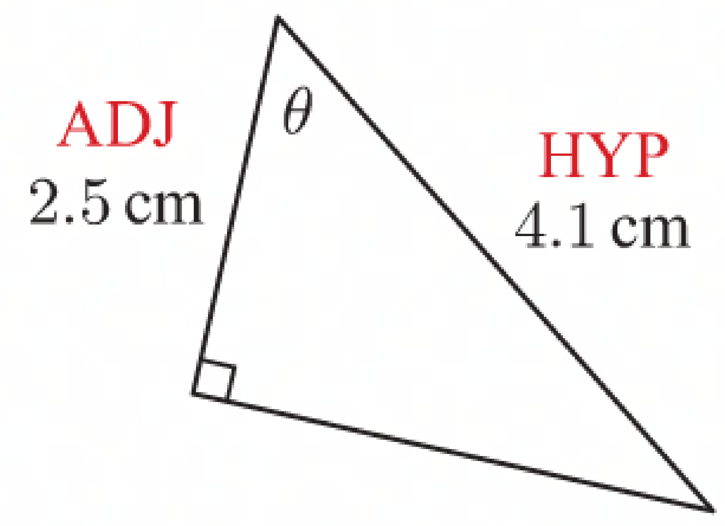


$\tan\theta = \frac{1}{4.1}$ $\quad\{\tan\theta = \frac{\text{OPP}}{\text{ADJ}}\}$

$\therefore\ \theta = \tan^{-1}\left(\frac{1}{4.1}\right)$

$\therefore\ \theta \approx 13.7°$

d

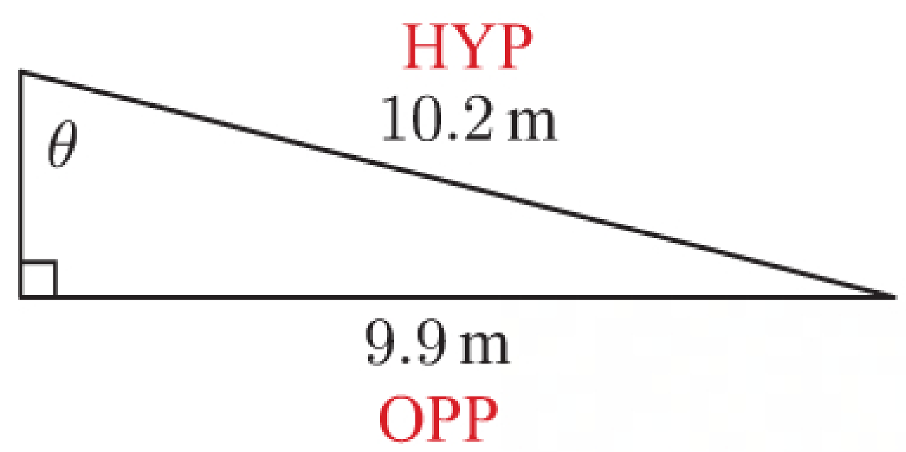


$\cos\theta = \frac{2.5}{4.1}$ $\quad\{\cos\theta = \frac{\text{ADJ}}{\text{HYP}}\}$

$\therefore\ \theta = \cos^{-1}\left(\frac{2.5}{4.1}\right)$

$\therefore\ \theta \approx 52.4°$

e

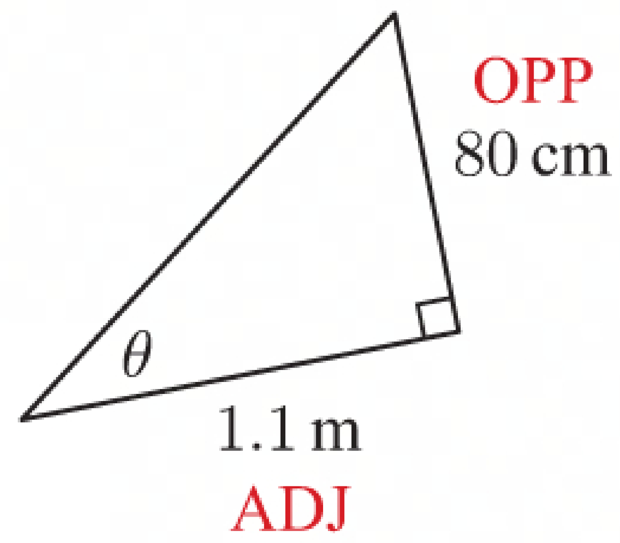


$\sin\theta = \frac{9.9}{10.2}$ $\quad\{\sin\theta = \frac{\text{OPP}}{\text{HYP}}\}$

$\therefore\ \theta = \sin^{-1}\left(\frac{9.9}{10.2}\right)$

$\therefore\ \theta \approx 76.1°$

f



$\tan\theta = \frac{80}{110}$ $\quad\{\tan\theta = \frac{\text{OPP}}{\text{ADJ}},$

$1.1 \text{ m} \equiv 110 \text{ cm}\}$

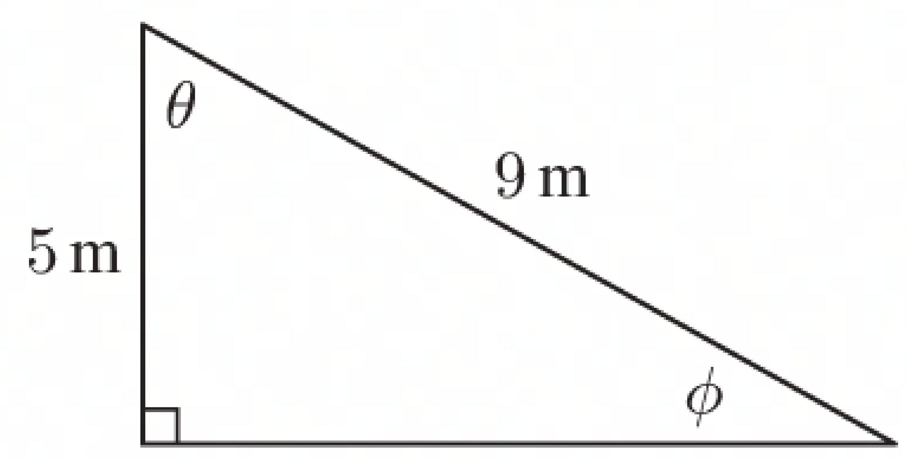
$\therefore\ \theta = \tan^{-1}\left(\frac{80}{110}\right)$

$\therefore\ \theta \approx 36.0°$

2 **a** $\cos\theta = \frac{5}{9}$ $\quad\{\cos\theta = \frac{\text{ADJ}}{\text{HYP}}\}$

$\therefore\ \theta = \cos^{-1}\left(\frac{5}{9}\right)$

$\therefore\ \theta \approx 56.3°$



b **i** $90° + \theta + \phi = 180°$ $\quad$ {angles in a triangle}

$\therefore\ \phi \approx 180° - 90° - 56.3°$ $\quad$ {using **a**}

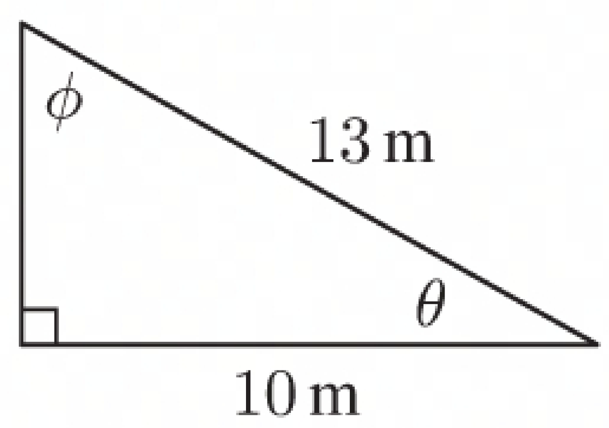
$\therefore\ \phi \approx 33.7°$

ii $\sin\phi = \frac{5}{9}$ $\quad\{\sin\phi = \frac{\text{OPP}}{\text{HYP}}\}$

$\therefore\ \phi = \sin^{-1}\left(\frac{5}{9}\right)$

$\therefore\ \phi \approx 33.7°$

3 **a**



$\cos\theta = \frac{10}{13}$ $\quad\{\cos\theta = \frac{\text{ADJ}}{\text{HYP}}\}$

$\therefore\ \theta = \cos^{-1}\left(\frac{10}{13}\right)$

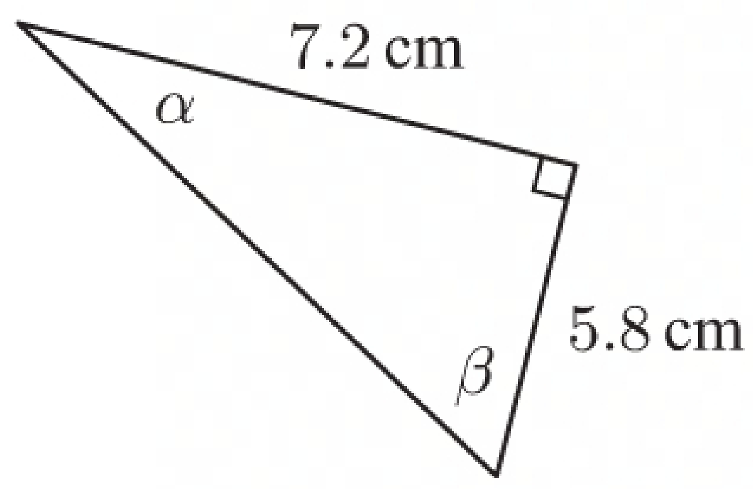
$\therefore\ \theta \approx 39.7°$

$\sin\phi = \frac{10}{13}$ $\quad\{\sin\phi = \frac{\text{OPP}}{\text{HYP}}\}$

$\therefore\ \phi = \sin^{-1}\left(\frac{10}{13}\right)$

$\therefore\ \phi \approx 50.3°$

b



$\tan\alpha = \frac{5.8}{7.2}$ $\quad\{\tan\alpha = \frac{\text{OPP}}{\text{ADJ}}\}$

$\therefore\ \alpha = \tan^{-1}\left(\frac{5.8}{7.2}\right)$

$\therefore\ \alpha \approx 38.9°$

$\tan\beta = \frac{7.2}{5.8}$ $\quad\{\tan\beta = \frac{\text{OPP}}{\text{ADJ}}\}$

$\therefore\ \beta = \tan^{-1}\left(\frac{7.2}{5.8}\right)$

$\therefore\ \beta \approx 51.1°$

c $\cos\theta = \frac{2.1}{4.4}$ $\{\cos\theta = \frac{\text{ADJ}}{\text{HYP}}\}$

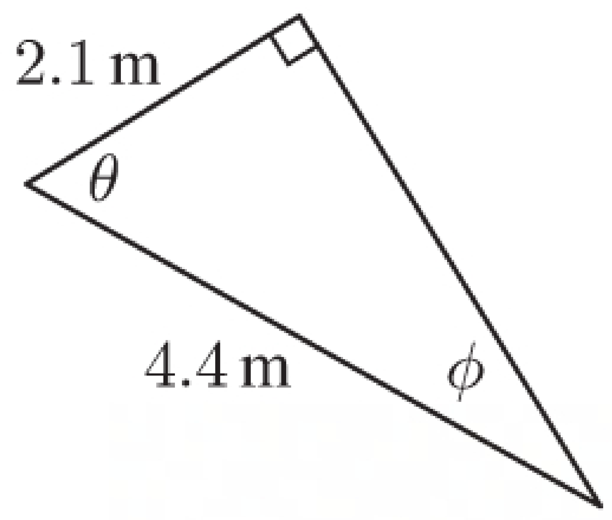
$\therefore\ \theta = \cos^{-1}\left(\frac{2.1}{4.4}\right)$

$\therefore\ \theta \approx 61.5°$

$\sin\phi = \frac{2.1}{4.4}$ $\{\sin\phi = \frac{\text{OPP}}{\text{HYP}}\}$

$\therefore\ \phi = \sin^{-1}\left(\frac{2.1}{4.4}\right)$

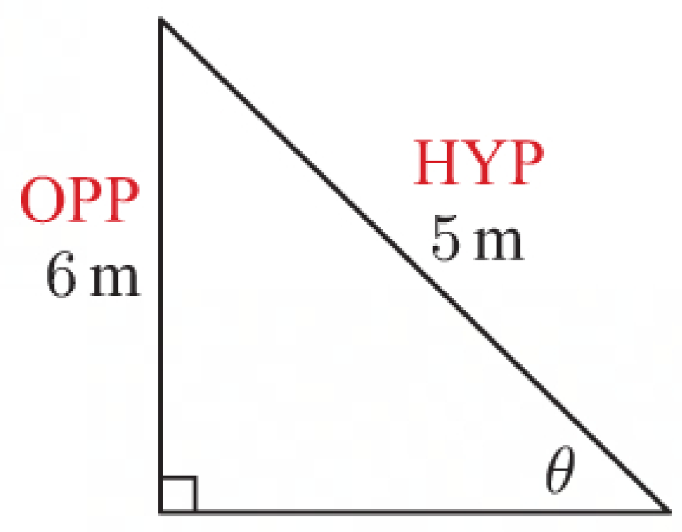
$\therefore\ \phi \approx 28.5°$



4 **a** $\sin\theta = \frac{6}{5}$ $\{\sin\theta = \frac{\text{OPP}}{\text{HYP}}\}$

$\therefore\ \theta = \sin^{-1}\left(\frac{6}{5}\right)$ which is undefined

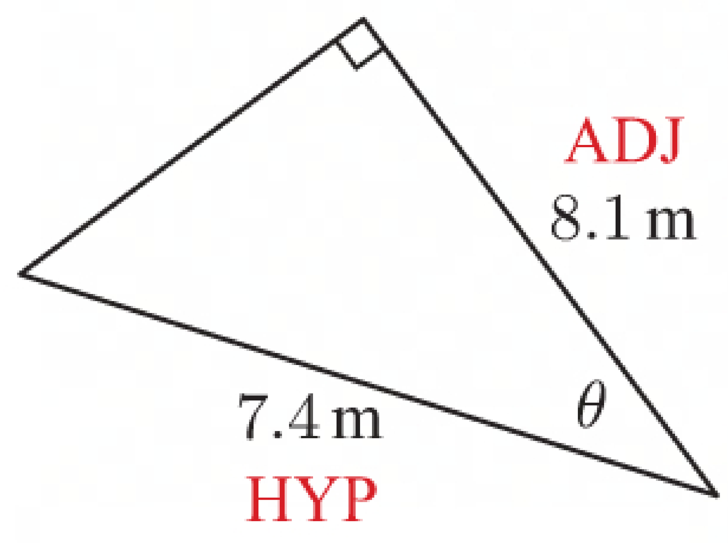
This triangle cannot be drawn with the given dimensions. (In any right angled triangle, the hypotenuse is the longest side.)



b $\cos\theta = \frac{8.1}{7.4}$ $\{\cos\theta = \frac{\text{ADJ}}{\text{HYP}}\}$

$\therefore\ \theta = \cos^{-1}\left(\frac{8.1}{7.4}\right)$ which is undefined

This triangle cannot be drawn with the given dimensions. (In any right angled triangle, the hypotenuse is the longest side.)

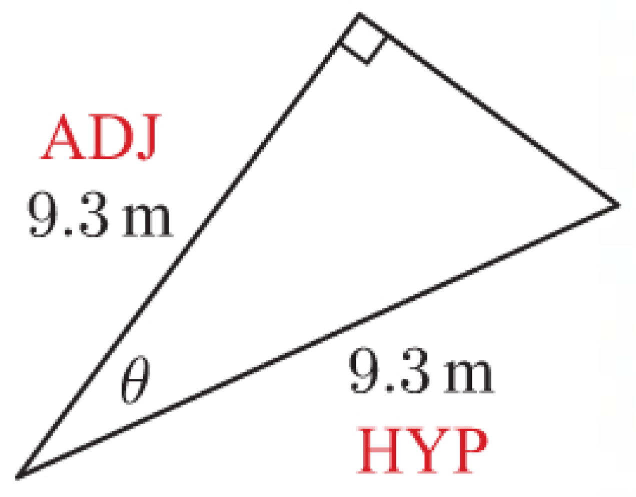


c $\cos\theta = \frac{9.3}{9.3}$ $\{\cos\theta = \frac{\text{ADJ}}{\text{HYP}}\}$

$\therefore\ \theta = \cos^{-1}\left(\frac{9.3}{9.3}\right)$

$\therefore\ \theta = 0°$

The resultant figure is not a triangle, but a straight line of length 9.3 m.



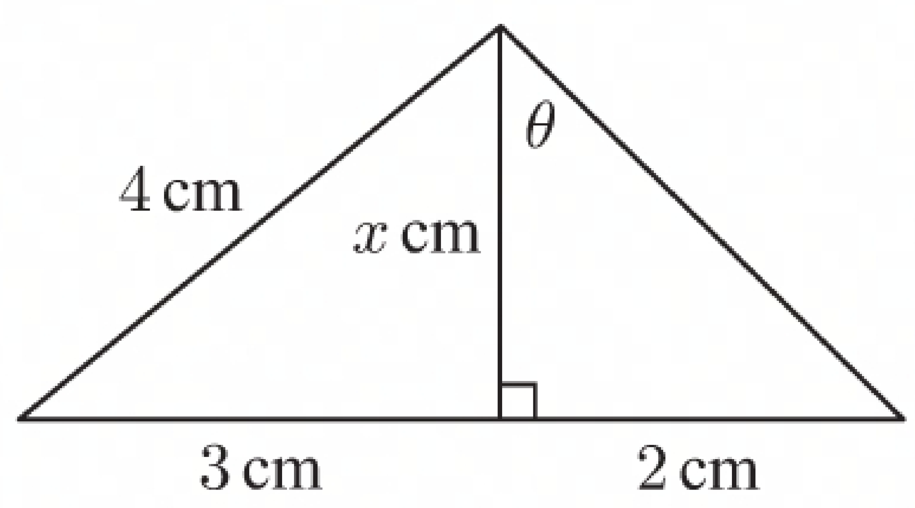
5 **a** $x^2 + 3^2 = 4^2$ {Pythagoras}

$\therefore\ x^2 + 9 = 16$

$\therefore\ x^2 = 7$

$\therefore\ x = \sqrt{7}$ {as $x > 0$}

$\therefore\ x \approx 2.65$



$\tan\theta = \frac{2}{x}$ $\{\tan\theta = \frac{\text{OPP}}{\text{ADJ}}\}$

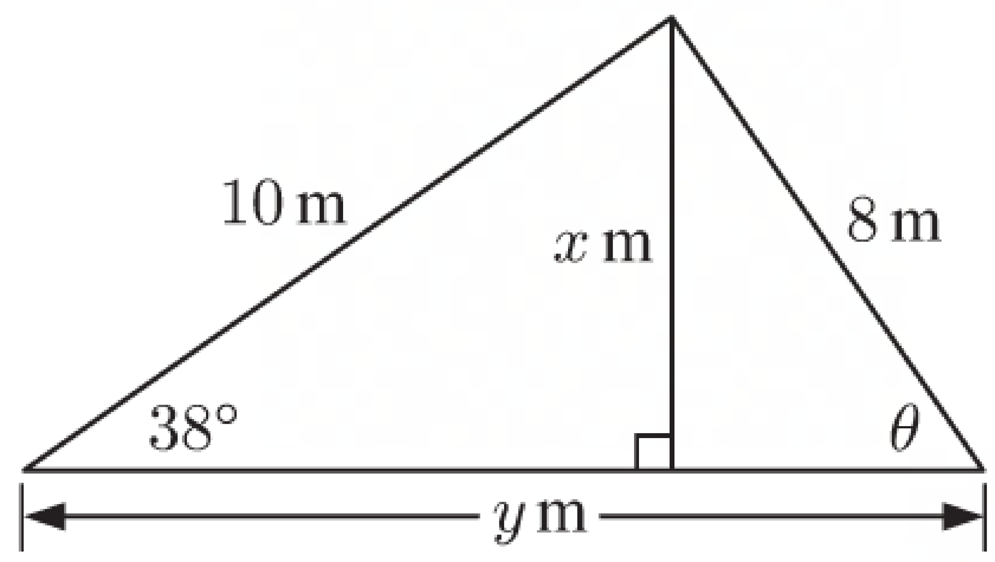
$\therefore\ \theta = \tan^{-1}\left(\frac{2}{\sqrt{7}}\right)$

$\therefore\ \theta \approx 37.1°$

b $\sin 38^\circ = \dfrac{x}{10}$ $\quad\{\sin\theta = \dfrac{\text{OPP}}{\text{HYP}}\}$

$\therefore\ 10 \times \sin 38^\circ = x$

$\therefore\ x \approx 6.16$



$\sin\theta = \dfrac{x}{8}$ $\quad\{\sin\theta = \dfrac{\text{OPP}}{\text{HYP}}\}$

$\therefore\ \sin\theta = \dfrac{10 \times \sin 38^\circ}{8}$

$\therefore\ \theta = \sin^{-1}\left(\frac{10\times\sin 38^\circ}{8}\right)$

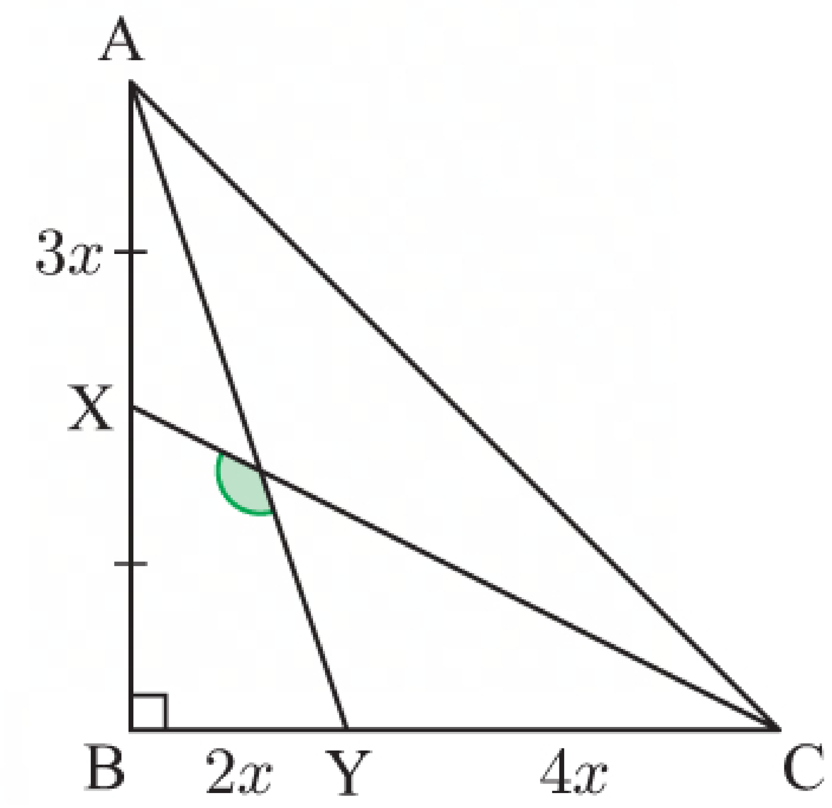
$\therefore\ \theta \approx 50.3^\circ$

$$\begin{aligned} y &= \sqrt{10^2 - x^2} + \sqrt{8^2 - x^2} \quad \{\text{Pythagoras}\} \\ &\approx \sqrt{100 - 6.16^2} + \sqrt{64 - 6.16^2} \\ &\approx 13.0 \end{aligned}$$

6 We are given that AB = BC, AX = BX,

and BY : YC = 1 : 2 $\therefore$ BY = $\frac{1}{2}$ × YC.

Let BY = $2x$ $\therefore$ YC = $4x$ and AX = BX = $3x$.



In $\triangle$ABY, $\tan \widehat{AYB} = \dfrac{6x}{2x}$ $\quad\{\tan\theta = \dfrac{\text{OPP}}{\text{ADJ}}\}$

$\therefore\ \tan \widehat{AYB} = 3$

$\therefore\ \widehat{AYB} = \tan^{-1}(3)$

In $\triangle$XBC, $\tan \widehat{BXC} = \dfrac{6x}{3x}$ $\quad\{\tan\theta = \dfrac{\text{OPP}}{\text{ADJ}}\}$

$\therefore\ \tan \widehat{BXC} = 2$

$\therefore\ \widehat{BXC} = \tan^{-1}(2)$

$$\begin{aligned} \text{Shaded angle} &= 360^\circ - 90^\circ - \widehat{AYB} - \widehat{BXC} \quad \{\text{angles in a quadrilateral}\} \\ &= 270^\circ - \tan^{-1}(3) - \tan^{-1}(2) \\ &= 135^\circ \end{aligned}$$

7 $\widehat{AFB} = 180^\circ - 60^\circ - 30^\circ$ $\quad$ {angles in a triangle}

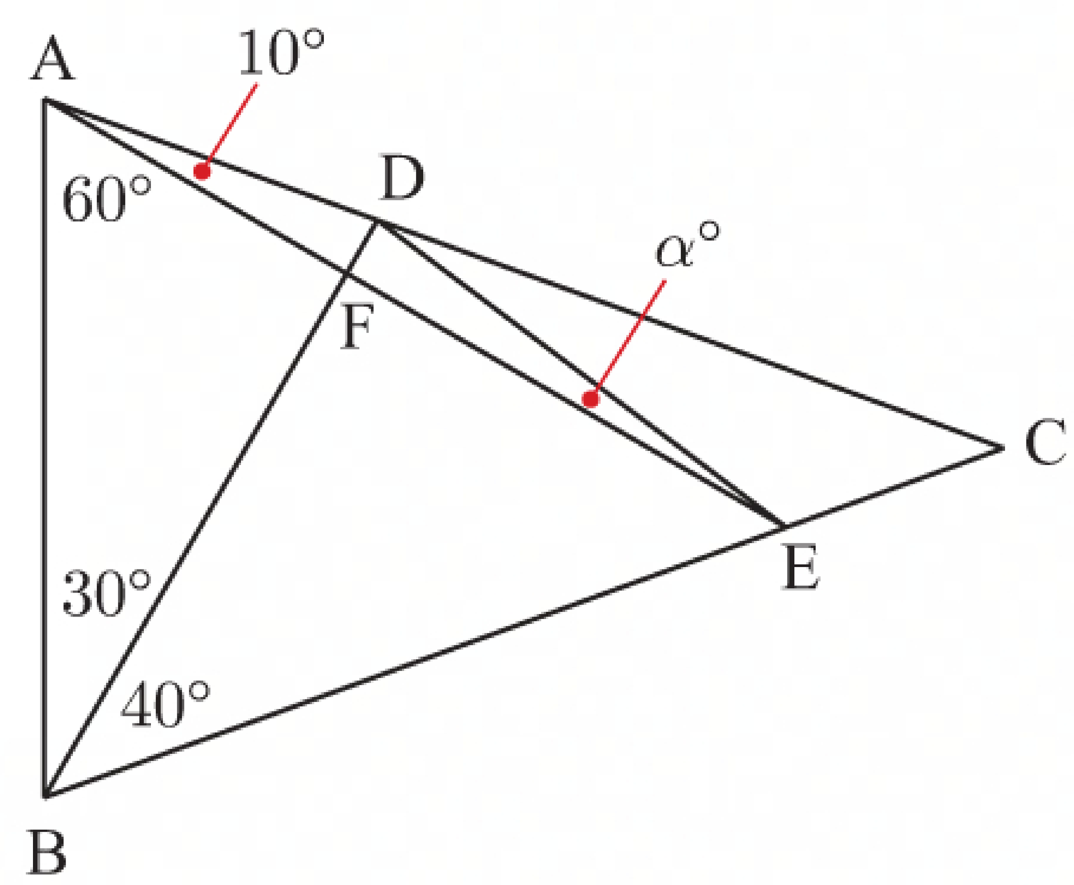
$= 90^\circ$

$\therefore\ \widehat{AFD} = 180^\circ - 90^\circ$ $\quad$ {angles on a line}

$= 90^\circ$

$\therefore\ \widehat{BFE} = 90^\circ$ $\quad$ {vertically opposite angles}

and $\widehat{DFE} = 90^\circ$ $\quad$ {vertically opposite angles}



$\therefore$ $\triangle$AFB, $\triangle$AFD, $\triangle$BFE, and $\triangle$DFE are all right angled at F.

In $\triangle$AFB, $\tan 30^\circ = \dfrac{\text{AF}}{\text{BF}}$ $\quad\{\tan\theta = \dfrac{\text{OPP}}{\text{ADJ}}\}$

In $\triangle$BFE, $\tan 40^\circ = \dfrac{\text{EF}}{\text{BF}}$

In $\triangle$AFD, $\tan 10^\circ = \dfrac{\text{DF}}{\text{AF}}$

$$\text{Now} \quad \tan \alpha^\circ = \frac{\text{DF}}{\text{EF}} \qquad \left\{\tan\theta = \frac{\text{OPP}}{\text{ADJ}}\right\}$$

$$= \frac{\text{DF}}{\text{AF}} \times \frac{\text{AF}}{\text{BF}} \times \frac{\text{BF}}{\text{EF}}$$

$$= \tan 10^\circ \times \tan 30^\circ \times \frac{1}{\tan 40^\circ}$$

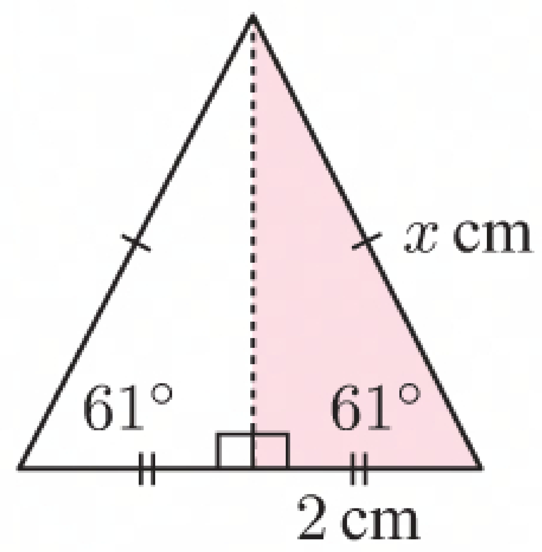
$$= \frac{\tan 10^\circ \times \tan 30^\circ}{\tan 40^\circ}$$

$$\therefore \quad \alpha^\circ = \tan^{-1}\left(\frac{\tan 10^\circ \times \tan 30^\circ}{\tan 40^\circ}\right)$$

$$\therefore \quad \alpha \approx 6.92$$

EXERCISE 7C

1 **a**



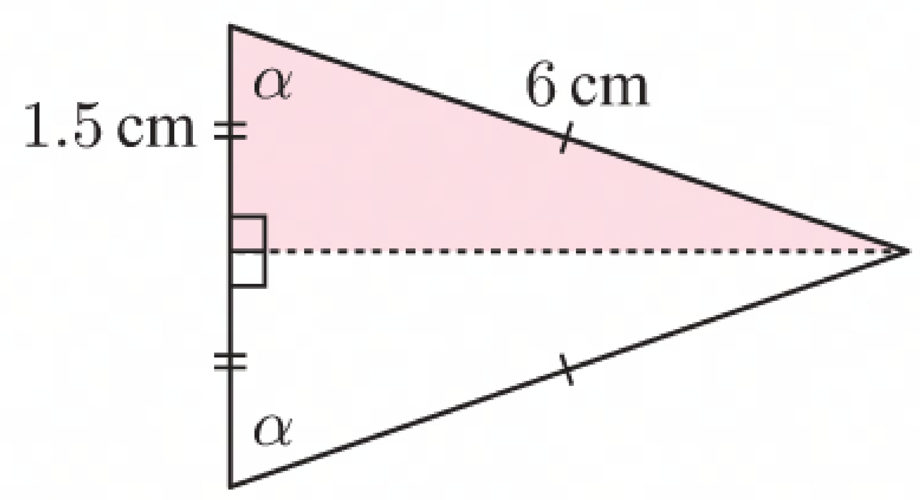
In the shaded right angled triangle,

$$\cos 61^\circ = \frac{2}{x}$$

$$\therefore \quad x = \frac{2}{\cos 61^\circ}$$

$$\therefore \quad x \approx 4.13$$

b



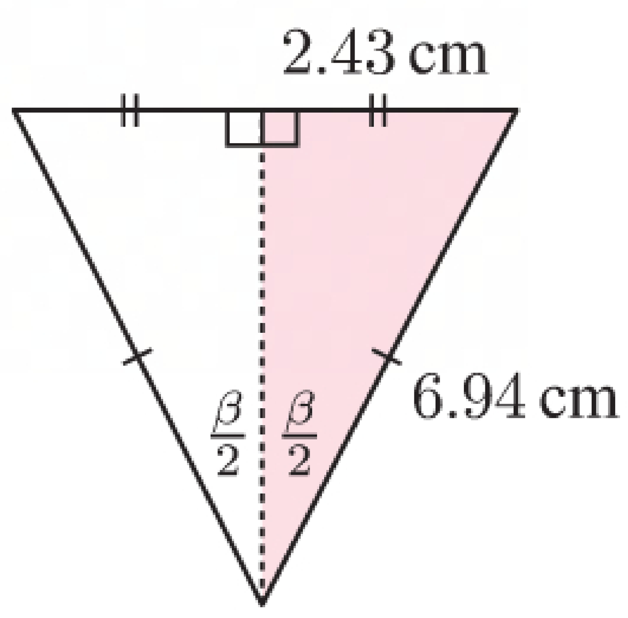
In the shaded right angled triangle,

$$\cos\alpha = \frac{1.5}{6}$$

$$\therefore \quad \alpha = \cos^{-1}\left(\tfrac{1.5}{6}\right)$$

$$\therefore \quad \alpha \approx 75.5^\circ$$

c



In the shaded right angled triangle,

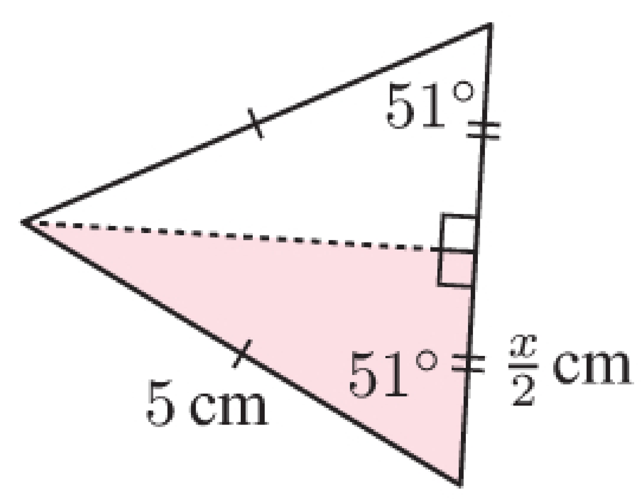
$$\sin\frac{\beta}{2} = \frac{2.43}{6.94}$$

$$\therefore \quad \frac{\beta}{2} = \sin^{-1}\left(\tfrac{2.43}{6.94}\right)$$

$$\therefore \quad \beta = 2\sin^{-1}\left(\tfrac{2.43}{6.94}\right)$$

$$\therefore \quad \beta \approx 41.0^\circ$$

d



In the shaded right angled triangle,

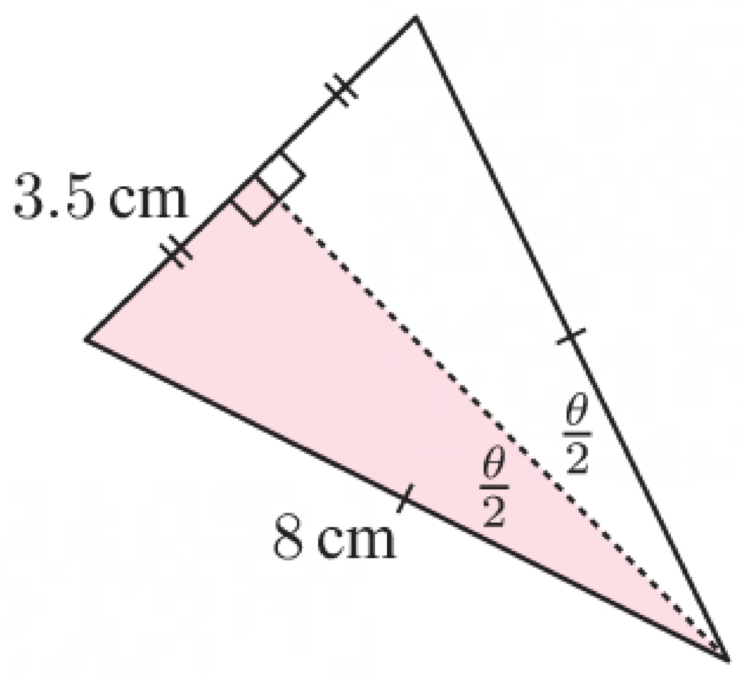
$$\cos 51^\circ = \frac{\left(\frac{x}{2}\right)}{5}$$

$$\therefore \quad 5 \times \cos 51^\circ = \frac{x}{2}$$

$$\therefore \quad x = 2 \times 5 \times \cos 51^\circ$$

$$\therefore \quad x \approx 6.29$$

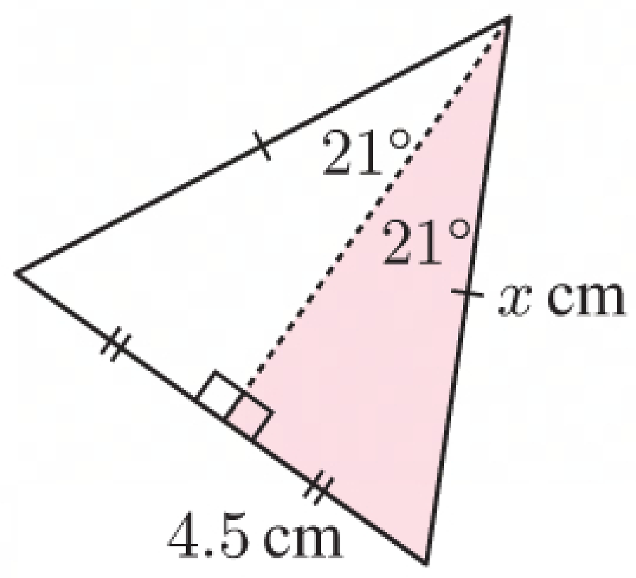
e



In the shaded right angled triangle,

$$\sin\frac{\theta}{2} = \frac{3.5}{8}$$
$$\therefore \quad \frac{\theta}{2} = \sin^{-1}\left(\tfrac{3.5}{8}\right)$$
$$\therefore \quad \theta = 2\sin^{-1}\left(\tfrac{3.5}{8}\right)$$
$$\therefore \quad \theta \approx 51.9°$$

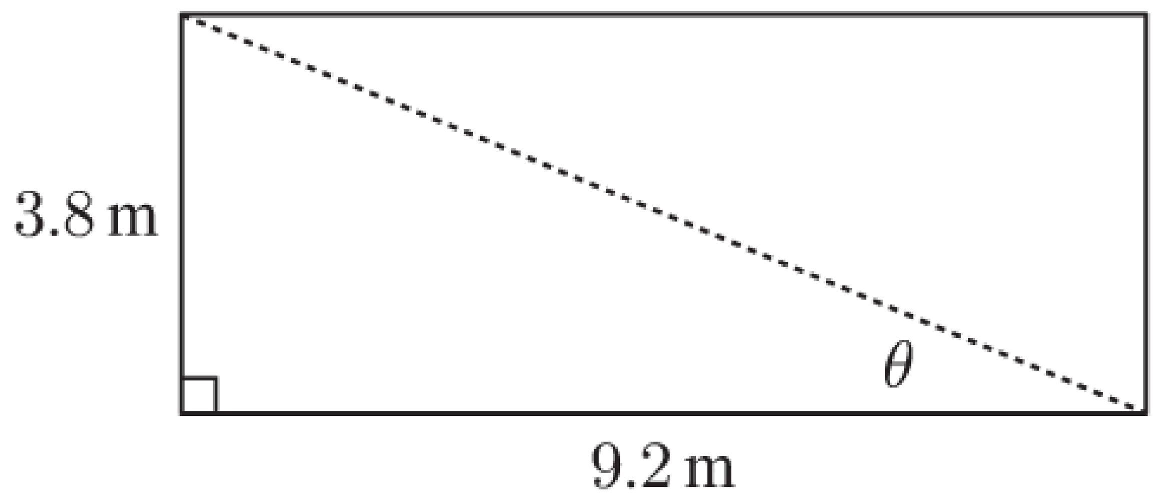
f



In the shaded right angled triangle,

$$\sin 21° = \frac{4.5}{x}$$
$$\therefore \quad x = \frac{4.5}{\sin 21°}$$
$$\therefore \quad x \approx 12.6$$

2

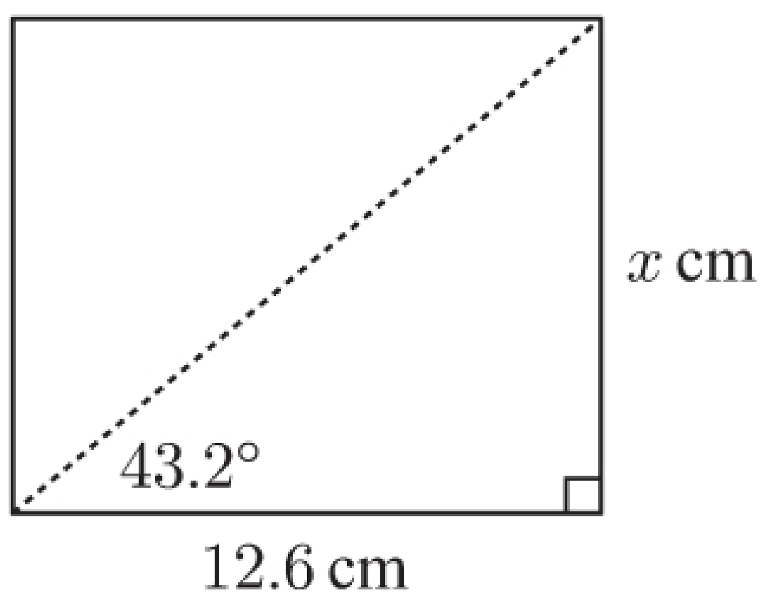


Let the angle between the diagonal and the longer side be θ.

$$\tan\theta = \frac{3.8}{9.2}$$
$$\therefore \quad \theta = \tan^{-1}\left(\tfrac{3.8}{9.2}\right)$$
$$\therefore \quad \theta \approx 22.4°$$

So, the angle between the diagonal and the longer side is about 22.4°.

3



Let the shorter side have length x cm.

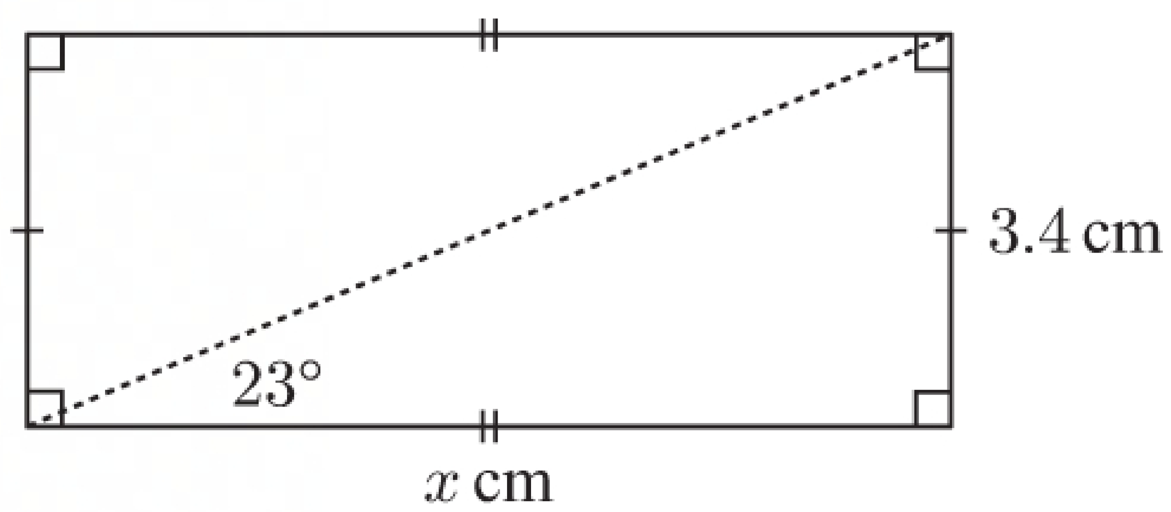
$$\tan 43.2° = \frac{x}{12.6}$$
$$\therefore \quad 12.6 \times \tan 43.2° = x$$
$$\therefore \quad x \approx 11.8$$

So, the length of the shorter side is about 11.8 cm.

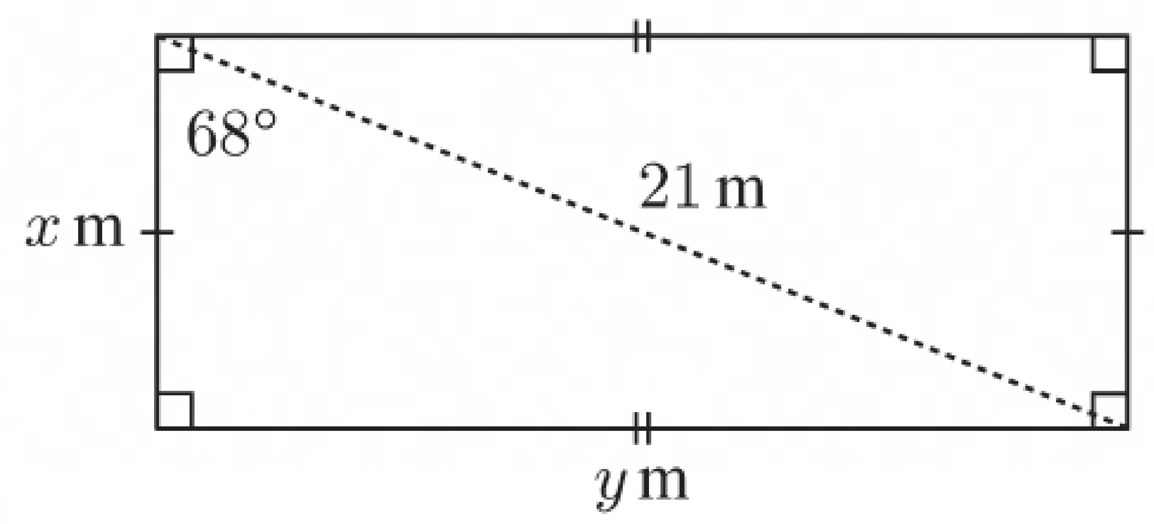
4 **a**

$$\tan 23° = \frac{3.4}{x}$$
$$\therefore \quad x = \frac{3.4}{\tan 23°}$$
$$\approx 8.01$$

Area of rectangle $=$ length $\times$ width
$\approx 8.01 \times 3.4\ \text{cm}^2$
$\approx 27.2\ \text{cm}^2$



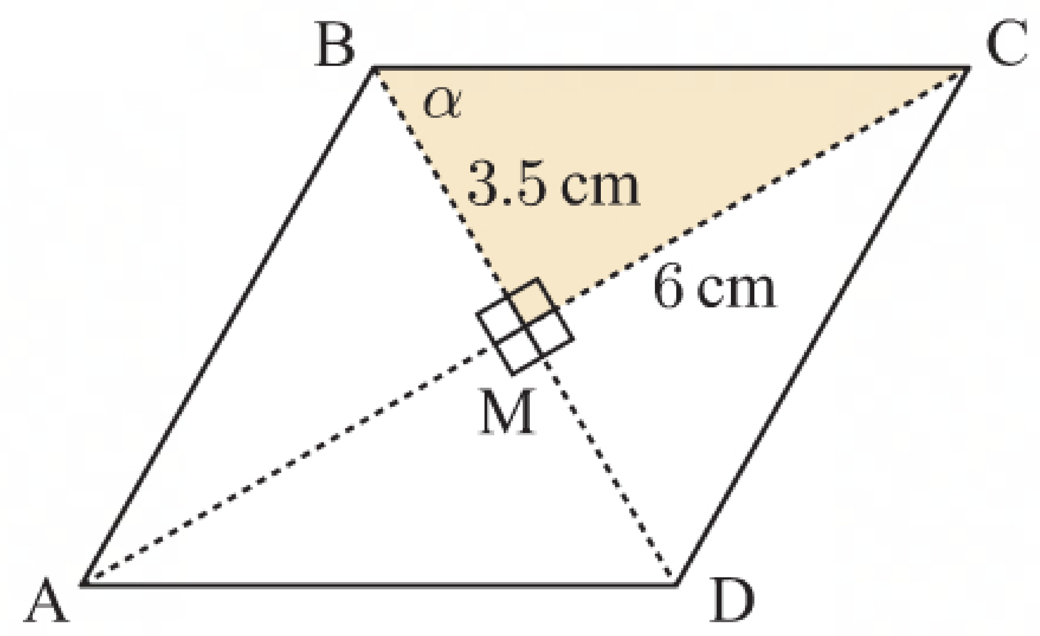
b



$$\cos 68° = \frac{x}{21}$$
$$\therefore\ 21 \times \cos 68° = x$$
$$\therefore\ x \approx 7.87$$
$$\sin 68° = \frac{y}{21}$$
$$\therefore\ 21 \times \sin 68° = y$$
$$\therefore\ y \approx 19.5$$

$$\begin{aligned}\text{Area of rectangle} &= \text{length} \times \text{width}\\ &= y \times x\\ &\approx 19.5 \times 7.87\ \text{m}^2\\ &\approx 153\ \text{m}^2\end{aligned}$$

5

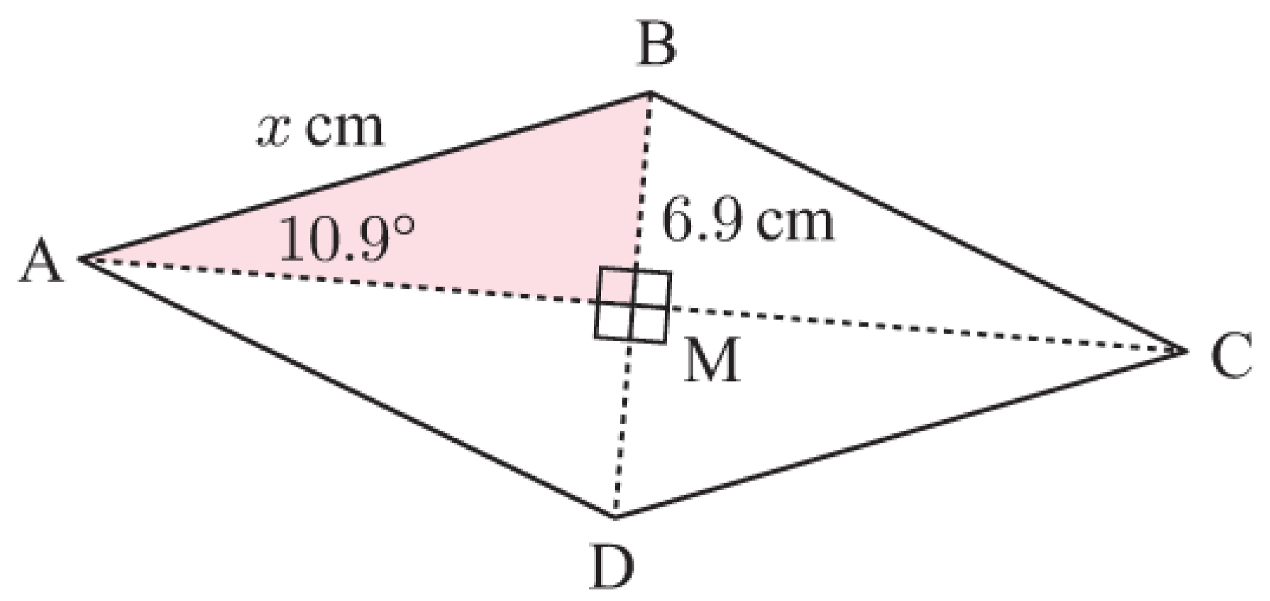


The diagonals bisect each other at right angles, so BM = 3.5 cm and CM = 6 cm.
In △BCM, α will be the larger non-right angle as it is opposite the longer side that is not the hypotenuse.

$$\tan\alpha = \frac{6}{3.5}$$
$$\therefore\ \alpha = \tan^{-1}\left(\tfrac{6}{3.5}\right)$$
$$\therefore\ \alpha \approx 59.74°$$

The required angle is 2α as the diagonals bisect the angles at each vertex.
So, the angle is about $2 \times 59.74° \approx 119°$.

6



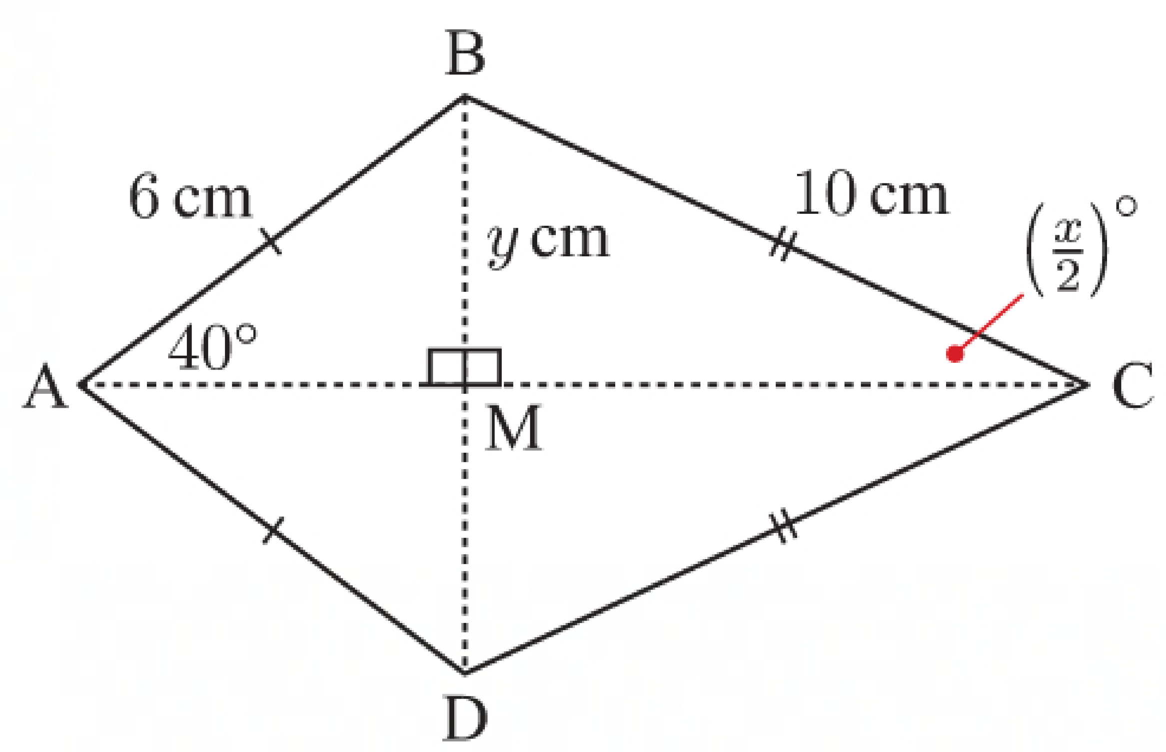
The diagonals bisect each other at right angles, so BM = 6.9 cm. The diagonals also bisect the angles at the vertices, so $B\widehat{A}M = 10.9°$.

In △ABM, $\sin 10.9° = \dfrac{6.9}{x}$

$$\therefore\ x = \frac{6.9}{\sin 10.9°}$$
$$\therefore\ x \approx 36.5$$

So, the lengths of the sides of the rhombus are about 36.5 cm.

7 a



The diagonals divide the kite into four right angled triangles.
Let BM = y cm.

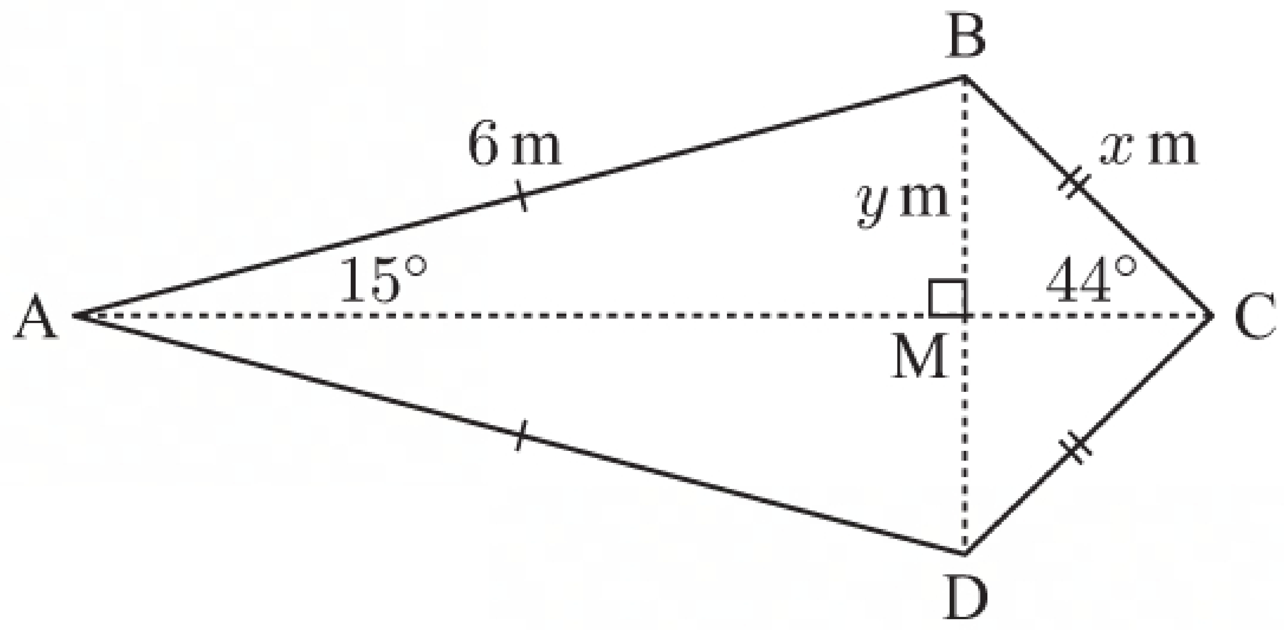
In △ABM, $\sin 40° = \dfrac{y}{6}$

$$\therefore\ y = 6 \times \sin 40°$$

In △BCM, $\sin\left(\frac{x}{2}\right)^\circ = \dfrac{y}{10} = \dfrac{6 \times \sin 40°}{10}$

$$\therefore\ \left(\frac{x}{2}\right)^\circ = \sin^{-1}\left(\frac{6 \times \sin 40°}{10}\right)$$
$$\therefore\ x° = 2 \times \sin^{-1}\left(\frac{6 \times \sin 40°}{10}\right)$$
$$\therefore\ x \approx 45.4$$

b



The diagonals divide the kite into four right angled triangles.

Let BM $= y$ m.

In $\triangle$ABM, $\sin 15° = \dfrac{y}{6}$

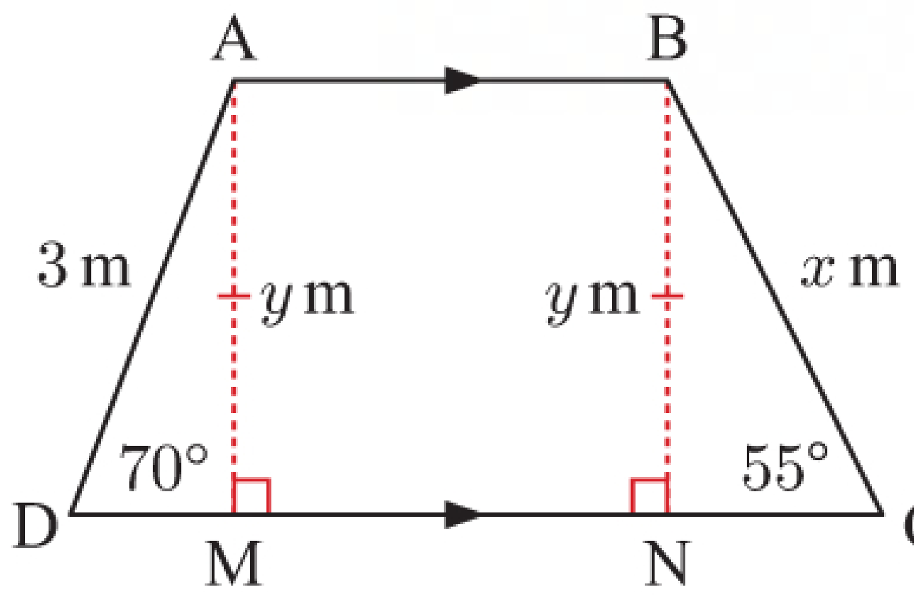
$\therefore\ y = 6 \times \sin 15°$

In $\triangle$BCM, $\sin 44° = \dfrac{y}{x} = \dfrac{6 \times \sin 15°}{x}$

$\therefore\ x = \dfrac{6 \times \sin 15°}{\sin 44°}$

$\therefore\ x \approx 2.24$

8 **a**



We draw perpendiculars [AM] and [BN] to [DC], creating two right angled triangles and the rectangle ABNM.

In $\triangle$ADM, $\sin 70° = \dfrac{y}{3}$

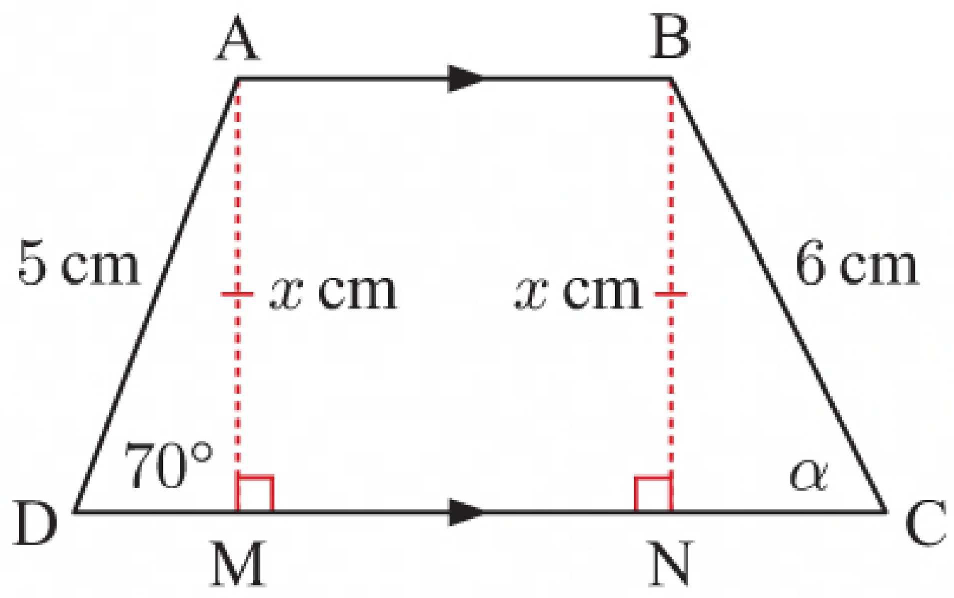
$\therefore\ y = 3 \times \sin 70°$

In $\triangle$BCN, $\sin 55° = \dfrac{y}{x} = \dfrac{3 \times \sin 70°}{x}$

$\therefore\ x = \dfrac{3 \times \sin 70°}{\sin 55°}$

$\therefore\ x \approx 3.44$

b



We draw perpendiculars [AM] and [BN] to [DC], creating two right angled triangles and the rectangle ABNM.

In $\triangle$ADM, $\sin 70° = \dfrac{x}{5}$

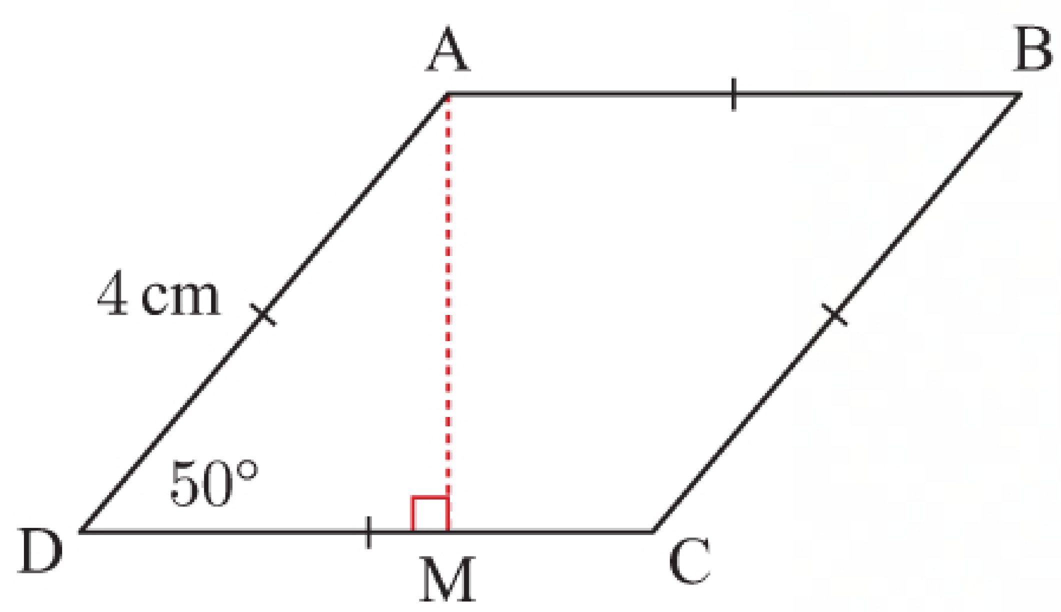
$\therefore\ x = 5 \times \sin 70°$

In $\triangle$BCN, $\sin \alpha = \dfrac{x}{6} = \dfrac{5 \times \sin 70°}{6}$

$\therefore\ \alpha = \sin^{-1}\left(\dfrac{5 \times \sin 70°}{6}\right)$

$\therefore\ \alpha \approx 51.5°$

9 **a** We draw perpendicular [AM] to [DC], creating a right angled triangle AMD.

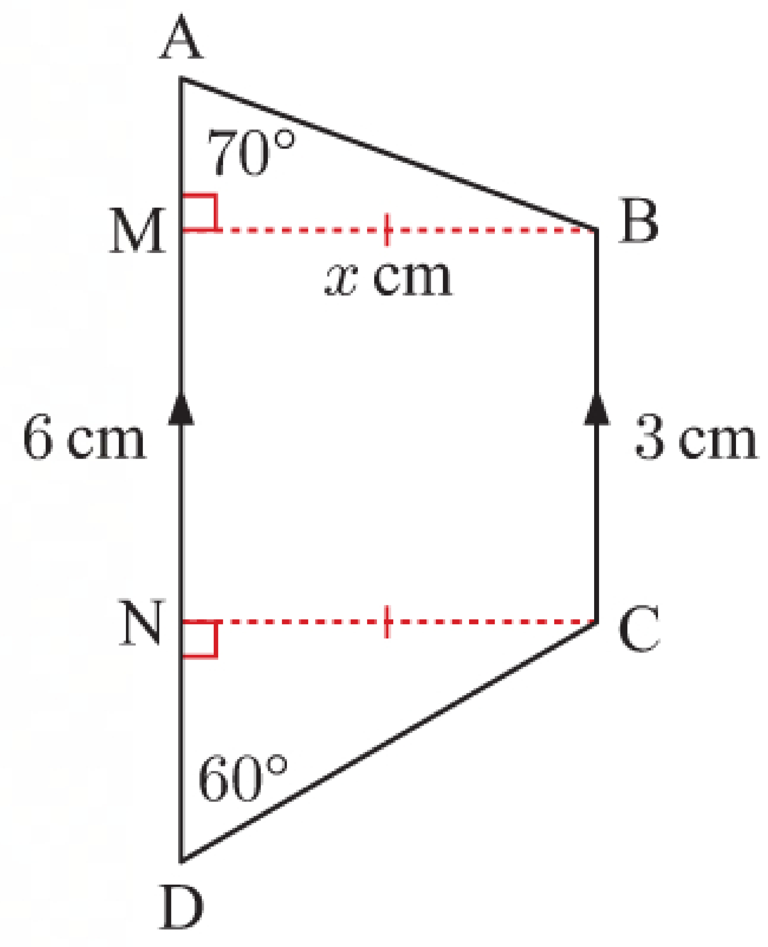


In $\triangle$AMD, $\sin 50° = \dfrac{\text{AM}}{4}$

$\therefore\ \text{AM} = 4 \sin 50°$

$$\begin{aligned}\text{Area of rhombus} &= \text{base} \times \text{height}\\ &= \text{DC} \times \text{AM}\\ &= 4 \times 4 \sin 50°\\ &\approx 12.3 \text{ cm}^2\end{aligned}$$

b We draw perpendiculars [BM] and [CN] to [AD], creating two right angled triangles and the rectangle BCNM.



Now, $\quad \text{AD} = \text{AM} + \text{MN} + \text{DN} = 6$

$$\therefore \quad \text{AM} + 3 + \text{DN} = 6 \qquad \{\text{MN} = \text{BC}\}$$

$$\therefore \quad \text{AM} + \text{DN} = 3$$

$$\therefore \quad \text{DN} = 3 - \text{AM} \qquad \text{.... (1)}$$

In $\triangle$ABM, $\quad \tan 70° = \dfrac{x}{\text{AM}}$

$$\therefore \quad x = \text{AM} \times \tan 70° \qquad \text{.... (2)}$$

In $\triangle$CDN, $\quad \tan 60° = \dfrac{x}{\text{DN}}$

$$\therefore \quad x = \text{DN} \times \tan 60°$$

$$\therefore \quad x = (3 - \text{AM}) \times \tan 60° \qquad \{\text{using (1)}\}$$

$$= 3\tan 60° - \text{AM} \times \tan 60°$$

So, $\quad \text{AM} \times \tan 70° = 3\tan 60° - \text{AM} \times \tan 60° \qquad \{\text{equating } x\}$

$$\therefore \quad \text{AM} \times \tan 70° + \text{AM} \times \tan 60° = 3\tan 60°$$

$$\therefore \quad \text{AM}(\tan 70° + \tan 60°) = 3\tan 60°$$

$$\therefore \quad \text{AM} = \frac{3\tan 60°}{\tan 70° + \tan 60°}$$

$$\therefore \quad \text{AM} \approx 1.16 \text{ cm}$$

$$\therefore \quad x \approx 1.160 \times \tan 70° \qquad \{\text{using (2)}\}$$

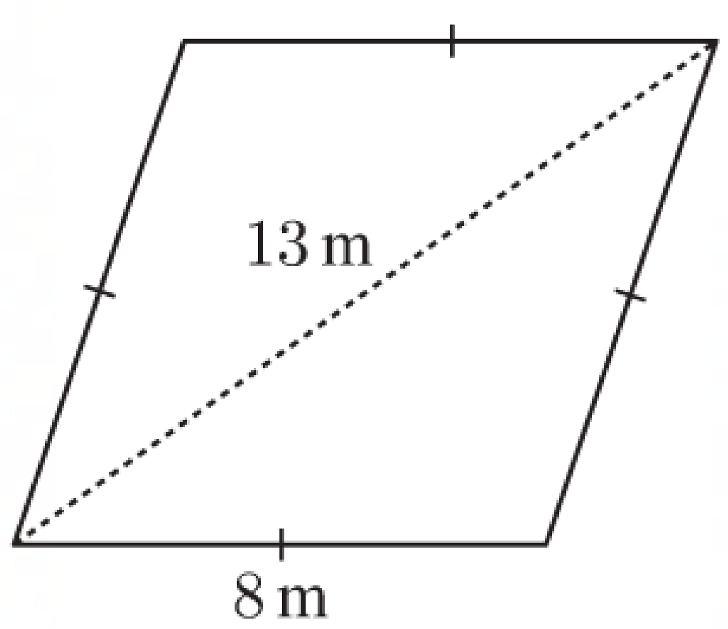
$$\approx 3.187$$

Area of trapezium $= \left(\dfrac{6+3}{2}\right) \times x$

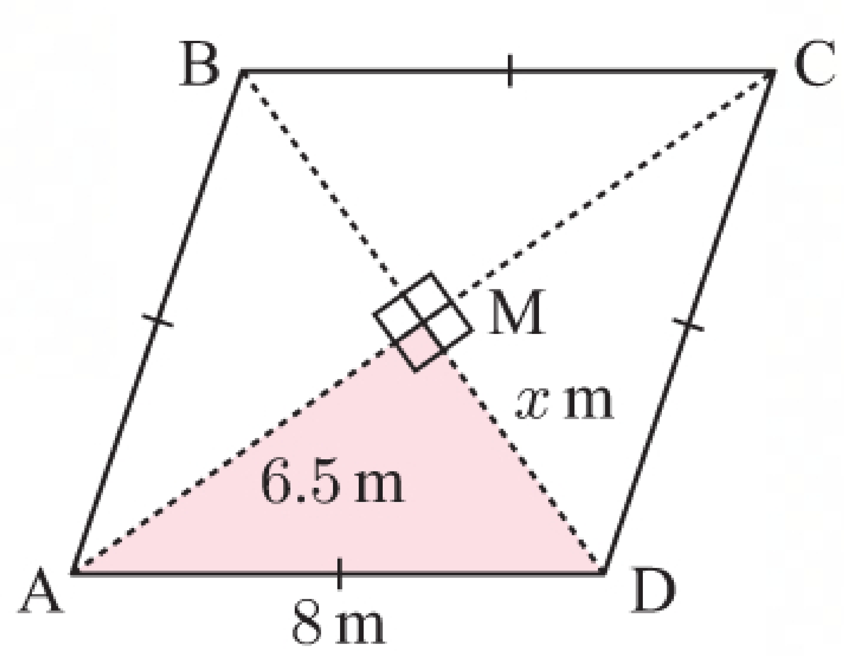
$$\approx \tfrac{9}{2} \times 3.187 \text{ cm}^2$$

$$\approx 14.3 \text{ cm}^2$$

10 **a**



b



The diagonals bisect each other at right angles, so AM $= 6.5$ m.

In $\triangle$AMD, $\quad 6.5^2 + x^2 = 8^2 \qquad \{\text{Pythagoras}\}$

$$\therefore \quad 42.25 + x^2 = 64$$

$$\therefore \quad x^2 = 21.75$$

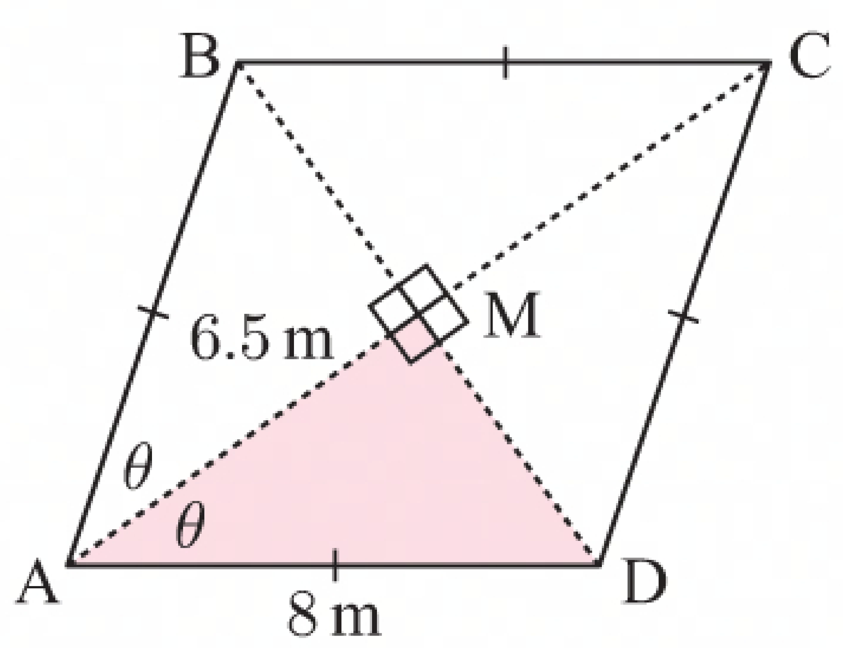
$$\therefore \quad x = \sqrt{21.75} \qquad \{\text{as } x > 0\}$$

$$\therefore \quad x \approx 4.663$$

The required length is $2x$ as the diagonals bisect each other.

So, the length of the shorter diagonal of the rhombus is approximately $2 \times 4.663 \approx 9.33$ m.

c



In $\triangle$AMD, θ will be the smallest non-right angle as it is opposite the shortest side.

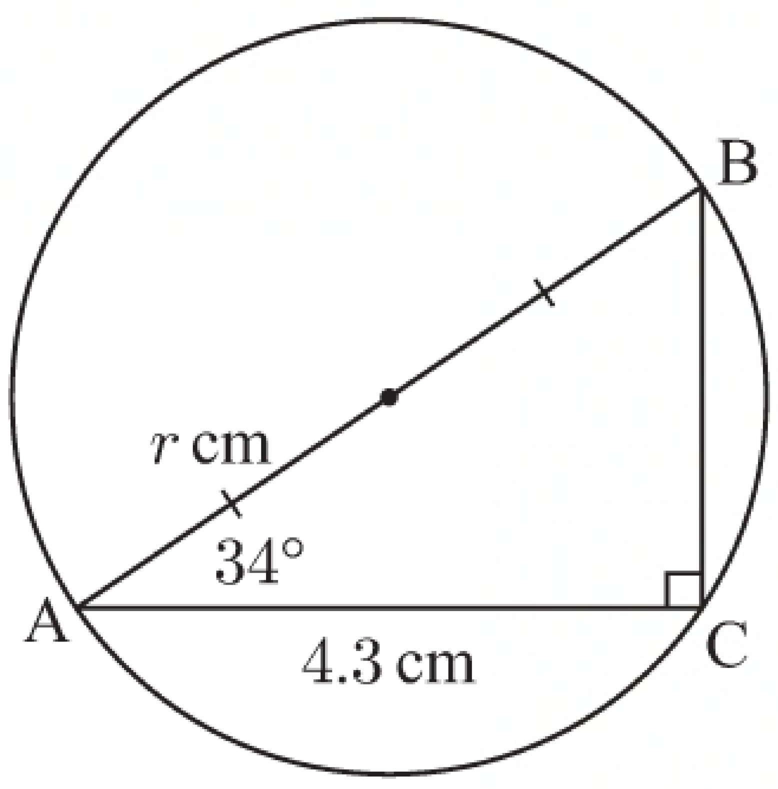
$$\cos\theta = \frac{6.5}{8}$$

$$\therefore \quad \theta = \cos^{-1}\left(\tfrac{6.5}{8}\right)$$

$$\therefore \quad \theta \approx 35.659°$$

The required angle is 2θ as the diagonals bisect the angles at each vertex.
So, the angle is about $2 \times 35.659° \approx 71.3°$.

11 **a**



$\widehat{ACB} = 90°$ {angle in a semi-circle}
$\therefore$ $\triangle$ABC is right angled at C.

$$\cos 34° = \frac{4.3}{AB}$$

$$\therefore \quad AB = \frac{4.3}{\cos 34°}$$

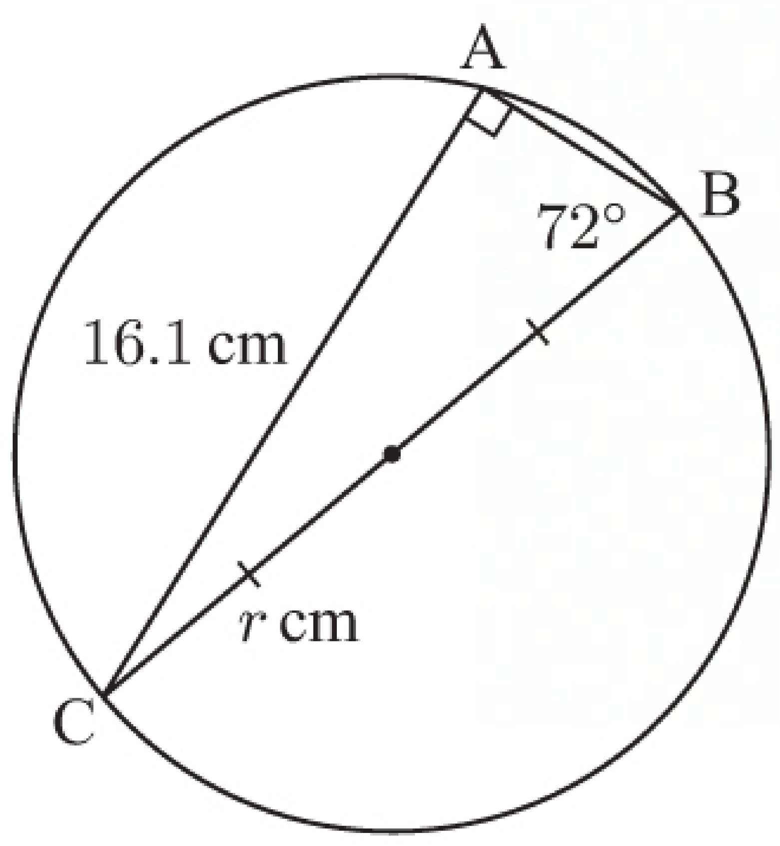
$$\therefore \quad 2r = \frac{4.3}{\cos 34°}$$

$$\therefore \quad r = \frac{4.3}{2 \times \cos 34°}$$

$$\approx 2.59$$

So, the radius is approximately 2.59 cm.

b



$\widehat{BAC} = 90°$ {angle in a semi-circle}
$\therefore$ $\triangle$ABC is right angled at A.

$$\sin 72° = \frac{16.1}{BC}$$

$$\therefore \quad BC = \frac{16.1}{\sin 72°}$$

$$\therefore \quad 2r = \frac{16.1}{\sin 72°}$$

$$\therefore \quad r = \frac{16.1}{2 \times \sin 72°}$$

$$\approx 8.46$$

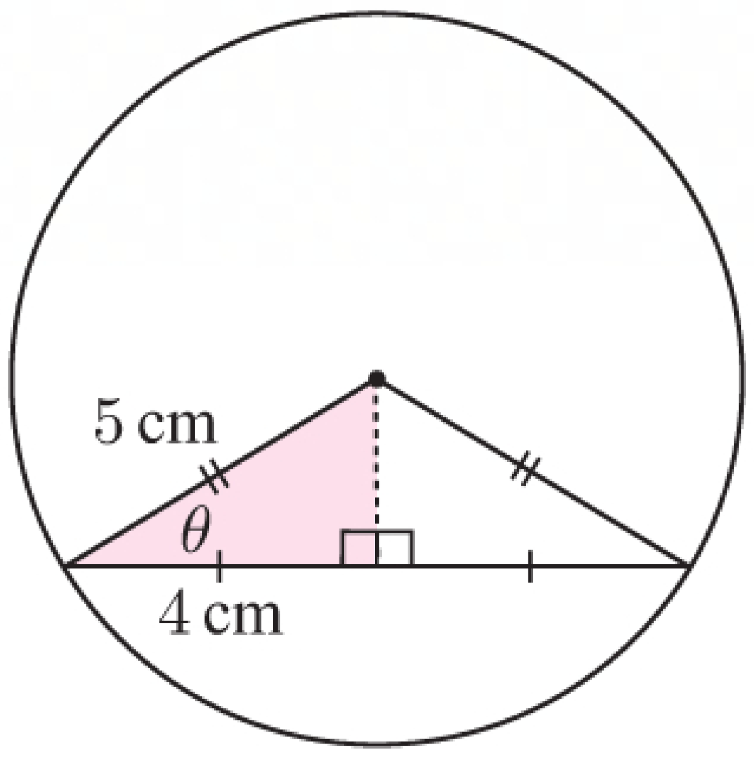
So, the radius is approximately 8.46 cm.

12 **a** We complete the isosceles triangle.
For the shaded triangle,

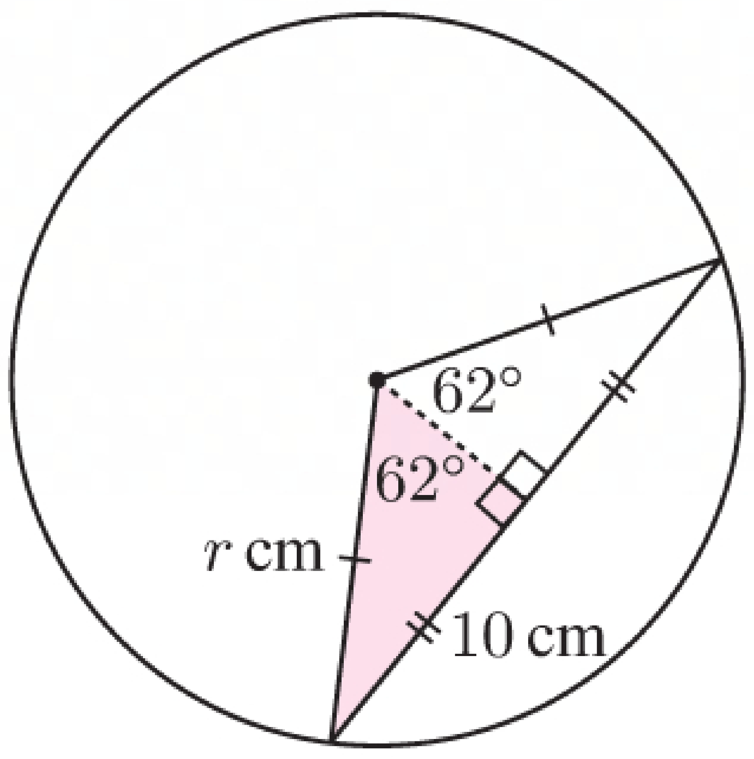
$$\cos\theta = \frac{4}{5}$$

$$\therefore \quad \theta = \cos^{-1}\left(\tfrac{4}{5}\right)$$

$$\therefore \quad \theta \approx 36.9°$$



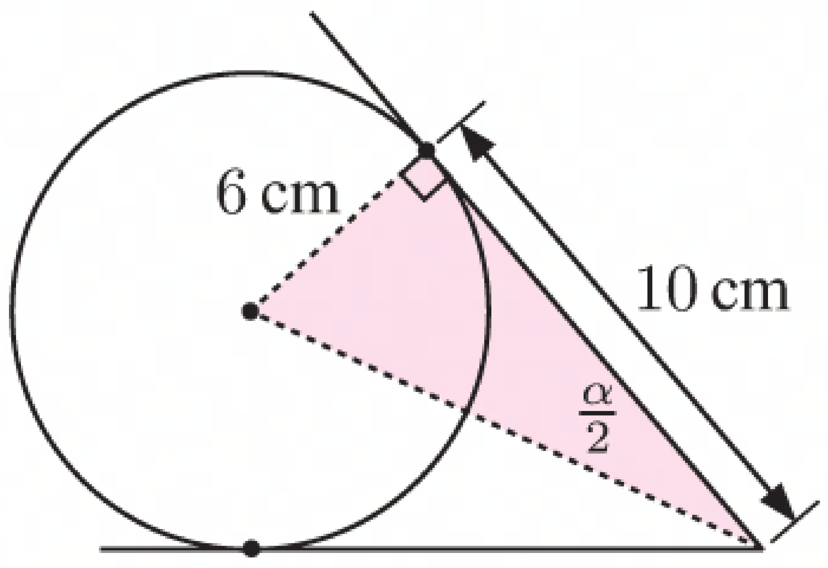
b



We construct the altitude as shown.
For the shaded triangle,

$$\sin 62^\circ = \frac{10}{r}$$
$$\therefore\ r = \frac{10}{\sin 62^\circ}$$
$$\therefore\ r \approx 11.3$$

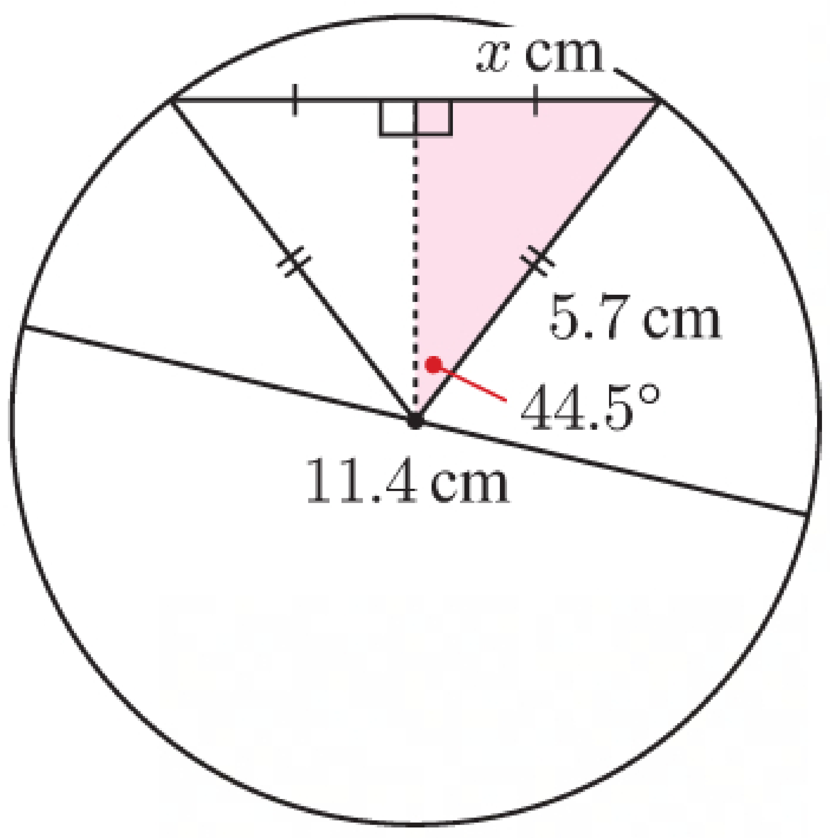
c



We construct the right angled triangle as shown.
For the shaded triangle,

$$\tan \tfrac{\alpha}{2} = \frac{6}{10}$$
$$\therefore\ \frac{\alpha}{2} = \tan^{-1}\left(\tfrac{6}{10}\right)$$
$$\therefore\ \alpha = 2\tan^{-1}\left(\tfrac{6}{10}\right)$$
$$\therefore\ \alpha \approx 61.9^\circ$$

13



We complete an isosceles triangle and add the perpendicular bisector of the base.

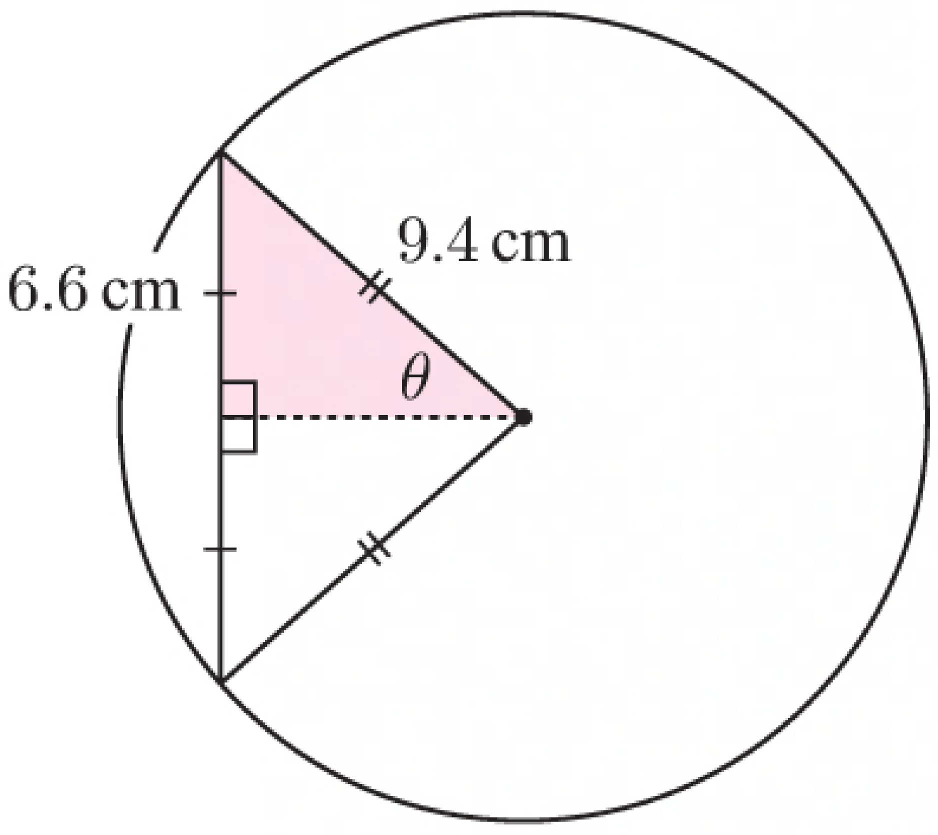
$\frac{1}{2} \times 11.4$ cm $= 5.7$ cm, $\frac{1}{2} \times 89^\circ = 44.5^\circ$

For the shaded triangle, $\sin 44.5^\circ = \dfrac{x}{5.7}$

$$\therefore\ 5.7 \times \sin 44.5^\circ = x$$
$$\therefore\ x \approx 3.995$$
$$\therefore\ 2x \approx 7.99$$

$\therefore$ the chord is about 7.99 cm long.

14



We complete an isosceles triangle and add the perpendicular bisector of the base.

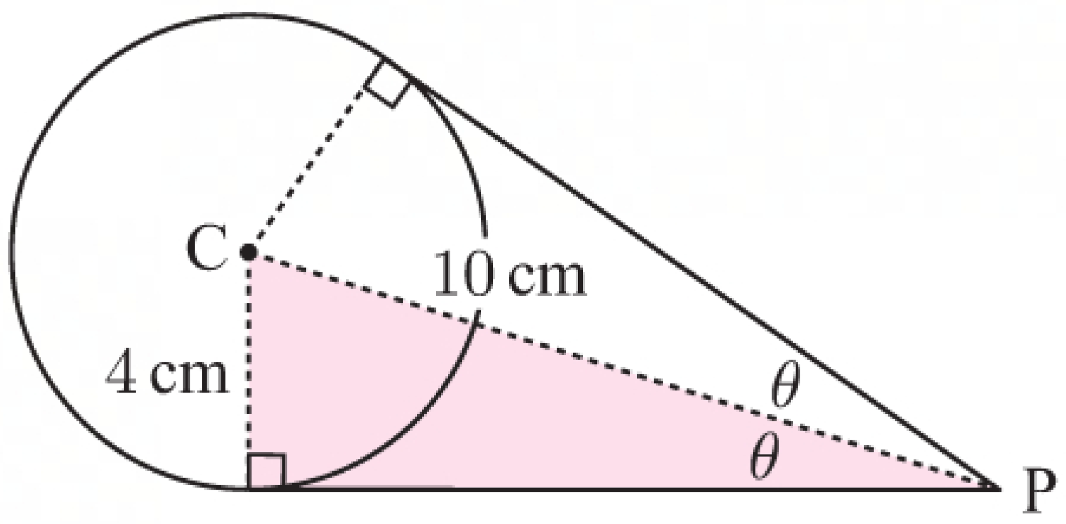
$\frac{1}{2} \times 13.2$ cm $= 6.6$ cm

For the shaded triangle, $\sin\theta = \dfrac{6.6}{9.4}$

$$\therefore\ \theta = \sin^{-1}\left(\tfrac{6.6}{9.4}\right)$$
$$\therefore\ 2\theta = 2\sin^{-1}\left(\tfrac{6.6}{9.4}\right)$$
$$\therefore\ 2\theta \approx 89.2^\circ$$

So, the angle subtended by the chord is about 89.2°.

15



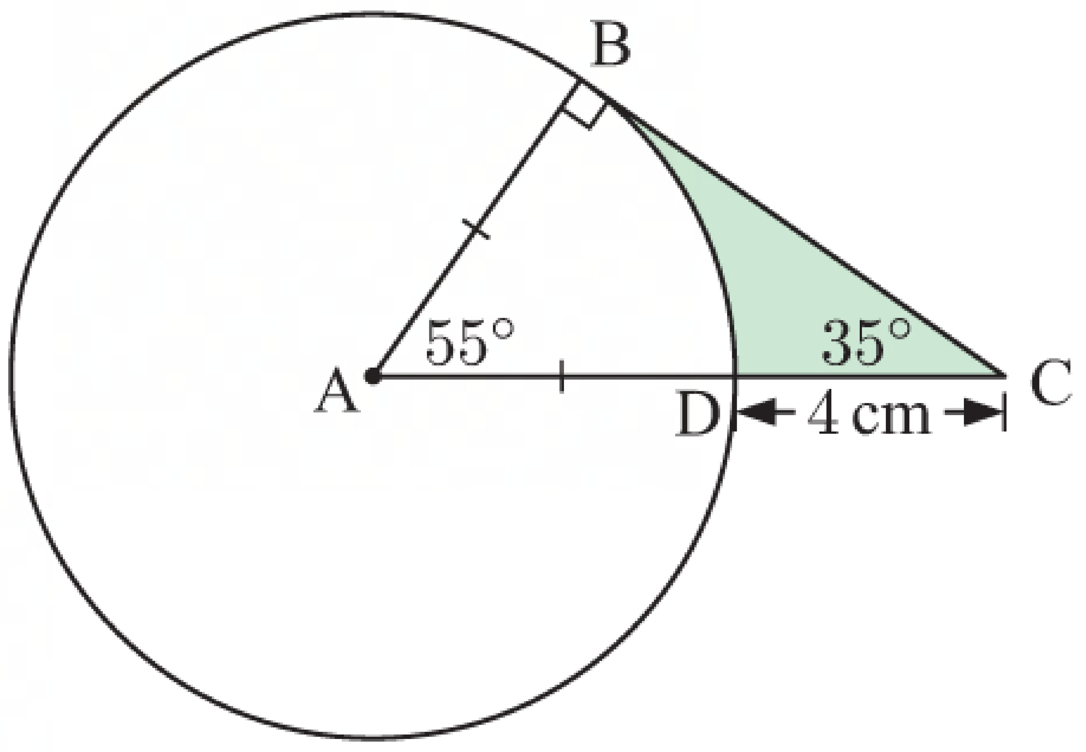
We draw the line from C to P.

For the shaded triangle, $\sin\theta = \dfrac{4}{10}$

$$\therefore\ \theta = \sin^{-1}\left(\tfrac{4}{10}\right)$$
$$\therefore\ 2\theta = 2\sin^{-1}\left(\tfrac{4}{10}\right)$$
$$\therefore\ 2\theta \approx 47.2^\circ$$

So, the angle between the tangents is about 47.2°.

16 We construct the right angled triangle ABC as shown, where AB = AD = radius of circle.



$$\sin 35° = \frac{AB}{AC}$$
$$= \frac{AB}{AD + DC}$$
$$= \frac{AB}{AB + 4} \quad \{AB = AD\}$$
$$\therefore \quad \sin 35°(AB + 4) = AB$$
$$\therefore \quad \sin 35° \times AB + 4 \times \sin 35° = AB$$
$$\therefore \quad AB(\sin 35° - 1) = -4 \times \sin 35°$$
$$\therefore \quad AB = \frac{-4 \times \sin 35°}{\sin 35° - 1}$$
$$\therefore \quad AB \approx 5.380 \text{ cm}$$

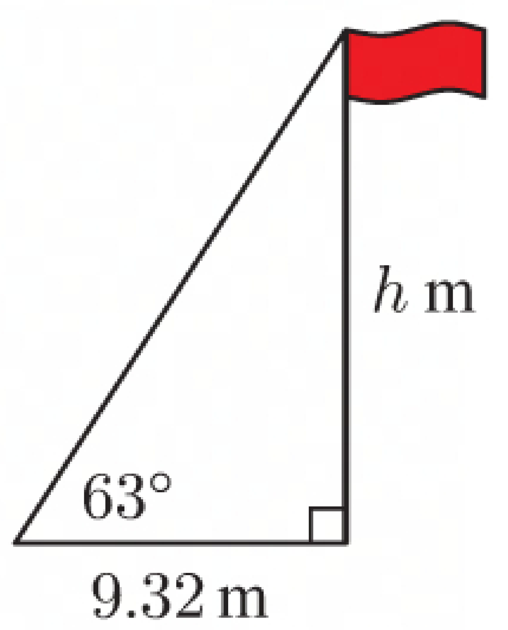
$$\tan 35° = \frac{AB}{BC}$$
$$\therefore \quad BC = \frac{AB}{\tan 35°}$$
$$\approx \frac{5.380}{\tan 35°}$$
$$\therefore \quad BC \approx 7.684 \text{ cm}$$

Shaded area = area of $\triangle$ABC − area of sector

$$= \tfrac{1}{2} \times \text{base} \times \text{height} \;-\; \frac{\theta}{360} \times \pi r^2$$
$$= \tfrac{1}{2} \times AB \times BC \;-\; \tfrac{55}{360} \times \pi \times AB^2$$
$$\approx \tfrac{1}{2} \times 5.380 \times 7.684 \;-\; \tfrac{55}{360} \times \pi \times (5.380)^2$$
$$\approx 6.78 \text{ cm}^2$$

EXERCISE 7D

1



Let the flagpole's height be h m.

For the 63° angle, OPP = h m, ADJ = 9.32 m

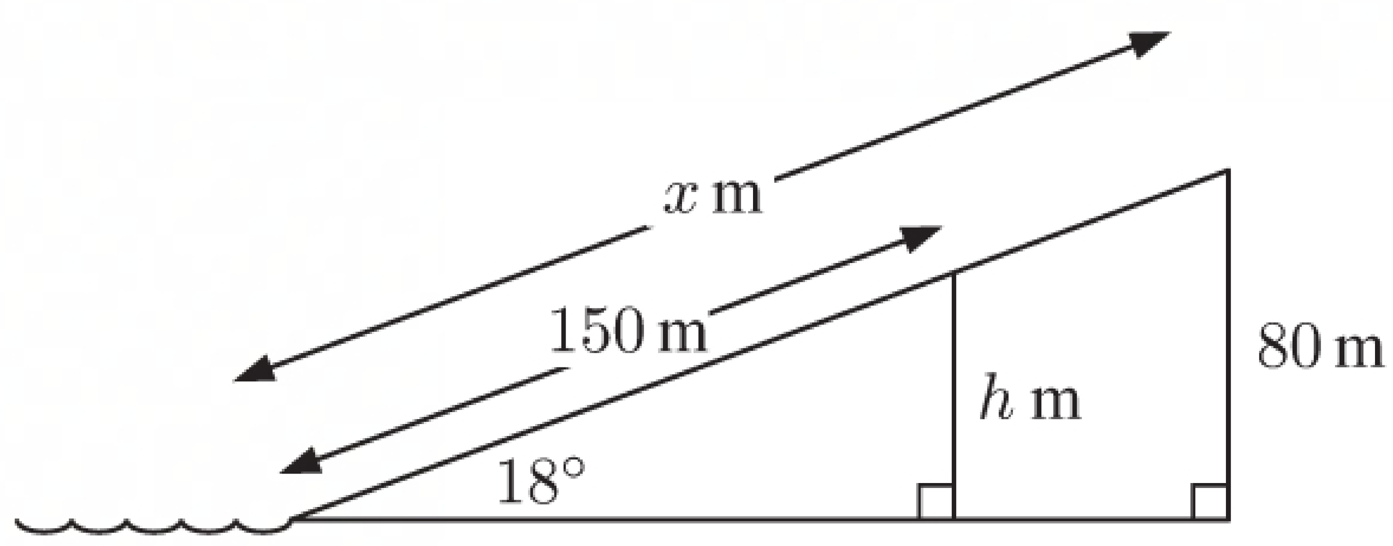
$$\therefore \quad \tan 63° = \frac{h}{9.32}$$
$$\therefore \quad 9.32 \times \tan 63° = h$$
$$\therefore \quad h \approx 18.3$$

So, the flagpole is about 18.3 m high.

2 **a** Let the height above sea level be h m.

For the 18° angle, OPP = h m,

HYP = 150 m.



$$\therefore \quad \sin 18° = \frac{h}{150}$$
$$\therefore \quad 150 \times \sin 18° = h$$
$$\therefore \quad h \approx 46.4$$

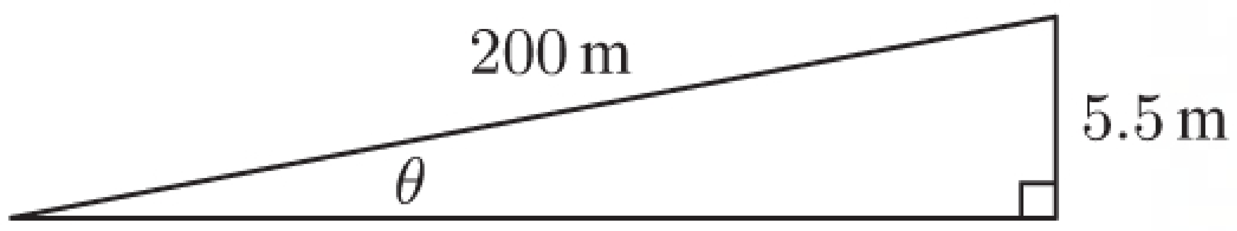
So, you are about 46.4 m above sea level.

b Let the distance walked be x m.
For the $18°$ angle, OPP = 80 m, HYP = x m.

$$\therefore \sin 18° = \frac{80}{x}$$
$$\therefore x = \frac{80}{\sin 18°}$$
$$\therefore x \approx 259$$

So, you have walked about 259 m up the hill.

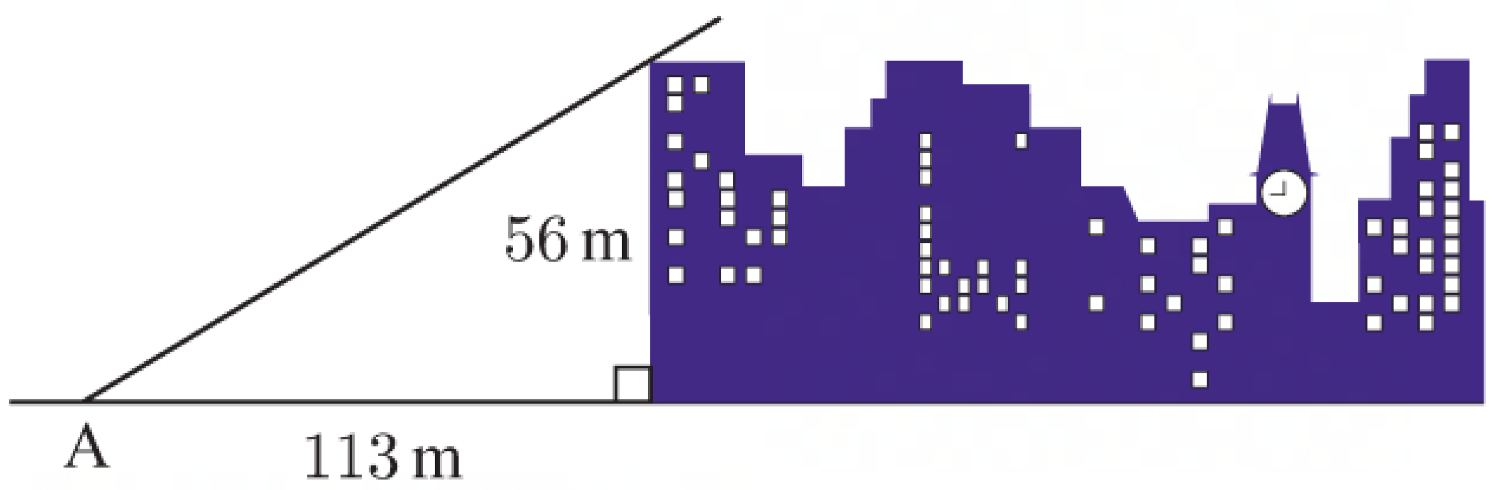
3 Let the angle of incline be θ.
For the angle θ, OPP = 5.5 m, HYP = 200 m.



$$\therefore \sin\theta = \frac{5.5}{200}$$
$$\therefore \theta = \sin^{-1}\left(\tfrac{5.5}{200}\right)$$
$$\therefore \theta \approx 1.58°$$

So, the angle of incline is about $1.58°$.

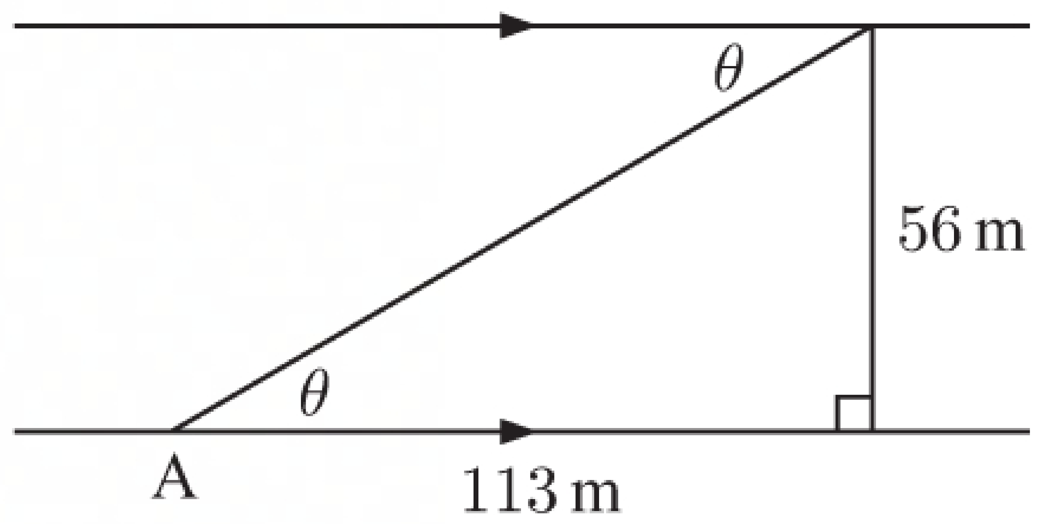
4 **a** Let the angle of elevation at A be θ.
For the angle θ, OPP = 56 m,
ADJ = 113 m.



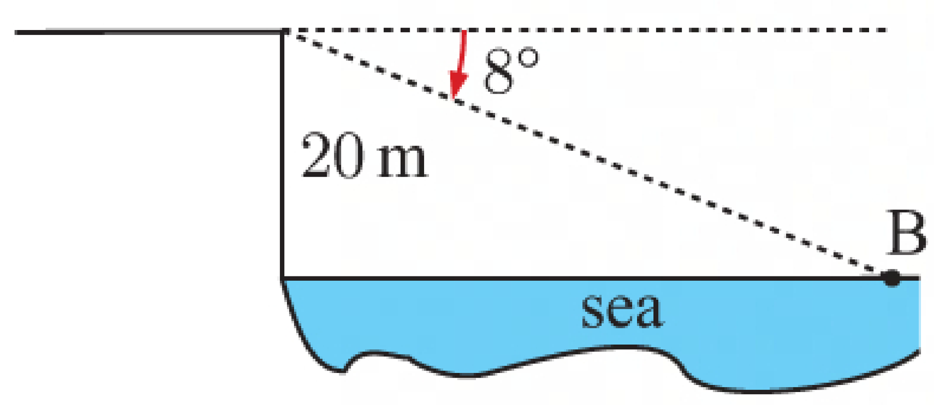
$$\therefore \tan\theta = \frac{56}{113}$$
$$\therefore \theta = \tan^{-1}\left(\tfrac{56}{113}\right)$$
$$\therefore \theta \approx 26.4°$$

So, the angle of elevation from A to the top of the building is about $26.4°$.

b The angle of depression from the top of the building to A is an alternate angle to θ, so it is also about $26.4°$.

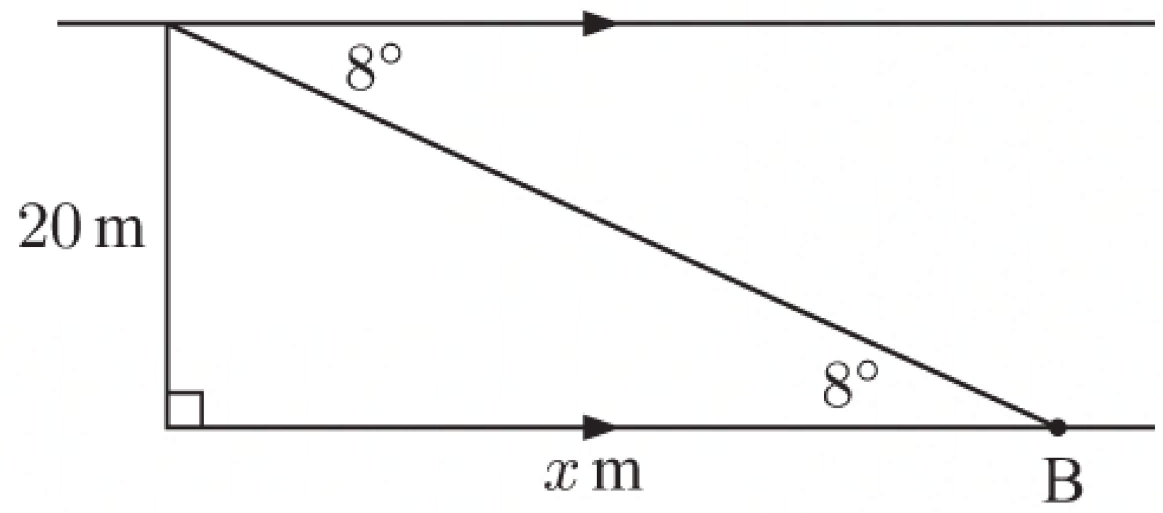


5 By alternate angles, the angle of elevation of the cliff from B is also $8°$.
So, if the distance of the boat from the base of the cliff is x m, then



$$\tan 8° = \frac{20}{x}$$
$$\therefore x = \frac{20}{\tan 8°}$$
$$\therefore x \approx 142$$

The boat is about 142 m from the base of the cliff.



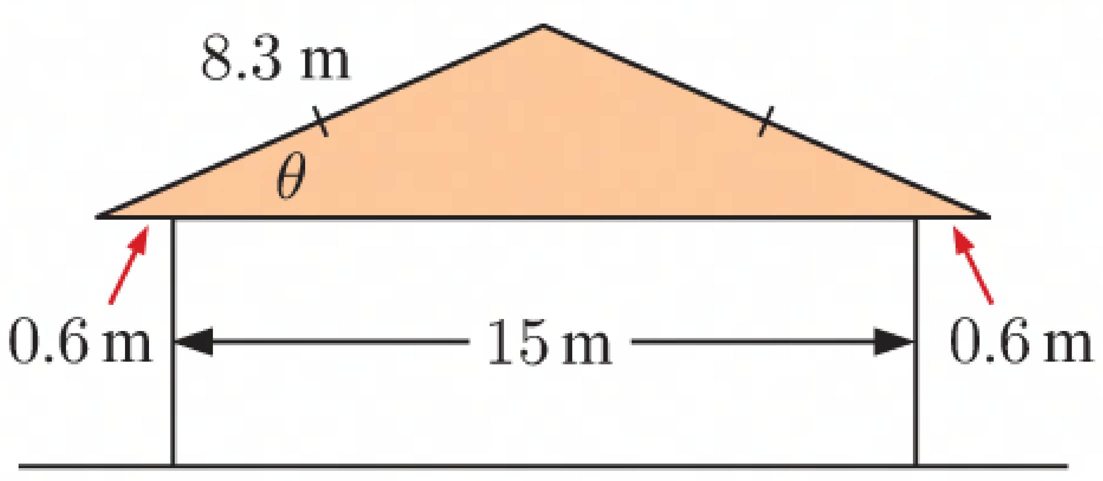
6 By constructing an altitude of the isosceles triangle, we form two right angled triangles.

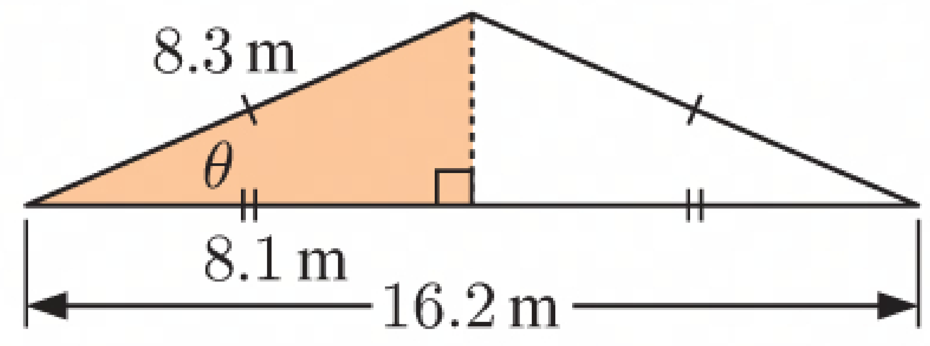
For the shaded triangle, $\cos\theta = \dfrac{8.1}{8.3}$

$\therefore \quad \theta = \cos^{-1}\left(\frac{8.1}{8.3}\right)$

$\therefore \quad \theta \approx 12.6°$

So, the pitch of the roof is approximately 12.6°.





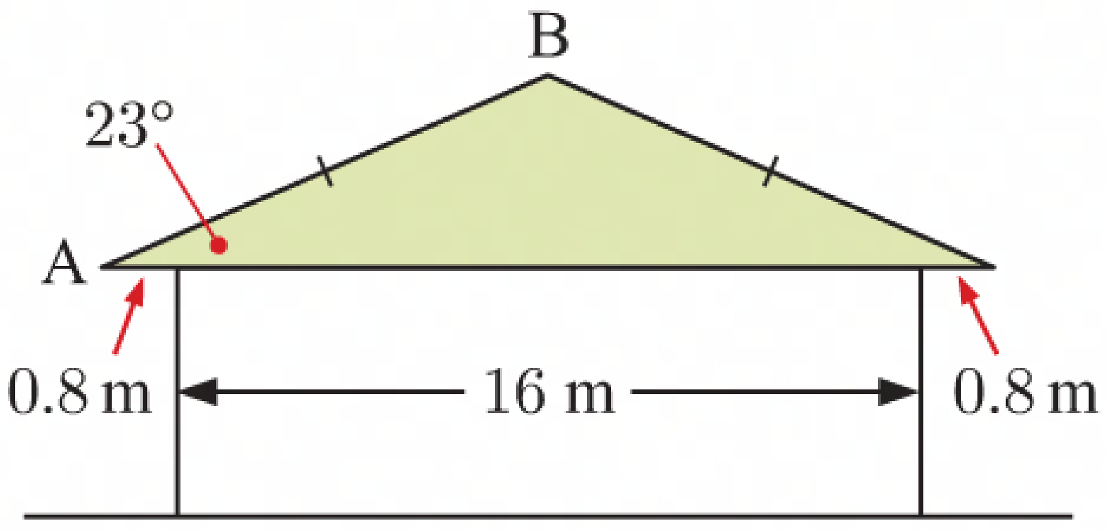
7 By constructing an altitude of the isosceles triangle, we form two right angled triangles.

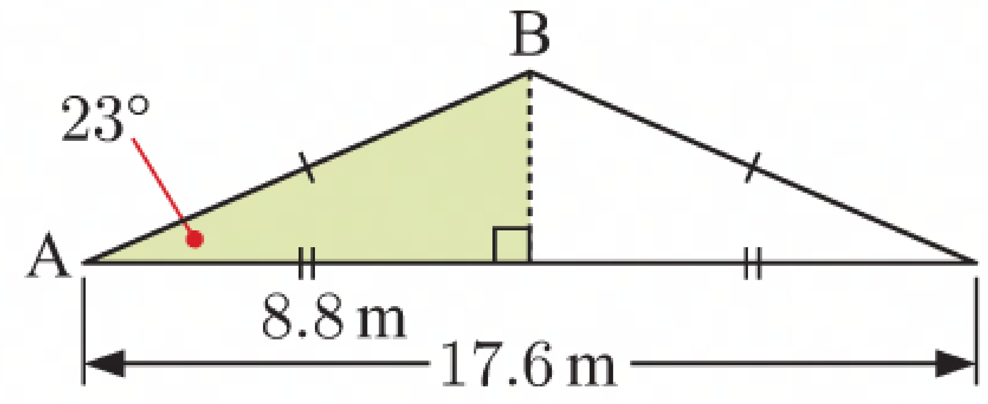
For the shaded triangle, $\cos 23° = \dfrac{8.8}{\text{AB}}$

$\therefore \quad \text{AB} = \dfrac{8.8}{\cos 23°}$

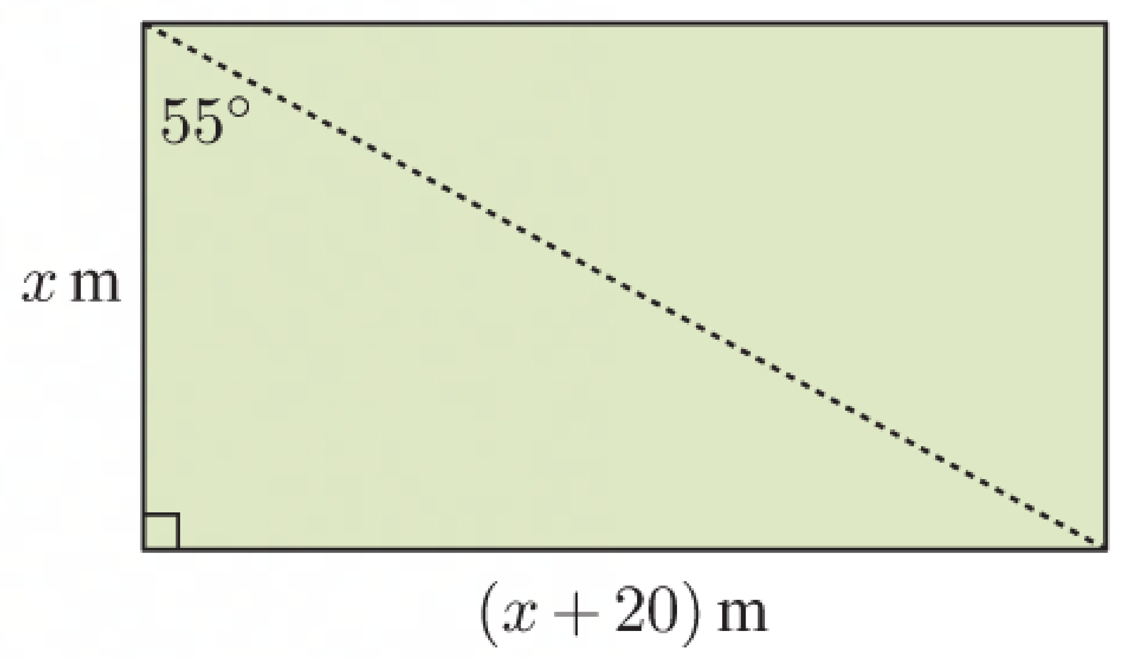
$\therefore \quad \text{AB} \approx 9.56$

So, the timber beam AB is about 9.56 m long.





8



Let the width of the field be x m.

$\therefore$ the length of the field is $(x+20)$ m.

$\therefore \quad \tan 55° = \dfrac{x+20}{x}$

$\therefore \quad x \times \tan 55° = x + 20$

$\therefore \quad x \times \tan 55° - x = 20$

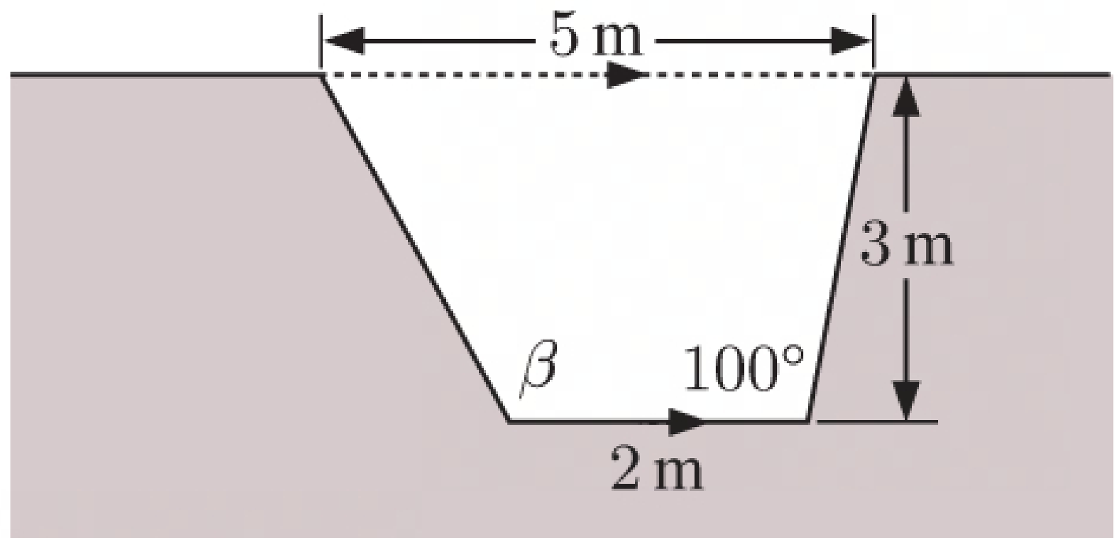
$\therefore \quad x(\tan 55° - 1) = 20$

$\therefore \quad x = \dfrac{20}{\tan 55° - 1}$

$\therefore \quad x \approx 46.7$

So, the shorter side of the field is about 46.7 m.

9 We draw perpendiculars [DM] and [CN] to [AB].



In $\triangle$BCN, $\tan 10° = \dfrac{x}{3}$

$$\therefore\ x = 3\tan 10°$$

So, $\text{AM} = 5 - 2 - (3\tan 10°)$ m

$$= 3 - 3\tan 10° \text{ m}$$

$\therefore$ in $\triangle$AMD,

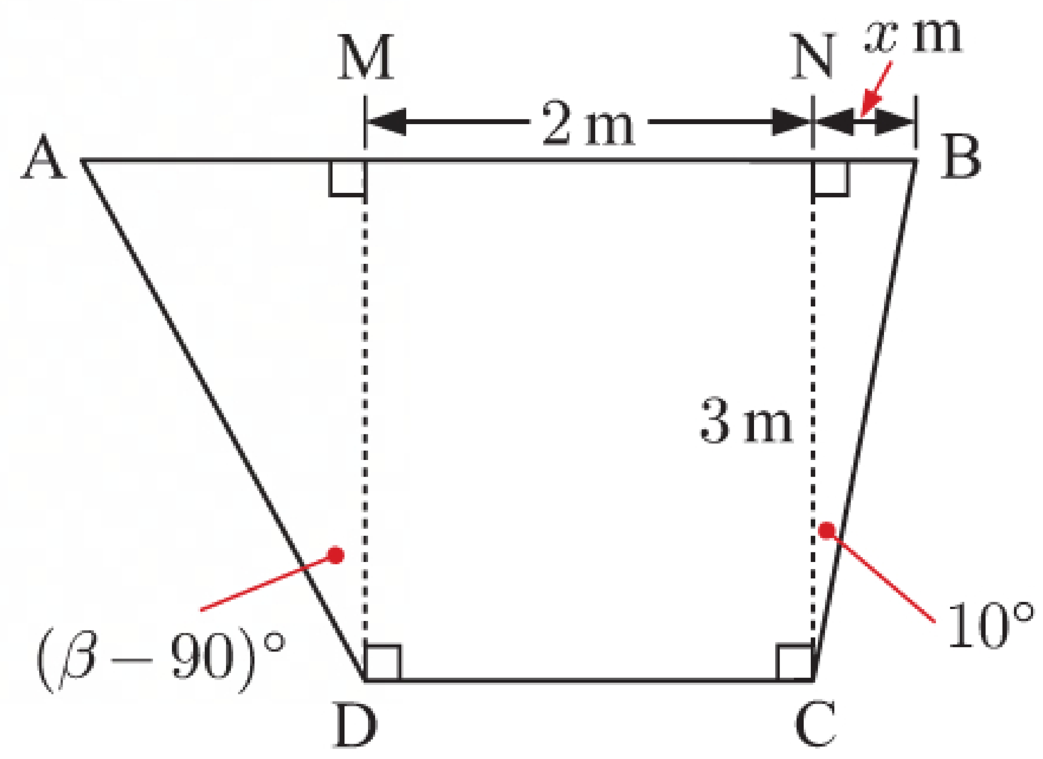
$$\tan(\beta - 90°) = \frac{\text{AM}}{3}$$

$$= \frac{3 - 3\tan 10°}{3}$$

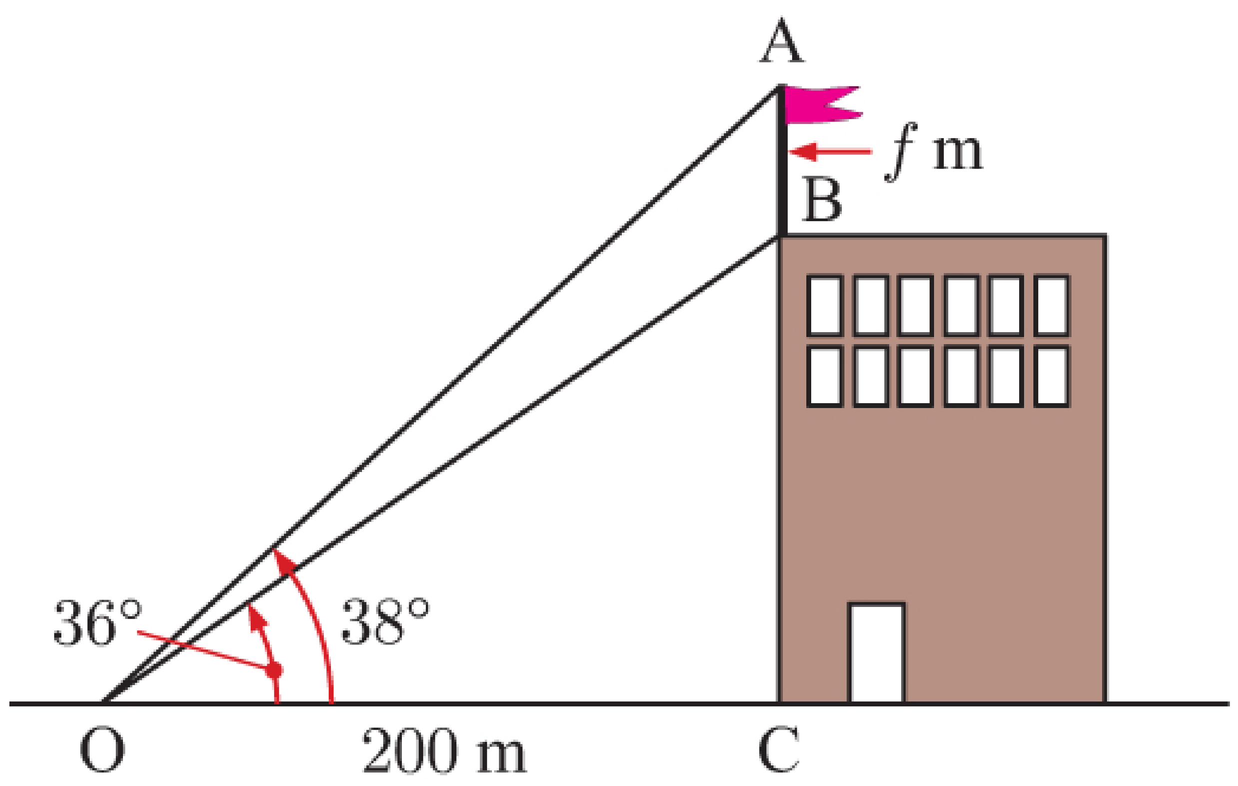
$$\therefore\ \beta - 90° = \tan^{-1}\left(\frac{3 - 3\tan 10°}{3}\right)$$

$$\therefore\ \beta - 90° \approx 39.48°$$

$$\therefore\ \beta \approx 129°$$



10



Let the height of the flagpole be f m.

In $\triangle$OAC, $\tan 38° = \dfrac{\text{AC}}{200}$

$$\therefore\ 200 \times \tan 38° = \text{AC}$$

In $\triangle$OBC, $\tan 36° = \dfrac{\text{BC}}{200}$

$$\therefore\ 200 \times \tan 36° = \text{BC}$$

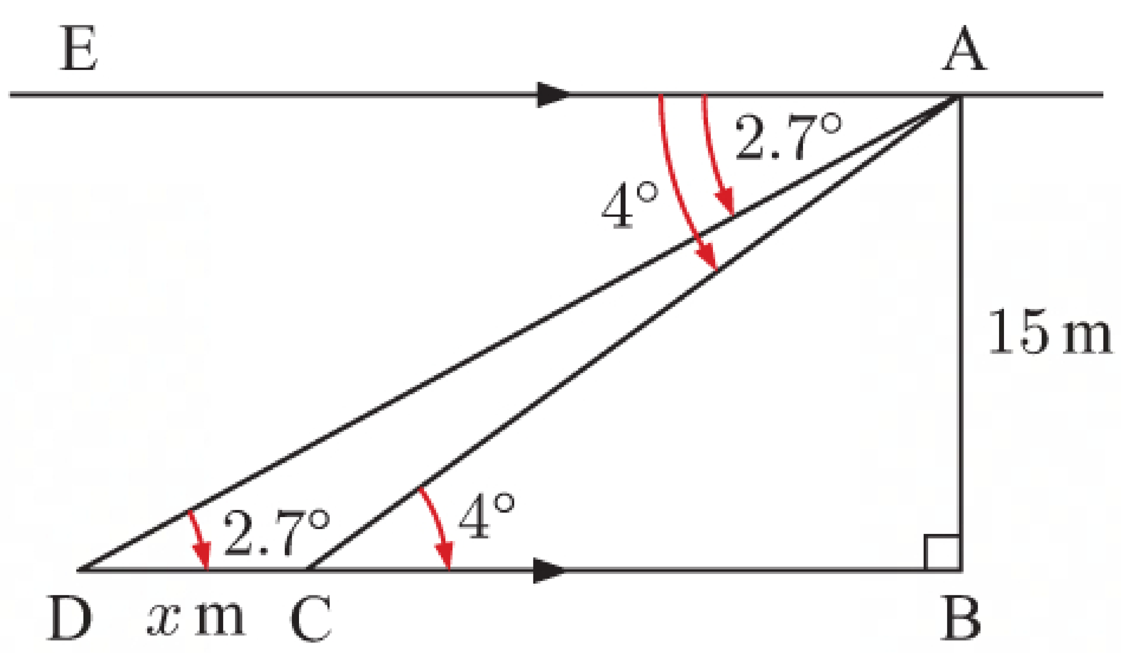
Now $f = \text{AC} - \text{BC}$

$$= 200\tan 38° - 200\tan 36°$$

$$\therefore\ f \approx 10.9$$

So, the flagpole is about 10.9 m high.

11



Let the distance the boat has to move be x m.

$\widehat{\text{ACB}} = \widehat{\text{CAE}} = 4°$ {equal alternate angles}

and $\widehat{\text{ADB}} = \widehat{\text{DAE}} = 2.7°$ {equal alternate angles}

In $\triangle$ABD, $\tan 2.7° = \dfrac{15}{\text{BD}}$

$$\therefore\ \text{BD} = \frac{15}{\tan 2.7°}$$

In $\triangle$ABC, $\tan 4° = \dfrac{15}{\text{BC}}$

$$\therefore\ \text{BC} = \frac{15}{\tan 4°}$$

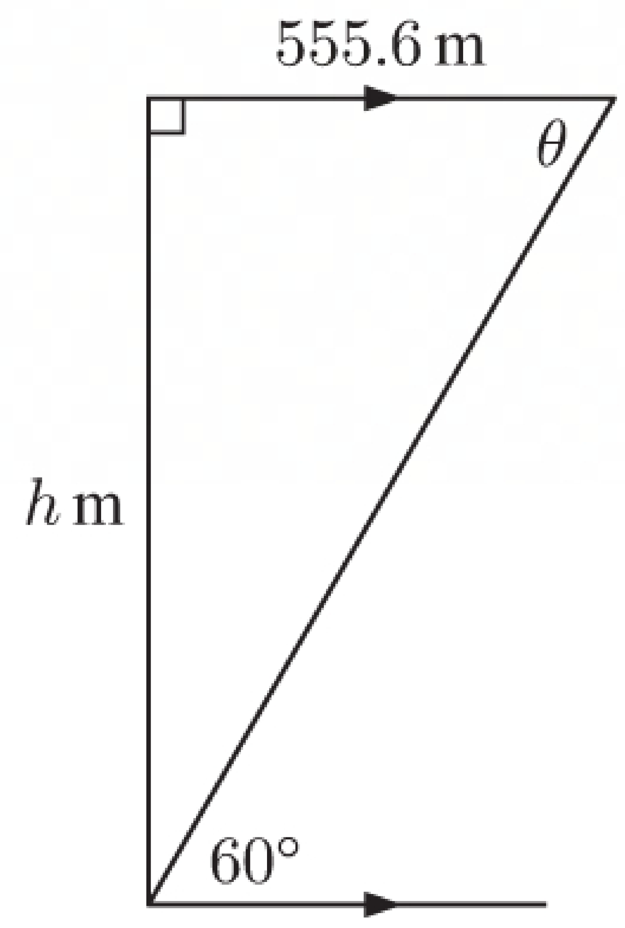
Now $x = \text{BD} - \text{BC}$

$$= \frac{15}{\tan 2.7°} - \frac{15}{\tan 4°}$$

$$\therefore\ x \approx 104$$

So, the boat must move about 104 m closer to the cliff.

12 The helicopter flies horizontally at 100 km h^{-1}.



Distance travelled = speed × time

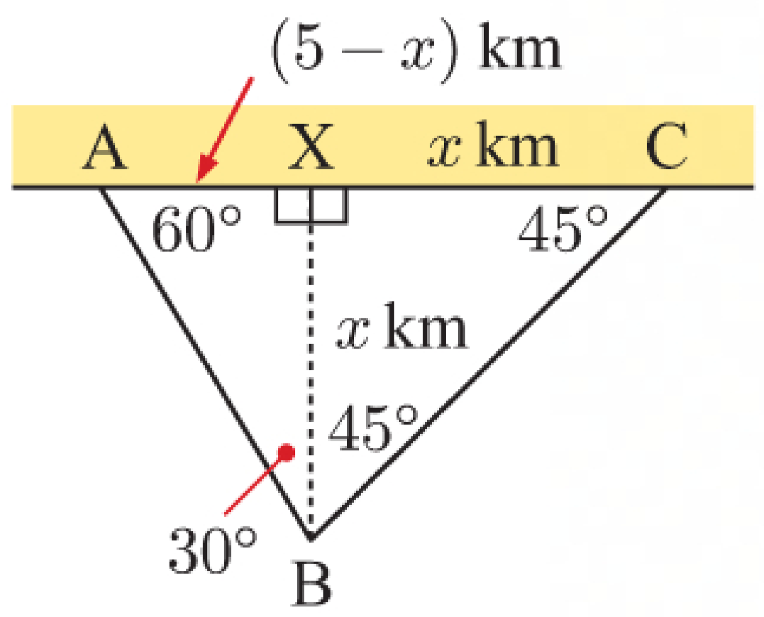
$$= 100 \text{ km h}^{-1} \times 20 \text{ s}$$
$$= 100 \text{ km h}^{-1} \times \frac{20}{60 \times 60} \text{ h}$$
$$\approx 0.5556 \text{ km}$$
$$\approx 555.6 \text{ m}$$

Now $\theta = 60°$ {equal alternate angles}

$$\therefore \tan 60° \approx \frac{h}{555.6}$$
$$\therefore 555.6 \times \tan 60° \approx h$$
$$\therefore h \approx 962$$

So, the helicopter is about 962 m above the ground.

13 By constructing an altitude from B to [AC], we form two right angled triangles.



Let the distance of B from the shore be x km.

$\widehat{XBC} = 45°$ and $\widehat{ABX} = 30°$ {angles in a triangle}

So, △XBC is isosceles.

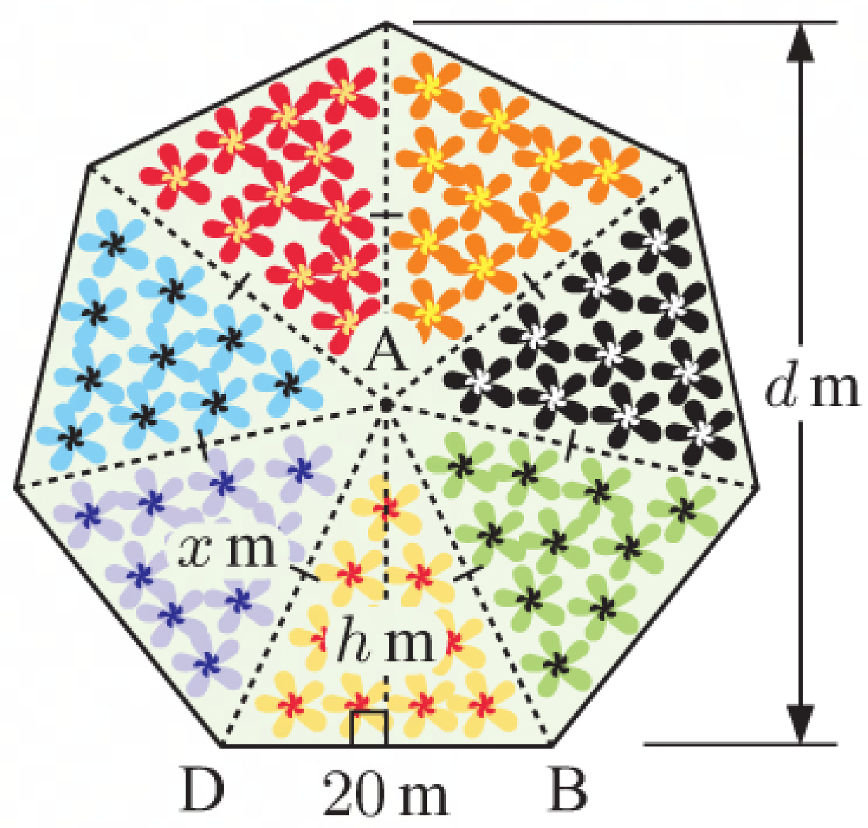
$\therefore$ XC = BX = x km and AX = $(5 - x)$ km

In △ABX, $\tan 30° = \dfrac{5 - x}{x}$

$$\therefore x \times \tan 30° = 5 - x$$
$$\therefore x(\tan 30° + 1) = 5$$
$$\therefore x = \frac{5}{\tan 30° + 1}$$
$$\therefore x \approx 3.17$$

So, the shortest distance from the boat to the shore is about 3.17 km.

14 The angle at the centre of the heptagon is 360°. {angles at a point}



$\therefore$ $\widehat{DAB} = \frac{360°}{7}$

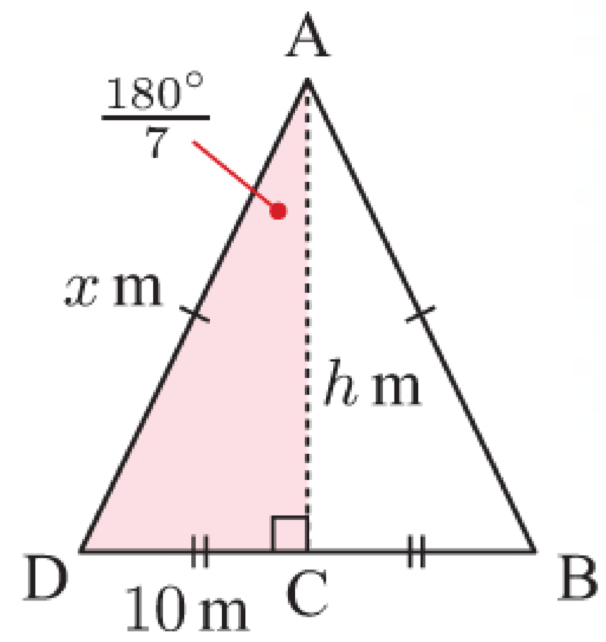
△ABD is isosceles with equal sides AD and AB.

[AC] perpendicularly bisects [BD].

$\therefore$ $\widehat{DAC} = \frac{1}{2} \times \frac{360°}{7} = \frac{180°}{7}$

Consider the shaded triangle shown.

Let the altitude of this triangle be h m and the hypotenuse be x m.



$$\therefore \tan\left(\frac{180°}{7}\right) = \frac{10}{h}$$
$$\therefore h = \frac{10}{\tan\left(\frac{180°}{7}\right)}$$

and $\sin\left(\frac{180°}{7}\right) = \dfrac{10}{x}$

$$\therefore x = \frac{10}{\sin\left(\frac{180°}{7}\right)}$$

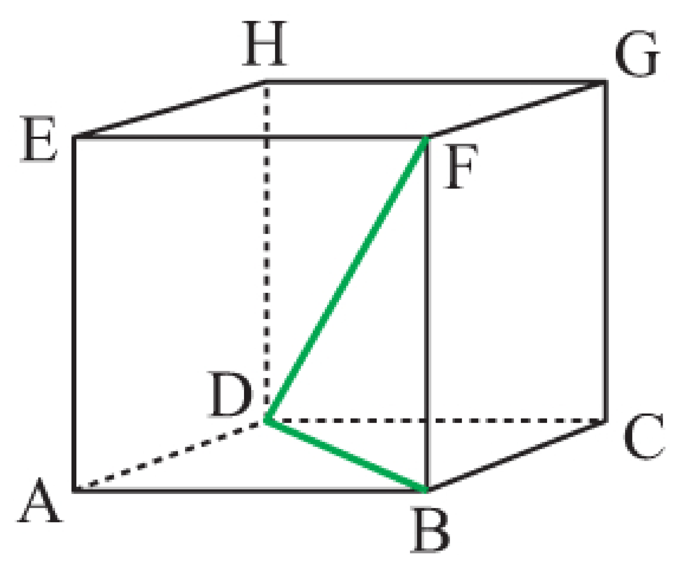
Now $d = x + h$

$$= \frac{10}{\sin\left(\frac{180^\circ}{7}\right)} + \frac{10}{\tan\left(\frac{180^\circ}{7}\right)}$$

$\therefore\ d \approx 43.8$

$\therefore$ the width of land required for the plot is about 43.8 m.

15



a Consider the base of the cube, letting BD be x cm.

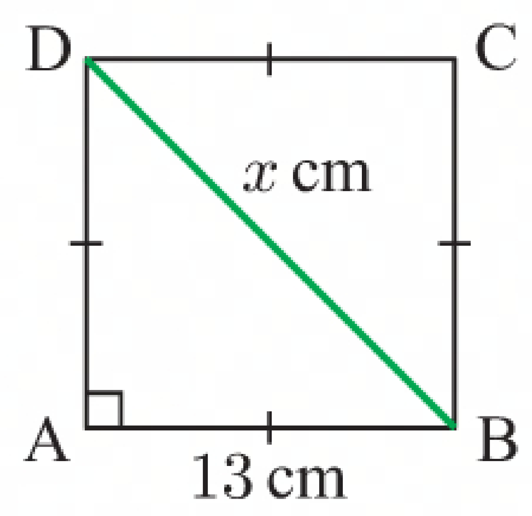
Using Pythagoras, $x^2 = 13^2 + 13^2$

$\therefore\ x^2 = 338$

$\therefore\ x = \sqrt{338}$ $\{$as $x > 0\}$

≈ 18.4

So, BD is about 18.4 cm long.



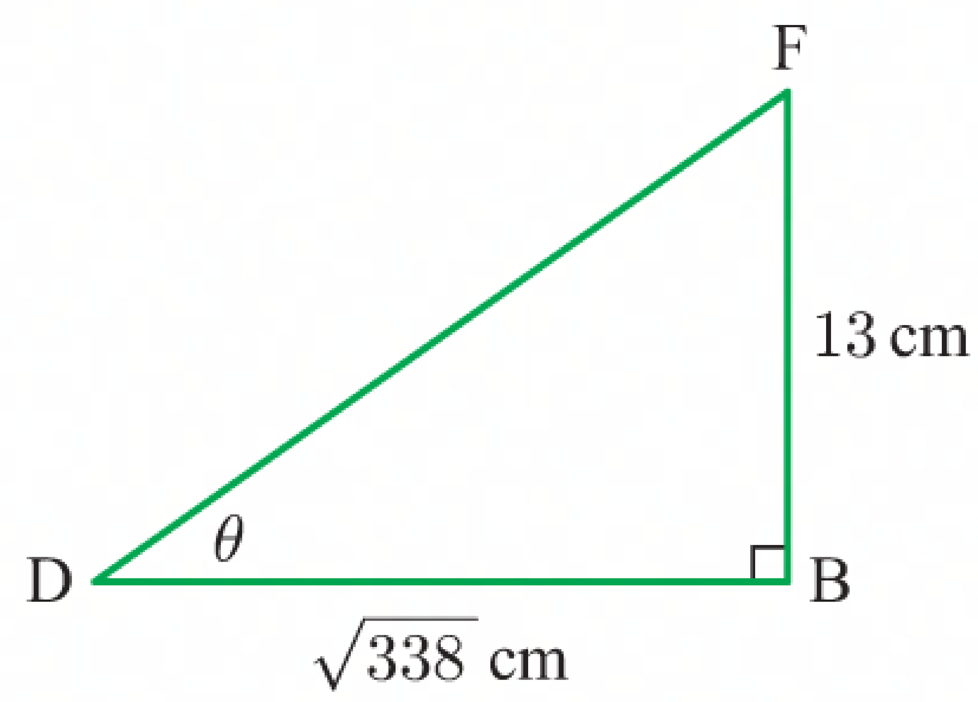
b $\triangle$DBF is right angled at B.

$\tan\theta = \frac{13}{\sqrt{338}}$

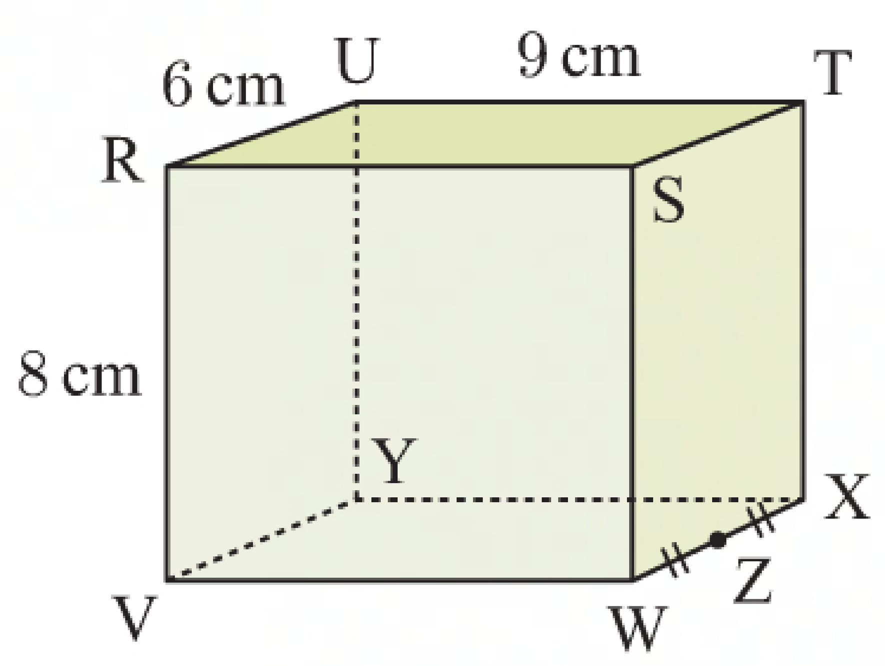
$\therefore\ \theta = \tan^{-1}\left(\frac{13}{\sqrt{338}}\right)$

$\therefore\ \theta \approx 35.3^\circ$

So, $\widehat{FDB}$ is about 35.3°.



16



a Consider the base of the prism, letting VX be x cm.

Using Pythagoras,

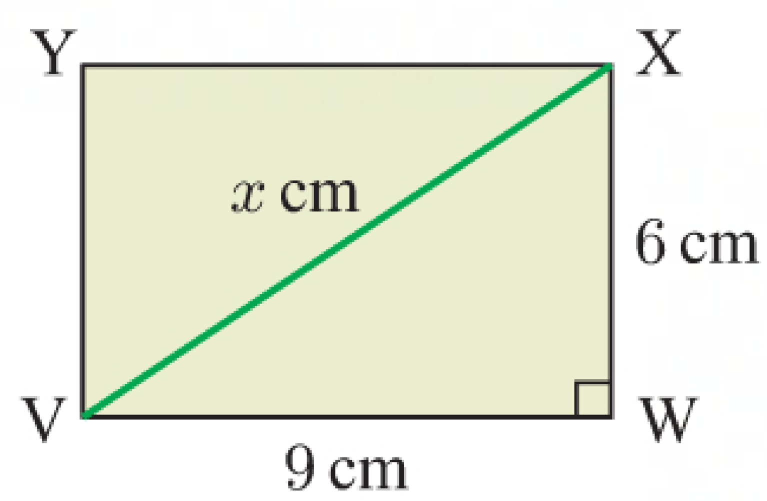
$x^2 = 6^2 + 9^2$

$\therefore\ x^2 = 117$

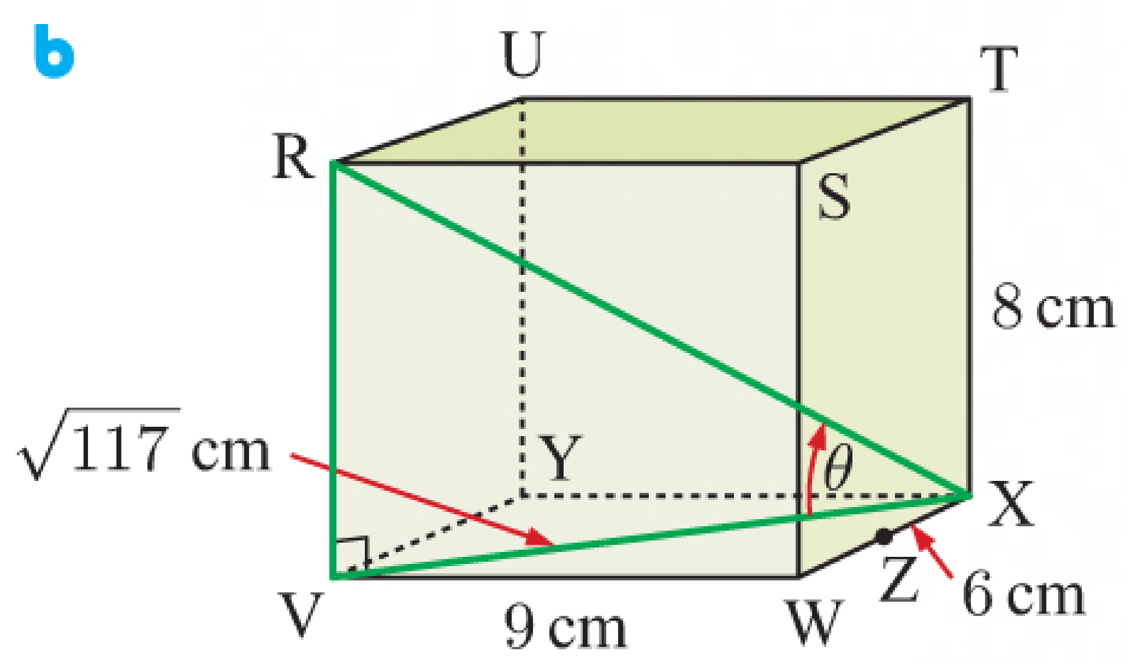
$\therefore\ x = \sqrt{117}$ $\{$as $x > 0\}$

≈ 10.8

So, VX is about 10.8 cm long.



b



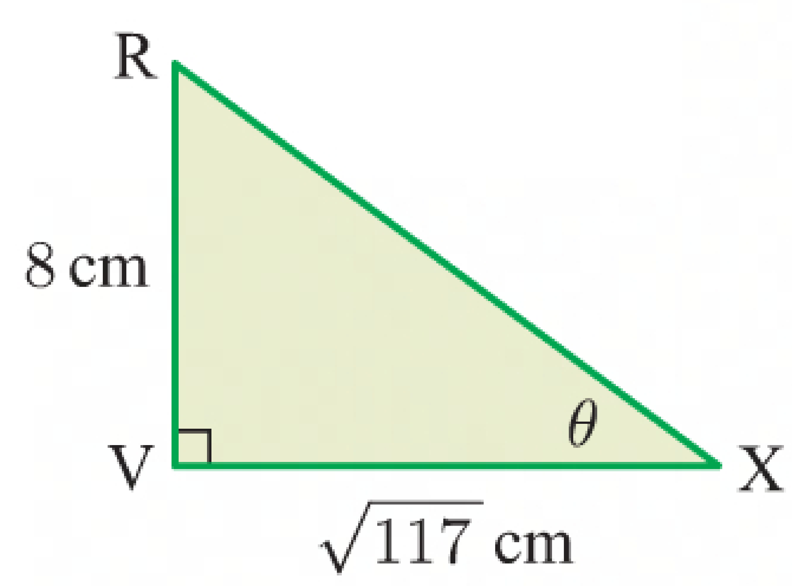
$\triangle$RVX is right angled at V.

$\tan\theta = \frac{8}{\sqrt{117}}$

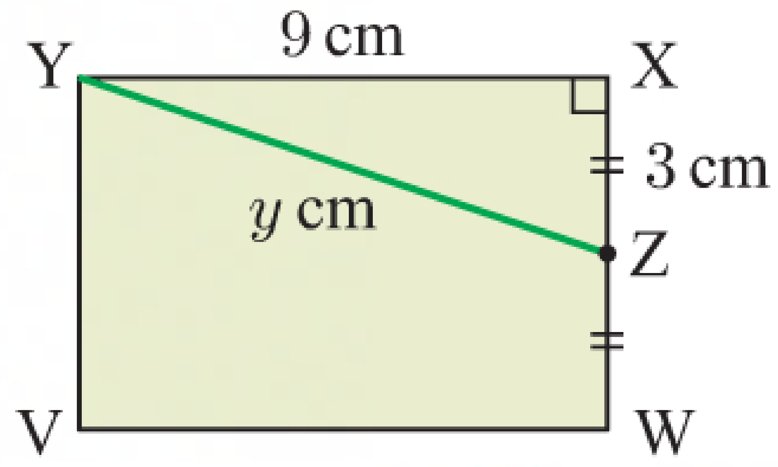
$\therefore\ \theta = \tan^{-1}\left(\frac{8}{\sqrt{117}}\right)$

$\therefore\ \theta \approx 36.5^\circ$

So, $\widehat{RXV}$ is about 36.5°.



c



Considering again the base of the prism, let YZ be y cm.

Using Pythagoras, $y^2 = 9^2 + 3^2$

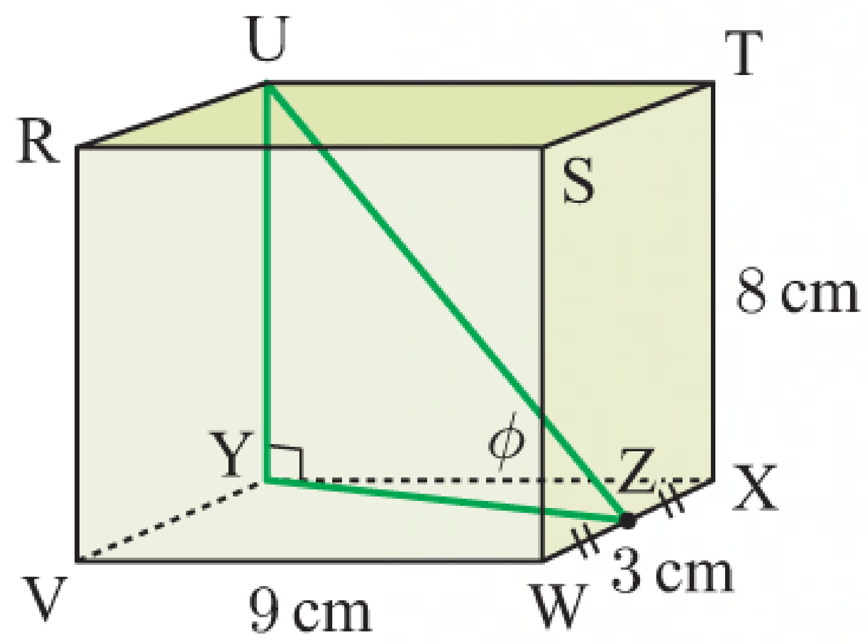
$\therefore\ y^2 = 90$

$\therefore\ y = \sqrt{90}$ {as $y > 0$}

≈ 9.49

So, YZ is about 9.49 cm long.

d



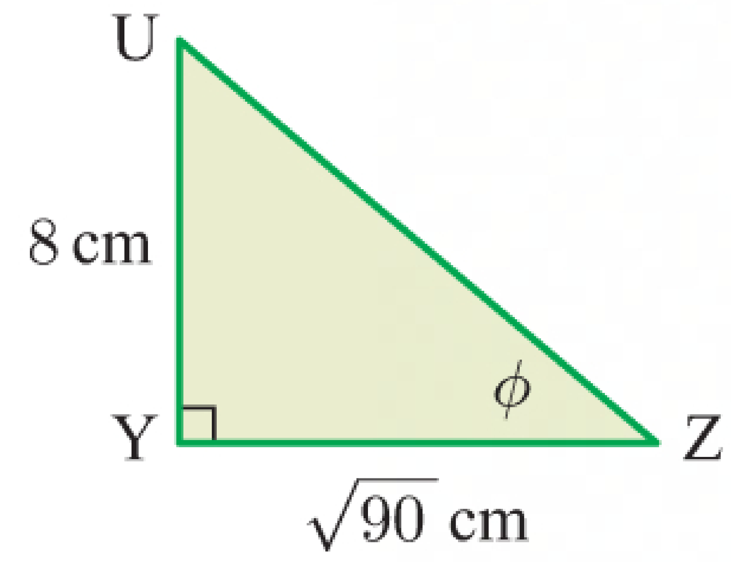
$\triangle$UYZ is right angled at Y.

$\tan\phi = \dfrac{8}{\sqrt{90}}$

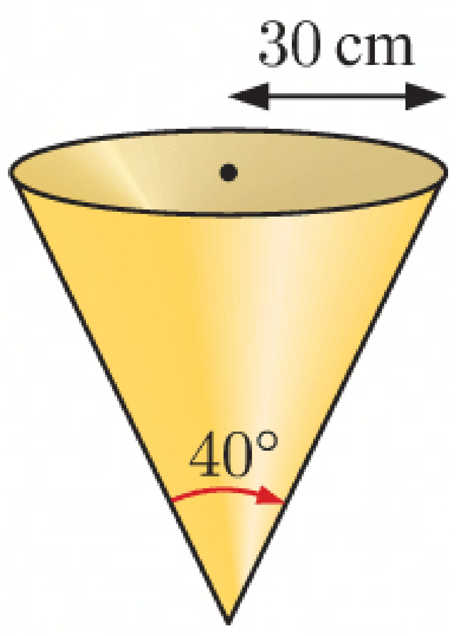
$\therefore\ \phi = \tan^{-1}\left(\frac{8}{\sqrt{90}}\right)$

$\therefore\ \phi \approx 40.1°$

So, $\text{Y}\widehat{\text{Z}}\text{U}$ is about 40.1°.



17



a We draw the isosceles triangle cross-section of the cone as shown.

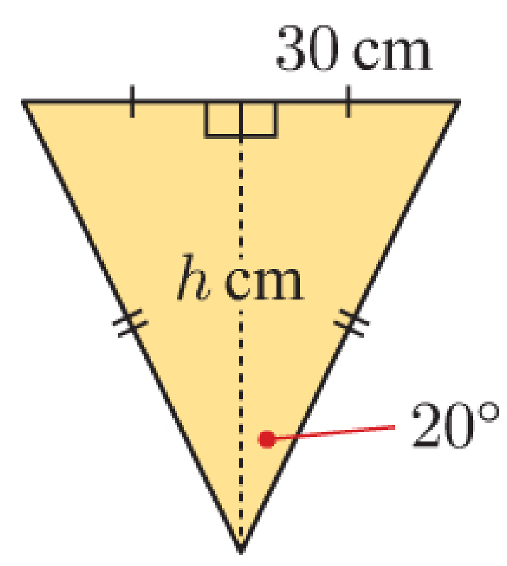
Let the height of the cone be h cm.

$\therefore\ \tan 20° = \dfrac{30}{h}$

$\therefore\ h = \dfrac{30}{\tan 20°}$

$\therefore\ h \approx 82.4$

So, the cone is about 82.4 cm high.



b $V = \frac{1}{3}\pi r^2 h$

$\approx \frac{1}{3} \times \pi \times 30^2 \times 82.4\ \text{cm}^3$

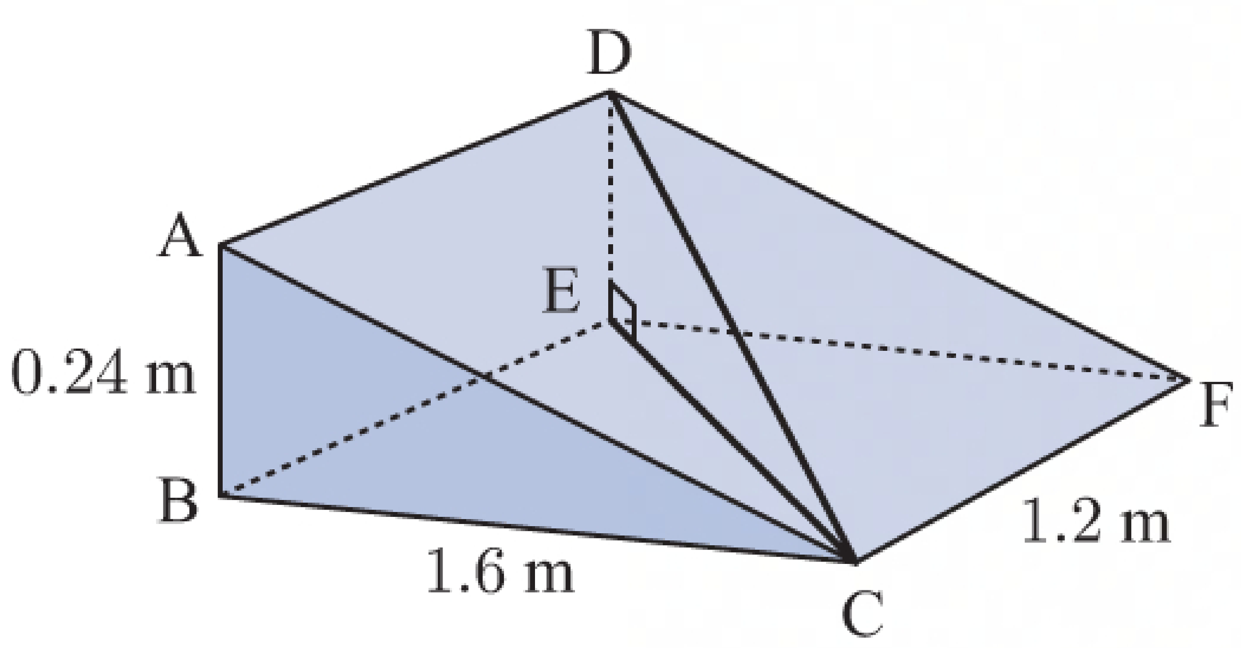
$\approx 77\,700\ \text{cm}^3$

$\approx 77\,700$ mL

≈ 77.7 L

The cone has a capacity of about 77.7 L.

18



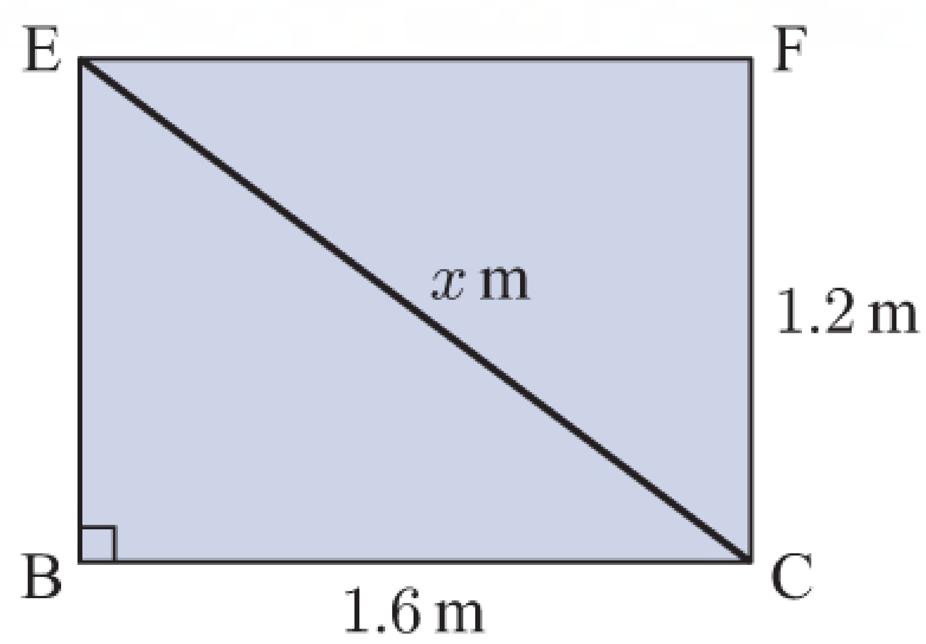
a **i** Consider the base of the prism, letting CE be x m.

Using Pythagoras, $x^2 = 1.2^2 + 1.6^2$

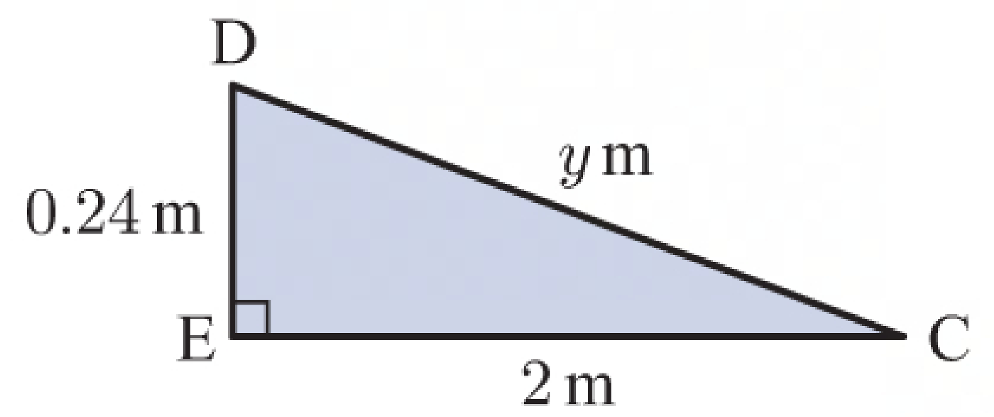
$\therefore\ x^2 = 4$

$\therefore\ x = 2$ {as $x > 0$}

So, CE is 2 m long.



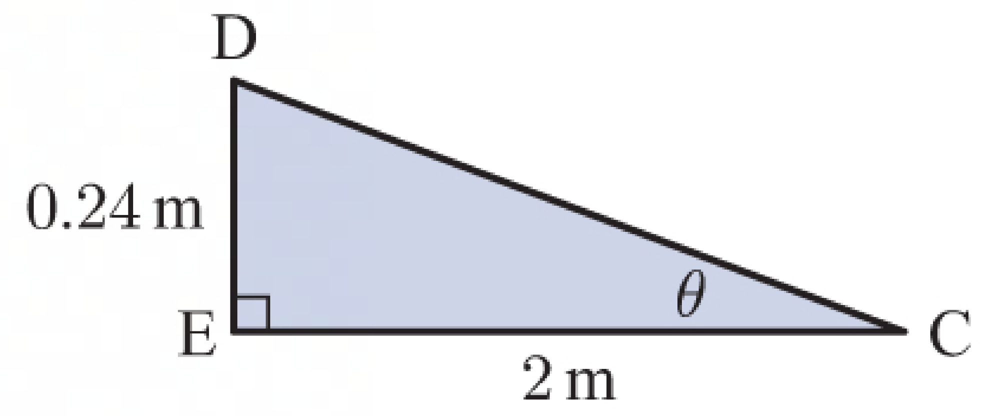
ii $\triangle$DEC is right angled at E.
Let CD be y m.



Using Pythagoras, $y^2 = 0.24^2 + 2^2$
$\therefore\ y^2 = 4.0576$
$\therefore\ y = \sqrt{4.0576}$ {as $y > 0$}
≈ 2.01

So, CD is about 2.01 m long.

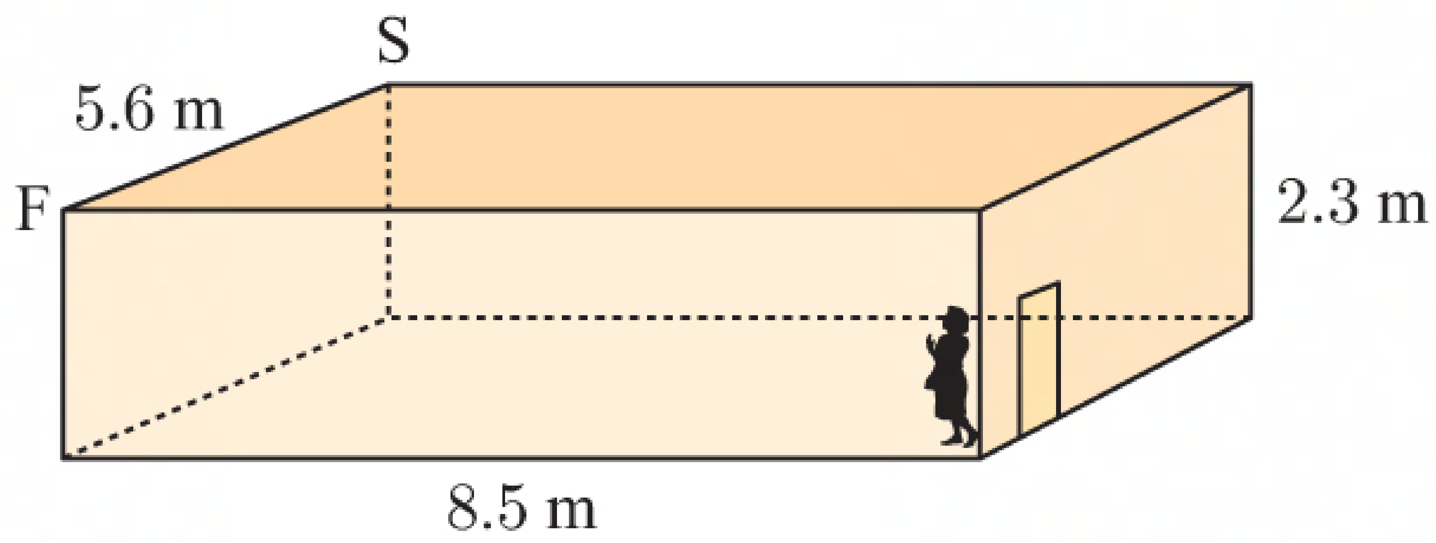
b Let D$\widehat{C}$E be θ.



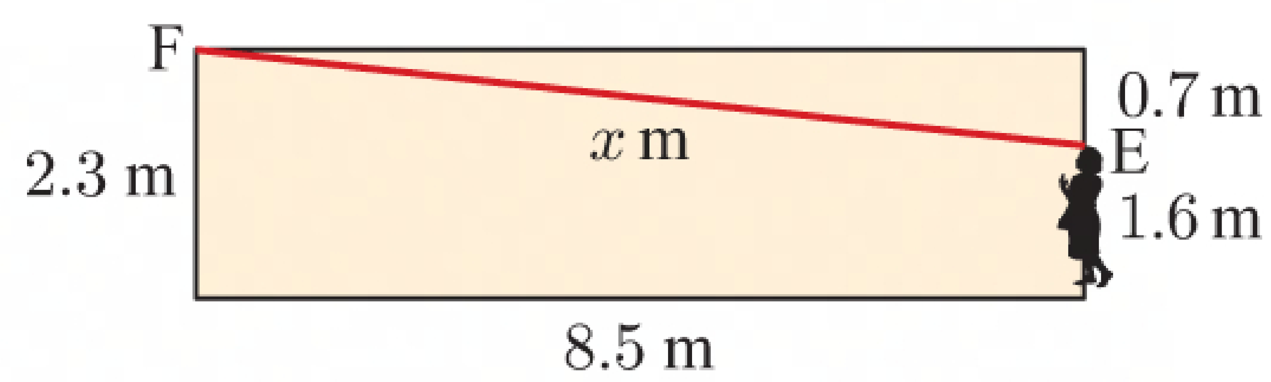
$\tan\theta = \dfrac{0.24}{2}$
$\therefore\ \theta = \tan^{-1}\left(\frac{0.24}{2}\right)$
$\therefore\ \theta \approx 6.84°$

So, D$\widehat{C}$E is about 6.84°.

19

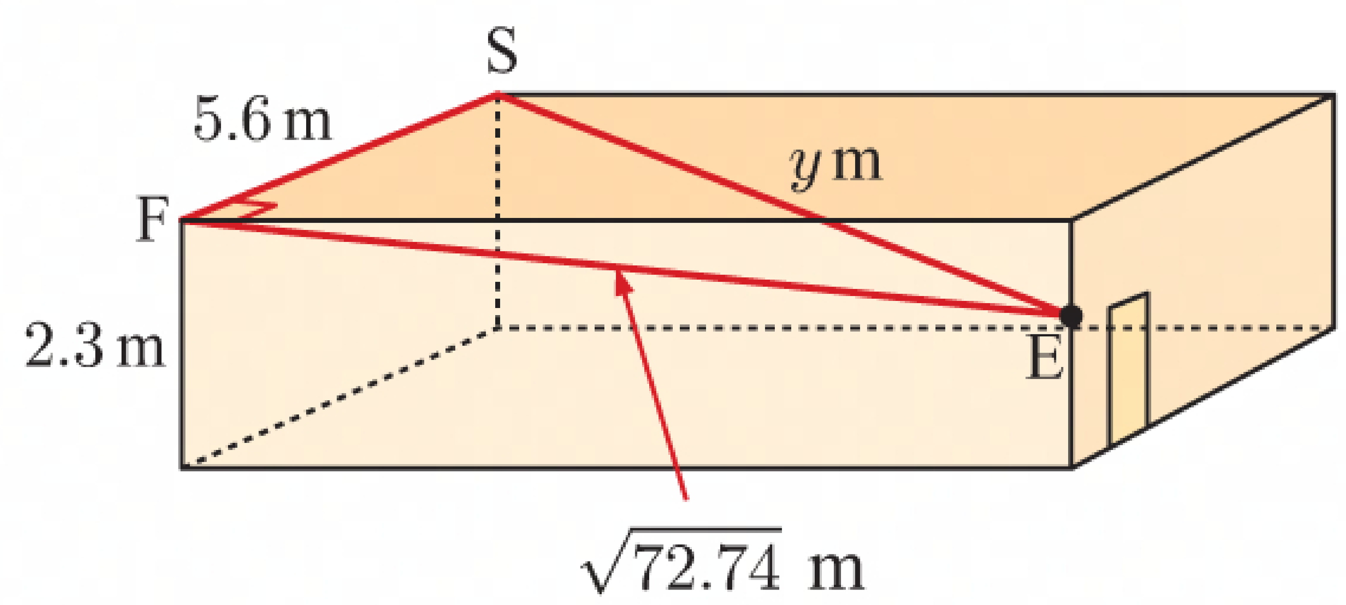


a Consider the side of the room containing point F and not containing S.
Let the distance from F to the top of Elizabeth's head E, be x m.



Using Pythagoras, $x^2 = 0.7^2 + 8.5^2$
$\therefore\ x^2 = 72.74$
$\therefore\ x = \sqrt{72.74}$ {as $x > 0$}

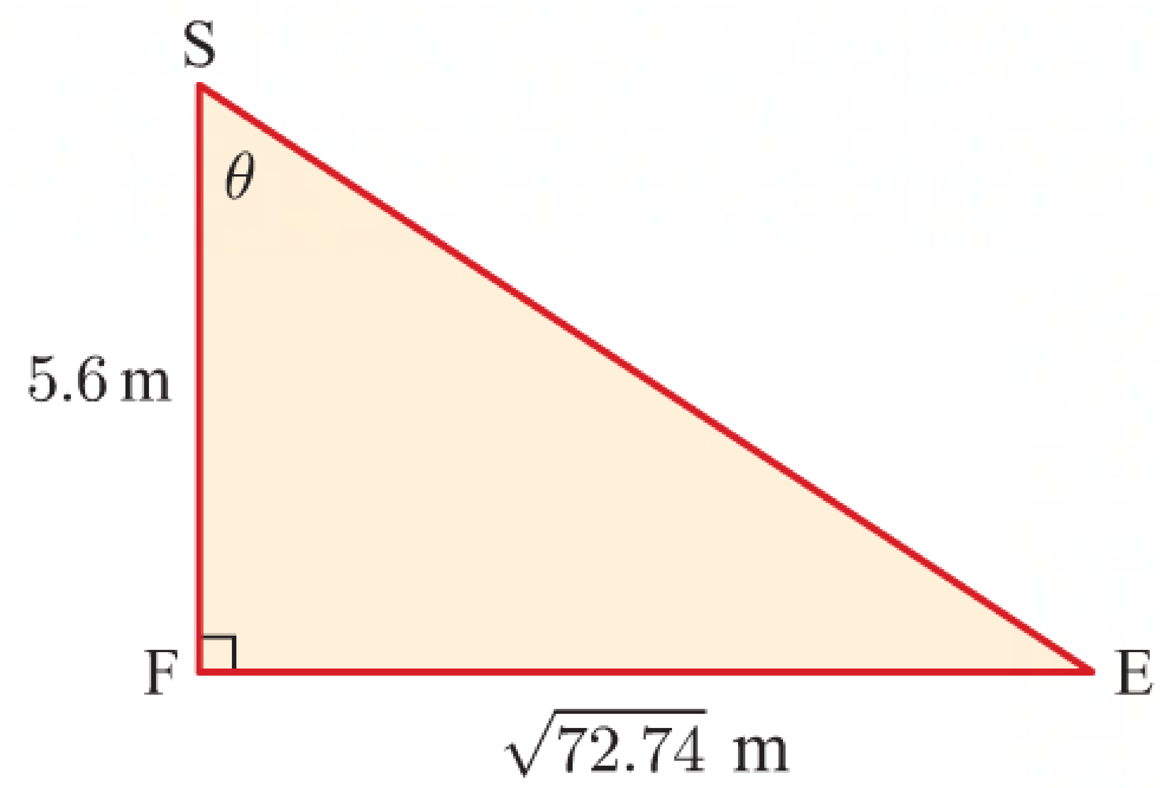
$\triangle$SFE is right angled at F.
Let SE be y m.



Using Pythagoras, $y^2 = 5.6^2 + (\sqrt{72.74})^2$
$\therefore\ y^2 = 104.1$
$\therefore\ y = \sqrt{104.1}$ {as $y > 0$}
≈ 10.2

So, the spider is about 10.2 m from Elizabeth's head.

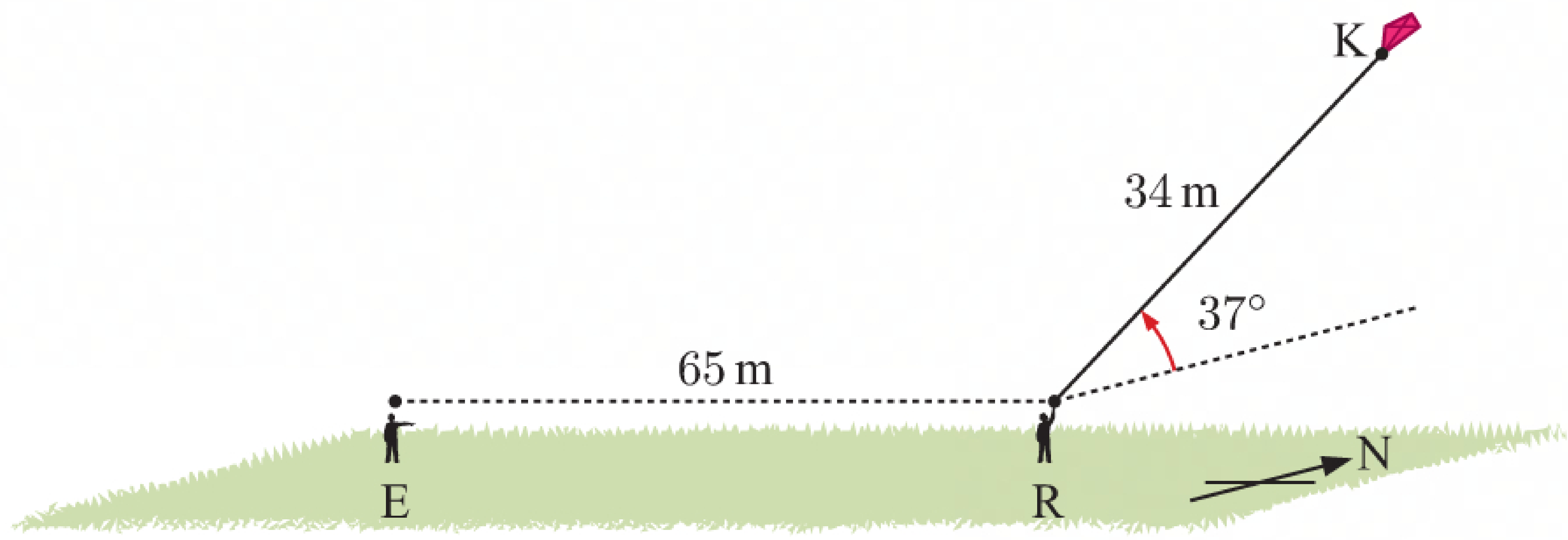
b Let F$\widehat{S}$E be θ.



$\tan\theta = \dfrac{\sqrt{72.74}}{5.6}$
$\therefore\ \theta = \tan^{-1}\left(\dfrac{\sqrt{72.74}}{5.6}\right)$
$\therefore\ \theta \approx 56.7°$

No, the spider cannot see Elizabeth because the angle F$\widehat{S}$E is about 56.7°, and the spider can only see up to an angle of 42°.

20



a $\triangle$ERK is right angled at R.

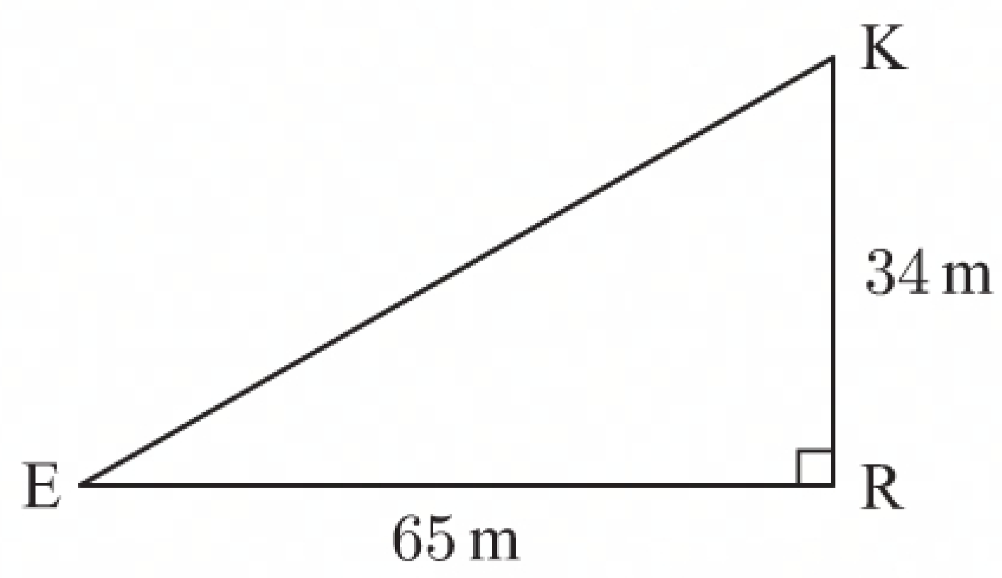
Using Pythagoras, $\text{EK}^2 = 65^2 + 34^2$

$\therefore \text{EK}^2 = 5381$

$\therefore \text{EK} = \sqrt{5381}$ {as EK > 0}

≈ 73.4 m

So, Edward is approximately 73.4 m from the kite.



b Let the point directly below the kite at a height level with Edward be X.

$\triangle$KXR is right angled at X.

For the 37° angle, OPP = KX, HYP = 34 m

$$\therefore \sin 37° = \frac{\text{KX}}{34}$$

$$\therefore \text{KX} = 34 \times \sin 37°$$

$\triangle$KXE is right angled at X.

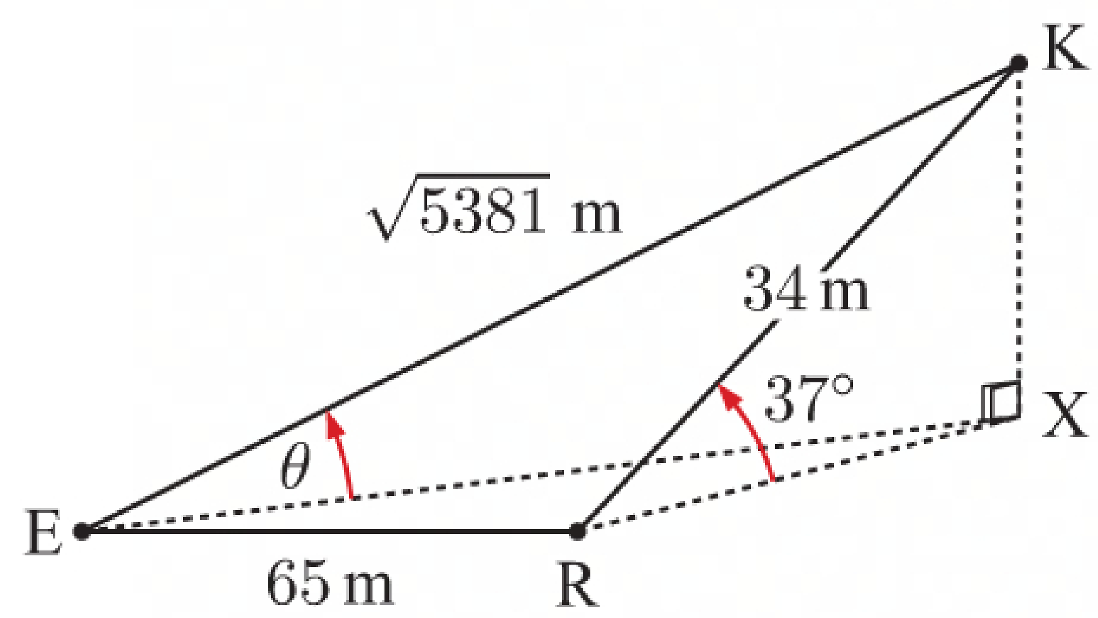
For angle θ, OPP = KX, HYP = $\sqrt{5381}$ m

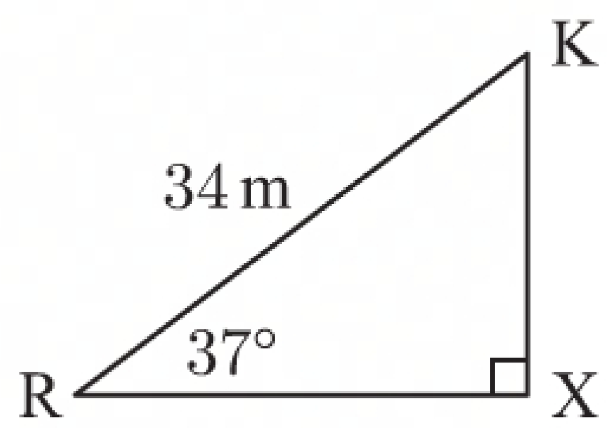
$$\therefore \sin\theta = \frac{34\sin 37°}{\sqrt{5381}}$$

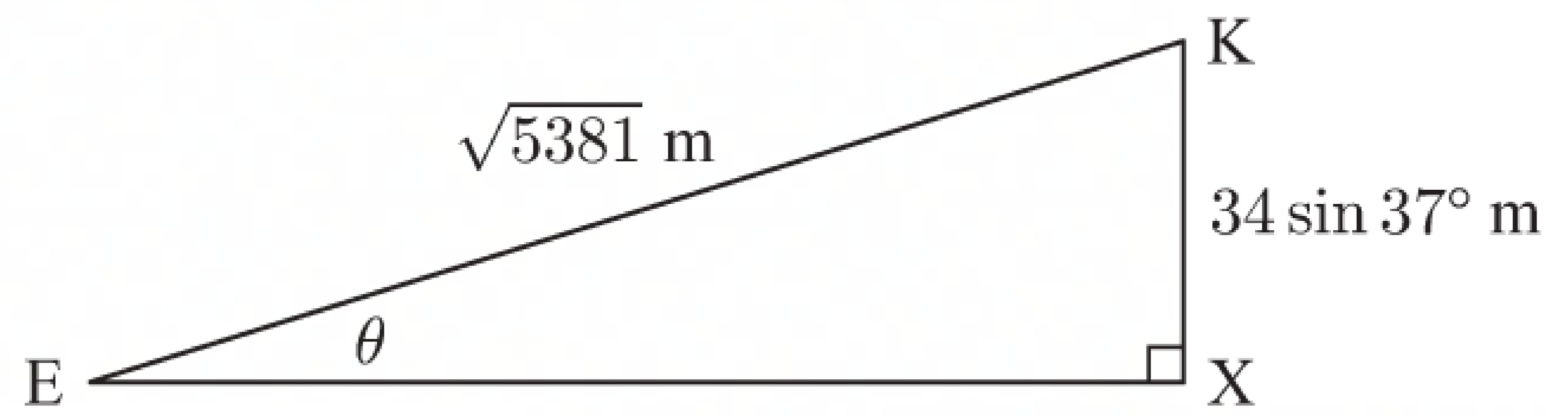
$$\therefore \theta = \sin^{-1}\left(\frac{34\sin 37°}{\sqrt{5381}}\right)$$

$$\therefore \theta \approx 16.2°$$

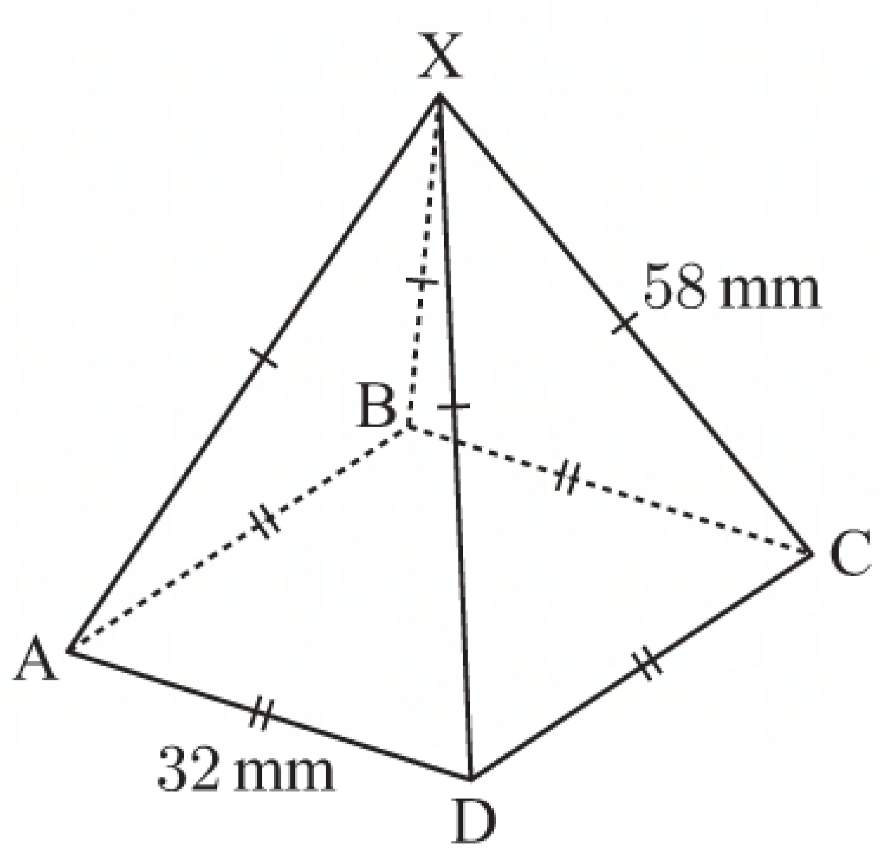
So, the angle of elevation from Edward to the kite is about 16.2°.







21

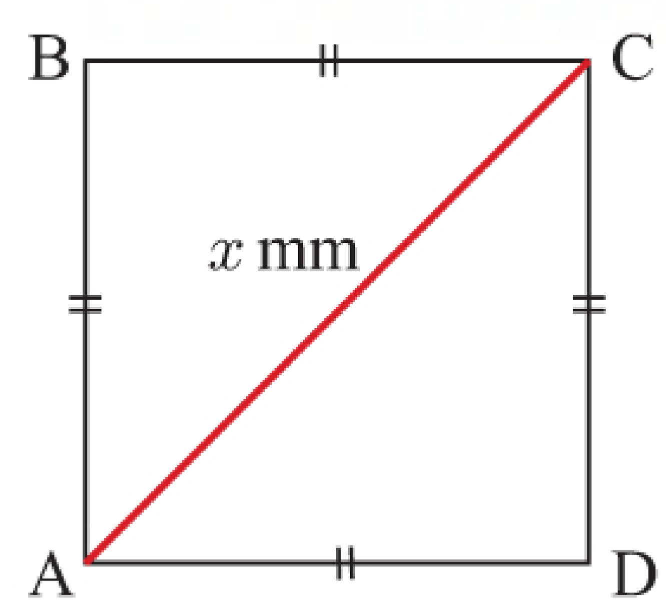


Consider the base of the pyramid, letting AC be x mm.

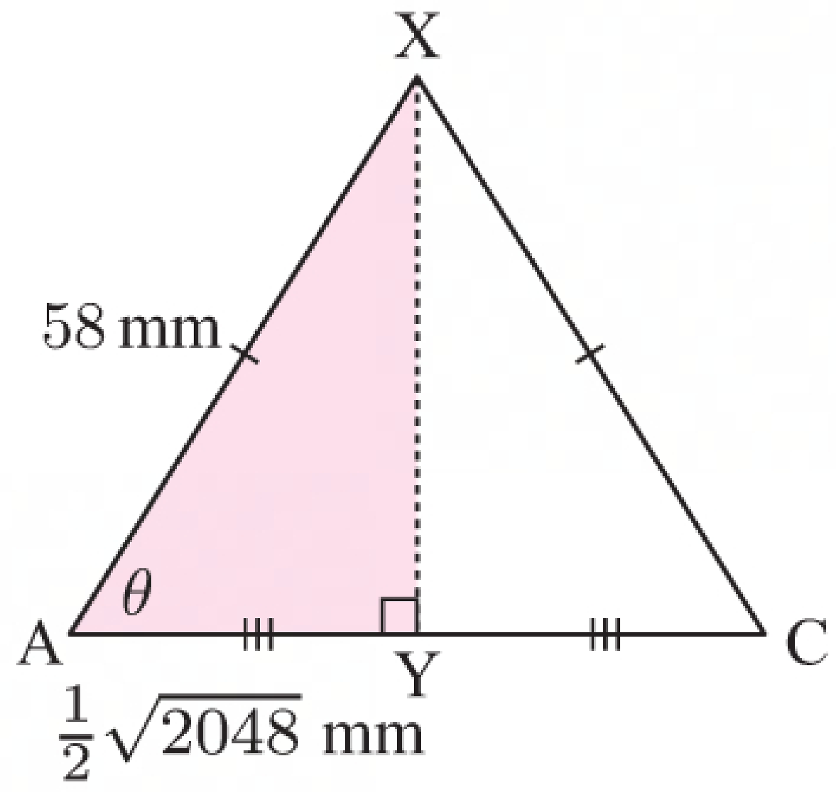
Using Pythagoras, $x^2 = 32^2 + 32^2$

$= 2048$

$\therefore x = \sqrt{2048}$ {as $x > 0$}



$\triangle ACX$ is isosceles as $AX = CX$.
We draw the perpendicular bisector [XY] of [AC].



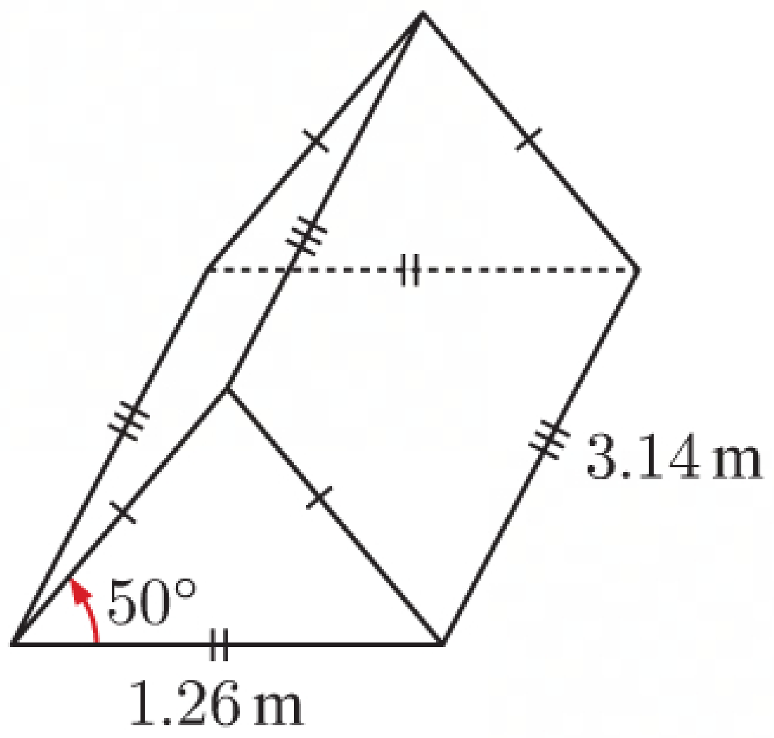
In $\triangle AXY$, $\cos\theta = \dfrac{\frac{1}{2}\sqrt{2048}}{58}$

$\therefore \quad \theta = \cos^{-1}\left(\dfrac{\frac{1}{2}\sqrt{2048}}{58}\right)$

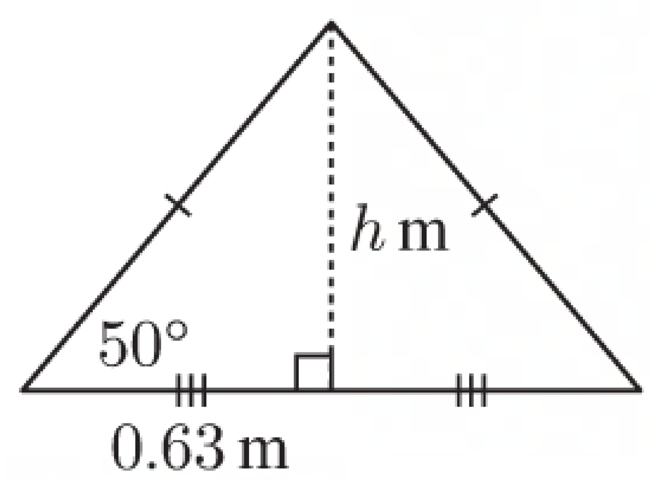
$\therefore \quad \theta \approx 67.0°$

So, the angle between [AX] and [AC] is about 67.0°.

22 **a**



Consider the triangular end of the prism.
Let the height of the triangular end be h m.



$$\tan 50° = \frac{h}{0.63}$$

$$\therefore \quad 0.63 \times \tan 50° = h$$

$$\therefore \quad h \approx 0.751$$

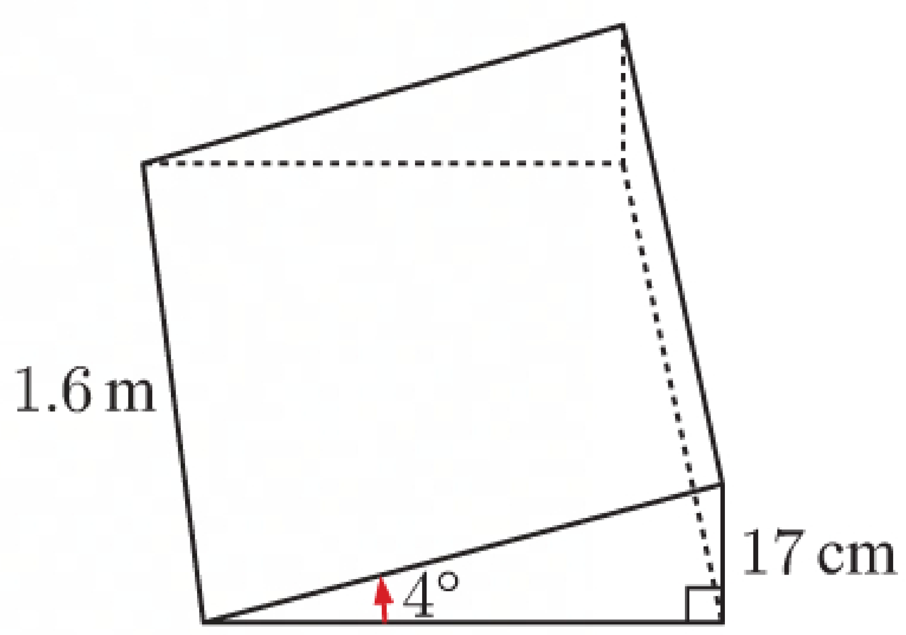
Volume of solid = area of triangular end × length

$= \frac{1}{2} \times \text{base} \times \text{height} \times \text{length}$

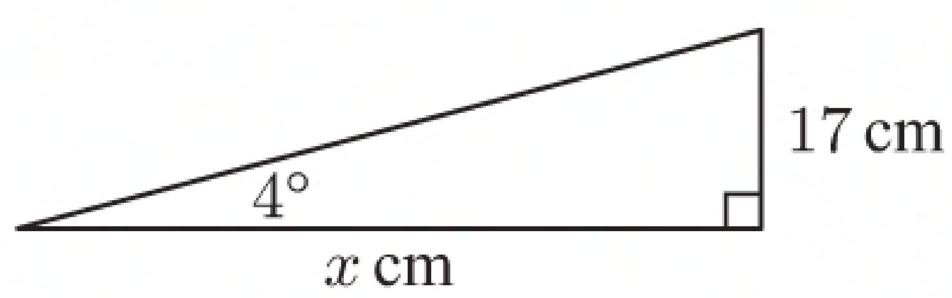
$\approx \frac{1}{2} \times 1.26 \times 0.751 \times 3.14 \text{ m}^3$

$\approx 1.49 \text{ m}^3$

b



Consider the triangular end of the prism.
Let the base of the triangular end be x cm.



$$\tan 4° = \frac{17}{x}$$

$$\therefore \quad x = \frac{17}{\tan 4°}$$

$$\approx 243.1$$

So, the base of the triangular end is about 243.1 cm long, or about 2.431 m long.

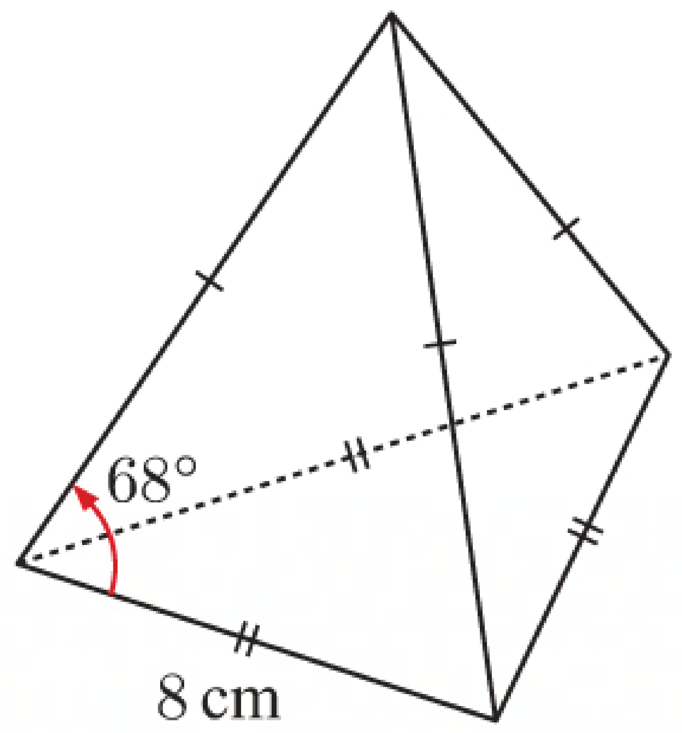
Volume of solid = area of triangular end × length

$= \frac{1}{2} \times \text{base} \times \text{height} \times \text{length}$

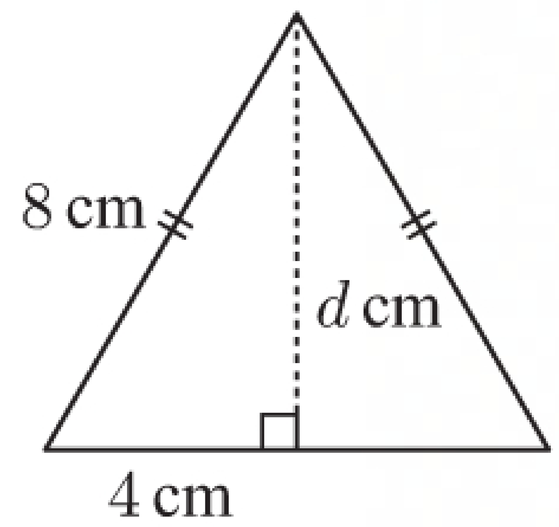
$\approx \frac{1}{2} \times 2.431 \times 0.17 \times 1.6$ {17 cm ≡ 0.17 m}

$\approx 0.331 \text{ m}^3$

c



Consider the base of the pyramid, which is an equilateral triangle with sides 8 cm.



Let the height of this triangle be d cm.

Using Pythagoras, $d^2 + 4^2 = 8^2$

$\therefore\ d^2 = 64 - 16$

$\therefore\ d^2 = 48$

$\therefore\ d = \sqrt{48}$

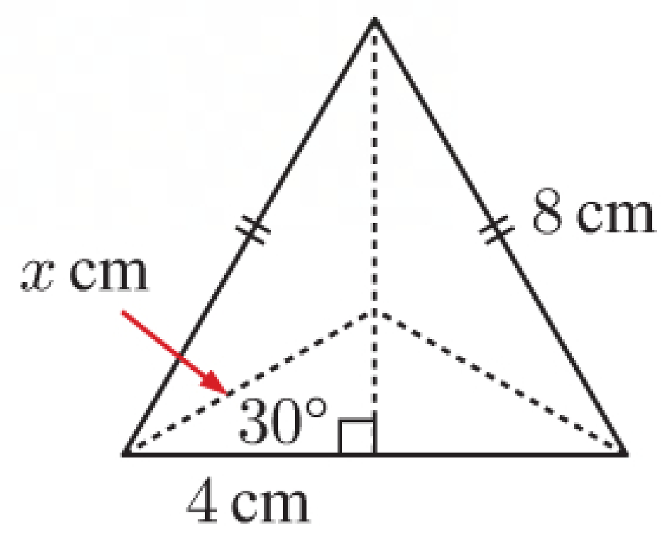
We can now find the area of the base of the pyramid.

Area $= \frac{1}{2} \times \text{base} \times \text{height}$

$= \frac{1}{2} \times 8 \times \sqrt{48}\ \text{cm}^2$

$= 4\sqrt{48}\ \text{cm}^2$

Now we need to find the height of the pyramid.

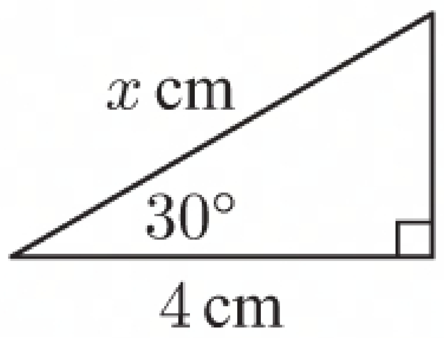


We need to find the distance from each corner of the base to the centre of the base. Let this distance be x cm.

We divide the base into 3 equal isosceles triangles, each with equal base angles of $\dfrac{60°}{2} = 30°$.

$\cos 30° = \dfrac{4}{x}$

$\therefore\ x = \dfrac{4}{\cos 30°}$

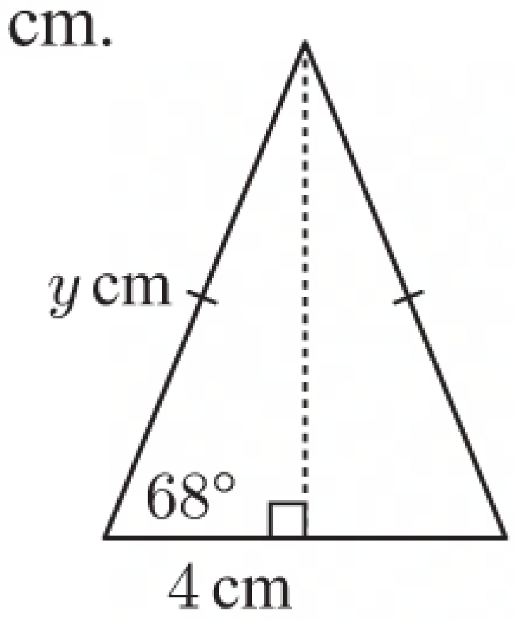


Consider the side faces of the pyramid, which are isosceles triangles with base 8 cm.

Let the equal side lengths of the isosceles triangle be y cm.

$\cos 68° = \dfrac{4}{y}$

$\therefore\ y = \dfrac{4}{\cos 68°}$



We can now find the height of the pyramid.

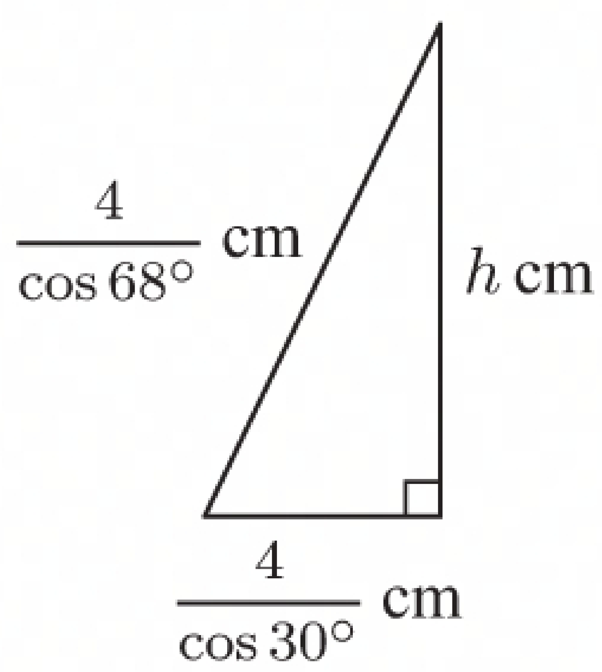
Let the height of the pyramid be h cm.

Using Pythagoras,

$$h^2 + \left(\frac{4}{\cos 30°}\right)^2 = \left(\frac{4}{\cos 68°}\right)^2$$

$$\therefore\ h = \sqrt{\left(\frac{4}{\cos 68°}\right)^2 - \left(\frac{4}{\cos 30°}\right)^2} \quad \{\text{as } h > 0\}$$

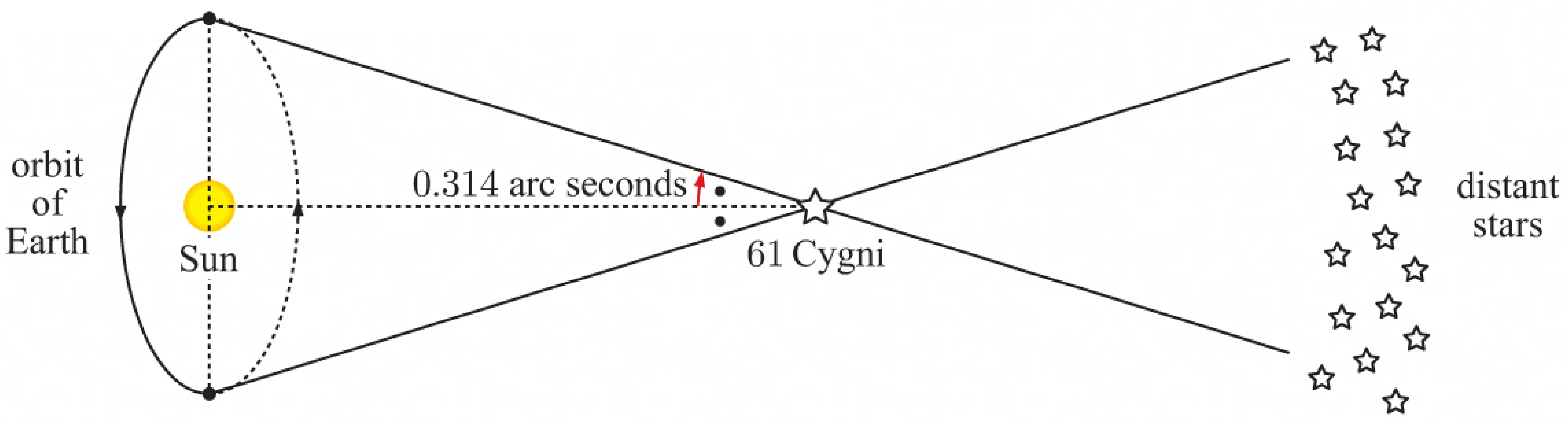
$$\therefore\ h \approx 9.627$$



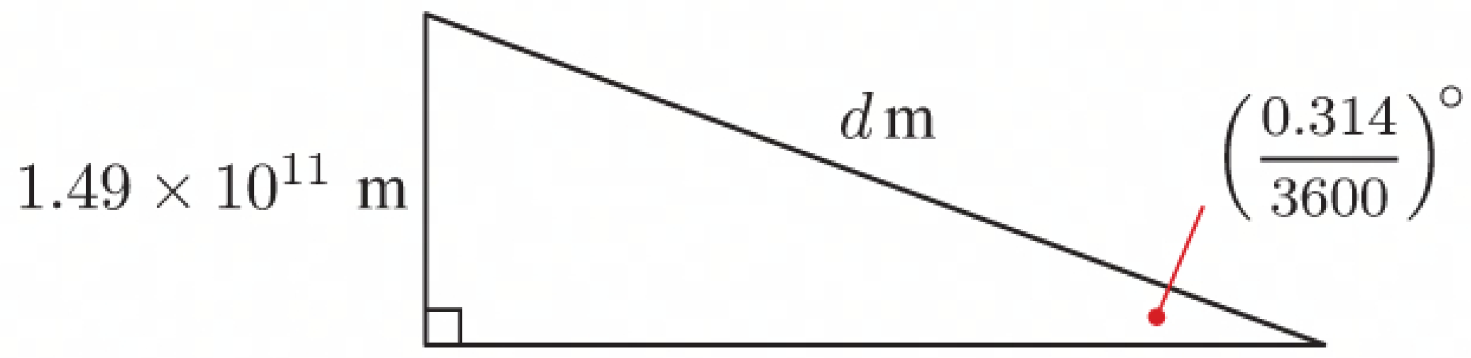
So, the pyramid's height is about 9.627 cm.

$$\begin{aligned}\text{Volume of solid} &= \tfrac{1}{3}(\text{area of base} \times \text{height})\\ &\approx \tfrac{1}{3} \times 4\sqrt{48} \times 9.627\ \text{cm}^3\\ &\approx 88.9\ \text{cm}^3\end{aligned}$$

23



a We need to find the distance from the Earth (in orbit) to the star 61 Cygni.



The radius of the Earth's orbit is $\approx 1.49 \times 10^{11}$ m.

The parallax of 61 Cygni is about 0.314 arc seconds, or about $\dfrac{0.314}{3600}$ degrees.

Let the distance from Earth to 61 Cygni be d m.

$$\sin\left(\frac{0.314}{3600}\right)^\circ = \frac{1.49 \times 10^{11}}{d}$$

$$\therefore\ d = \frac{1.49 \times 10^{11}}{\sin\left(\frac{0.314}{3600}\right)^\circ}$$

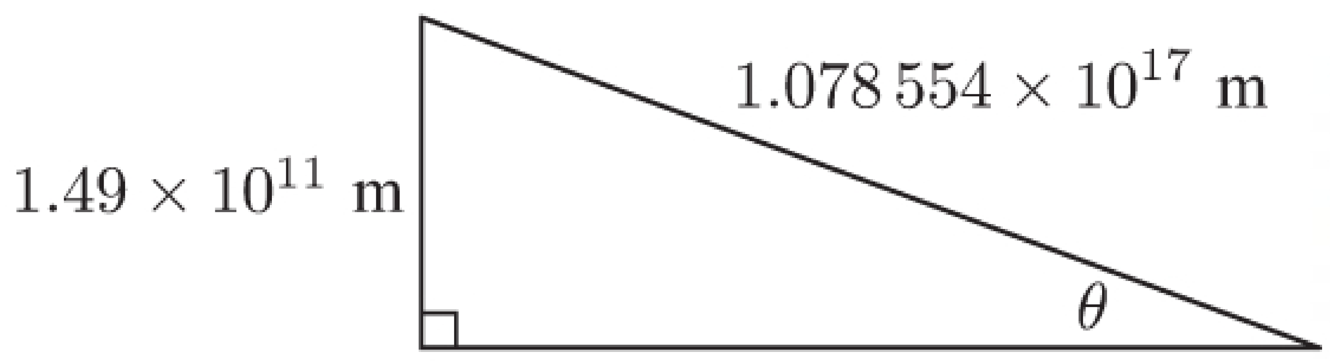
$$\therefore\ d \approx 9.7877 \times 10^{16}\ \text{m}$$

$$\therefore\ d \approx \frac{9.7877 \times 10^{16}}{9.461 \times 10^{15}}\ \text{light-years} \qquad \{1\ \text{light-year} \approx 9.461 \times 10^{15}\ \text{m}\}$$

$$\therefore\ d \approx 10.3\ \text{light-years}$$

b
$$\begin{aligned}11.4\ \text{light-years} &\approx 11.4 \times 9.461 \times 10^{15}\ \text{m}\\ &\approx 1.078\,554 \times 10^{17}\ \text{m}\end{aligned}$$

Let the parallax of 61 Cygni be θ.



$$\sin\theta = \frac{1.49 \times 10^{11}}{1.078\,554 \times 10^{17}}$$

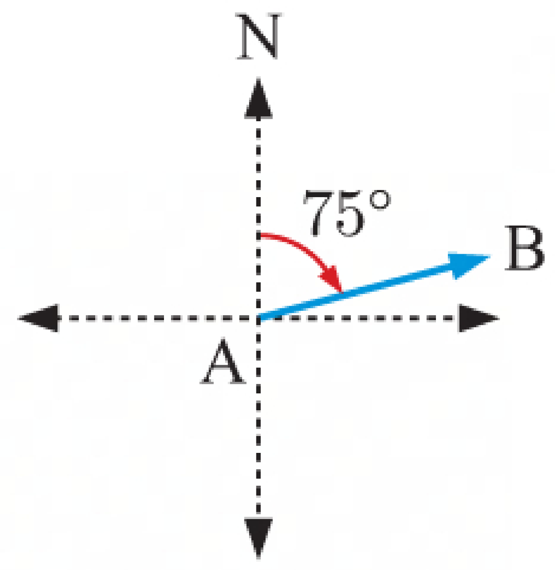
$$\therefore\ \theta = \sin^{-1}\left(\frac{1.49 \times 10^{11}}{1.078\,554 \times 10^{17}}\right)$$

$$\begin{aligned}\therefore\ \theta &\approx 7.9153 \times 10^{-5}\ \text{degrees}\\ &\approx 7.9153 \times 10^{-5} \times 3600\ \text{arc seconds} \qquad \{1\ \text{arc second} = \tfrac{1}{3600}\ \text{degree}\}\\ &\approx 0.285\ \text{arc seconds}\end{aligned}$$

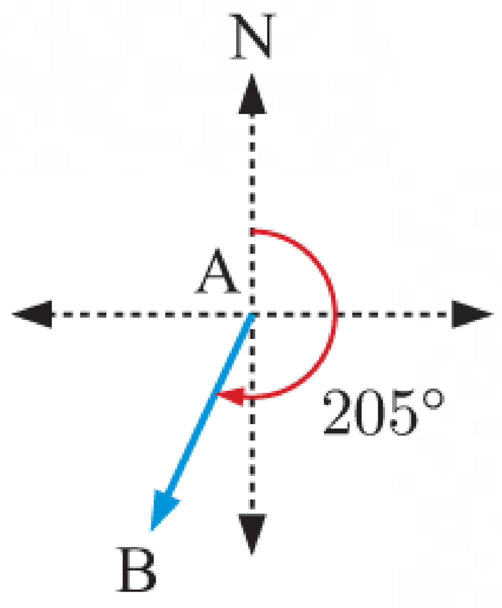
So, the parallax of 61 Cygni is about 0.285 arc seconds.

EXERCISE 7E

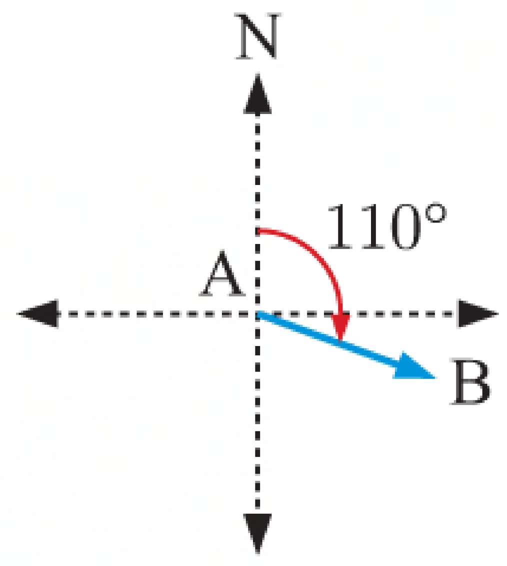
1 **a**



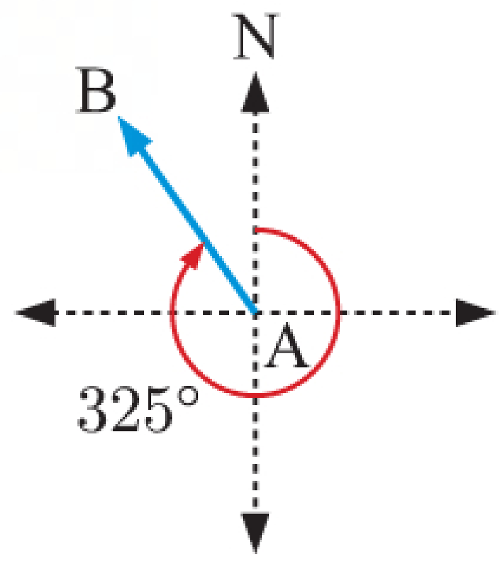
b



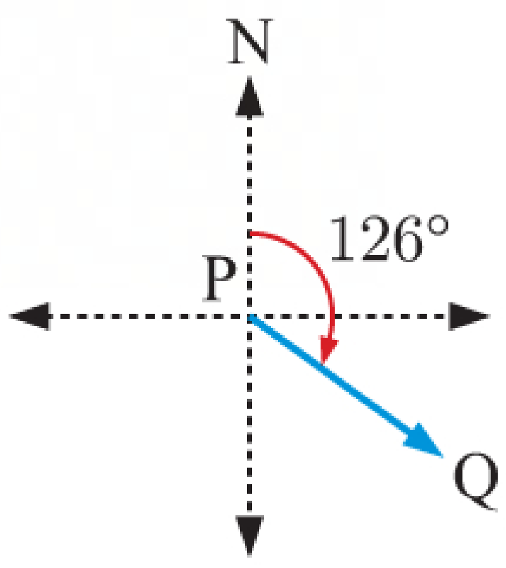
c



d

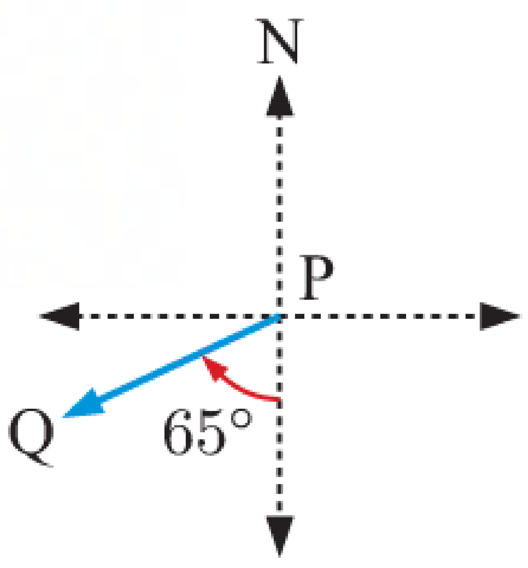


2 **a**



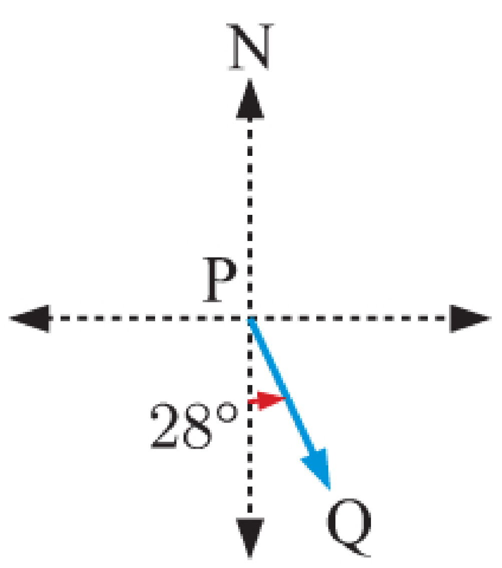
The bearing of Q from P
$= 126°$

b



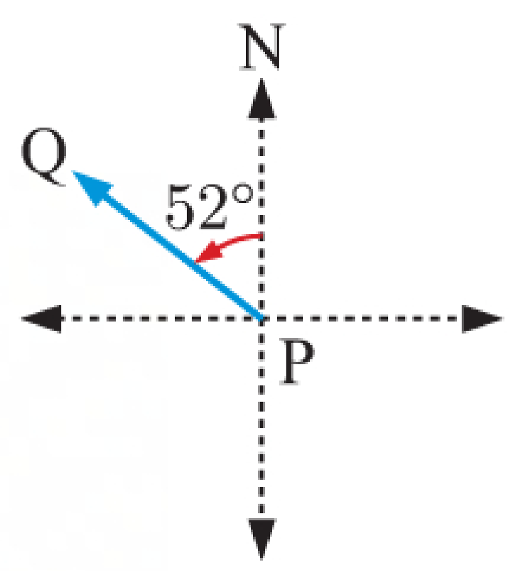
The bearing of Q from P
$= 180° + 65°$
$= 245°$

c



The bearing of Q from P
$= 180° - 28°$
$= 152°$

d



The bearing of Q from P
$= 360° - 52°$
$= 308°$

3 **a** The bearing of B from A is 072°.

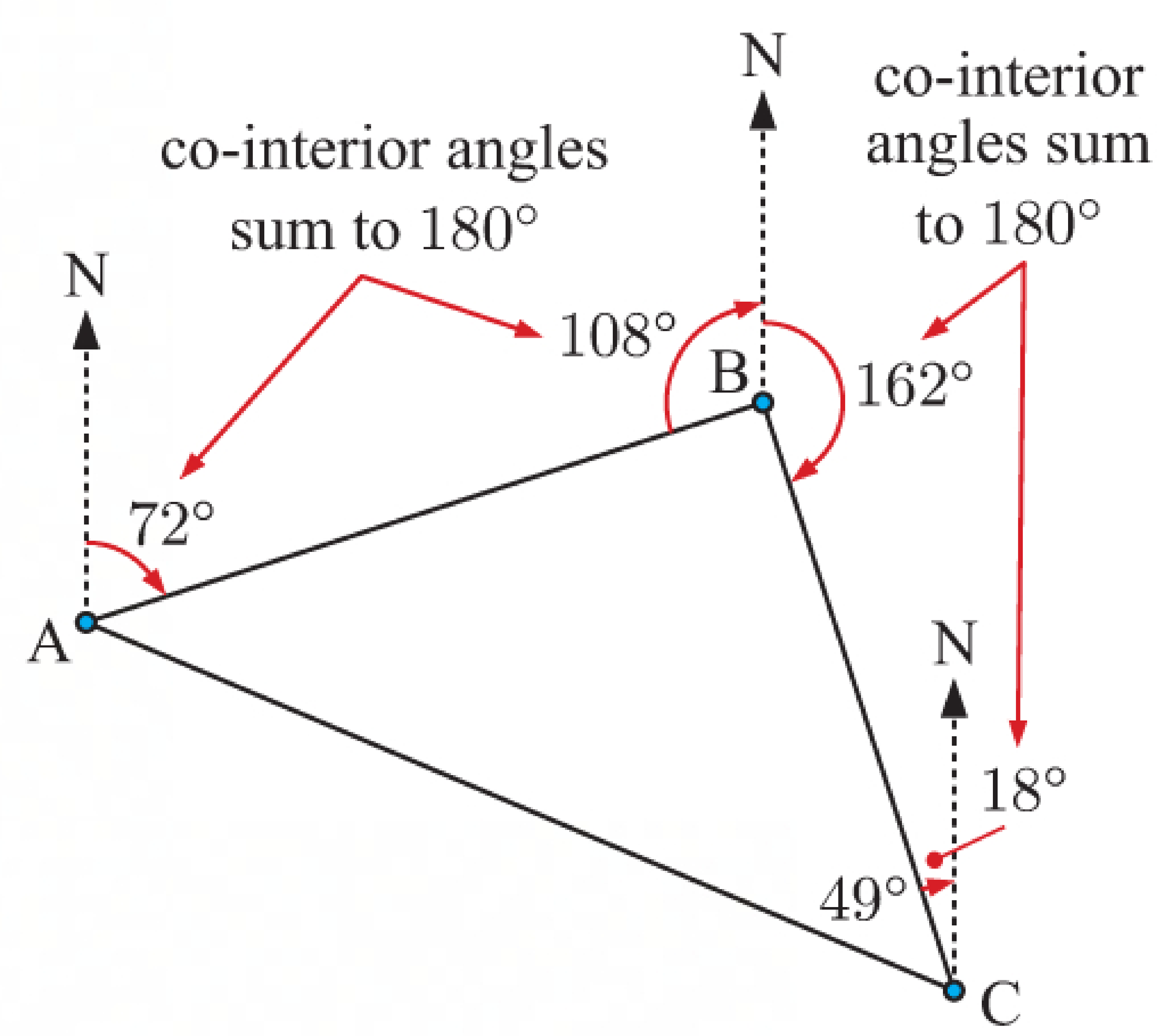
b The bearing of A from B $= 360° - 108°$
$= 252°$

c The bearing of C from B is 162°.

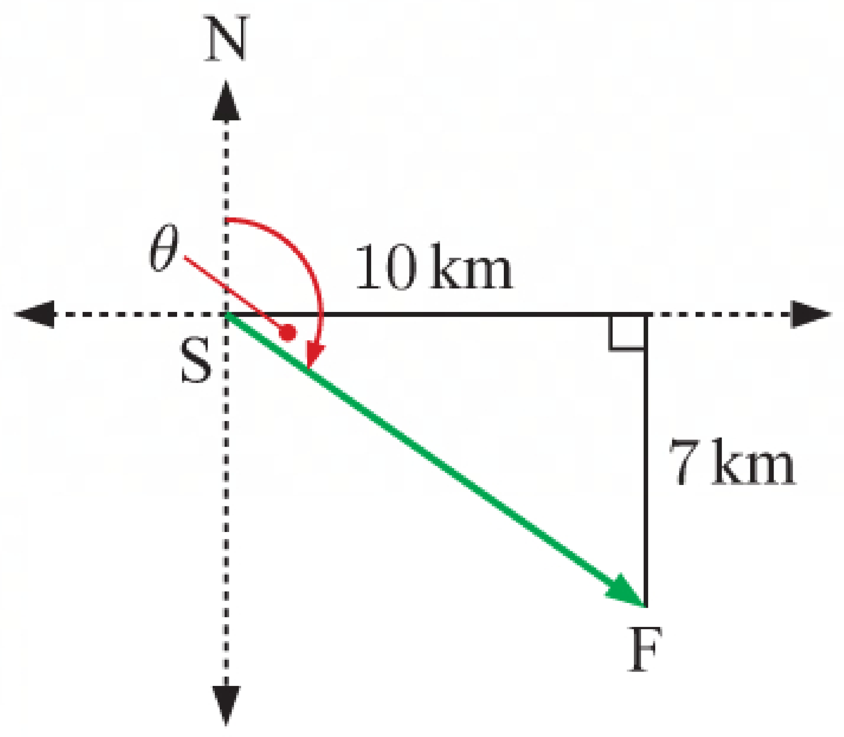
d The bearing of B from C $= 360° - 18°$
$= 342°$

e The bearing of C from A $= 72° + 41°$
$= 113°$

f The bearing of A from C $= 360° - 49° - 18°$
$= 293°$



4 Suppose Walter starts at S and finishes at F.

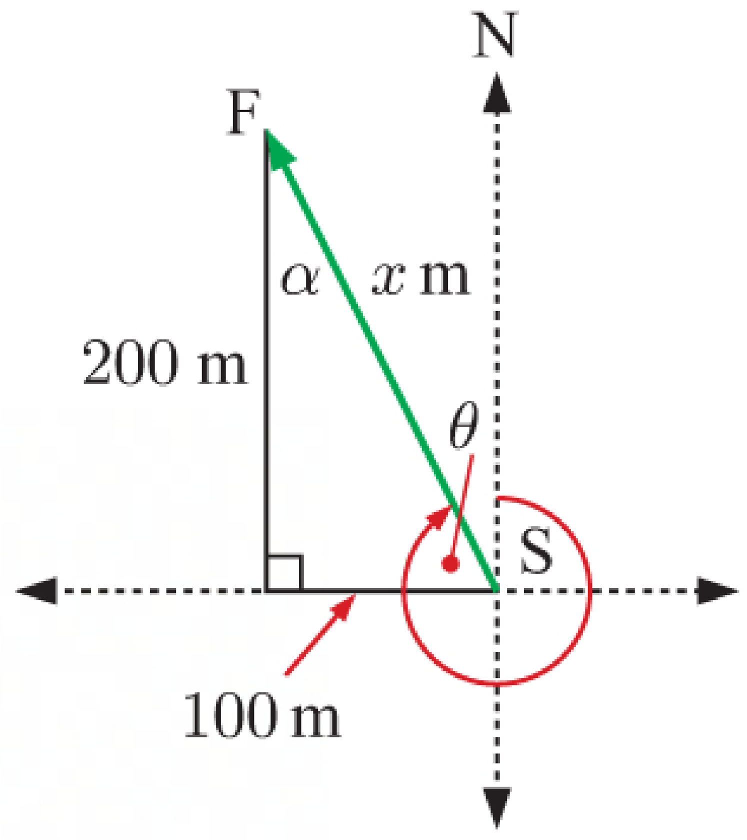


$\tan\theta = \dfrac{7}{10}$

$\therefore \quad \theta = \tan^{-1}\left(\frac{7}{10}\right) \approx 35.0°$

So, the bearing $\approx 90° + 35°$
$\approx 125°$

5 **a** Suppose Julia starts at S and finishes at F.



Now $\quad x^2 = 200^2 + 100^2 \quad$ {Pythagoras}
$= 50\,000$
$\therefore \quad x \approx 224 \quad$ {as $x > 0$}

$\therefore$ Julia is about 224 m from her starting point.

b $\tan\theta = \dfrac{200}{100} = 2$

$\therefore \quad \theta = \tan^{-1}(2) \approx 63.4°$

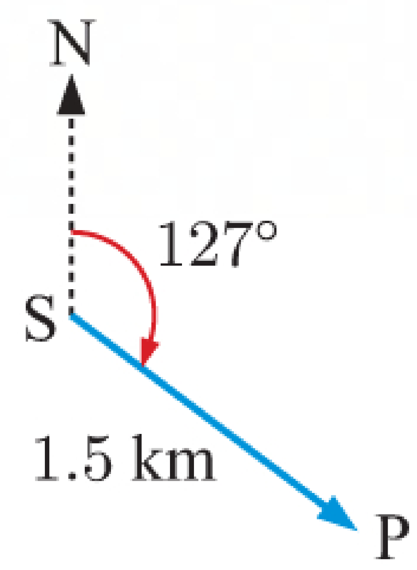
So, the bearing $\approx 270° + 63.4°$
$\approx 333°$

c $\alpha = 180° - 90° - \theta \quad$ {angles in a triangle}
$\approx 90° - 63.4°$
$\approx 26.6°$

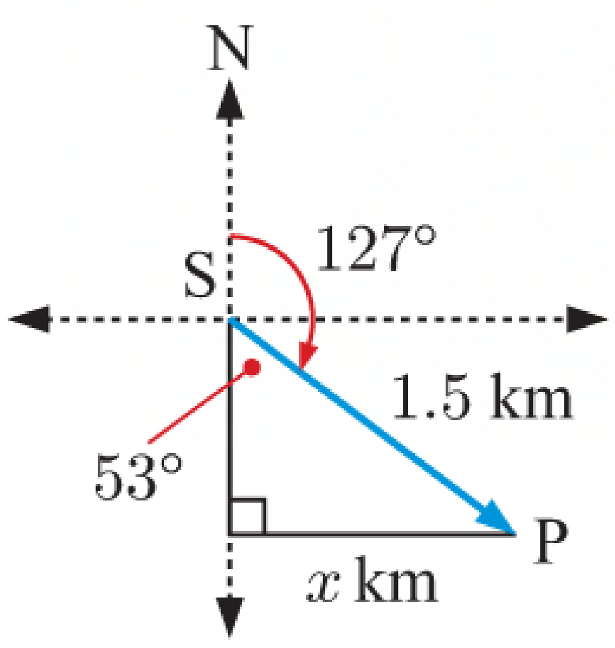
The bearing of S from F $\approx 180° - 26.6°$
$\approx 153.4°$

So, the starting point is at a bearing of approximately 153° from where Julia is now.

6 **a** Suppose Paul starts at S and finishes at P.



b Let the distance Paul has travelled east be x km.



$\sin 53° = \dfrac{x}{1.5}$

$\therefore \quad 1.5 \times \sin 53° = x$

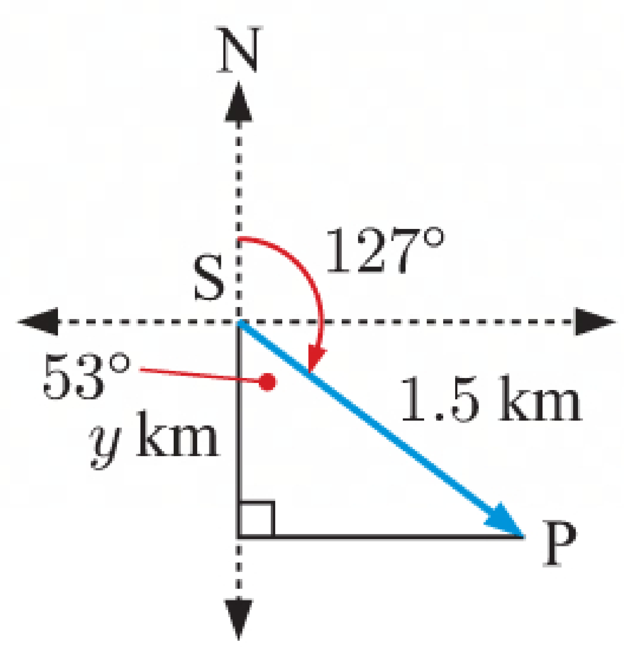
$\therefore \quad x \approx 1.20$

$\therefore$ Paul is about 1.20 km east from his starting point.

c Let the distance Paul has travelled south be y km.

$$\cos 53° = \frac{y}{1.5}$$
$$\therefore \quad 1.5 \times \cos 53° = y$$
$$\therefore \quad y \approx 0.903$$

$\therefore$ Paul is about 0.903 km south of his starting point.

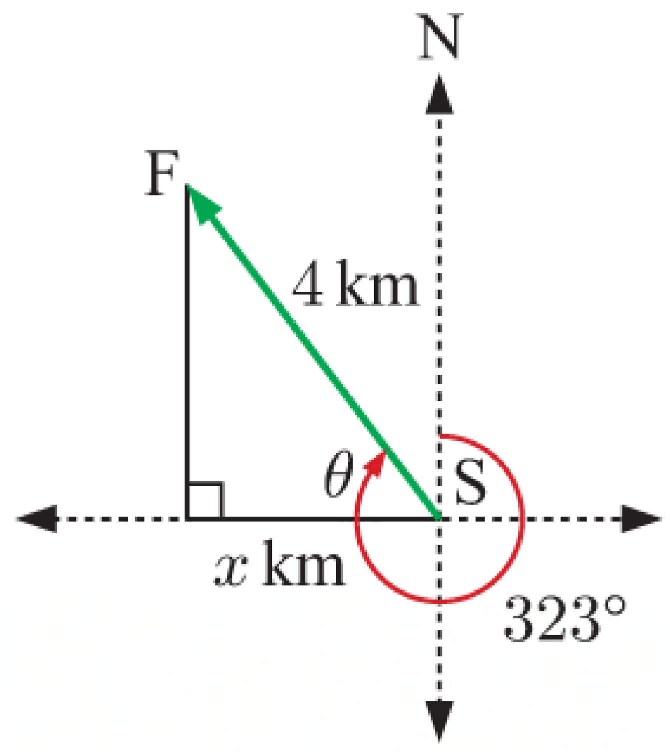


7 Suppose Tiffany starts at S and finishes at F.
Let the distance Tiffany has travelled west be x km.

$\theta = 323° - 270° = 53°$

$$\therefore \quad \cos 53° = \frac{x}{4}$$
$$\therefore \quad 4 \times \cos 53° = x$$
$$\therefore \quad x \approx 2.41$$

$\therefore$ Tiffany is about 2.41 km west from her starting point.

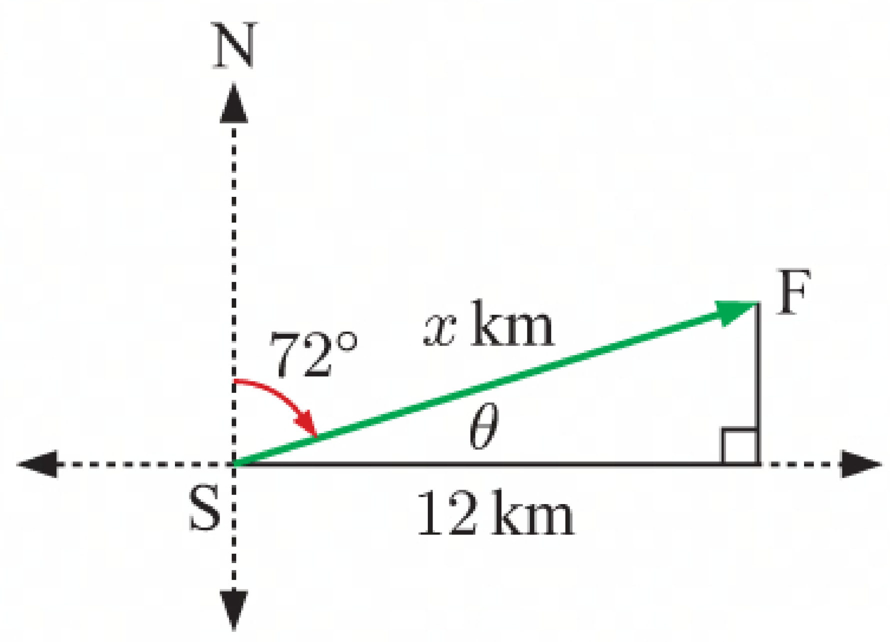


8 Suppose the train starts at S and finishes at F.
Let the distance travelled by the train be x km.

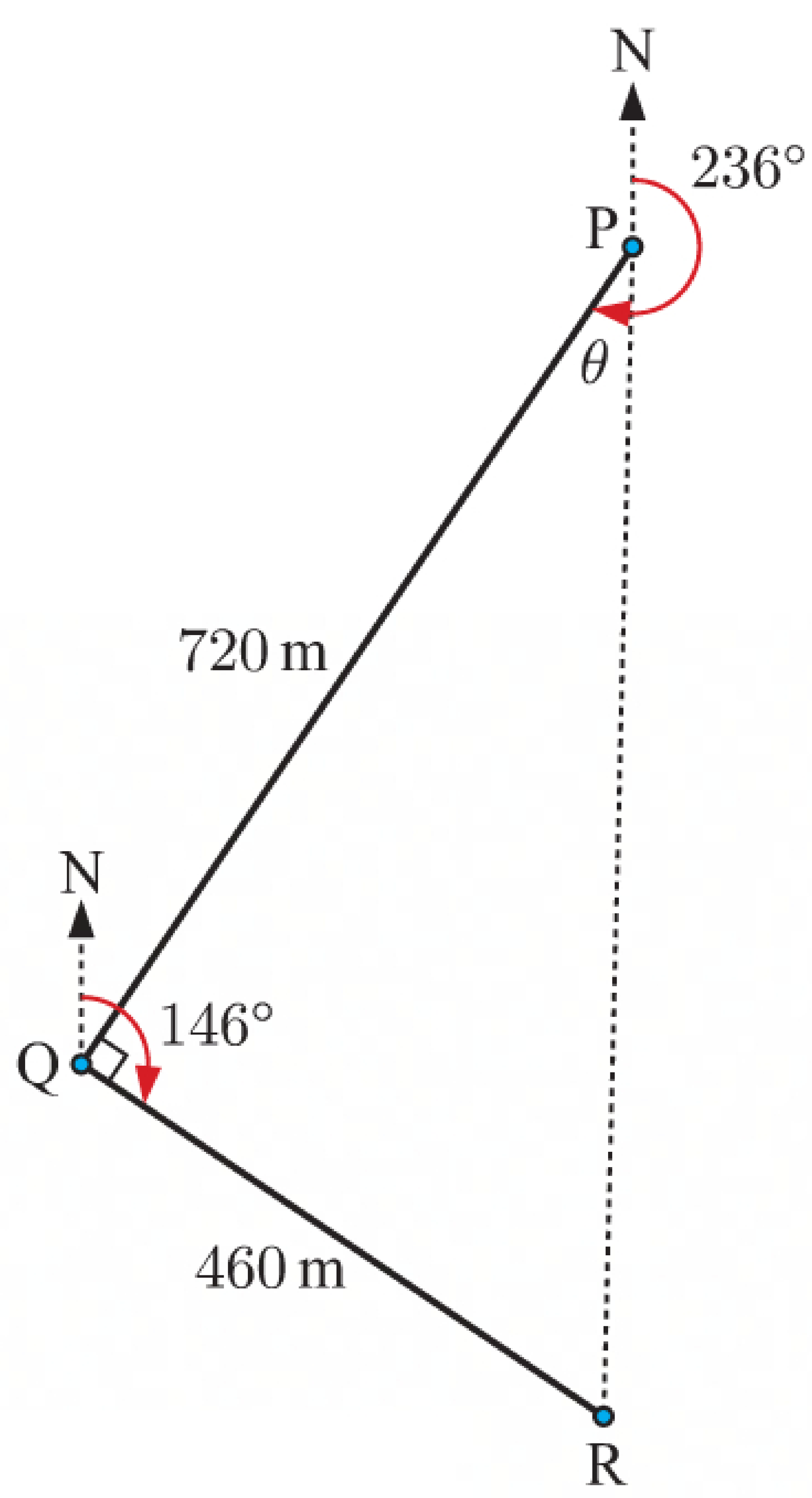
$\theta = 90° - 72° = 18°$

$$\therefore \quad \cos 18° = \frac{12}{x}$$
$$\therefore \quad x = \frac{12}{\cos 18°} \approx 12.6$$

$\therefore$ the train travelled about 12.6 km on the bearing 072°.



9



Suppose the orienteer starts at P, then travels to Q, and finishes at R.

$$\begin{aligned} N\widehat{P}Q &= 360° - 236° && \{\text{angles at a point}\} \\ &= 124° \\ N\widehat{Q}P &= 180° - 124° && \{\text{co-interior angles}\} \\ &= 56° \\ P\widehat{Q}R &= 146° - 56° \\ &= 90° \end{aligned}$$

$\therefore$ $\triangle$PQR is right angled at Q.

a $PR^2 = 720^2 + 460^2$ {Pythagoras}

$\therefore$ $PR = \sqrt{720^2 + 460^2}$ {as $PR > 0$}

≈ 854

So, the finishing point is about 854 km from the starting point.

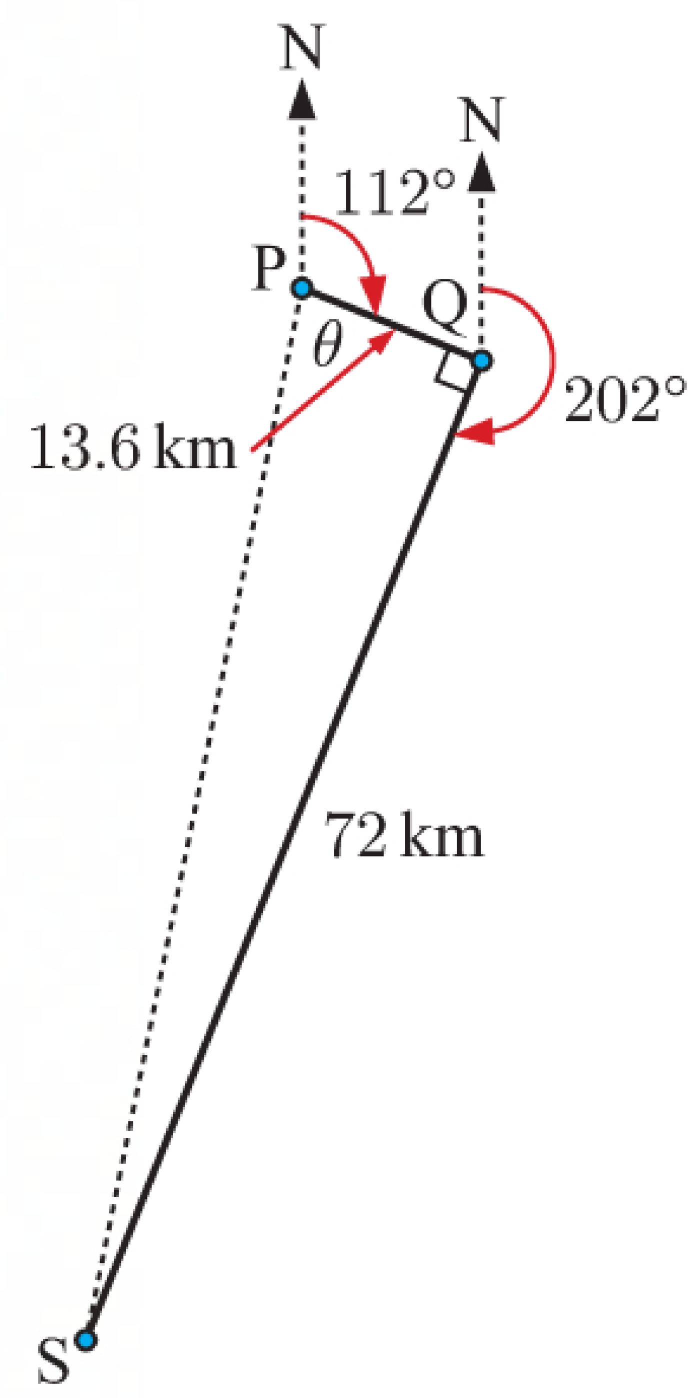
b $\tan\theta = \dfrac{460}{720}$

$\therefore \quad \theta = \tan^{-1}\left(\frac{460}{720}\right)$

$\therefore \quad \theta \approx 32.6°$

So, the bearing of the finishing point from the starting point is about $236° - 32.6° \approx 203°$.

10 Suppose the cruise ship first sails from P to Q, then from Q to S.



$\widehat{NQP} = 180° - 112°$ {co-interior angles}
$= 68°$

$\widehat{PQS} = 360° - 68° - 202°$ {angles at a point}
$= 90°$

$\therefore$ $\triangle$PQS is right angled at Q.

$\tan\theta = \dfrac{72}{13.6}$

$\therefore$ $\theta = \tan^{-1}\left(\frac{72}{13.6}\right)$

$\therefore$ $\theta \approx 79.3°$

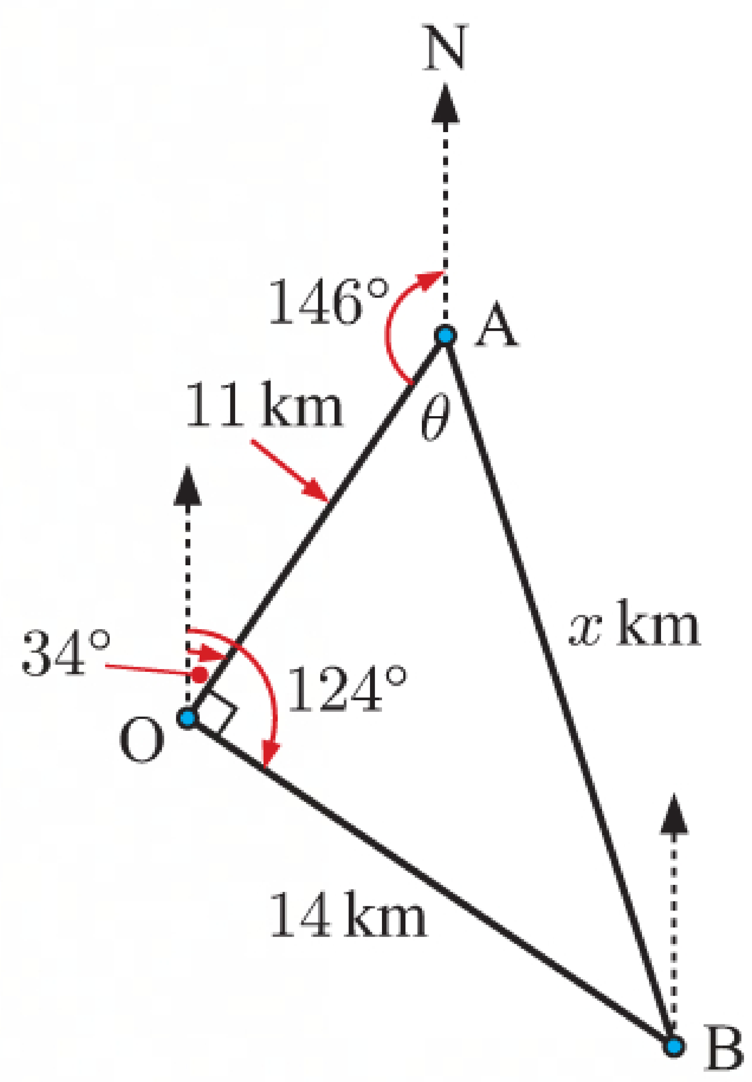
$\therefore$ the bearing of the cruise ship from P $\approx 112° + 79.3°$
$\approx 191°$

Now, $PS^2 = 13.6^2 + 72^2$ {Pythagoras}

$\therefore$ $PS = \sqrt{13.6^2 + 72^2}$ {as $PS > 0$}
≈ 73.3

So, the cruise ship is about 73.3 km on a bearing of 191° from P.

11 Suppose the yachts both depart from O.



$\widehat{OAN} = 180° - 34° = 146°$ {co-interior angles}

$\widehat{AOB} = 124° - 34° = 90°$

$\therefore$ $\triangle$AOB is right angled.

$\tan\theta = \dfrac{14}{11}$

$\therefore$ $\theta = \tan^{-1}\left(\frac{14}{11}\right) \approx 51.8°$

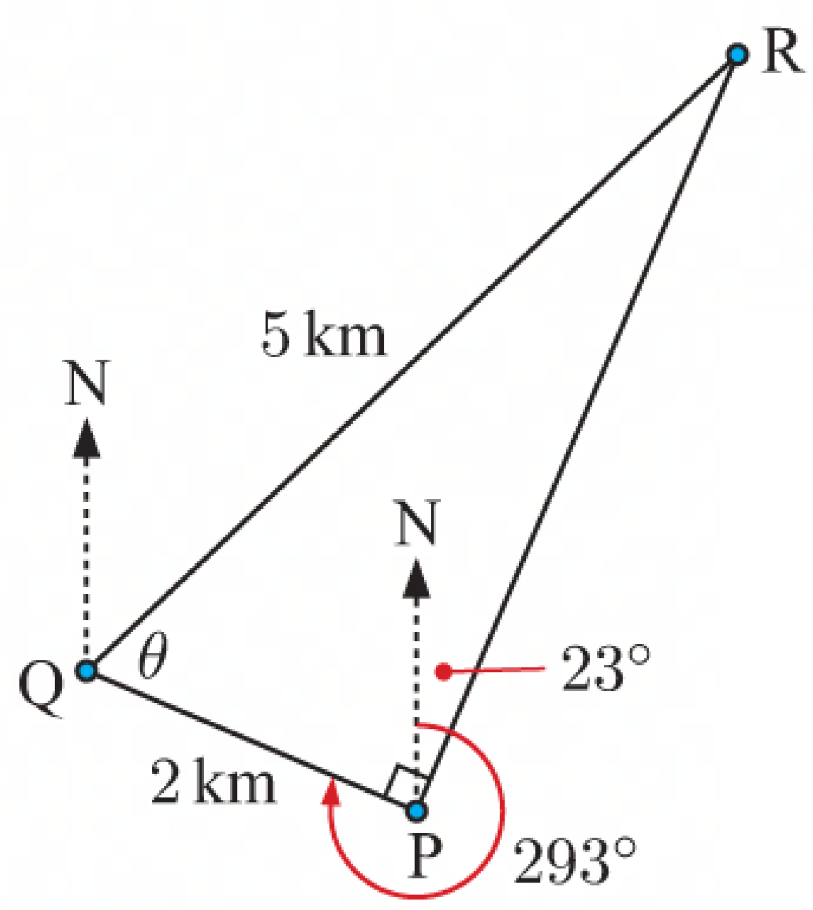
$\therefore$ the bearing of B from A $\approx 360° - 146° - 51.8°$
$\approx 162°$

Now, $AB^2 = 11^2 + 14^2$ {Pythagoras}

$\therefore$ $AB = \sqrt{11^2 + 14^2}$ {as $AB > 0$}
≈ 17.8

So, yacht B is about 17.8 km from yacht A on the bearing of about 162°.

12 Suppose the eagle starts at Q and flies to R, and the eagle's nest is at P.



$\widehat{NPQ} = 360° - 293°$ {angles at a point}
$= 67°$

$\widehat{QPR} = 67° + 23°$
$= 90°$

$\therefore$ $\triangle$PQR is right angled at P.

a $\cos\theta = \dfrac{2}{5}$

$\therefore \quad \theta = \cos^{-1}(\frac{2}{5})$

$\therefore \quad \theta \approx 66.4^\circ$

The eagle flew on a bearing of $\approx 180^\circ - \theta - \widehat{NPQ}$ {co-interior angles}

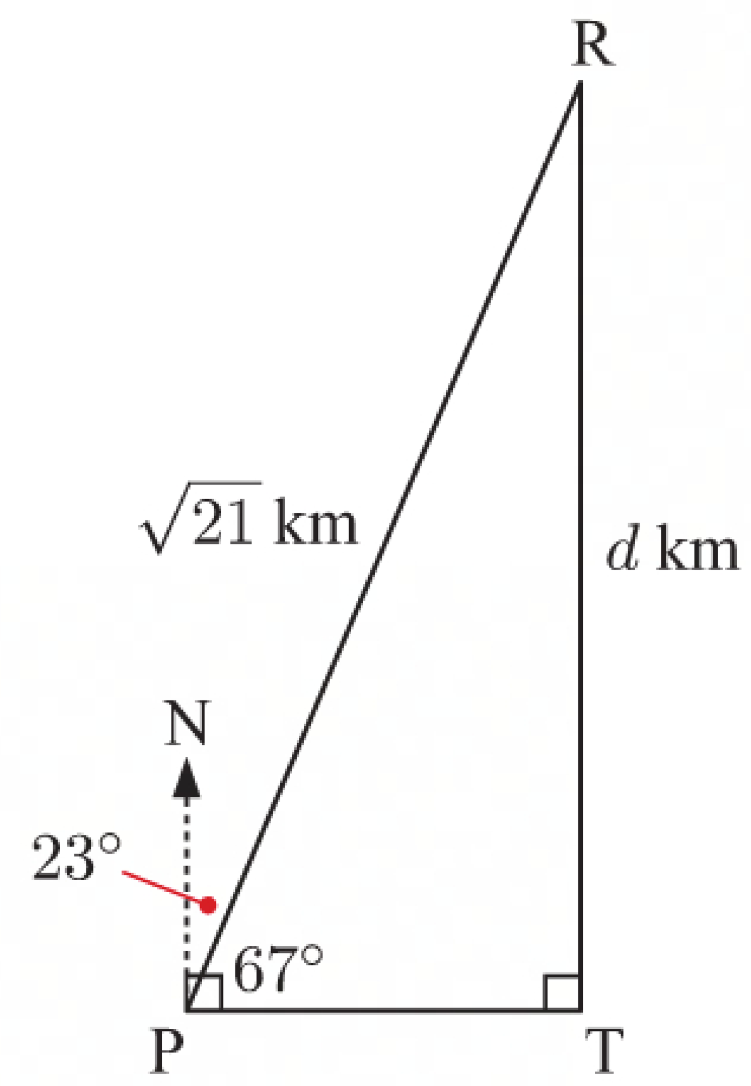
$\approx 180^\circ - 66.4^\circ - 67^\circ$

$\approx 046.6^\circ$

b In $\triangle PQR$, $PR^2 + 2^2 = 5^2$ {Pythagoras}

$\therefore \quad PR = \sqrt{5^2 - 2^2}$ {as $PR > 0$}

$\therefore \quad PR = \sqrt{21}$ km



Let T be the point directly south of R and directly east of P.

In $\triangle PRT$, $\widehat{RPT} = 90^\circ - 23^\circ$ {complementary angles}

$= 67^\circ$

$\sin 67^\circ = \dfrac{d}{\sqrt{21}}$

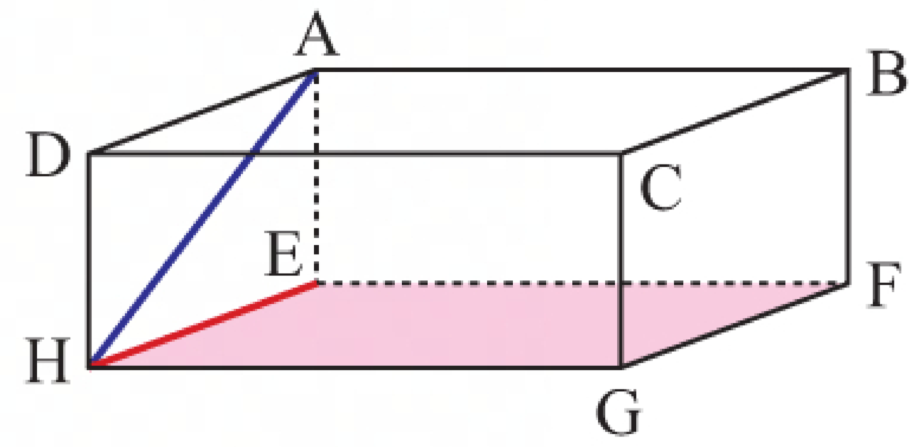
$\therefore \quad d = \sqrt{21} \times \sin 67^\circ$

≈ 4.22

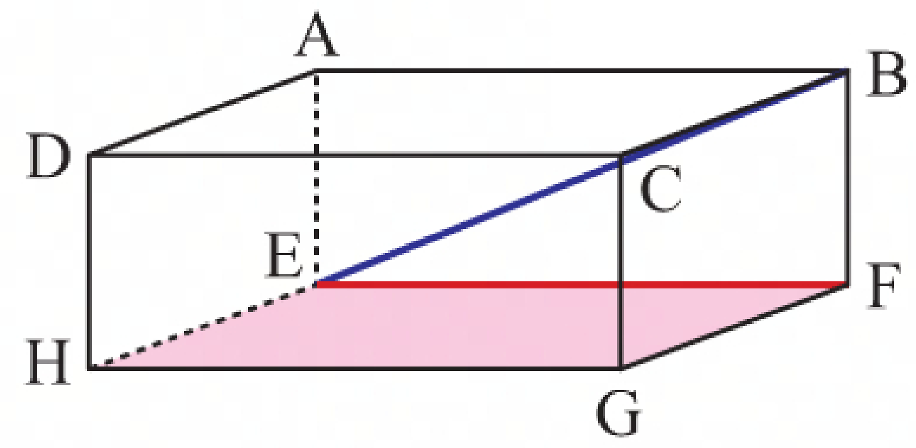
The eagle is now approximately 4.22 km north of its nest.

EXERCISE 7F

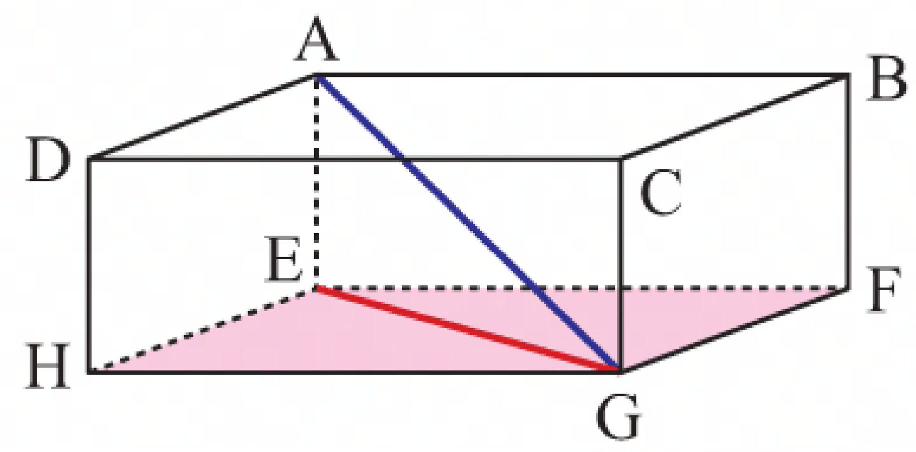
1 **a** **i** The projection of [AH] onto the base plane is [EH].



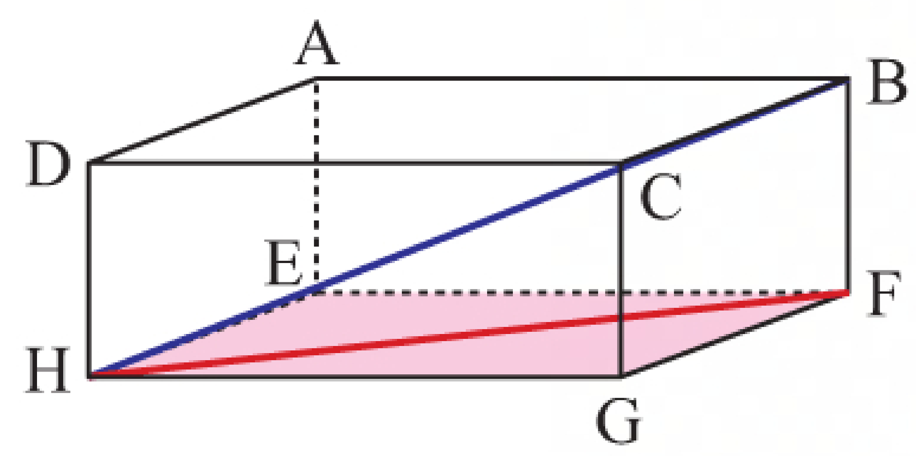
ii The projection of [BE] onto the base plane is [EF].



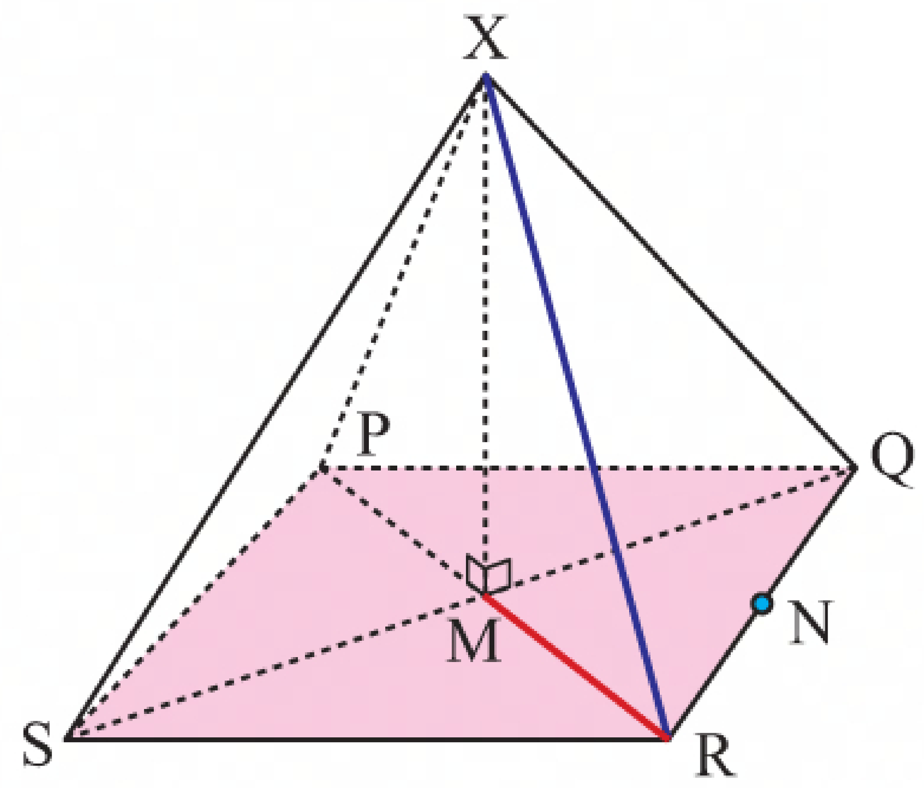
iii The projection of [AG] onto the base plane is [EG].



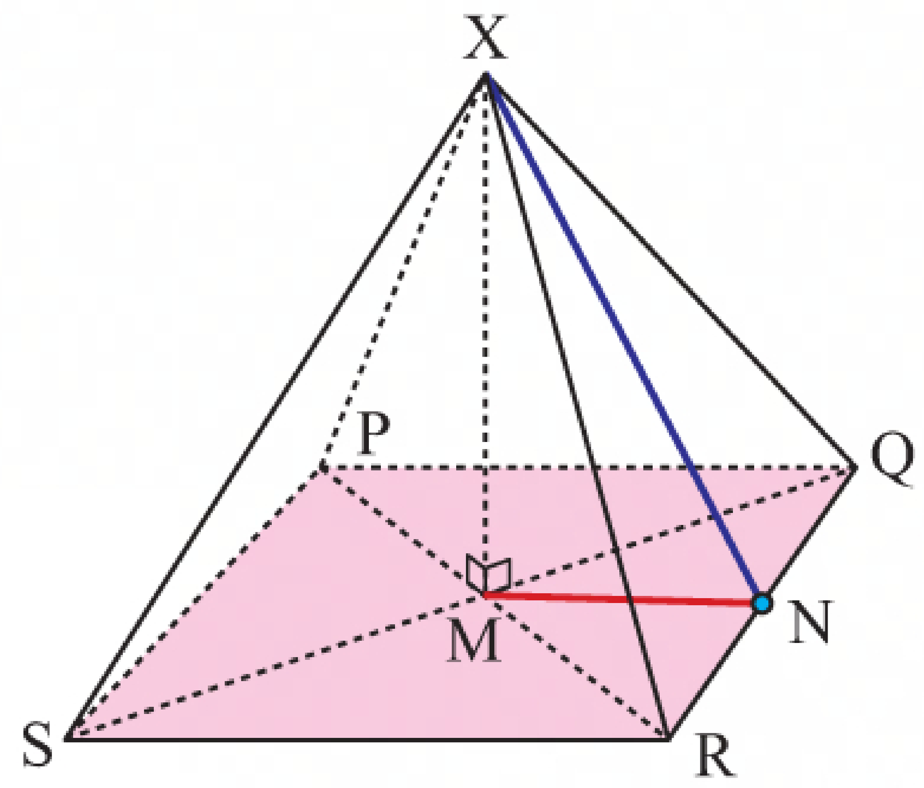
iv The projection of [BH] onto the base plane is [FH].



b **i** The projection of [RX] onto the base plane is [MR].

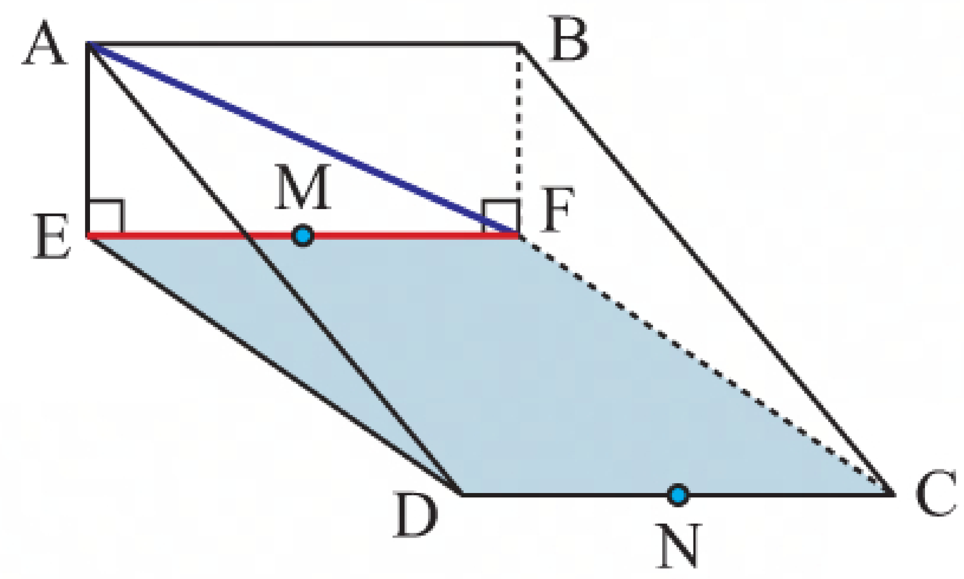


ii The projection of [NX] onto the base plane is [MN].



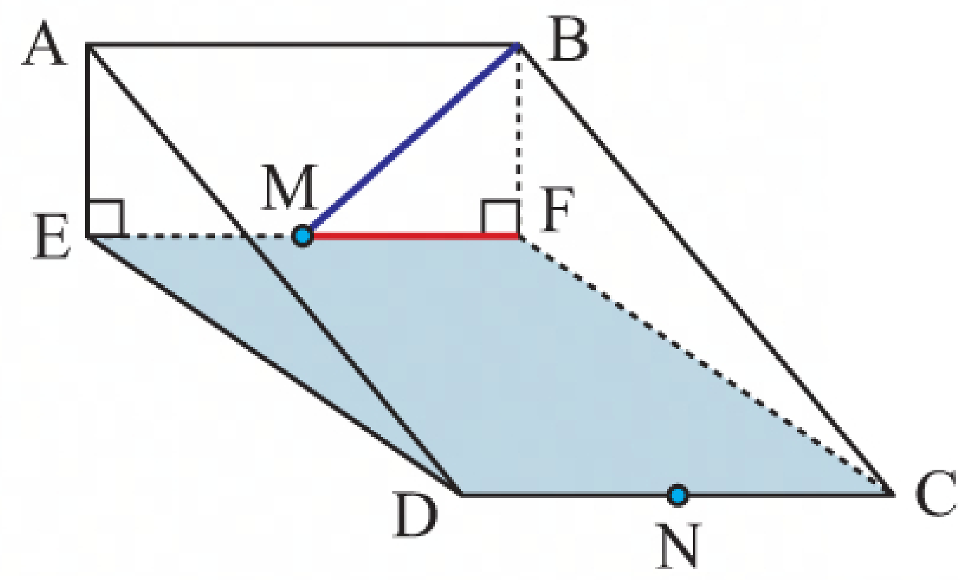
2 **a** **i** The projection of [AF] onto the base plane is [EF].

$\therefore$ the required angle is $A\widehat{F}E$.



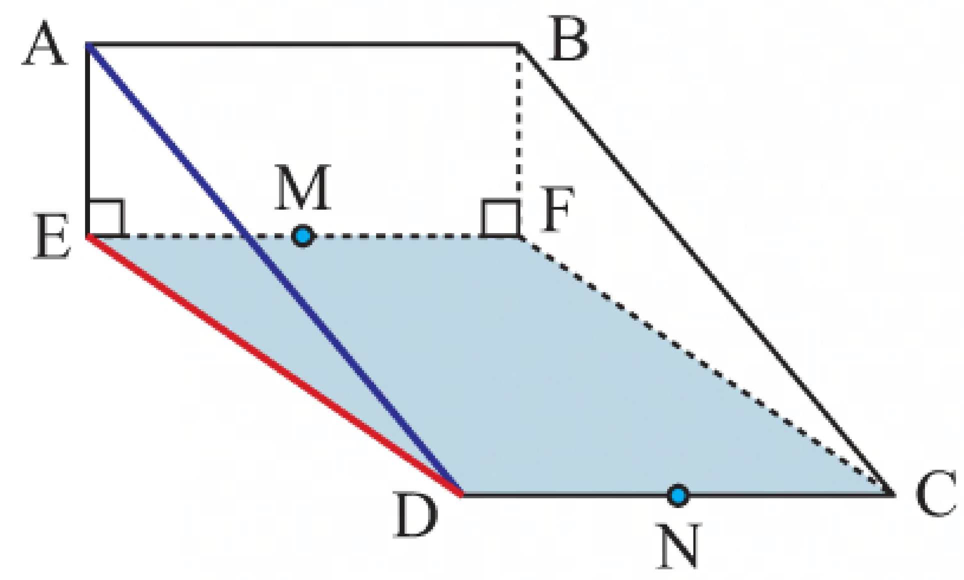
ii The projection of [BM] onto the base plane is [FM].

$\therefore$ the required angle is $B\widehat{M}F$.



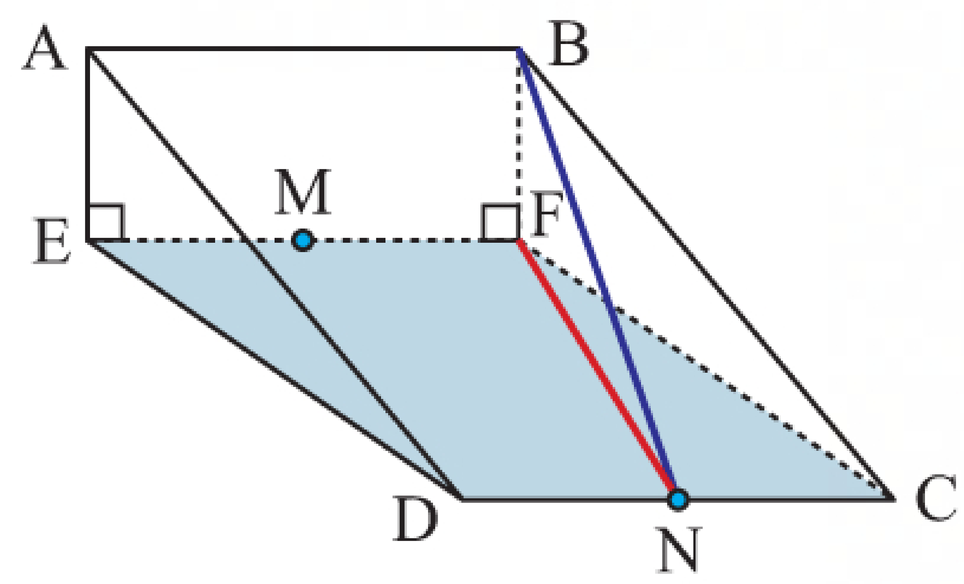
iii The projection of [AD] onto the base plane is [DE].

$\therefore$ the required angle is $A\widehat{D}E$.



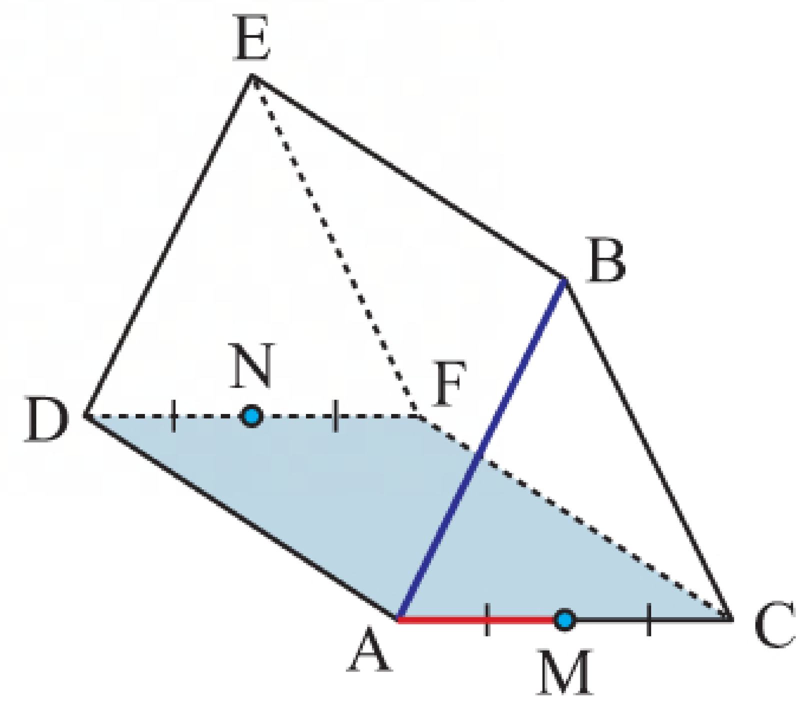
iv The projection of [BN] onto the base plane is [FN].

$\therefore$ the required angle is $B\widehat{N}F$.



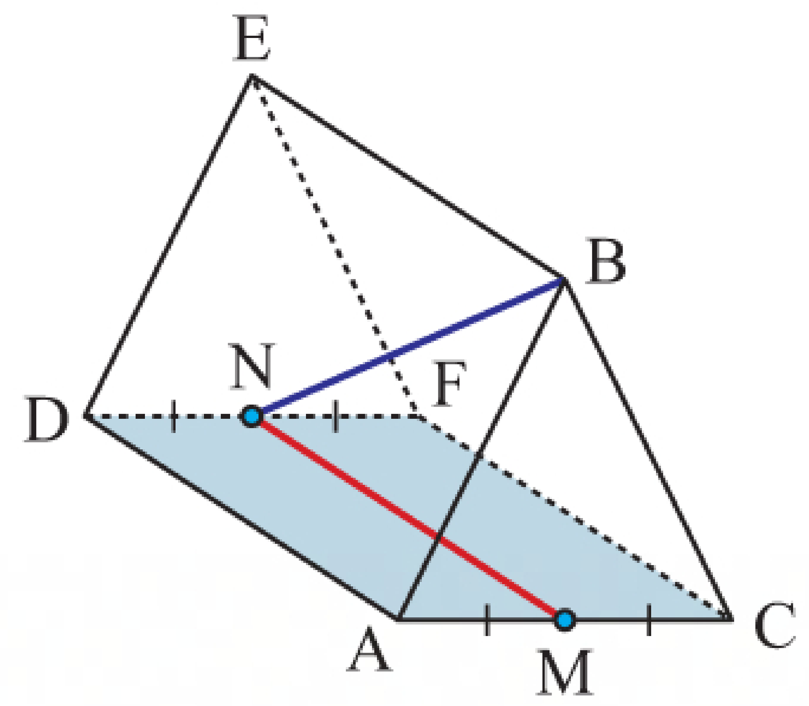
b **i** The projection of [AB] onto the base plane is [AM].

$\therefore$ the required angle is $\mathrm{B\widehat{A}M}$.



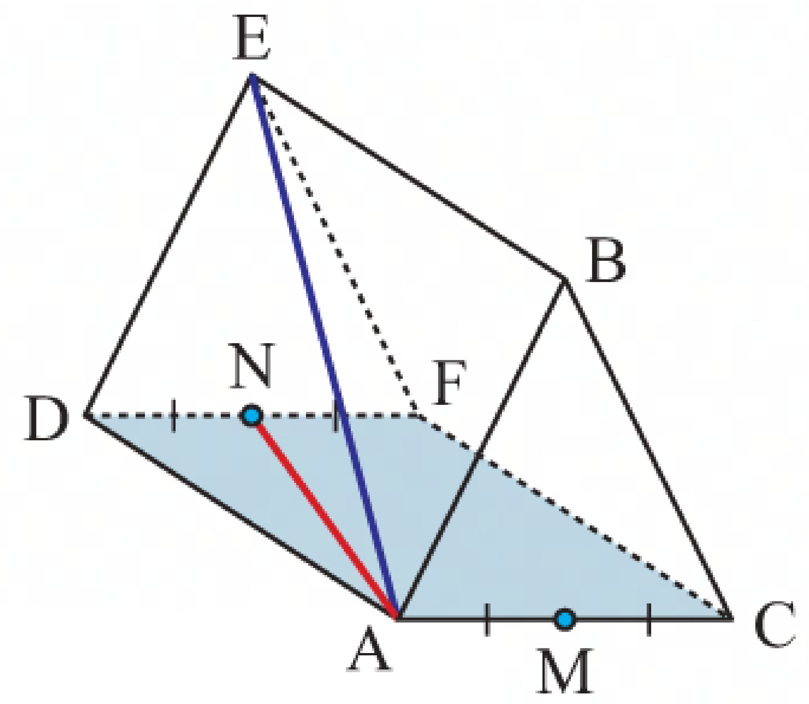
ii The projection of [BN] onto the base plane is [MN].

$\therefore$ the required angle is $\mathrm{B\widehat{N}M}$.



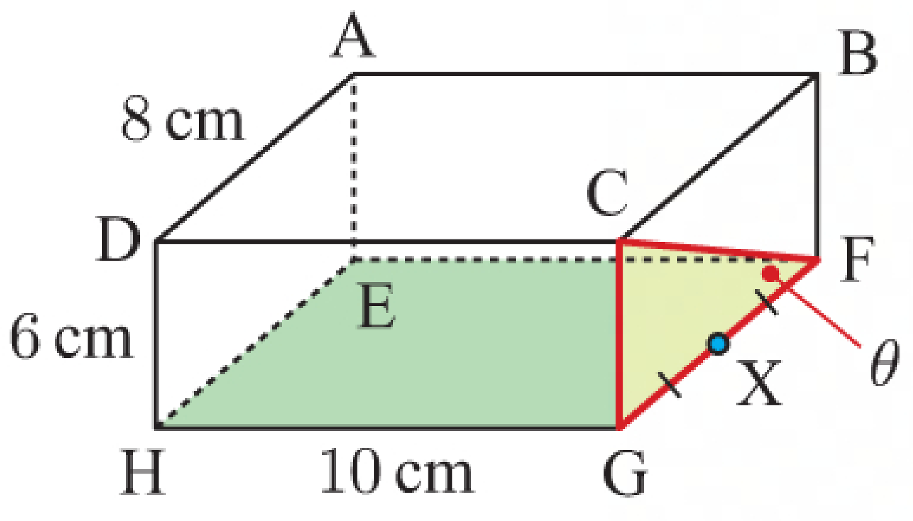
iii The projection of [AE] onto the base plane is [AN].

$\therefore$ the required angle is $\mathrm{E\widehat{A}N}$.



3 **a** **i** The projection of [CF] onto the base plane is [FG].

$\therefore$ the required angle is $\mathrm{C\widehat{F}G}$.



$\tan\theta = \frac{6}{8}$

$\therefore\ \theta = \tan^{-1}\left(\frac{6}{8}\right)$

$\therefore\ \theta \approx 36.9°$

The angle is about 36.9°.

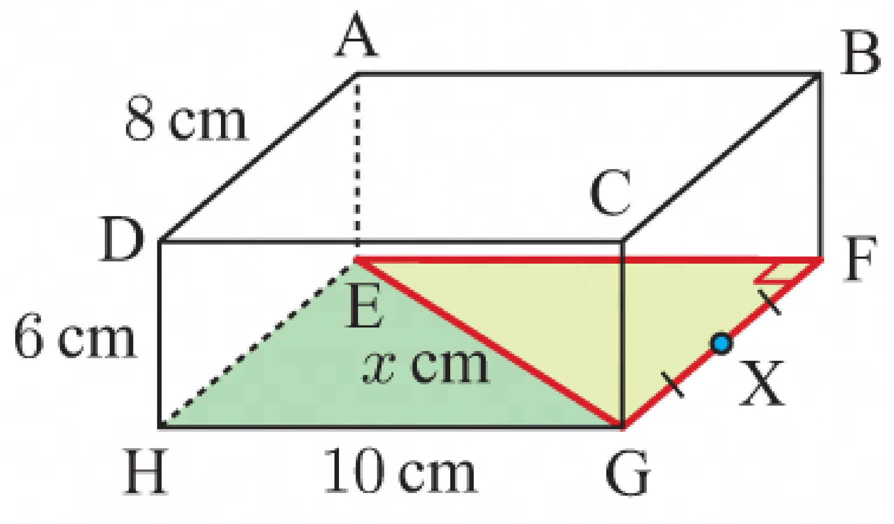
ii The projection of [AG] onto the base plane is [EG].

$\therefore$ the required angle is $\mathrm{A\widehat{G}E}$.

Let EG be x cm.

Using Pythagoras in $\triangle$EFG,

$$x^2 = 10^2 + 8^2$$
$$\therefore\ x^2 = 164$$
$$\therefore\ x = \sqrt{164} \qquad \{\text{as } x > 0\}$$



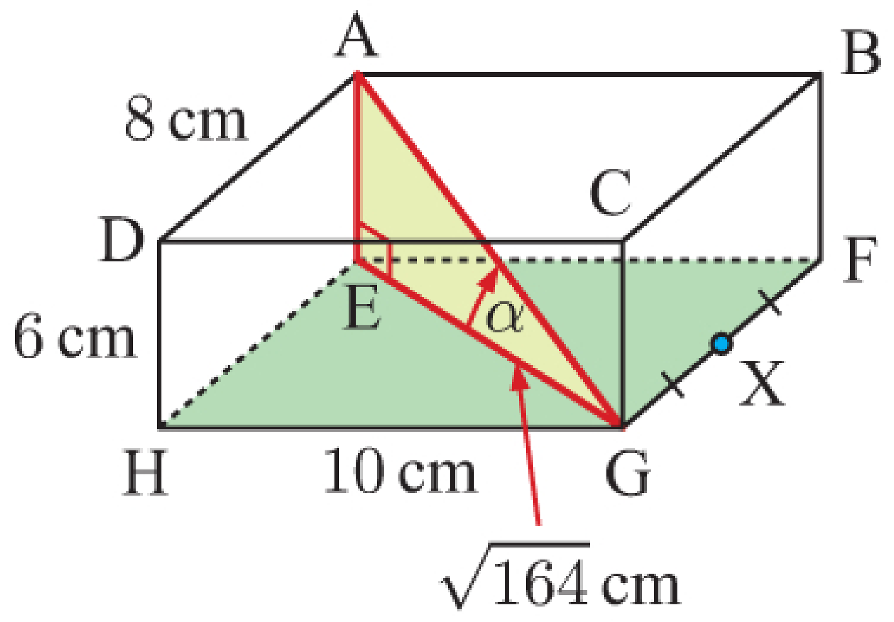
Let $\mathrm{A\widehat{G}E}$ be α.

$\therefore\ \tan\alpha = \frac{6}{\sqrt{164}}$

$\therefore\ \alpha = \tan^{-1}\left(\frac{6}{\sqrt{164}}\right)$

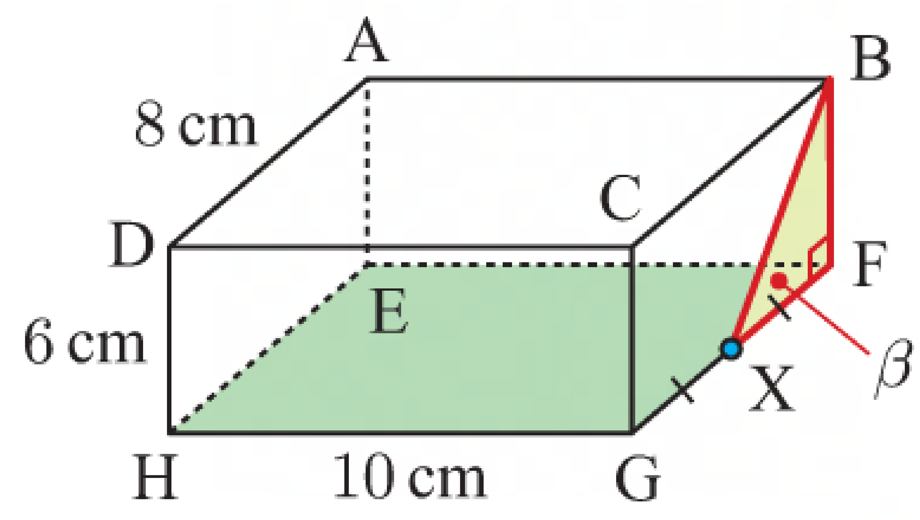
$\therefore\ \alpha \approx 25.1°$

The angle is about 25.1°.



iii The projection of [BX] onto the base plane is [FX].

The required angle is $B\widehat{X}F$.



$\tan\beta = \frac{6}{4}$

$\therefore\ \beta = \tan^{-1}\left(\frac{6}{4}\right)$

$\therefore\ \beta \approx 56.3°$

The angle is about 56.3°.

iv The projection of [DX] onto the base plane is [HX].

$\therefore$ the required angle is $D\widehat{X}H$.

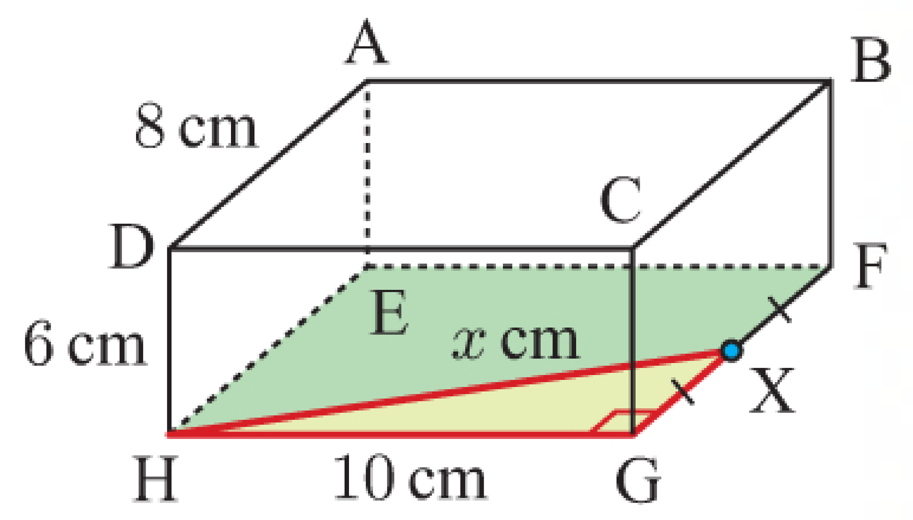
Let HX be x cm.

Using Pythagoras in $\triangle$HGX,

$$x^2 = 10^2 + 4^2$$

$$\therefore\ x^2 = 116$$

$$\therefore\ x = \sqrt{116} \qquad \{\text{as } x > 0\}$$



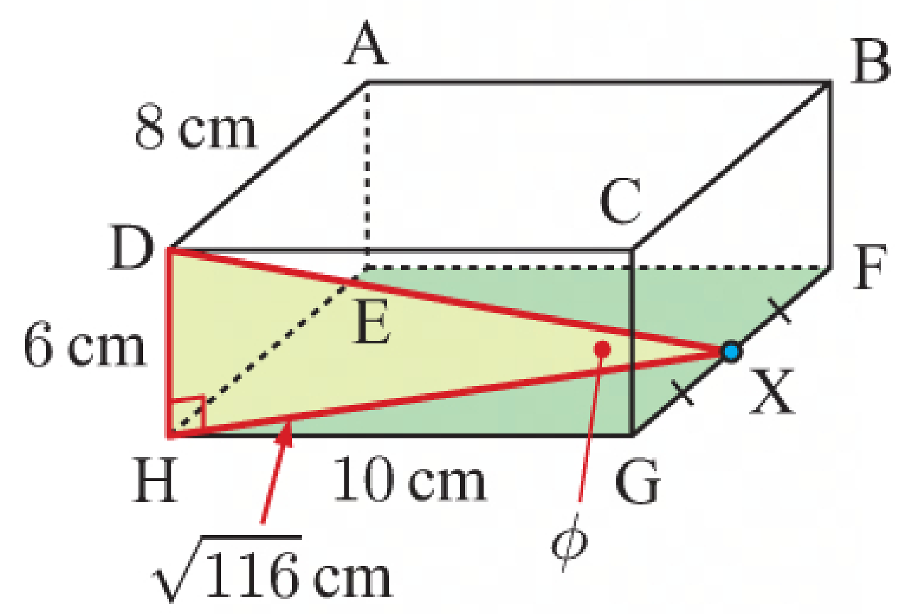
Let $D\widehat{X}H$ be ϕ.

$\therefore\ \tan\phi = \frac{6}{\sqrt{116}}$

$\therefore\ \phi = \tan^{-1}\left(\frac{6}{\sqrt{116}}\right)$

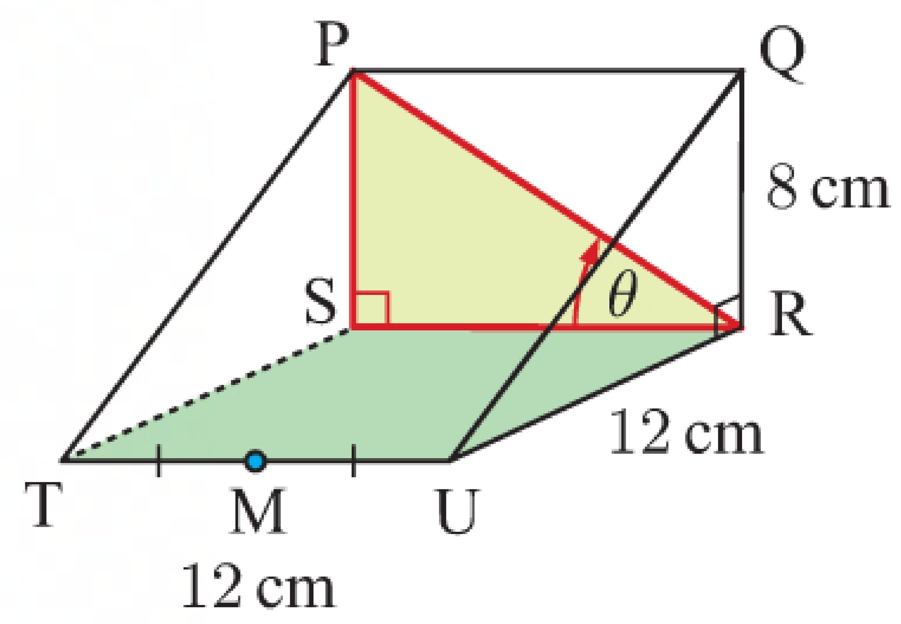
$\therefore\ \phi \approx 29.1°$

The angle is about 29.1°.



b **i** The projection of [PR] onto the base plane is [RS].

$\therefore$ the required angle is $P\widehat{R}S$.



$\tan\theta = \frac{8}{12}$

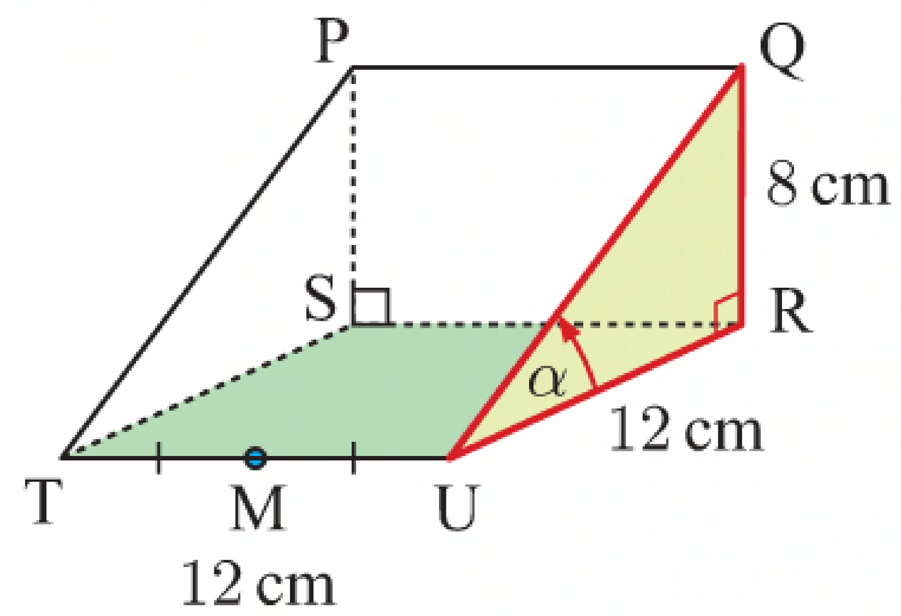
$\therefore\ \theta = \tan^{-1}\left(\frac{8}{12}\right)$

$\therefore\ \theta \approx 33.7°$

The angle is about 33.7°.

ii The projection of [QU] onto the base plane is [RU].

$\therefore$ the required angle is $Q\widehat{U}R$.



$\tan\alpha = \frac{8}{12}$

$\therefore\ \alpha = \tan^{-1}\left(\frac{8}{12}\right)$

$\therefore\ \alpha \approx 33.7°$

The angle is about 33.7°.

iii The projection of [PU] onto the base plane is [SU].

$\therefore$ the required angle is $P\widehat{U}S$.

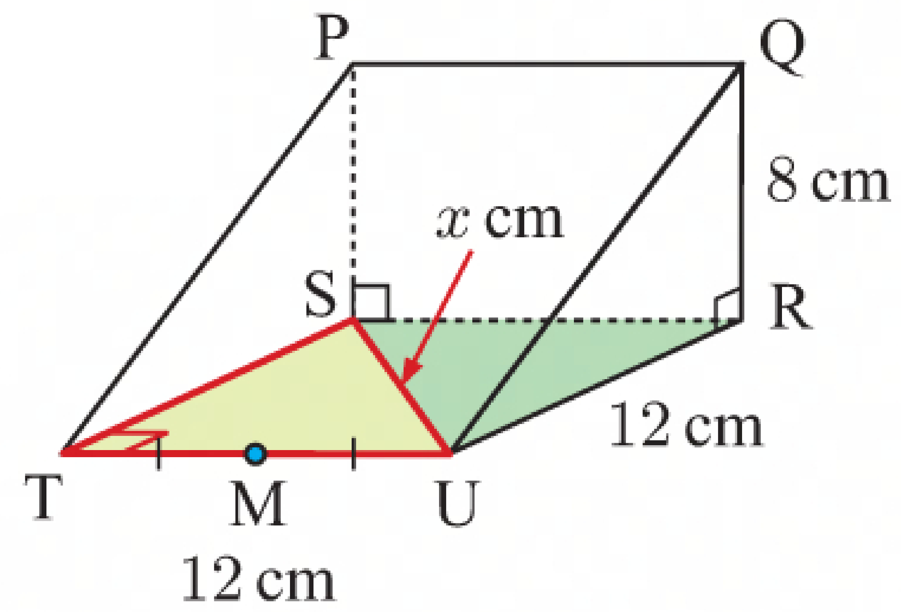
Let SU be x cm.

Using Pythagoras in $\triangle$STU,

$$x^2 = 12^2 + 12^2$$

$$\therefore\ x^2 = 288$$

$$\therefore\ x = \sqrt{288} \qquad \{\text{as } x > 0\}$$



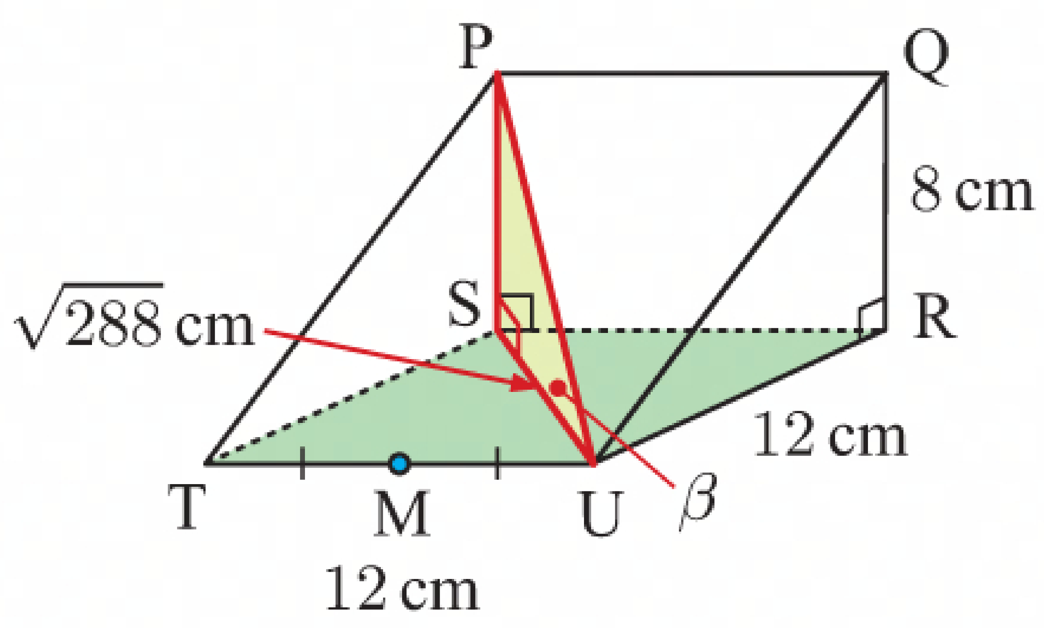
Let $\mathrm{P\widehat{U}S}$ be β.

$\therefore \quad \tan\beta = \dfrac{8}{\sqrt{288}}$

$\therefore \quad \beta = \tan^{-1}\left(\frac{8}{\sqrt{288}}\right)$

$\therefore \quad \beta \approx 25.2^\circ$

The angle is about 25.2°.



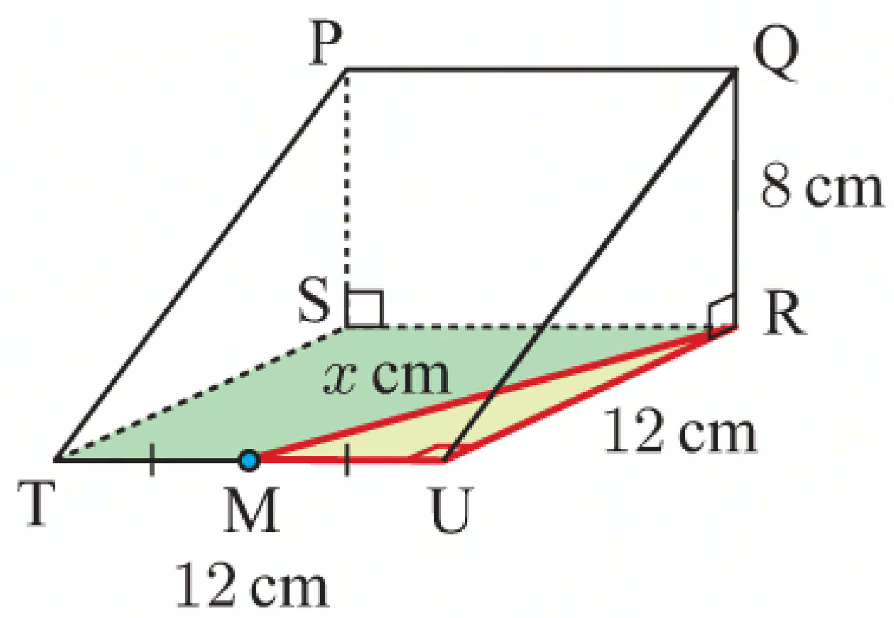
iv The projection of [QM] onto the base plane is [MR].

$\therefore$ the required angle is $\mathrm{Q\widehat{M}R}$.

Let MR be x cm.

Using Pythagoras in $\triangle$MUR,

$$x^2 = 6^2 + 12^2$$
$$\therefore \quad x^2 = 180$$
$$\therefore \quad x = \sqrt{180} \qquad \{\text{as } x > 0\}$$



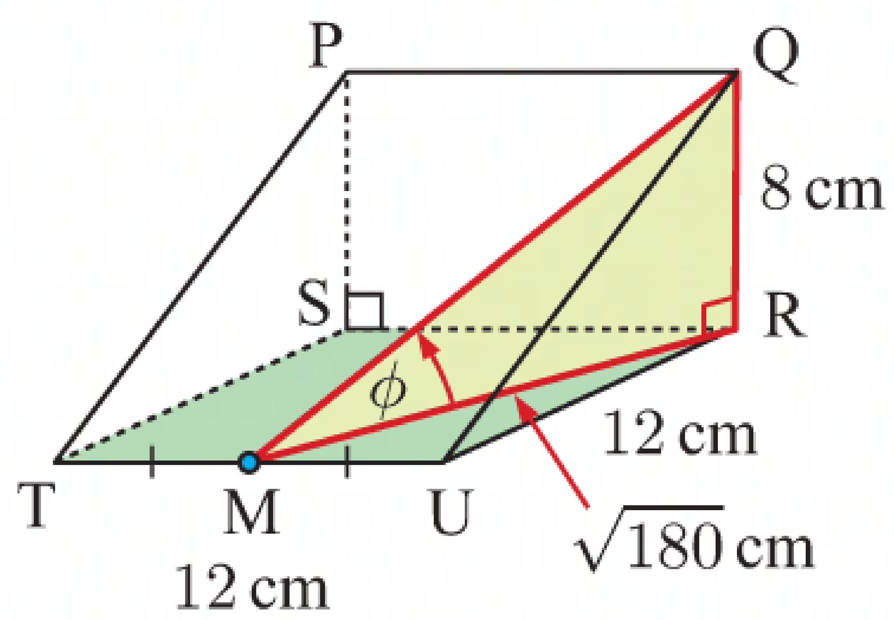
Let $\mathrm{Q\widehat{M}R}$ be ϕ.

$\tan\phi = \dfrac{8}{\sqrt{180}}$

$\therefore \quad \phi = \tan^{-1}\left(\frac{8}{\sqrt{180}}\right)$

$\therefore \quad \phi \approx 30.8^\circ$

The angle is about 30.8°.



c **i** The projection of [QR] onto the base plane is [MR].

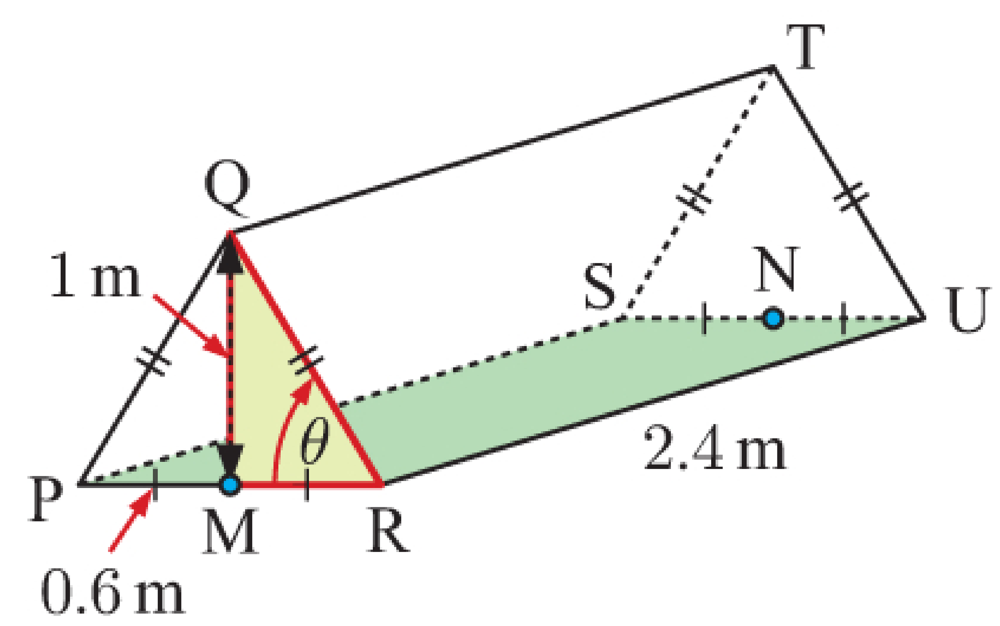
$\therefore$ the required angle is $\mathrm{M\widehat{R}Q}$.

$\tan\theta = \dfrac{1}{0.6}$

$\therefore \quad \theta = \tan^{-1}\left(\frac{1}{0.6}\right)$

$\therefore \quad \theta \approx 59.0^\circ$

The angle is about 59.0°.



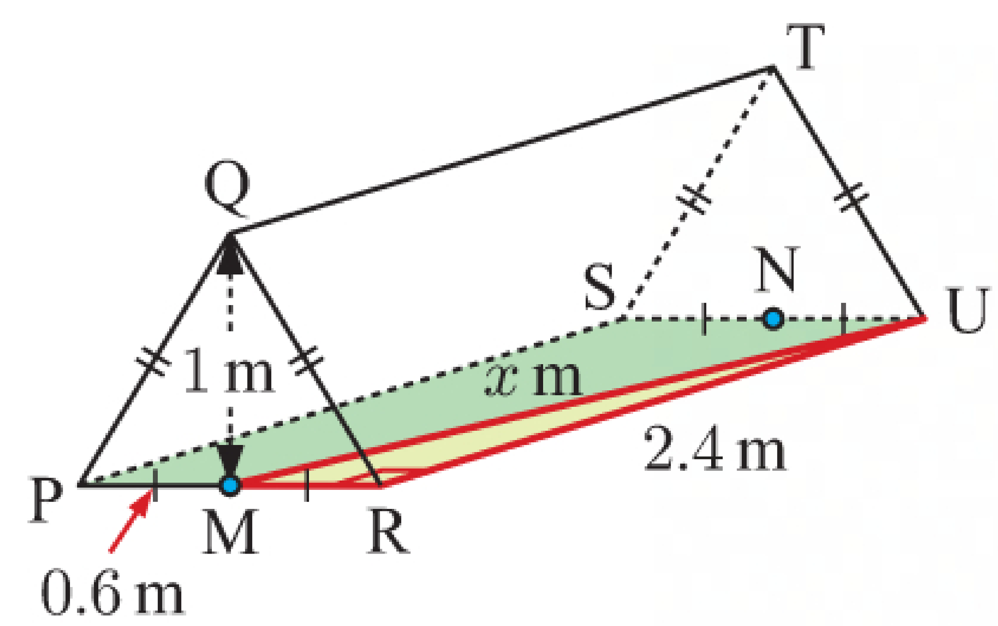
ii The projection of [QU] onto the base plane is [MU].

$\therefore$ the required angle is $\mathrm{Q\widehat{U}M}$.

Let MU be x m.

Using Pythagoras in $\triangle$MRU,

$$x^2 = 0.6^2 + 2.4^2$$
$$\therefore \quad x^2 = 6.12$$
$$\therefore \quad x = \sqrt{6.12} \qquad \{\text{as } x > 0\}$$



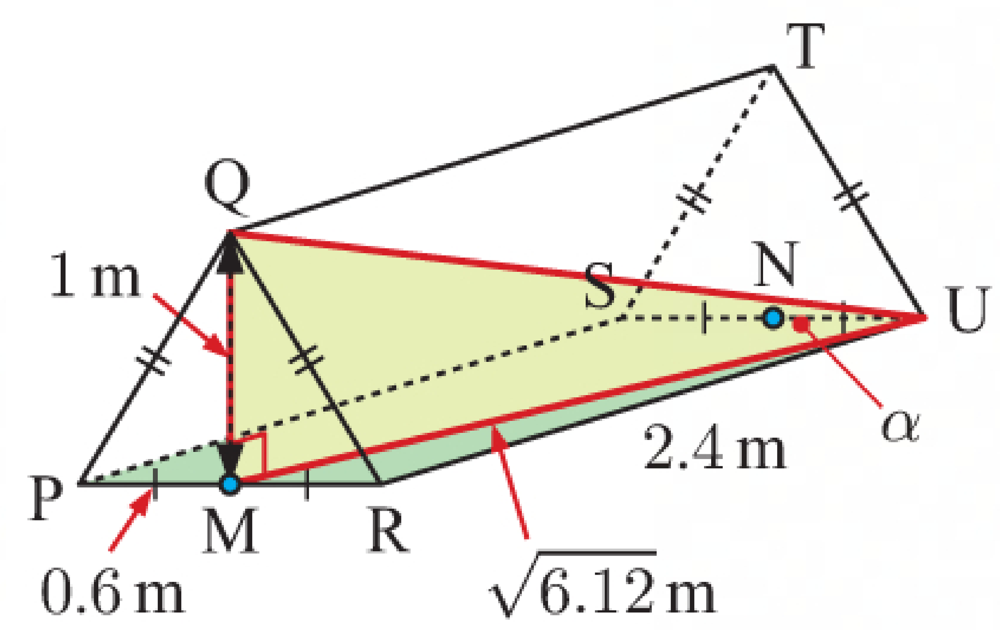
Let $\mathrm{Q\widehat{U}M}$ be α.

$\therefore \quad \tan\alpha = \dfrac{1}{\sqrt{6.12}}$

$\therefore \quad \alpha = \tan^{-1}\left(\frac{1}{\sqrt{6.12}}\right)$

$\therefore \quad \alpha \approx 22.0^\circ$

The angle is about 22.0°.



iii The projection of [QN] onto the base plane is [MN].

$\therefore$ the required angle is $\mathrm{Q\widehat{N}M}$.

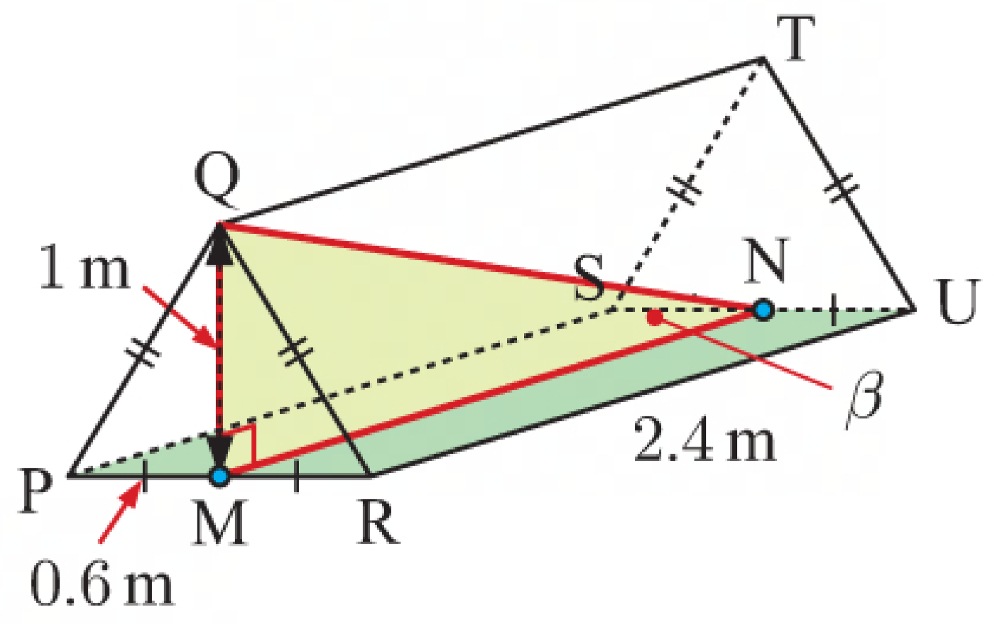
$\mathrm{MN} \parallel \mathrm{RU}$, so $\mathrm{MN} = \mathrm{RU} = 2.4$ m

$$\tan\beta = \frac{1}{2.4}$$

$$\therefore \quad \beta = \tan^{-1}\left(\tfrac{1}{2.4}\right)$$

$$\therefore \quad \beta \approx 22.6°$$

The angle is about 22.6°.



d **i** The projection of [AX] onto the base plane is [AM].

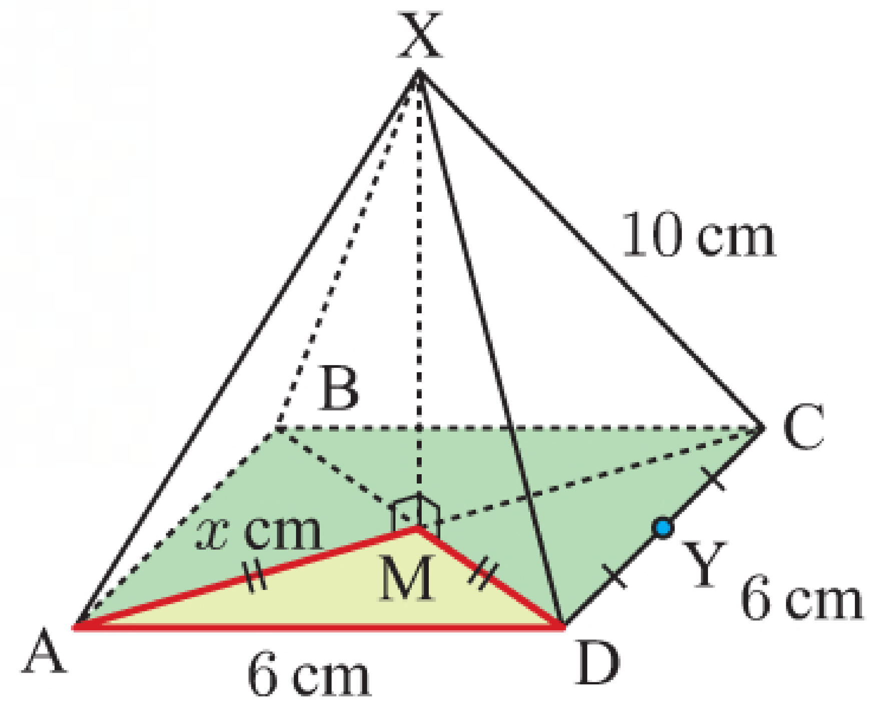
$\therefore$ the required angle is $\mathrm{X\widehat{A}M}$.

Let AM = DM be x cm.

(The base of the figure is a square, so its diagonals [AC] and [BD] perpendicularly bisect each other.)

Using Pythagoras in $\triangle$AMD,

$$x^2 + x^2 = 6^2$$
$$\therefore \quad 2x^2 = 36$$
$$\therefore \quad x^2 = 18$$
$$\therefore \quad x = \sqrt{18} \qquad \{\text{as } x > 0\}$$



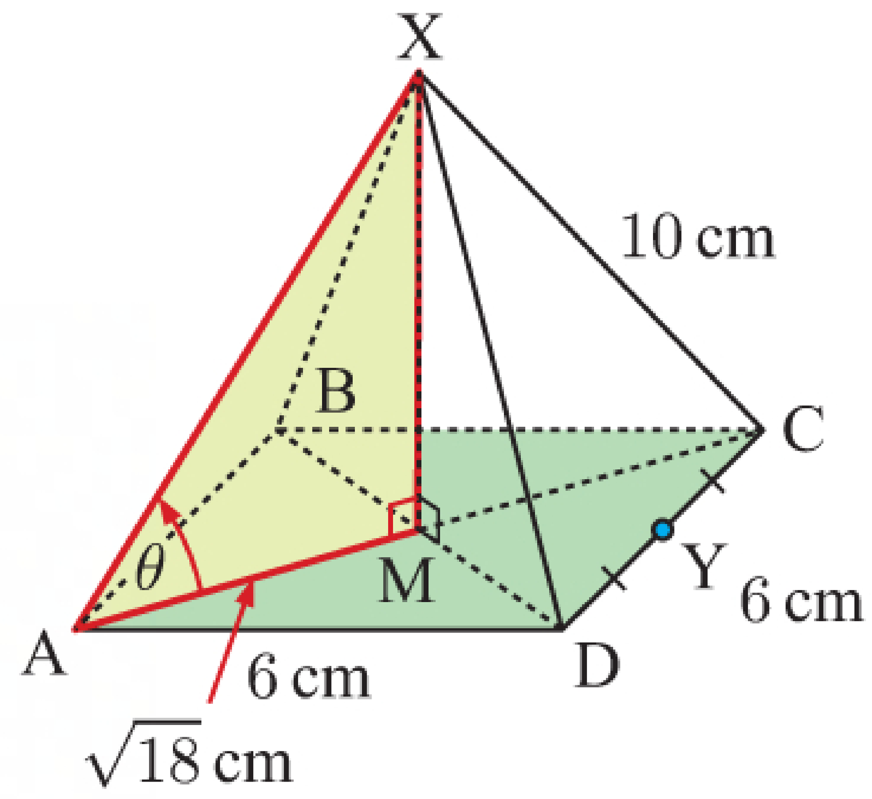
Let $\mathrm{X\widehat{A}M}$ be θ.

$$\therefore \quad \cos\theta = \frac{\sqrt{18}}{10}$$

$$\therefore \quad \theta = \cos^{-1}\left(\frac{\sqrt{18}}{10}\right)$$

$$\therefore \quad \theta \approx 64.9°$$

The angle is about 64.9°.



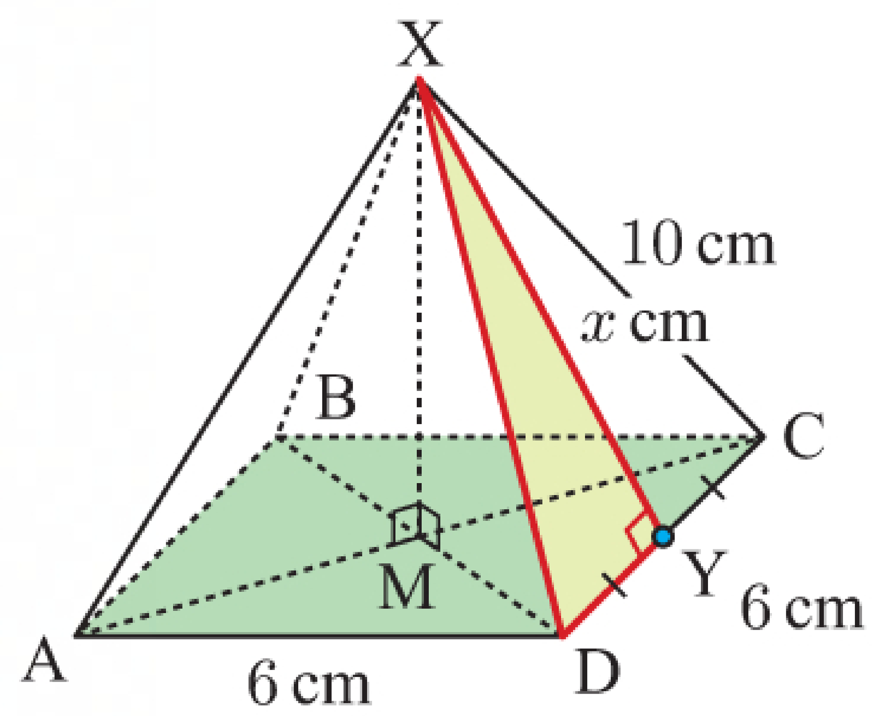
ii The projection of [XY] onto the base plane is [MY].

$\therefore$ the required angle is $\mathrm{X\widehat{Y}M}$.

Let XY be x cm.

Using Pythagoras in $\triangle$XYD,

$$x^2 + 3^2 = 10^2$$
$$\therefore \quad x^2 = 100 - 9$$
$$\therefore \quad x^2 = 91$$
$$\therefore \quad x = \sqrt{91} \qquad \{\text{as } x > 0\}$$



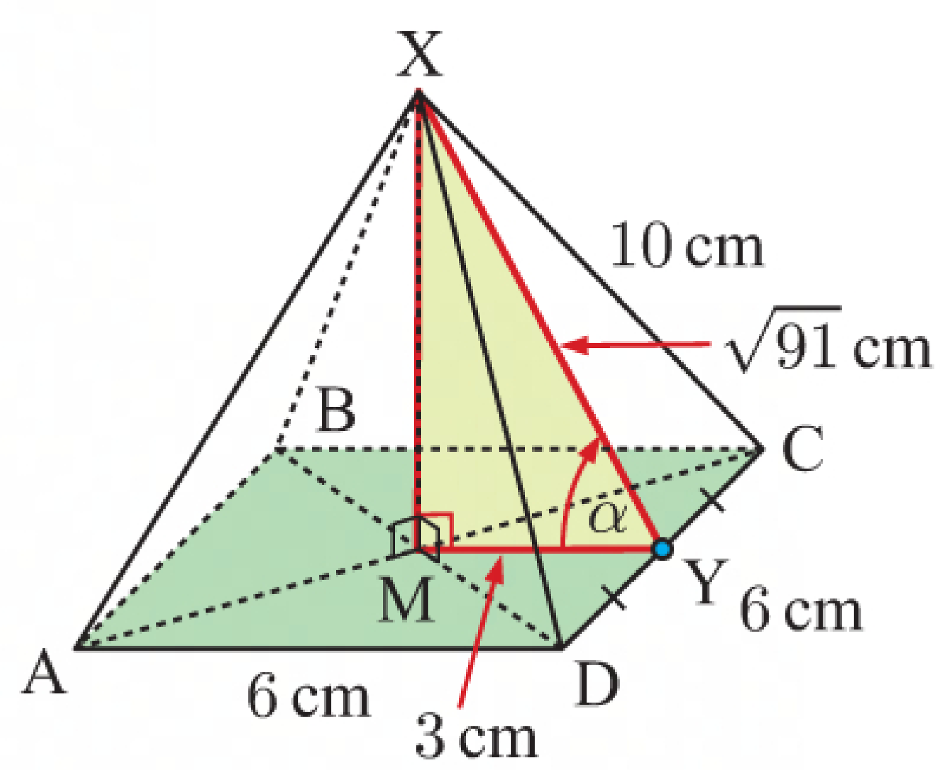
Let $\mathrm{X\widehat{Y}M}$ be α.

$$\therefore \quad \cos\alpha = \frac{3}{\sqrt{91}}$$

$$\therefore \quad \alpha = \cos^{-1}\left(\frac{3}{\sqrt{91}}\right)$$

$$\therefore \quad \alpha \approx 71.7°$$

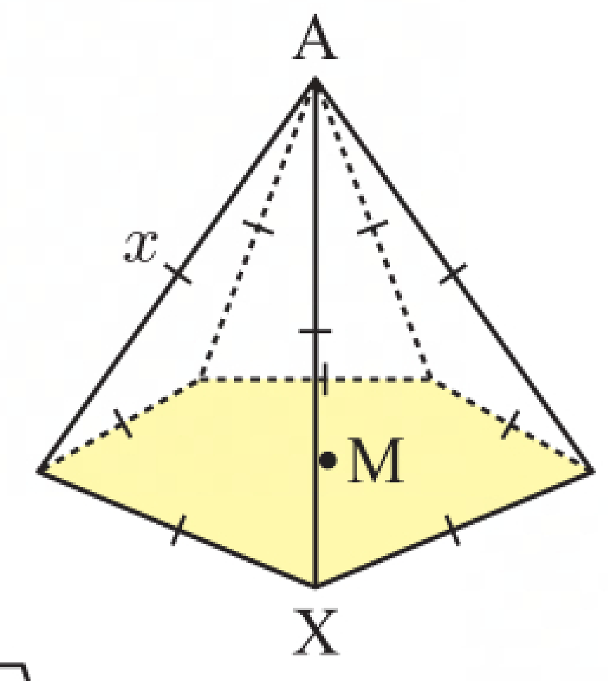
The angle is about 71.7°.



4 Let the equal sides of the pyramid be x.

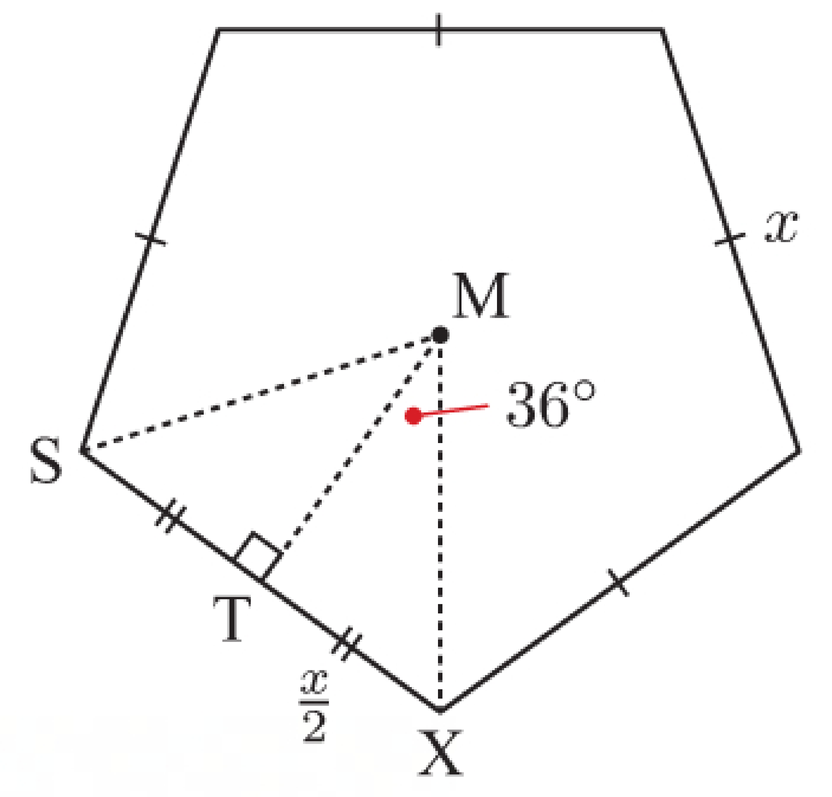
The projection of [AX] onto the base plane is [AM], where M is the point directly below A on the base plane.

$\therefore$ the required angle is $\mathrm{A\widehat{X}M}$.



Consider the pentagonal base of the pyramid.

The angle at the centre of the pentagon is $360°$. {angles at a point}



$\therefore\ \mathrm{S\widehat{M}X} = \dfrac{360°}{5} = 72°$

$\therefore\ \mathrm{T\widehat{M}X} = \dfrac{72°}{2} = 36°$

In $\triangle$TMX, $\quad \sin 36° = \dfrac{\frac{x}{2}}{\mathrm{MX}}$

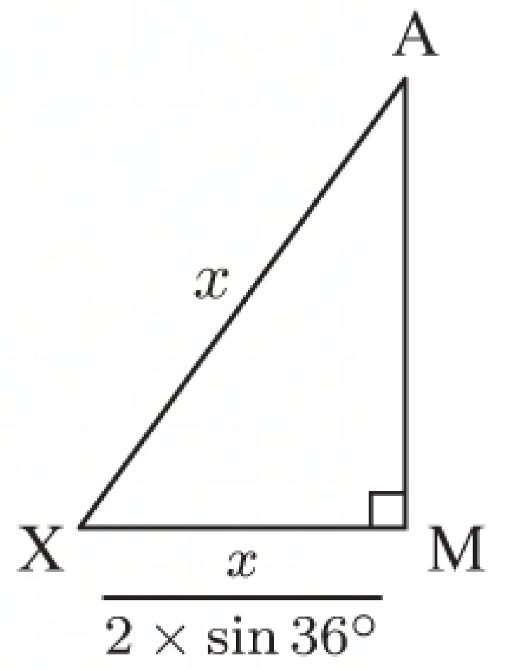
$\therefore\ \mathrm{MX} = \dfrac{x}{2 \times \sin 36°}$

In $\triangle$AXM, $\quad \cos \mathrm{A\widehat{X}M} = \dfrac{\frac{x}{2\times\sin 36°}}{x}$

$= \dfrac{1}{2 \times \sin 36°}$

$\therefore\ \mathrm{A\widehat{X}M} = \cos^{-1}\left(\dfrac{1}{2 \times \sin 36°}\right)$

$\approx 31.7°$



So, the angle between [AX] and the base plane is approximately $31.7°$.

REVIEW SET 7A

1 **a** $\mathrm{AC}^2 = \mathrm{AB}^2 + \mathrm{BC}^2$ {Pythagoras}

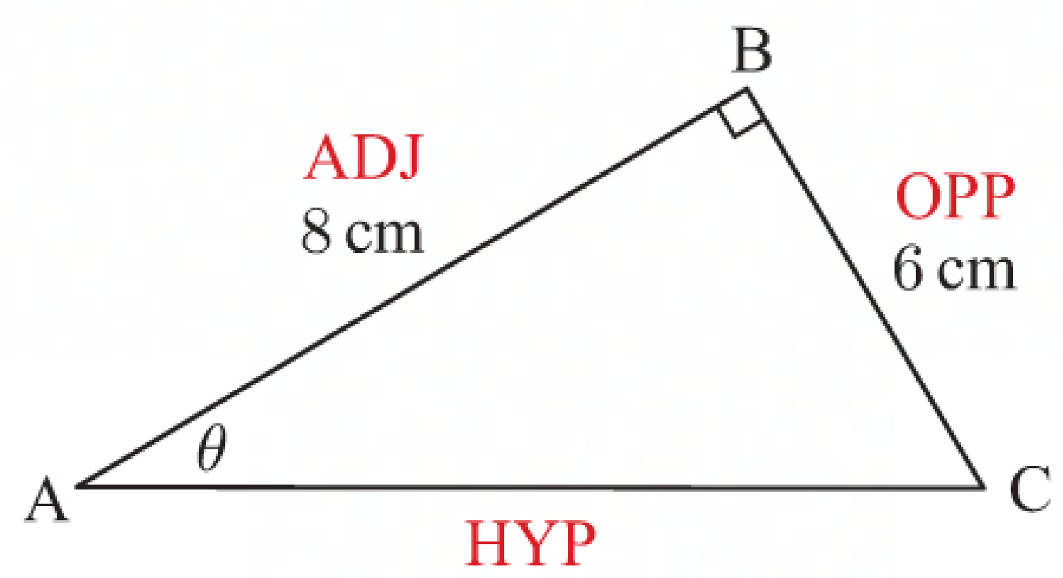
$= 8^2 + 6^2$

$= 100$

$\therefore\ \mathrm{AC} = \sqrt{100}$ {as AC > 0}

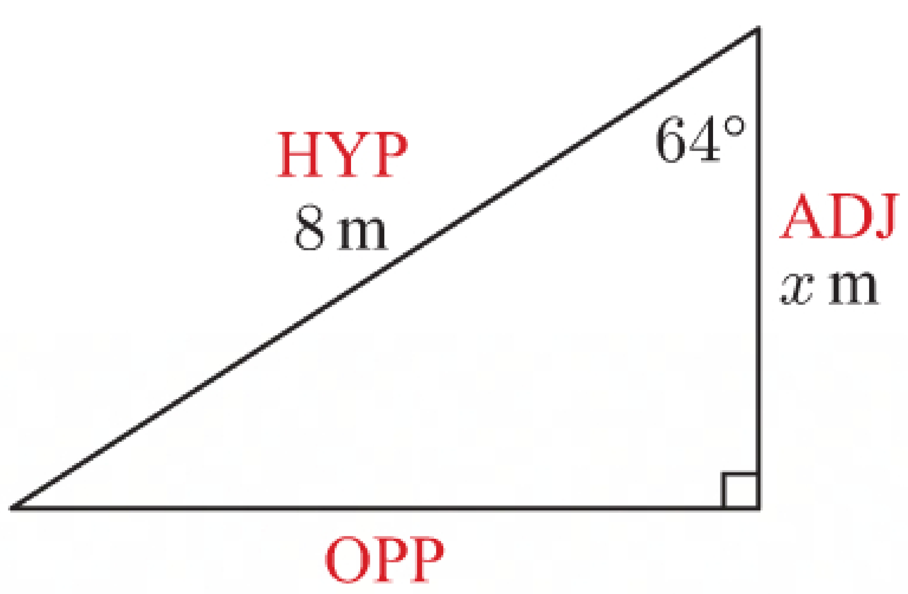
$\therefore\ \mathrm{AC} = 10$

The hypotenuse is 10 cm long.



b $\sin\theta = \dfrac{\mathrm{OPP}}{\mathrm{HYP}} = \dfrac{6}{10} = \frac{3}{5}$ **c** $\cos\theta = \dfrac{\mathrm{ADJ}}{\mathrm{HYP}} = \dfrac{8}{10} = \frac{4}{5}$ **d** $\tan\theta = \dfrac{\mathrm{OPP}}{\mathrm{ADJ}} = \dfrac{6}{8} = \frac{3}{4}$

2 **a**

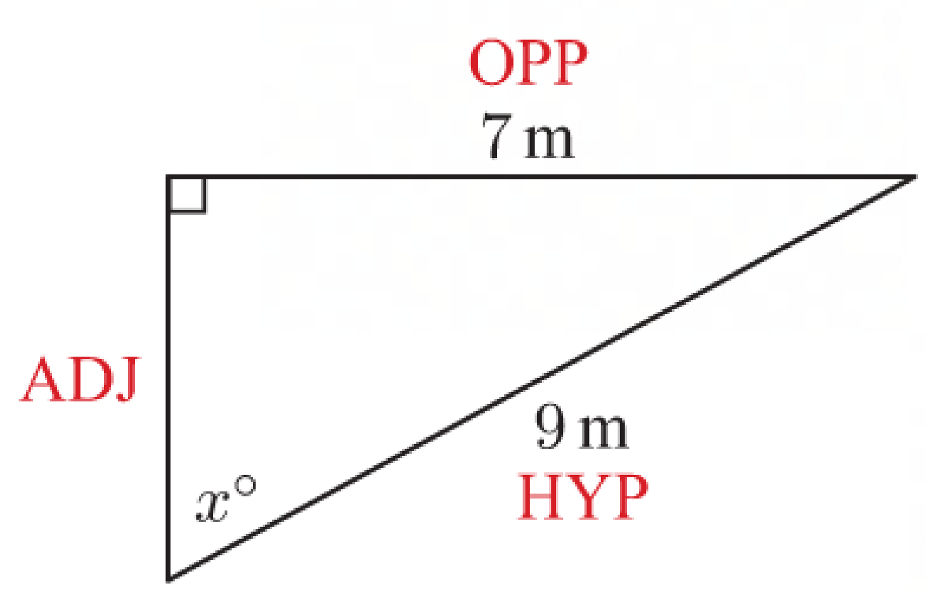


$\cos 64° = \dfrac{x}{8} \quad \left\{\cos\theta = \dfrac{\mathrm{ADJ}}{\mathrm{HYP}}\right\}$

$\therefore\ 8 \times \cos 64° = x$

$\therefore\ x \approx 3.51$

b

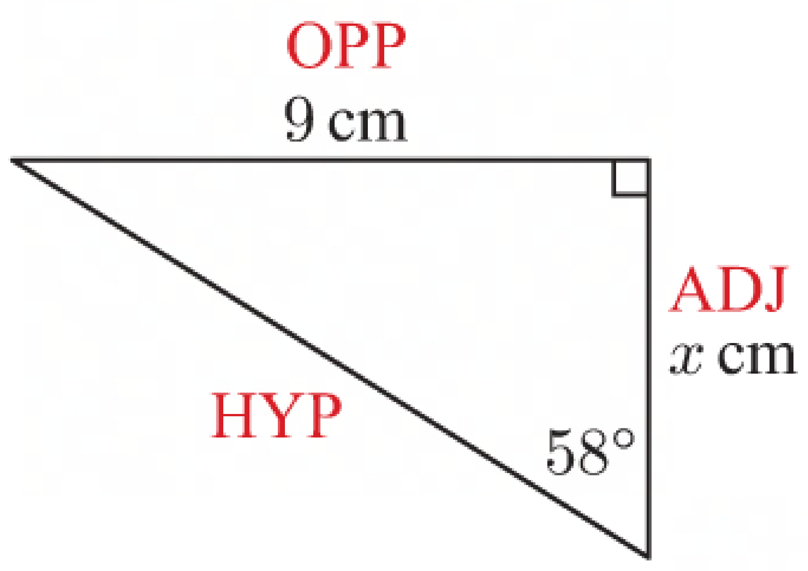


$\sin x° = \dfrac{7}{9} \quad \left\{\sin\theta = \dfrac{\mathrm{OPP}}{\mathrm{HYP}}\right\}$

$\therefore\ x° = \sin^{-1}\left(\frac{7}{9}\right)$

$\therefore\ x \approx 51.1$

c



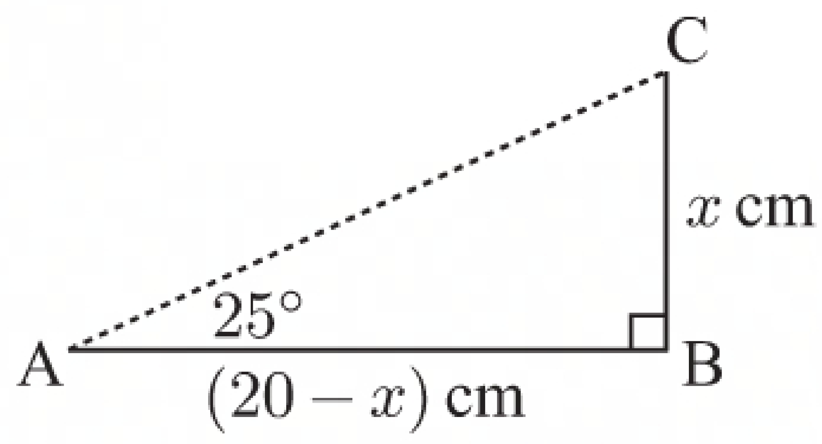
$$\tan 58° = \frac{9}{x} \qquad \{\tan\theta = \frac{\text{OPP}}{\text{ADJ}}\}$$

$$\therefore\ x = \frac{9}{\tan 58°}$$

$$\therefore\ x \approx 5.62$$

3 Let BC be x cm.

$\therefore$ AB $= (20 - x)$ cm {since AB + BC = 20 cm}



$$\tan 25° = \frac{x}{20 - x} \qquad \{\tan\theta = \frac{\text{OPP}}{\text{ADJ}}\}$$

$$\therefore\ \tan 25°(20 - x) = x$$

$$\therefore\ 20 \times \tan 25° - x \times \tan 25° = x$$

$$\therefore\ x(1 + \tan 25°) = 20 \times \tan 25°$$

$$\therefore\ x = \frac{20 \times \tan 25°}{1 + \tan 25°}$$

$$\approx 6.36$$

Area of $\triangle$ABC $= \frac{1}{2} \times$ base $\times$ height

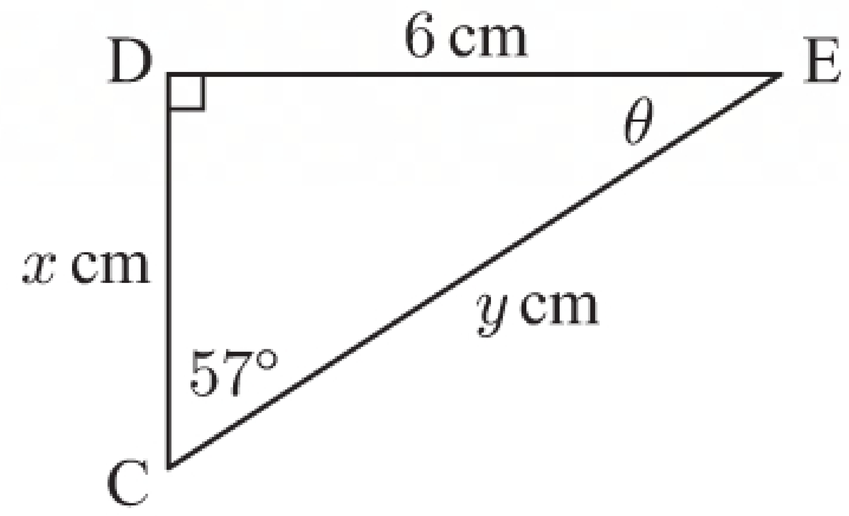
$= \frac{1}{2} \times (20 - x) \times x$

$\approx \frac{1}{2} \times (20 - 6.36) \times 6.36$

≈ 43.4 cm^2

4 $\theta = 180° - 90° - 57°$ {angles in a triangle}

$\therefore\ \theta = 33°$



$$\tan 57° = \frac{6}{x} \qquad \{\tan\theta = \frac{\text{OPP}}{\text{ADJ}}\}$$

$$\therefore\ x = \frac{6}{\tan 57°}$$

$$\therefore\ x \approx 3.90$$

Now $y^2 = x^2 + 6^2$ {Pythagoras}

$\therefore\ y \approx \sqrt{3.90^2 + 6^2}$ {as $y > 0$}

$\therefore\ y \approx 7.15$

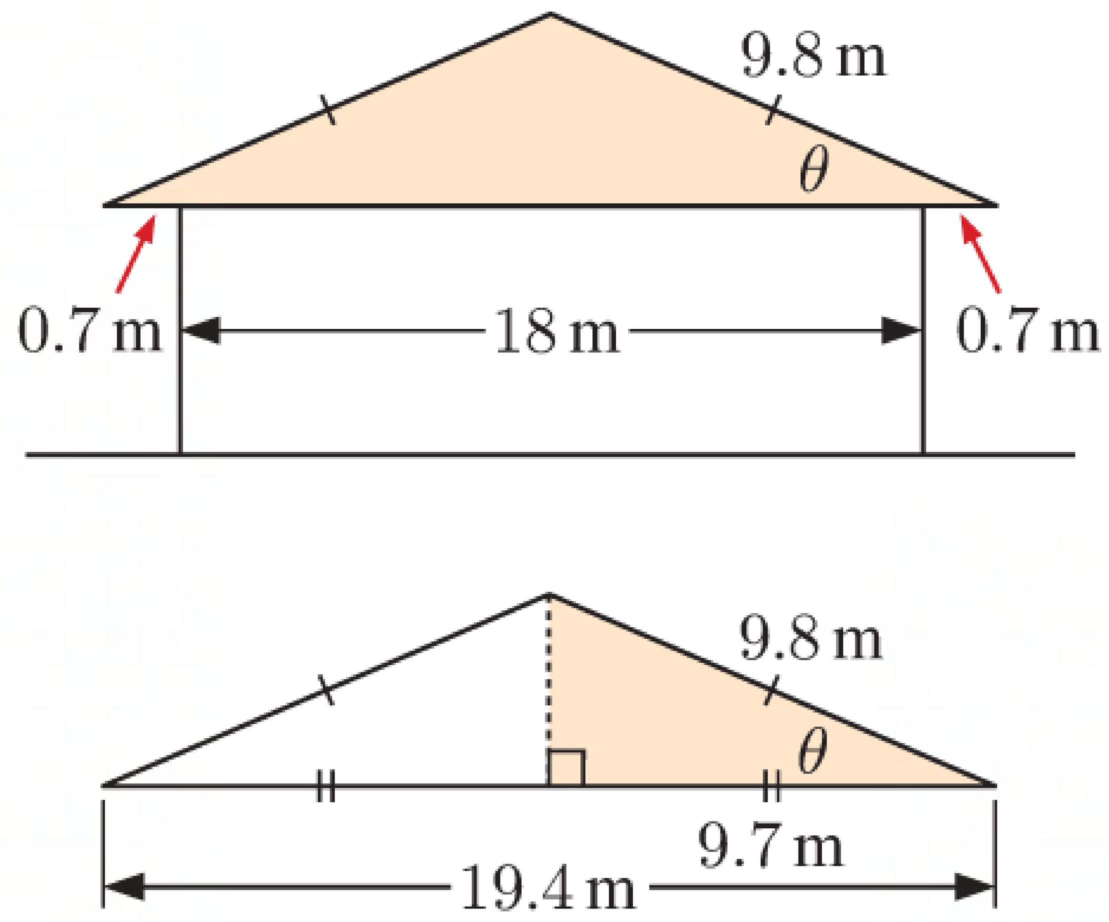
5 By constructing the altitude of the isosceles triangle, we form two right angled triangles.

For angle θ, $\cos\theta = \frac{9.7}{9.8}$

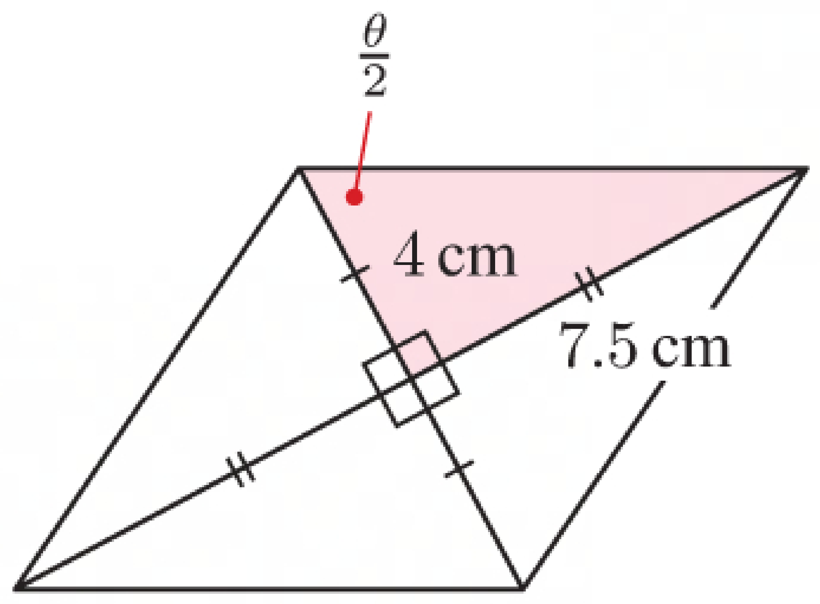
$\therefore\ \theta = \cos^{-1}\left(\frac{9.7}{9.8}\right)$

$\therefore\ \theta \approx 8.19°$

So, the pitch of the roof is approximately 8.19°.



6



The diagonals of a rhombus bisect each other at right angles.

So, $\tan\frac{\theta}{2} = \frac{7.5}{4}$ $\quad \{\tan\theta = \frac{\text{OPP}}{\text{ADJ}}\}$

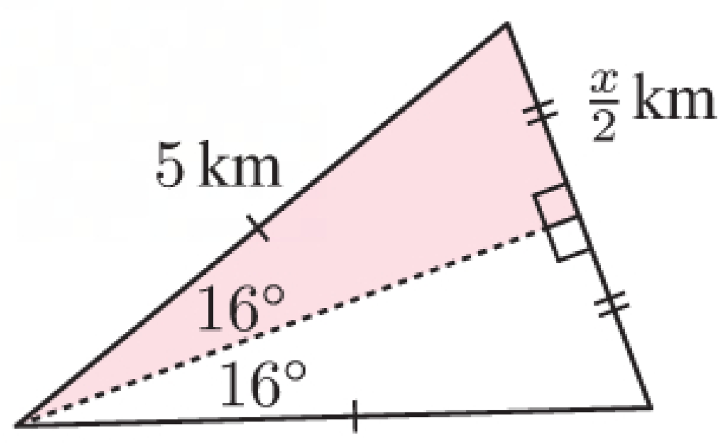
$\therefore\ \frac{\theta}{2} = \tan^{-1}\left(\frac{7.5}{4}\right)$

$\therefore\ \theta = 2 \times \tan^{-1}\left(\frac{7.5}{4}\right)$

$\approx 124°$

So the larger angle of the rhombus is about 124°.

7 **a**



We construct the altitude to form two right angled triangles.

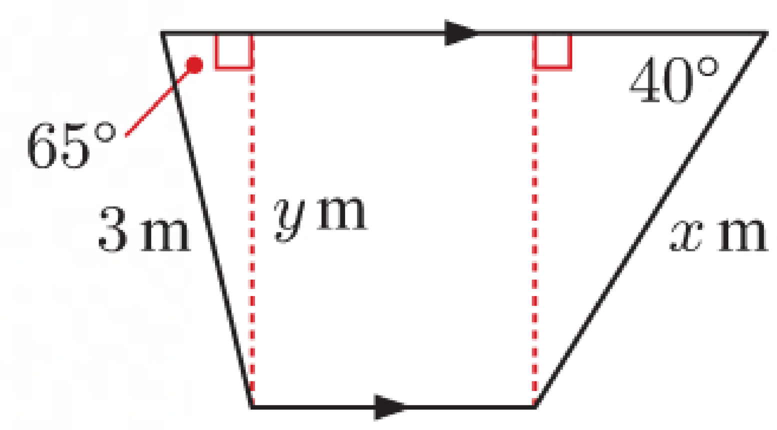
$\sin 16° = \frac{\left(\frac{x}{2}\right)}{5}$

$\therefore\ \frac{x}{2} = 5 \times \sin 16°$

$\therefore\ x = 2 \times 5 \times \sin 16°$

$\therefore\ x \approx 2.8$

b



We construct perpendiculars to form two right angled triangles and a rectangle.

$\sin 65° = \frac{y}{3}$

$\therefore\ y = 3 \times \sin 65°$

$\sin 40° = \frac{y}{x}$

$\therefore\ x \times \sin 40° = 3 \times \sin 65°$

$\therefore\ x = \frac{3\sin 65°}{\sin 40°}$

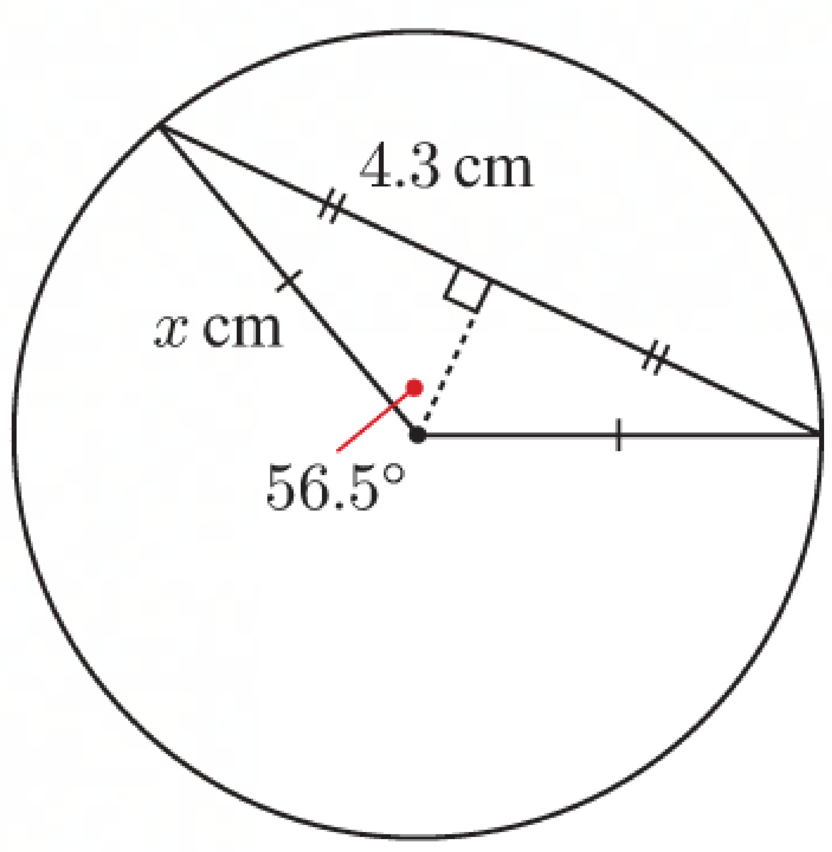
$\therefore\ x \approx 4.2$

c We construct the altitude to form two right angled triangles.

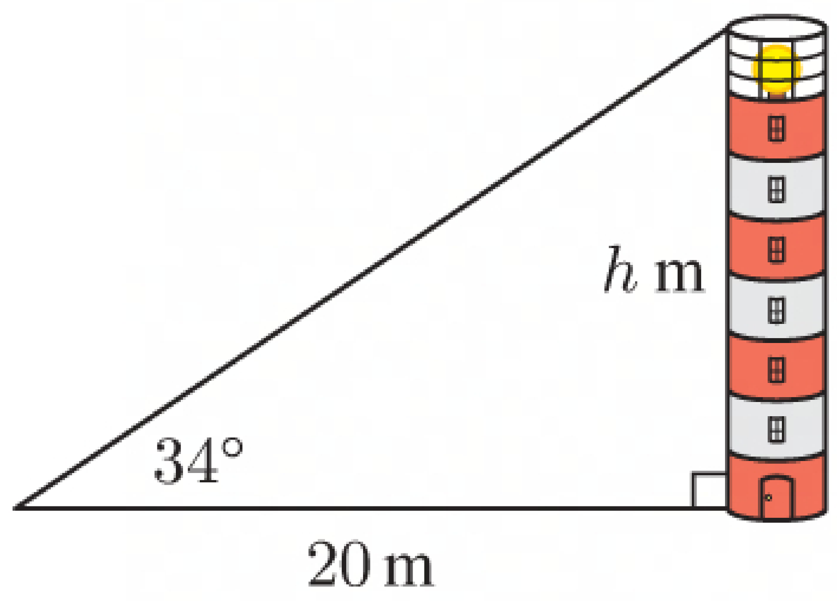
$\sin 56.5° = \frac{4.3}{x}$

$\therefore\ x = \frac{4.3}{\sin 56.5°}$

$\therefore\ x \approx 5.2$



8



Let the height of the lighthouse be h m.

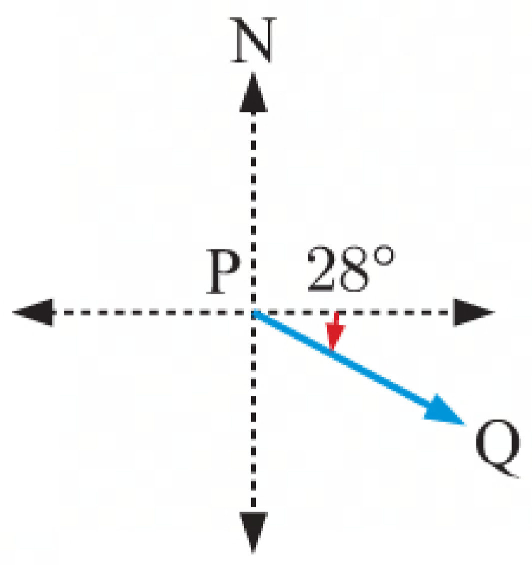
$$\tan 34° = \frac{h}{20}$$

$$\therefore\ 20 \times \tan 34° = h$$

$$\therefore\ h \approx 13.5$$

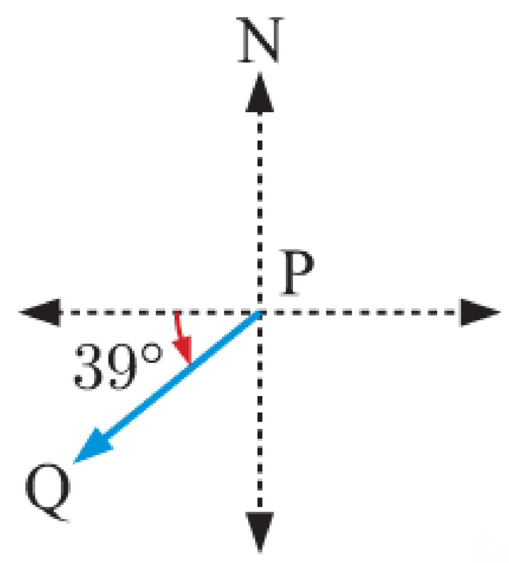
$\therefore$ the height of the lighthouse is about 13.5 m.

9 **a**



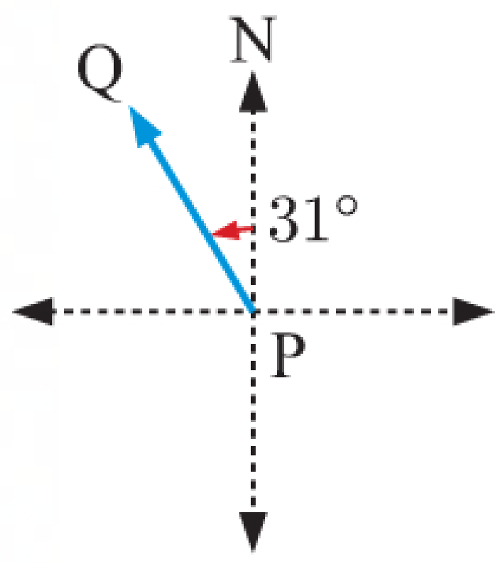
The bearing of Q from P
$= 90° + 28°$
$= 118°$

b



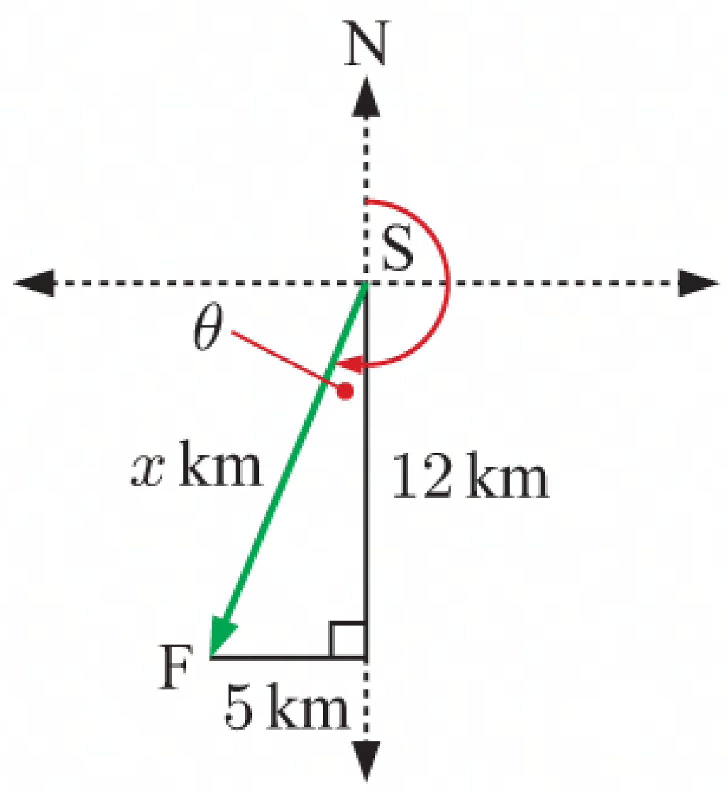
The bearing of Q from P
$= 270° - 39°$
$= 231°$

c



The bearing of Q from P
$= 360° - 31°$
$= 329°$

10 Suppose the helicopter starts at S and finishes at F.



$\tan\theta = \frac{5}{12}$

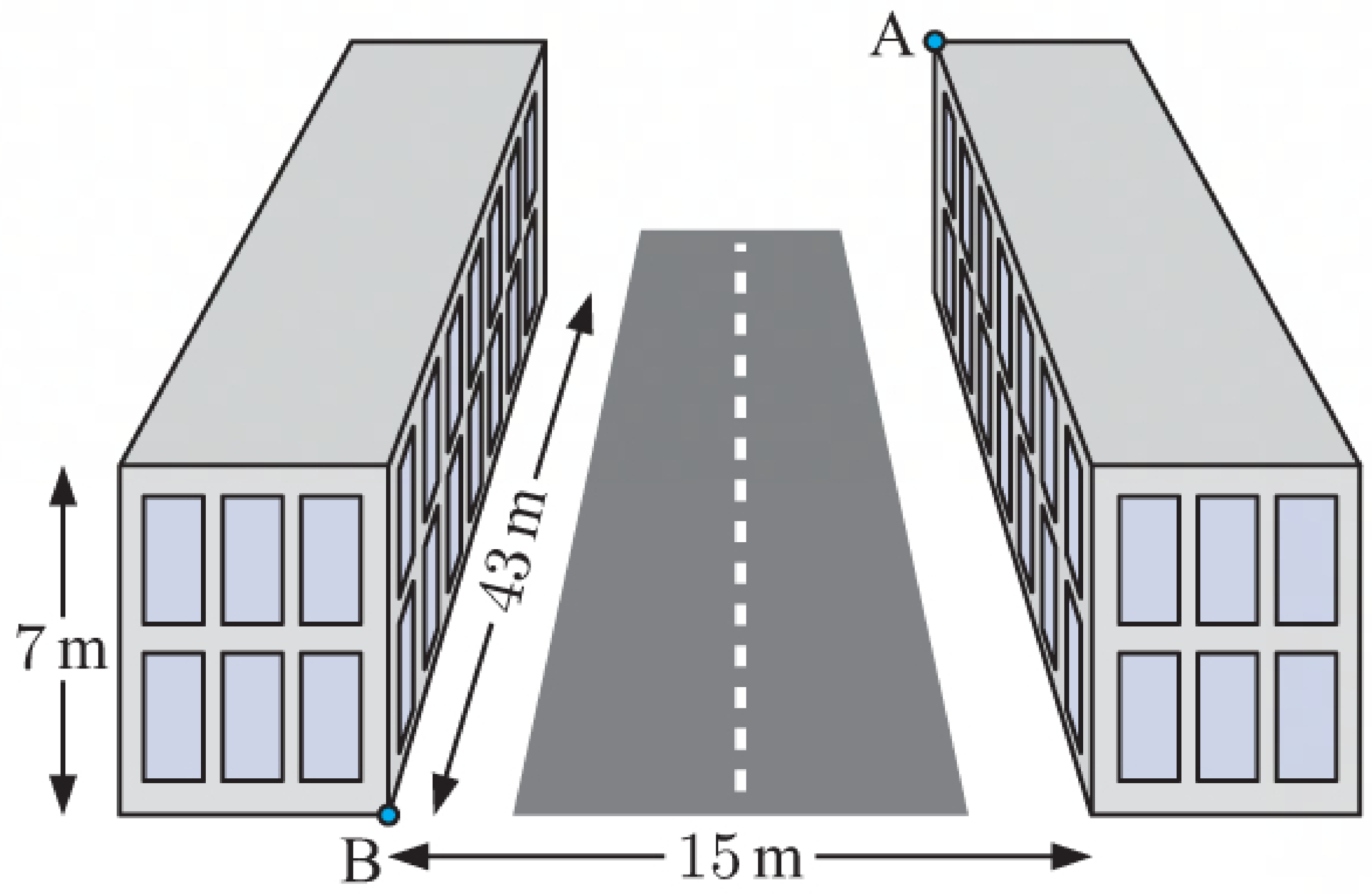
$\therefore\ \theta = \tan^{-1}\left(\frac{5}{12}\right) \approx 22.6°$

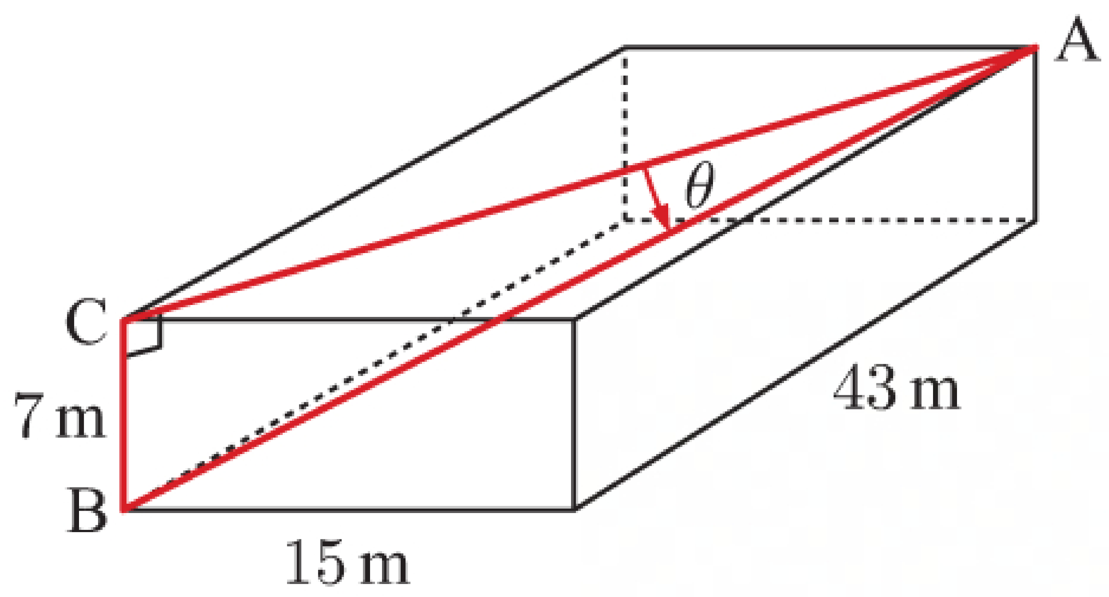
$\therefore$ the bearing of F from S $\approx 180° + 22.6°$
$\approx 203°$

Now, $x^2 = 12^2 + 5^2$ {Pythagoras}
$\therefore\ x^2 = 169$
$\therefore\ x = 13$ {as $x > 0$}

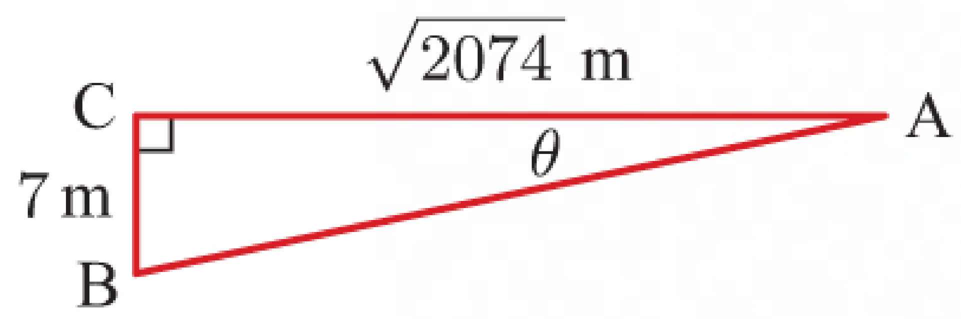
So, the helicopter is 13 km on the bearing of about 203° from the helipad.

11





$$\begin{aligned} AC^2 &= 15^2 + 43^2 && \{\text{Pythagoras}\} \\ \therefore\ AC &= \sqrt{15^2+43^2} && \{\text{as } AC > 0\} \\ &= \sqrt{2074}\text{ m} \end{aligned}$$

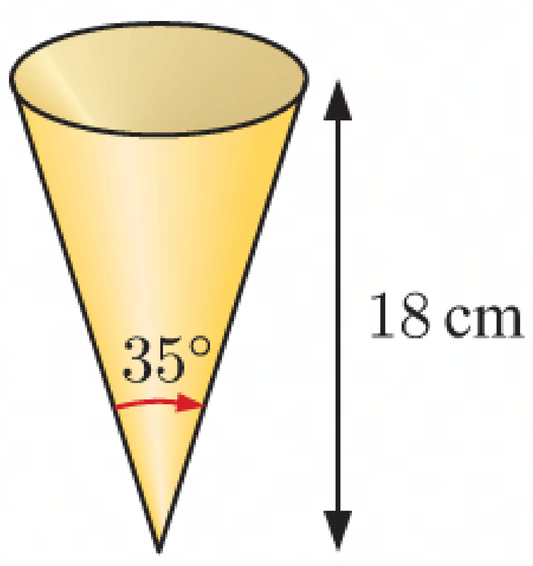


$$\tan\theta = \frac{7}{\sqrt{2074}}$$

$$\begin{aligned} \therefore\ \theta &= \tan^{-1}\left(\frac{7}{\sqrt{2074}}\right) \\ &\approx 8.74° \end{aligned}$$

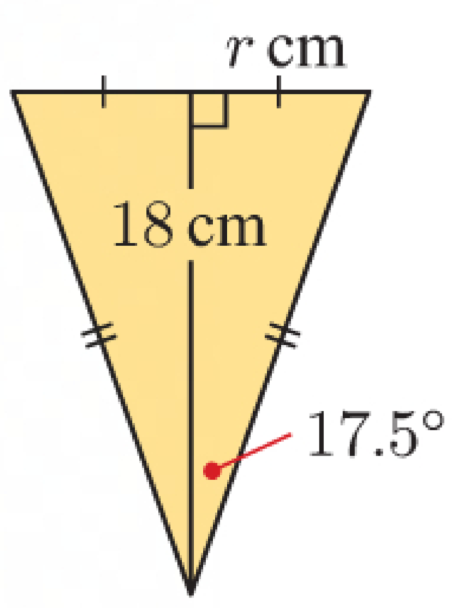
So, the angle of depression from A to B is approximately $8.74°$.

12



We draw the isosceles triangle cross-section of the cone as shown.

Let the radius of the cone be r cm.



$$\tan 17.5° = \frac{r}{18}$$

$$\therefore\ 18 \times \tan 17.5° = r$$

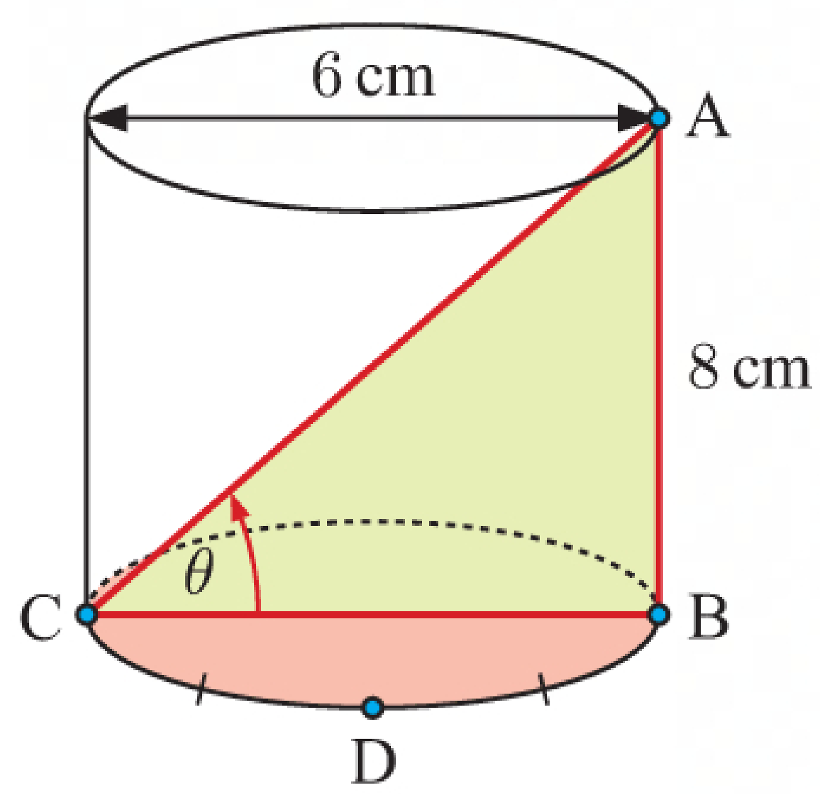
$$\therefore\ r \approx 5.68$$

$$\begin{aligned} V &= \tfrac{1}{3}\pi r^2 h \\ &\approx \tfrac{1}{3} \times \pi \times 5.68^2 \times 18\ \text{cm}^3 \\ &\approx 607\ \text{cm}^3 \\ &\approx 607\ \text{mL} \\ &\approx 0.607\ \text{L} \end{aligned}$$

The cone has a capacity of about 0.607 L.

13 **a** The projection of [AC] onto the base plane is [BC].

$\therefore$ the required angle is $A\widehat{C}B$.



$$\tan\theta = \frac{8}{6}$$

$$\therefore\ \theta = \tan^{-1}\left(\tfrac{8}{6}\right)$$

$$\therefore\ \theta \approx 53.1°$$

The angle is about $53.1°$.

b The projection of [AD] onto the base plane is [BD].

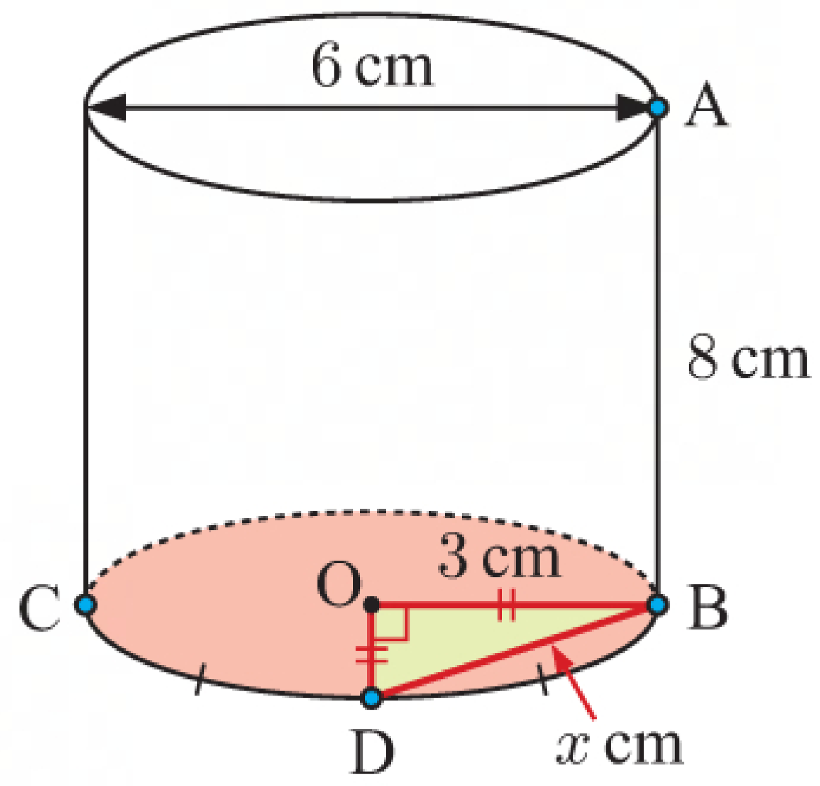
$\therefore$ the required angle is $\widehat{ADB}$.

Let DB be x cm, and the centre of the circular base be O.

OB = OD = 3 cm {both radii of circle}

Using Pythagoras in $\triangle$BOD,

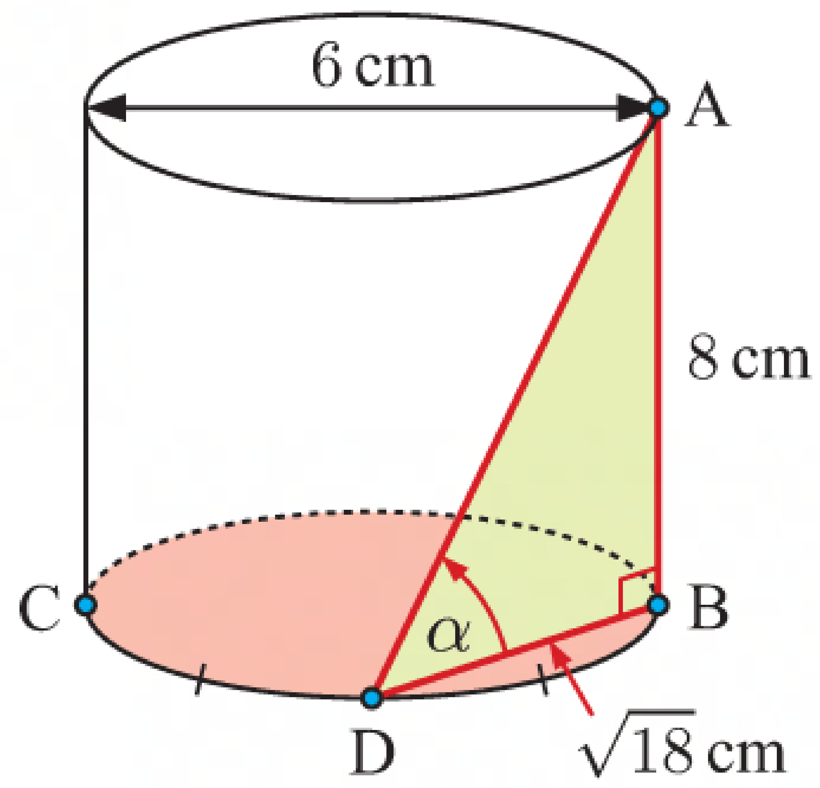
$$x^2 = 3^2 + 3^2$$
$$\therefore\ x^2 = 18$$
$$\therefore\ x = \sqrt{18} \qquad \{\text{as } x > 0\}$$



Let $\widehat{ADB}$ be α.

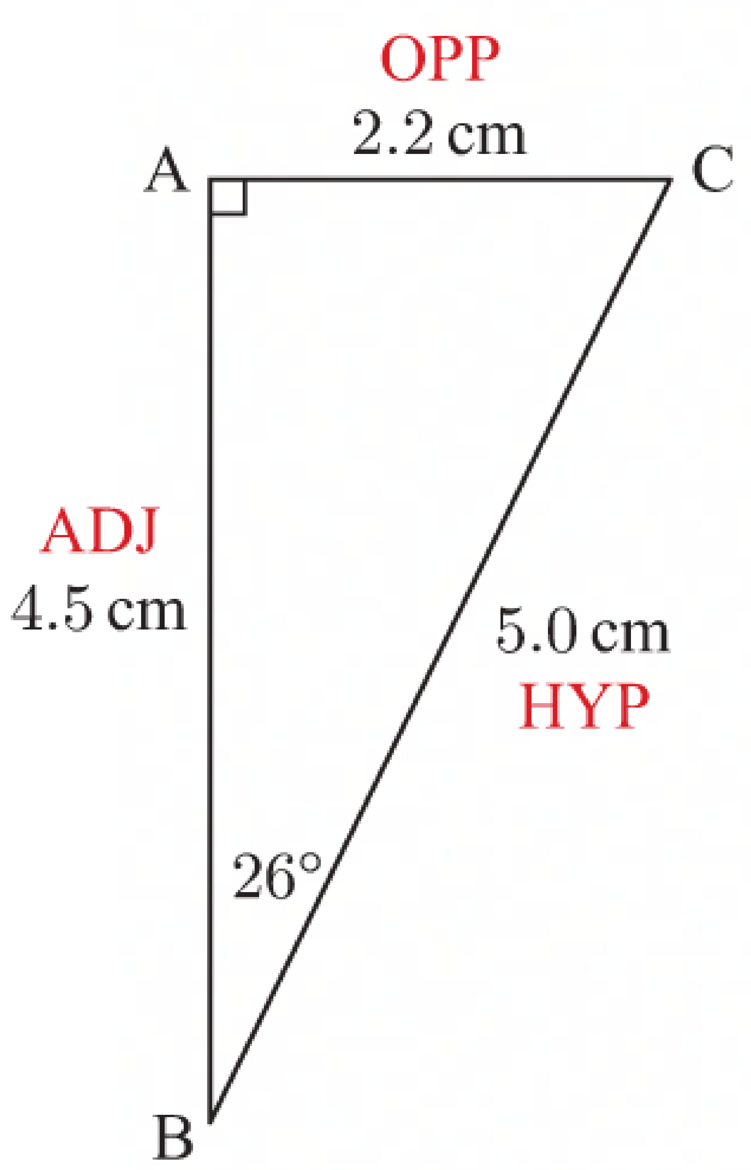
$$\therefore\ \tan\alpha = \frac{8}{\sqrt{18}}$$
$$\therefore\ \alpha = \tan^{-1}\left(\frac{8}{\sqrt{18}}\right)$$
$$\therefore\ \alpha \approx 62.1^\circ$$

The angle is about 62.1°.



REVIEW SET 7B

1 **a**



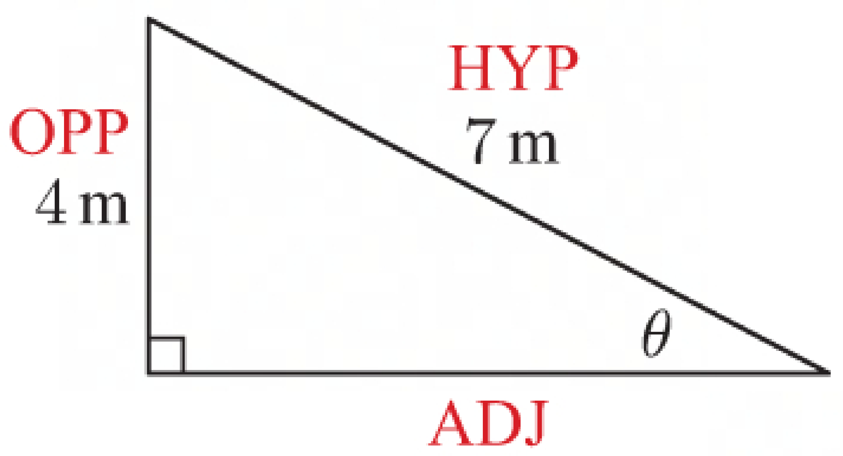
b **i** $\sin 26^\circ = \dfrac{\text{OPP}}{\text{HYP}} \approx \dfrac{2.2}{5.0} \approx 0.44$

ii $\cos 26^\circ = \dfrac{\text{ADJ}}{\text{HYP}} \approx \dfrac{4.5}{5.0} \approx 0.90$

iii $\tan 26^\circ = \dfrac{\text{OPP}}{\text{ADJ}} \approx \dfrac{2.2}{4.5} \approx 0.49$

c $\sin 26^\circ \approx 0.44,\quad \cos 26^\circ \approx 0.90,\quad \tan 26^\circ \approx 0.49$

2 **a**

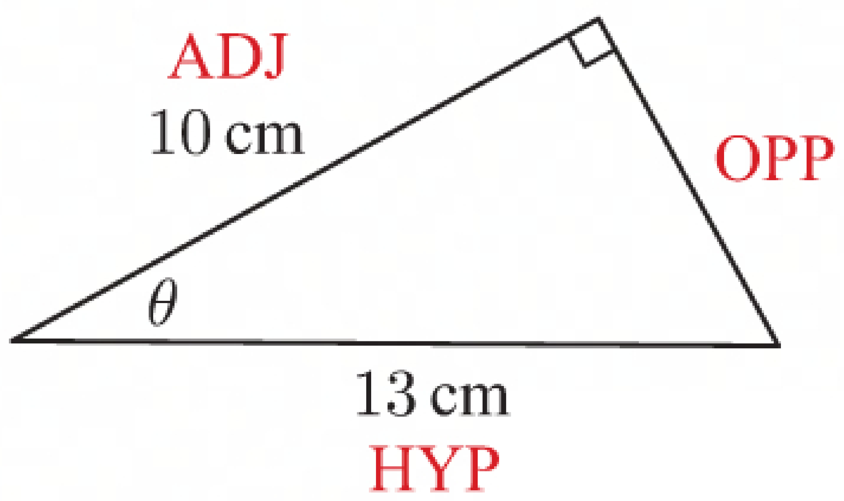


$\sin\theta = \dfrac{4}{7} \qquad \left\{\sin\theta = \dfrac{\text{OPP}}{\text{HYP}}\right\}$

$\therefore\ \theta = \sin^{-1}\left(\frac{4}{7}\right)$

$\approx 34.8^\circ$

b



$\cos\theta = \dfrac{10}{13} \qquad \left\{\cos\theta = \dfrac{\text{ADJ}}{\text{HYP}}\right\}$

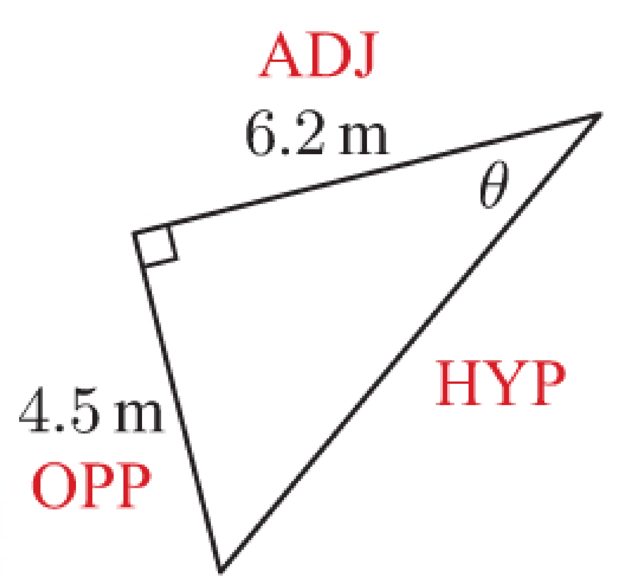
$\therefore\ \theta = \cos^{-1}\left(\frac{10}{13}\right)$

$\approx 39.7^\circ$

c $\tan\theta = \frac{4.5}{6.2}$ $\quad\{\tan\theta = \frac{\text{OPP}}{\text{ADJ}}\}$

$\therefore\ \theta = \tan^{-1}\left(\frac{4.5}{6.2}\right)$

$\approx 36.0°$



3 $\sin 23° = \frac{47}{\text{AB}}$ $\quad\{\sin\theta = \frac{\text{OPP}}{\text{HYP}}\}$

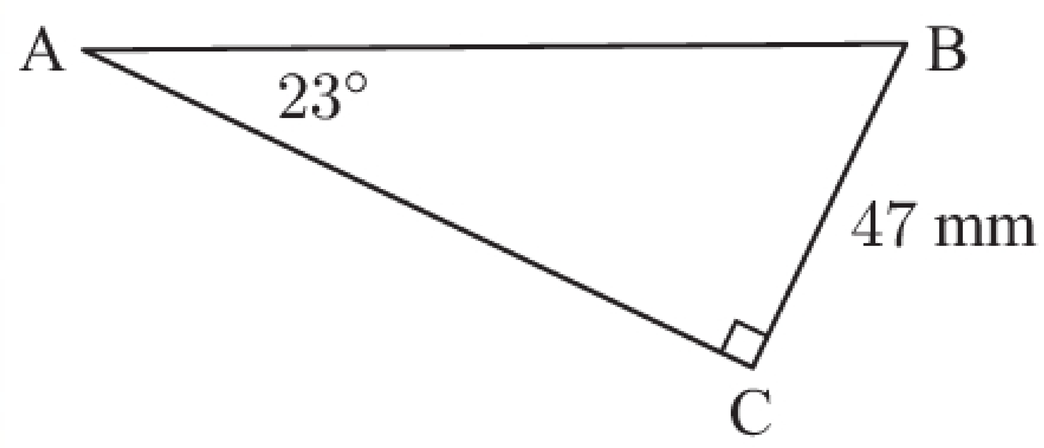
$\therefore\ \text{AB} = \frac{47}{\sin 23°}$

≈ 120 mm

$\tan 23° = \frac{47}{\text{AC}}$ $\quad\{\tan\theta = \frac{\text{OPP}}{\text{ADJ}}\}$

$\therefore\ \text{AC} = \frac{47}{\tan 23°}$

≈ 111 mm



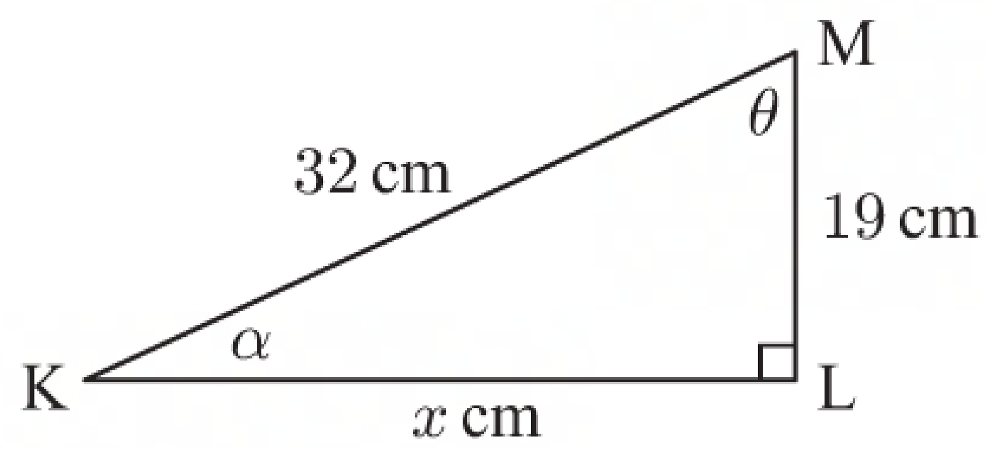
4 $x^2 + 19^2 = 32^2$ $\quad$ {Pythagoras}

$\therefore\ x^2 = 32^2 - 19^2$

$\therefore\ x^2 = 663$

$\therefore\ x = \sqrt{663}$ $\quad\{\text{as } x > 0\}$

≈ 25.7



$\cos\theta = \frac{19}{32}$ $\quad\{\cos\theta = \frac{\text{ADJ}}{\text{HYP}}\}$

$\therefore\ \theta = \cos^{-1}\left(\frac{19}{32}\right)$

$\therefore\ \theta \approx 53.6°$

$\sin\alpha = \frac{19}{32}$ $\quad\{\sin\theta = \frac{\text{OPP}}{\text{HYP}}\}$

$\therefore\ \alpha = \sin^{-1}\left(\frac{19}{32}\right)$

$\therefore\ \alpha \approx 36.4°$

5 **a** $\tan 28° = \frac{40}{\text{PR}}$ $\quad\{\tan\theta = \frac{\text{OPP}}{\text{ADJ}}\}$

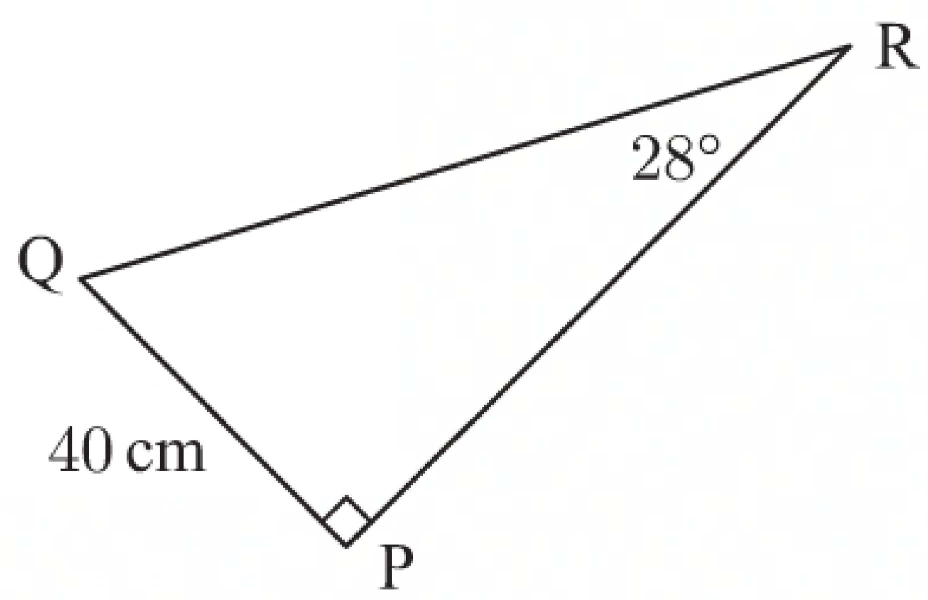
$\therefore\ \text{PR} = \frac{40}{\tan 28°}$

$\therefore\ \text{PR} \approx 75.2$ cm

$\sin 28° = \frac{40}{\text{QR}}$ $\quad\{\sin\theta = \frac{\text{OPP}}{\text{HYP}}\}$

$\therefore\ \text{QR} = \frac{40}{\sin 28°}$

$\therefore\ \text{QR} \approx 85.2$ cm



Perimeter of triangle PQR $=$ PQ $+$ PR $+$ QR

$\approx 40 + 75.2 + 85.2$ cm

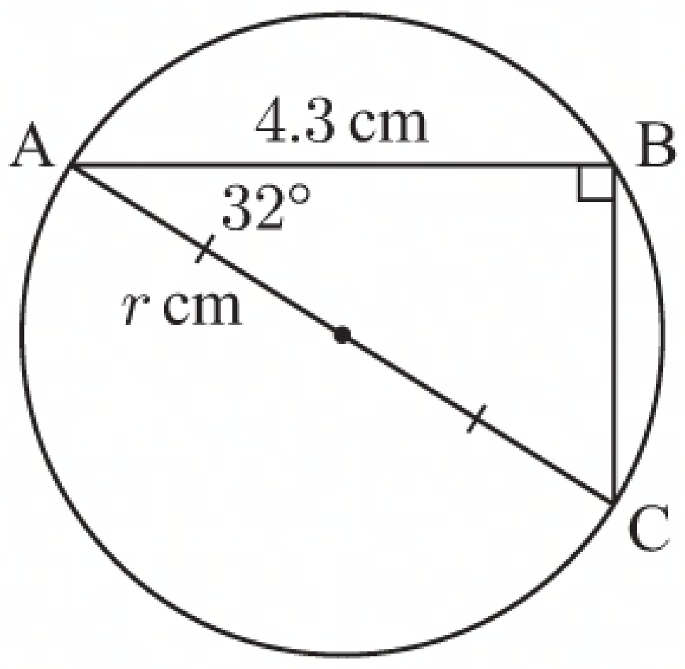
≈ 200 cm

b Area of triangle PQR $= \frac{1}{2} \times \text{base} \times \text{height}$

$$= \frac{1}{2} \times \text{PQ} \times \text{PR}$$
$$\approx \frac{1}{2} \times 40 \times 75.2 \text{ cm}^2$$
$$\approx 1500 \text{ cm}^2$$

6 $\text{A}\widehat{\text{B}}\text{C} = 90°$ {angle in a semi-circle}

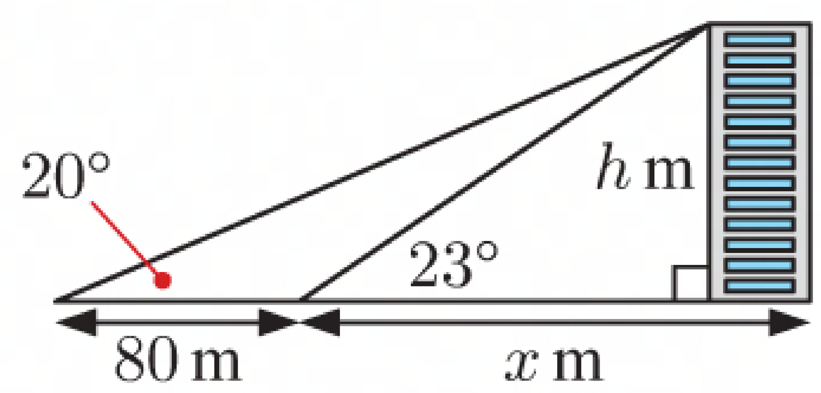
$\therefore$ $\triangle$ABC is right angled at B.



$$\cos 32° = \frac{4.3}{\text{AC}}$$
$$\therefore \text{AC} = \frac{4.3}{\cos 32°}$$
$$\therefore 2r = \frac{4.3}{\cos 32°}$$
$$\therefore r = \frac{4.3}{2 \times \cos 32°}$$
$$\approx 2.54$$

The radius is approximately 2.54 cm.

7



Let the height of the building be h m.

$$\tan 20° = \frac{h}{x+80}$$
$$\therefore h = (x+80)\tan 20°$$

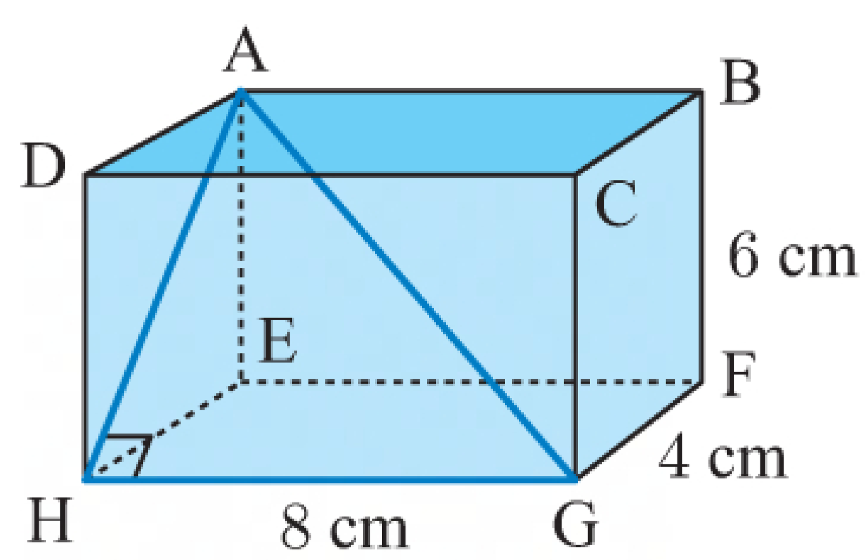
Also $\tan 23° = \frac{h}{x}$

$$\therefore \tan 23° = \frac{(x+80)\tan 20°}{x}$$
$$\therefore x\tan 23° = x\tan 20° + 80\tan 20°$$
$$\therefore x(\tan 23° - \tan 20°) = 80\tan 20°$$
$$\therefore x = \frac{80\tan 20°}{\tan 23° - \tan 20°}$$
$$\approx 481.25$$

$$\therefore h \approx (481.25 + 80)\tan 20° \text{ m}$$
$$\approx 204 \text{ m}$$

The building is about 204 m tall.

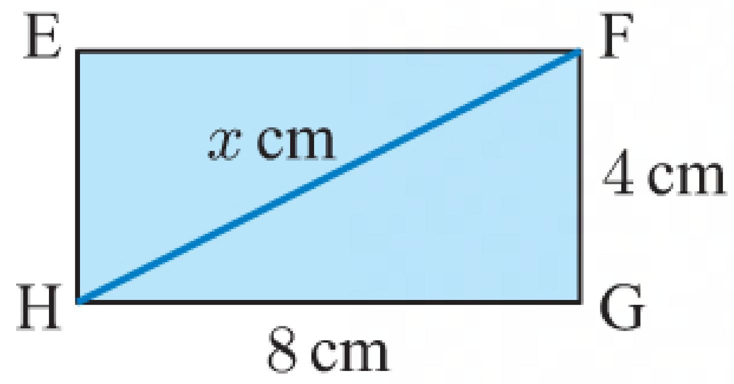
8 **a**

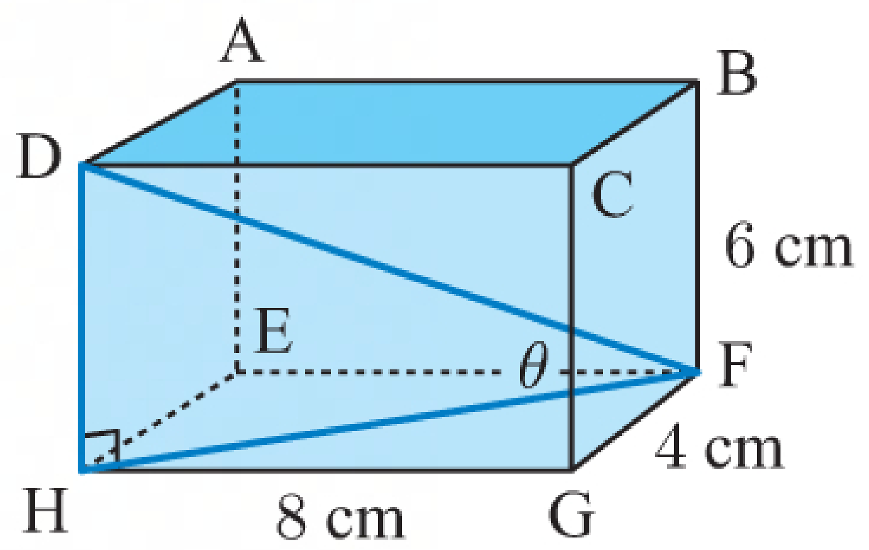


$\triangle$AHG is right angled at H.

$\text{A}\widehat{\text{H}}\text{G} = 90°$

b





Consider the base of the prism.

Let FH be x cm.

Using Pythagoras, $x^2 = 4^2 + 8^2$

$\therefore\ x^2 = 80$

$\therefore\ x = \sqrt{80}$ $\{$as $x > 0\}$

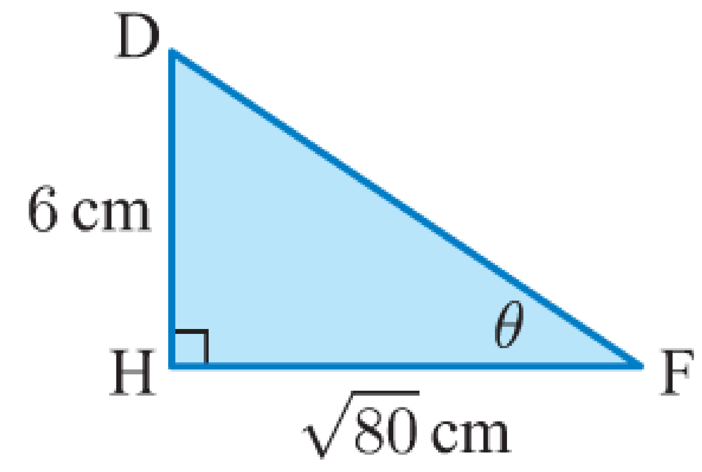
$\triangle$DFH is right angled at H.

$\tan\theta = \dfrac{6}{\sqrt{80}}$

$\therefore\ \theta = \tan^{-1}\left(\frac{6}{\sqrt{80}}\right)$

$\therefore\ \theta \approx 33.9°$

So, $\widehat{DFH} \approx 33.9°$.



9 Suppose Aaron starts at S, travels to O, and finishes at F.

$\widehat{FON_1} = 360° - 303° = 57°$ $\{$angles at a point$\}$

$\widehat{OSN_2} = 360° - 213° = 147°$ $\{$angles at a point$\}$

$\therefore\ \widehat{N_1OS} = 180° - 147° = 33°$ $\{$co-interior angles$\}$

$\therefore\ \widehat{FOS} = 57° + 33° = 90°$

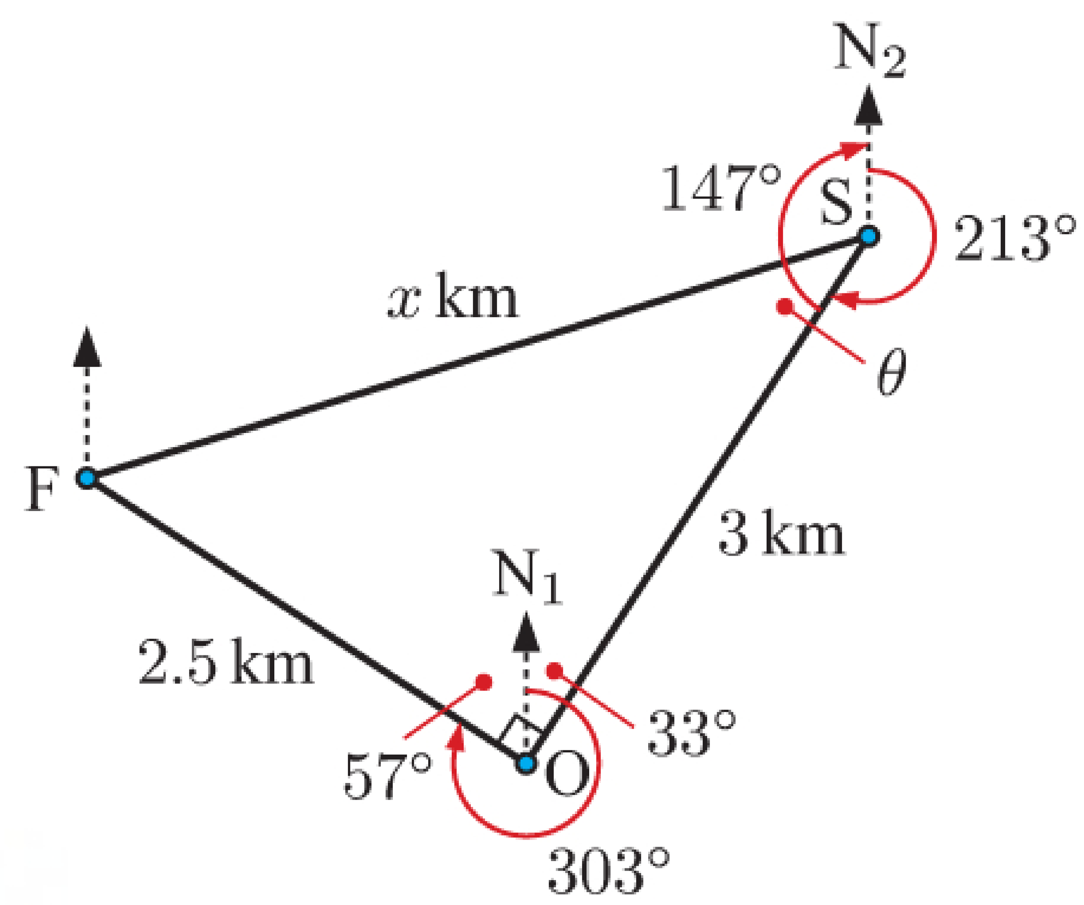
$\therefore$ $\triangle$FOS is right angled at O.

$\tan\theta = \dfrac{2.5}{3}$

$\therefore\ \theta = \tan^{-1}\left(\frac{2.5}{3}\right) \approx 39.8°$

$\therefore$ the bearing of F from S $\approx 213° + 39.8°$

$\approx 253°$



Now, $x^2 = 2.5^2 + 3^2$ $\{$Pythagoras$\}$

$\therefore\ x = \sqrt{2.5^2 + 3^2}$ $\{$as $x > 0\}$

≈ 3.91

So, Aaron is about 3.91 km on a bearing of about 253° from his starting point.

10 Suppose Amelia and Kristos both depart from point S. Kristos jogs x km to point K, and Amelia jogs $(x + 2)$ km to point A.

$\widehat{ASK} = 164° - 74° = 90°$

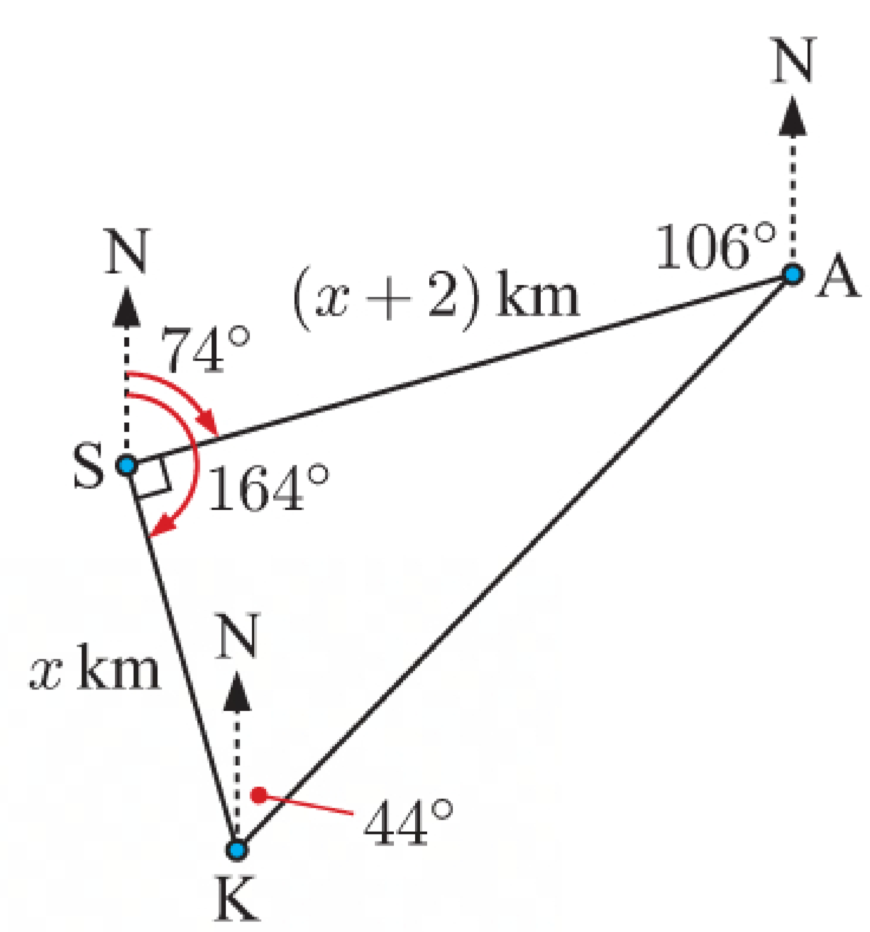
$\therefore$ $\triangle$AKS is right angled at S.

$\widehat{SAN} = 180° - 74°$ $\{$co-interior angles$\}$

$= 106°$

$\widehat{SAK} = 180° - 106° - 44°$ $\{$co-interior angles$\}$

$= 30°$



In $\triangle$AKS, $\tan 30^\circ = \dfrac{x}{x+2}$

$$\therefore \quad \frac{1}{\sqrt{3}} = \frac{x}{x+2}$$

$$\therefore \quad x + 2 = \sqrt{3}x$$

$$\therefore \quad x(1-\sqrt{3}) = -2$$

$$\therefore \quad x = \frac{-2}{1-\sqrt{3}}$$

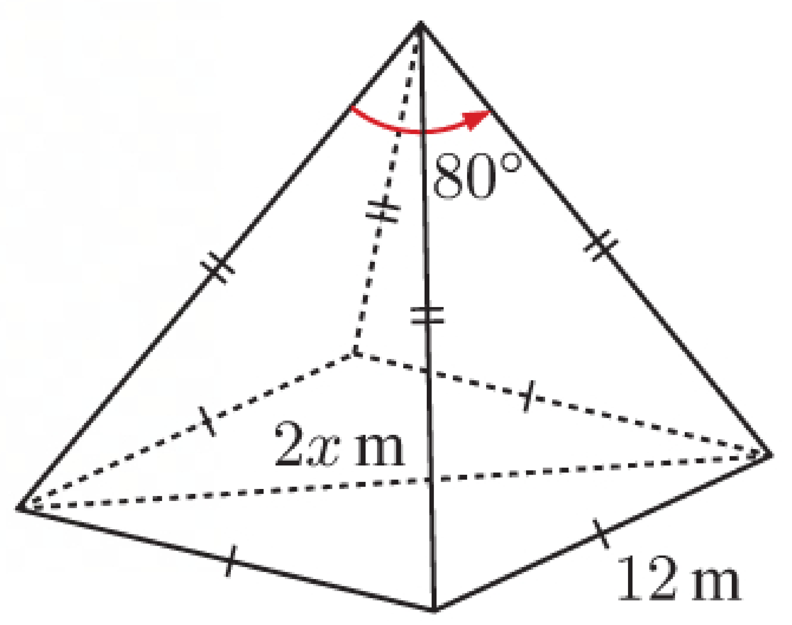
Also, $\sin 30^\circ = \dfrac{x}{\text{AK}}$

$$\therefore \quad \tfrac{1}{2} = \frac{\frac{-2}{1-\sqrt{3}}}{\text{AK}}$$

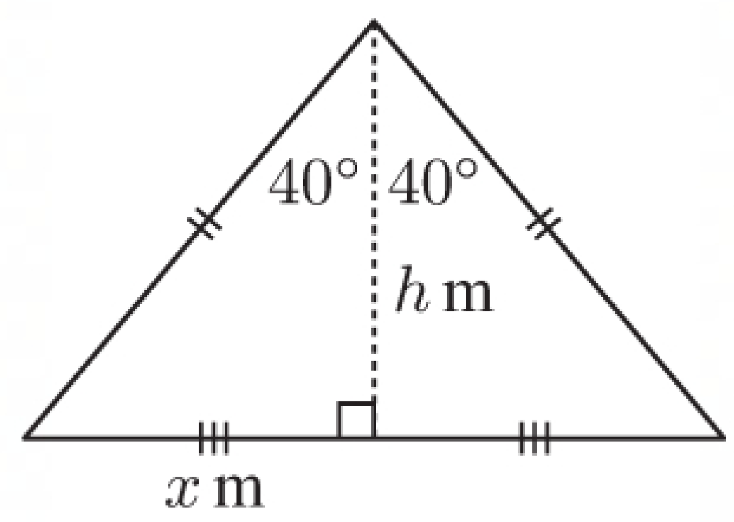
$$\therefore \quad \text{AK} = \frac{-4}{1-\sqrt{3}} \approx 5.46$$

So, the joggers are approximately 5.46 km apart at this time.

11



Let the height of the pyramid be h m, and the diagonal of the base of the pyramid be $2x$ m.



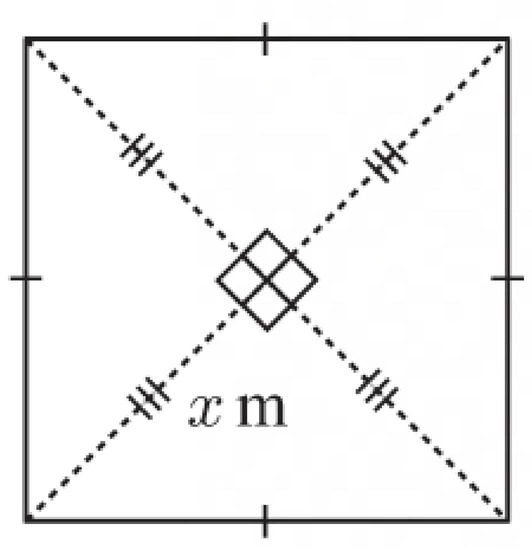
Consider the base of the pyramid.

$x^2 + x^2 = 12^2$ {Pythagoras}

$\therefore \quad 2x^2 = 144$

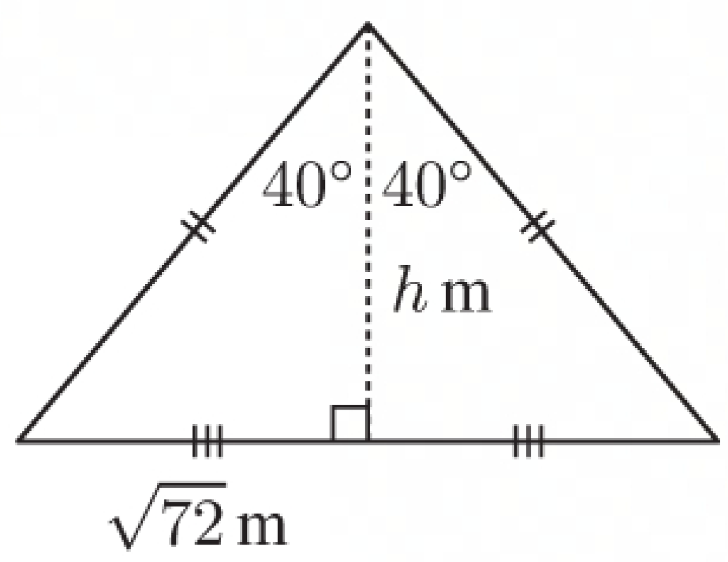
$\therefore \quad x^2 = 72$

$\therefore \quad x = \sqrt{72}$ {as $x > 0$}



$$\tan 40^\circ = \frac{\sqrt{72}}{h}$$

$$\therefore \quad h = \frac{\sqrt{72}}{\tan 40^\circ} \approx 10.1$$



$$\text{Volume of pyramid} = \tfrac{1}{3} \times (\text{area of base}) \times \text{height}$$
$$\approx \tfrac{1}{3} \times 12 \times 12 \times 10.1 \text{ m}^3$$
$$\approx 485 \text{ m}^3$$

12 **a** The projection of [BH] onto the base plane is [FH].

$\therefore$ the required angle is B$\widehat{\text{H}}$F.

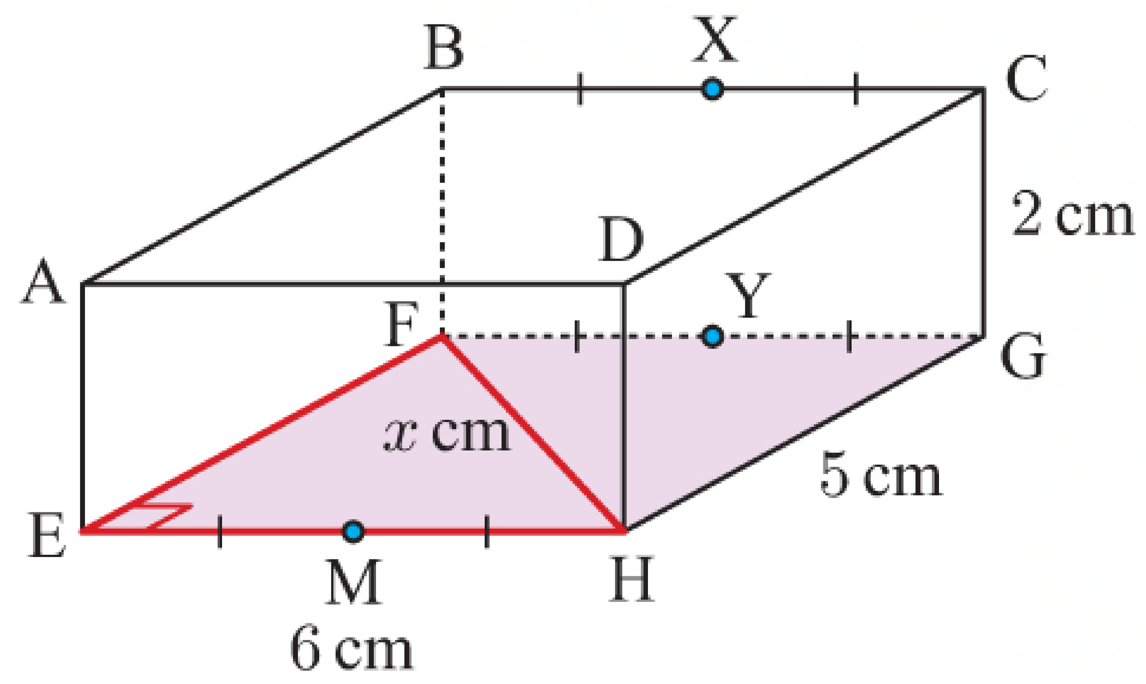
Let FH be x cm.

Using Pythagoras in $\triangle$FEH,

$$x^2 = 5^2 + 6^2$$

$$\therefore \quad x^2 = 61$$

$$\therefore \quad x = \sqrt{61} \qquad \{\text{as } x > 0\}$$



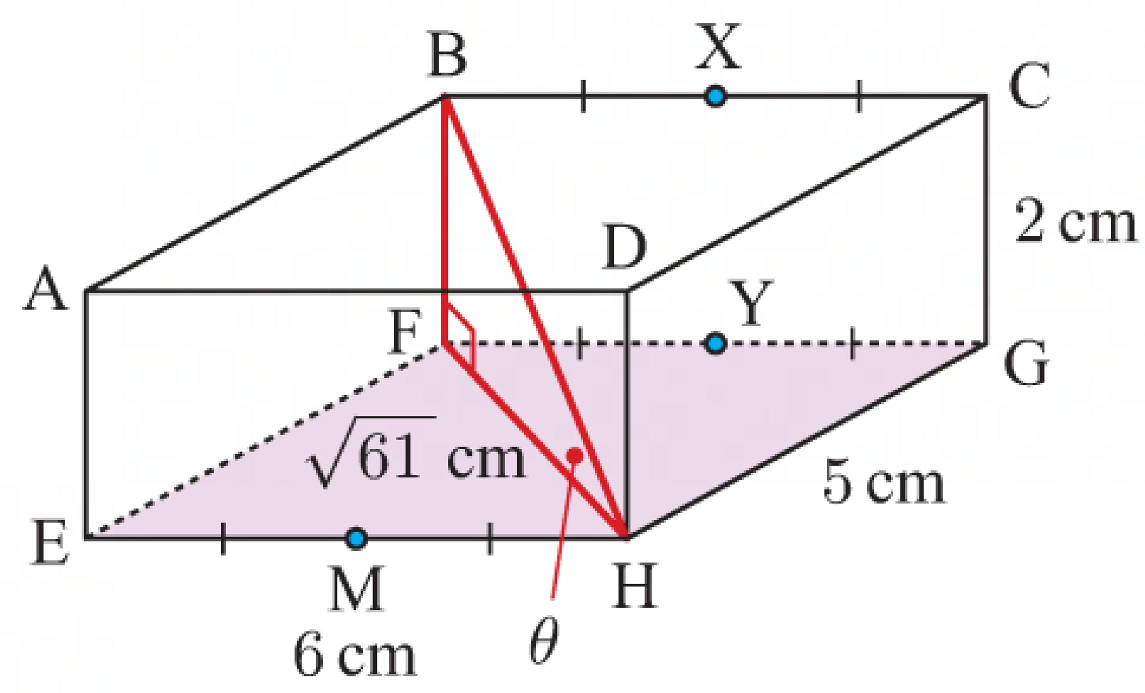
Let $B\widehat{H}F$ be θ.

$\therefore \tan\theta = \frac{2}{\sqrt{61}}$

$\therefore \theta = \tan^{-1}\left(\frac{2}{\sqrt{61}}\right)$

$\therefore \theta \approx 14.4°$

The angle is about $14.4°$.



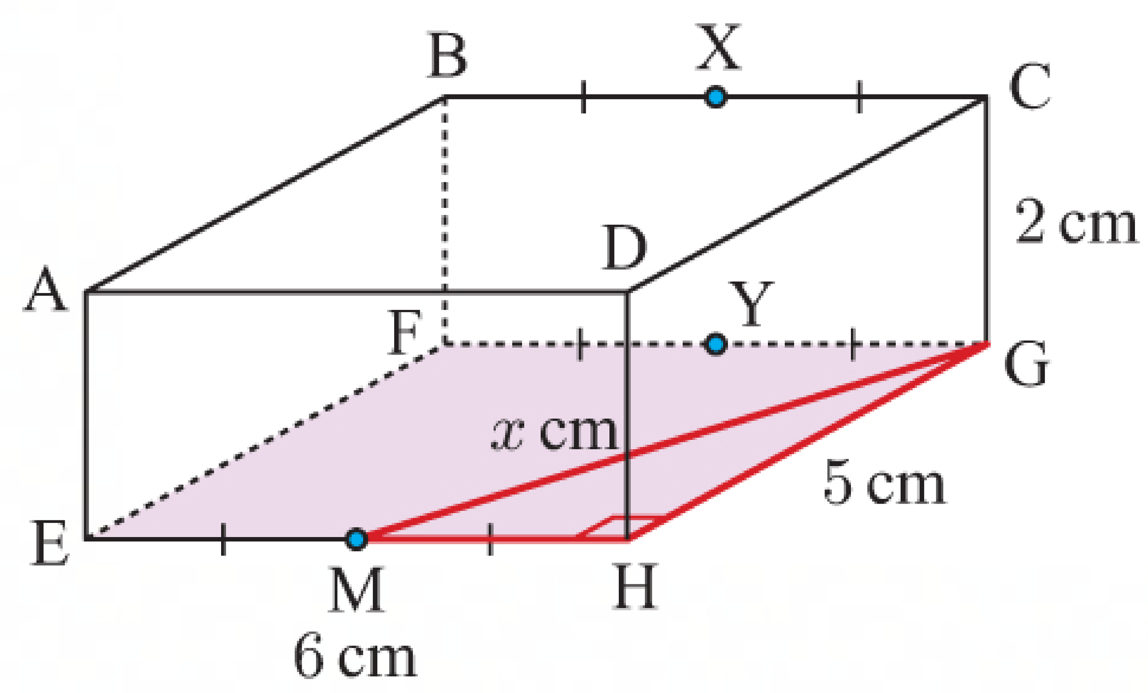
b The projection of [CM] onto the base plane is [GM].

$\therefore$ the required angle is $C\widehat{M}G$.

Let GM be x cm.

Using Pythagoras in $\triangle$GHM,

$$x^2 = 5^2 + 3^2$$
$$\therefore x^2 = 34$$
$$\therefore x = \sqrt{34} \quad \{\text{as } x > 0\}$$



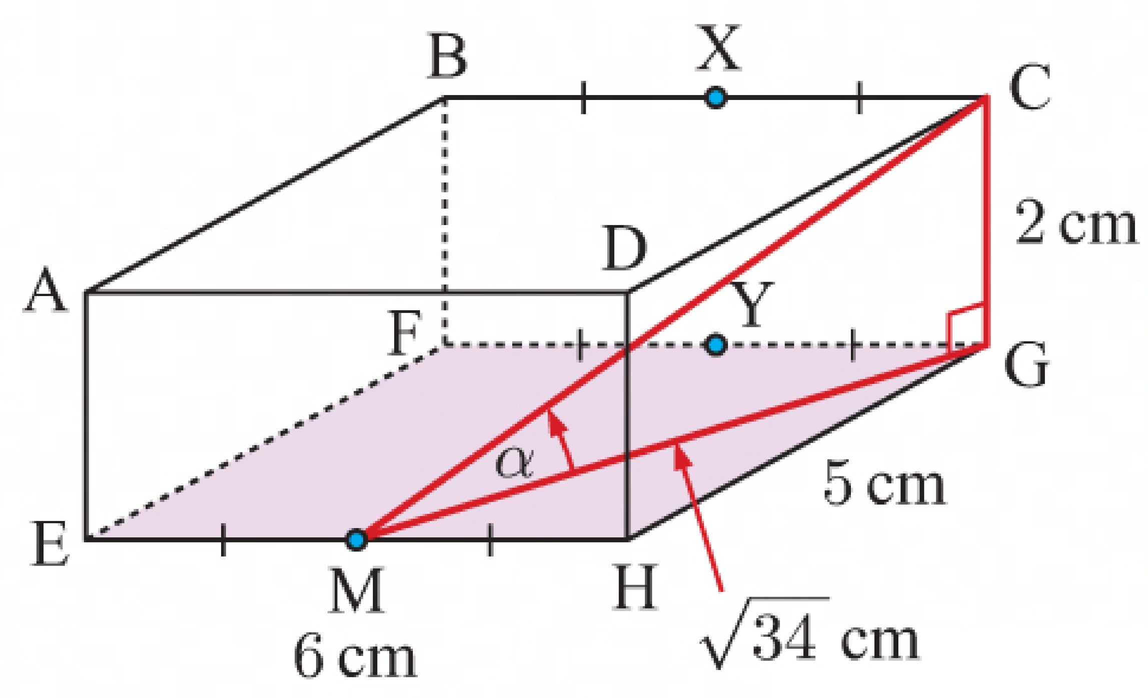
Let $C\widehat{M}G$ be α.

$\therefore \tan\alpha = \frac{2}{\sqrt{34}}$

$\therefore \alpha = \tan^{-1}\left(\frac{2}{\sqrt{34}}\right)$

$\therefore \alpha \approx 18.9°$

The angle is about $18.9°$.



c The projection of [XM] onto the base plane is [MY].

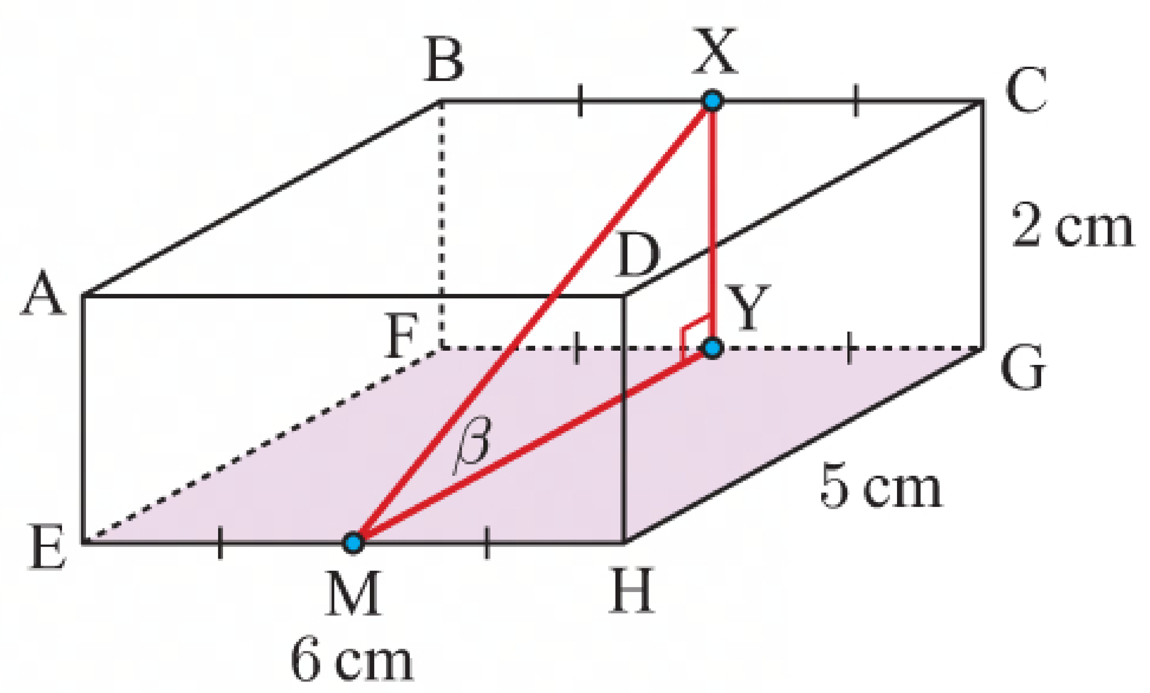
$\therefore$ the required angle is $X\widehat{M}Y$.

$\tan\beta = \frac{2}{5}$

$\therefore \beta = \tan^{-1}\left(\frac{2}{5}\right)$

$\therefore \beta \approx 21.8°$

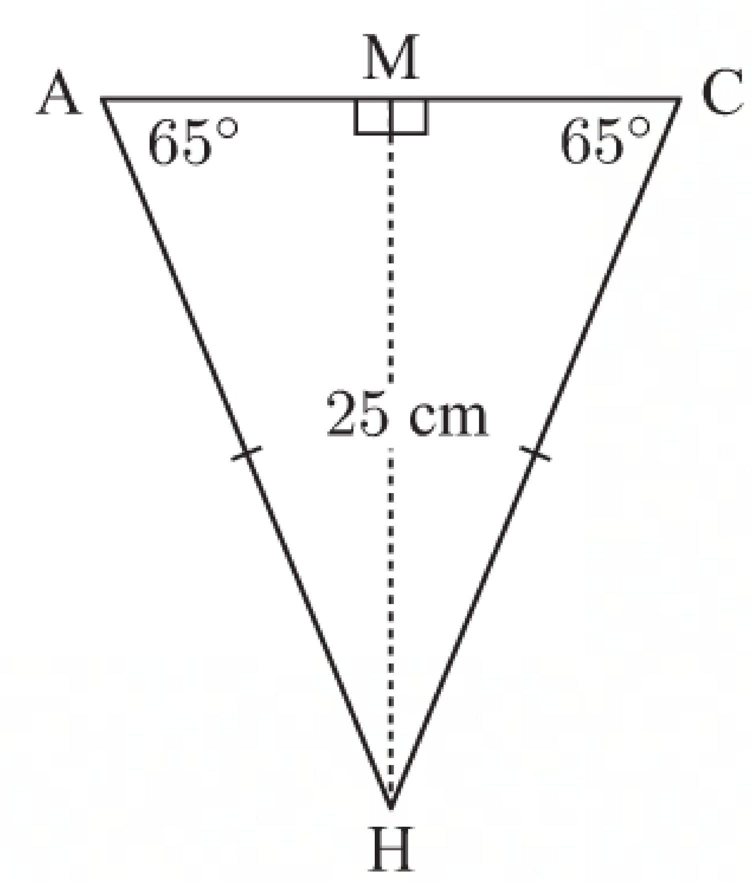
The angle is about $21.8°$.



13 **a** **i** In $\triangle$AMH, $\sin 65° = \frac{25}{\text{AH}}$ $\{\sin\theta = \frac{\text{OPP}}{\text{HYP}}\}$

$\therefore \text{AH} = \frac{25}{\sin 65°}$

$\therefore \text{AH} \approx 27.6$ cm



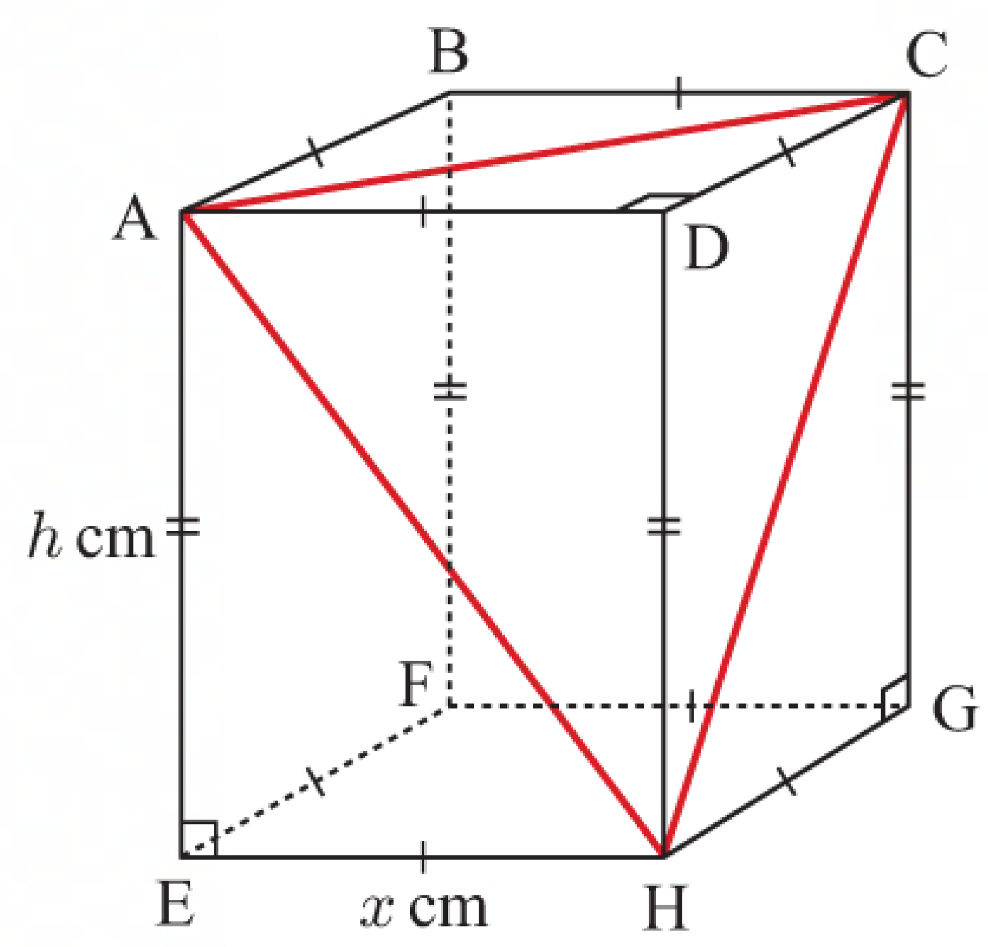
ii In $\triangle$AMH, $\tan 65^\circ = \dfrac{25}{\text{AM}}$ $\qquad \{\tan\theta = \dfrac{\text{OPP}}{\text{ADJ}}\}$

$$\therefore \ \text{AM} = \frac{25}{\tan 65^\circ}$$
$$\therefore \ \text{AM} \approx 11.66 \text{ cm}$$
$$\therefore \ \text{CM} \approx 11.66 \text{ cm} \qquad \{\text{altitude of isosceles triangle bisects the base}\}$$
$$\therefore \ \text{AC} \approx 2 \times 11.66 \text{ cm}$$
$$\therefore \ \text{AC} \approx 23.3 \text{ cm}$$

b $\text{CH} = \text{AH} = \dfrac{25}{\sin 65^\circ}$ $\qquad$ {from **a i**}

and $\text{AC} = 2 \times \dfrac{25}{\tan 65^\circ}$ $\qquad$ {from **a ii**}

$$\therefore \ \text{AC} = \frac{50}{\tan 65^\circ}$$



Let the height of the prism, AE, be h cm and the length of the prism, EH, be x cm.

In $\triangle$ADC, $\text{AD}^2 + \text{CD}^2 = \text{AC}^2$ $\qquad$ {Pythagoras}

$$\therefore \ x^2 + x^2 = \left(\frac{50}{\tan 65^\circ}\right)^2$$
$$\therefore \ 2x^2 = \left(\frac{50}{\tan 65^\circ}\right)^2$$
$$\therefore \ x^2 = \frac{\left(\frac{50}{\tan 65^\circ}\right)^2}{2} \qquad \text{.... (1)}$$
$$\therefore \ x = \sqrt{\frac{\left(\frac{50}{\tan 65^\circ}\right)^2}{2}}$$
$$\therefore \ x \approx 16.49$$

In $\triangle$AEH, $\text{AE}^2 + \text{EH}^2 = \text{AH}^2$ $\qquad$ {Pythagoras}

$$\therefore \ h^2 + x^2 = \left(\frac{25}{\sin 65^\circ}\right)^2$$
$$\therefore \ h^2 + \frac{\left(\frac{50}{\tan 65^\circ}\right)^2}{2} = \left(\frac{25}{\sin 65^\circ}\right)^2 \qquad \{\text{using (1)}\}$$
$$\therefore \ h = \sqrt{\left(\frac{25}{\sin 65^\circ}\right)^2 - \frac{\left(\frac{50}{\tan 65^\circ}\right)^2}{2}} \qquad \{\text{as } h > 0\}$$
$$\therefore \ h \approx 22.11$$

$$\begin{aligned}\text{Volume of prism} &= \text{length} \times \text{width} \times \text{height} \\ &\approx 16.49 \times 16.49 \times 22.11 \text{ cm}^3 \\ &\approx 6010 \text{ cm}^3\end{aligned}$$

Chapter 8

THE UNIT CIRCLE AND RADIAN MEASURE

EXERCISE 8A

1 **a** $90^\circ = \left(90 \times \frac{\pi}{180}\right)$ radians
$= \frac{\pi}{2}$ radians

b $60^\circ = \left(60 \times \frac{\pi}{180}\right)$ radians
$= \frac{\pi}{3}$ radians

c $30^\circ = \left(30 \times \frac{\pi}{180}\right)$ radians
$= \frac{\pi}{6}$ radians

d $18^\circ = \left(18 \times \frac{\pi}{180}\right)$ radians
$= \frac{\pi}{10}$ radians

e $9^\circ = \left(9 \times \frac{\pi}{180}\right)$ radians
$= \frac{\pi}{20}$ radians

f $135^\circ = \left(135 \times \frac{\pi}{180}\right)$ radians
$= \frac{3\pi}{4}$ radians

g $225^\circ = \left(225 \times \frac{\pi}{180}\right)$ radians
$= \frac{5\pi}{4}$ radians

h $270^\circ = \left(270 \times \frac{\pi}{180}\right)$ radians
$= \frac{3\pi}{2}$ radians

i $360^\circ = \left(360 \times \frac{\pi}{180}\right)$ radians
$= 2\pi$ radians

j $720^\circ = \left(720 \times \frac{\pi}{180}\right)$ radians
$= 4\pi$ radians

k $315^\circ = \left(315 \times \frac{\pi}{180}\right)$ radians
$= \frac{7\pi}{4}$ radians

l $540^\circ = \left(540 \times \frac{\pi}{180}\right)$ radians
$= 3\pi$ radians

m $36^\circ = \left(36 \times \frac{\pi}{180}\right)$ radians
$= \frac{\pi}{5}$ radians

n $80^\circ = \left(80 \times \frac{\pi}{180}\right)$ radians
$= \frac{4\pi}{9}$ radians

o $230^\circ = \left(230 \times \frac{\pi}{180}\right)$ radians
$= \frac{23\pi}{18}$ radians

2 **a** $36.7^\circ = \left(36.7 \times \frac{\pi}{180}\right)$ radians
≈ 0.641 radians

b $137.2^\circ = \left(137.2 \times \frac{\pi}{180}\right)$ radians
≈ 2.39 radians

c $317.9^\circ = \left(317.9 \times \frac{\pi}{180}\right)$ radians
≈ 5.55 radians

d $219.6^\circ = \left(219.6 \times \frac{\pi}{180}\right)$ radians
≈ 3.83 radians

e $396.7^\circ = \left(396.7 \times \frac{\pi}{180}\right)$ radians
≈ 6.92 radians

3 **a** $\frac{\pi}{5} = \left(\frac{\pi}{5} \times \frac{180}{\pi}\right)^\circ$
$= 36^\circ$

b $\frac{3\pi}{5} = \left(\frac{3\pi}{5} \times \frac{180}{\pi}\right)^\circ$
$= 108^\circ$

c $\frac{3\pi}{4} = \left(\frac{3\pi}{4} \times \frac{180}{\pi}\right)^\circ$
$= 135^\circ$

d $\frac{\pi}{18} = \left(\frac{\pi}{18} \times \frac{180}{\pi}\right)^\circ$
$= 10^\circ$

e $\frac{\pi}{9} = \left(\frac{\pi}{9} \times \frac{180}{\pi}\right)^\circ$
$= 20^\circ$

f $\frac{7\pi}{9} = \left(\frac{7\pi}{9} \times \frac{180}{\pi}\right)^\circ$
$= 140^\circ$

g $\frac{\pi}{10} = \left(\frac{\pi}{10} \times \frac{180}{\pi}\right)^\circ$
$= 18^\circ$

h $\frac{3\pi}{20} = \left(\frac{3\pi}{20} \times \frac{180}{\pi}\right)^\circ$
$= 27^\circ$

i $\frac{7\pi}{6} = \left(\frac{7\pi}{6} \times \frac{180}{\pi}\right)^\circ$
$= 210^\circ$

j $\frac{\pi}{8} = \left(\frac{\pi}{8} \times \frac{180}{\pi}\right)^\circ$
$= 22.5^\circ$

4 **a** 2 radians $= \left(2 \times \frac{180}{\pi}\right)^\circ$
$\approx 114.59^\circ$

b 1.53 radians $= \left(1.53 \times \frac{180}{\pi}\right)^\circ$
$\approx 87.66^\circ$

c 0.867 radians $= \left(0.867 \times \frac{180}{\pi}\right)^\circ$
$\approx 49.68^\circ$

d 3.179 radians $= \left(3.179 \times \frac{180}{\pi}\right)^\circ$
$\approx 182.14^\circ$

e 5.267 radians $= \left(5.267 \times \frac{180}{\pi}\right)^\circ$
$\approx 301.78^\circ$

5 **a**

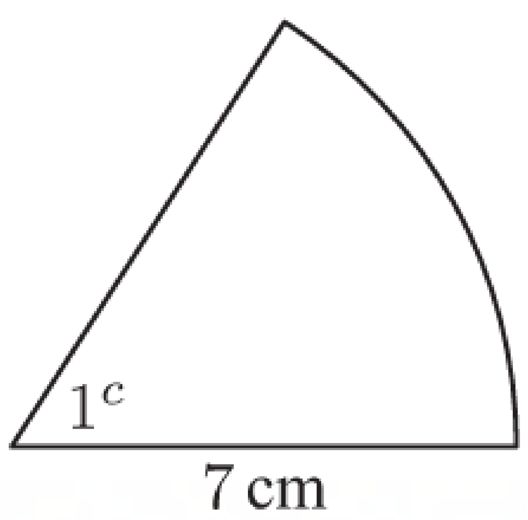
Degrees	0	45	90	135	180	225	270	315	360
Radians	0	$\frac{\pi}{4}$	$\frac{\pi}{2}$	$\frac{3\pi}{4}$	π	$\frac{5\pi}{4}$	$\frac{3\pi}{2}$	$\frac{7\pi}{4}$	2π

b

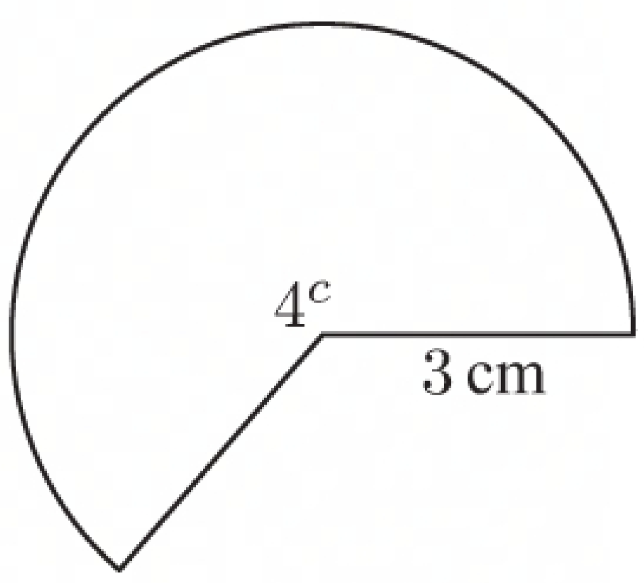
Degrees	0	30	60	90	120	150	180	210	240	270	300	330	360
Radians	0	$\frac{\pi}{6}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{5\pi}{6}$	π	$\frac{7\pi}{6}$	$\frac{4\pi}{3}$	$\frac{3\pi}{2}$	$\frac{5\pi}{3}$	$\frac{11\pi}{6}$	2π

EXERCISE 8B

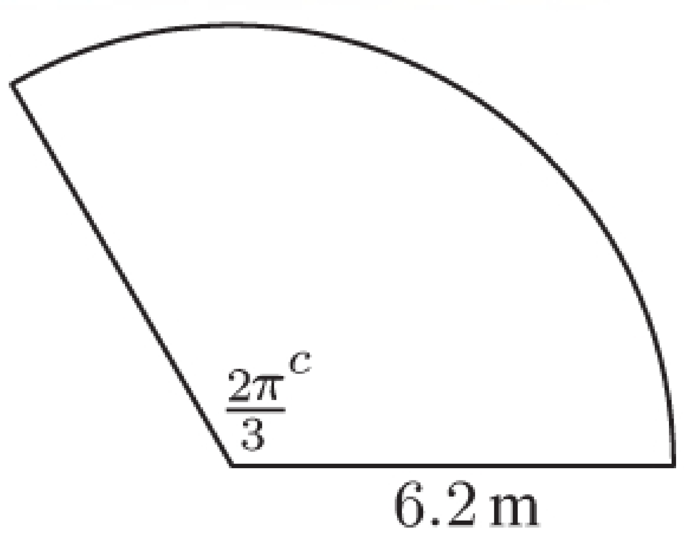
1 **a** arc length $= \theta r$
$= 1 \times 7$
$= 7$ cm



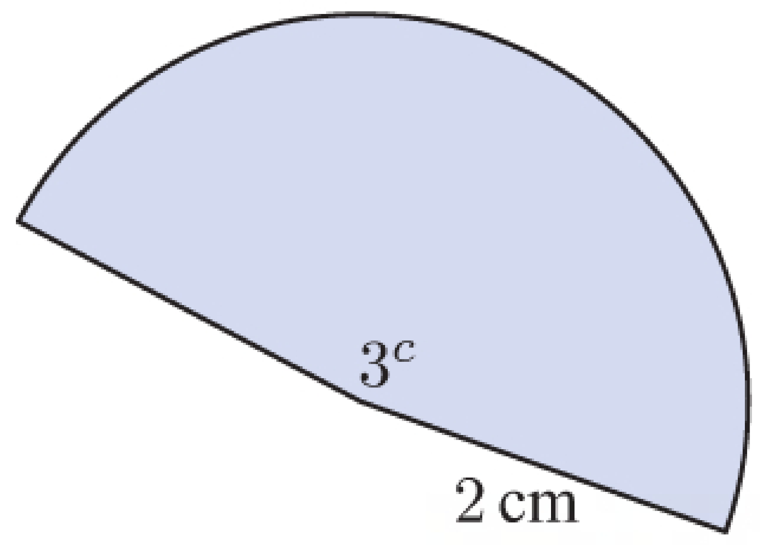
b arc length $= \theta r$
$= 4 \times 3$
$= 12$ cm



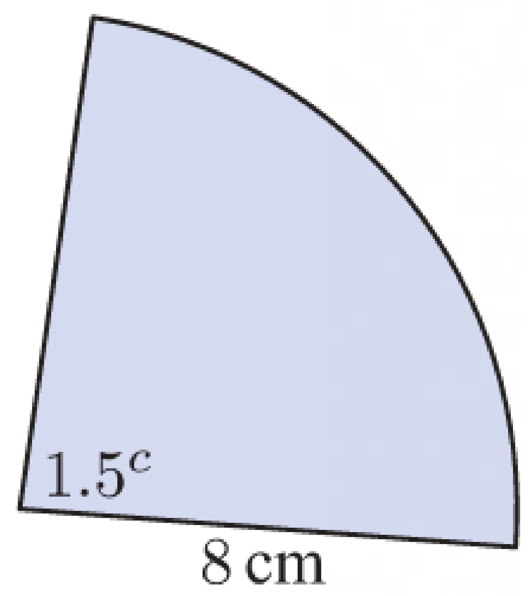
c arc length $= \theta r$
$= \frac{2\pi}{3} \times 6.2$
≈ 13.0 m



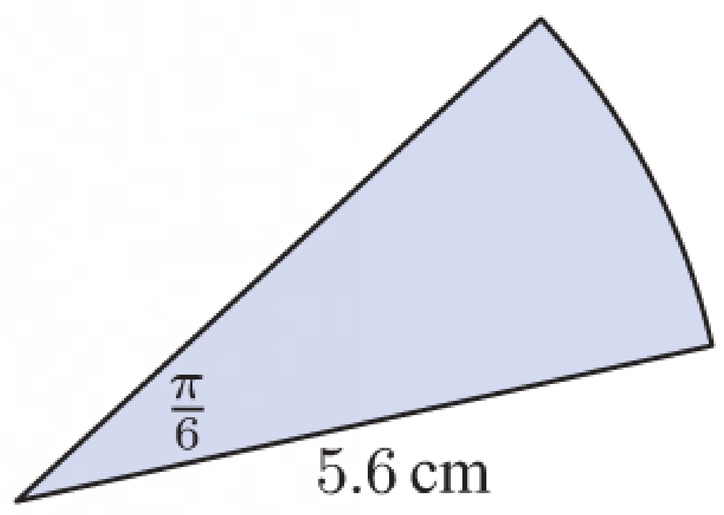
2 **a** $\text{area} = \frac{1}{2}\theta r^2$
$= \frac{1}{2} \times 3 \times 2^2$
$= 6 \text{ cm}^2$



b $\text{area} = \frac{1}{2}\theta r^2$
$= \frac{1}{2} \times 1.5 \times 8^2$
$= 48 \text{ cm}^2$



c $\text{area} = \frac{1}{2}\theta r^2$
$= \frac{1}{2} \times \frac{\pi}{6} \times 5.6^2$
$\approx 8.21 \text{ cm}^2$



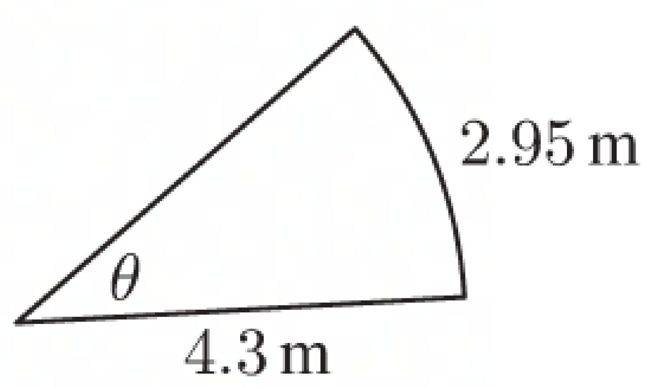
3 **a** $\text{arc length} = \theta r$
$= \frac{7\pi}{4} \times 9$
$\approx 49.5 \text{ cm}$

$\text{area} = \frac{1}{2}\theta r^2$
$= \frac{1}{2} \times \frac{7\pi}{4} \times 9^2$
$\approx 223 \text{ cm}^2$

b $\text{arc length} = \theta r$
$= 4.67 \times 4.93$
$\approx 23.0 \text{ cm}$

$\text{area} = \frac{1}{2}\theta r^2$
$= \frac{1}{2} \times 4.67 \times 4.93^2$
$\approx 56.8 \text{ cm}^2$

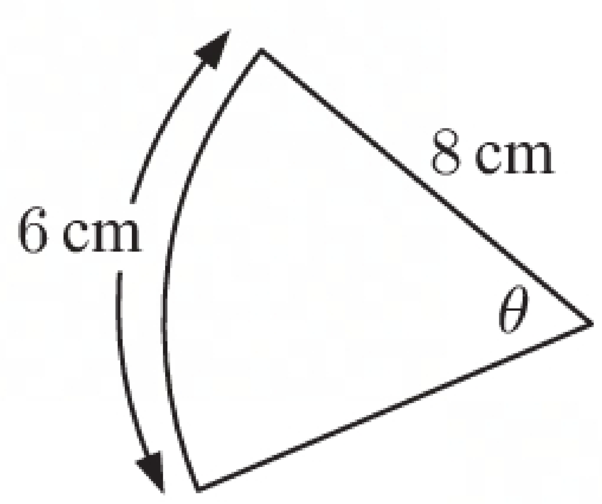
4 **a**



$l = \theta r$
$\therefore\ 2.95 = \theta \times 4.3$
$\therefore\ \theta = \frac{2.95}{4.3}$
$\therefore\ \theta \approx 0.686^c$

b $\text{area} = \frac{1}{2}\theta r^2$
$\therefore\ 30 = \frac{1}{2} \times \theta \times 10^2$
$\therefore\ \theta = \frac{30 \times 2}{100}$
$\therefore\ \theta = 0.6^c$

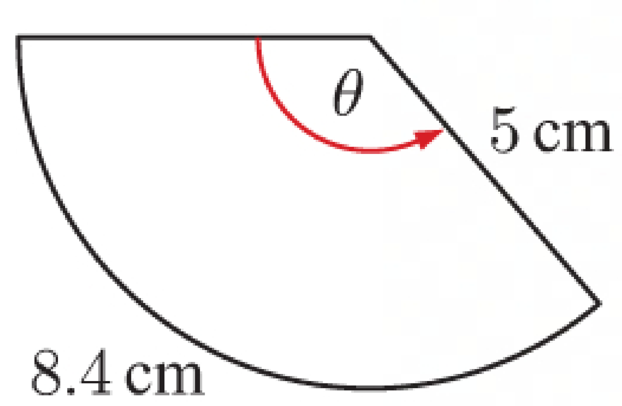
5 **a**



$l = \theta r$
$\therefore\ 6 = \theta \times 8$
$\therefore\ \theta = \frac{6}{8}$
$\therefore\ \theta = 0.75^c$

$\text{area} = \frac{1}{2}\theta r^2$
$= \frac{1}{2} \times 0.75 \times 8^2$
$= 24 \text{ cm}^2$

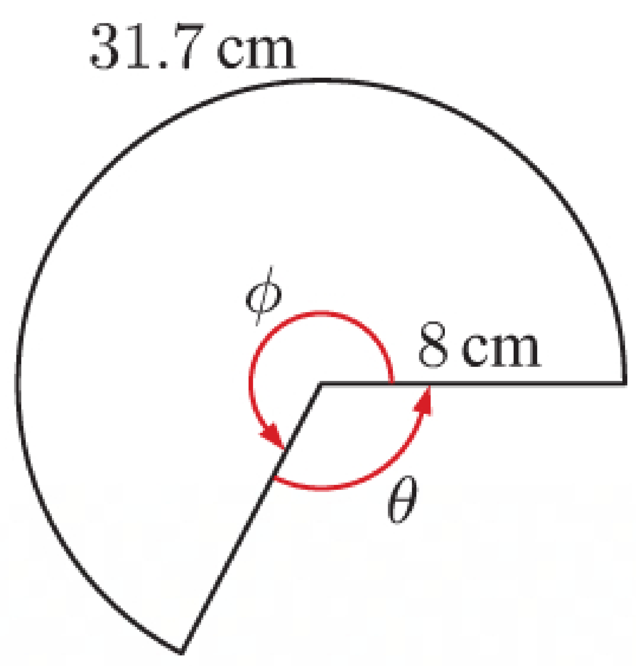
b



$l = \theta r$
$\therefore\ 8.4 = \theta \times 5$
$\therefore\ \theta = \frac{8.4}{5}$
$\therefore\ \theta = 1.68^c$

$\text{area} = \frac{1}{2}\theta r^2$
$= \frac{1}{2} \times 1.68 \times 5^2$
$= 21 \text{ cm}^2$

c



$$l = \phi r$$
$$\therefore\ 31.7 = \phi \times 8$$
$$\therefore\ \phi = \frac{31.7}{8}$$
$$\therefore\ \phi \approx 3.96^c$$
$$\text{But}\quad \theta = 2\pi - \phi$$
$$\therefore\ \theta \approx 2.32^c$$

$$\text{area} = \tfrac{1}{2}\phi r^2$$
$$= \tfrac{1}{2} \times \frac{31.7}{8} \times 8^2$$
$$= 126.8\ \text{cm}^2$$

6 **a**

$$l = \theta r$$
$$\therefore\ r = \frac{l}{\theta}$$
$$\therefore\ r = \frac{5.92}{1.88}$$
$$\therefore\ r \approx 3.15$$

The radius is about 3.15 m.

b

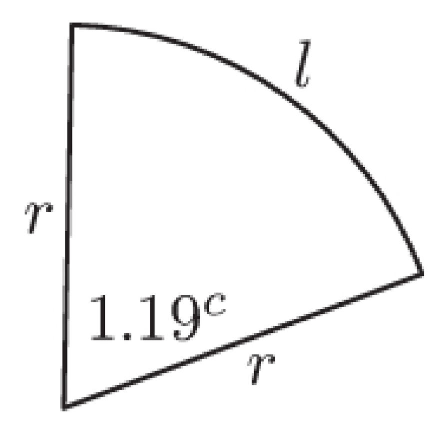
$$\text{area} = \tfrac{1}{2}\theta r^2$$
$$= \tfrac{1}{2} \times 1.88 \times \left(\frac{5.92}{1.88}\right)^2$$
$$\approx 9.32\ \text{m}^2$$

7 **a**

$$\text{area} = \tfrac{1}{2}\theta r^2$$
$$\therefore\ 20.8 = \tfrac{1}{2} \times 1.19 \times r^2$$
$$\therefore\ \frac{20.8 \times 2}{1.19} = r^2$$
$$\therefore\ r = \sqrt{\frac{20.8 \times 2}{1.19}} \quad \{r > 0\}$$
$$\therefore\ r \approx 5.91$$

The radius is about 5.91 cm.

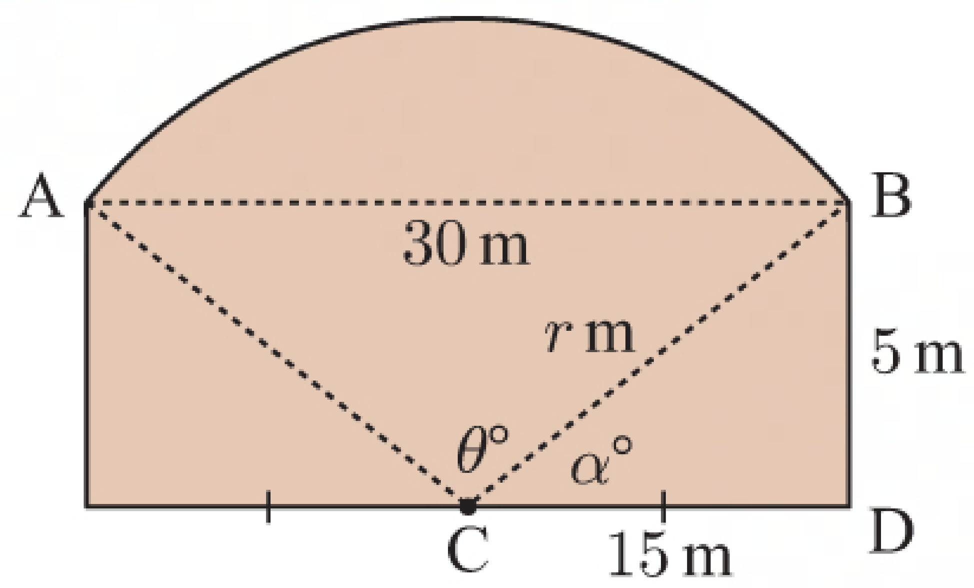
b



perimeter
$$= l + 2r$$
$$\approx 1.19 \times 5.912 + 2 \times 5.912$$
$$\approx 18.9\ \text{cm}$$

8 **a**

$$\tan\alpha = \tfrac{5}{15}$$
$$\therefore\ \alpha = \tan^{-1}(\tfrac{1}{3})$$
$$\therefore\ \alpha \approx 0.3218^c$$



b

$$\theta + 2\alpha = \pi \quad \{\text{angles on a line}\}$$
$$\therefore\ \theta \approx \pi - 2 \times 0.3218$$
$$\therefore\ \theta \approx 2.498^c$$

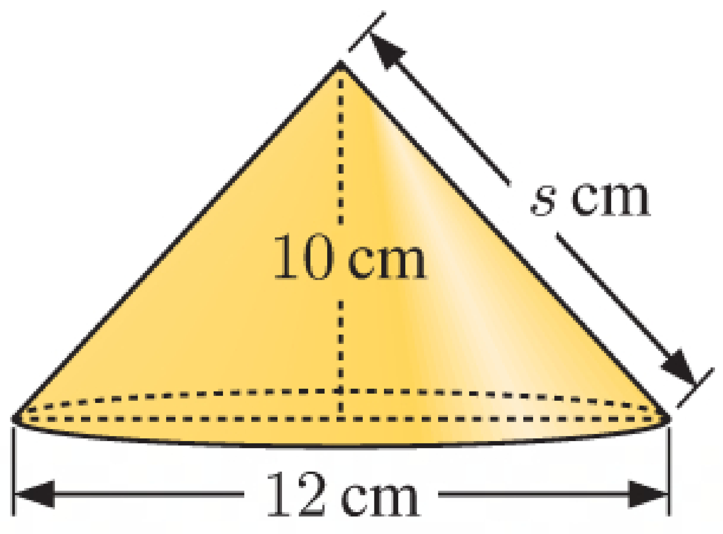
c

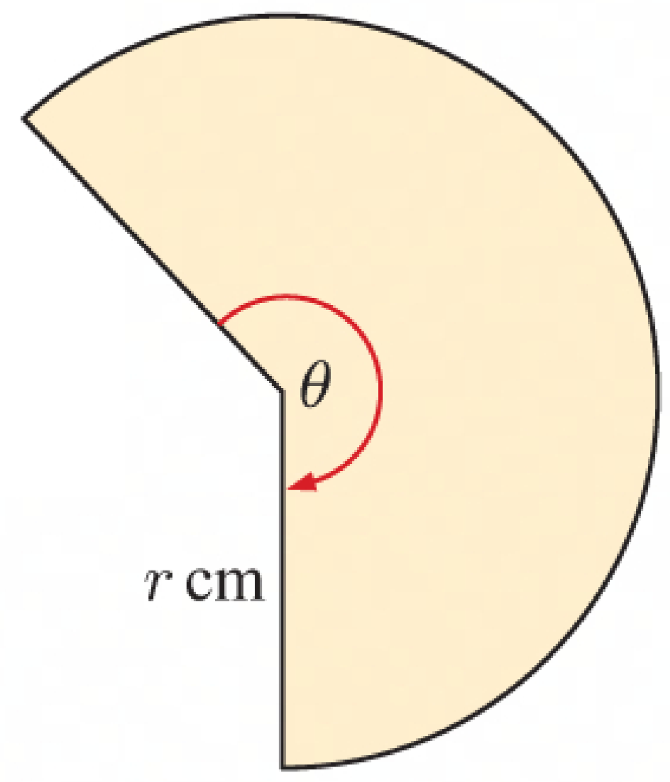
$$\text{area} = 2 \times \text{area of } \triangle\text{CDB} + \text{area of sector}$$
$$= 2 \times \tfrac{1}{2} \times \text{CD} \times \text{BD} + \left(\tfrac{\theta}{2\pi}\right) \times \pi \times r^2$$

Now $r^2 = 5^2 + 15^2 = 250$

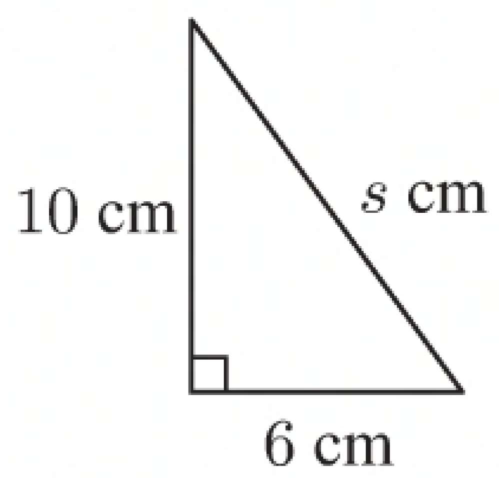
$$\therefore\ \text{area} \approx 2 \times \tfrac{1}{2} \times 15 \times 5 + \left(\frac{2.498}{2\pi}\right) \times \pi \times 250$$
$$\approx 387\ \text{m}^2$$

9





a



$$s^2 = 6^2 + 10^2 \quad \{\text{Pythagoras}\}$$
$$\therefore \quad s = \sqrt{6^2 + 10^2}$$
$$\therefore \quad s = \sqrt{136}$$
$$\therefore \quad s \approx 11.7$$

$\therefore$ slant length is about 11.7 cm.

b $r = s \approx 11.7$

c arc length = circumference of cone base

$$= 2\pi \times 6$$
$$= 12\pi$$
$$\approx 37.7 \text{ cm}$$

d arc length $= \theta r$

$$\therefore \quad 12\pi = \theta \times \sqrt{136}$$
$$\therefore \quad \theta = \frac{12\pi}{\sqrt{136}}$$
$$\therefore \quad \theta \approx 3.23^c$$

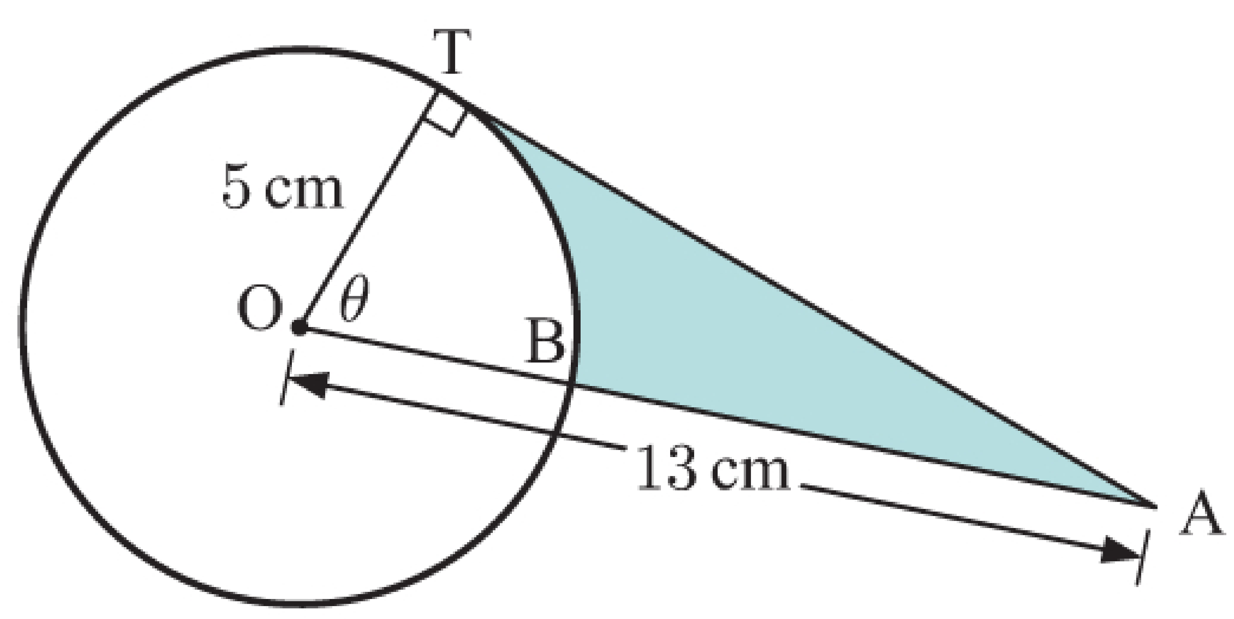
10 Since [AT] is a tangent, $\widehat{OTA}$ is a right angle.

We let $\widehat{TOA} = \theta$.

$$\therefore \quad \cos\theta = \tfrac{5}{13}$$
$$\therefore \quad \theta \approx 1.176^c$$

arc length BT $= \theta r$

$$\approx 1.176 \times 5$$
$$\approx 5.88 \text{ cm}$$



$$\text{AT}^2 + \text{OT}^2 = \text{OA}^2 \quad \{\text{Pythagoras}\}$$
$$\therefore \quad \text{AT}^2 = 13^2 - 5^2$$
$$\therefore \quad \text{AT} = 12 \text{ cm}$$

perimeter = AT + arc length BT + AB

$$\approx 12 + 5.88 + (13 - 5)$$
$$\approx 25.9 \text{ cm}$$

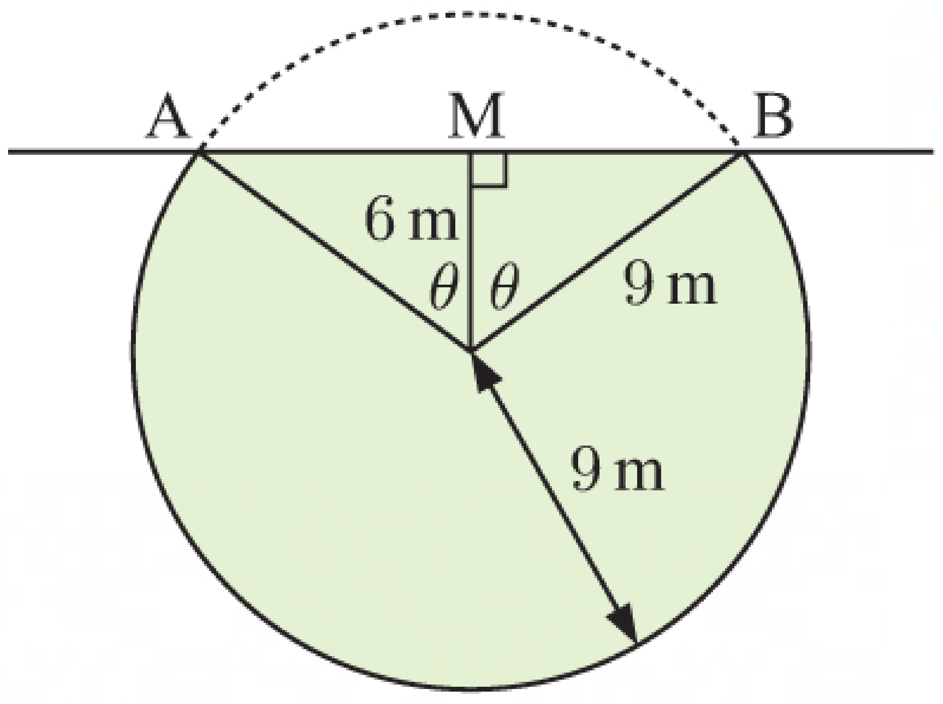
11 **a** $l = \left(\frac{\theta}{360}\right) \times 2\pi r$

$$= \frac{\frac{1}{60}}{360} \times 2 \times \pi \times 6370 \text{ km}$$
$$\approx 1.853 \text{ km}$$

b $\text{time} = \dfrac{\text{distance}}{\text{speed}}$

$$= \frac{2130 \text{ km}}{480 \text{ n miles h}^{-1}}$$
$$\approx \frac{2130 \text{ km}}{480 \times 1.853 \text{ km h}^{-1}}$$
$$\approx 2.395 \text{ hours}$$
$$\approx 2 \text{ hours } 24 \text{ min}$$

12



$\cos\theta = \frac{6}{9} = \frac{2}{3}$

$\therefore \quad \theta = \cos^{-1}\left(\frac{2}{3}\right)$

$\therefore \quad \theta \approx 0.841^c$

So, $\quad 2\pi - 2\theta \approx 4.601^c$

Now $\quad MB = \sqrt{9^2 - 6^2}$ {Pythagoras}

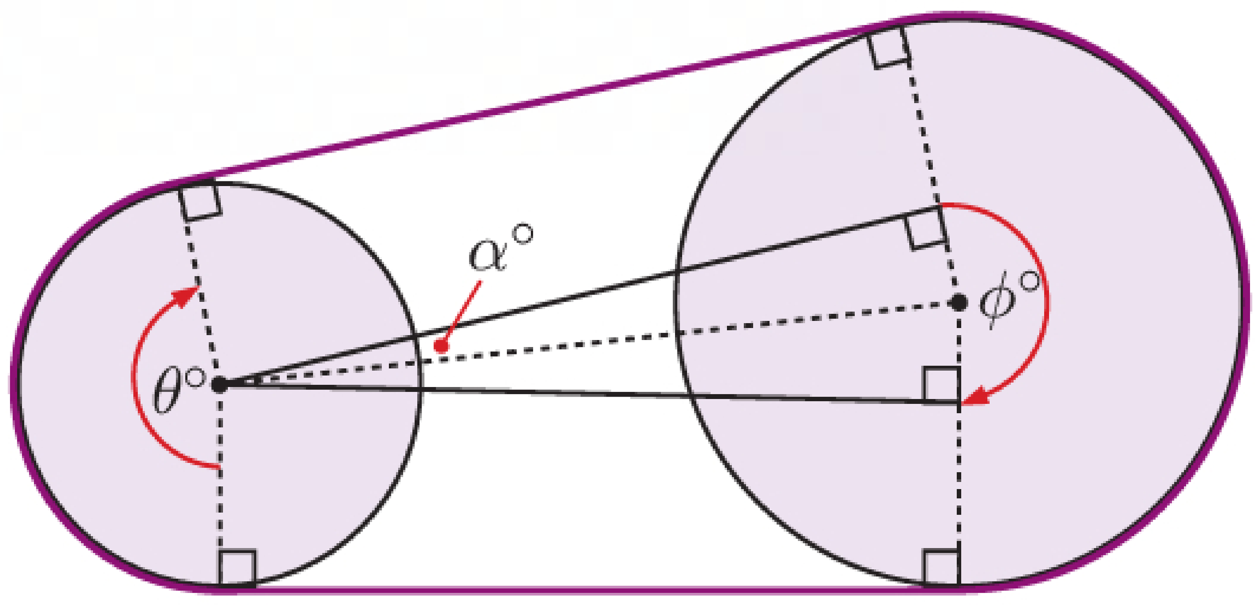
$= \sqrt{45}$

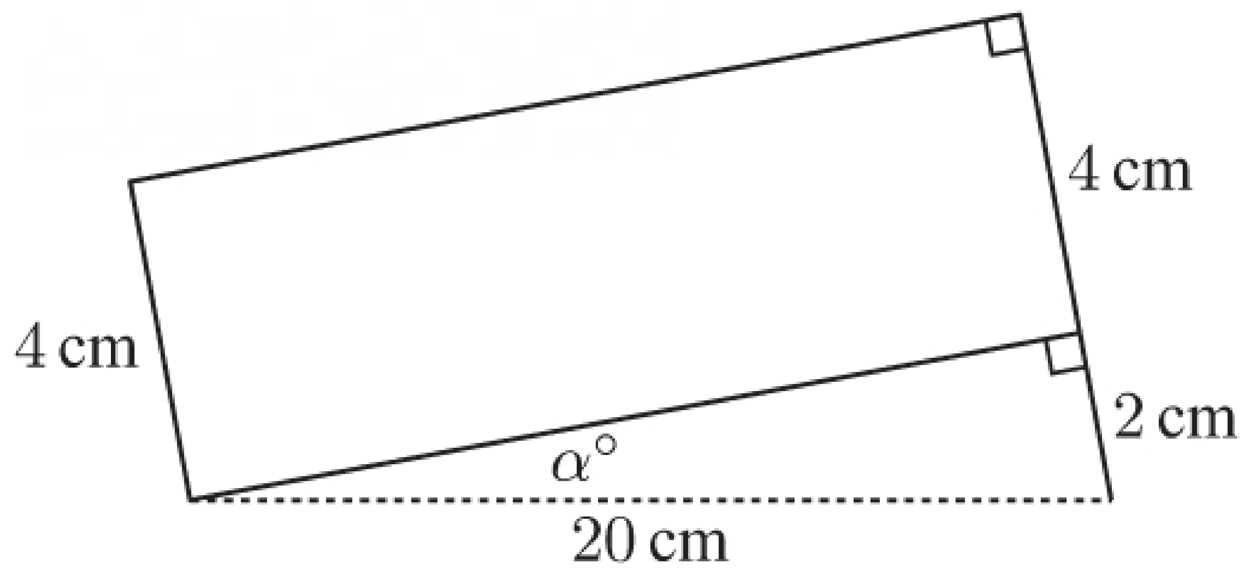
$\therefore$ available feeding area = area of $\triangle$ + area of sector

$\approx \frac{1}{2} \times 2 \times \sqrt{45} \times 6 + \frac{1}{2} \times 4.601 \times 9^2$

$\approx 227 \text{ m}^2$

13





a $\sin\alpha = \frac{2}{20} = 0.1$

$\therefore \quad \alpha = \sin^{-1}(0.1)$

$\therefore \quad \alpha \approx 5.739$

b $\theta + 90 + 90 + 2\alpha = 360$

$\therefore \quad \theta = 180 - 2\alpha$

$\approx 180 - 2 \times 5.739$

≈ 168.5

c $\phi + \theta = 360$

$\therefore \quad \phi \approx 360 - 168.5$

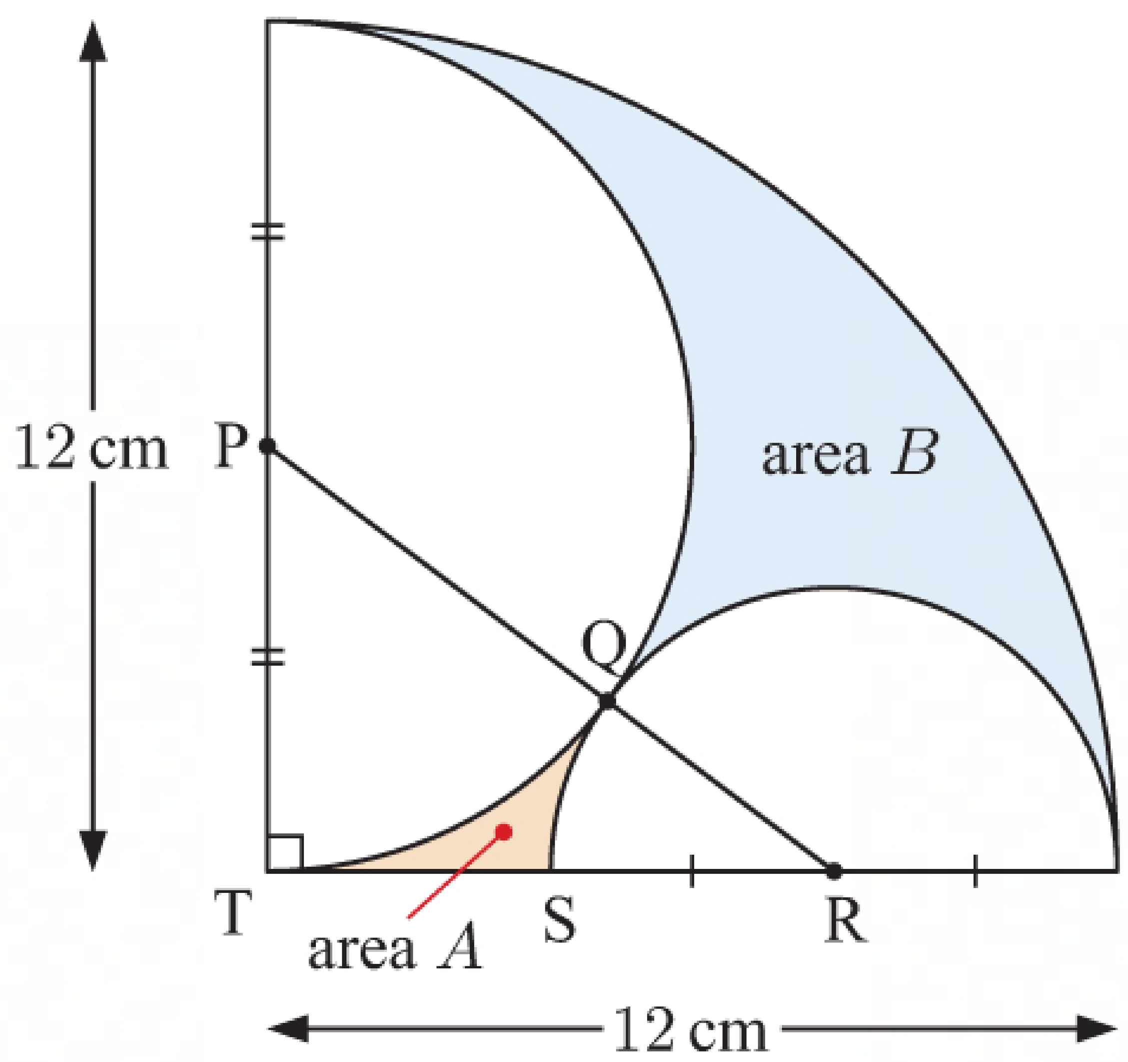
$\therefore \quad \phi \approx 191.5$

d length of belt

$= 2 \times \sqrt{20^2 - 2^2} + \left(\frac{\theta}{360}\right) \times 2\pi \times 4 + \left(\frac{\phi}{360}\right) \times 2\pi \times 6$

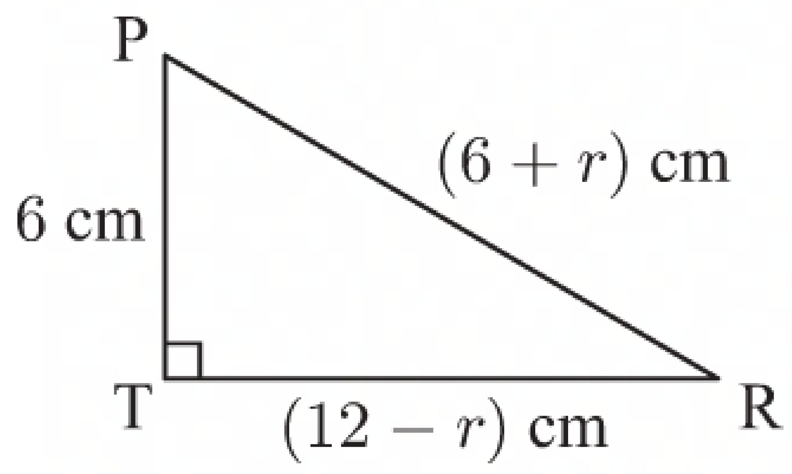
≈ 71.62 cm

14



a Let the smaller semi-circle have radius r cm.

In $\triangle$PTR, we have:



Thus $\quad (6+r)^2 = 6^2 + (12-r)^2$

$\therefore \quad 36 + 12r + r^2 = 36 + 144 - 24r + r^2$

$\therefore \quad 36r = 144$

$\therefore \quad r = 4$

b $\cos(\widehat{TPR}) = \frac{6}{10} = 0.6$

$\therefore \widehat{TPR} = \cos^{-1}(0.6) \approx 0.927^c$

$\widehat{PRT} = \frac{\pi}{2} - 0.927^c \approx 0.644^c$

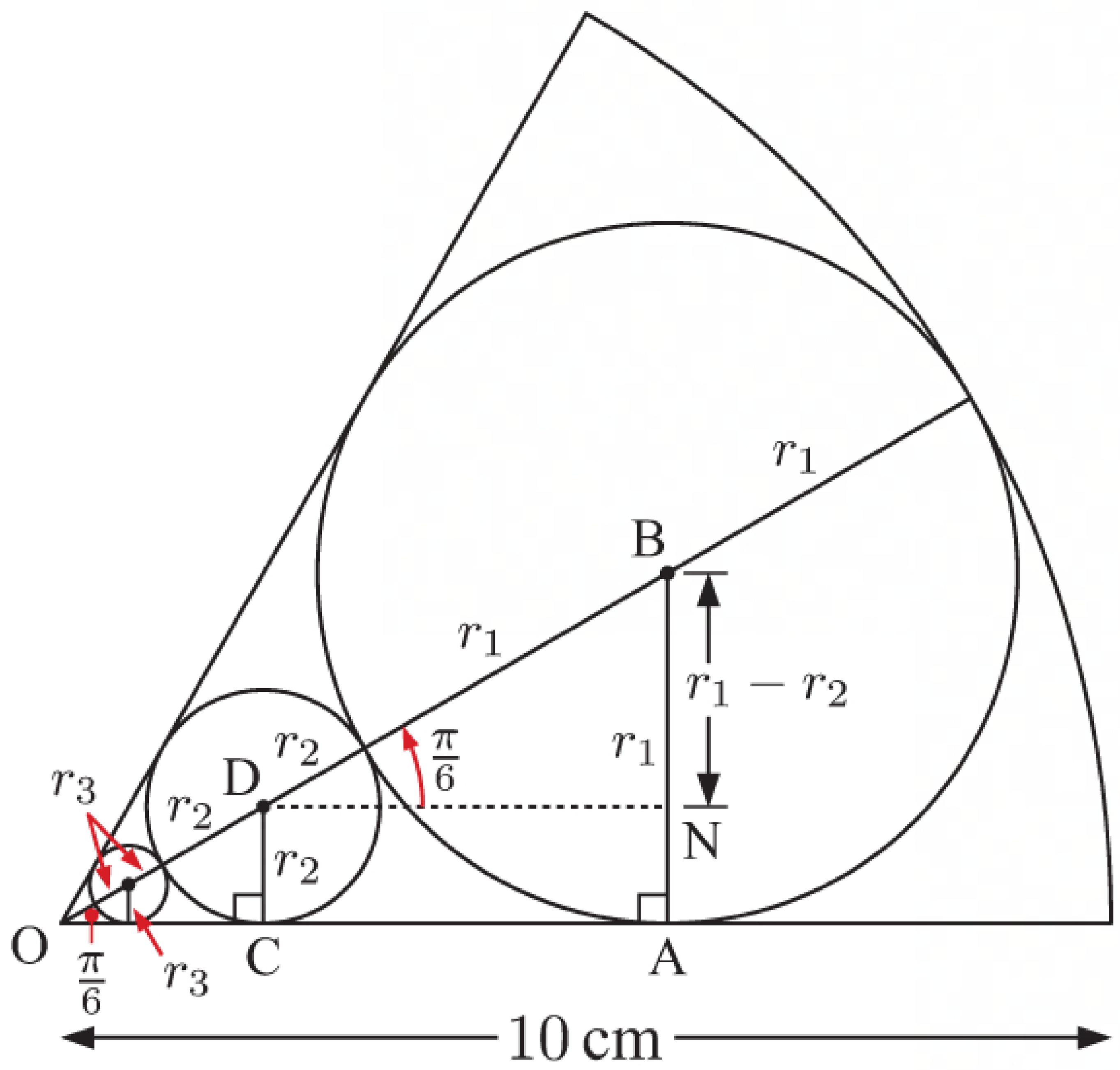
i area A = area $\triangle PTR$ − (area sector PQT + area sector RQS)

$$\approx \tfrac{1}{2}(8 \times 6) - \left(\tfrac{1}{2}(0.927)(6^2) + \tfrac{1}{2}(0.644)(4^2)\right)$$
$$\approx 24 - 16.69 - 5.15$$
$$\approx 2.16 \text{ cm}^2$$

ii area B = area of quarter circle − area of semi-circles − area A

$$\approx \tfrac{1}{4}\pi(12^2) - \tfrac{1}{2}\pi(6^2) - \tfrac{1}{2}\pi(4^2) - 2.16$$
$$\approx 36\pi - 18\pi - 8\pi - 2.16$$
$$\approx 10\pi - 2.16$$
$$\approx 29.3 \text{ cm}^2$$

15 **a**



$\sin\frac{\pi}{6} = \dfrac{r_1}{10 - r_1}$ {in $\triangle$OAB}

$\therefore \frac{1}{2} = \dfrac{r_1}{10 - r_1}$

$\therefore 10 - r_1 = 2r_1$

$\therefore 3r_1 = 10$

$\therefore r_1 = \frac{10}{3}$

$\therefore$ the largest circle has radius $\frac{10}{3}$ cm.

b In $\triangle$DBN, $\sin\frac{\pi}{6} = \dfrac{r_1 - r_2}{r_1 + r_2} = \frac{1}{2}$

$\therefore 2r_1 - 2r_2 = r_1 + r_2$

$\therefore r_1 = 3r_2$

$\therefore r_2 = \frac{1}{3}r_1$

So, in successive circles, radii are reduced by a factor of 3.

$\therefore r_2 = \frac{10}{9}, \; r_3 = \frac{10}{27}, \; r_4 = \frac{10}{81}$, and so on.

Thus the total area of the circles

$$= \pi {r_1}^2 + \pi {r_2}^2 + \pi {r_3}^2 + \pi {r_4}^2 + \ldots$$
$$= \pi\left(\left(\tfrac{10}{3}\right)^2 + \left(\tfrac{10}{9}\right)^2 + \left(\tfrac{10}{27}\right)^2 + \left(\tfrac{10}{81}\right)^2 + \ldots\right)$$
$$= \pi \times \left(\tfrac{10}{3}\right)^2 \left[1 + \left(\tfrac{1}{3}\right)^2 + \left(\tfrac{1}{3}\right)^4 + \left(\tfrac{1}{3}\right)^6 + \ldots\right]$$
$$= \pi \times \frac{100}{9} \times \left(\frac{1}{1 - \frac{1}{9}}\right) \quad \{\text{infinite geometric series with } u_1 = 1, \; r = \tfrac{1}{9}\}$$
$$= \frac{25\pi}{2} \text{ units}^2$$

c area of sector $= \frac{1}{2}\theta r^2$

$$= \frac{1}{2} \times \frac{\pi}{3} \times 10^2$$

$$= \frac{50\pi}{3}$$

$\therefore$ fraction of sector occupied by circles $= \dfrac{\text{total area of circles}}{\text{area of sector}}$

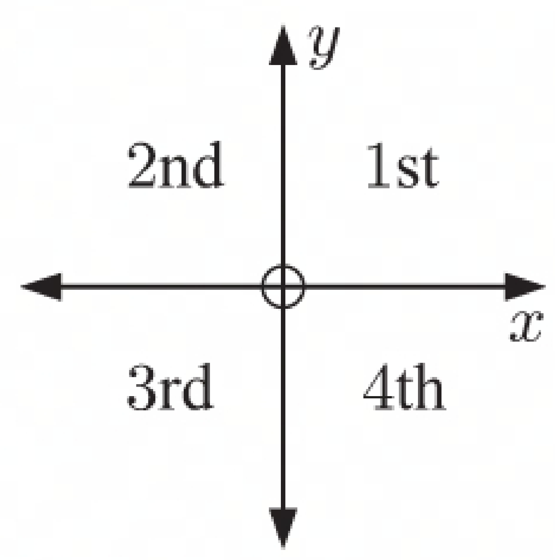
$$= \frac{\frac{25\pi}{2}}{\frac{50\pi}{3}}$$

$$= \frac{3}{4}$$

INVESTIGATION THE TRIGONOMETRIC RATIOS

1

Quadrant	$\cos\theta$	$\sin\theta$	$\tan\theta$
1	positive	positive	positive
2	negative	positive	negative
3	negative	negative	positive
4	positive	negative	negative

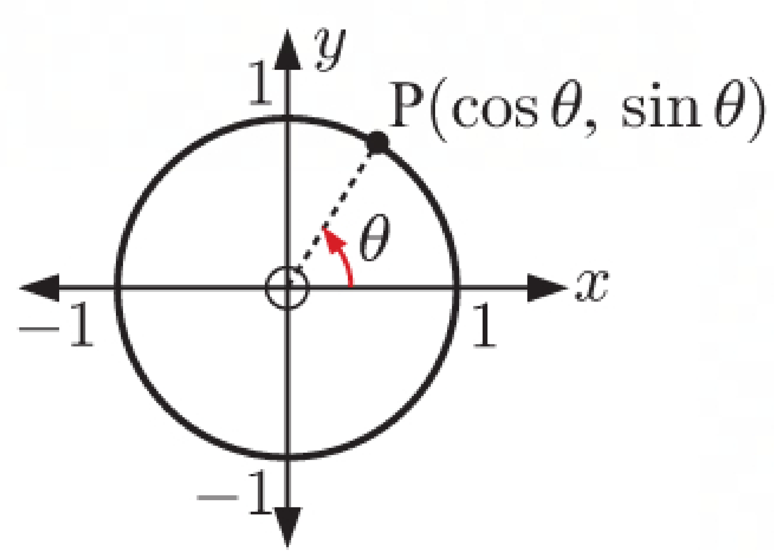


2
- $\sin\theta$, $\cos\theta$, and $\tan\theta$ are all positive in quadrant 1
- only $\sin\theta$ is positive in quadrant 2
- only $\tan\theta$ is positive in quadrant 3
- only $\cos\theta$ is positive in quadrant 4.

EXERCISE 8C

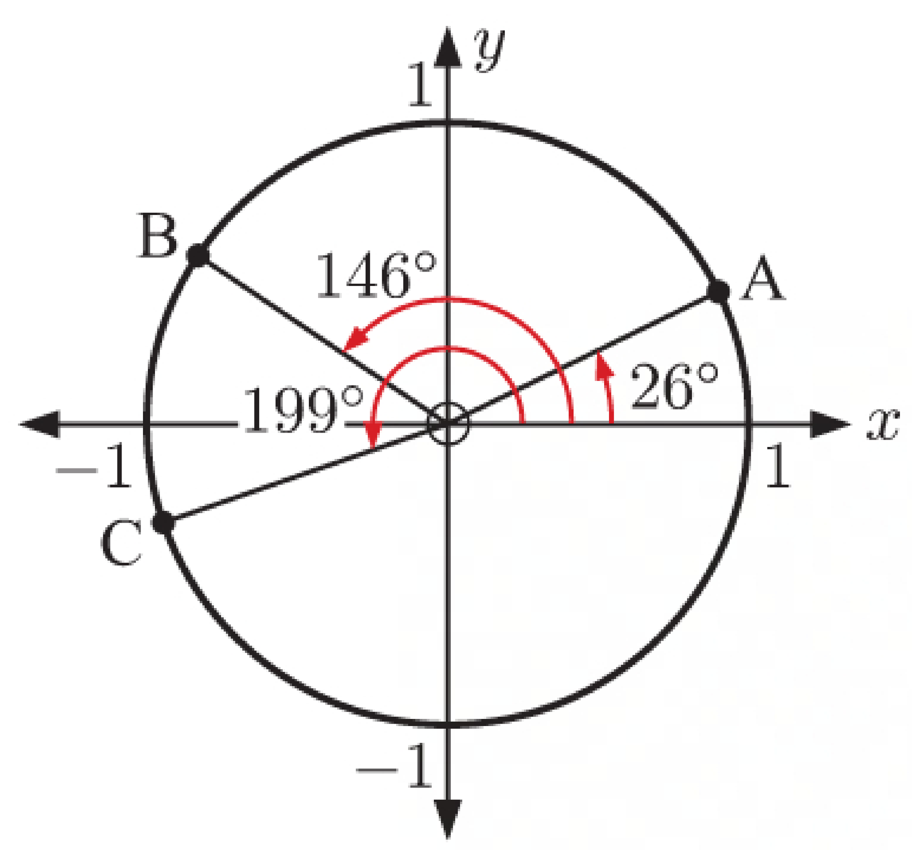
1 Since any point P on the unit circle has coordinates $(\cos\theta, \sin\theta)$, then:

θ (degrees)	$0°$	$90°$	$180°$	$270°$	$360°$	$450°$
θ (radians)	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π	$\frac{5\pi}{2}$
sine	0	1	0	-1	0	1
cosine	1	0	-1	0	1	0
tangent	0	undefined	0	undefined	0	undefined



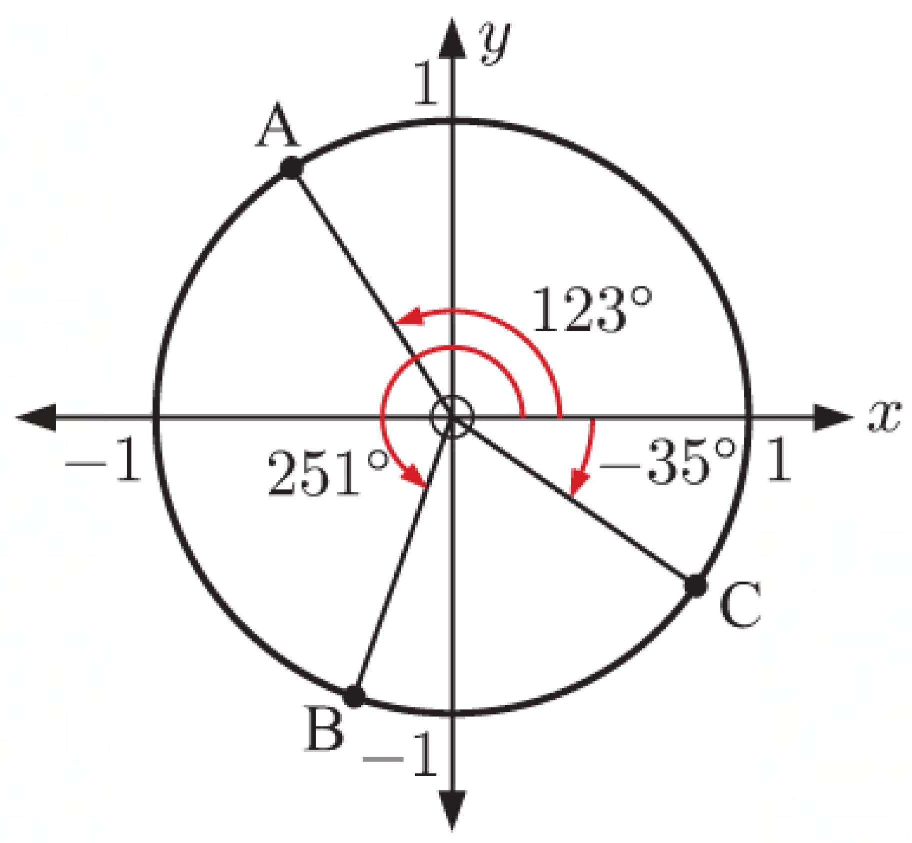
2 **a** **i** $A(\cos 26°, \sin 26°)$, $B(\cos 146°, \sin 146°)$, $C(\cos 199°, \sin 199°)$

ii $A(0.899, 0.438)$, $B(-0.829, 0.559)$, $C(-0.946, -0.326)$



b **i** A($\cos 123°$, $\sin 123°$), B($\cos 251°$, $\sin 251°$), C($\cos(-35°)$, $\sin(-35°)$)

ii A$(-0.545, 0.839)$, B$(-0.326, -0.946)$, C$(0.819, -0.574)$



3 **a** **i** $\frac{1}{\sqrt{2}} \approx 0.707$ **ii** $\frac{\sqrt{3}}{2} \approx 0.866$

b

θ (degrees)	$30°$	$45°$	$60°$	$135°$	$150°$	$240°$	$315°$
θ (radians)	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	$\frac{4\pi}{3}$	$\frac{7\pi}{4}$
sine	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	$-\frac{\sqrt{3}}{2}$	$-\frac{1}{\sqrt{2}}$
cosine	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	$-\frac{1}{\sqrt{2}}$	$-\frac{\sqrt{3}}{2}$	$-\frac{1}{2}$	$\frac{1}{\sqrt{2}}$
tangent	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	-1	$-\frac{1}{\sqrt{3}}$	$\sqrt{3}$	-1

4

Quadrant	*Degree measure*	*Radian measure*	$\cos\theta$	$\sin\theta$	$\tan\theta$
1	$0° < \theta < 90°$	$0 < \theta < \frac{\pi}{2}$	positive	positive	positive
2	$90° < \theta < 180°$	$\frac{\pi}{2} < \theta < \pi$	negative	positive	negative
3	$180° < \theta < 270°$	$\pi < \theta < \frac{3\pi}{2}$	negative	negative	positive
4	$270° < \theta < 360°$	$\frac{3\pi}{2} < \theta < 2\pi$	positive	negative	negative

5 **a** $\cos\theta$ is positive in quadrants 1 and 4.

b $\cos\theta$ is negative in quadrants 2 and 3.

c $\cos\theta$ and $\sin\theta$ are both negative in quadrant 3.

d $\cos\theta$ is negative and $\sin\theta$ is positive in quadrant 2.

6 **a**
$$\begin{aligned} &\cos 400° \\ &= \cos(360° + 40°) \\ &= \cos 40° \end{aligned}$$

b
$$\begin{aligned} &\sin \tfrac{5\pi}{7} \\ &= \sin\left(\tfrac{5\pi}{7} + 2\pi\right) \\ &= \sin \tfrac{19\pi}{7} \end{aligned}$$

c
$$\begin{aligned} &\tan \tfrac{13\pi}{8} \\ &= \tan\left(\tfrac{13\pi}{8} - 3\pi\right) \\ &= \tan\left(-\tfrac{11\pi}{8}\right) \end{aligned}$$

7
$$\begin{aligned} \tan 230° &= \tan(180° + 50°) \\ &= \tan 50° \end{aligned}$$

$\therefore$ **B** and **D** have the same value.

8
$$\begin{aligned} \sin 40° &= \sin \tfrac{40\times\pi}{180} \\ &= \sin \tfrac{2\pi}{9} \end{aligned}$$

$\therefore$ **B** and **E** have the same value.

9 **a** **i** $\sin 100° \approx 0.985$ **ii** $\sin 80° \approx 0.985$ **iii** $\sin 120° \approx 0.866$

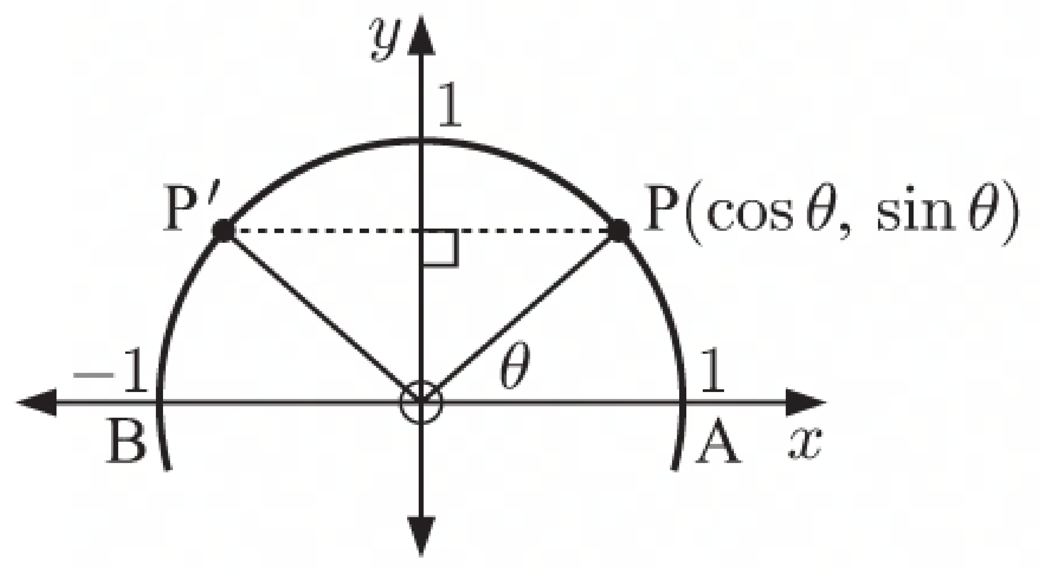
iv $\sin 60° \approx 0.866$ **v** $\sin 150° = 0.5$ **vi** $\sin 30° = 0.5$

vii $\sin 45° \approx 0.707$ **viii** $\sin 135° \approx 0.707$

b $\sin(180° - \theta) = \sin\theta$ as the points have the same y-coordinate.

c $\sin(\pi - \theta) = \sin\theta$

d



The diagram shows P reflected in the y-axis to P′, so $P'\widehat{O}B = P\widehat{O}A = \theta$, and P′ has coordinates $(-\cos\theta, \sin\theta)$.

But $A\widehat{O}P' = 180° - \theta$ $\{A\widehat{O}P' + P'\widehat{O}B = 180°\}$, so P′ has coordinates $(\cos(180° - \theta), \sin(180° - \theta))$.

$\therefore$ $\sin(180° - \theta) = \sin\theta$ {equating y-coordinates of P′}

e **i** $180° - 45° = 135°$ **ii** $180° - 51° = 129°$ **iii** $\pi - \frac{\pi}{3} = \frac{2\pi}{3}$

iv $\pi - \frac{\pi}{6} = \frac{5\pi}{6}$ {using $\sin(\pi - \theta) = \theta$}

10 **a** **i** $\cos 70° \approx 0.342$ **ii** $\cos 110° \approx -0.342$ **iii** $\cos 60° = 0.5$

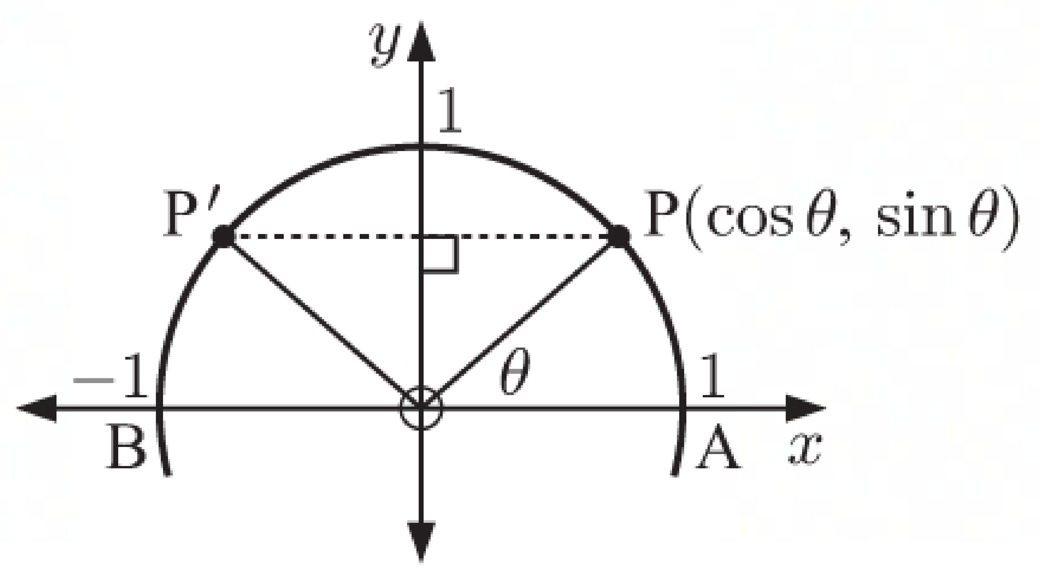
iv $\cos 120° = -0.5$ **v** $\cos 25° \approx 0.906$ **vi** $\cos 155° \approx -0.906$

vii $\cos 80° \approx 0.174$ **viii** $\cos 100° \approx -0.174$

b $\cos(180° - \theta) = -\cos\theta$

c $\cos(\pi - \theta) = -\cos\theta$

d



The diagram shows P reflected in the y-axis to P′, so $P'\widehat{O}B = P\widehat{O}A = \theta$, and P′ has coordinates $(-\cos\theta, \sin\theta)$.

But $A\widehat{O}P' = 180° - \theta$, so P′ has coordinates $(\cos(180° - \theta), \sin(180° - \theta))$.

$\therefore$ $\cos(180° - \theta) = -\cos\theta$ {equating x-coordinates of P′}

e **i** $180° - 40° = 140°$ **ii** $180° - 19° = 161°$ **iii** $\pi - \frac{\pi}{5} = \frac{4\pi}{5}$

iv $\pi - \frac{2\pi}{5} = \frac{3\pi}{5}$ {using $\cos(\pi - \theta) = -\cos\theta$}

11 $\tan\theta = \dfrac{\sin\theta}{\cos\theta}$

$$\begin{aligned}\therefore\ \tan(\pi - \theta) &= \frac{\sin(\pi - \theta)}{\cos(\pi - \theta)}\\ &= \frac{\sin\theta}{-\cos\theta}\\ &= -\frac{\sin\theta}{\cos\theta}\\ &= -\tan\theta\end{aligned}$$

12 **a** $\sin 137°$
$= \sin(180 - 137)°$
$= \sin 43°$
≈ 0.6820

b $\sin 59°$
$= \sin(180 - 59)°$
$= \sin 121°$
≈ 0.8572

c $\cos 143°$
$= -\cos(180 - 143)°$
$= -\cos 37°$
≈ -0.7986

d $\cos 24°$
$= -\cos(180 - 24)°$
$= -\cos 156°$
≈ 0.9135

e $\sin 115°$
$= \sin(180 - 115)°$
$= \sin 65°$
≈ 0.9063

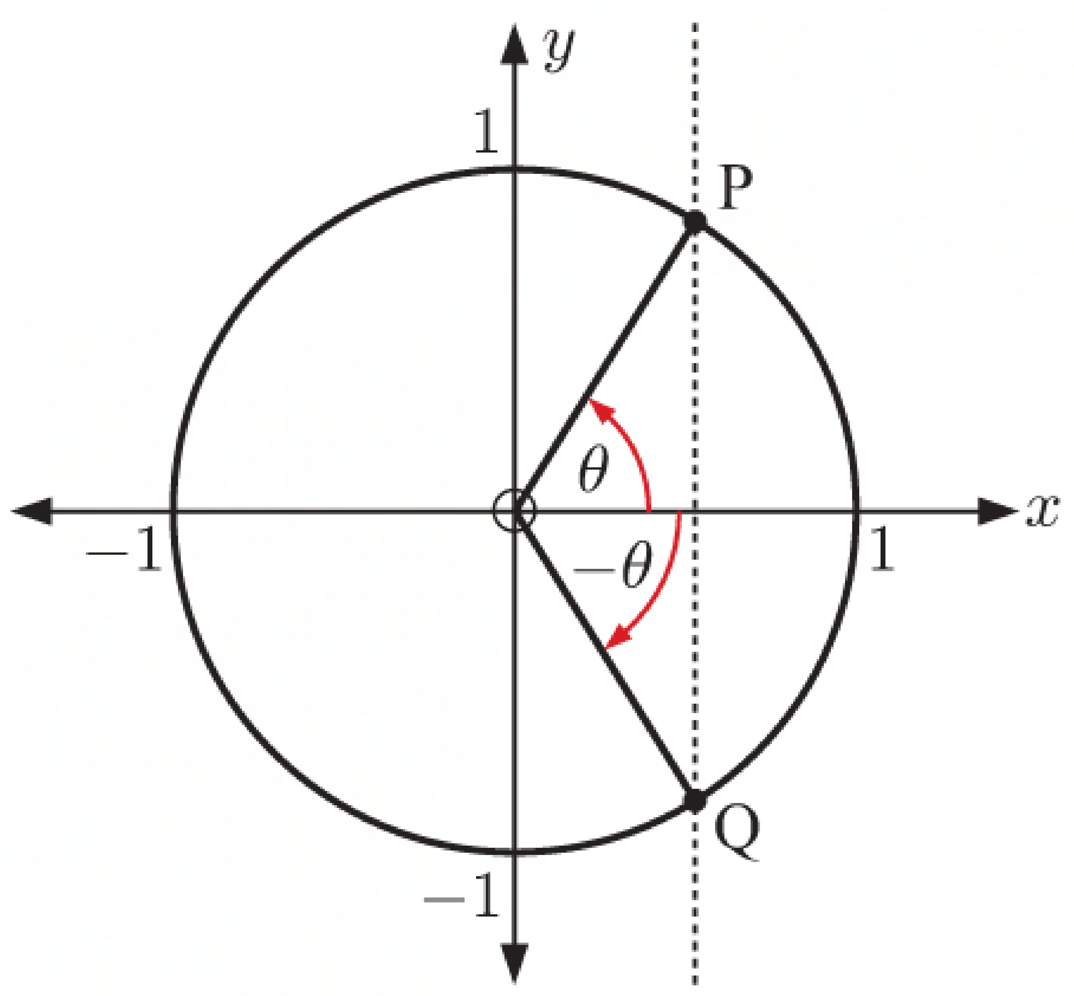
f $\cos 132°$
$= -\cos(180 - 132)°$
$= -\cos 48°$
≈ -0.6691

13 **a**

θ (radians)	$\sin\theta$	$\sin(-\theta)$	$\cos\theta$	$\cos(-\theta)$
0.75	≈ 0.682	≈ -0.682	≈ 0.732	≈ 0.732
1.772	≈ 0.980	≈ -0.980	≈ -0.200	≈ -0.200
3.414	≈ -0.269	≈ 0.269	≈ -0.963	≈ -0.963
6.25	≈ -0.0332	≈ 0.0332	≈ 0.999	≈ 0.999
-1.17	≈ -0.921	≈ 0.921	≈ 0.390	≈ 0.390

b We deduce that $\sin(-\theta) = -\sin\theta$ and $\cos(-\theta) = \cos\theta$.

c



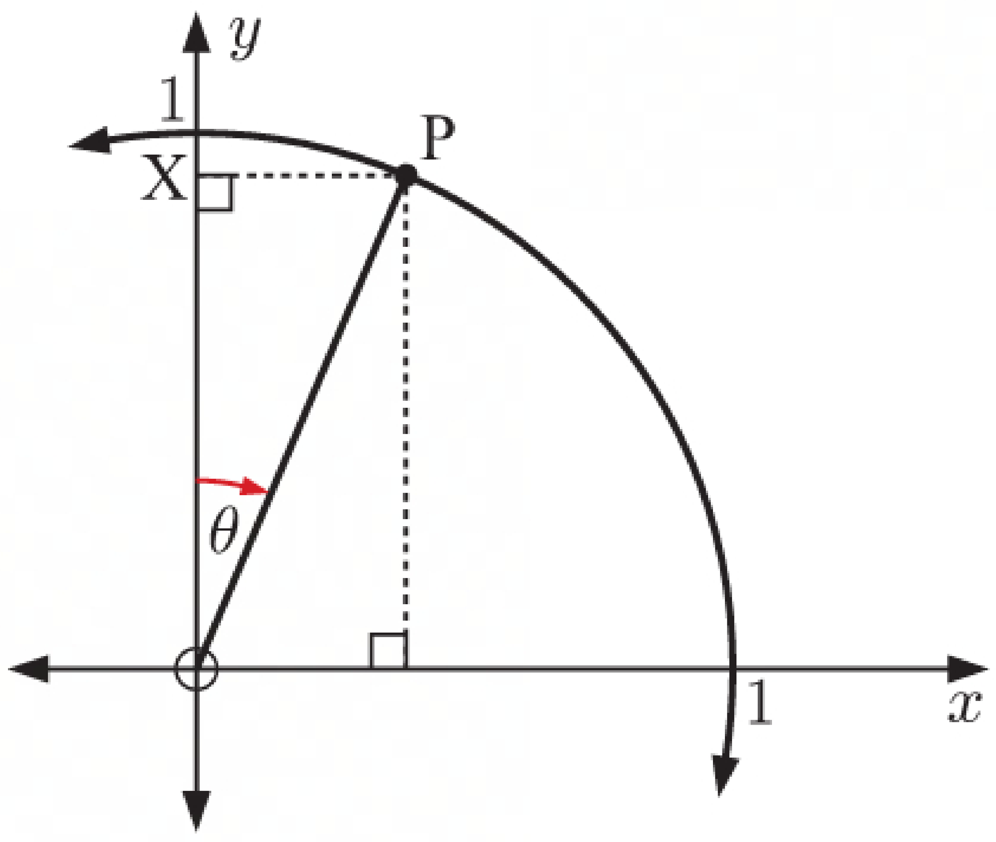
P is reflected in the x-axis to Q, so Q has coordinates $(\cos\theta, -\sin\theta)$.
But Q has coordinates $(\cos(-\theta), \sin(-\theta))$.
$\therefore$ $Q(\cos(-\theta), \sin(-\theta)) = Q(\cos\theta, -\sin\theta)$.
$\therefore$ $\cos(-\theta) = \cos\theta$ and $\sin(-\theta) = -\sin\theta$
So the deduction is correct.

d There are 2π radians in a circle.
$\therefore$ we could write Q with coordinates $(\cos(2\pi - \theta), \sin(2\pi - \theta))$.
From **c** we know that Q has coordinates $(\cos\theta, -\sin\theta)$
$\therefore$ $\cos(2\pi - \theta) = \cos\theta$ and $\sin(2\pi - \theta) = -\sin\theta$

e $$\tan(2\pi - \theta) = \frac{\sin(2\pi - \theta)}{\cos(2\pi - \theta)} = \frac{-\sin\theta}{\cos\theta} = -\frac{\sin\theta}{\cos\theta} = -\tan\theta$$

14 **a** The angle between [OP] and the positive x-axis is $\left(\frac{\pi}{2}-\theta\right)$.

$\therefore$ P is $\left(\cos\left(\frac{\pi}{2}-\theta\right),\ \sin\left(\frac{\pi}{2}-\theta\right)\right)$



b **i** In $\triangle$OXP, $\sin\theta = \dfrac{\text{XP}}{\text{OP}} = \dfrac{\text{XP}}{1}$

$\therefore$ XP $= \sin\theta$

ii In $\triangle$OXP, $\cos\theta = \dfrac{\text{OX}}{\text{OP}} = \dfrac{\text{OX}}{1}$

$\therefore$ OX $= \cos\theta$

c **i** $\cos\left(\frac{\pi}{2}-\theta\right) = \text{XP} = \sin\theta$

ii $\sin\left(\frac{\pi}{2}-\theta\right) = \text{OX} = \cos\theta$

d **i** $\cos\frac{\pi}{5} = \sin\left(\frac{\pi}{2}-\frac{\pi}{5}\right) = \sin\frac{3\pi}{10} \approx 0.809$

ii $\sin\frac{\pi}{8} = \cos\left(\frac{\pi}{2}-\frac{\pi}{8}\right) = \cos\frac{3\pi}{8} \approx 0.383$

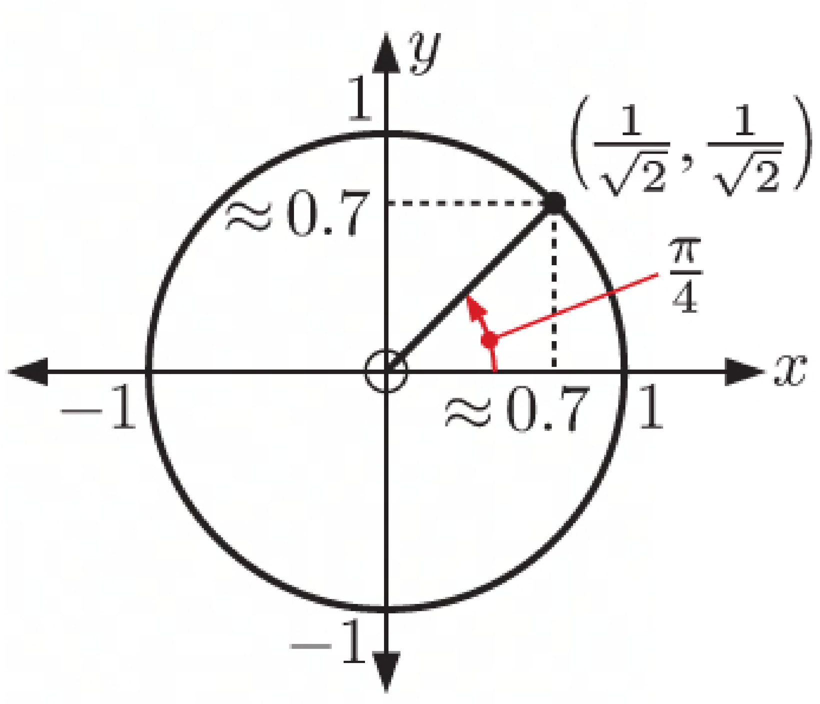
e $\tan\left(\frac{\pi}{2}-\theta\right) = \dfrac{\sin(\frac{\pi}{2}-\theta)}{\cos(\frac{\pi}{2}-\theta)}$

$= \dfrac{\cos\theta}{\sin\theta}$

$= \dfrac{1}{\tan\theta}$

EXERCISE 8D

1 **a**

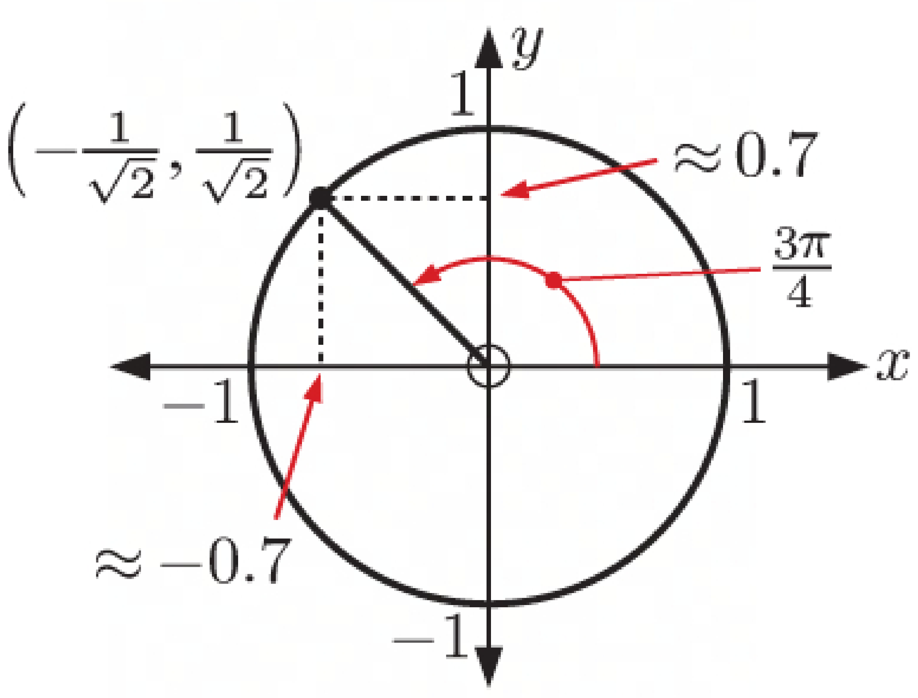


$\sin\frac{\pi}{4} = \frac{1}{\sqrt{2}}$

$\cos\frac{\pi}{4} = \frac{1}{\sqrt{2}}$

$\tan\frac{\pi}{4} = \dfrac{\frac{1}{\sqrt{2}}}{\frac{1}{\sqrt{2}}} = 1$

b

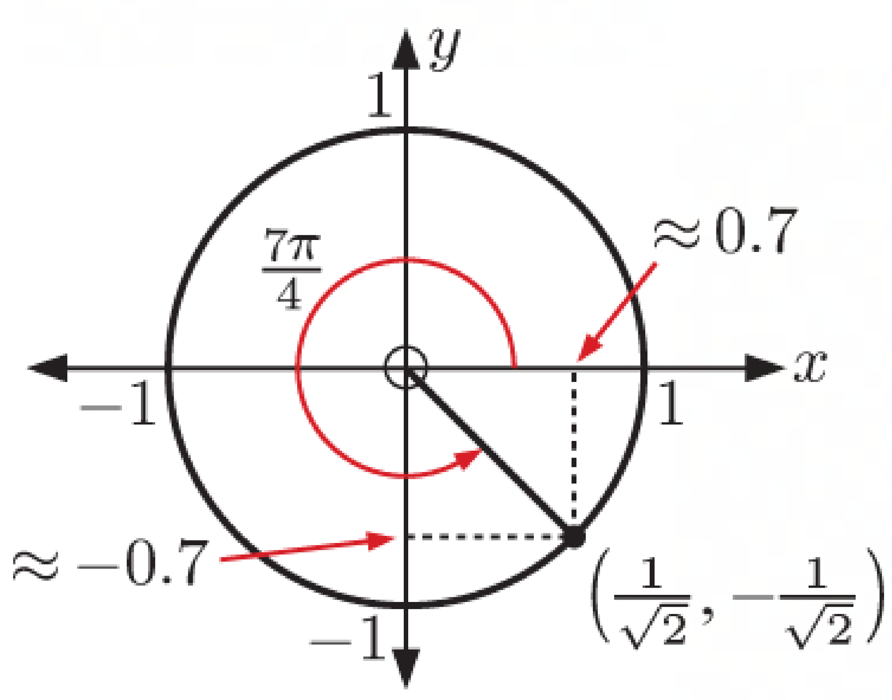


$\sin\frac{3\pi}{4} = \frac{1}{\sqrt{2}}$

$\cos\frac{3\pi}{4} = -\frac{1}{\sqrt{2}}$

$\tan\frac{3\pi}{4} = \dfrac{\frac{1}{\sqrt{2}}}{-\frac{1}{\sqrt{2}}} = -1$

c

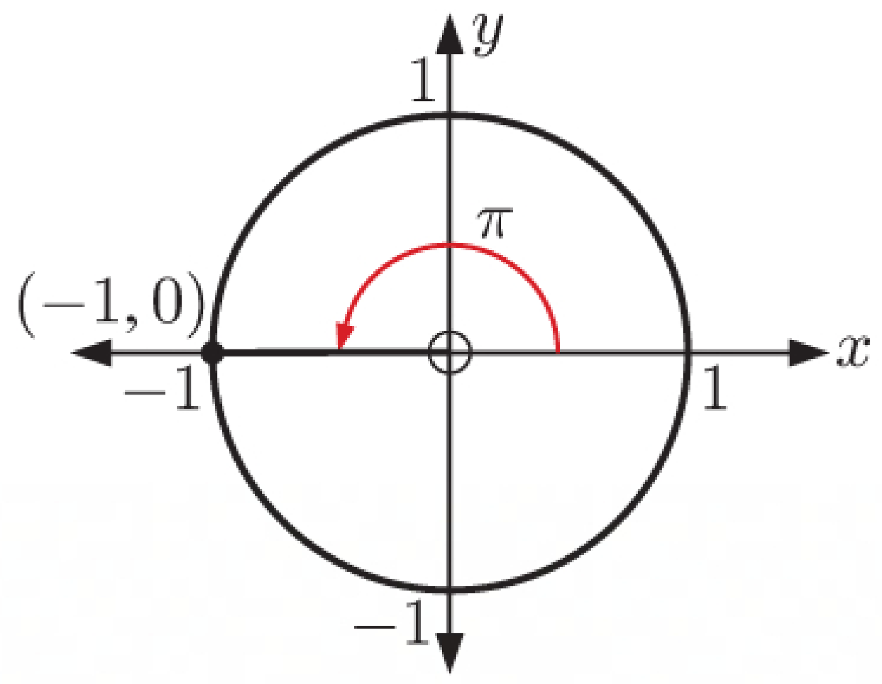


$\sin\frac{7\pi}{4} = -\frac{1}{\sqrt{2}}$

$\cos\frac{7\pi}{4} = \frac{1}{\sqrt{2}}$

$\tan\frac{7\pi}{4} = \dfrac{-\frac{1}{\sqrt{2}}}{\frac{1}{\sqrt{2}}} = -1$

d

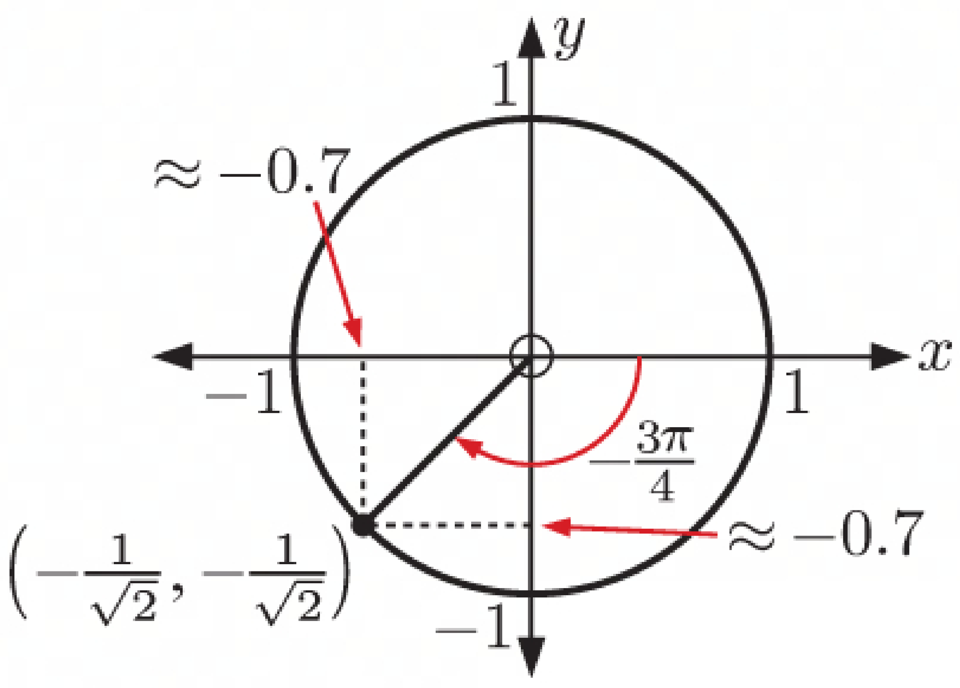


$$\sin \pi = 0$$

$$\cos \pi = -1$$

$$\tan \pi = \frac{0}{-1} = 0$$

e

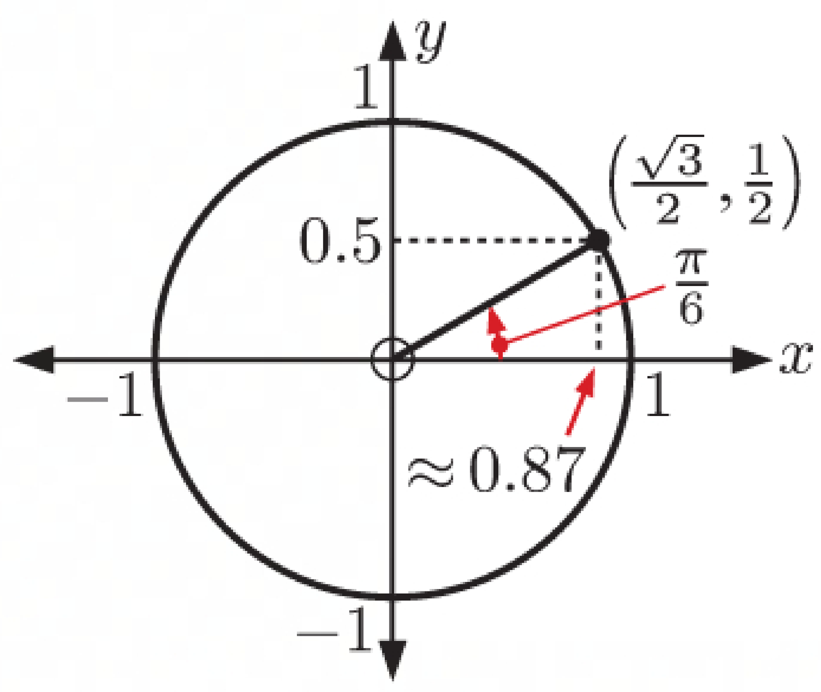


$$\sin\left(-\tfrac{3\pi}{4}\right) = -\tfrac{1}{\sqrt{2}}$$

$$\cos\left(-\tfrac{3\pi}{4}\right) = -\tfrac{1}{\sqrt{2}}$$

$$\tan\left(-\tfrac{3\pi}{4}\right) = \frac{-\frac{1}{\sqrt{2}}}{-\frac{1}{\sqrt{2}}} = 1$$

2 **a**

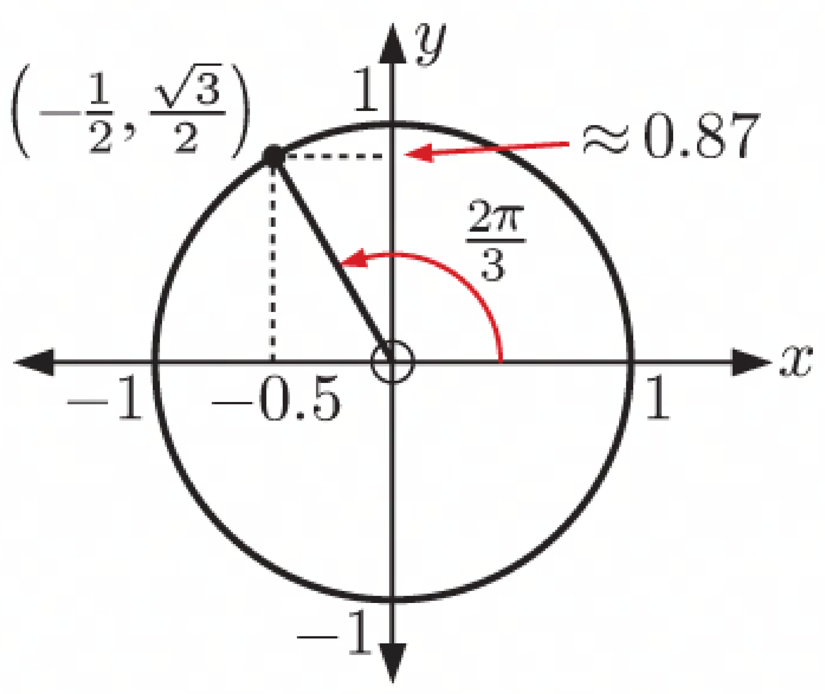


$$\sin \tfrac{\pi}{6} = \tfrac{1}{2}$$

$$\cos \tfrac{\pi}{6} = \tfrac{\sqrt{3}}{2}$$

$$\tan \tfrac{\pi}{6} = \frac{\frac{1}{2}}{\frac{\sqrt{3}}{2}} = \frac{1}{\sqrt{3}}$$

b

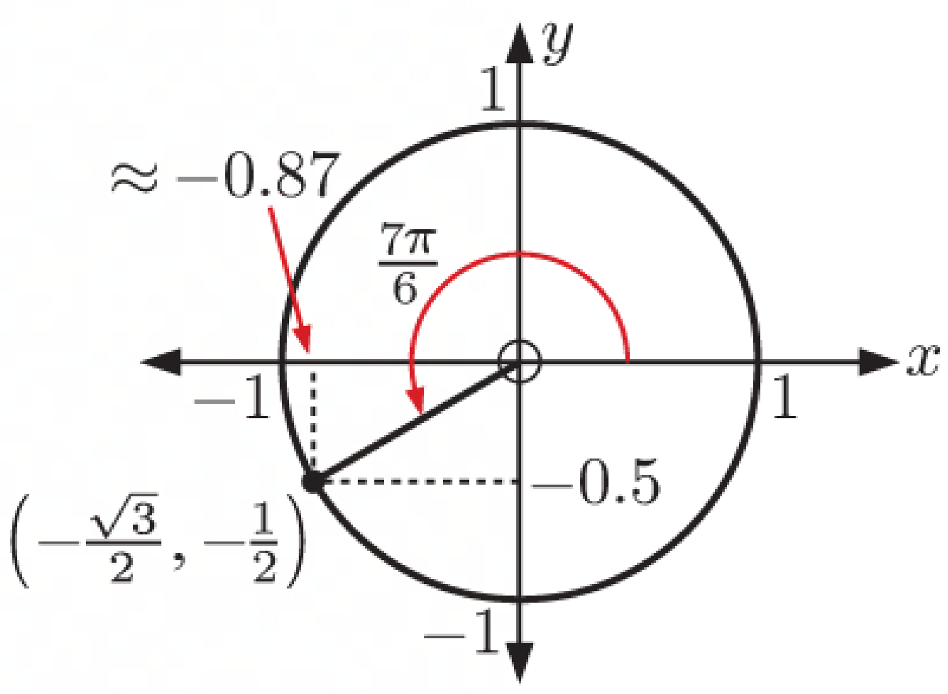


$$\sin \tfrac{2\pi}{3} = \tfrac{\sqrt{3}}{2}$$

$$\cos \tfrac{2\pi}{3} = -\tfrac{1}{2}$$

$$\tan \tfrac{2\pi}{3} = \frac{\frac{\sqrt{3}}{2}}{-\frac{1}{2}} = -\sqrt{3}$$

c

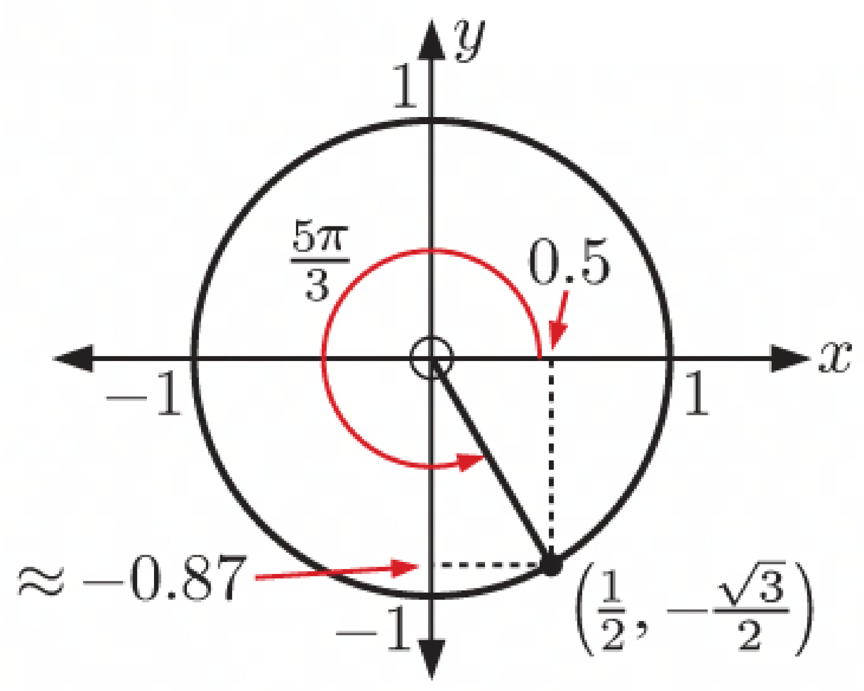


$$\sin \tfrac{7\pi}{6} = -\tfrac{1}{2}$$

$$\cos \tfrac{7\pi}{6} = -\tfrac{\sqrt{3}}{2}$$

$$\tan \tfrac{7\pi}{6} = \frac{-\frac{1}{2}}{-\frac{\sqrt{3}}{2}} = \frac{1}{\sqrt{3}}$$

d

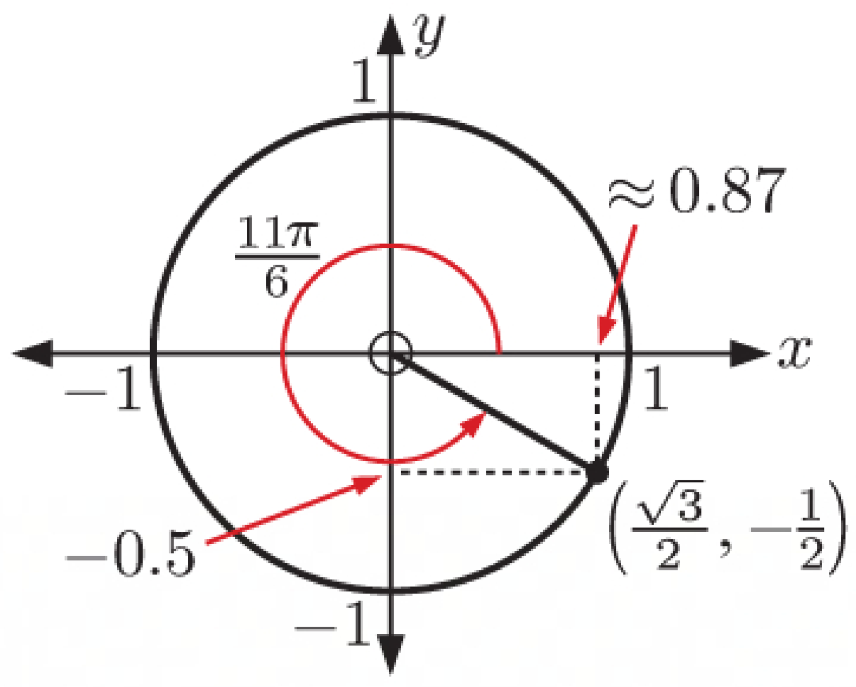


$$\sin \tfrac{5\pi}{3} = -\tfrac{\sqrt{3}}{2}$$

$$\cos \tfrac{5\pi}{3} = \tfrac{1}{2}$$

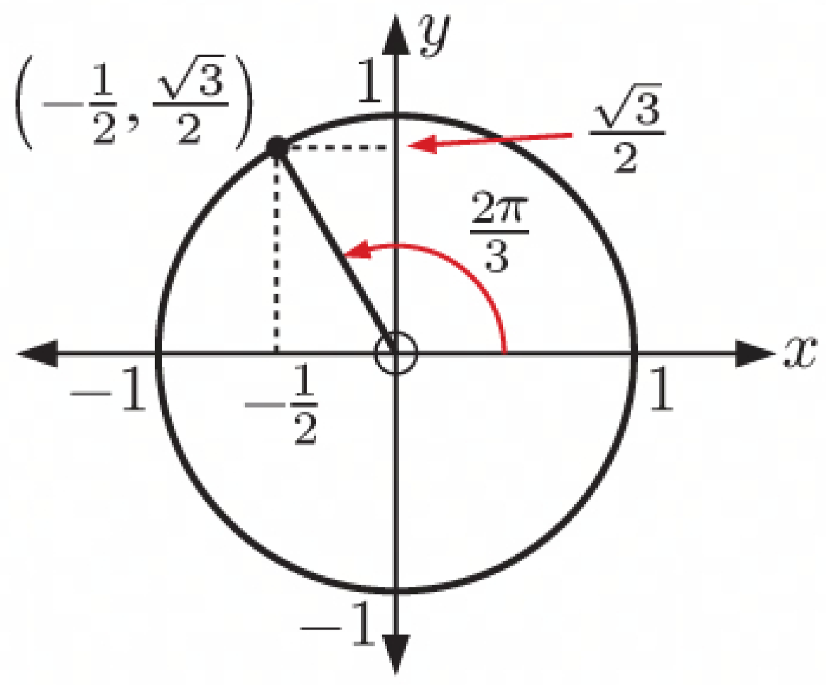
$$\tan \tfrac{5\pi}{3} = \frac{-\frac{\sqrt{3}}{2}}{\frac{1}{2}} = -\sqrt{3}$$

e



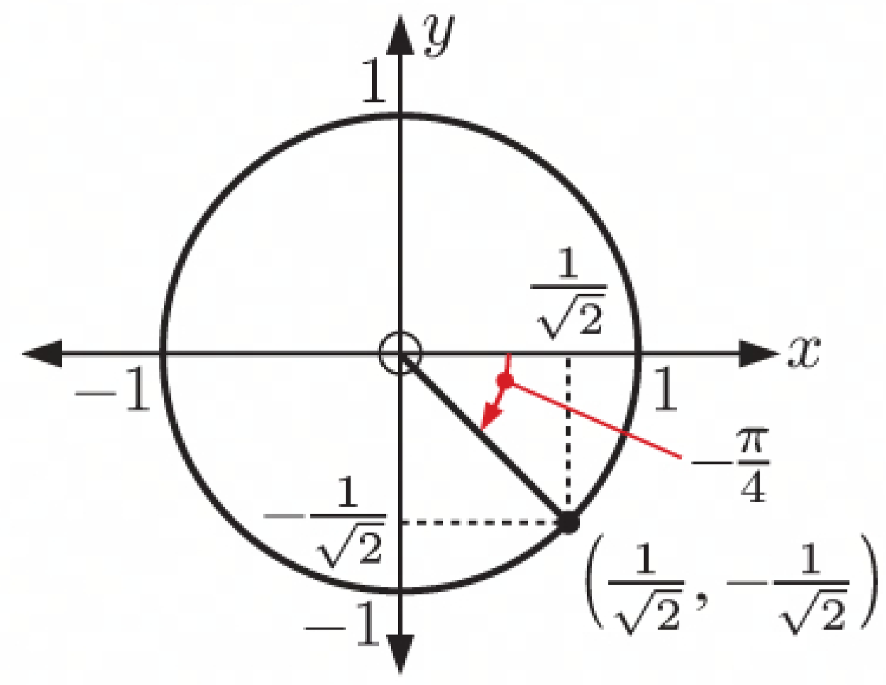
$$\sin \frac{11\pi}{6} = -\frac{1}{2}$$
$$\cos \frac{11\pi}{6} = \frac{\sqrt{3}}{2}$$
$$\tan \frac{11\pi}{6} = \frac{-\frac{1}{2}}{\frac{\sqrt{3}}{2}} = -\frac{1}{\sqrt{3}}$$

3 **a** $\frac{2\pi}{3}$ is a multiple of $\frac{\pi}{6}$



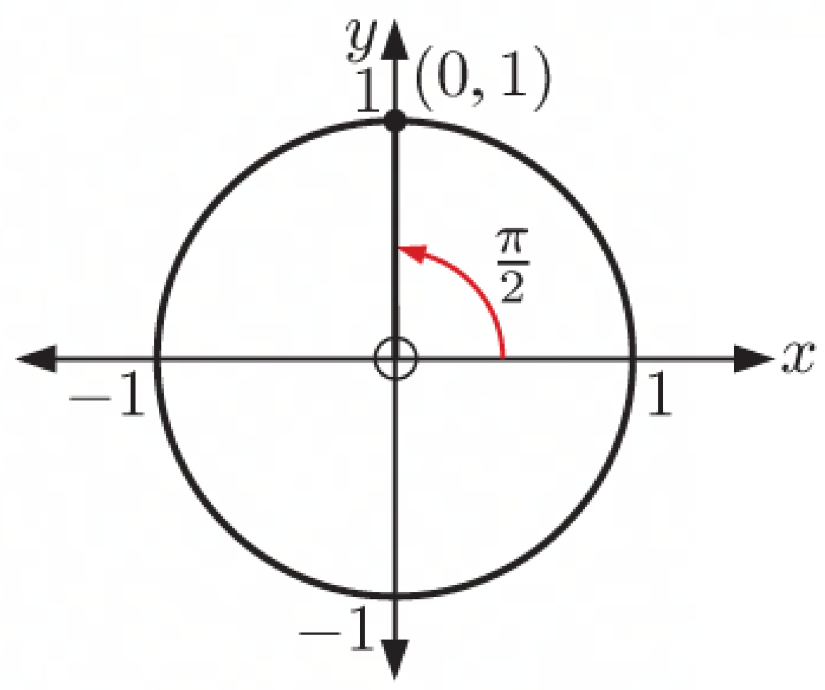
So,
$$\cos \frac{2\pi}{3} = -\frac{1}{2}$$
$$\sin \frac{2\pi}{3} = \frac{\sqrt{3}}{2}$$
$$\tan \frac{2\pi}{3} = -\sqrt{3}$$

b $-\frac{\pi}{4}$ is a multiple of $\frac{\pi}{4}$



So,
$$\cos\left(-\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}$$
$$\sin\left(-\frac{\pi}{4}\right) = -\frac{1}{\sqrt{2}}$$
$$\tan\left(-\frac{\pi}{4}\right) = -1$$

4 **a**

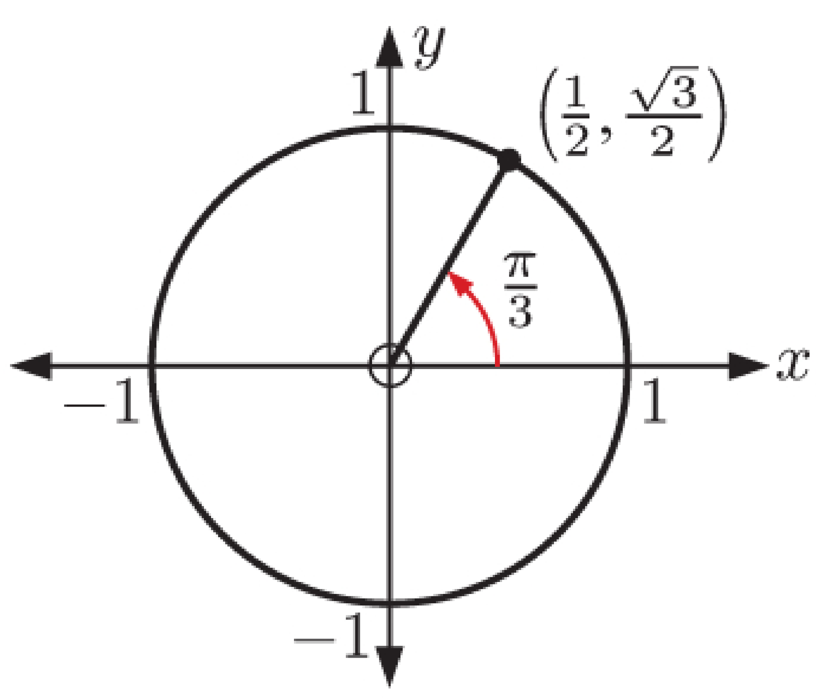


$\cos \frac{\pi}{2} = 0, \quad \sin \frac{\pi}{2} = 1$

b $\tan \frac{\pi}{2} = \dfrac{\sin \frac{\pi}{2}}{\cos \frac{\pi}{2}} = \dfrac{1}{0}$

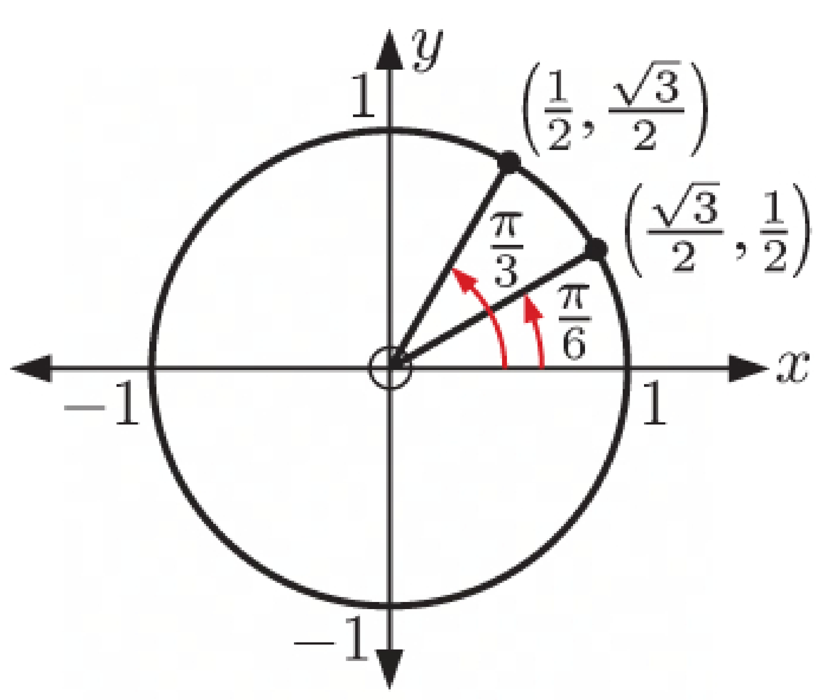
$\therefore \quad \tan \frac{\pi}{2}$ is undefined.

5 **a**



$$\sin \frac{\pi}{3} = \frac{\sqrt{3}}{2}$$
$$\therefore \quad \sin^2\left(\frac{\pi}{3}\right) = \sin \frac{\pi}{3} \times \sin \frac{\pi}{3}$$
$$= \frac{\sqrt{3}}{2} \times \frac{\sqrt{3}}{2}$$
$$= \frac{3}{4}$$

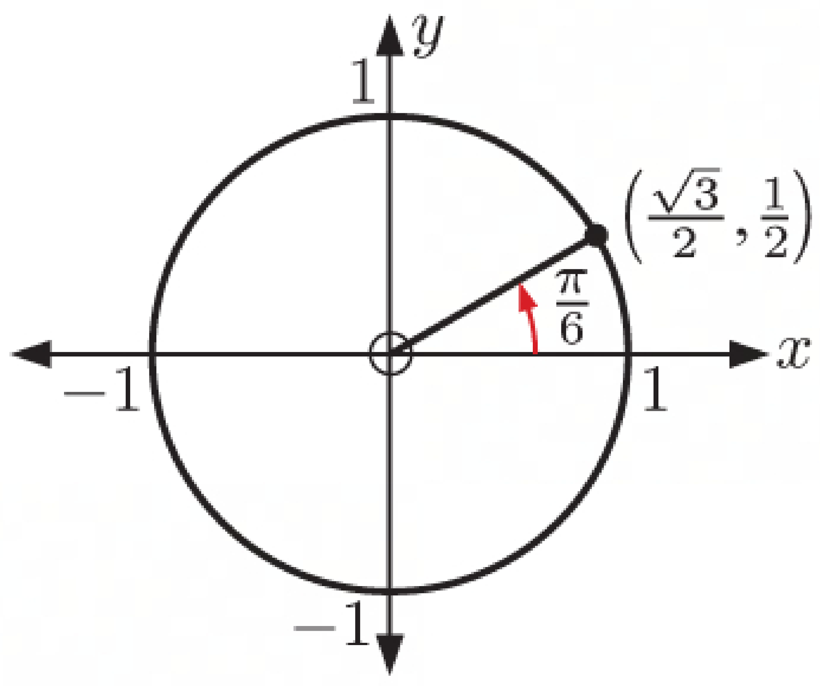
b



$\sin \frac{\pi}{6} = \frac{1}{2}$ and $\cos \frac{\pi}{3} = \frac{1}{2}$

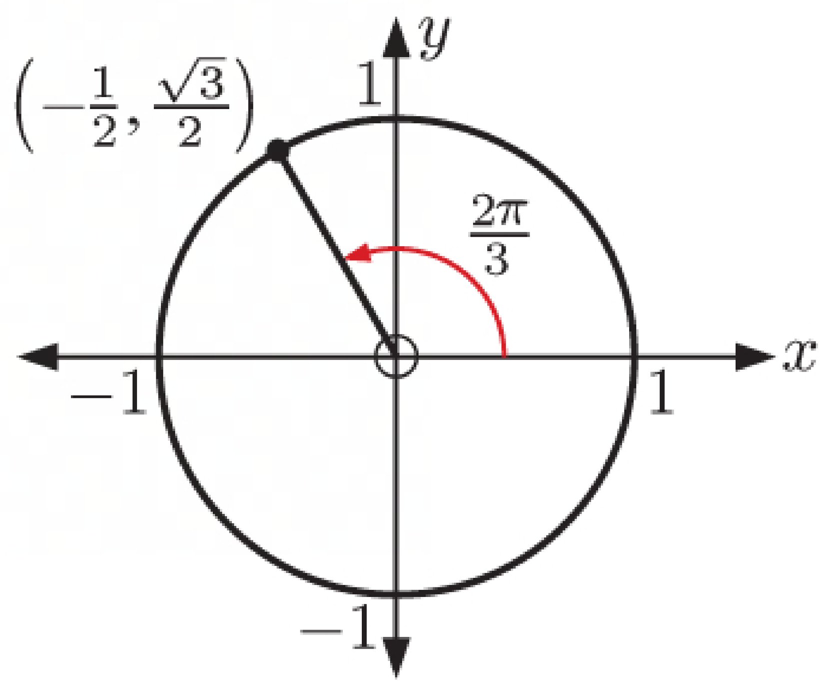
$$\therefore \quad \sin \frac{\pi}{6} \cos \frac{\pi}{3} = \frac{1}{2} \times \frac{1}{2}$$
$$= \frac{1}{4}$$

c



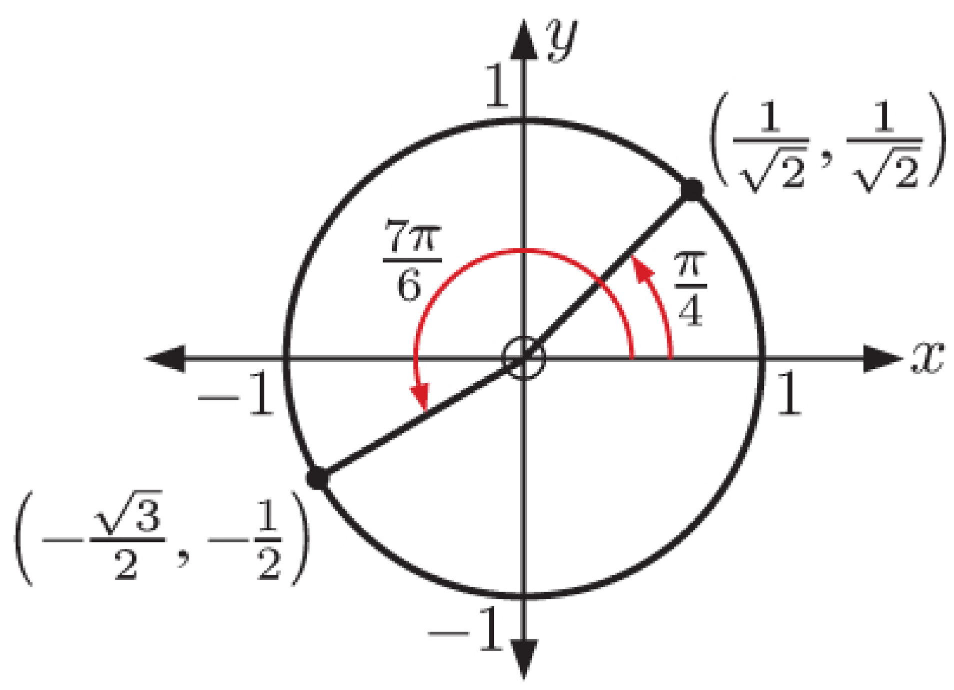
$$\begin{aligned} \cos\tfrac{\pi}{6} &= \tfrac{\sqrt{3}}{2} \\ \therefore\ 1 - \cos^2\left(\tfrac{\pi}{6}\right) &= 1 - \left(\tfrac{\sqrt{3}}{2}\right)^2 \\ &= 1 - \tfrac{3}{4} \\ &= \tfrac{1}{4} \end{aligned}$$

d



$$\begin{aligned} \sin\tfrac{2\pi}{3} &= \tfrac{\sqrt{3}}{2} \\ \therefore\ \sin^2\left(\tfrac{2\pi}{3}\right) - 1 &= \left(\tfrac{\sqrt{3}}{2}\right)^2 - 1 \\ &= \tfrac{3}{4} - 1 \\ &= -\tfrac{1}{4} \end{aligned}$$

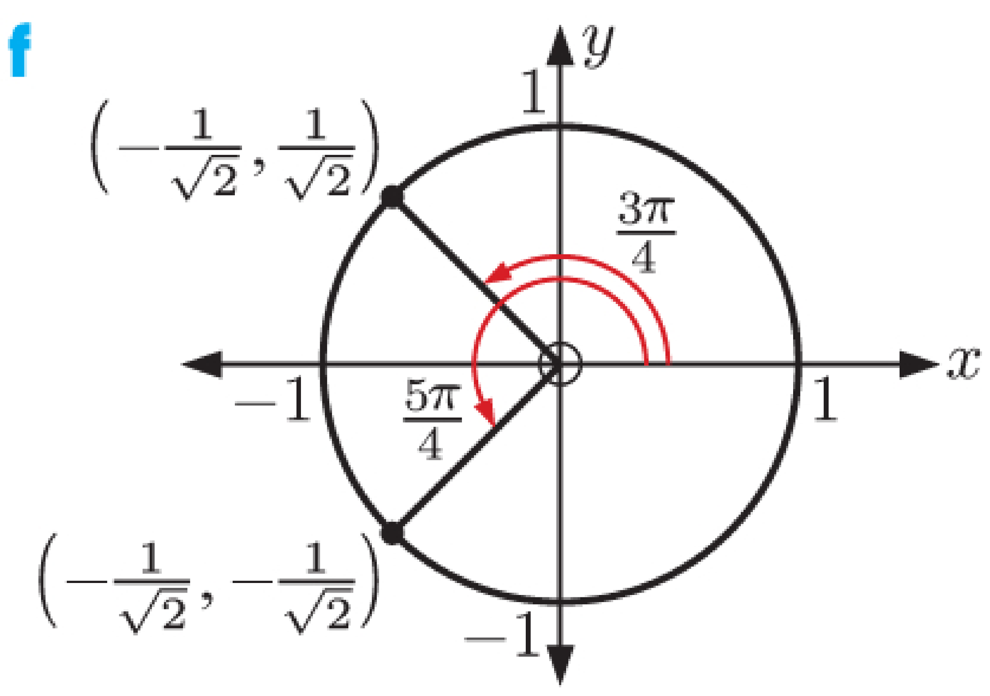
e



$\cos\frac{\pi}{4} = \frac{1}{\sqrt{2}}$ and $\sin\frac{7\pi}{6} = -\frac{1}{2}$

$$\begin{aligned} \therefore\ \cos^2\left(\tfrac{\pi}{4}\right) - \sin\tfrac{7\pi}{6} &= \left(\tfrac{1}{\sqrt{2}}\right)^2 - \left(-\tfrac{1}{2}\right) \\ &= \tfrac{1}{2} + \tfrac{1}{2} \\ &= 1 \end{aligned}$$

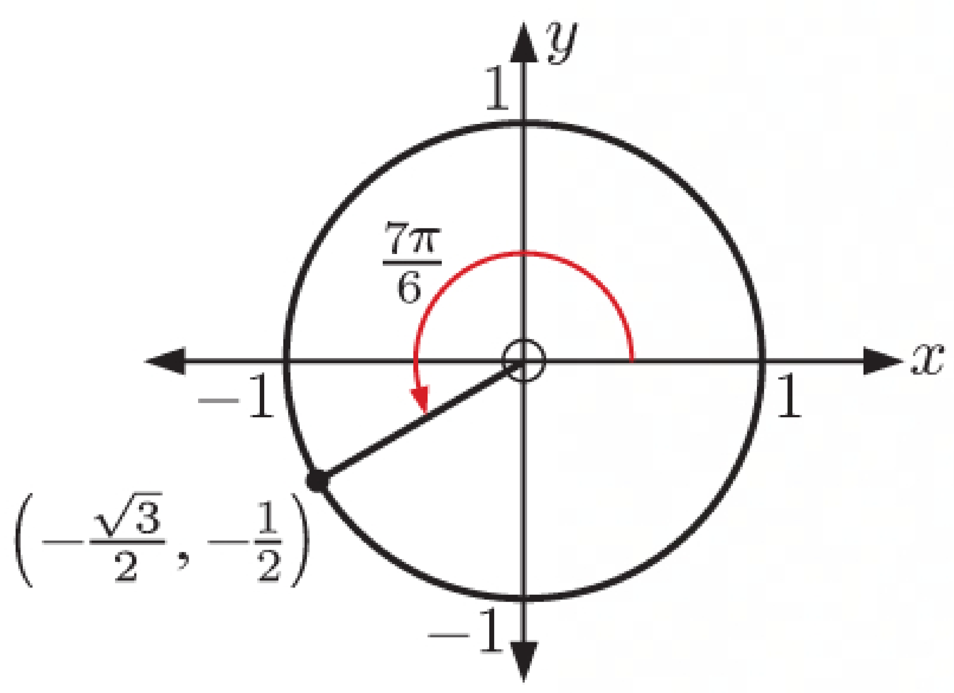
f



$\sin\frac{3\pi}{4} = \frac{1}{\sqrt{2}}$ and $\cos\frac{5\pi}{4} = -\frac{1}{\sqrt{2}}$

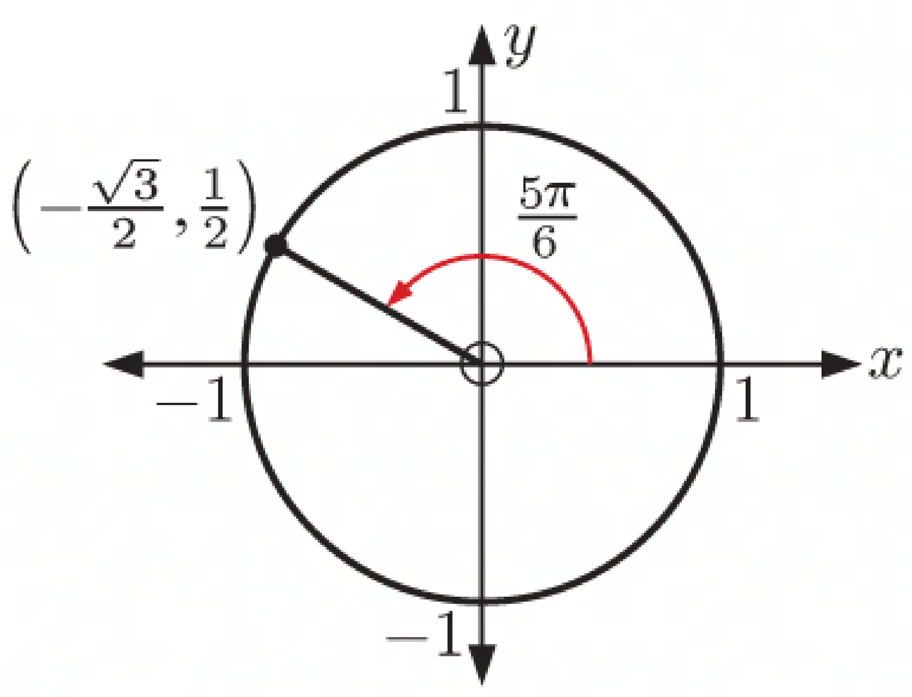
$$\begin{aligned} \therefore\ \sin\tfrac{3\pi}{4} - \cos\tfrac{5\pi}{4} &= \tfrac{1}{\sqrt{2}} - \left(-\tfrac{1}{\sqrt{2}}\right) \\ &= \tfrac{1}{\sqrt{2}} + \tfrac{1}{\sqrt{2}} \\ &= \tfrac{2}{\sqrt{2}} \ \text{ or } \ \sqrt{2} \end{aligned}$$

g



$$\begin{aligned} \sin\tfrac{7\pi}{6} &= -\tfrac{1}{2} \\ \therefore\ 1 - 2\sin^2\left(\tfrac{7\pi}{6}\right) &= 1 - 2\left(-\tfrac{1}{2}\right)^2 \\ &= 1 - 2 \times \tfrac{1}{4} \\ &= \tfrac{1}{2} \end{aligned}$$

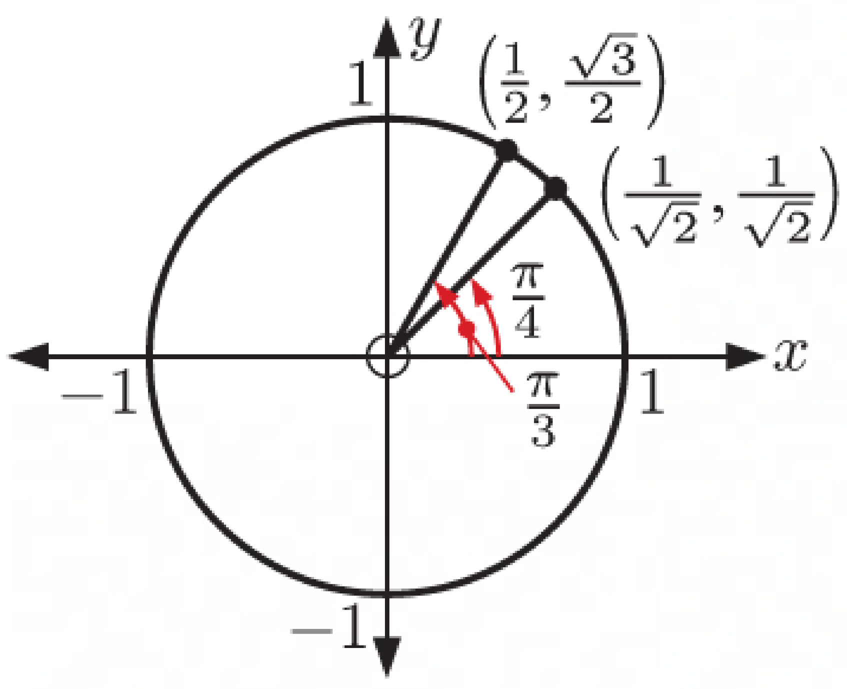
h



$\cos\frac{5\pi}{6} = -\frac{\sqrt{3}}{2}$ and $\sin\frac{5\pi}{6} = \frac{1}{2}$

$$\begin{aligned} \therefore\ \cos^2\left(\tfrac{5\pi}{6}\right) - \sin^2\left(\tfrac{5\pi}{6}\right) &= \left(-\tfrac{\sqrt{3}}{2}\right)^2 - \left(\tfrac{1}{2}\right)^2 \\ &= \tfrac{3}{4} - \tfrac{1}{4} \\ &= \tfrac{1}{2} \end{aligned}$$

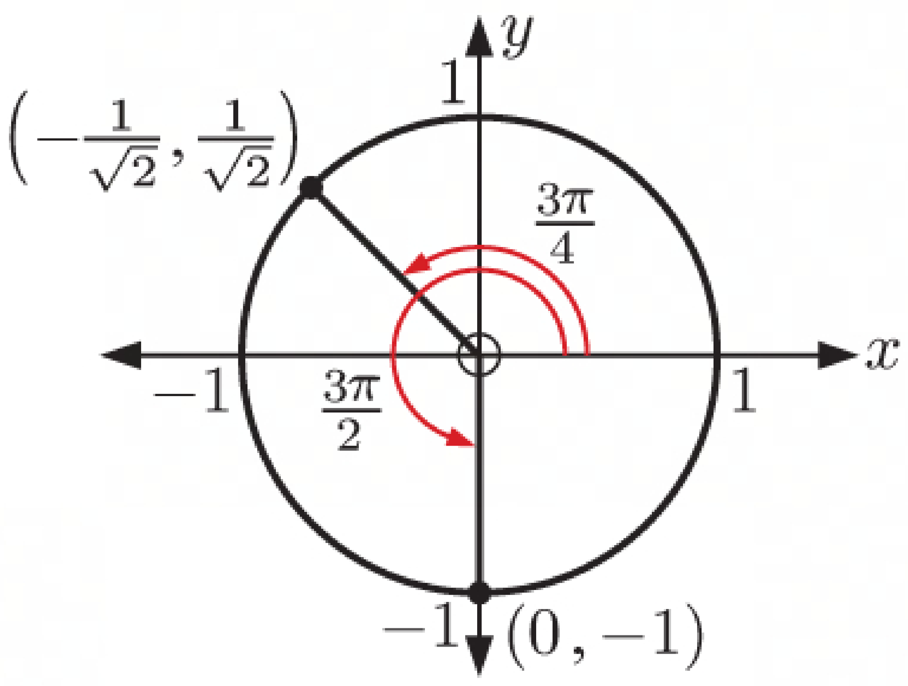
i



$$\tan\frac{\pi}{3} = \frac{\frac{\sqrt{3}}{2}}{\frac{1}{2}} = \sqrt{3} \quad \text{and} \quad \sin\frac{\pi}{4} = \frac{1}{\sqrt{2}}$$

$$\begin{aligned}\therefore\ \tan^2\left(\tfrac{\pi}{3}\right) - 2\sin^2\left(\tfrac{\pi}{4}\right) &= \left(\sqrt{3}\right)^2 - 2\left(\tfrac{1}{\sqrt{2}}\right)^2 \\ &= 3 - 2\left(\tfrac{1}{2}\right) \\ &= 2\end{aligned}$$

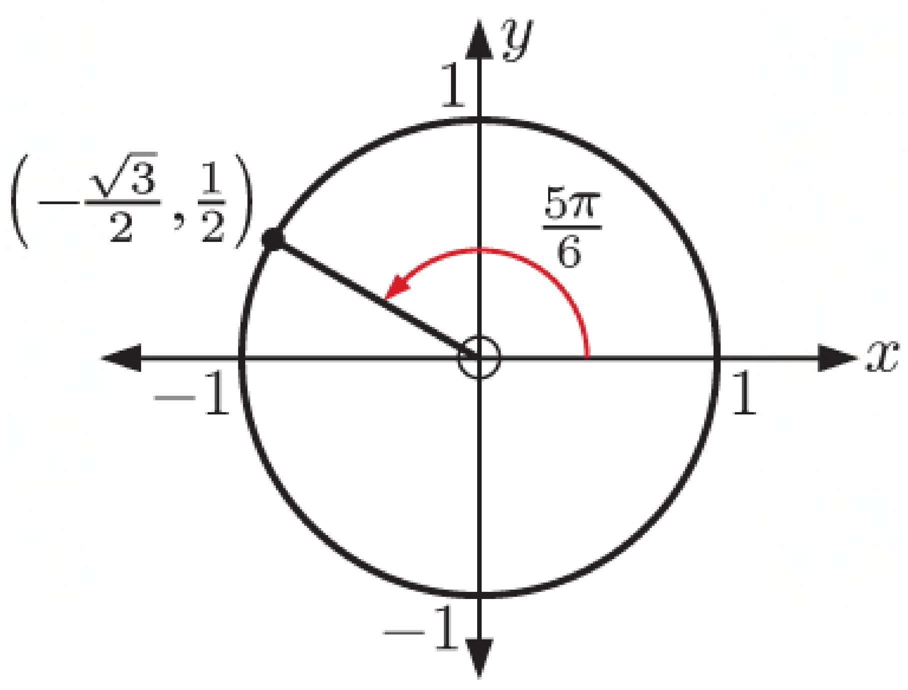
j



$$\tan\left(-\tfrac{5\pi}{4}\right) = \frac{\frac{1}{\sqrt{2}}}{-\frac{1}{\sqrt{2}}} = -1 \quad \text{and} \quad \sin\frac{3\pi}{2} = -1$$

$$\begin{aligned}\therefore\ 2\tan\left(-\tfrac{5\pi}{4}\right) - \sin\tfrac{3\pi}{2} &= 2(-1) - (-1) \\ &= -1\end{aligned}$$

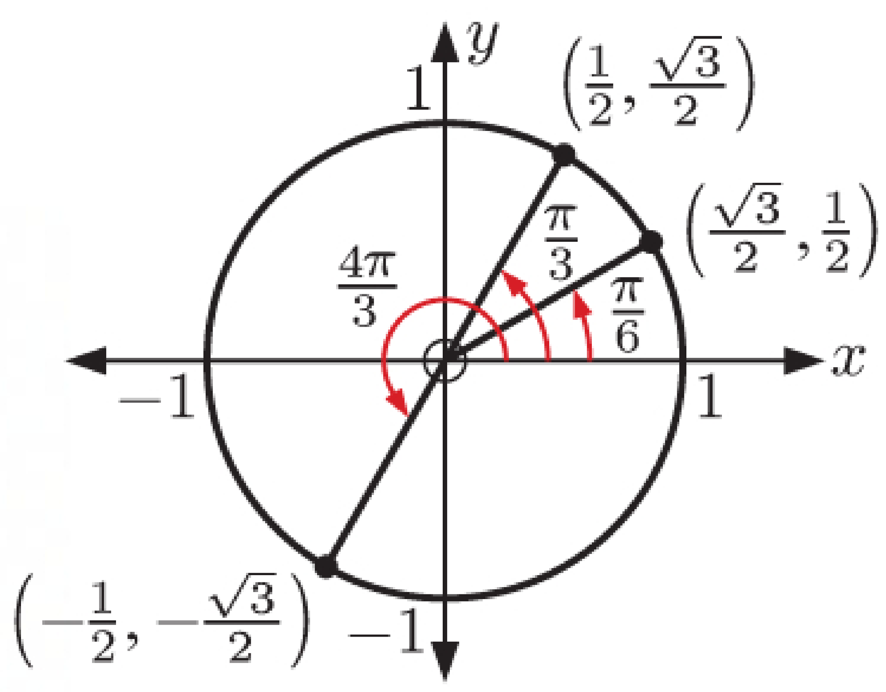
k



$$\tan\frac{5\pi}{6} = \frac{\frac{1}{2}}{-\frac{\sqrt{3}}{2}} = -\frac{1}{\sqrt{3}}$$

$$\begin{aligned}\therefore\ \frac{2\tan\frac{5\pi}{6}}{1 - \tan^2(\frac{5\pi}{6})} &= \frac{2(-\frac{1}{\sqrt{3}})}{1 - (-\frac{1}{\sqrt{3}})^2} \\ &= \frac{-\frac{2}{\sqrt{3}}}{1 - \frac{1}{3}} \\ &= \frac{-\frac{2}{\sqrt{3}}}{\frac{2}{3}} \\ &= -\frac{3}{\sqrt{3}} = -\sqrt{3}\end{aligned}$$

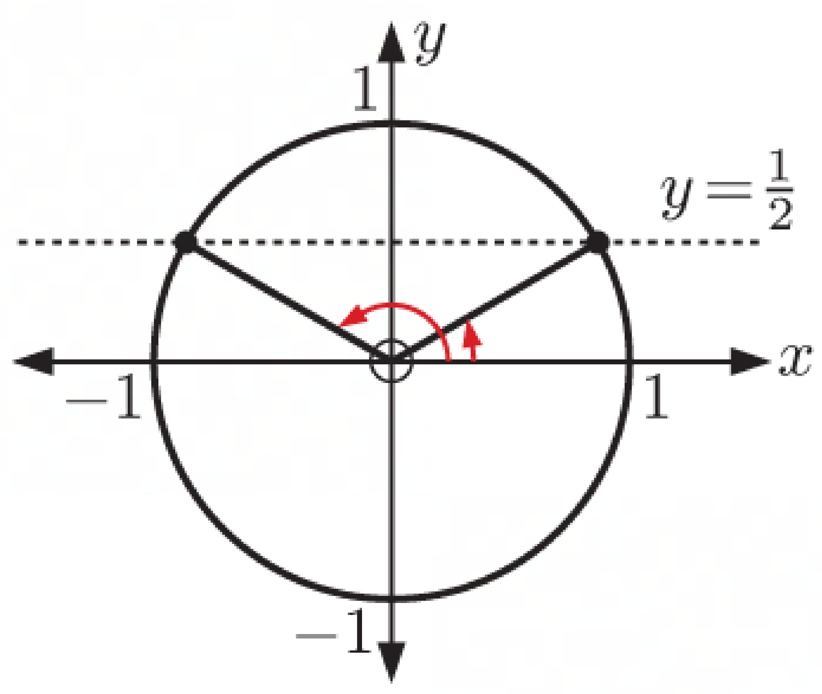
l



$$\cos\frac{\pi}{3} = \frac{1}{2}, \quad \sin\frac{4\pi}{3} = -\frac{\sqrt{3}}{2}, \quad \text{and} \quad \tan\frac{\pi}{6} = \frac{\frac{1}{2}}{\frac{\sqrt{3}}{2}} = \frac{1}{\sqrt{3}}$$

$$\begin{aligned}\therefore\ \frac{\cos\frac{\pi}{3}}{\sin\frac{4\pi}{3} + \tan\frac{\pi}{6}} &= \frac{\frac{1}{2}}{-\frac{\sqrt{3}}{2} + \frac{1}{\sqrt{3}}} \\ &= \frac{\frac{1}{2}}{-\frac{3}{2\sqrt{3}} + \frac{2}{2\sqrt{3}}} \\ &= \frac{\frac{1}{2}}{-\frac{1}{2\sqrt{3}}} \\ &= -\sqrt{3}\end{aligned}$$

6 **a**

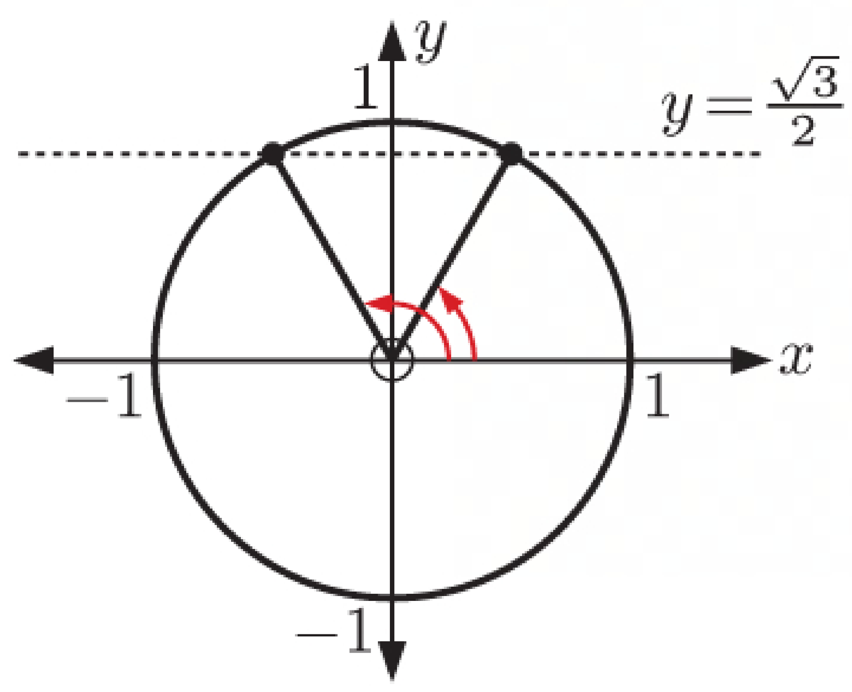


Since the sine is $\frac{1}{2}$, we draw the horizontal line $y = \frac{1}{2}$.

Because $\frac{1}{2}$ is involved, we know the required angles are multiples of $\frac{\pi}{6}$.

They are $\frac{\pi}{6}$ and $\frac{5\pi}{6}$.

b

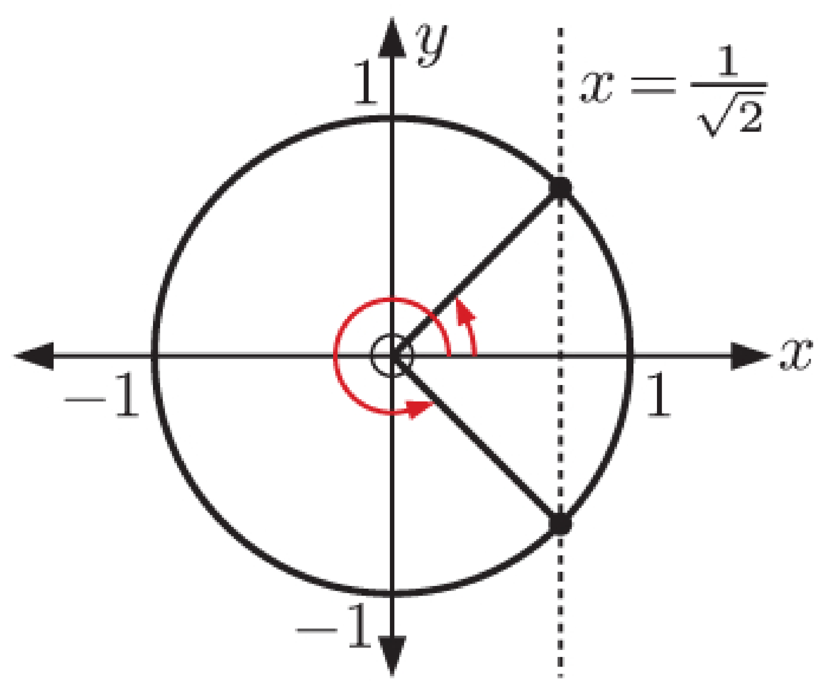


Since the sine is $\frac{\sqrt{3}}{2}$, we draw the horizontal line $y = \frac{\sqrt{3}}{2}$.

Because $\frac{\sqrt{3}}{2}$ is involved, we know the required angles are multiples of $\frac{\pi}{6}$.

They are $\frac{\pi}{3}$ and $\frac{2\pi}{3}$.

c

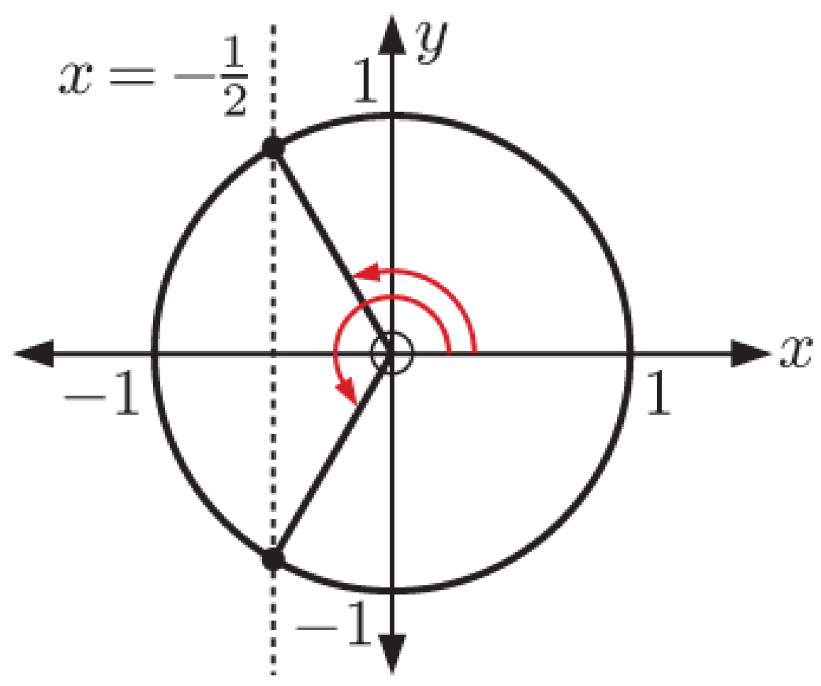


Since the cosine is $\frac{1}{\sqrt{2}}$, we draw the vertical line $x = \frac{1}{\sqrt{2}}$.

Because $\frac{1}{\sqrt{2}}$ is involved, we know the required angles are multiples of $\frac{\pi}{4}$.

They are $\frac{\pi}{4}$ and $\frac{7\pi}{4}$.

d

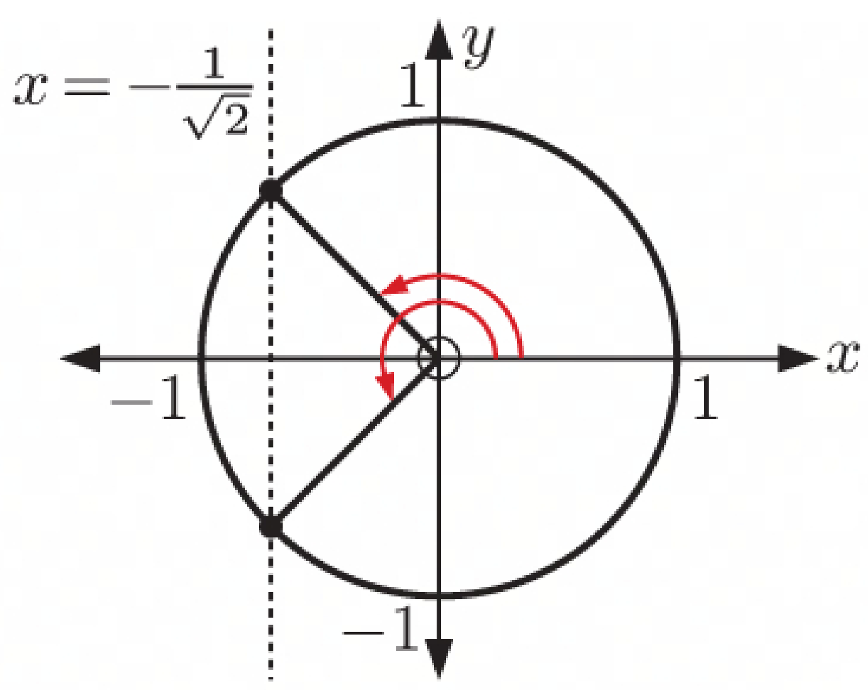


Since the cosine is $-\frac{1}{2}$, we draw the vertical line $x = -\frac{1}{2}$.

Because $\frac{1}{2}$ is involved, we know the required angles are multiples of $\frac{\pi}{6}$.

They are $\frac{2\pi}{3}$ and $\frac{4\pi}{3}$.

e

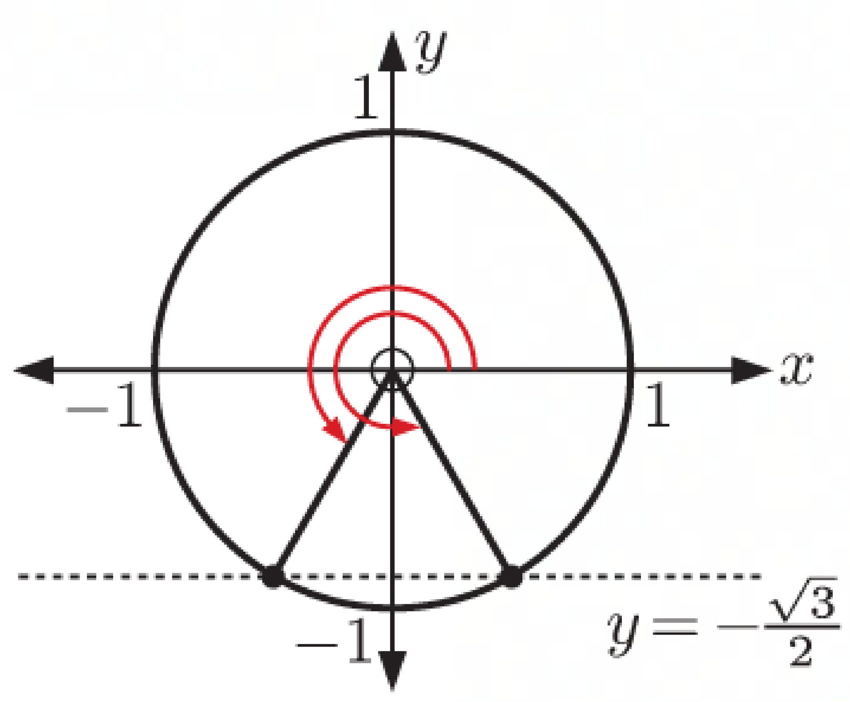


Since the cosine is $-\frac{1}{\sqrt{2}}$, we draw the vertical line $x = -\frac{1}{\sqrt{2}}$.

Because $\frac{1}{\sqrt{2}}$ is involved, we know the required angles are multiples of $\frac{\pi}{4}$.

They are $\frac{3\pi}{4}$ and $\frac{5\pi}{4}$.

f

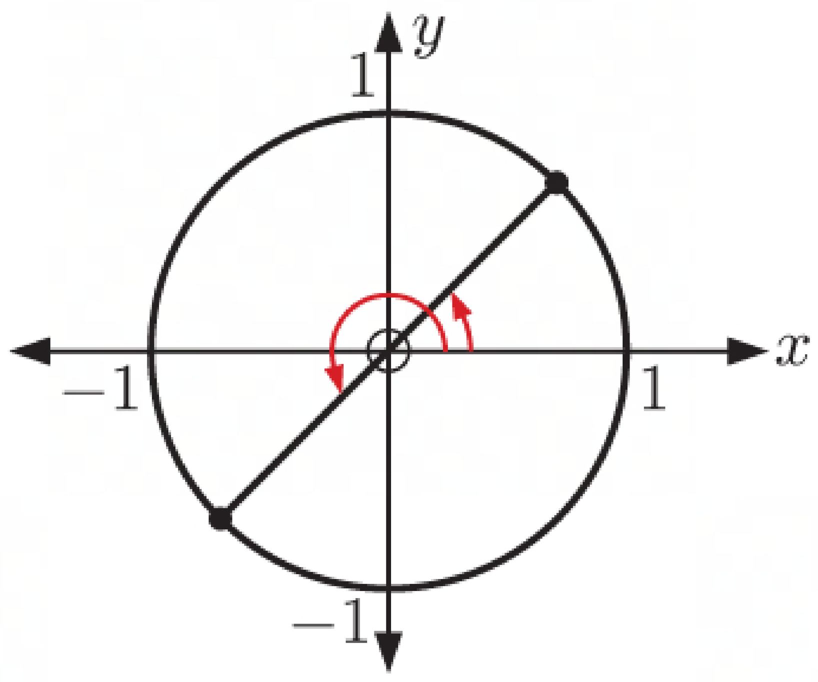


Since the sine is $-\frac{\sqrt{3}}{2}$, we draw the horizontal line $y = -\frac{\sqrt{3}}{2}$.

Because $\frac{\sqrt{3}}{2}$ is involved, we know the required angles are multiples of $\frac{\pi}{6}$.

They are $\frac{4\pi}{3}$ and $\frac{5\pi}{3}$.

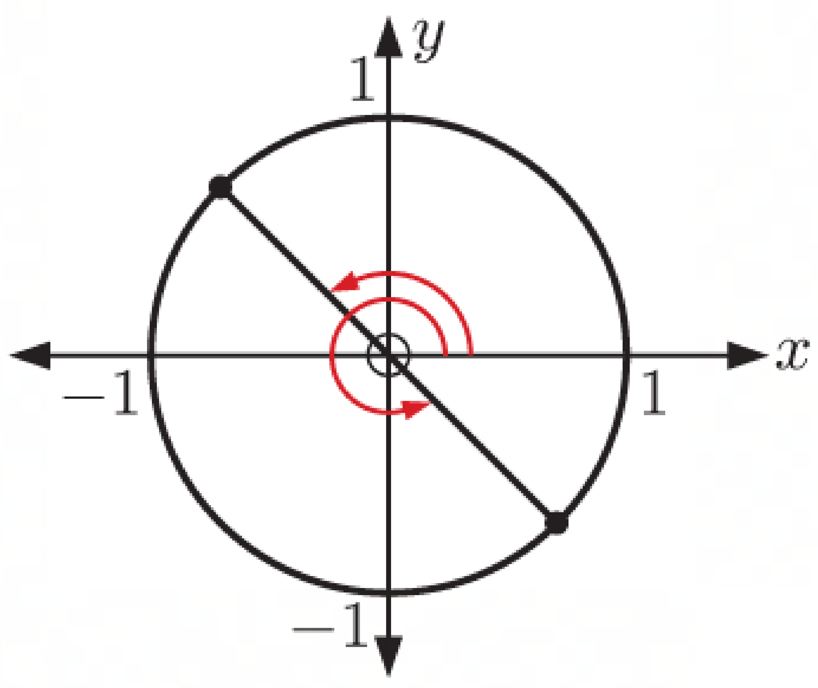
7 **a**



Since the tangent is 1, the sine and cosine must be identical (since $\tan\theta = \dfrac{\sin\theta}{\cos\theta}$).

This is only true when the angles are $\frac{\pi}{4}$ and $\frac{5\pi}{4}$.

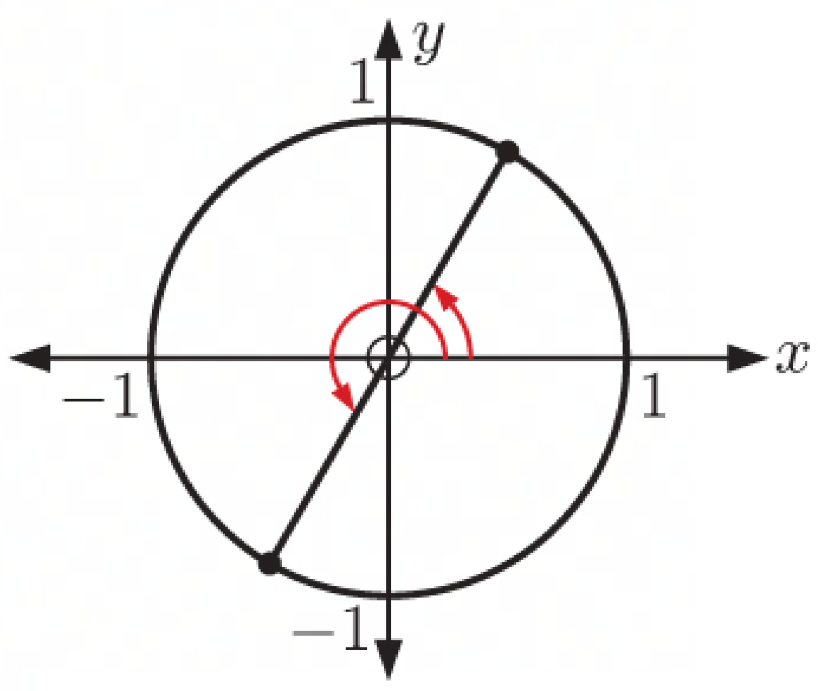
b



Since the tangent is -1, the sine and cosine must be equal in value but opposite in sign (since $\tan\theta = \dfrac{\sin\theta}{\cos\theta}$).

This is only true when the angles are $\frac{3\pi}{4}$ and $\frac{7\pi}{4}$.

c

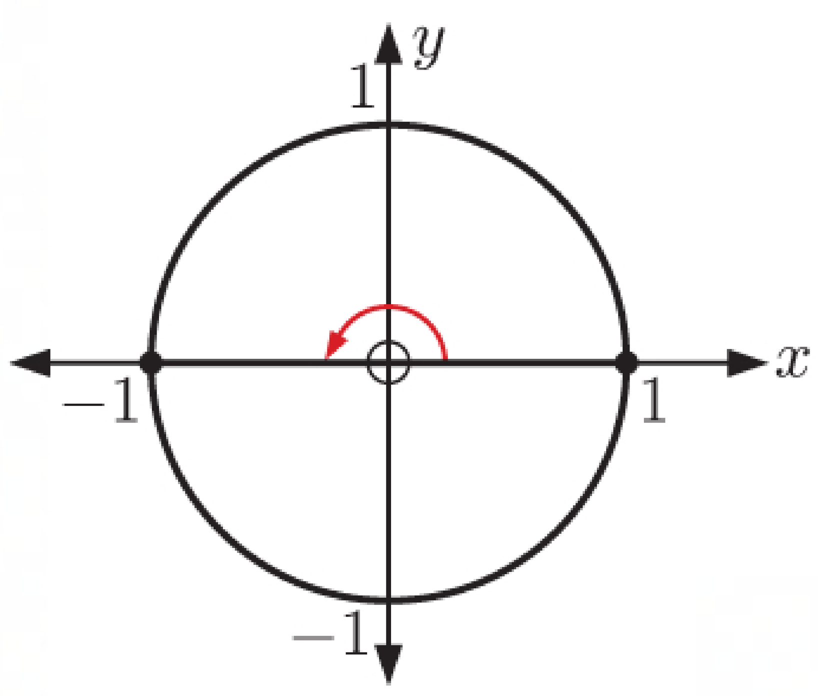


Since the tangent is $\sqrt{3} = \dfrac{\frac{\sqrt{3}}{2}}{\frac{1}{2}} = \dfrac{-\frac{\sqrt{3}}{2}}{-\frac{1}{2}}$, the sine must be $\pm\frac{\sqrt{3}}{2}$, and the cosine must be $\pm\frac{1}{2}$ (since $\tan\theta = \dfrac{\sin\theta}{\cos\theta}$).

Because $\frac{\sqrt{3}}{2}$ is involved, we know the required angles are multiples of $\frac{\pi}{6}$.

They are $\frac{\pi}{3}$ and $\frac{4\pi}{3}$.

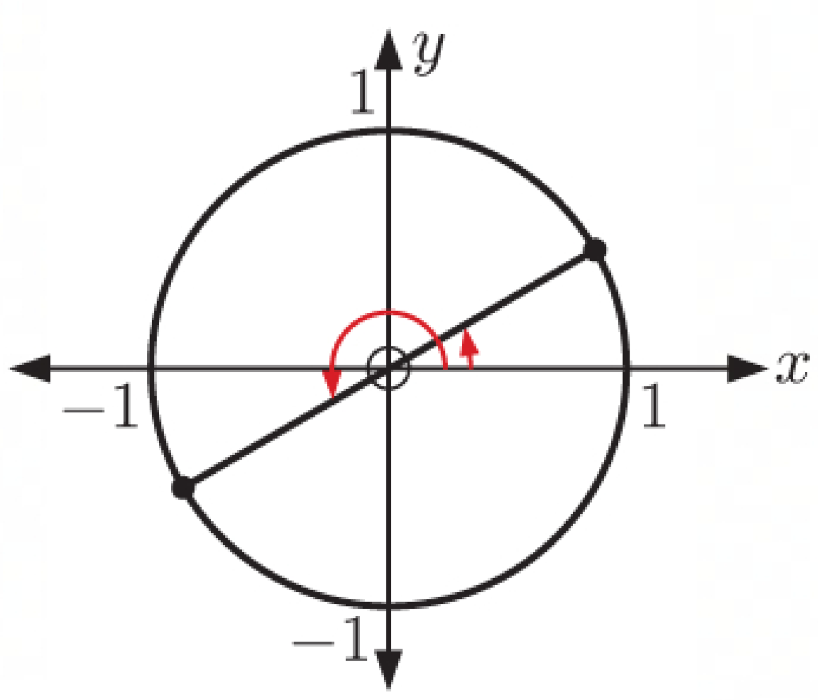
d



Since the tangent is 0, the sine must be 0 (since $\tan\theta = \dfrac{\sin\theta}{\cos\theta}$).

This is only true when the cosine is ± 1, and the angles are 0, π, and 2π.

e

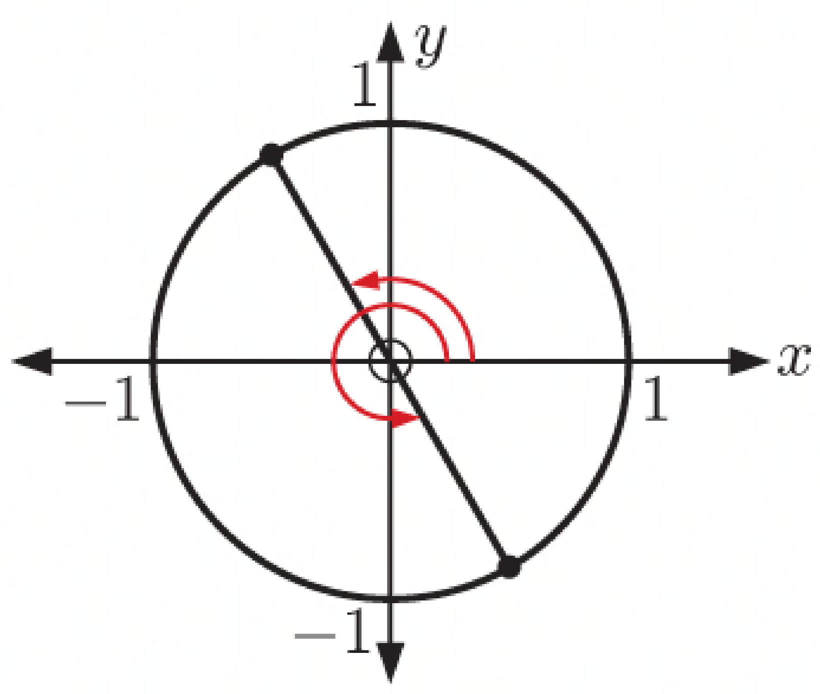


Since the tangent is $\frac{1}{\sqrt{3}} = \dfrac{\frac{1}{2}}{\frac{\sqrt{3}}{2}} = \dfrac{-\frac{1}{2}}{-\frac{\sqrt{3}}{2}}$, the sine must be $\pm\frac{1}{2}$, and the cosine must be $\pm\frac{\sqrt{3}}{2}$ (since $\tan\theta = \dfrac{\sin\theta}{\cos\theta}$).

Because $\frac{1}{2}$ and $\frac{\sqrt{3}}{2}$ are both involved, we know the required angles are multiples of $\frac{\pi}{6}$.

They are $\frac{\pi}{6}$ and $\frac{7\pi}{6}$.

f

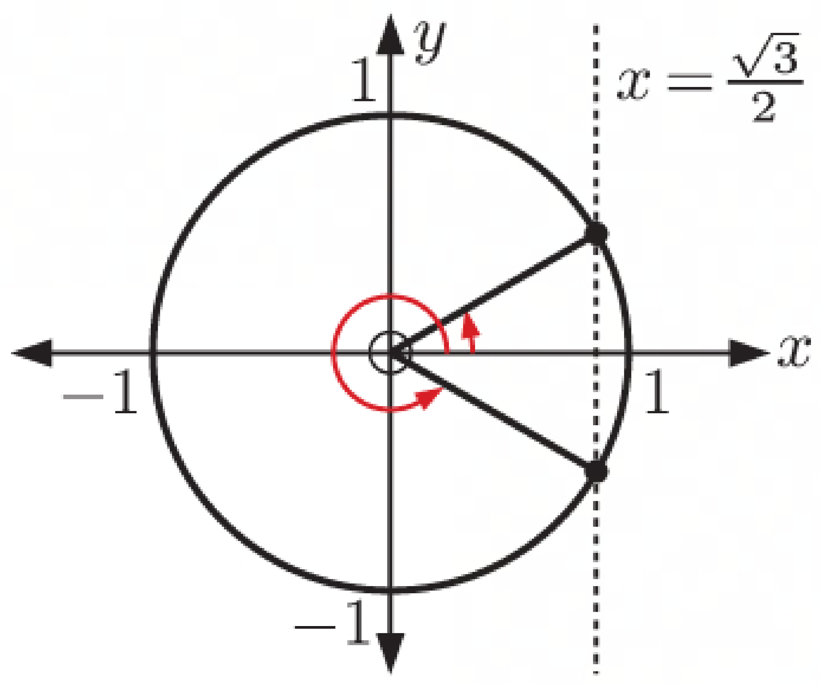


Since the tangent is $-\sqrt{3} = \dfrac{\frac{\sqrt{3}}{2}}{-\frac{1}{2}} = \dfrac{-\frac{\sqrt{3}}{2}}{\frac{1}{2}}$, the sine must be $\pm\frac{\sqrt{3}}{2}$, and the cosine must be $\mp\frac{1}{2}$ (since $\tan\theta = \dfrac{\sin\theta}{\cos\theta}$).

Because $\frac{1}{2}$ and $\frac{\sqrt{3}}{2}$ are both involved, we know the required angles are multiples of $\frac{\pi}{6}$.

They are $\frac{2\pi}{3}$ and $\frac{5\pi}{3}$.

8 a

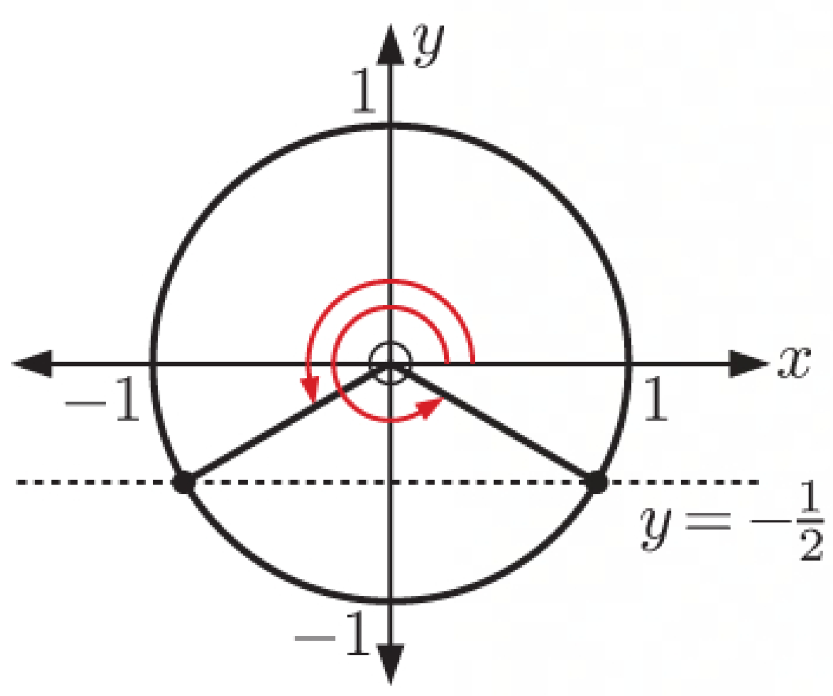


Since the cosine is $\frac{\sqrt{3}}{2}$, we draw the vertical line $x = \frac{\sqrt{3}}{2}$.

Because $\frac{\sqrt{3}}{2}$ is involved, we know the required angles are multiples of $\frac{\pi}{6}$.

They are $\frac{\pi}{6}$, $\frac{11\pi}{6}$, $\frac{13\pi}{6}$, and $\frac{23\pi}{6}$.

b

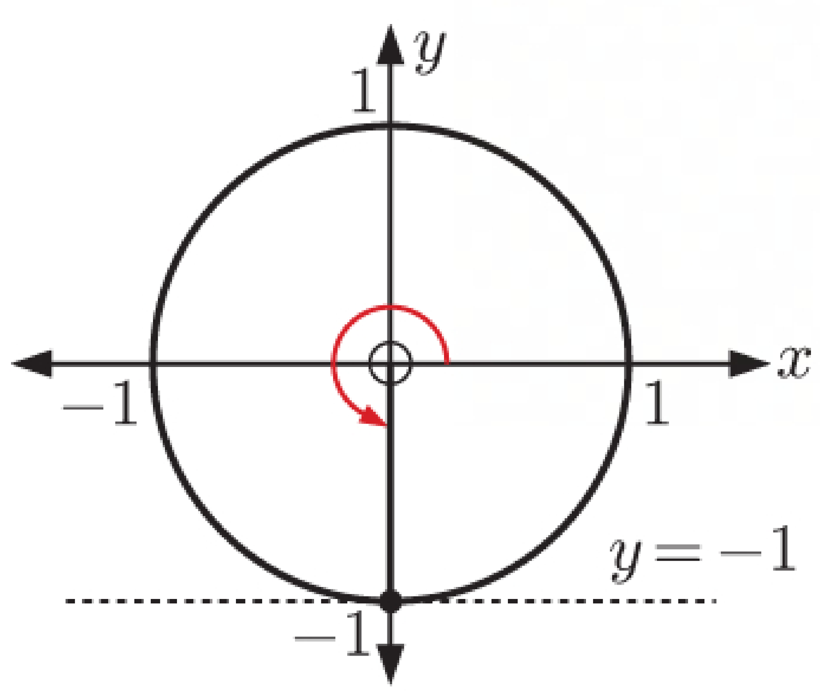


Since the sine is $-\frac{1}{2}$, we draw the horizontal line $y = -\frac{1}{2}$.

Because $\frac{1}{2}$ is involved, we know the required angles are multiples of $\frac{\pi}{6}$.

They are $\frac{7\pi}{6}$, $\frac{11\pi}{6}$, $\frac{19\pi}{6}$, and $\frac{23\pi}{6}$.

c

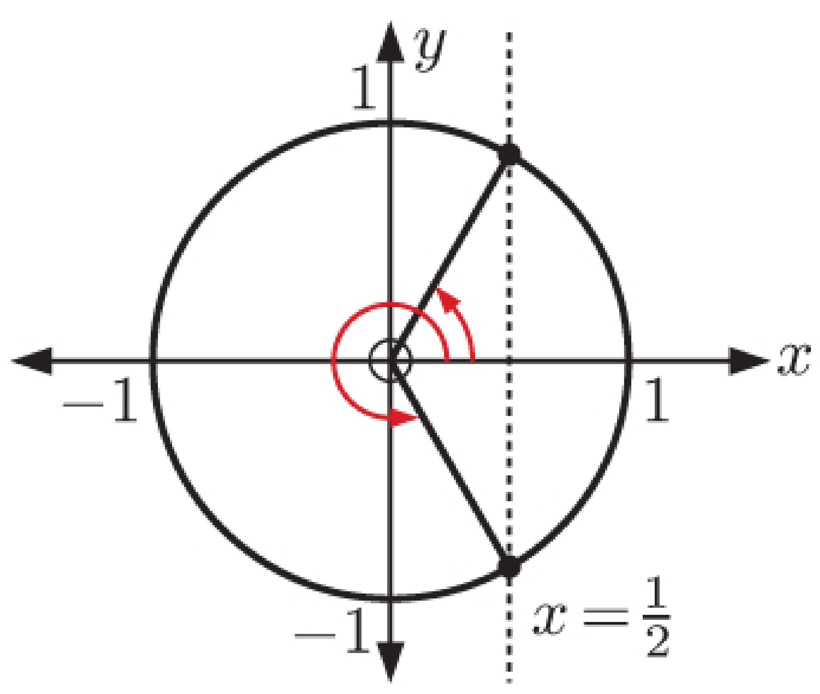


Since the sine is -1, we draw the horizontal line $y = -1$.

Because 1 is involved, we know the required angles are multiples of $\frac{\pi}{2}$.

They are $\frac{3\pi}{2}$ and $\frac{7\pi}{2}$.

9 a

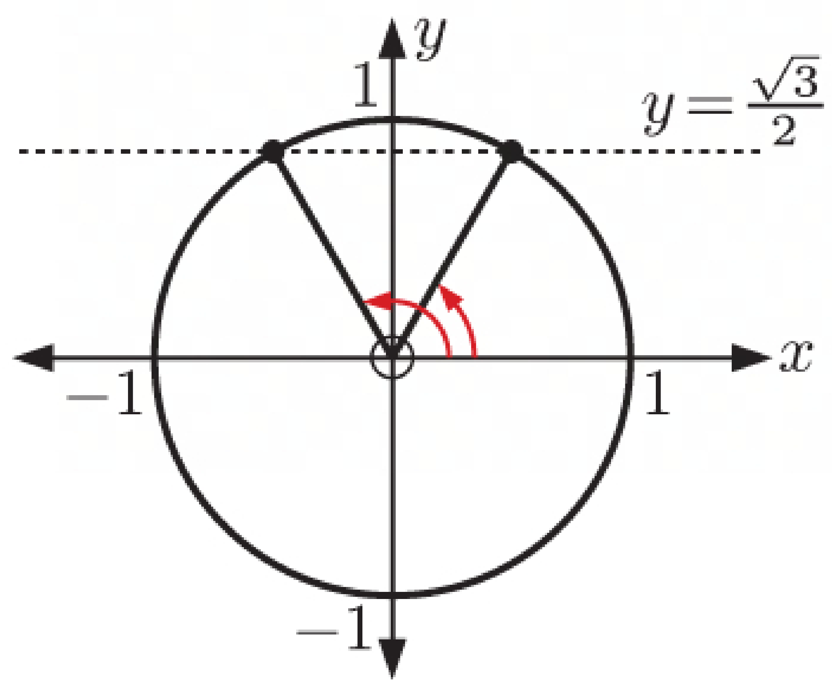


Since $\cos\theta = \frac{1}{2}$, we draw the vertical line $x = \frac{1}{2}$.

Because $\frac{1}{2}$ is involved, we know the required angles are multiples of $\frac{\pi}{6}$.

$\therefore \quad \theta = \frac{\pi}{3}, \frac{5\pi}{3}$

b

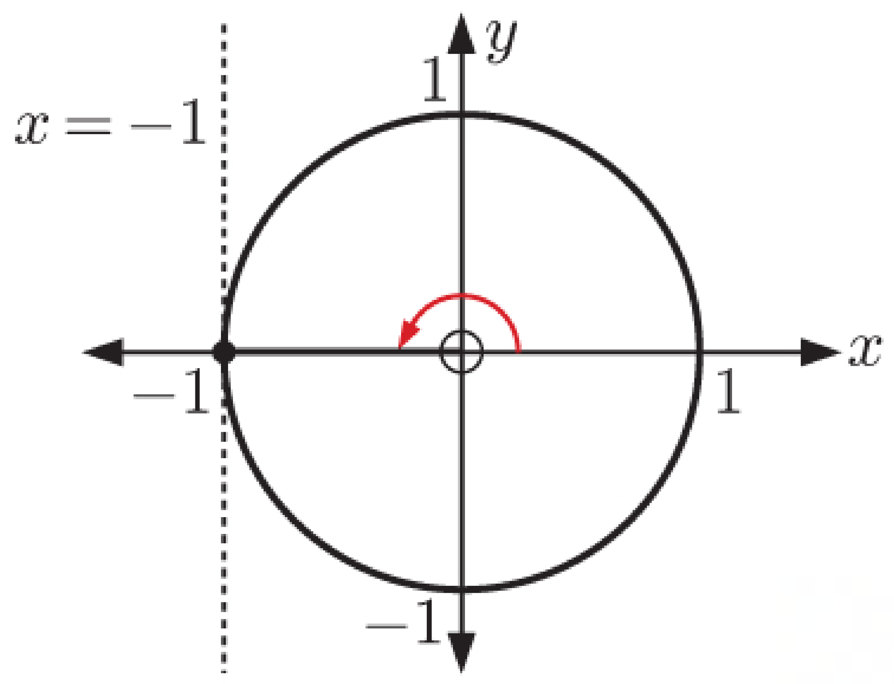


Since $\sin\theta = \frac{\sqrt{3}}{2}$, we draw the horizontal line $y = \frac{\sqrt{3}}{2}$.

Because $\frac{\sqrt{3}}{2}$ is involved, we know the required angles are multiples of $\frac{\pi}{6}$.

$\therefore\ \ \theta = \frac{\pi}{3}, \frac{2\pi}{3}$

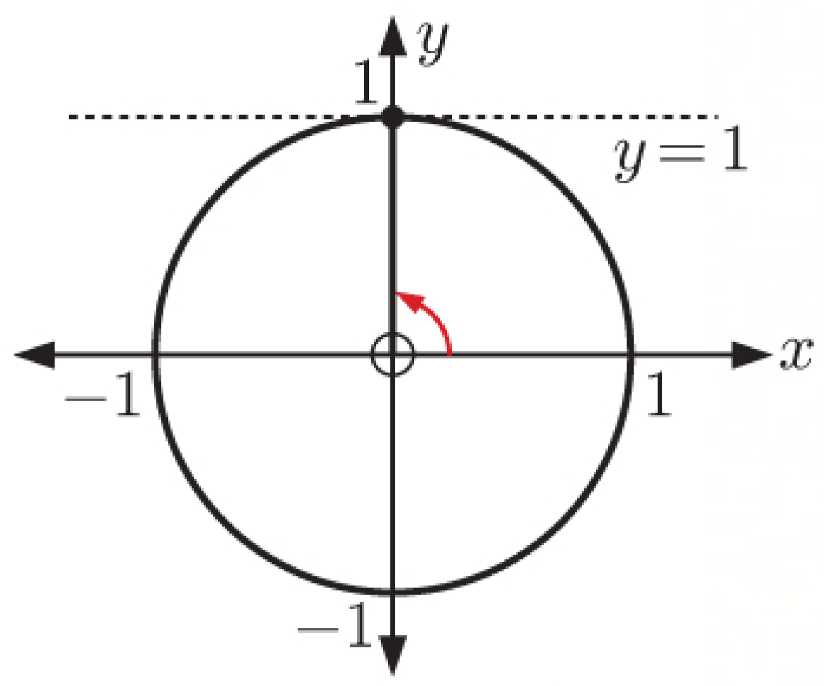
c



Since $\cos\theta = -1$, we draw the vertical line $x = -1$. Because 1 is involved, we know the required angles are multiples of $\frac{\pi}{2}$.

$\therefore\ \ \theta = \pi$

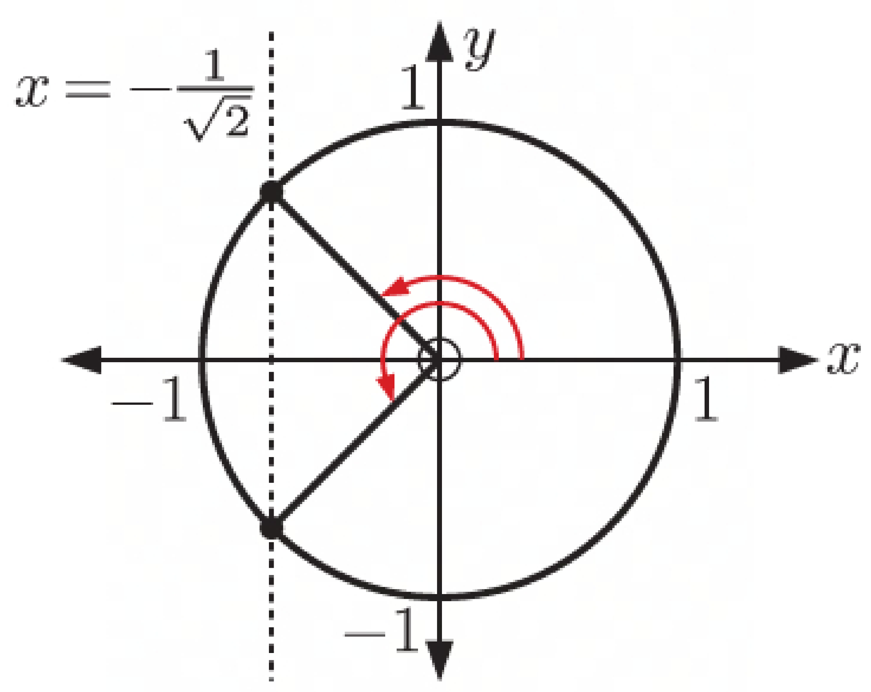
d



Since $\sin\theta = 1$, we draw the horizontal line $y = 1$. Because 1 is involved, we know the required angles are multiples of $\frac{\pi}{2}$.

$\therefore\ \ \theta = \frac{\pi}{2}$

e

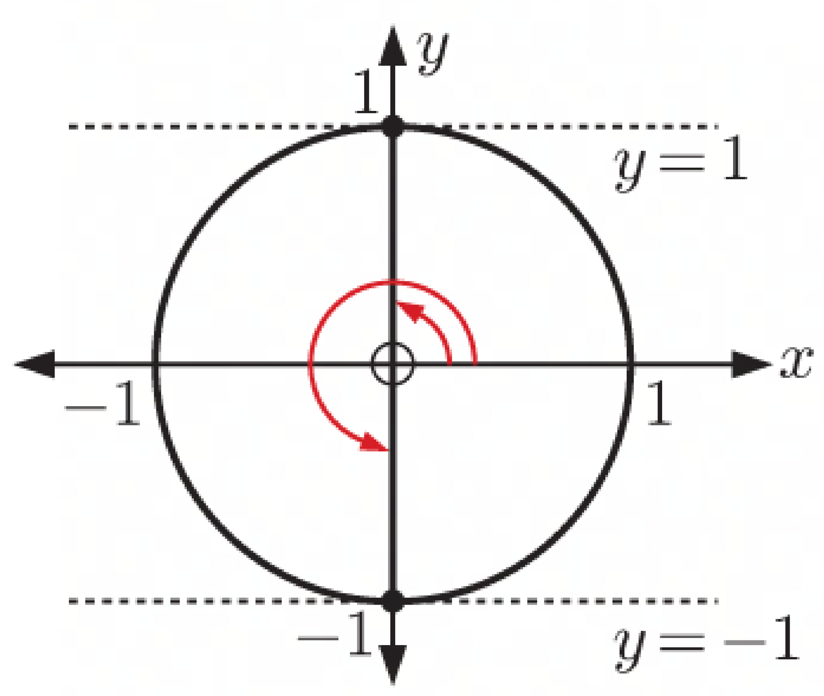


Since $\cos\theta = -\frac{1}{\sqrt{2}}$, we draw the vertical line $x = -\frac{1}{\sqrt{2}}$.

Because $\frac{1}{\sqrt{2}}$ is involved, we know the required angles are multiples of $\frac{\pi}{4}$.

$\therefore\ \ \theta = \frac{3\pi}{4}, \frac{5\pi}{4}$

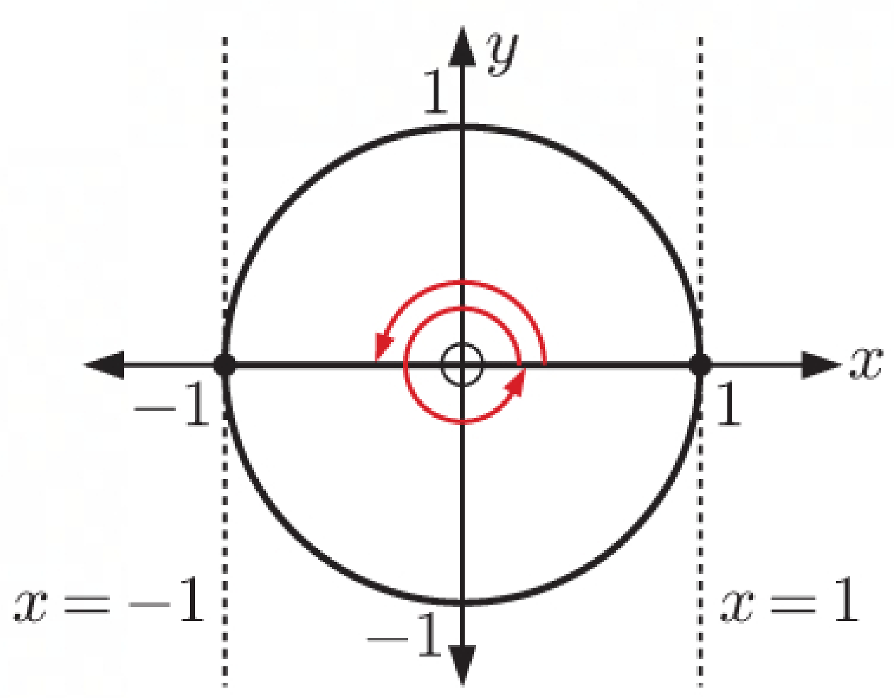
f



Since $\sin^2\theta = 1$, then $\sin\theta = \pm 1$, so we draw the horizontal lines $y = 1$ and $y = -1$.
Because 1 is involved, we know the required angles are multiples of $\frac{\pi}{2}$.

$\therefore\ \ \theta = \frac{\pi}{2}, \frac{3\pi}{2}$

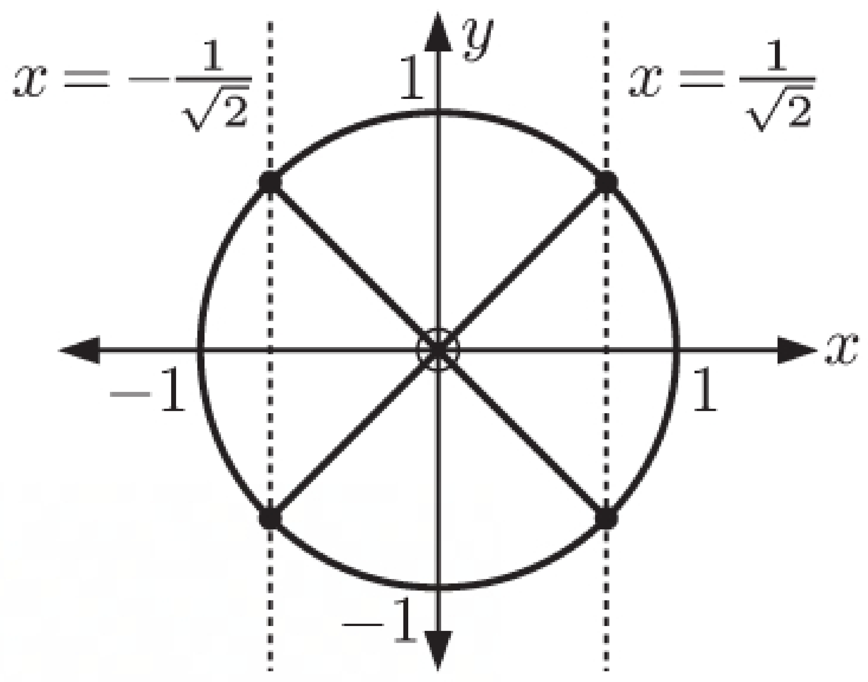
g



Since $\cos^2\theta = 1$, then $\cos\theta = \pm 1$, so we draw the vertical lines $x = -1$ and $x = 1$.
Because 1 is involved, we know the required angles are multiples of $\frac{\pi}{2}$.

$\therefore\ \ \theta = 0, \pi, 2\pi$

h

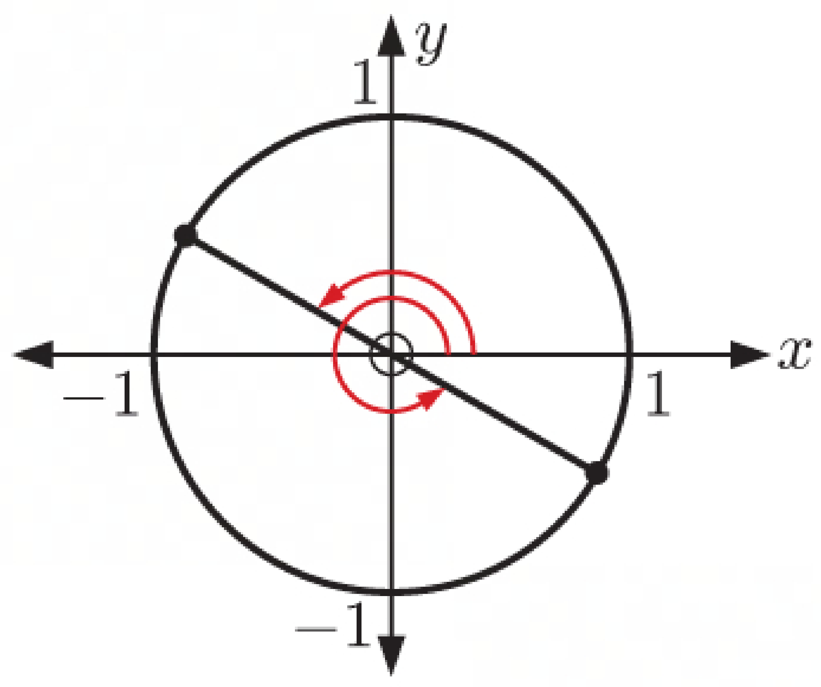


Since $\cos^2\theta = \frac{1}{2}$, then $\cos\theta = \pm\frac{1}{\sqrt{2}}$, so we draw the vertical lines $x = -\frac{1}{\sqrt{2}}$ and $x = \frac{1}{\sqrt{2}}$.

Because $\frac{1}{\sqrt{2}}$ is involved, we know the required angles are multiples of $\frac{\pi}{4}$.

$\therefore \quad \theta = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$

i



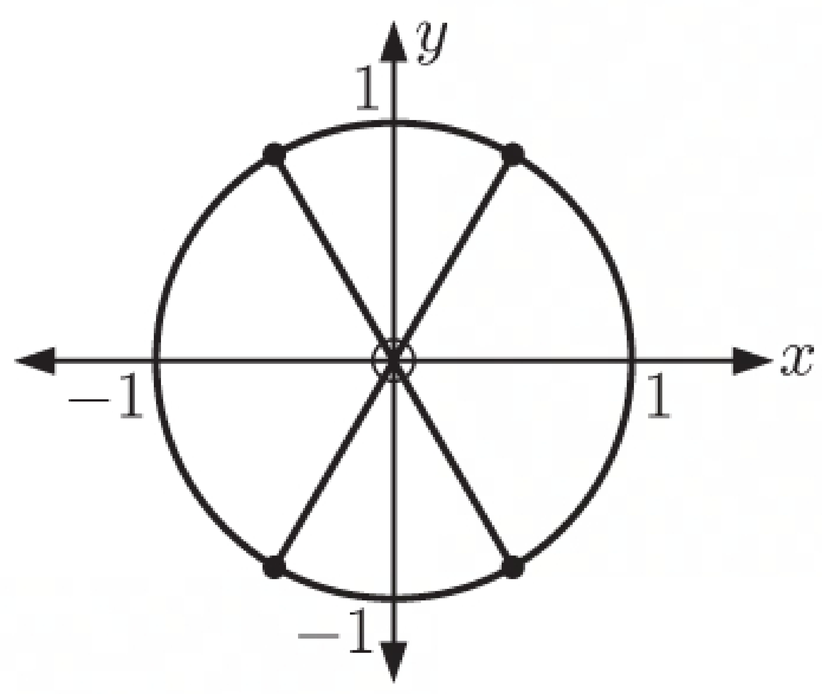
Since $\tan\theta = -\frac{1}{\sqrt{3}} = \dfrac{-\frac{1}{2}}{\frac{\sqrt{3}}{2}} = \dfrac{\frac{1}{2}}{-\frac{\sqrt{3}}{2}}$, then

$\sin\theta = \mp\frac{1}{2}$ and $\cos\theta = \pm\frac{\sqrt{3}}{2}$ (since $\tan\theta = \dfrac{\sin\theta}{\cos\theta}$).

Because $\frac{1}{2}$ and $\frac{\sqrt{3}}{2}$ are both involved, we know the required angles are multiples of $\frac{\pi}{6}$.

$\therefore \quad \theta = \frac{5\pi}{6}, \frac{11\pi}{6}$

j



Since $\tan^2\theta = 3$, then $\tan\theta = \pm\sqrt{3} = \pm\dfrac{\frac{\sqrt{3}}{2}}{\frac{1}{2}}$.

So, $\sin\theta = \pm\frac{\sqrt{3}}{2}$ and $\cos\theta = \pm\frac{1}{2}$

(since $\tan\theta = \dfrac{\sin\theta}{\cos\theta}$).

Because $\frac{\sqrt{3}}{2}$ and $\frac{1}{2}$ are both involved, we know the required angles are multiples of $\frac{\pi}{6}$.

$\therefore \quad \theta = \frac{\pi}{3}, \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{5\pi}{3}$

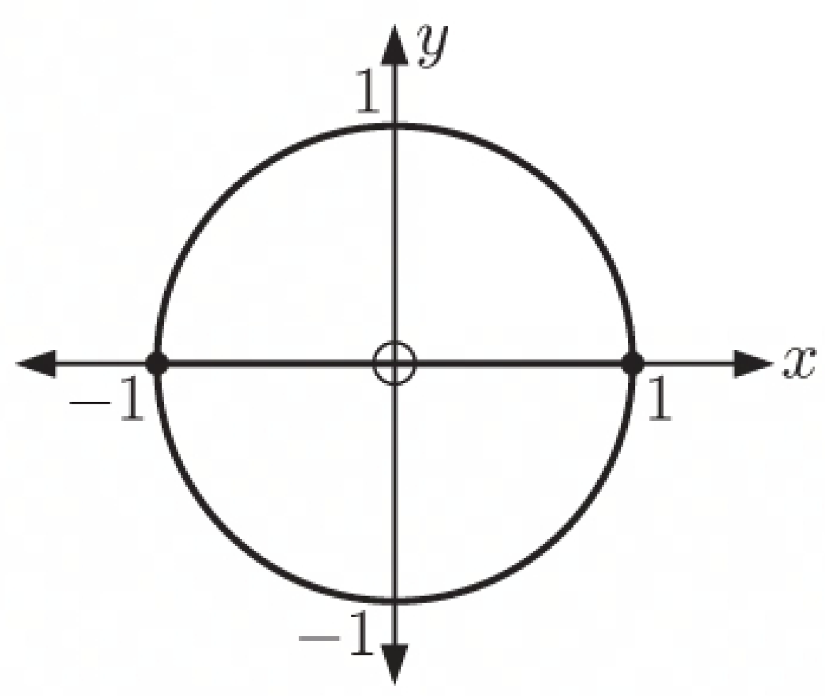
10 **a** $\tan\theta$ is zero when

$$\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{0}{\cos\theta}$$

$\therefore$ when $\sin\theta = 0$

$\therefore \quad \theta = ..., -\pi, 0, \pi, 2\pi, ...$

$\therefore \quad \theta = k\pi$, for $k \in \mathbb{Z}$



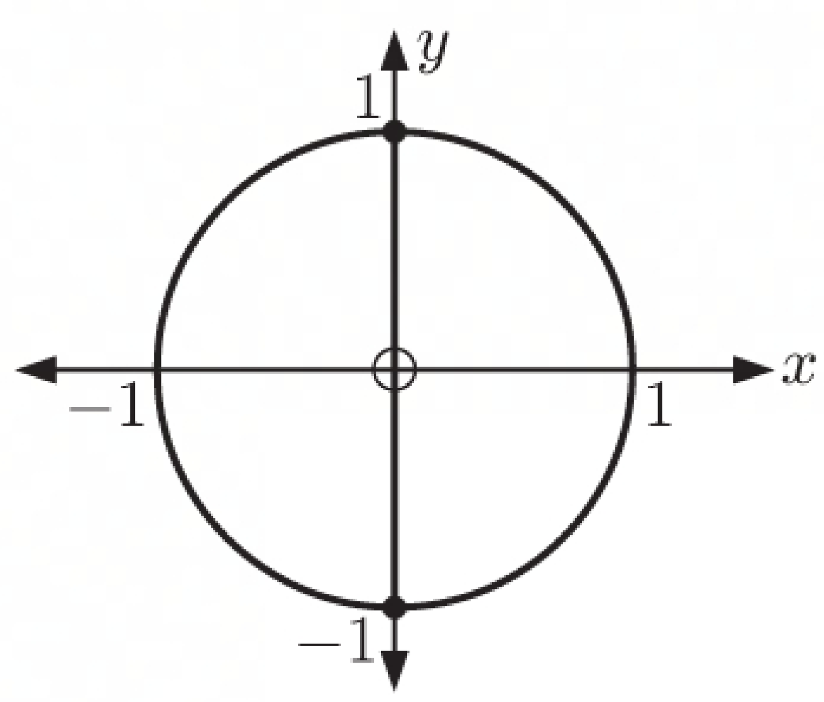
b $\tan\theta$ is undefined when

$$\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{\sin\theta}{0}$$

$\therefore$ when $\cos\theta = 0$

$\therefore \quad \theta = ..., -\frac{\pi}{2}, \frac{\pi}{2}, \frac{3\pi}{2}, ...$

$\therefore \quad \theta = \frac{\pi}{2} + k\pi$, for $k \in \mathbb{Z}$



EXERCISE 8E

1 a
$$\cos^2\theta+\sin^2\theta=1$$
$$\therefore\ \cos^2\theta+(\tfrac{1}{2})^2=1$$
$$\therefore\ \cos^2\theta=\tfrac{3}{4}$$
$$\therefore\ \cos\theta=\pm\tfrac{\sqrt{3}}{2}$$

b
$$\cos^2\theta+\sin^2\theta=1$$
$$\therefore\ \cos^2\theta+(-\tfrac{1}{3})^2=1$$
$$\therefore\ \cos^2\theta=\tfrac{8}{9}$$
$$\therefore\ \cos\theta=\pm\tfrac{\sqrt{8}}{3}$$
$$\therefore\ \cos\theta=\pm\tfrac{2\sqrt{2}}{3}$$

c
$$\cos^2\theta+\sin^2\theta=1$$
$$\therefore\ \cos^2\theta+0^2=1$$
$$\therefore\ \cos\theta=\pm1$$

d
$$\cos^2\theta+\sin^2\theta=1$$
$$\therefore\ \cos^2\theta+(-1)^2=1$$
$$\therefore\ \cos\theta=0$$

2 a
$$\cos^2\theta+\sin^2\theta=1$$
$$\therefore\ (\tfrac{4}{5})^2+\sin^2\theta=1$$
$$\therefore\ \sin^2\theta=\tfrac{9}{25}$$
$$\therefore\ \sin\theta=\pm\tfrac{3}{5}$$

b
$$\cos^2\theta+\sin^2\theta=1$$
$$\therefore\ (-\tfrac{3}{4})^2+\sin^2\theta=1$$
$$\therefore\ \sin^2\theta=\tfrac{7}{16}$$
$$\therefore\ \sin\theta=\pm\tfrac{\sqrt{7}}{4}$$

c
$$\cos^2\theta+\sin^2\theta=1$$
$$\therefore\ 1^2+\sin^2\theta=1$$
$$\therefore\ \sin^2\theta=0$$
$$\therefore\ \sin\theta=0$$

d
$$\cos^2\theta+\sin^2\theta=1$$
$$\therefore\ 0^2+\sin^2\theta=1$$
$$\therefore\ \sin\theta=\pm1$$

3 a
$$\cos^2\theta+\sin^2\theta=1$$
$$\therefore\ \tfrac{4}{9}+\sin^2\theta=1$$
$$\therefore\ \sin^2\theta=\tfrac{5}{9}$$
$$\therefore\ \sin\theta=\pm\tfrac{\sqrt{5}}{3}$$

But θ is in quadrant 1 where $\sin\theta>0$
$$\therefore\ \sin\theta=\tfrac{\sqrt{5}}{3}$$

b
$$\cos^2\theta+\sin^2\theta=1$$
$$\therefore\ \cos^2\theta+\tfrac{4}{25}=1$$
$$\therefore\ \cos^2\theta=\tfrac{21}{25}$$
$$\therefore\ \cos\theta=\pm\tfrac{\sqrt{21}}{5}$$

But θ is in quadrant 2 where $\cos\theta<0$
$$\therefore\ \cos\theta=-\tfrac{\sqrt{21}}{5}$$

c
$$\cos^2\theta+\sin^2\theta=1$$
$$\therefore\ \cos^2\theta+\tfrac{9}{25}=1$$
$$\therefore\ \cos^2\theta=\tfrac{16}{25}$$
$$\therefore\ \cos\theta=\pm\tfrac{4}{5}$$

But θ is in quadrant 4 where $\cos\theta>0$
$$\therefore\ \cos\theta=\tfrac{4}{5}$$

d
$$\cos^2\theta+\sin^2\theta=1$$
$$\therefore\ \tfrac{25}{169}+\sin^2\theta=1$$
$$\therefore\ \sin^2\theta=\tfrac{144}{169}$$
$$\therefore\ \sin\theta=\pm\tfrac{12}{13}$$

But θ is in quadrant 3 where $\sin\theta<0$
$$\therefore\ \sin\theta=-\tfrac{12}{13}$$

4 **a**

$$\cos^2\theta + \sin^2\theta = 1$$
$$\therefore\ \cos^2\theta + \tfrac{1}{9} = 1$$
$$\therefore\ \cos^2\theta = \tfrac{8}{9}$$
$$\therefore\ \cos\theta = \pm\tfrac{2\sqrt{2}}{3}$$

But θ is in quadrant 2 where $\cos\theta < 0$

$$\therefore\ \cos\theta = -\tfrac{2\sqrt{2}}{3}$$

and so
$$\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{\frac{1}{3}}{-\frac{2\sqrt{2}}{3}} = -\frac{1}{2\sqrt{2}}$$

b

$$\cos^2\theta + \sin^2\theta = 1$$
$$\therefore\ \tfrac{1}{25} + \sin^2\theta = 1$$
$$\therefore\ \sin^2\theta = \tfrac{24}{25}$$
$$\therefore\ \sin\theta = \pm\tfrac{2\sqrt{6}}{5}$$

But θ is in quadrant 4 where $\sin\theta < 0$

$$\therefore\ \sin\theta = -\tfrac{2\sqrt{6}}{5}$$

and so
$$\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{-\frac{2\sqrt{6}}{5}}{\frac{1}{5}} = -2\sqrt{6}$$

c

$$\cos^2\theta + \sin^2\theta = 1$$
$$\therefore\ \cos^2\theta + \tfrac{1}{3} = 1$$
$$\therefore\ \cos^2\theta = \tfrac{2}{3}$$
$$\therefore\ \cos\theta = \pm\tfrac{\sqrt{2}}{\sqrt{3}}$$

But θ is in quadrant 3 where $\cos\theta < 0$

$$\therefore\ \cos\theta = -\tfrac{\sqrt{2}}{\sqrt{3}}$$

and so
$$\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{-\frac{1}{\sqrt{3}}}{-\frac{\sqrt{2}}{\sqrt{3}}} = \frac{1}{\sqrt{2}}$$

d

$$\cos^2\theta + \sin^2\theta = 1$$
$$\therefore\ \tfrac{9}{16} + \sin^2\theta = 1$$
$$\therefore\ \sin^2\theta = \tfrac{7}{16}$$
$$\therefore\ \sin\theta = \pm\tfrac{\sqrt{7}}{4}$$

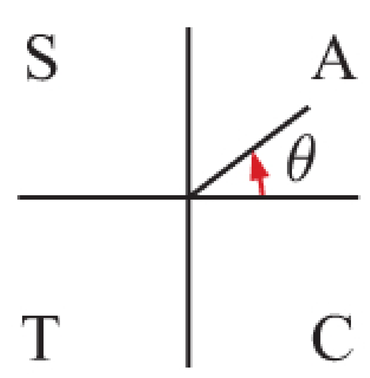
But θ is in quadrant 2 where $\sin\theta > 0$

$$\therefore\ \sin\theta = \tfrac{\sqrt{7}}{4}$$

and so
$$\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{\frac{\sqrt{7}}{4}}{-\frac{3}{4}} = -\frac{\sqrt{7}}{3}$$

5 **a**

$$\tan\theta = \frac{\sin\theta}{\cos\theta} = \tfrac{2}{3}$$
$$\therefore\ \sin\theta = \tfrac{2}{3}\cos\theta$$



Now $\cos^2\theta + \sin^2\theta = 1$

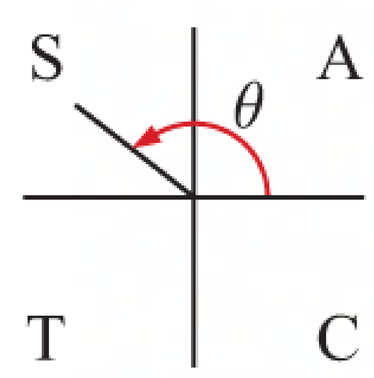
$$\therefore\ \cos^2\theta + \tfrac{4}{9}\cos^2\theta = 1$$
$$\therefore\ \tfrac{13}{9}\cos^2\theta = 1$$
$$\therefore\ \cos^2\theta = \tfrac{9}{13}$$
$$\therefore\ \cos\theta = \pm\tfrac{3}{\sqrt{13}}$$

But θ is in quadrant 1 where $\cos\theta$ and $\sin\theta$ are positive.

$$\therefore\ \cos\theta = \tfrac{3}{\sqrt{13}},\quad \sin\theta = \tfrac{2}{\sqrt{13}}$$

b

$$\tan\theta = \frac{\sin\theta}{\cos\theta} = -\tfrac{4}{3}$$
$$\therefore\ \sin\theta = -\tfrac{4}{3}\cos\theta$$



Now $\cos^2\theta + \sin^2\theta = 1$

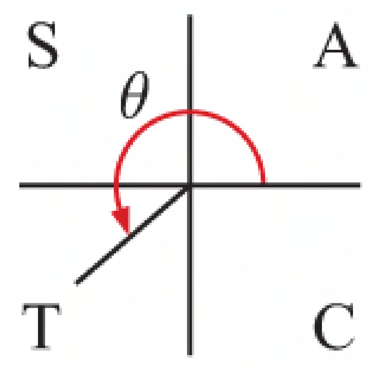
$$\therefore\ \cos^2\theta + \tfrac{16}{9}\cos^2\theta = 1$$
$$\therefore\ \tfrac{25}{9}\cos^2\theta = 1$$
$$\therefore\ \cos^2\theta = \tfrac{9}{25}$$
$$\therefore\ \cos\theta = \pm\tfrac{3}{5}$$

But θ is in quadrant 2 where $\cos\theta$ is negative and $\sin\theta$ is positive.

$$\therefore\ \cos\theta = -\tfrac{3}{5},\quad \sin\theta = \tfrac{4}{5}$$

c $\tan\theta = \dfrac{\sin\theta}{\cos\theta} = \dfrac{\sqrt{5}}{3}$

$\therefore\ \sin\theta = \frac{\sqrt{5}}{3}\cos\theta$



Now $\cos^2\theta + \sin^2\theta = 1$

$\therefore\ \cos^2\theta + \frac{5}{9}\cos^2\theta = 1$

$\therefore\ \frac{14}{9}\cos^2\theta = 1$

$\therefore\ \cos^2\theta = \frac{9}{14}$

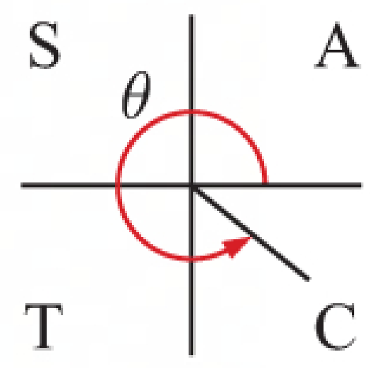
$\therefore\ \cos\theta = \pm\frac{3}{\sqrt{14}}$

But θ is in quadrant 3 where $\cos\theta$ and $\sin\theta$ are both negative.

$\therefore\ \cos\theta = -\frac{3}{\sqrt{14}},\quad \sin\theta = -\frac{\sqrt{5}}{\sqrt{14}}$

d $\tan\theta = \dfrac{\sin\theta}{\cos\theta} = -\frac{12}{5}$

$\therefore\ \sin\theta = -\frac{12}{5}\cos\theta$



Now $\cos^2\theta + \sin^2\theta = 1$

$\therefore\ \cos^2\theta + \frac{144}{25}\cos^2\theta = 1$

$\therefore\ \frac{169}{25}\cos^2\theta = 1$

$\therefore\ \cos^2\theta = \frac{25}{169}$

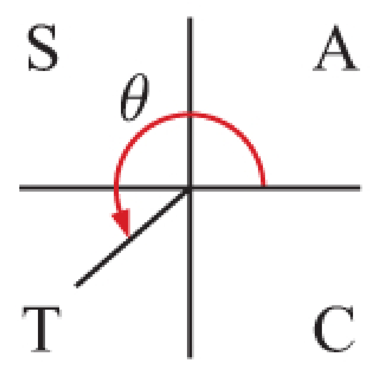
$\therefore\ \cos\theta = \pm\frac{5}{13}$

But θ is in quadrant 4 where $\cos\theta$ is positive and $\sin\theta$ is negative.

$\therefore\ \cos\theta = \frac{5}{13},\quad \sin\theta = -\frac{12}{13}$

6 $\tan\theta = \dfrac{\sin\theta}{\cos\theta} = k$

$\therefore\ \sin\theta = k\cos\theta$



Now $\cos^2\theta + \sin^2\theta = 1$

$\therefore\ \cos^2\theta + k^2\cos^2\theta = 1$

$\therefore\ (k^2+1)\cos^2\theta = 1$

$\therefore\ \cos\theta = \dfrac{\pm 1}{\sqrt{k^2+1}}$

But θ is in quadrant 3 where $\cos\theta$ and $\sin\theta$ are both negative, and $\tan\theta$ is positive

$\therefore\ k$ is positive.

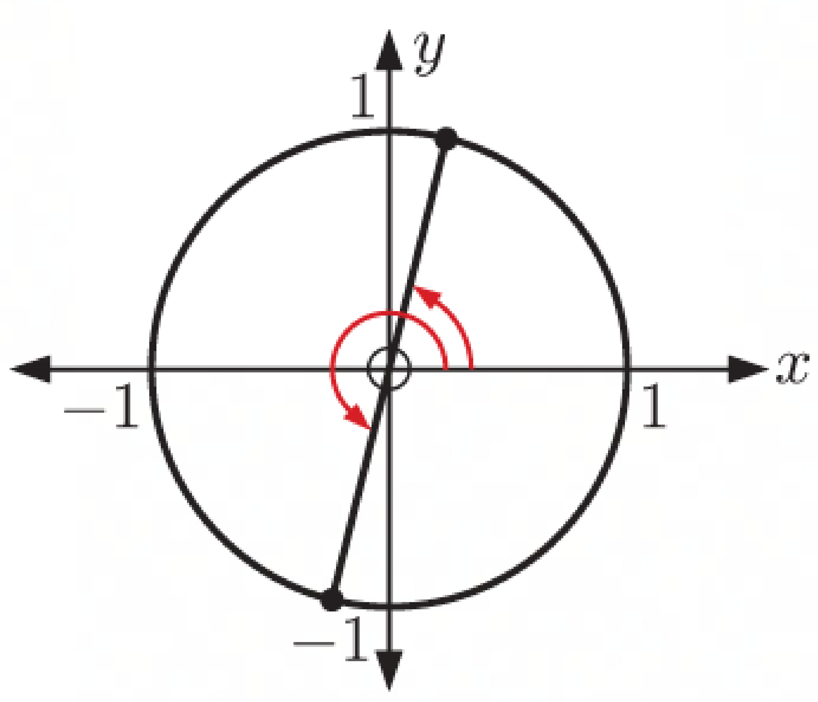
$\therefore\ \cos\theta = \dfrac{-1}{\sqrt{k^2+1}},\quad \sin\theta = \dfrac{-k}{\sqrt{k^2+1}}$

EXERCISE 8F

1 **a** $\tan\theta = 4$

Using technology,

$\tan^{-1}(4) \approx 75.96°$



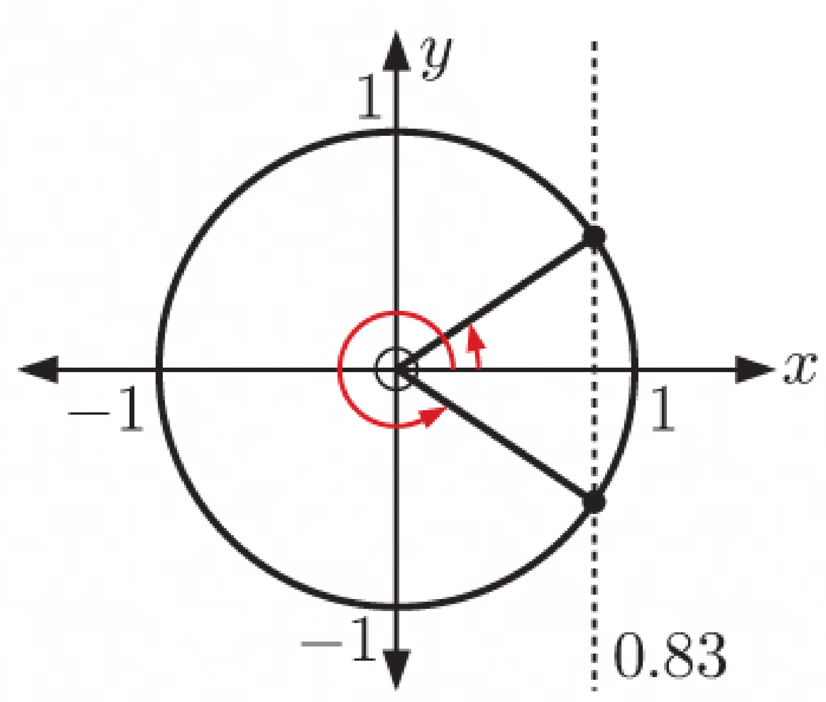
$\therefore\ \theta \approx 75.96°$ or $180° + 75.96°$

$\therefore\ \theta \approx 76.0°$ or $256°$

b $\cos\theta = 0.83$

Using technology,

$\cos^{-1}(0.83) \approx 33.90°$



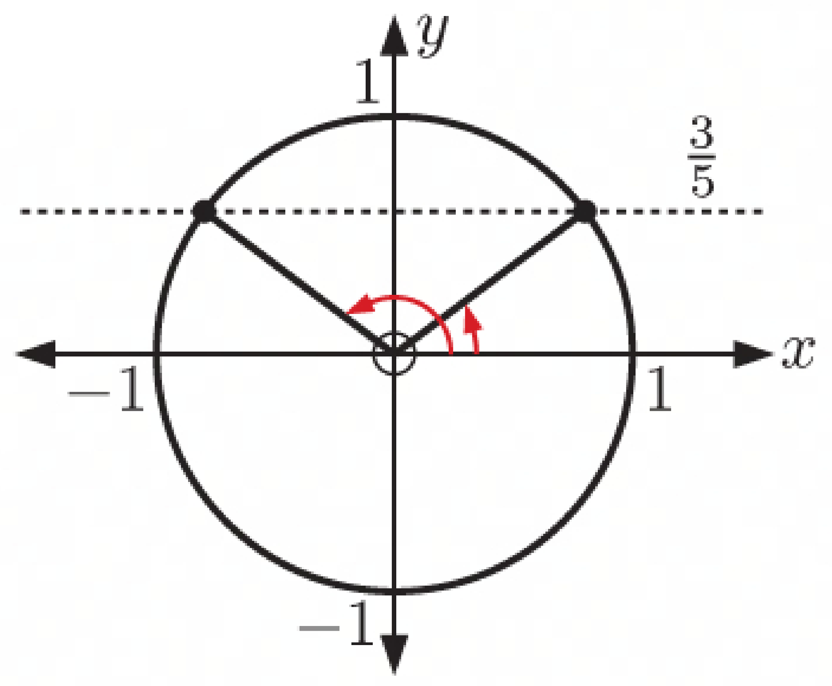
$\therefore\ \theta \approx 33.90°$ or $360° - 33.90°$

$\therefore\ \theta \approx 33.9°$ or $326.1°$

c $\sin\theta = \frac{3}{5}$

Using technology,

$\sin^{-1}\left(\frac{3}{5}\right) \approx 36.87°$

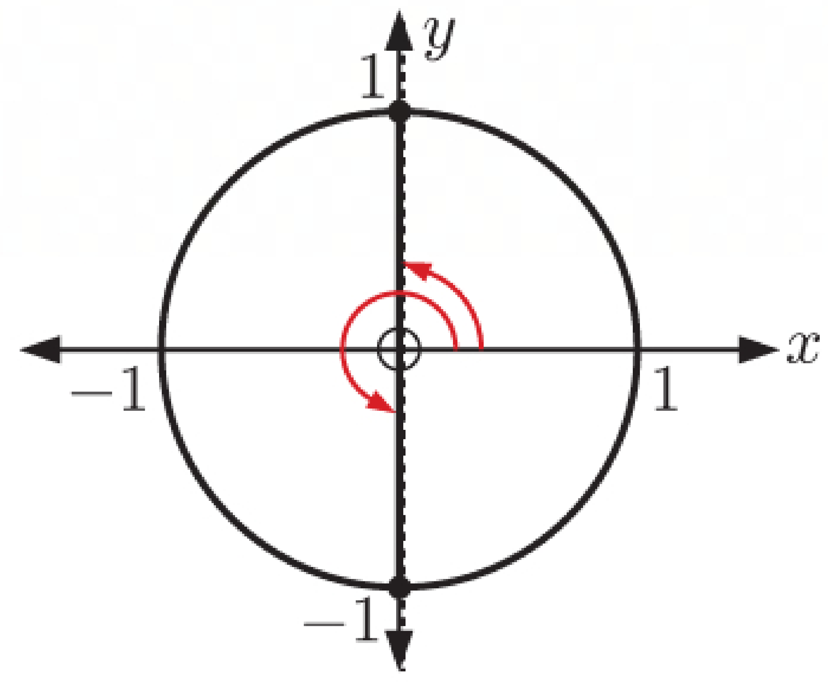


$\therefore\ \theta \approx 36.87°$ or $180° - 36.87°$

$\therefore\ \theta \approx 36.9°$ or $143.1°$

d $\cos\theta = 0$

$\cos^{-1}(0) = 90°$



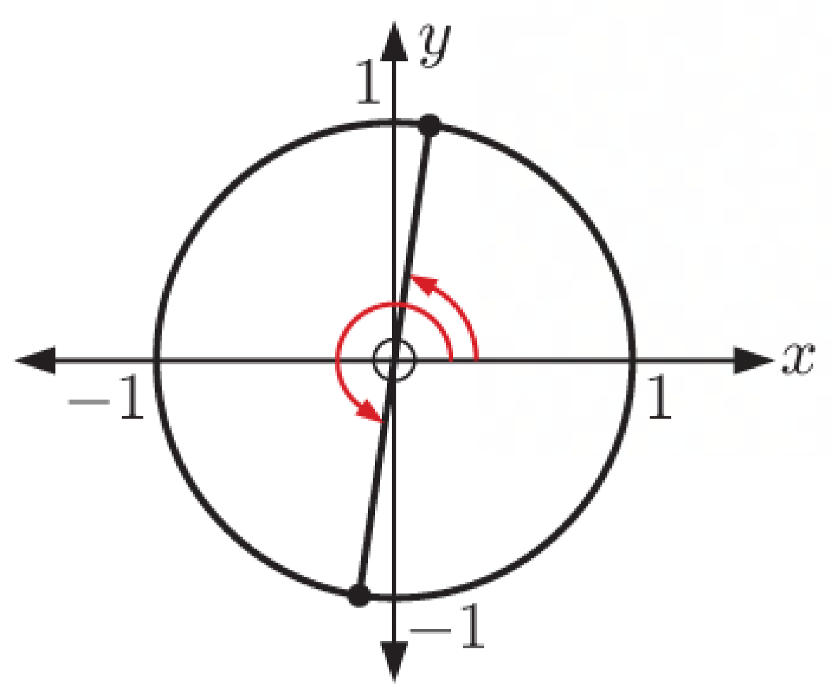
$\therefore\ \theta = 90°$ or $360° - 90°$

$\therefore\ \theta = 90°$ or $270°$

e $\tan\theta = 6.67$

Using technology,

$\tan^{-1}(6.67) \approx 81.47°$



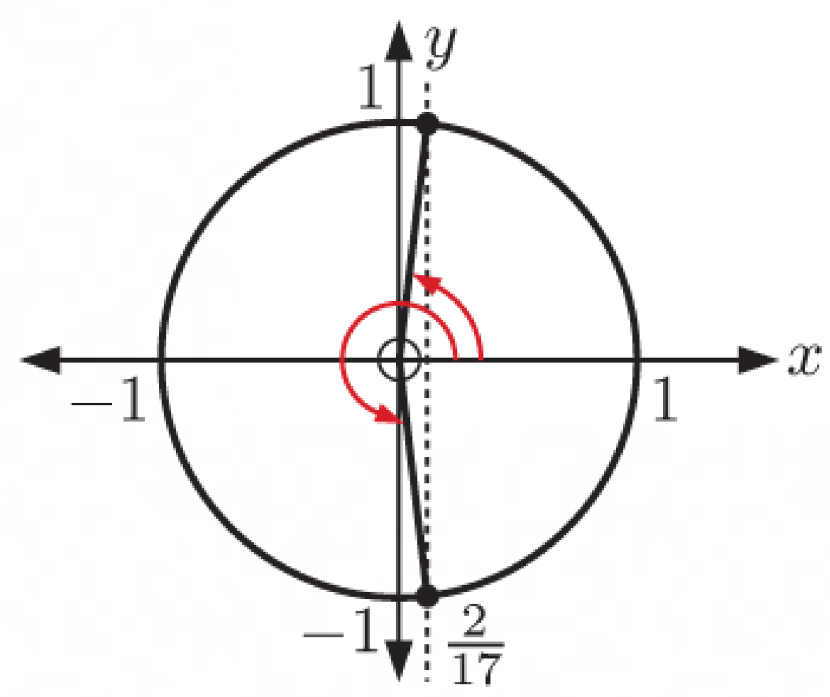
$\therefore\ \theta \approx 81.47°$ or $180° + 81.47°$

$\therefore\ \theta \approx 81.5°$ or $261.5°$

f $\cos\theta = \frac{2}{17}$

Using technology,

$\cos^{-1}\left(\frac{2}{17}\right) \approx 83.24°$



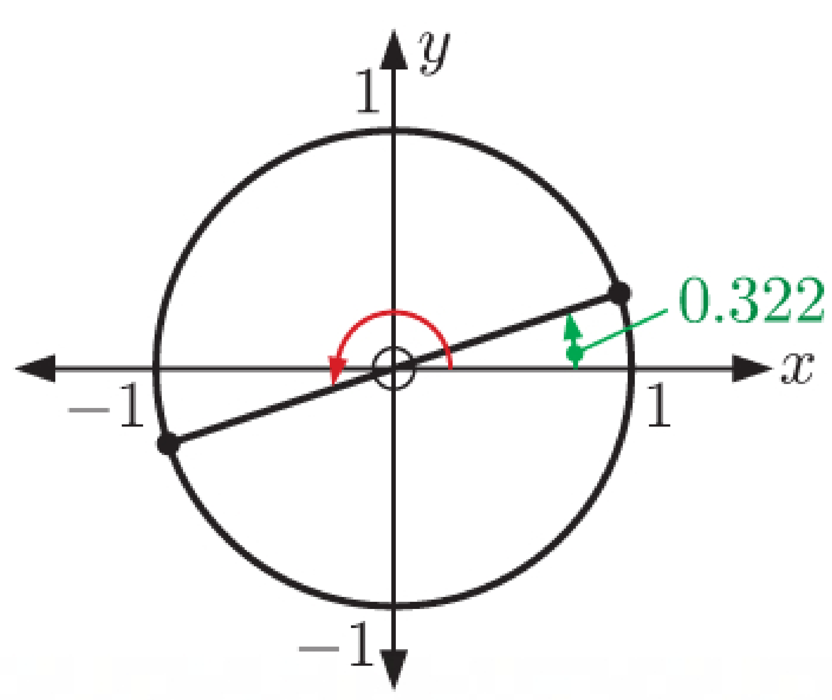
$\therefore\ \theta \approx 83.24°$ or $360° - 83.24°$

$\therefore\ \theta \approx 83.2°$ or $276.8°$

2 **a** $\tan\theta = \frac{1}{3}$

Using technology,

$\tan^{-1}\left(\frac{1}{3}\right) \approx 0.322$



But $0 \leqslant \theta \leqslant 2\pi$

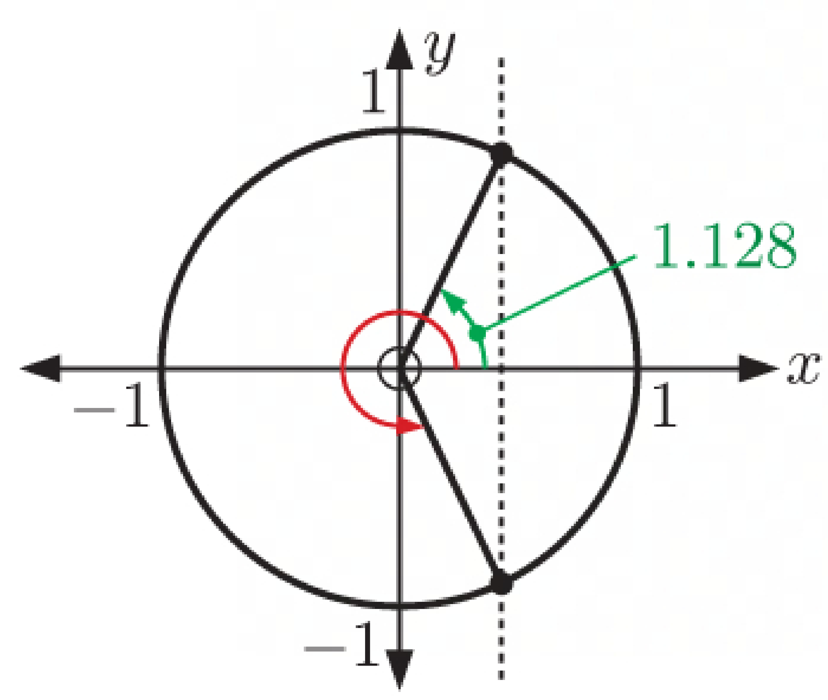
$\therefore\ \theta \approx 0.322$ or $\pi + 0.322$

$\therefore\ \theta \approx 0.322$ or 3.46

b $\cos\theta = \frac{3}{7}$

Using technology,

$\cos^{-1}\left(\frac{3}{7}\right) \approx 1.128$



But $0 \leqslant \theta \leqslant 2\pi$

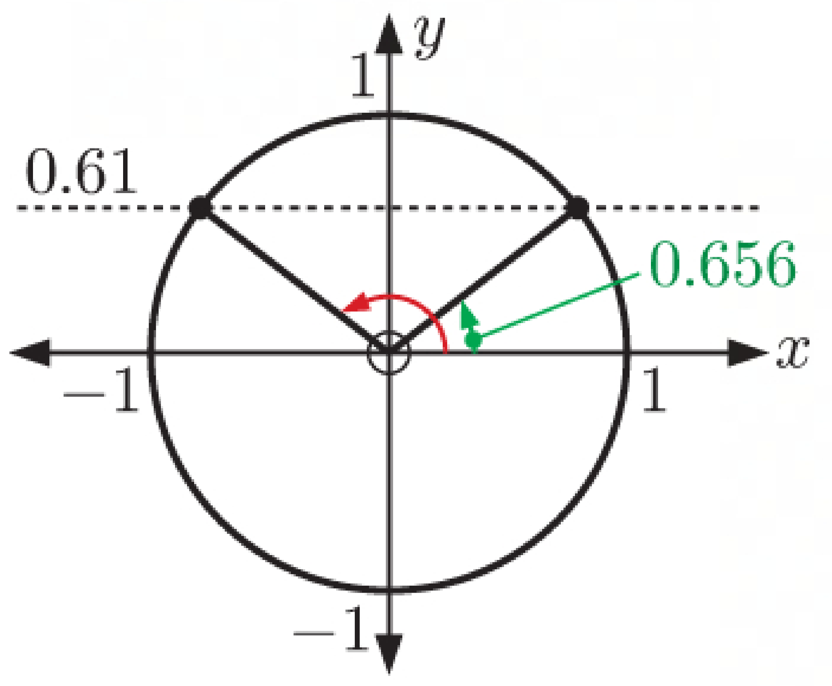
$\therefore\ \theta \approx 1.128$ or $2\pi - 1.128$

$\therefore\ \theta \approx 1.13$ or 5.16

c $\sin\theta = 0.61$

Using technology,

$\sin^{-1}(0.61) \approx 0.656$



But $0 \leqslant \theta \leqslant 2\pi$

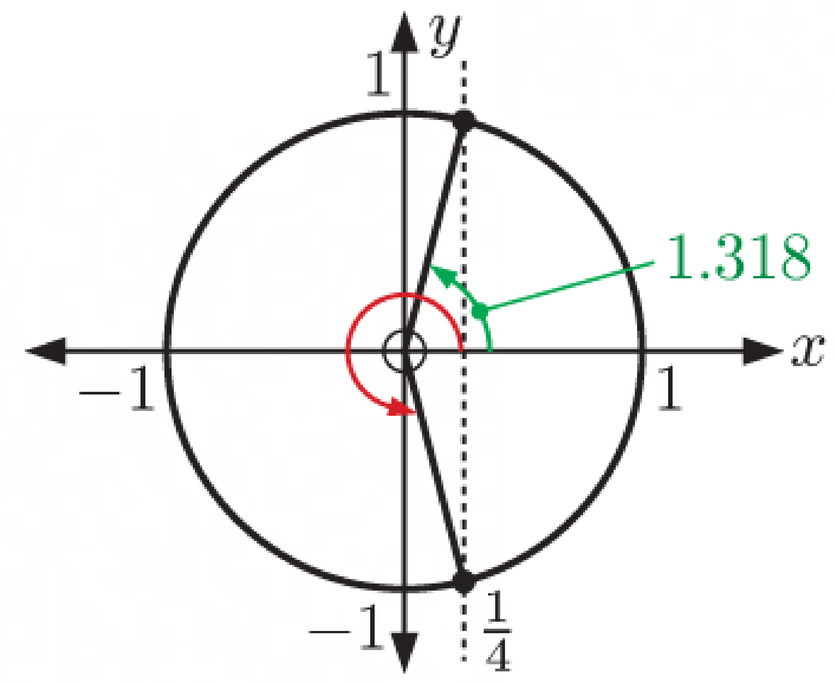
$\therefore \quad \theta \approx 0.656 \quad \text{or} \quad \pi - 0.656$

$\therefore \quad \theta \approx 0.656 \quad \text{or} \quad 2.49$

d $\cos\theta = \frac{1}{4}$

Using technology,

$\cos^{-1}\left(\frac{1}{4}\right) \approx 1.318$



But $0 \leqslant \theta \leqslant 2\pi$

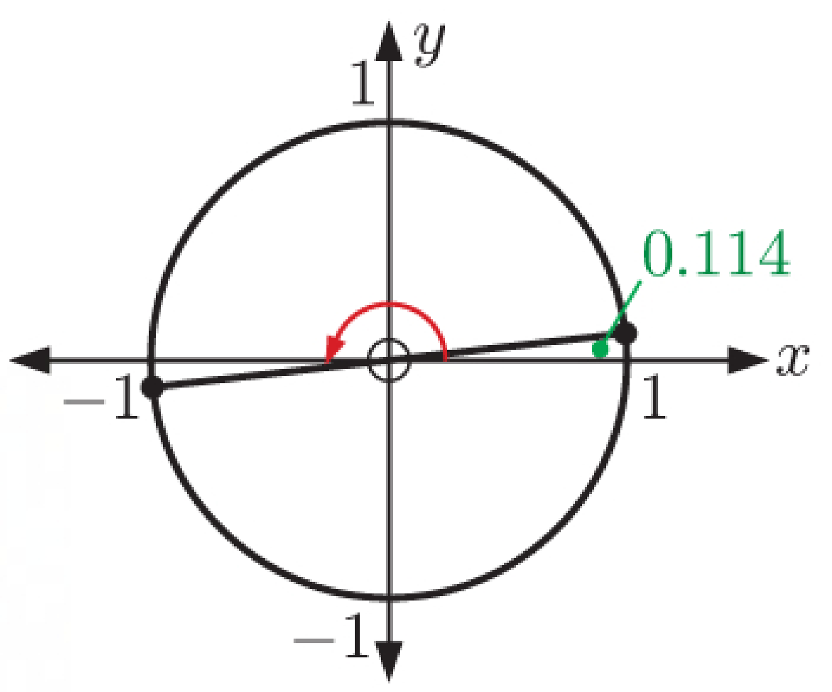
$\therefore \quad \theta \approx 1.318 \quad \text{or} \quad 2\pi - 1.318$

$\therefore \quad \theta \approx 1.32 \quad \text{or} \quad 4.97$

e $\tan\theta = 0.114$

Using technology,

$\tan^{-1}(0.114) \approx 0.114$



But $0 \leqslant \theta \leqslant 2\pi$

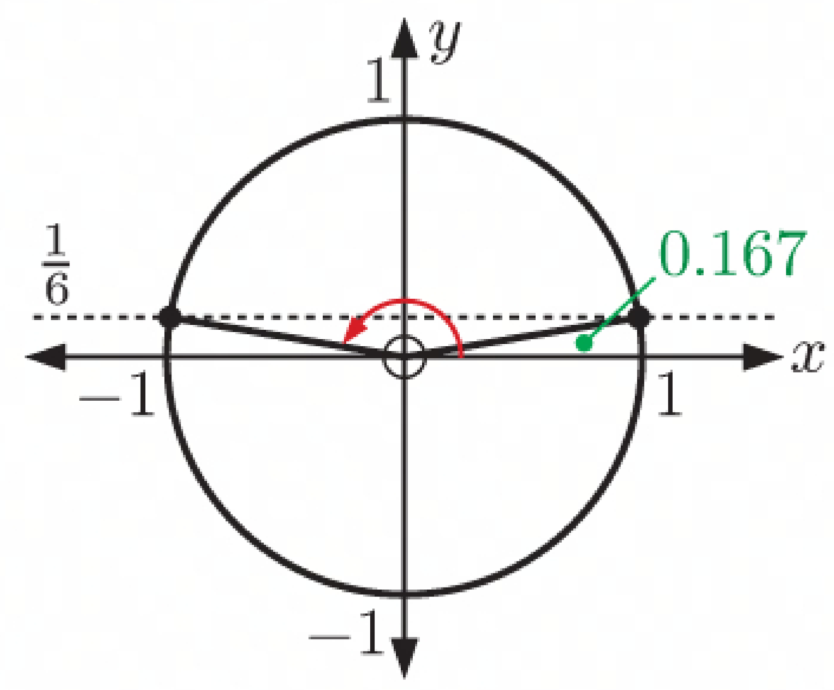
$\therefore \quad \theta \approx 0.114 \quad \text{or} \quad \pi + 0.114$

$\therefore \quad \theta \approx 0.114 \quad \text{or} \quad 3.26$

f $\sin\theta = \frac{1}{6}$

Using technology,

$\sin^{-1}\left(\frac{1}{6}\right) \approx 0.167$



But $0 \leqslant \theta \leqslant 2\pi$

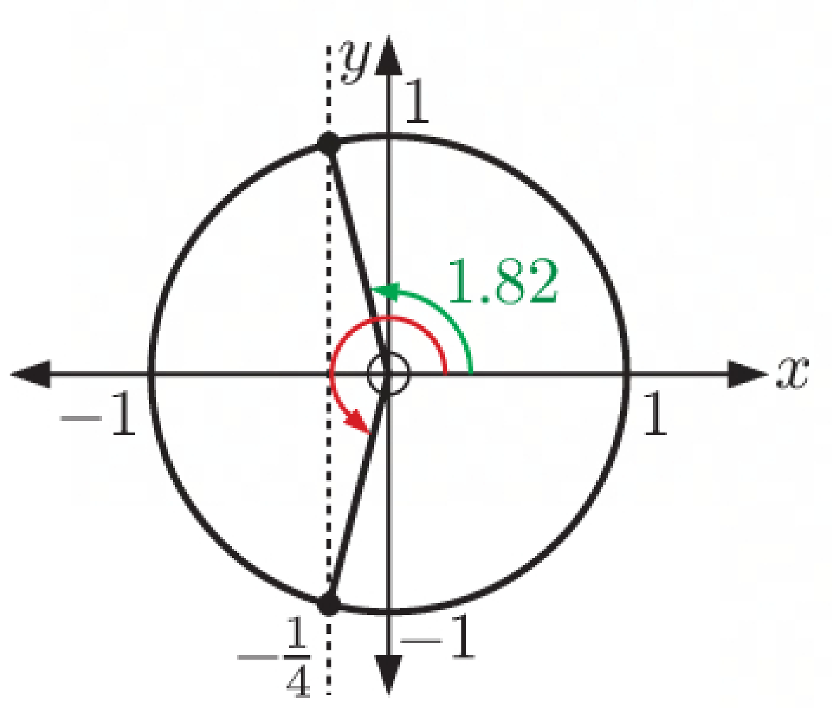
$\therefore \quad \theta \approx 0.167 \quad \text{or} \quad \pi - 0.167$

$\therefore \quad \theta \approx 0.167 \quad \text{or} \quad 2.97$

3 **a** $\cos\theta = -\frac{1}{4}$

Using technology,

$\cos^{-1}\left(-\frac{1}{4}\right) \approx 1.82$



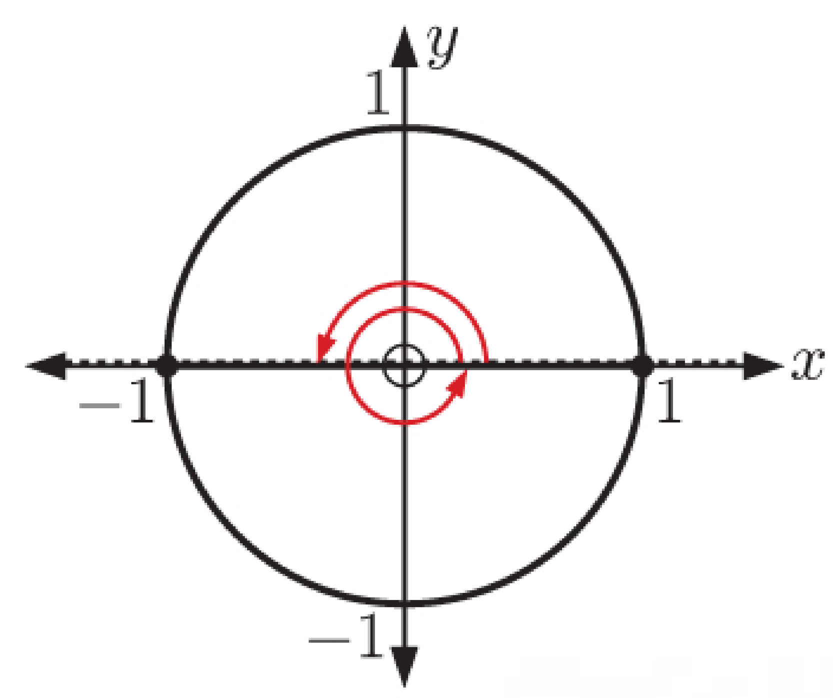
But $0 \leqslant \theta \leqslant 2\pi$

$\therefore \quad \theta \approx 1.82 \quad \text{or} \quad 2\pi - 1.82$

$\therefore \quad \theta \approx 1.82 \quad \text{or} \quad 4.46$

b $\sin\theta = 0$

$\therefore \quad \sin^{-1}(0) = 0$



But $0 \leqslant \theta \leqslant 2\pi$

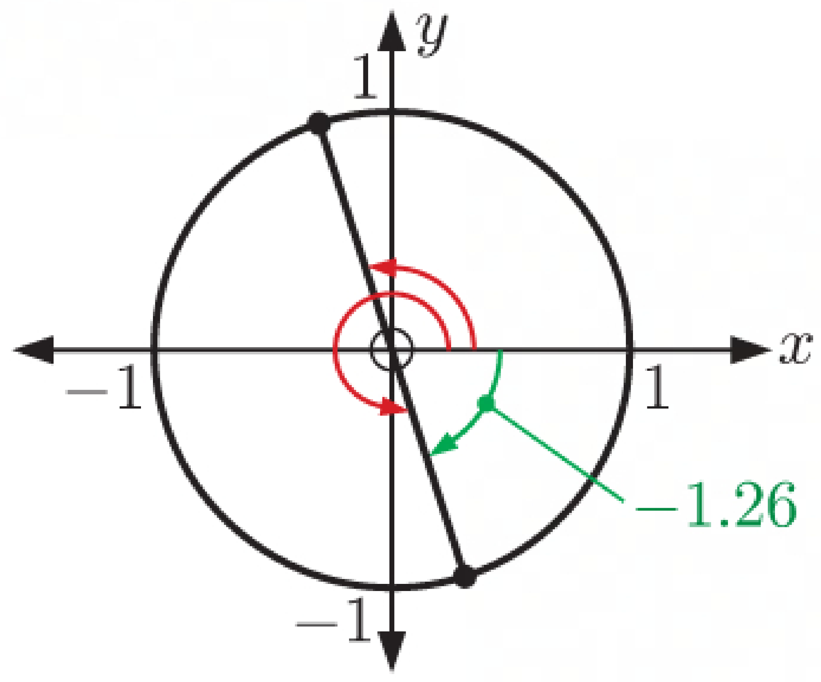
$\therefore \quad \theta = 0 \quad \text{or} \quad \pi - 0 \quad \text{or} \quad 2\pi$

$\therefore \quad \theta = 0,\ \pi,\ \text{or}\ 2\pi$

c $\tan\theta = -3.1$

Using technology,

$\tan^{-1}(-3.1) \approx -1.26$



But $0 \leqslant \theta \leqslant 2\pi$

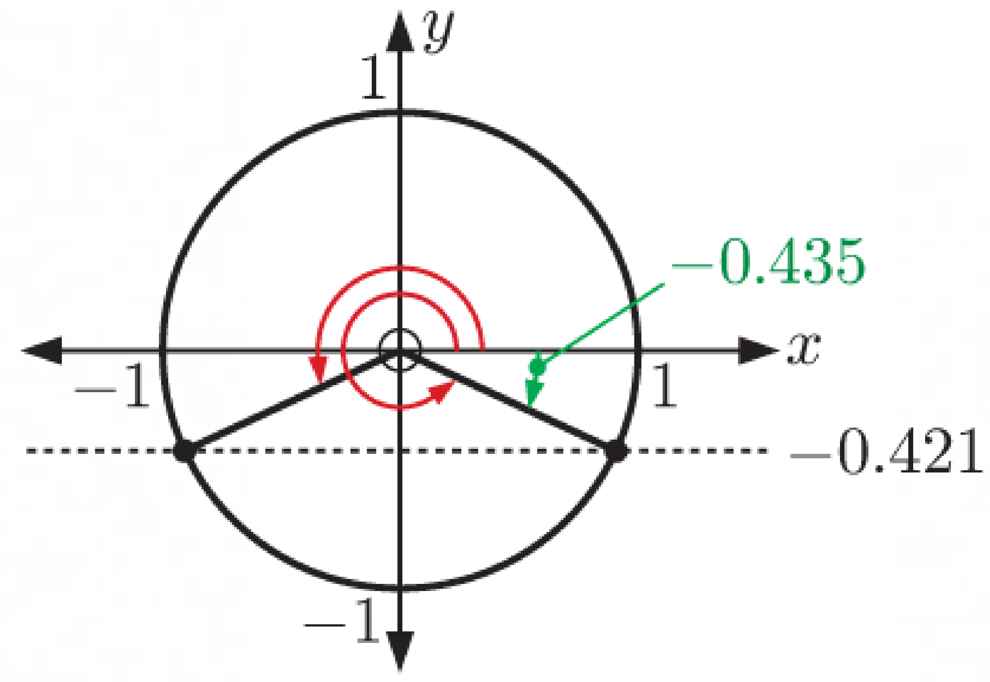
$\therefore \quad \theta \approx \pi - 1.26 \quad \text{or} \quad 2\pi - 1.26$

$\therefore \quad \theta \approx 1.88 \quad \text{or} \quad 5.02$

d $\sin\theta = -0.421$

Using technology,

$\sin^{-1}(-0.421) \approx -0.435$



But $0 \leqslant \theta \leqslant 2\pi$

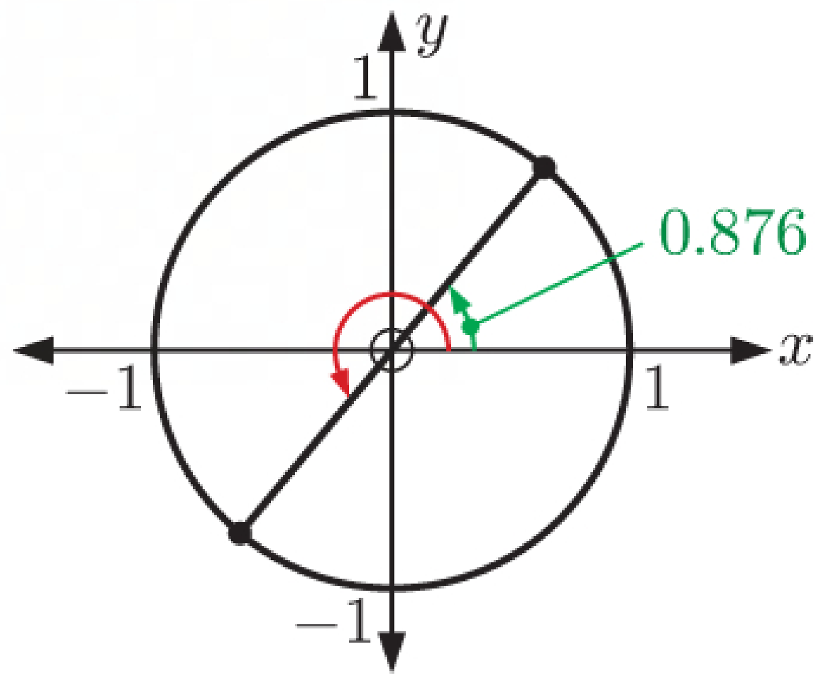
$\therefore \quad \theta \approx \pi + 0.435 \quad \text{or} \quad 2\pi - 0.435$

$\therefore \quad \theta \approx 3.58 \quad \text{or} \quad 5.85$

e $\tan\theta = 1.2$

Using technology,

$\tan^{-1}(1.2) \approx 0.876$



But $0 \leqslant \theta \leqslant 2\pi$

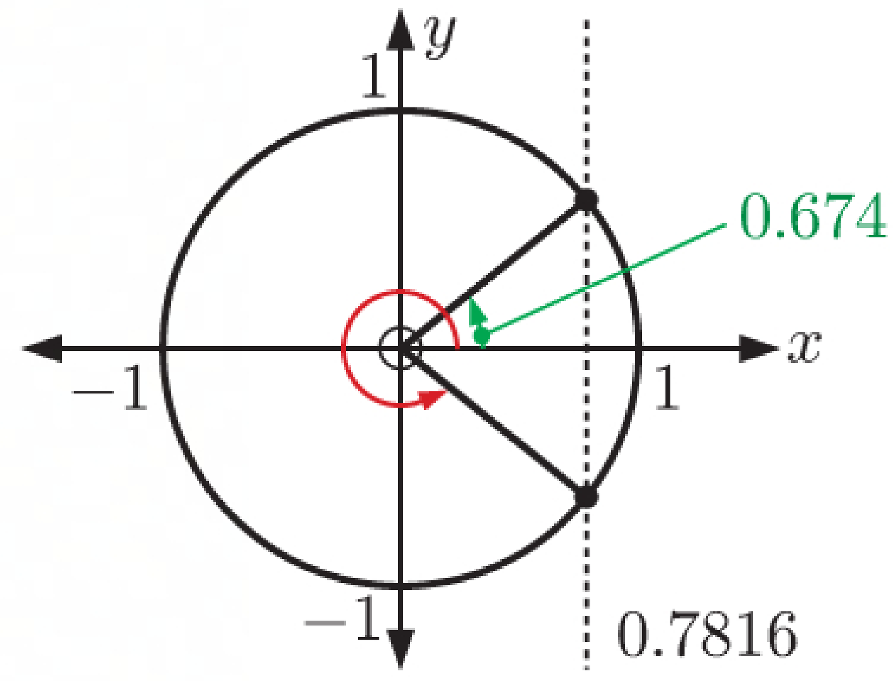
$\therefore \quad \theta \approx 0.876 \quad \text{or} \quad \pi + 0.876$

$\therefore \quad \theta \approx 0.876 \quad \text{or} \quad 4.02$

f $\cos\theta = 0.7816$

Using technology,

$\cos^{-1}(0.7816) \approx 0.674$



But $0 \leqslant \theta \leqslant 2\pi$

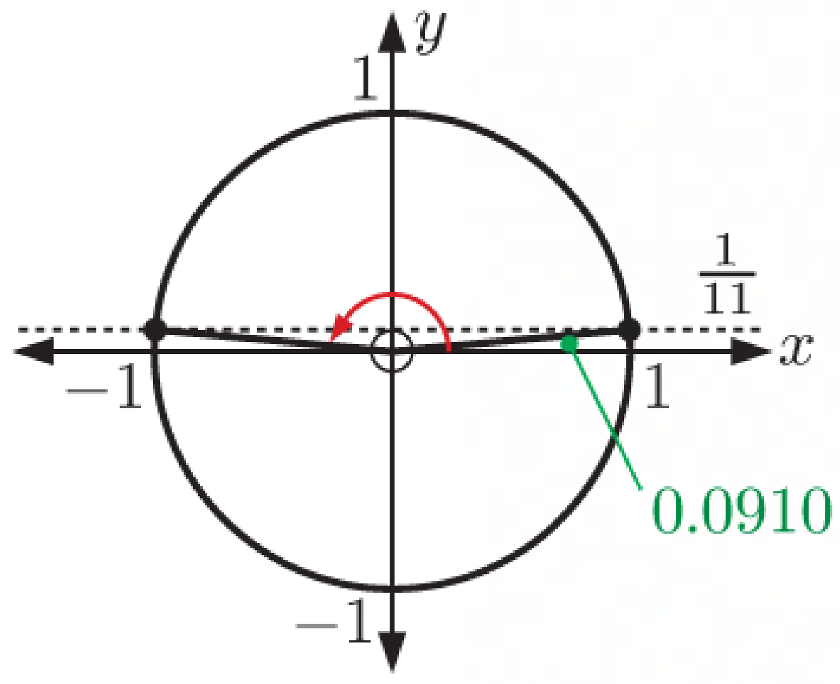
$\therefore \quad \theta \approx 0.674 \quad \text{or} \quad 2\pi - 0.674$

$\therefore \quad \theta \approx 0.674 \quad \text{or} \quad 5.61$

g $\sin\theta = \frac{1}{11}$

Using technology,

$\sin^{-1}\left(\frac{1}{11}\right) \approx 0.0910$



But $0 \leqslant \theta \leqslant 2\pi$

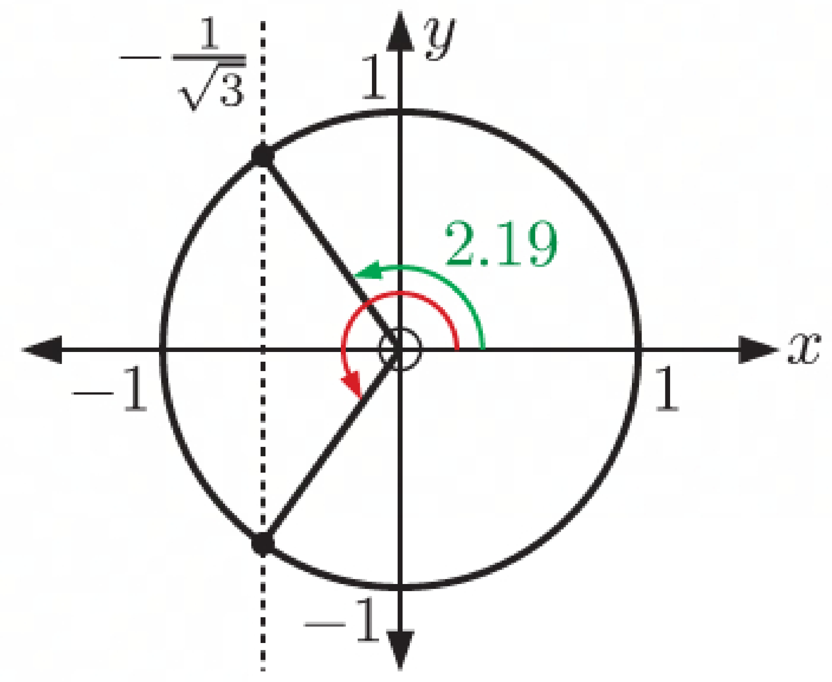
$\therefore \quad \theta \approx 0.0910 \quad \text{or} \quad \pi - 0.0910$

$\therefore \quad \theta \approx 0.0910 \quad \text{or} \quad 3.05$

h $\cos\theta = -\frac{1}{\sqrt{3}}$

Using technology,

$\cos^{-1}\left(-\frac{1}{\sqrt{3}}\right) \approx 2.19$



But $0 \leqslant \theta \leqslant 2\pi$

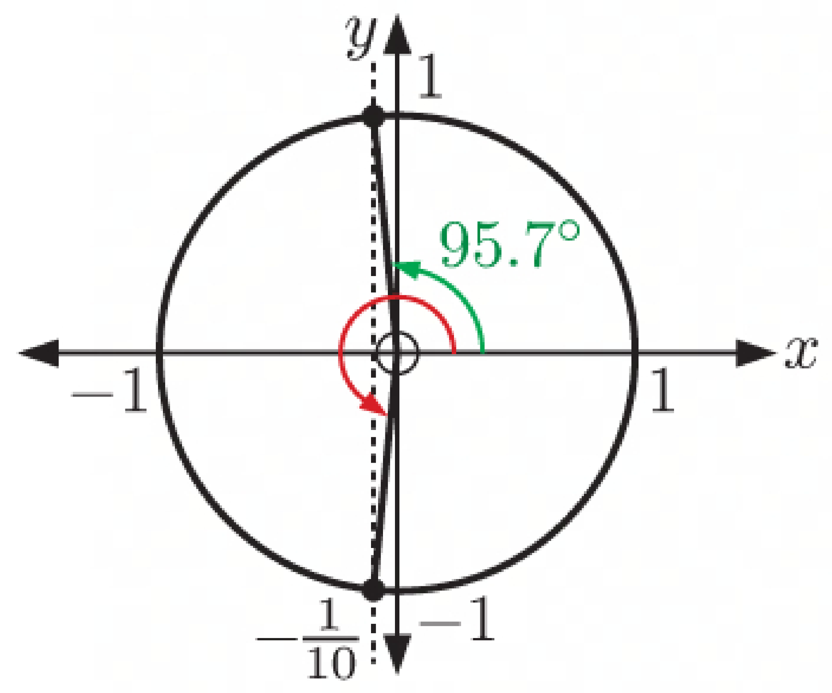
$\therefore \quad \theta \approx 2.19 \quad \text{or} \quad 2\pi - 2.19$

$\therefore \quad \theta \approx 2.19 \quad \text{or} \quad 4.10$

4 **a** $\cos\theta = -\frac{1}{10}$

Using technology,

$\cos^{-1}\left(-\frac{1}{10}\right) \approx 95.7°$



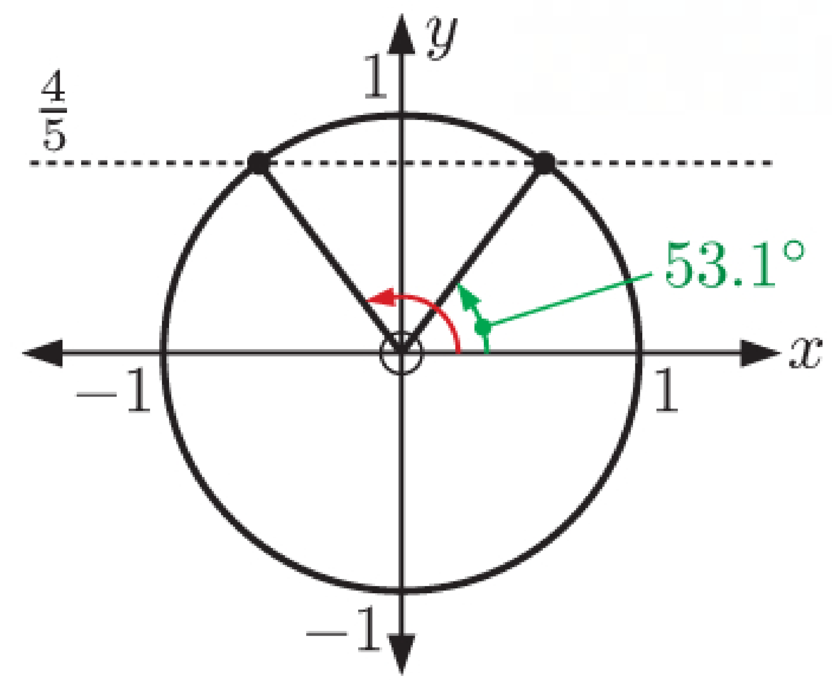
But $-180° \leqslant \theta \leqslant 180°$

$\therefore\ \theta \approx -95.7°$ or $95.7°$

b $\sin\theta = \frac{4}{5}$

Using technology,

$\sin^{-1}\left(\frac{4}{5}\right) \approx 53.1°$



But $-180° \leqslant \theta \leqslant 180°$

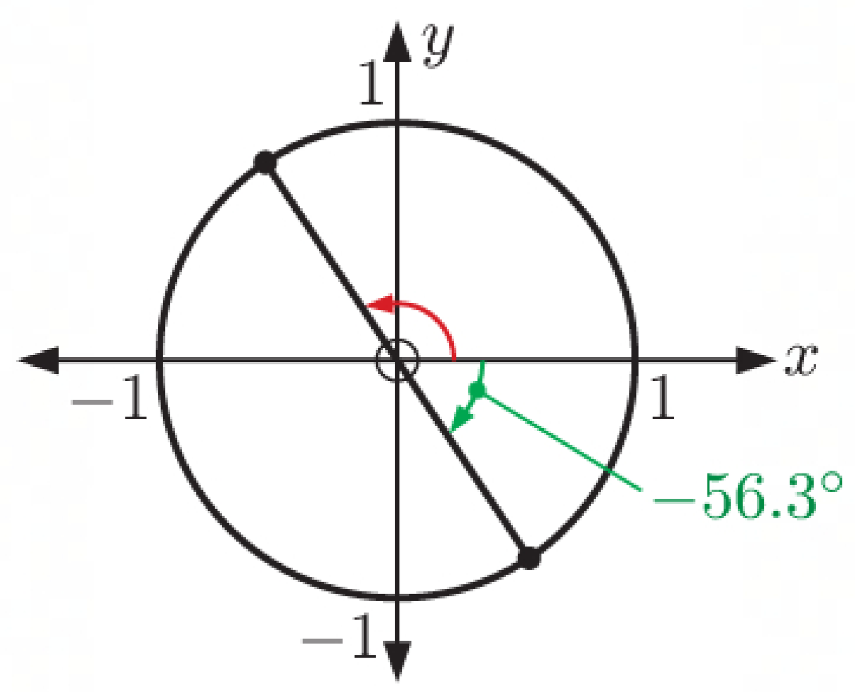
$\therefore\ \theta \approx 53.1°$ or $180° - 53.1°$

$\therefore\ \theta \approx 53.1°$ or $126.9°$

c $\tan\theta = -\frac{3}{2}$

Using technology,

$\tan^{-1}\left(-\frac{3}{2}\right) \approx -56.3°$



But $-180° \leqslant \theta \leqslant 180°$

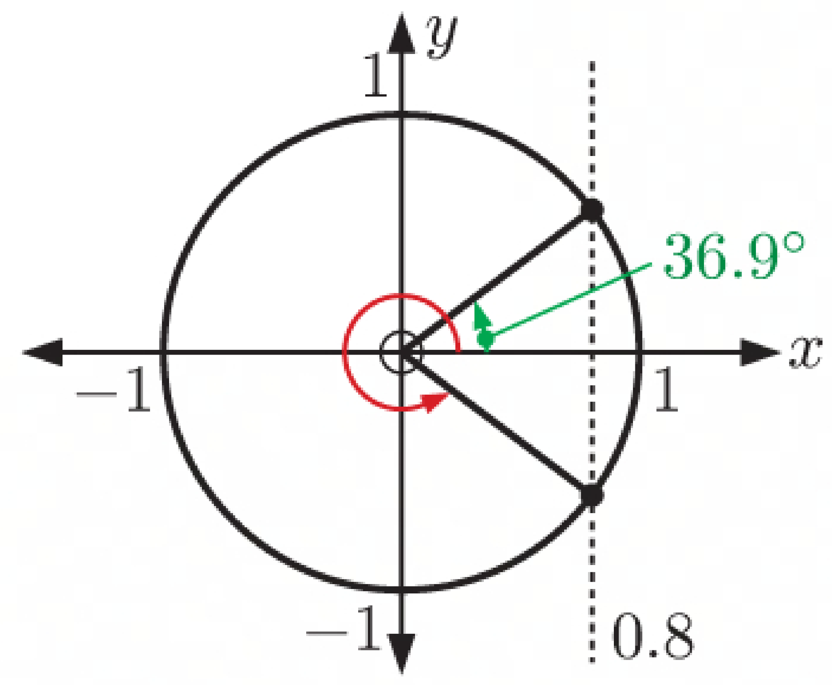
$\therefore\ \theta \approx -56.3°$ or $180° - 56.3°$

$\therefore\ \theta \approx -56.3°$ or $123.7°$

d $\cos\theta = 0.8$

Using technology,

$\cos^{-1}(0.8) \approx 36.9°$



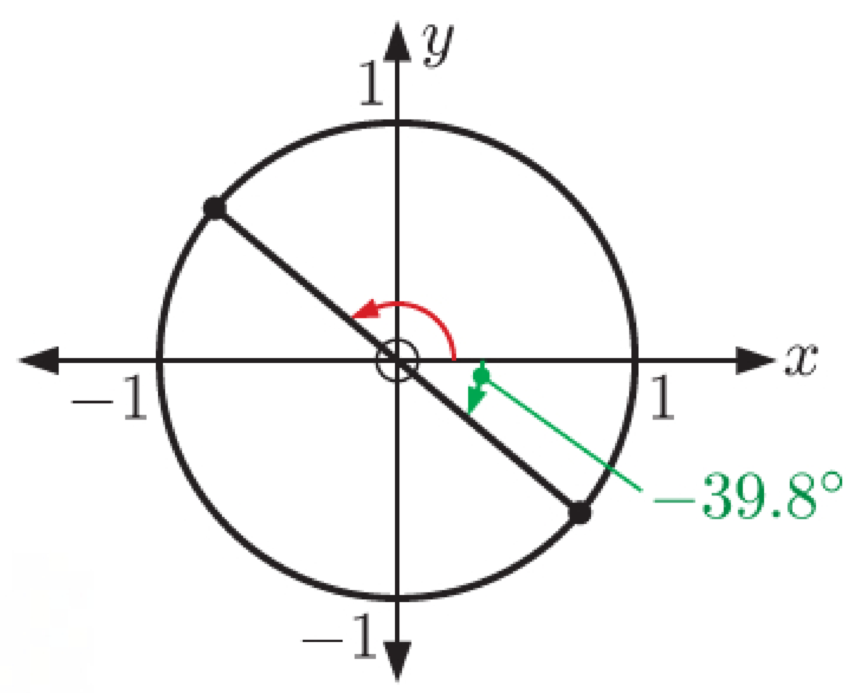
But $-180° \leqslant \theta \leqslant 180°$

$\therefore\ \theta \approx -36.9°$ or $36.9°$

e $\tan\theta = -\frac{5}{6}$

Using technology,

$\tan^{-1}\left(-\frac{5}{6}\right) \approx -39.8°$



But $-180° \leqslant \theta \leqslant 180°$

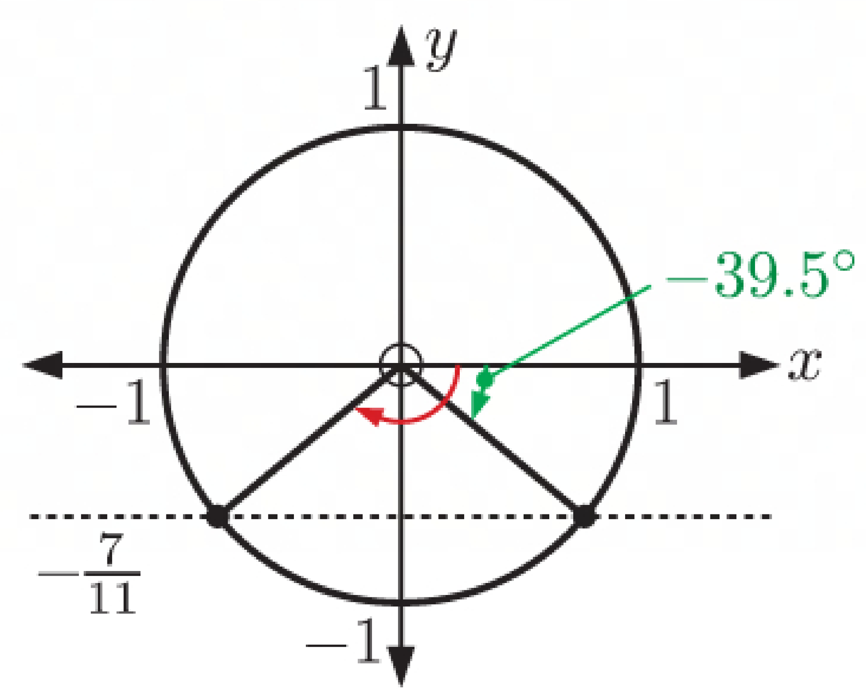
$\therefore\ \theta \approx -39.8°$ or $180° - 39.8°$

$\therefore\ \theta \approx -39.8°$ or $140.2°$

f $\sin\theta = -\frac{7}{11}$

Using technology,

$\sin^{-1}\left(-\frac{7}{11}\right) \approx -39.5°$



But $-180° \leqslant \theta \leqslant 180°$

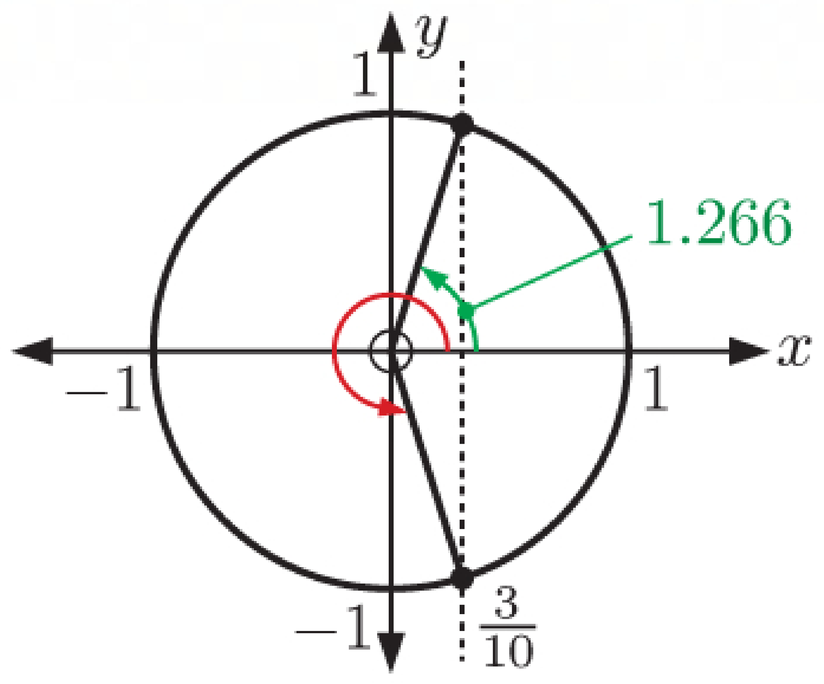
$\therefore\ \theta \approx 39.5° - 180°$ or $-39.5°$

$\therefore\ \theta \approx -140.5°$ or $-39.5°$

5 **a** $\cos\theta = \frac{3}{10}$

Using technology,

$\cos^{-1}\left(\frac{3}{10}\right) \approx 1.266$



But $0 \leqslant \theta \leqslant 2\pi$

$\therefore \quad \theta \approx 1.266 \quad \text{or} \quad 2\pi - 1.266$

$\therefore \quad \theta \approx 1.27 \quad \text{or} \quad 5.02$

b
$$\sin^2\theta + \cos^2\theta = 1$$
$$\therefore \quad \sin^2\theta + \left(\tfrac{3}{10}\right)^2 = 1$$
$$\therefore \quad \sin^2\theta + \tfrac{9}{100} = 1$$
$$\therefore \quad \sin^2\theta = \tfrac{91}{100}$$
$$\therefore \quad \sin\theta = \pm\tfrac{\sqrt{91}}{10}$$

$\theta \approx 1.27$ corresponds to the first quadrant, where $\sin\theta$ is positive.

So, for $\theta \approx 1.27$, $\sin\theta = \frac{\sqrt{91}}{10}$,

$$\tan\theta = \frac{\frac{\sqrt{91}}{10}}{\frac{3}{10}} = \frac{\sqrt{91}}{3}.$$

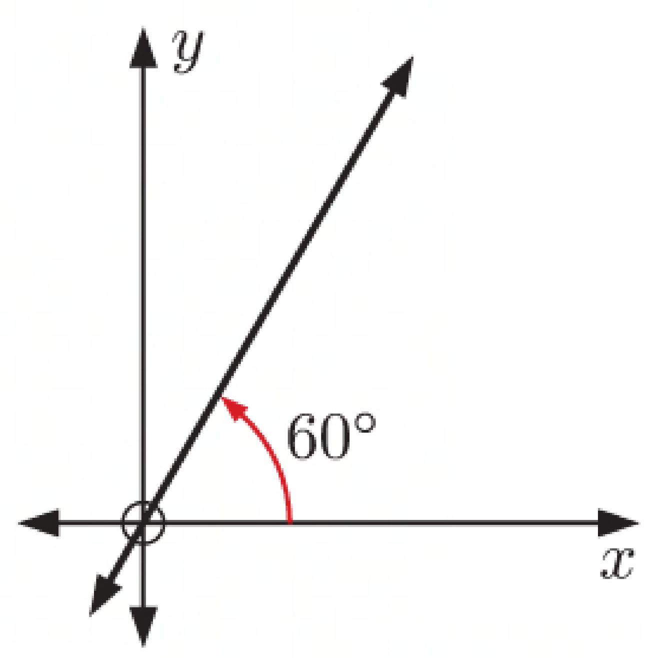
$\theta \approx 5.02$ corresponds to the second quadrant, where $\sin\theta$ is negative.

So, for $\theta \approx 5.02$, $\sin\theta = -\frac{\sqrt{91}}{10}$,

$$\tan\theta = \frac{-\frac{\sqrt{91}}{10}}{\frac{3}{10}} = -\frac{\sqrt{91}}{3}.$$

EXERCISE 8G

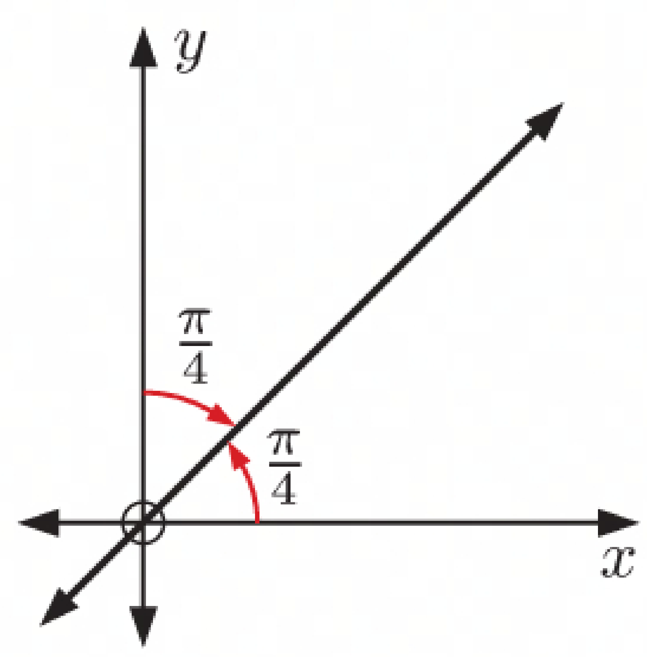
1 **a**



The line has gradient $m = \tan 60° = \sqrt{3}$ and y-intercept 0.

$\therefore$ the line has equation $y = \sqrt{3}x$.

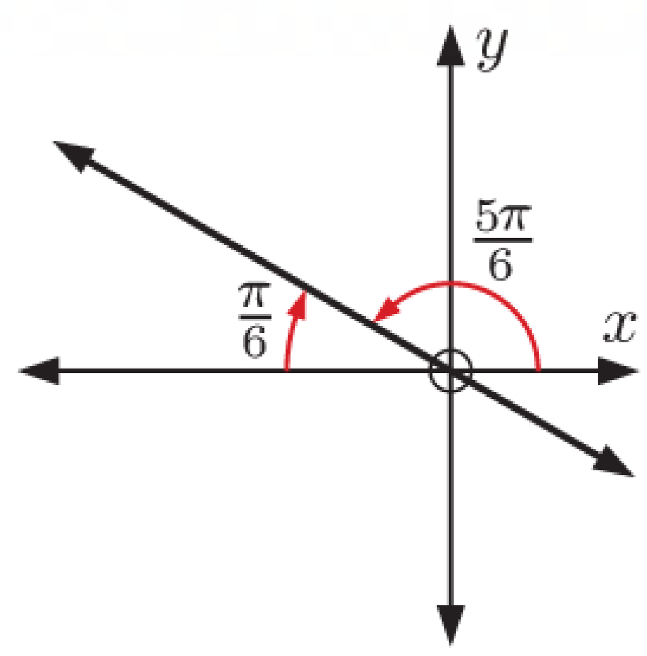
b



The line has gradient $m = \tan\frac{\pi}{4} = 1$ and y-intercept 0.

$\therefore$ the line has equation $y = x$.

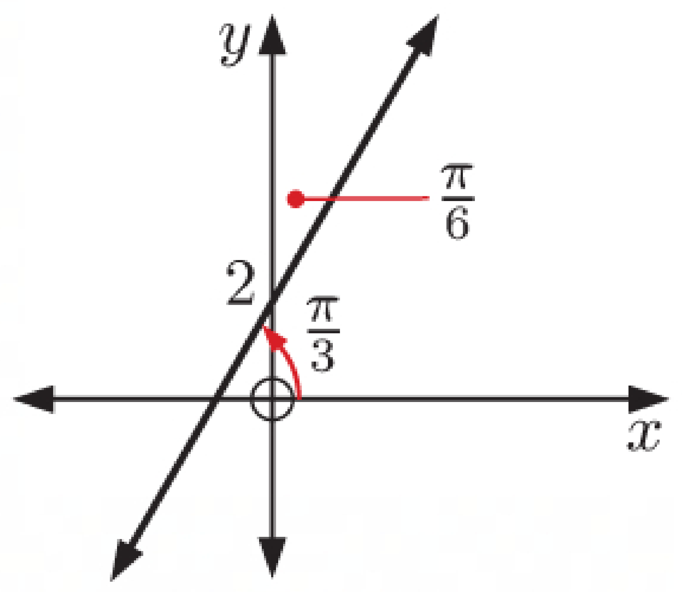
c



The line has gradient $m = \tan\frac{5\pi}{6} = -\frac{1}{\sqrt{3}}$ and y-intercept 0.

$\therefore$ the line has equation $y = -\frac{1}{\sqrt{3}}x$.

2 **a**

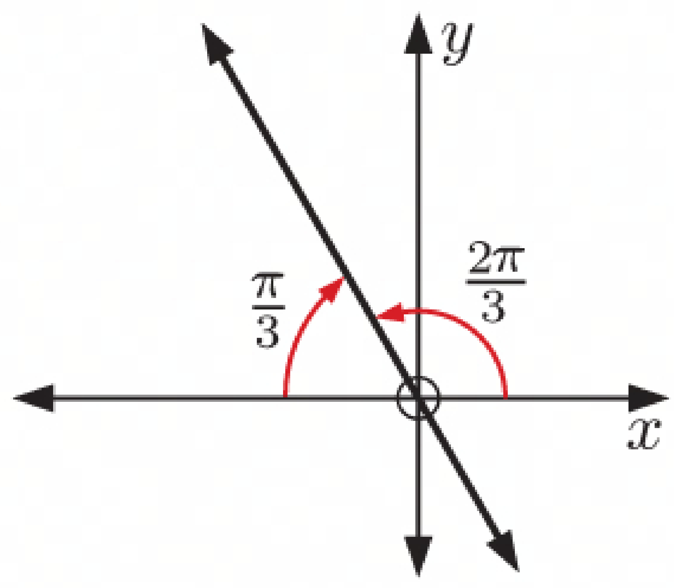


The line makes an angle of $\pi - \frac{\pi}{2} - \frac{\pi}{6} = \frac{\pi}{3}$ with the positive x-axis.

$\therefore$ the line has gradient $m = \tan\frac{\pi}{3} = \sqrt{3}$ and y-intercept 2.

$\therefore$ the line has equation $y = \sqrt{3}x + 2$.

b

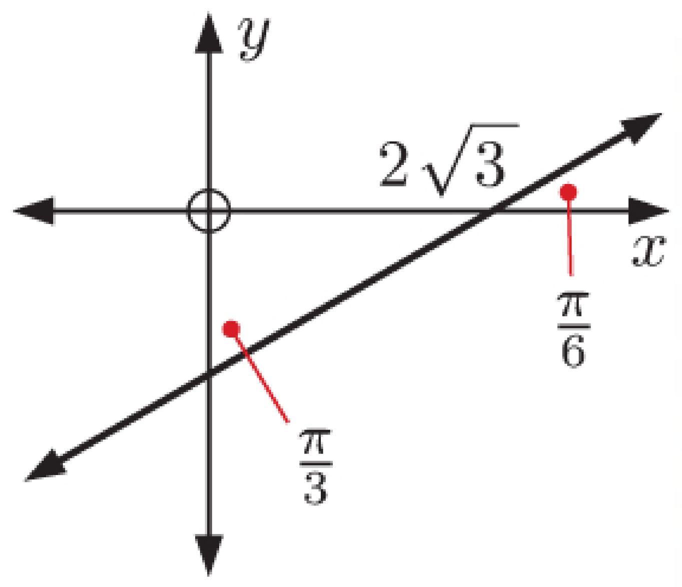


The line makes an angle of $\pi - \frac{\pi}{3} = \frac{2\pi}{3}$ with the positive x-axis.

$\therefore$ the line has gradient $m = \tan\frac{2\pi}{3} = -\sqrt{3}$ and y-intercept 0.

$\therefore$ the line has equation $y = -\sqrt{3}x$.

c



The line makes an angle of $\pi - \frac{\pi}{2} - \frac{\pi}{3} = \frac{\pi}{6}$ with the positive x-axis.

$\therefore$ the line has gradient $m = \tan\frac{\pi}{6} = \frac{1}{\sqrt{3}}$.

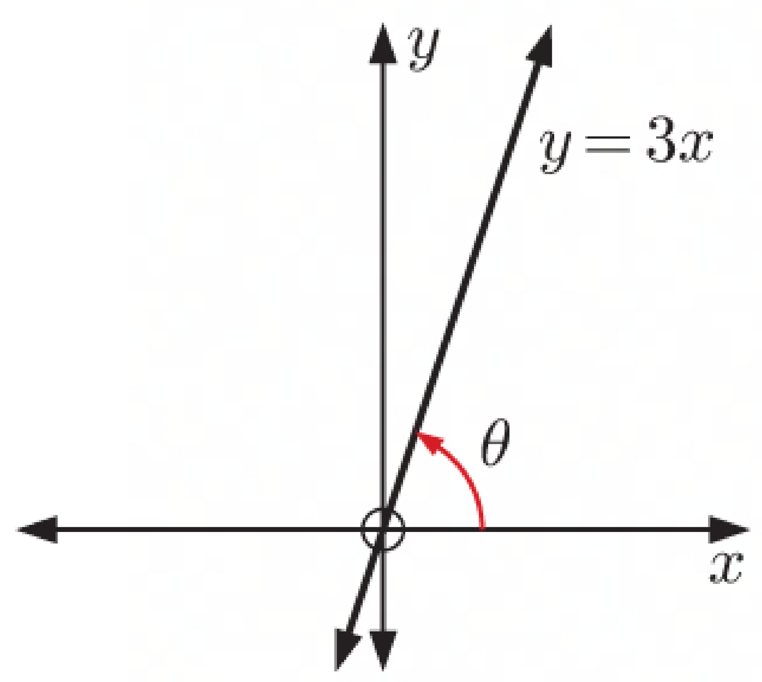
$\therefore$ the line has equation $y = \frac{1}{\sqrt{3}}x + c$.

When $x = 2\sqrt{3}$, $y = 0$, $\therefore$ $0 = \frac{1}{\sqrt{3}}(2\sqrt{3}) + c$

$\therefore$ $0 = 2 + c$

$\therefore$ $c = -2$

3 **a**

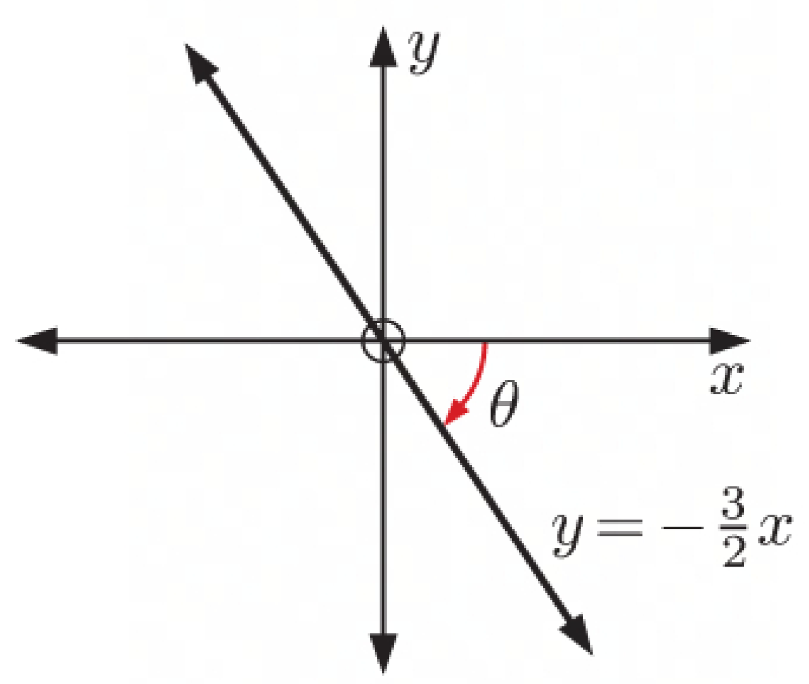


The line has gradient 3, so $\tan\theta = 3$.

Using technology, $\tan^{-1}(3) \approx 1.25$

$\therefore$ $\theta \approx 1.25$

b

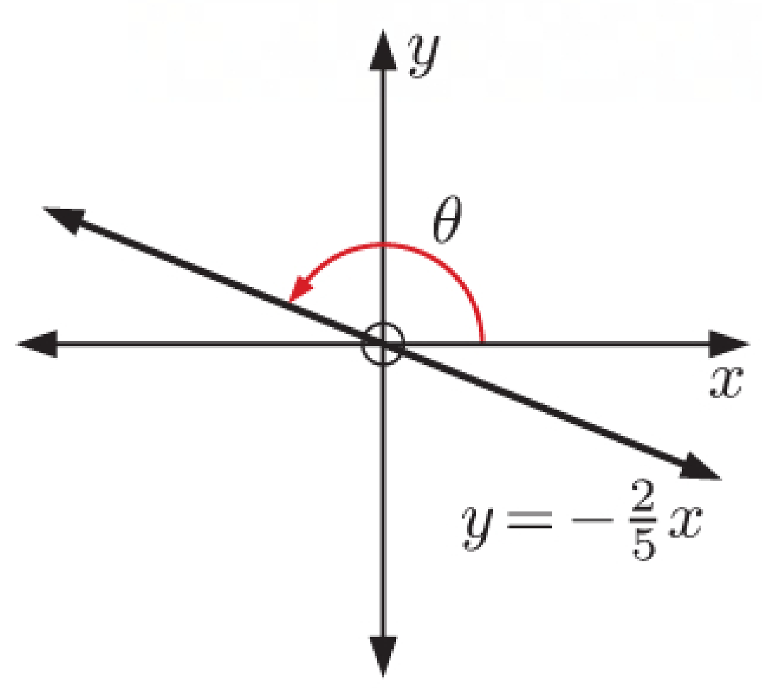


The line has gradient $-\frac{3}{2}$, so $\tan\theta = -\frac{3}{2}$.

Using technology, $\tan^{-1}\left(-\frac{3}{2}\right) \approx -0.983$

$\therefore$ $\theta \approx -0.983$

c

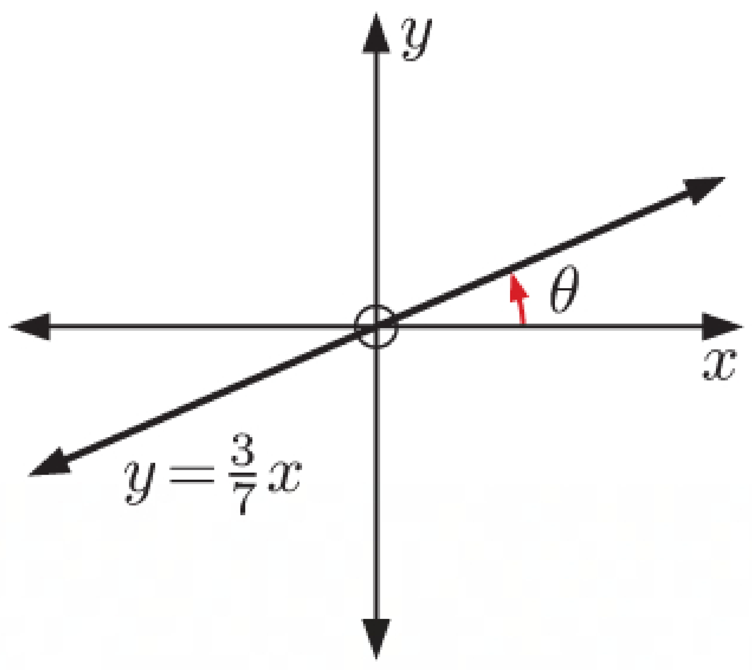


The line has gradient $-\frac{2}{5}$, so $\tan\theta = -\frac{2}{5}$.

Using technology, $\tan^{-1}\left(-\frac{2}{5}\right) \approx -0.381$

But $0 < \theta < \pi$, so $\theta \approx \pi - 0.381 \approx 2.76$

4 **a**

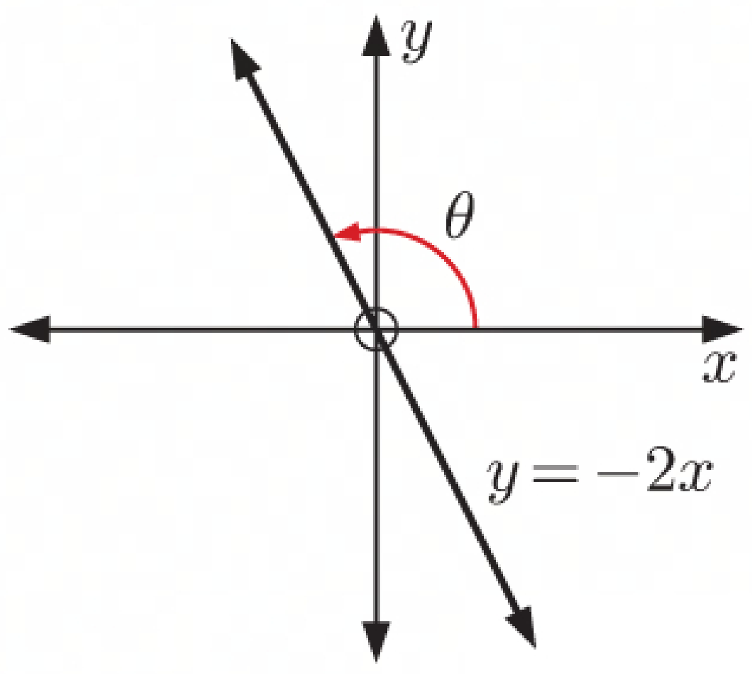


The line has gradient $\frac{3}{7}$, so $\tan\theta = \frac{3}{7}$.

Using technology, $\tan^{-1}\left(\frac{3}{7}\right) \approx 23.2°$

$\therefore \quad \theta \approx 23.2°$

b

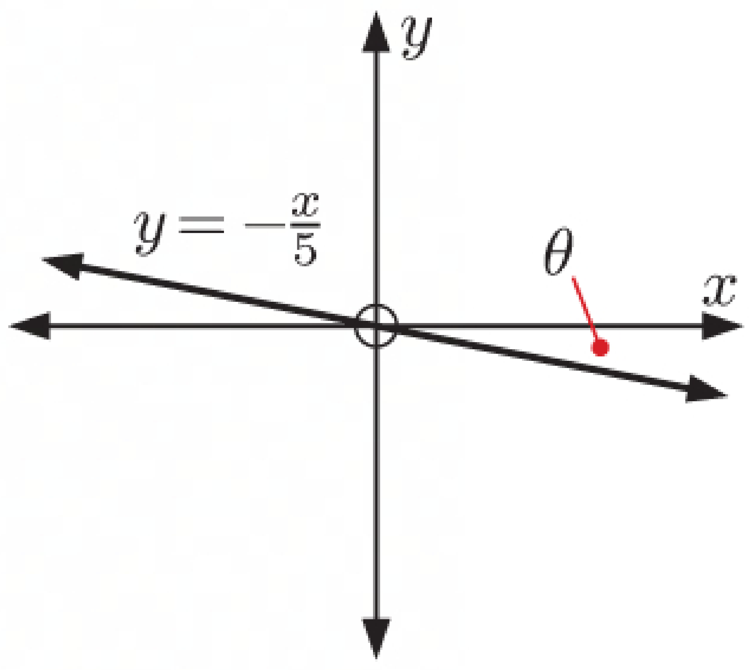


The line has gradient -2, so $\tan\theta = -2$.

Using technology, $\tan^{-1}(-2) \approx -63.4°$

But $0° < \theta < 180°$, so $\theta \approx 180° - 63.4° \approx 117°$

c



The line has gradient $-\frac{1}{5}$, so $\tan\theta = -\frac{1}{5}$.

Using technology, $\tan^{-1}\left(-\frac{1}{5}\right) \approx -11.3°$

$\therefore \quad \theta \approx -11.3°$

REVIEW SET 8A

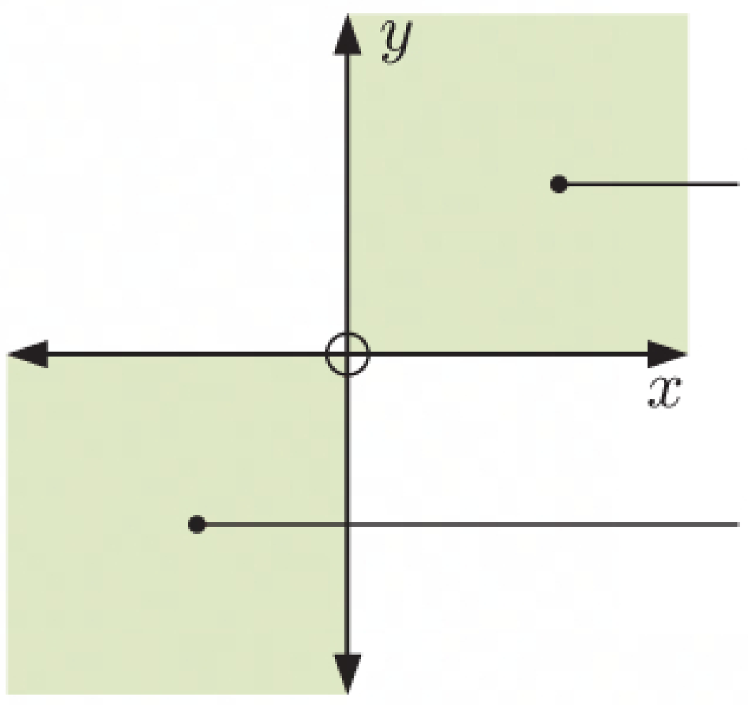
1 **a** $120° = \left(120 \times \frac{\pi}{180}\right)$ radians
$= \frac{2\pi}{3}$ radians

b $225° = \left(225 \times \frac{\pi}{180}\right)$ radians
$= \frac{5\pi}{4}$ radians

c $150° = \left(150 \times \frac{\pi}{180}\right)$ radians
$= \frac{5\pi}{6}$ radians

d $540° = \left(540 \times \frac{\pi}{180}\right)$ radians
$= 3\pi$ radians

2



3 **a** The point is $(\cos 320°, \sin 320°) \approx (0.766, -0.643)$.

b The point is $(\cos 163°, \sin 163°) \approx (-0.956, 0.292)$.

c The point is $(\cos 0.68^c, \sin 0.68^c) \approx (0.778, 0.629)$.

d The point is $\left(\cos \frac{11\pi}{6}, \sin \frac{11\pi}{6}\right) \approx (0.866, -0.5)$.

4 arc length $= \theta r$
$= 1.5 \times 8$
$= 12$ cm

5 **a** $\sin \frac{2\pi}{3} = \sin\left(\pi - \frac{2\pi}{3}\right)$
$= \sin \frac{\pi}{3}$
$\therefore \ \theta = \frac{\pi}{3}$

b $\sin 165^\circ = \sin(180 - 165)^\circ$
$= \sin 15^\circ$
$\therefore \ \theta = 15^\circ$

c $\cos 276^\circ = \cos(360 - 276)^\circ$
$= \cos 84^\circ$
$\therefore \ \theta = 84^\circ$

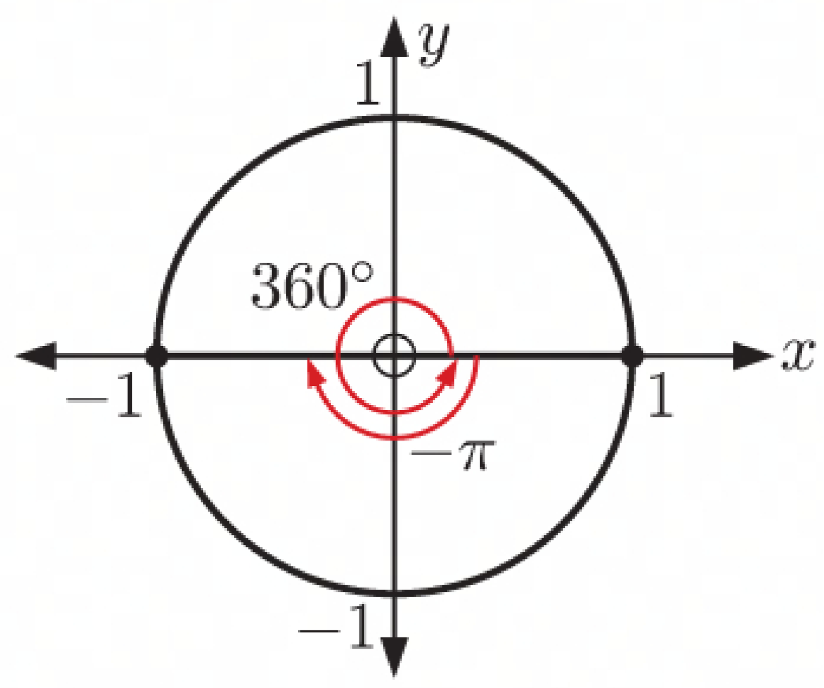
6 **a** $\sin 159^\circ = \sin(180 - 159)^\circ$
$= \sin 21^\circ$
≈ 0.358

b $\cos 92^\circ = -\cos(180 - 92)^\circ$
$= -\cos 88^\circ$
≈ -0.035

c $\cos 75^\circ = -\cos(180 - 75)^\circ$
$= -\cos 105^\circ$
≈ 0.259

d $\tan(-133^\circ) = \tan(180 - 133)^\circ$
$= \tan 47^\circ$
≈ 1.072

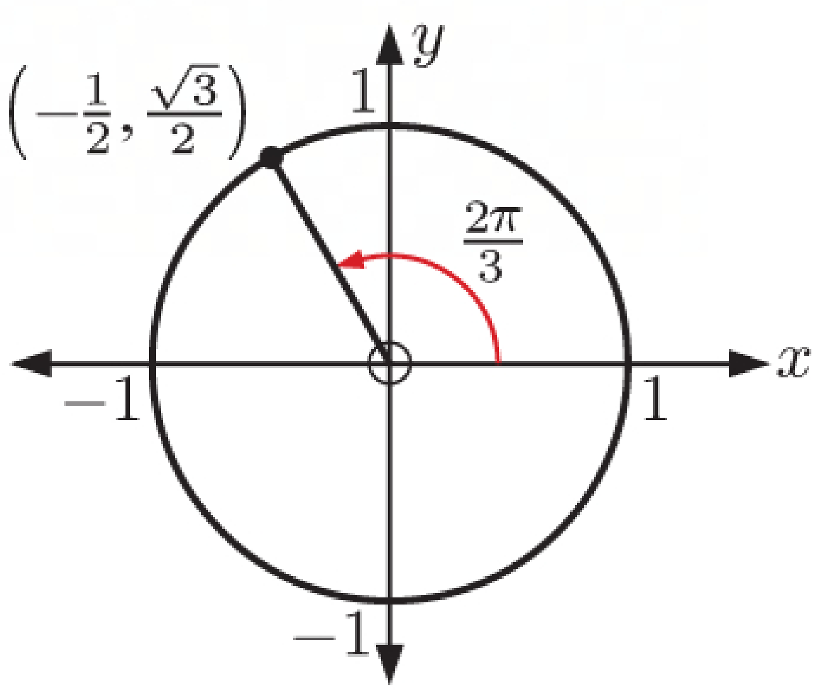
7



a $\cos 360^\circ = 1, \quad \sin 360^\circ = 0$

b $\cos(-\pi) = -1, \quad \sin(-\pi) = 0$

8 **a**

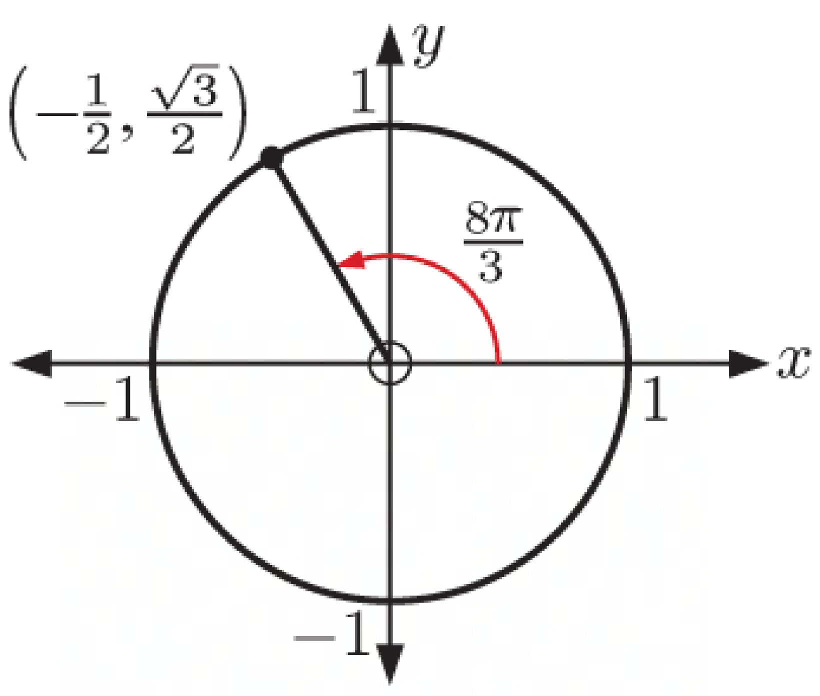


$\sin \frac{2\pi}{3} = \frac{\sqrt{3}}{2}$

$\cos \frac{2\pi}{3} = -\frac{1}{2}$

$\tan \frac{2\pi}{3} = \dfrac{\frac{\sqrt{3}}{2}}{-\frac{1}{2}}$
$= -\sqrt{3}$

b

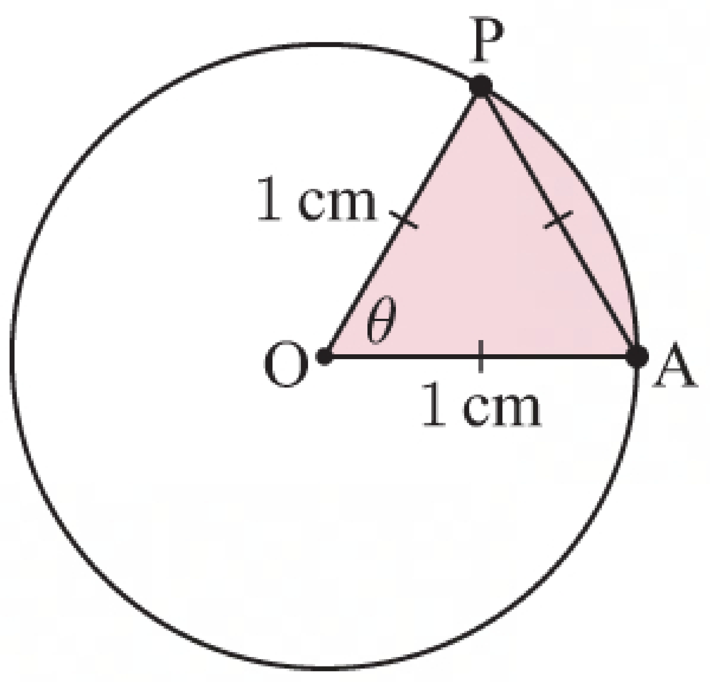


$\sin \frac{8\pi}{3} = \frac{\sqrt{3}}{2}$

$\cos \frac{8\pi}{3} = -\frac{1}{2}$

$\tan \frac{8\pi}{3} = \dfrac{\frac{\sqrt{3}}{2}}{-\frac{1}{2}}$
$= -\sqrt{3}$

9



a **i** $\theta = 60°$ {equilateral triangle}

ii $\theta = \frac{\pi}{3}$ radians

b arc length AP $= \theta r$
$= \frac{\pi}{3} \times 1$
$= \frac{\pi}{3}$ cm

c sector area $= \frac{1}{2}\theta r^2$
$= \frac{1}{2} \times \frac{\pi}{3} \times 1^2$
$= \frac{\pi}{6}$ cm^2

10 $\cos^2 x + \sin^2 x = 1$

$\therefore \cos^2 x + \frac{1}{16} = 1$

$\therefore \cos^2 x = \frac{15}{16}$

$\therefore \cos x = \pm\frac{\sqrt{15}}{4}$

But x is in quadrant 3 where $\cos x < 0$

$\therefore \cos x = -\frac{\sqrt{15}}{4}$

and so $\tan x = \dfrac{\sin x}{\cos x} = \dfrac{-\frac{1}{4}}{-\frac{\sqrt{15}}{4}} = \dfrac{1}{\sqrt{15}}$

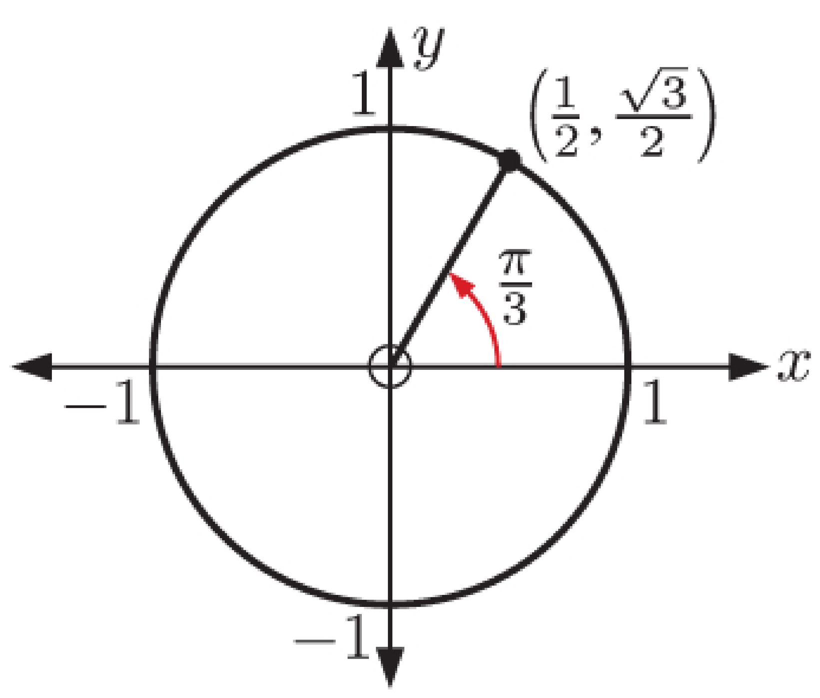
11 $\cos^2\theta + \sin^2\theta = 1$

$\therefore \frac{9}{16} + \sin^2\theta = 1$

$\therefore \sin^2\theta = \frac{7}{16}$

$\therefore \sin\theta = \pm\frac{\sqrt{7}}{4}$

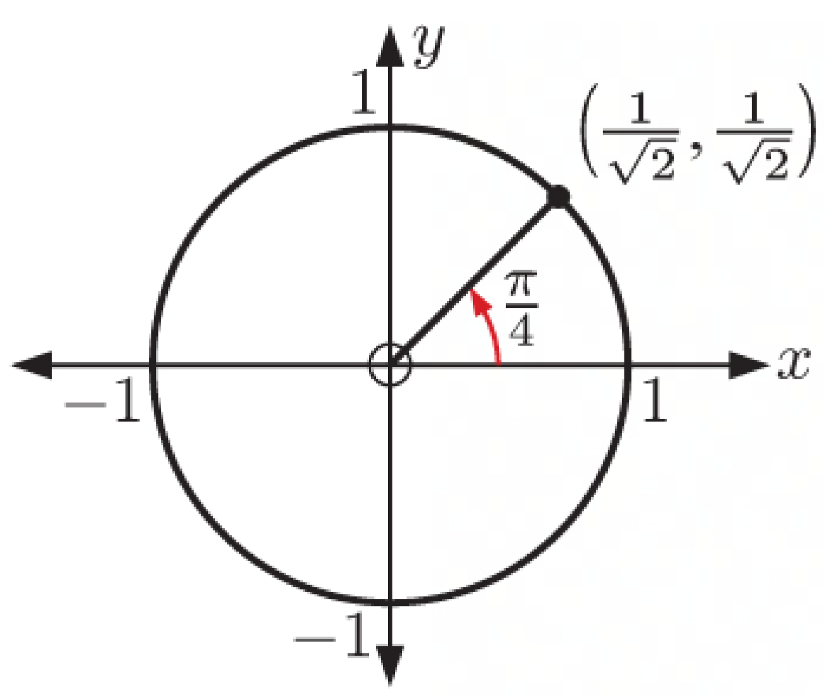
12 **a**



$\sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}$ and $\cos\frac{\pi}{3} = \frac{1}{2}$

$\therefore 2\sin\frac{\pi}{3}\cos\frac{\pi}{3} = 2 \times \frac{\sqrt{3}}{2} \times \frac{1}{2}$
$= \frac{\sqrt{3}}{2}$

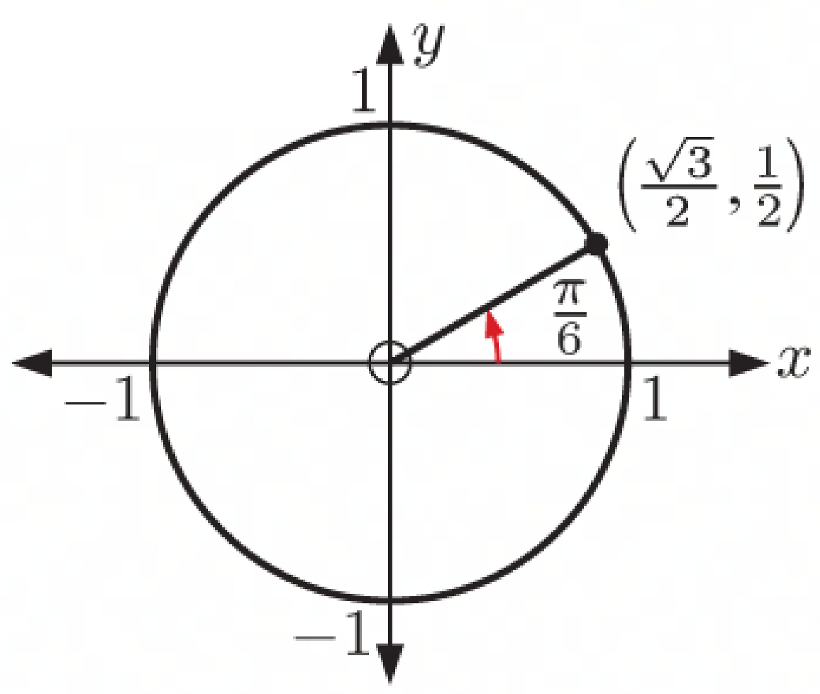
b



$\tan\frac{\pi}{4} = \dfrac{\frac{1}{\sqrt{2}}}{\frac{1}{\sqrt{2}}} = 1$

$\therefore \tan^2\left(\frac{\pi}{4}\right) - 1 = 1^2 - 1$
$= 0$

c

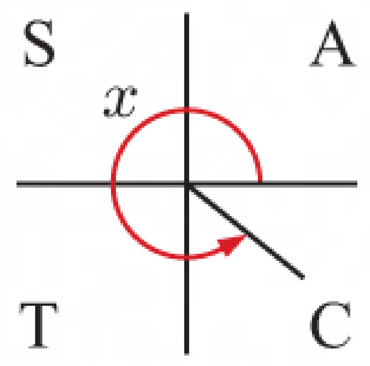


$\cos\frac{\pi}{6} = \frac{\sqrt{3}}{2}$ and $\sin\frac{\pi}{6} = \frac{1}{2}$

$\therefore \cos^2\left(\frac{\pi}{6}\right) - \sin^2\left(\frac{\pi}{6}\right) = \left(\frac{\sqrt{3}}{2}\right)^2 - \left(\frac{1}{2}\right)^2$
$= \frac{3}{4} - \frac{1}{4}$
$= \frac{1}{2}$

13 $\tan x = \dfrac{\sin x}{\cos x} = -\frac{3}{2}$

$\therefore \ \sin x = -\frac{3}{2}\cos x$



Now $\cos^2 x + \sin^2 x = 1$

$\therefore \ \cos^2 x + \frac{9}{4}\cos^2 x = 1$

$\therefore \ \frac{13}{4}\cos^2 x = 1$

$\therefore \ \cos^2 x = \frac{4}{13}$

$\therefore \ \cos x = \pm\frac{2}{\sqrt{13}}$

But x is in quadrant 4, so $\cos x$ is positive and $\sin x$ is negative.

$\therefore \ \cos x = \frac{2}{\sqrt{13}}, \quad \sin x = -\frac{3}{\sqrt{13}}$

a $\cos x = \frac{2}{\sqrt{13}}$ **b** $\sin x = -\frac{3}{\sqrt{13}}$

14 $\cos^2\theta + \sin^2\theta = 1$

$\therefore \ \left(\frac{\sqrt{11}}{\sqrt{17}}\right)^2 + \sin^2\theta = 1$

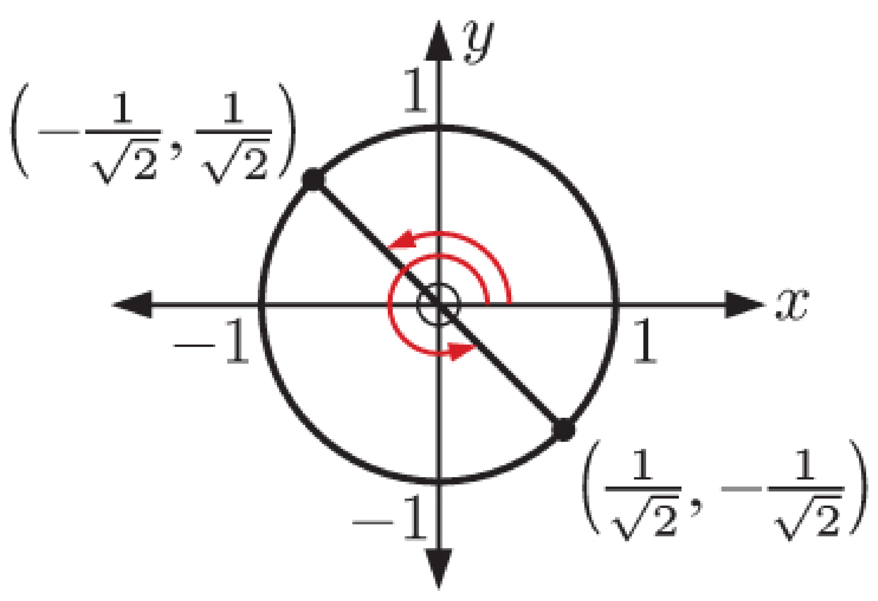
$\therefore \ \sin^2\theta = \frac{6}{17}$

$\therefore \ \sin\theta = \pm\frac{\sqrt{6}}{\sqrt{17}}$

But θ is acute, $\therefore \ \sin\theta = \frac{\sqrt{6}}{\sqrt{17}}$

$$\tan\theta = \frac{\frac{\sqrt{6}}{\sqrt{17}}}{\frac{\sqrt{11}}{\sqrt{17}}} = \frac{\sqrt{6}}{\sqrt{11}}$$

15



When $\cos\theta = -\sin\theta$,

$\dfrac{\sin\theta}{\cos\theta} = -1$

$\therefore \ \tan\theta = -1$

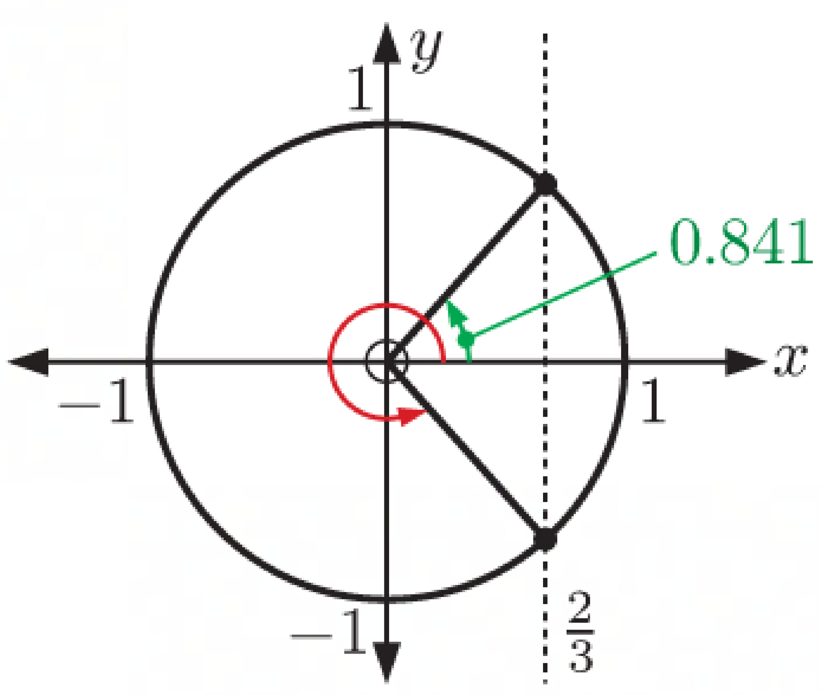
which only occurs at the two points shown.

So, $\theta = \frac{3\pi}{4}$ or $\frac{7\pi}{4}$.

16 **a** $\cos\theta = \frac{2}{3}$

Using technology,

$\cos^{-1}\left(\frac{2}{3}\right) \approx 0.841$



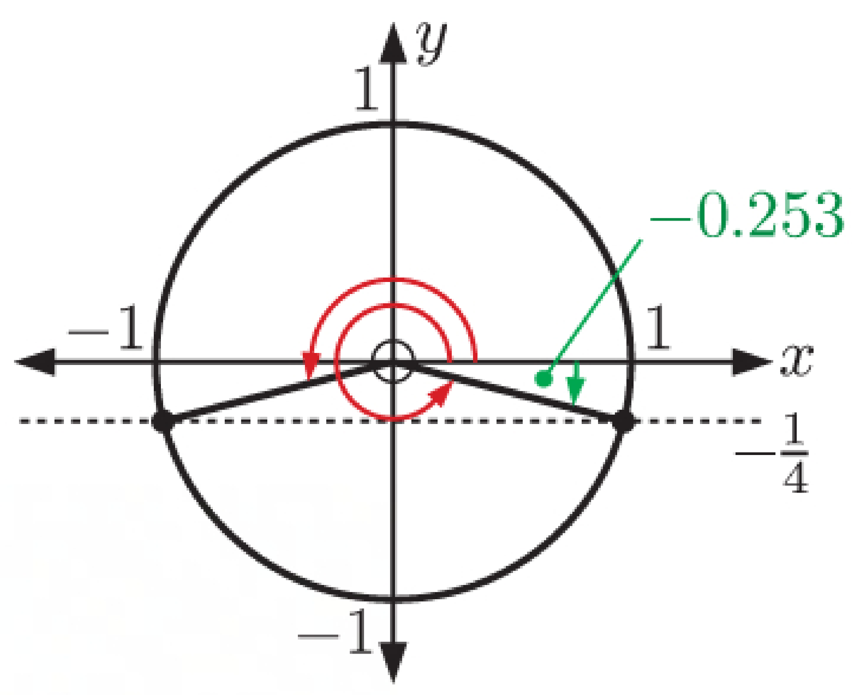
$\therefore \ \theta \approx 0.841$ or $2\pi - 0.841$

$\therefore \ \theta \approx 0.841$ or 5.44

b $\sin\theta = -\frac{1}{4}$

Using technology,

$\sin^{-1}\left(-\frac{1}{4}\right) \approx -0.253$



But $0 \leqslant \theta \leqslant 2\pi$

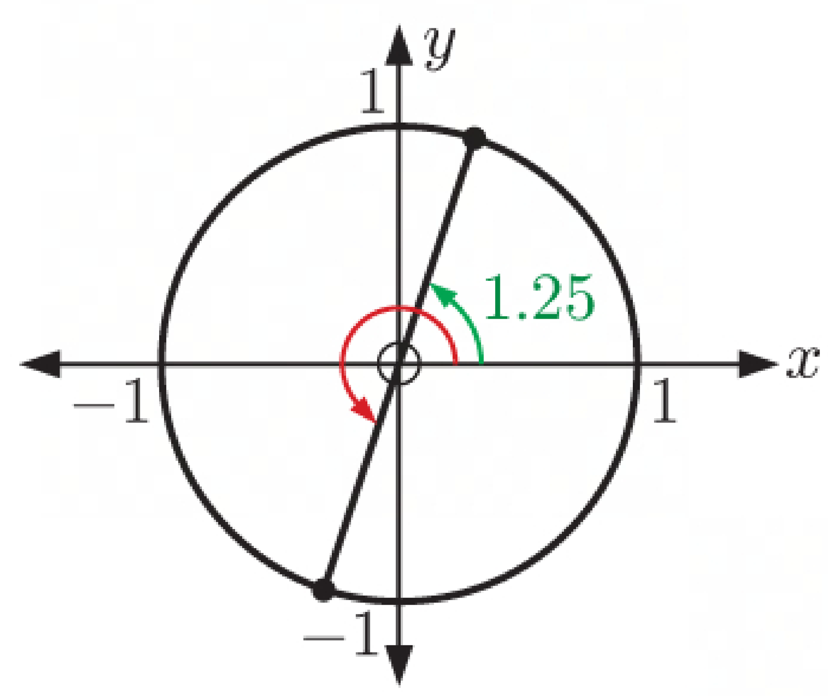
$\therefore \ \theta \approx \pi + 0.253$ or $2\pi - 0.253$

$\therefore \ \theta \approx 3.39$ or 6.03

c $\tan\theta = 3$

Using technology,

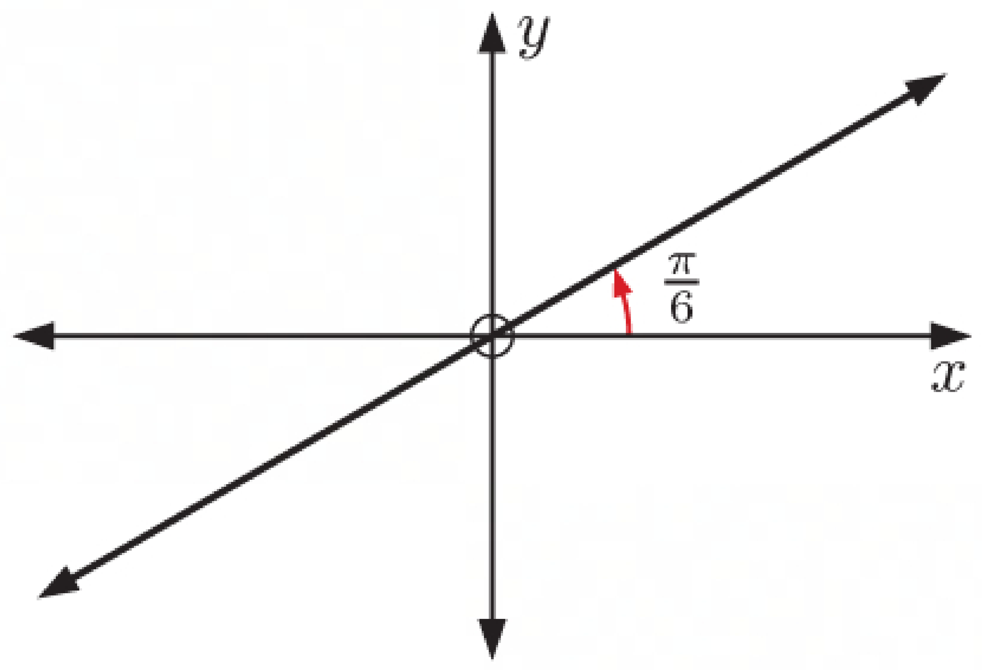
$\tan^{-1}(3) \approx 1.25$



$\therefore \ \theta \approx 1.25$ or $\pi + 1.25$

$\therefore \ \theta \approx 1.25$ or 4.39

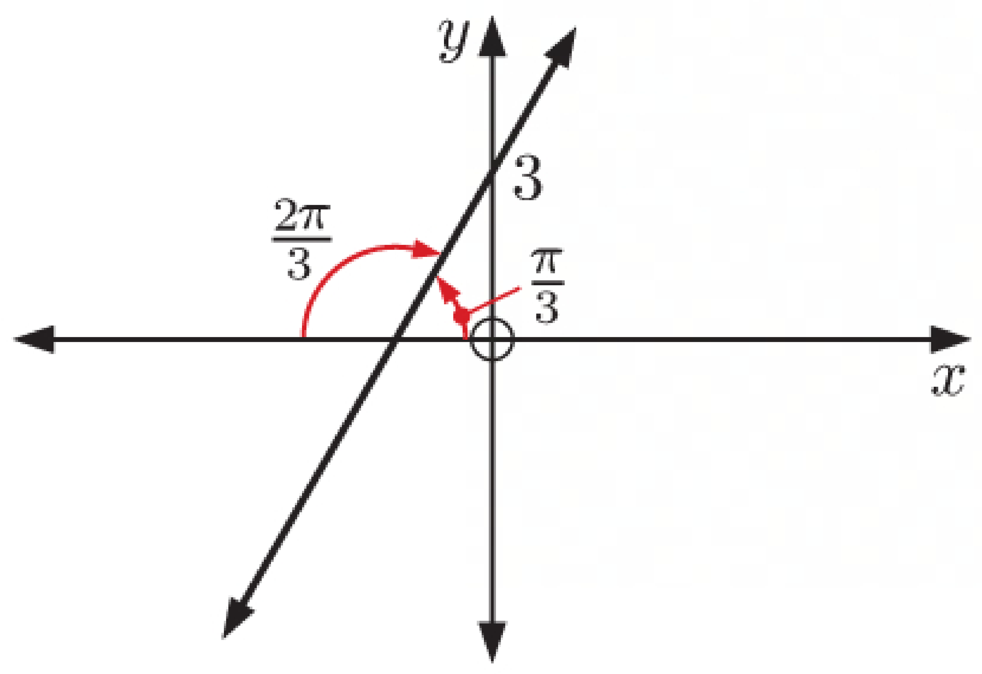
17 **a**



The line has gradient $m = \tan\frac{\pi}{6} = \frac{1}{\sqrt{3}}$ and y-intercept 0.

$\therefore$ the line has equation $y = \frac{1}{\sqrt{3}}x$.

b



The graph makes an angle of $\pi - \frac{2\pi}{3} = \frac{\pi}{3}$ with the positive x-axis.

$\therefore$ the line has gradient $m = \tan\frac{\pi}{3} = \sqrt{3}$ and y-intercept 3.

$\therefore$ the line has equation $y = \sqrt{3}x + 3$.

REVIEW SET 8B

1 **a** $\frac{2\pi}{5} = \left(\frac{2\pi}{5} \times \frac{180}{\pi}\right)^\circ$
$= 72^\circ$

b $1.46^c = \left(1.46 \times \frac{180}{\pi}\right)^\circ$
$\approx 83.65^\circ$

c $0.435^c = \left(0.435 \times \frac{180}{\pi}\right)^\circ$
$\approx 24.92^\circ$

d $-5.271^c = \left(-5.271 \times \frac{180}{\pi}\right)^\circ$
$\approx -302.01^\circ$

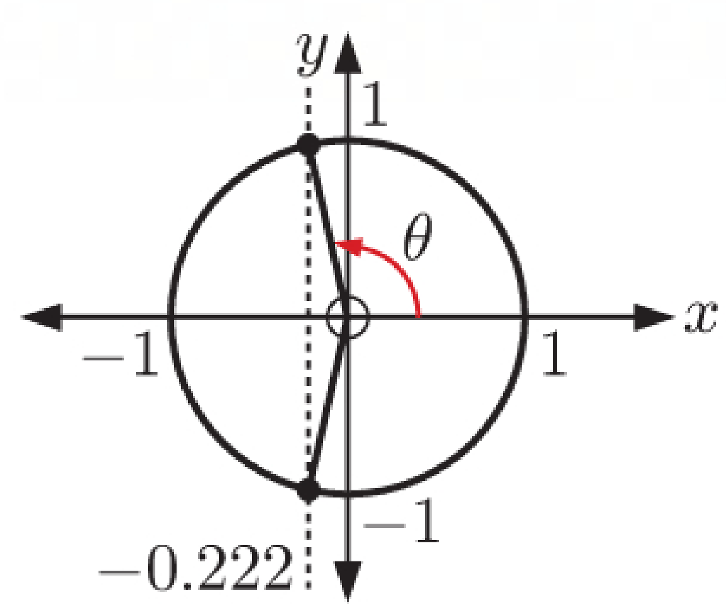
2 area $= \frac{1}{2}\theta r^2$
$= \frac{1}{2} \times \frac{5\pi}{12} \times 13^2$
$\approx 111 \text{ cm}^2$

3 The x-coordinate of A $= -0.222$

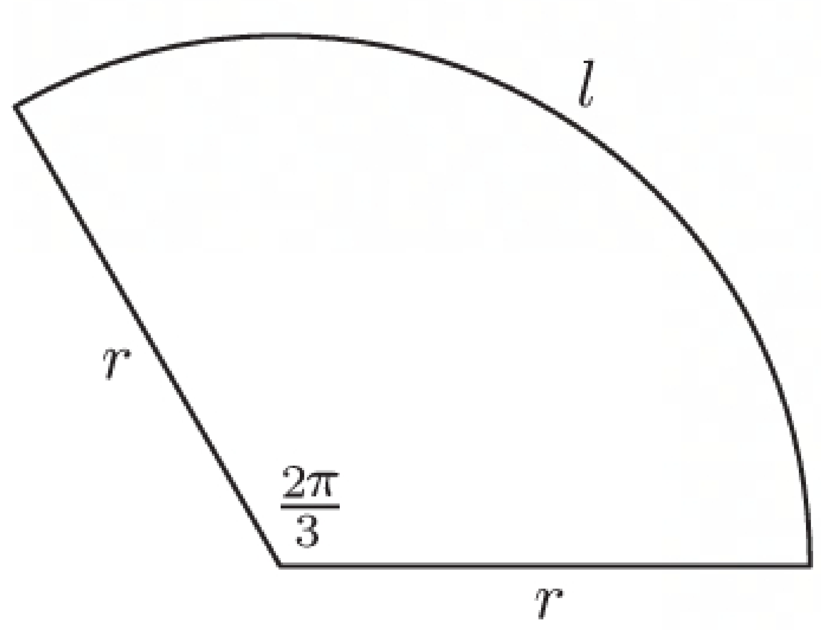
$\therefore \cos\theta = -0.222$

$\therefore \theta = \cos^{-1}(-0.222)$

$\therefore \theta \approx 103^\circ$



4



perimeter $= l + 2r$

$\therefore 36 = \frac{2\pi r}{3} + 2r$

$\therefore 36 = r\left(\frac{2\pi}{3} + 2\right)$

$\therefore r = \dfrac{36}{\frac{2\pi}{3} + 2}$

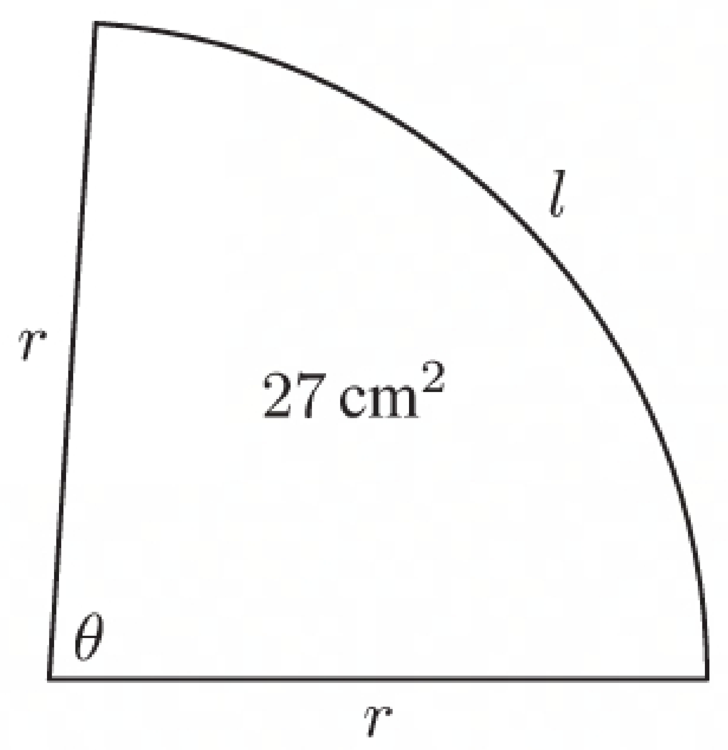
$\therefore r \approx 8.7925$

$\therefore$ the radius is ≈ 8.79 cm.

area $\approx \frac{1}{2} \times \frac{2\pi}{3} \times 8.7925^2$
$\approx 81.0 \text{ cm}^2$

5

$$\text{perimeter} = l + 2r$$
$$\therefore\ l + 2r = 21$$
$$\therefore\ \theta r + 2r = 21$$
$$\therefore\ r(\theta + 2) = 21$$
$$\therefore\ \theta + 2 = \frac{21}{r}$$
$$\therefore\ \theta = \frac{21}{r} - 2 \quad .... \text{ (1)}$$

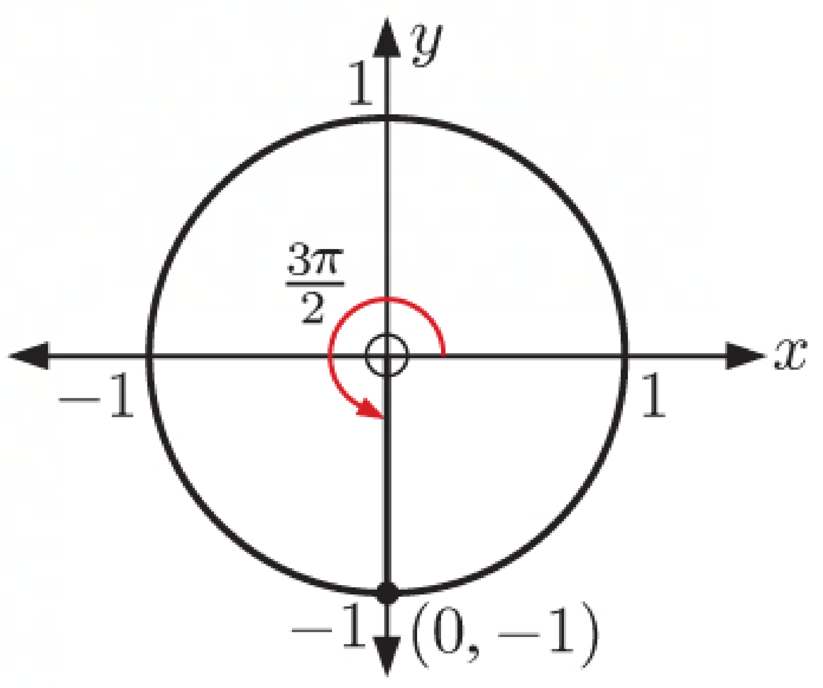


Now $\text{area} = \frac{1}{2}\theta r^2$

$$\therefore\ 27 = \frac{1}{2}\theta r^2$$
$$\therefore\ r^2\theta = 54$$
$$\therefore\ r^2\left(\frac{21}{r} - 2\right) = 54 \qquad \{\text{using (1)}\}$$
$$\therefore\ 21r - 2r^2 = 54$$
$$\therefore\ 2r^2 - 21r + 54 = 0$$
$$\therefore\ r = 4.5 \text{ or } 6 \qquad \{\text{using technology}\}$$

$\therefore$ the radius of the sector is either 4.5 cm or 6 cm.

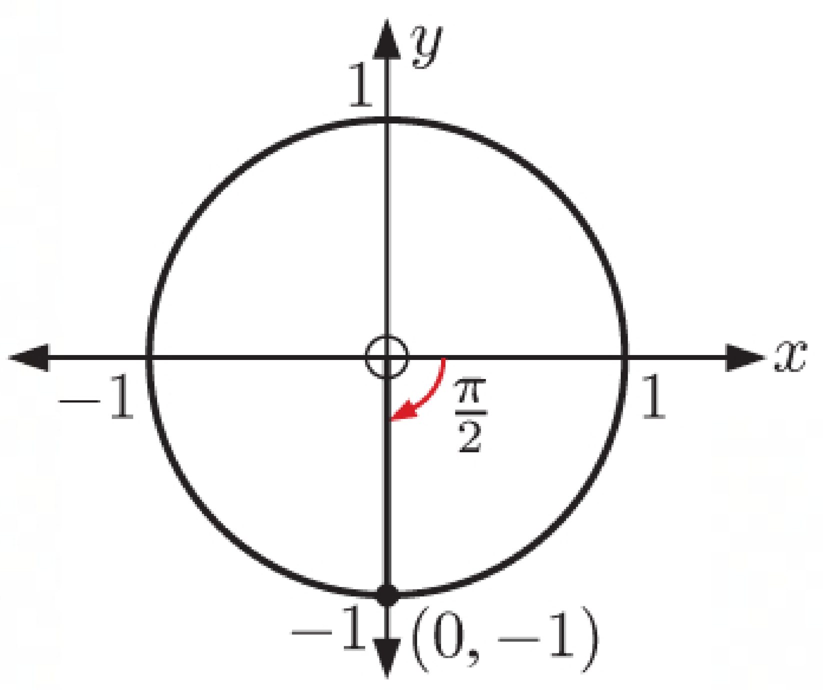
6 **a**



$\cos\frac{3\pi}{2} = 0$

$\sin\frac{3\pi}{2} = -1$

b



$\cos\left(-\frac{\pi}{2}\right) = 0$

$\sin\left(-\frac{\pi}{2}\right) = -1$

7 **a**

$$\sin(\pi - \theta) = \sin\theta$$
$$\therefore\ \sin(\pi - p) = \sin p$$
$$= m$$

b

$$\sin(\theta + 2\pi) = \sin\theta$$
$$\therefore\ \sin(p + 2\pi) = \sin p$$
$$= m$$

c

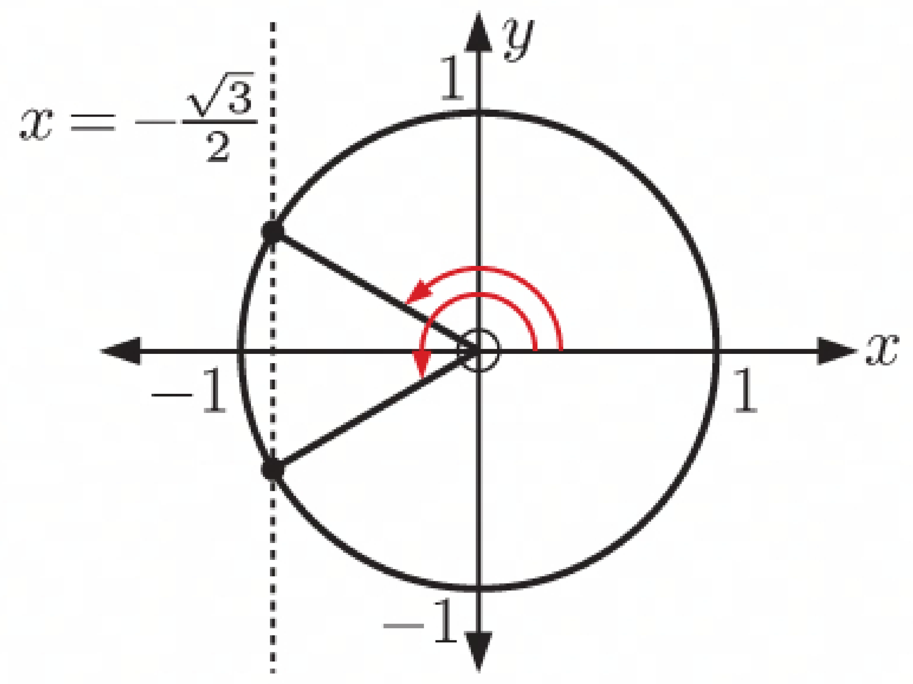
$$\cos^2 p + \sin^2 p = 1$$
$$\therefore\ \cos^2 p + m^2 = 1$$
$$\therefore\ \cos^2 p = 1 - m^2$$
$$\therefore\ \cos p = \pm\sqrt{1 - m^2}$$

But p is acute, $\therefore\ \cos p = \sqrt{1 - m^2}$

d

$$\tan p = \frac{\sin p}{\cos p}$$
$$= \frac{m}{\sqrt{1 - m^2}}$$

8 **a**

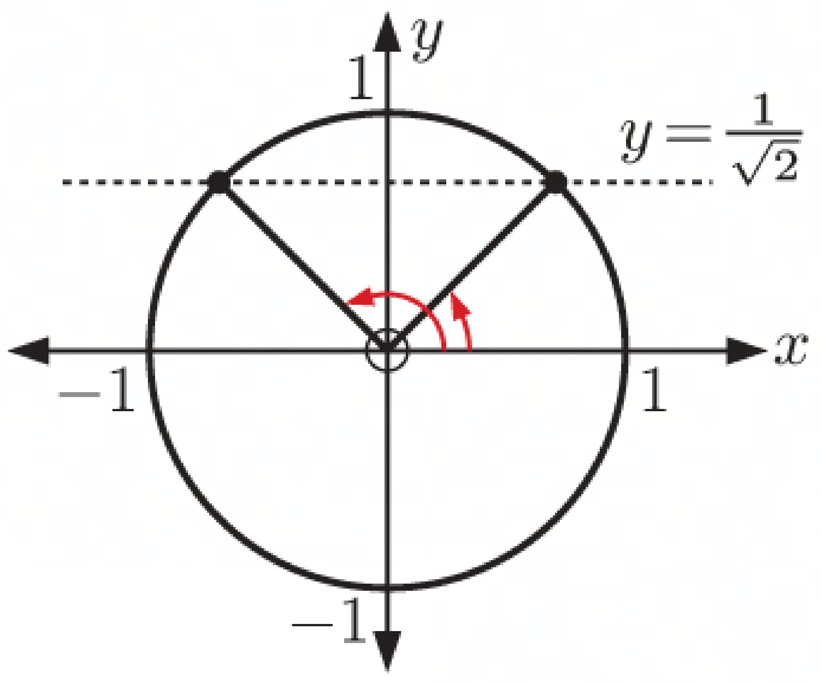


Since the cosine is $-\frac{\sqrt{3}}{2}$, we draw the vertical line $x = -\frac{\sqrt{3}}{2}$.

Because $\frac{\sqrt{3}}{2}$ is involved, we know the required angles are multiples of $30°$.

They are $150°$ and $210°$.

b

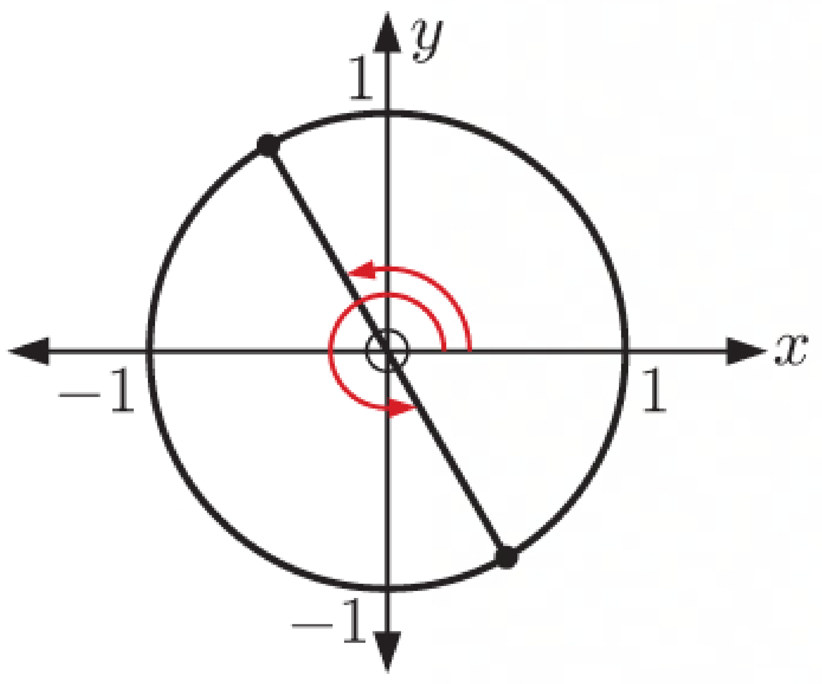


Since the sine is $\frac{1}{\sqrt{2}}$, we draw the horizontal line $y = \frac{1}{\sqrt{2}}$.

Because $\frac{1}{\sqrt{2}}$ is involved, we know the required angles are multiples of $45°$.

They are $45°$ and $135°$.

c

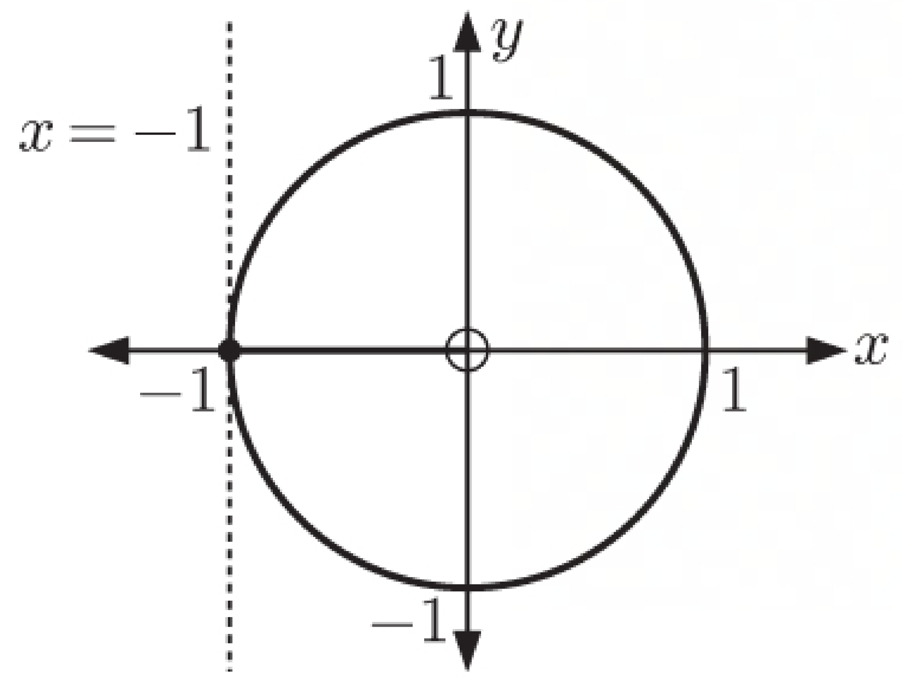


Since the tangent is $-\sqrt{3} = \dfrac{\frac{\sqrt{3}}{2}}{-\frac{1}{2}} = \dfrac{-\frac{\sqrt{3}}{2}}{\frac{1}{2}}$, the sine must be $\pm\frac{\sqrt{3}}{2}$, and the cosine must be $\mp\frac{1}{2}$ (since $\tan\theta = \dfrac{\sin\theta}{\cos\theta}$).

Because $\frac{\sqrt{3}}{2}$ and $\frac{1}{2}$ are both involved, we know the required angles are multiples of $30°$.

They are $120°$ and $300°$.

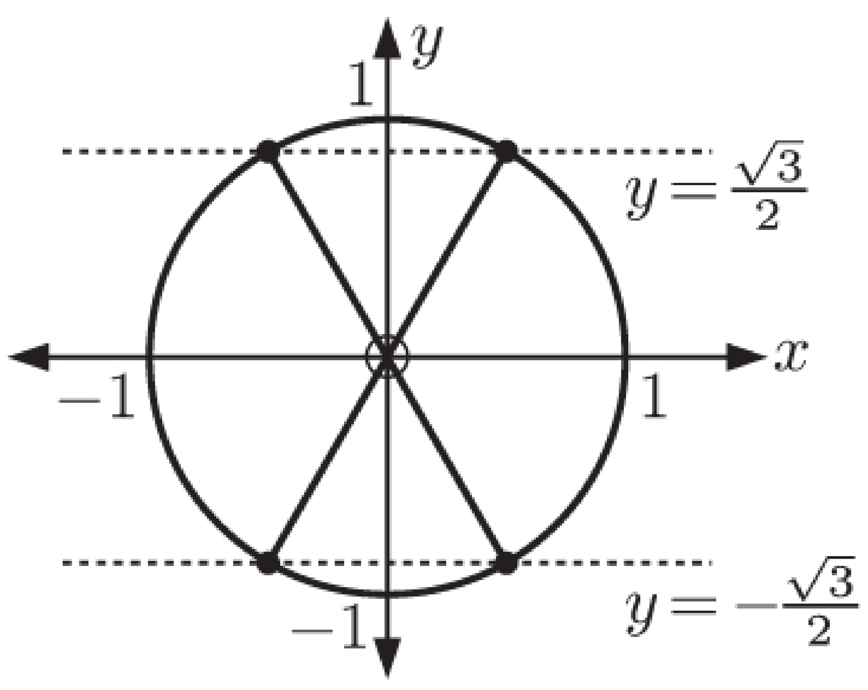
9 **a**



Since $\cos\theta = -1$, we draw the vertical line $x = -1$.
Because 1 is involved, we know the required angles are multiples of $\frac{\pi}{2}$.

$\therefore\ \theta = \pi$

b



Since $\sin^2\theta = \frac{3}{4}$, then $\sin\theta = \pm\frac{\sqrt{3}}{2}$, so we draw the horizontal lines $y = \frac{\sqrt{3}}{2}$ and $y = -\frac{\sqrt{3}}{2}$.

Because $\frac{\sqrt{3}}{2}$ is involved, we know the required angles are multiples of $\frac{\pi}{6}$.

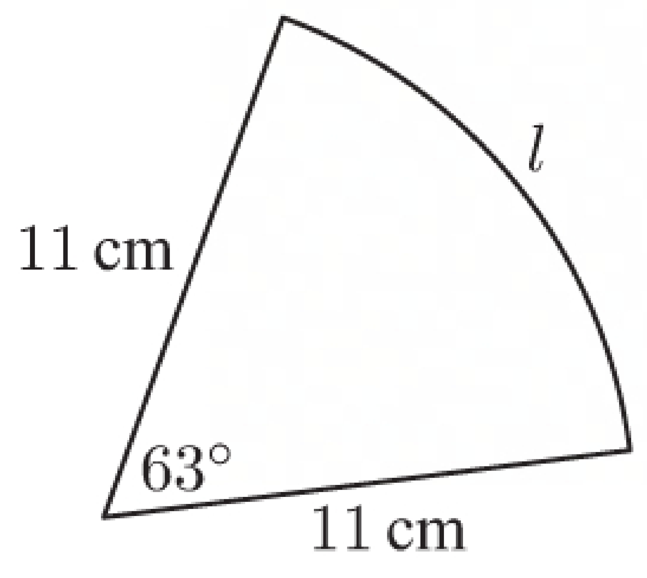
$\therefore\ \theta = \frac{\pi}{3}, \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{5\pi}{3}$

10 **a** $\sin 47° = \sin(180 - 47)°$
$= \sin 133°$

b $\sin\frac{\pi}{15} = \sin\left(\pi - \frac{\pi}{15}\right)$
$= \sin\frac{14\pi}{15}$

c $\cos 186° = \cos(360 - 186)°$
$= \cos 174°$

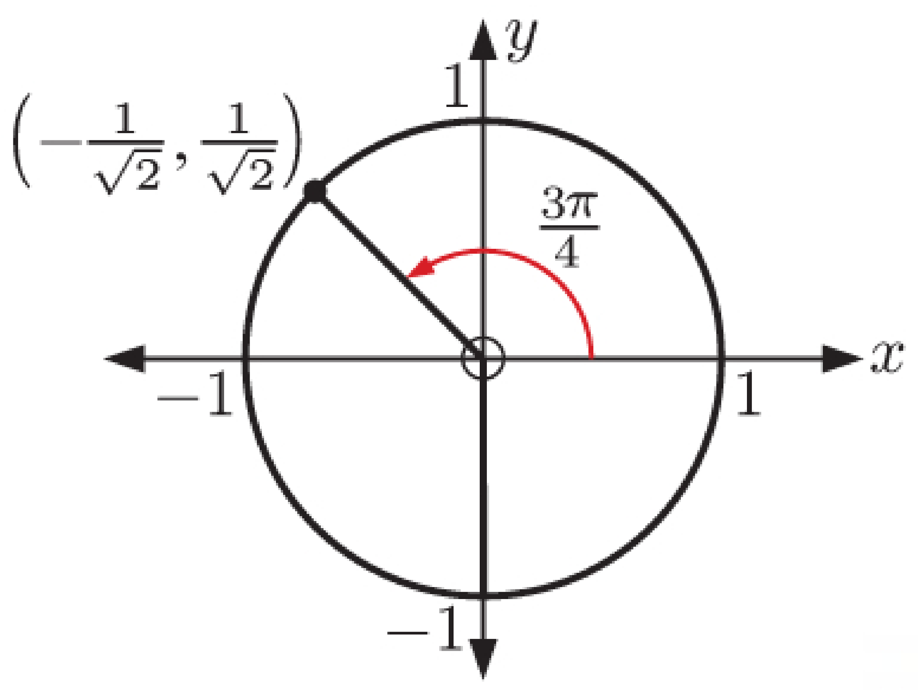
11



$$\begin{aligned}\text{perimeter} &= l + 2r\\ &= \left(\tfrac{63}{360}\right) \times 2\pi \times 11 + 2 \times 11\\ &\approx 34.1 \text{ cm}\end{aligned}$$

$$\begin{aligned}\text{area} &= \left(\tfrac{63}{360}\right) \times \pi \times 11^2\\ &\approx 66.5 \text{ cm}^2\end{aligned}$$

12



$$\begin{aligned}&\cos\tfrac{3\pi}{4} - \sin\tfrac{3\pi}{4}\\ &= -\tfrac{1}{\sqrt{2}} - \tfrac{1}{\sqrt{2}}\\ &= -\tfrac{2}{\sqrt{2}} \times \tfrac{\sqrt{2}}{\sqrt{2}}\\ &= -\sqrt{2}\end{aligned}$$

13 **a** $\cos^2\theta + \sin^2\theta = 1$

$\therefore \frac{9}{16} + \sin^2\theta = 1$

$\therefore \sin^2\theta = \frac{7}{16}$

$\therefore \sin\theta = \pm\frac{\sqrt{7}}{4}$

But θ is in quadrant 2, where $\sin\theta > 0$

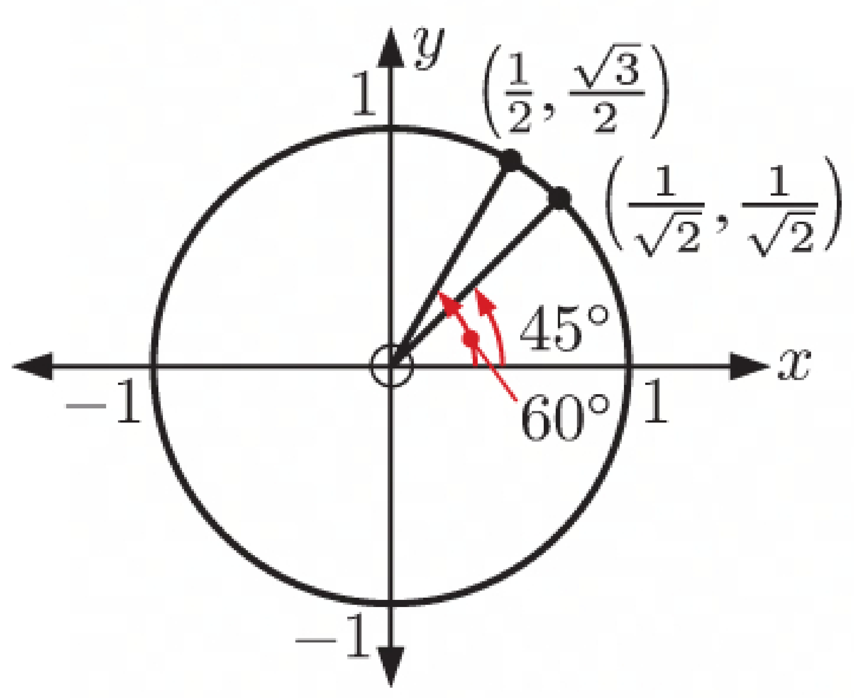
$\therefore \sin\theta = \frac{\sqrt{7}}{4}$

b $\tan\theta = \dfrac{\sin\theta}{\cos\theta} = \dfrac{\frac{\sqrt{7}}{4}}{-\frac{3}{4}} = -\dfrac{\sqrt{7}}{3}$

c $\cos(\pi - \theta) = -\cos\theta$

$= \frac{3}{4}$

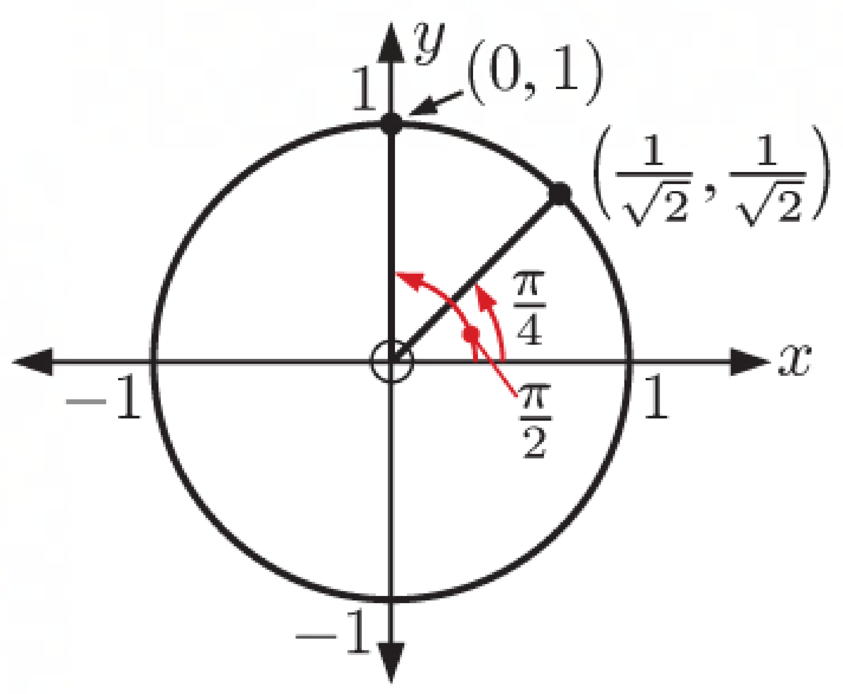
14 **a**



$\tan 60° = \dfrac{\frac{\sqrt{3}}{2}}{\frac{1}{2}} = \sqrt{3}$ and $\sin 45° = \frac{1}{\sqrt{2}}$

$$\begin{aligned}\therefore \tan^2 60° - \sin^2 45° &= (\sqrt{3})^2 - \left(\tfrac{1}{\sqrt{2}}\right)^2\\ &= 3 - \tfrac{1}{2}\\ &= 2\tfrac{1}{2}\end{aligned}$$

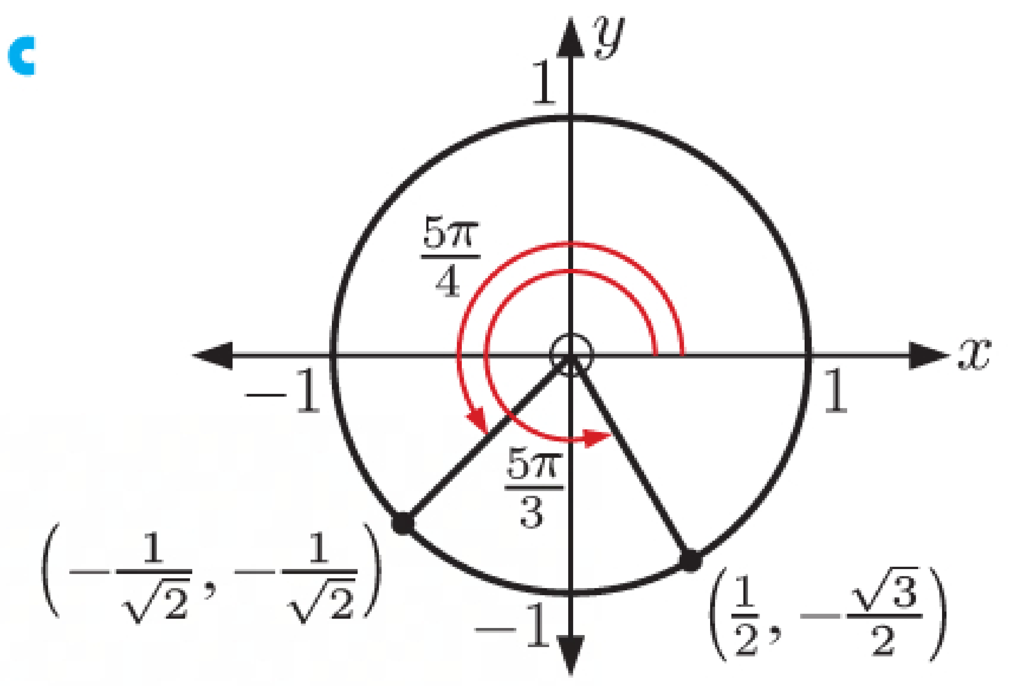
b



$\cos\frac{\pi}{4} = \frac{1}{\sqrt{2}}$ and $\sin\frac{\pi}{2} = 1$

$$\begin{aligned}\therefore \cos^2\left(\tfrac{\pi}{4}\right) + \sin\tfrac{\pi}{2} &= \left(\tfrac{1}{\sqrt{2}}\right)^2 + 1\\ &= \tfrac{1}{2} + 1\\ &= 1\tfrac{1}{2}\end{aligned}$$

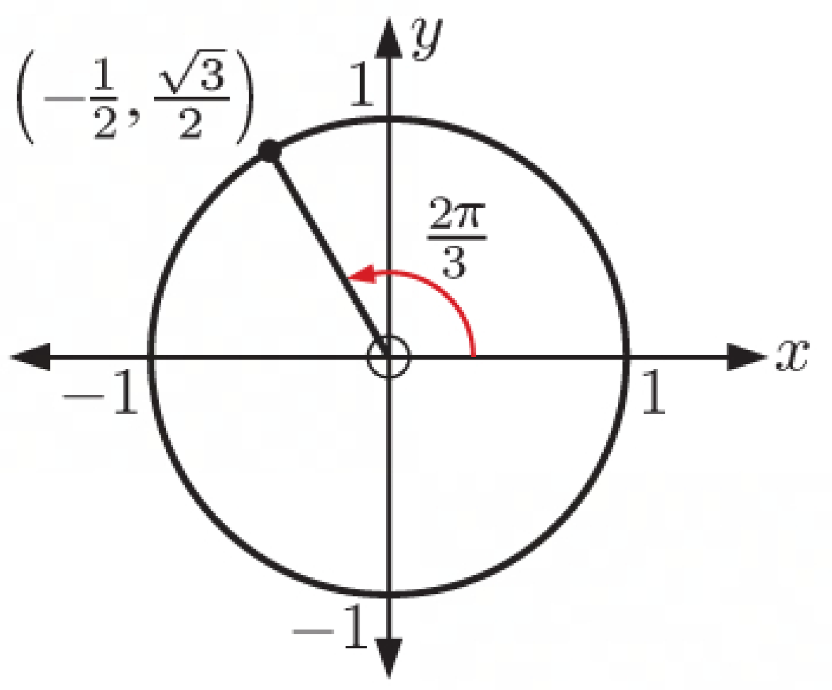
c



$$\cos\tfrac{5\pi}{3} = \tfrac{1}{2} \quad \text{and} \quad \tan\tfrac{5\pi}{4} = \frac{-\frac{1}{\sqrt{2}}}{-\frac{1}{\sqrt{2}}} = 1$$

$$\begin{aligned}\therefore\ \cos\tfrac{5\pi}{3} - \tan\tfrac{5\pi}{4} &= \tfrac{1}{2} - 1 \\ &= -\tfrac{1}{2}\end{aligned}$$

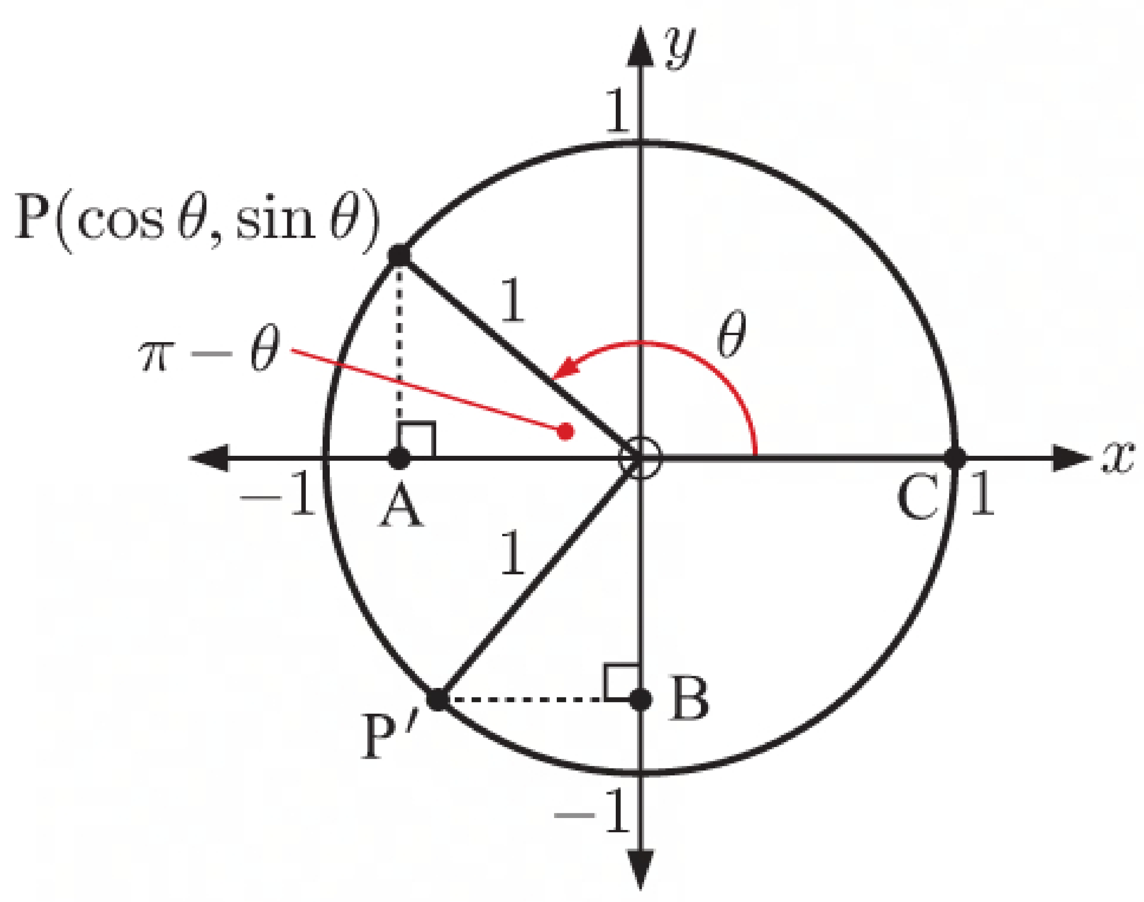
d



$$\tan\tfrac{2\pi}{3} = \frac{\frac{\sqrt{3}}{2}}{-\frac{1}{2}} = -\sqrt{3}$$

$$\begin{aligned}\therefore\ \tan^2\left(\tfrac{2\pi}{3}\right) &= (-\sqrt{3})^2 \\ &= 3\end{aligned}$$

15



For $\frac{\pi}{2} < \theta < \pi$:

The diagram shows P rotated through $\frac{\pi}{2}$ to P′, so OP′ makes an angle of $\frac{\pi}{2} + \theta$ with the positive x-axis.

$$\mathrm{P\widehat{O}A} = \pi - \theta \quad \text{and} \quad \begin{aligned}\mathrm{P'\widehat{O}B} &= \text{reflex } \mathrm{C\widehat{O}B} - \text{reflex } \mathrm{C\widehat{O}P'} \\ &= \tfrac{3\pi}{2} - \left(\tfrac{\pi}{2} + \theta\right) \\ &= \pi - \theta\end{aligned}$$

In triangles P′OB and POA:
- $\mathrm{OP'} = \mathrm{OP}$
- $\mathrm{P'\widehat{O}B} = \mathrm{P\widehat{O}A}$
- $\mathrm{P'\widehat{B}O} = \mathrm{P\widehat{A}O}$

$\therefore$ triangles P′OB and POA are congruent {AAcorS}

$\therefore$ $\mathrm{P'B} = \mathrm{PA} = \sin\theta$

So P′ has x-coordinate $-\sin\theta$

But P′ has x-coordinate $\cos\left(\frac{\pi}{2} + \theta\right)$

$\therefore$ $\cos\left(\frac{\pi}{2} + \theta\right) = -\sin\theta$

16 [AB], [AC], and [BC] are all radii,

so $AB = AC = BC = r$.

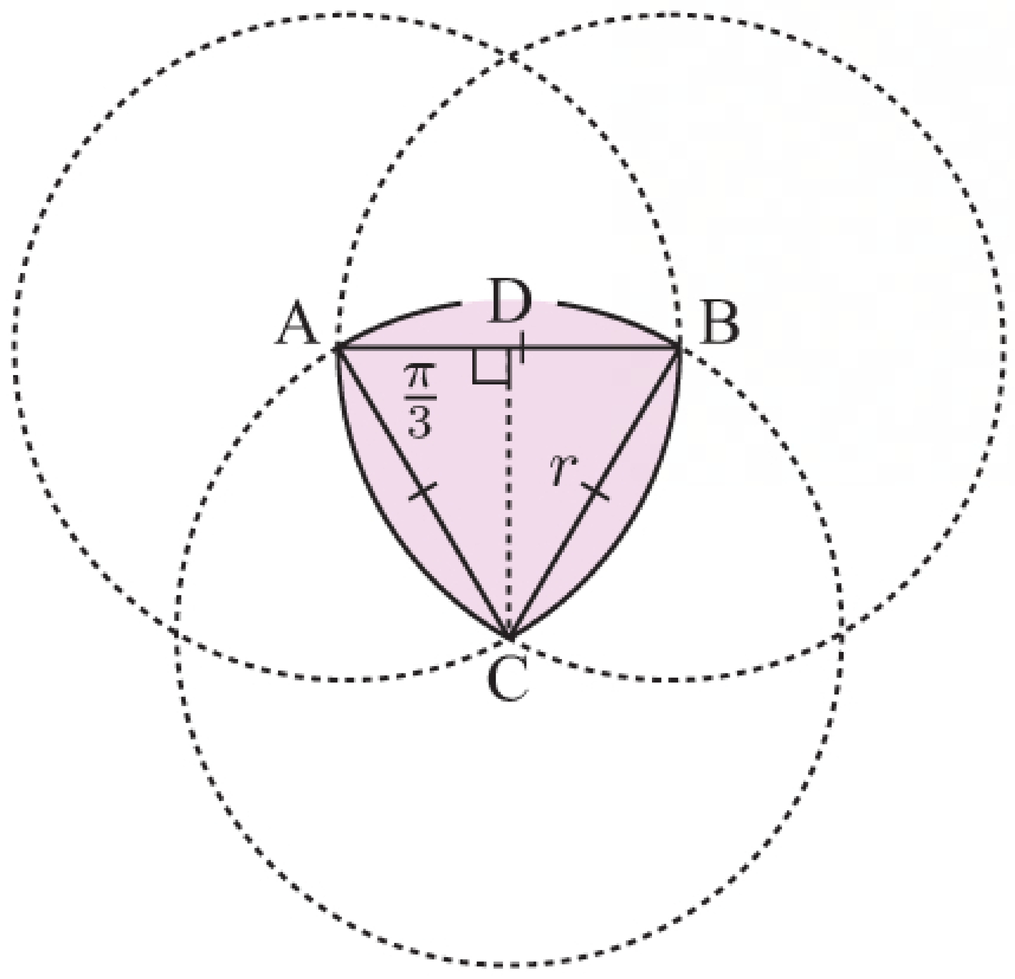
Hence $\triangle ABC$ is equilateral

and so $\widehat{CAB} = \frac{\pi}{3}$.

$$\sin\frac{\pi}{3} = \frac{CD}{AC}$$

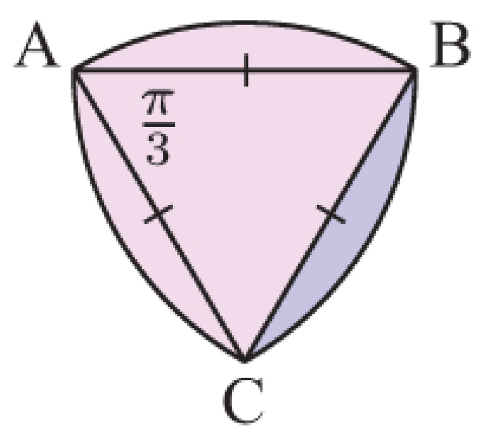
$$\therefore\ CD = \sin\frac{\pi}{3} \times AC = \frac{\sqrt{3}}{2}r$$

$$\therefore\ \text{area of } \triangle ABC = \tfrac{1}{2} \times r \times \tfrac{\sqrt{3}}{2}r = \tfrac{\sqrt{3}}{4}r^2$$



purple shaded area = area of sector − area of $\triangle$

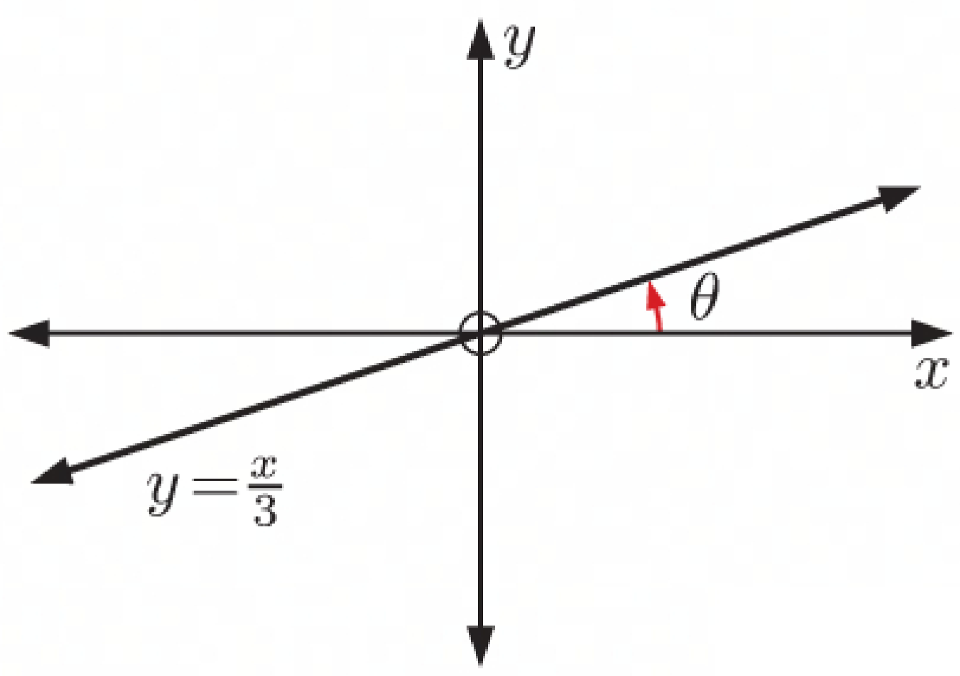
$$= \tfrac{1}{2} \times \tfrac{\pi}{3} \times r^2 - \tfrac{\sqrt{3}}{4}r^2$$
$$= \tfrac{\pi}{6}r^2 - \tfrac{\sqrt{3}}{4}r^2$$



$\therefore$ area of shaded region = 3 × purple shaded area + area of $\triangle$

$$= 3\left[\tfrac{\pi}{6}r^2 - \tfrac{\sqrt{3}}{4}r^2\right] + \tfrac{\sqrt{3}}{4}r^2$$
$$= \tfrac{\pi}{2}r^2 - \tfrac{3\sqrt{3}}{4}r^2 + \tfrac{\sqrt{3}}{4}r^2$$
$$= \tfrac{\pi}{2}r^2 - \tfrac{2}{4}\sqrt{3}r^2$$
$$\therefore\ A = \frac{r^2}{2}\left(\pi - \sqrt{3}\right)$$

17 **a**

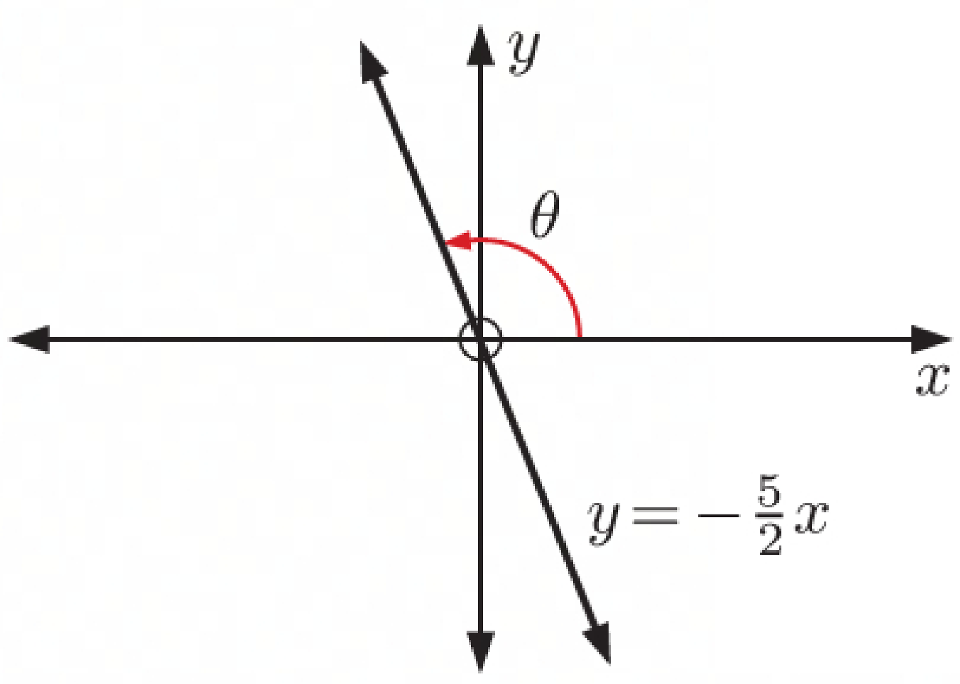


The line has gradient $\frac{1}{3}$, so $\tan\theta = \frac{1}{3}$.

Using technology, $\tan^{-1}\left(\frac{1}{3}\right) \approx 0.322$

$\therefore\ \theta \approx 0.322$

b



The line has gradient $-\frac{5}{2}$, so $\tan\theta = -\frac{5}{2}$.

Using technology, $\tan^{-1}\left(-\frac{5}{2}\right) \approx -1.19$

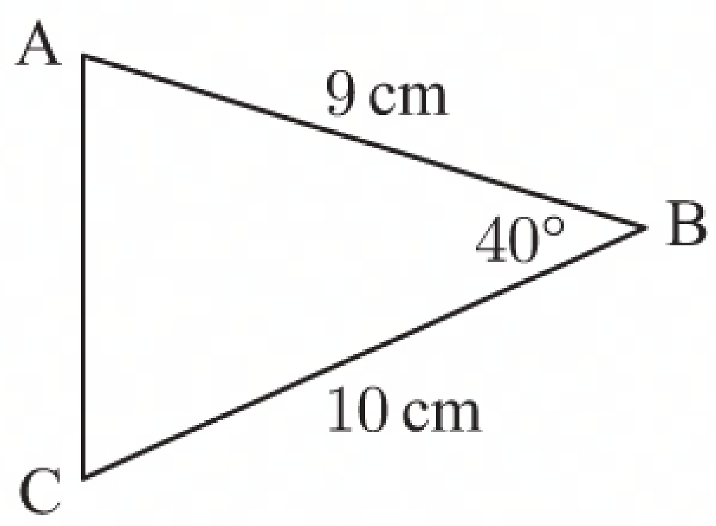
But $0 < \theta < \pi$, so $\theta \approx \pi - 1.19 \approx 1.95$

Chapter 9

NON-RIGHT ANGLED TRIANGLE TRIGONOMETRY

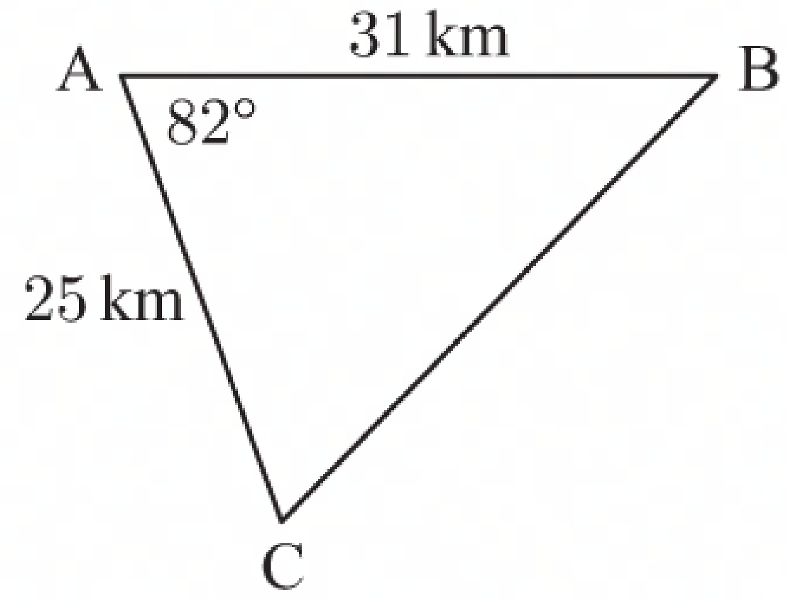
EXERCISE 9A

1 **a**



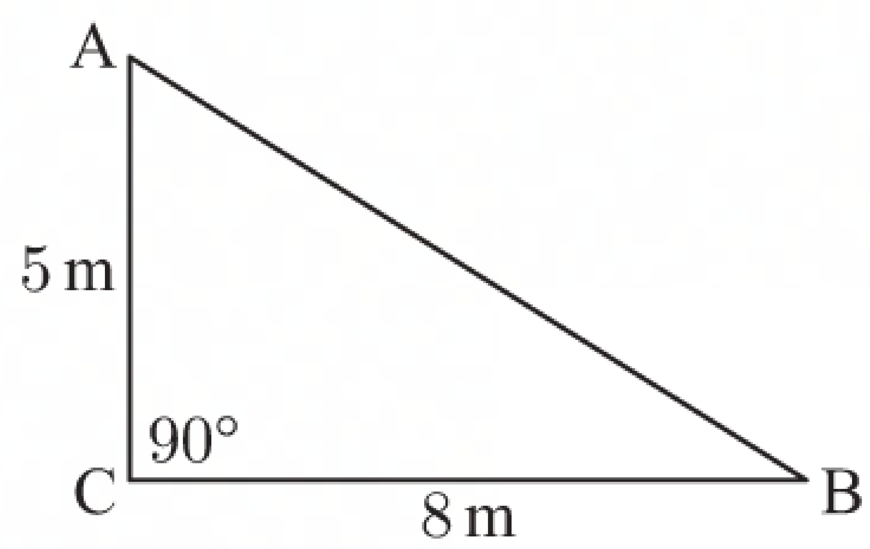
$$\begin{aligned}\text{Area} &= \tfrac{1}{2}ac\sin B \\ &= \tfrac{1}{2} \times 9 \times 10 \times \sin 40^\circ \\ &\approx 28.9 \text{ cm}^2\end{aligned}$$

b



$$\begin{aligned}\text{Area} &= \tfrac{1}{2}bc\sin A \\ &= \tfrac{1}{2} \times 25 \times 31 \times \sin 82^\circ \\ &\approx 384 \text{ km}^2\end{aligned}$$

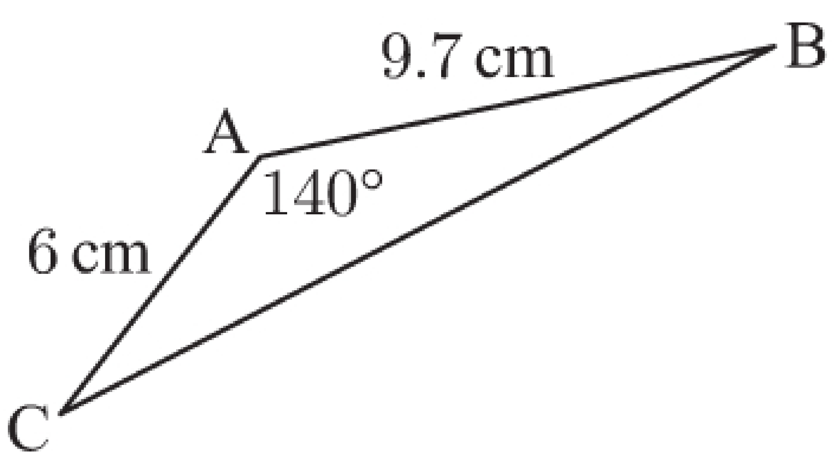
c



The triangle is right angled at C.

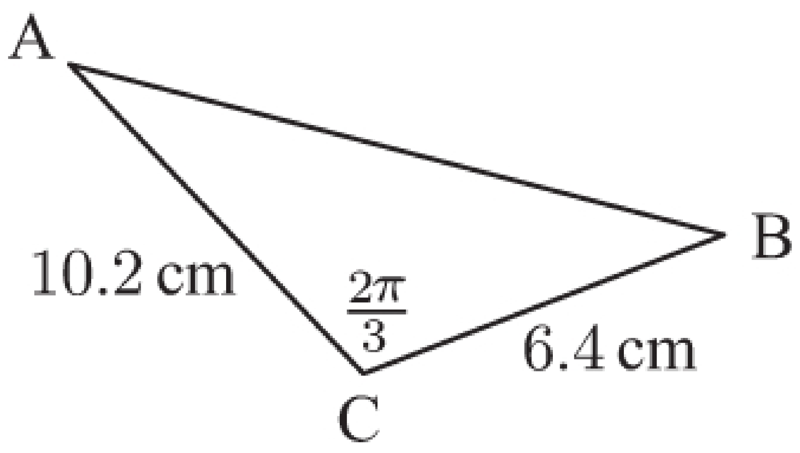
$$\begin{aligned}\text{Area} &= \tfrac{1}{2}(\text{base} \times \text{height}) \\ &= \tfrac{1}{2} \times 8 \times 5 \\ &= 20 \text{ m}^2\end{aligned}$$

2 **a**



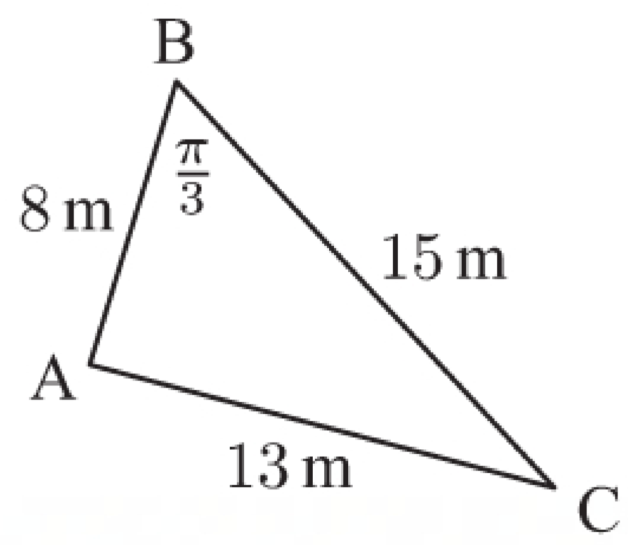
$$\begin{aligned}\text{Area} &= \tfrac{1}{2}bc\sin A \\ &= \tfrac{1}{2} \times 6 \times 9.7 \times \sin 140^\circ \\ &\approx 18.7 \text{ cm}^2\end{aligned}$$

b



$$\begin{aligned}\text{Area} &= \tfrac{1}{2}ab\sin C \\ &= \tfrac{1}{2} \times 6.4 \times 10.2 \times \sin \tfrac{2\pi}{3} \\ &\approx 28.3 \text{ cm}^2\end{aligned}$$

c

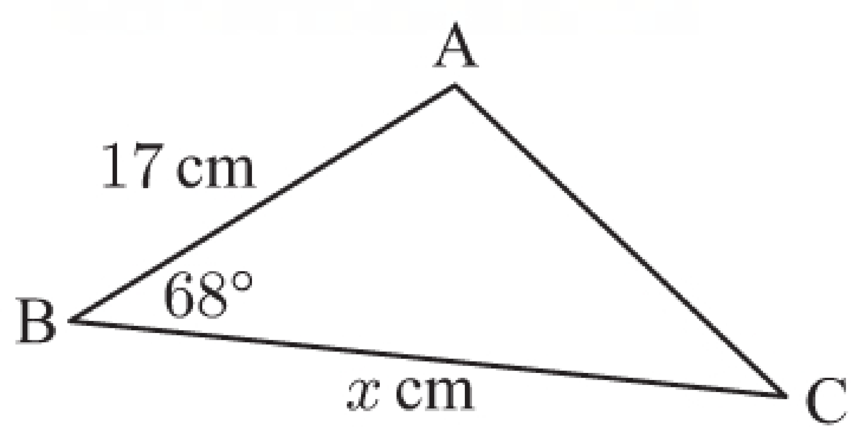


Using the sides adjacent to the included angle,

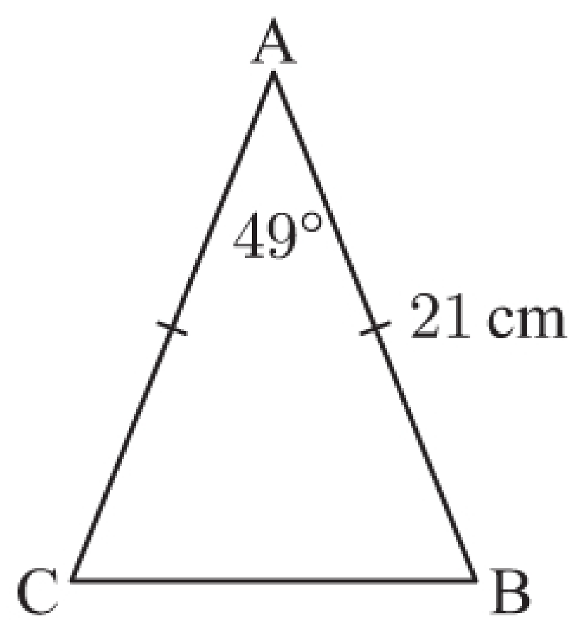
$$\text{area} = \tfrac{1}{2}ac\sin B$$
$$= \tfrac{1}{2} \times 15 \times 8 \times \sin\tfrac{\pi}{3}$$
$$\approx 52.0\ \text{m}^2$$

3

$$\text{Area} = 150\ \text{cm}^2$$
$$\therefore\ \tfrac{1}{2}ac\sin B = 150$$
$$\therefore\ \tfrac{1}{2} \times x \times 17 \times \sin 68^\circ = 150$$
$$\therefore\ x = \frac{2 \times 150}{17 \times \sin 68^\circ}$$
$$\therefore\ x \approx 19.0$$

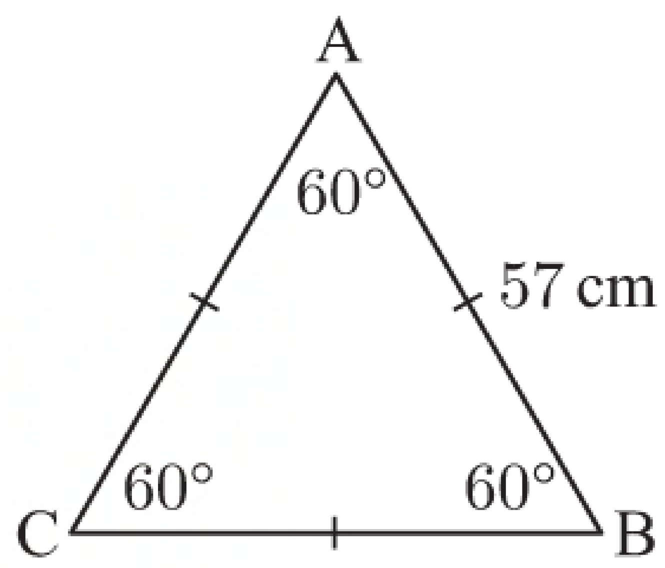


4 **a**



$$\text{Area} = \tfrac{1}{2}bc\sin A$$
$$= \tfrac{1}{2} \times 21 \times 21 \times \sin 49^\circ$$
$$\approx 166\ \text{cm}^2$$

b

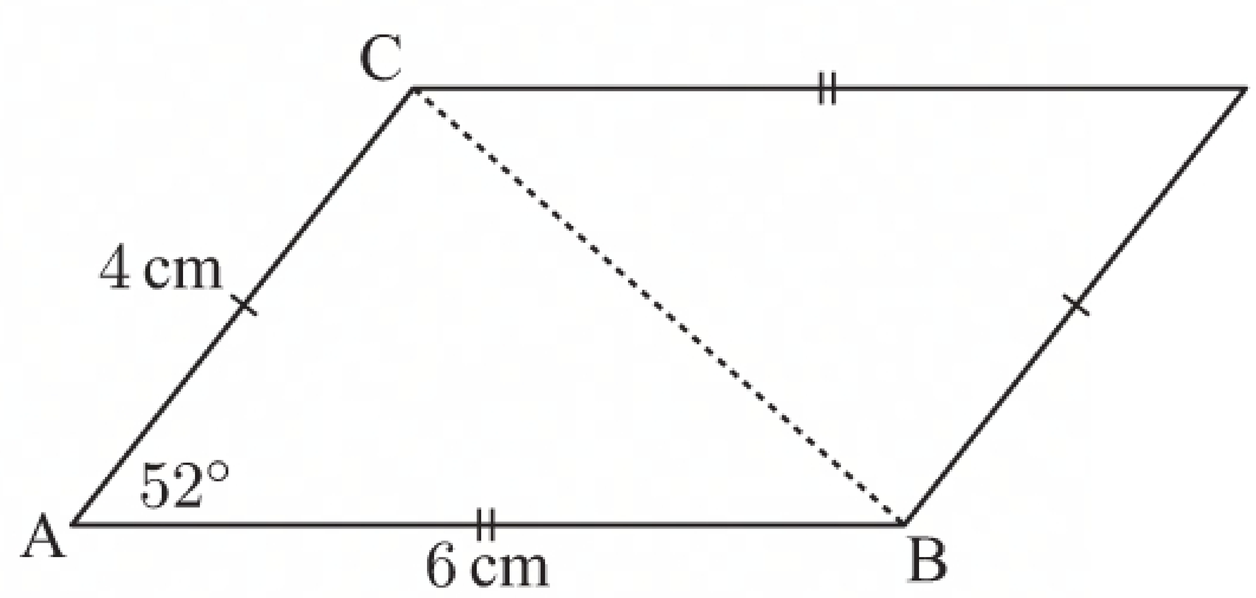


An equilateral triangle has all sides and angles equal.

$$\text{Area} = \tfrac{1}{2}ab\sin C$$
$$= \tfrac{1}{2} \times 57 \times 57 \times \sin 60^\circ$$
$$\approx 1410\ \text{cm}^2$$

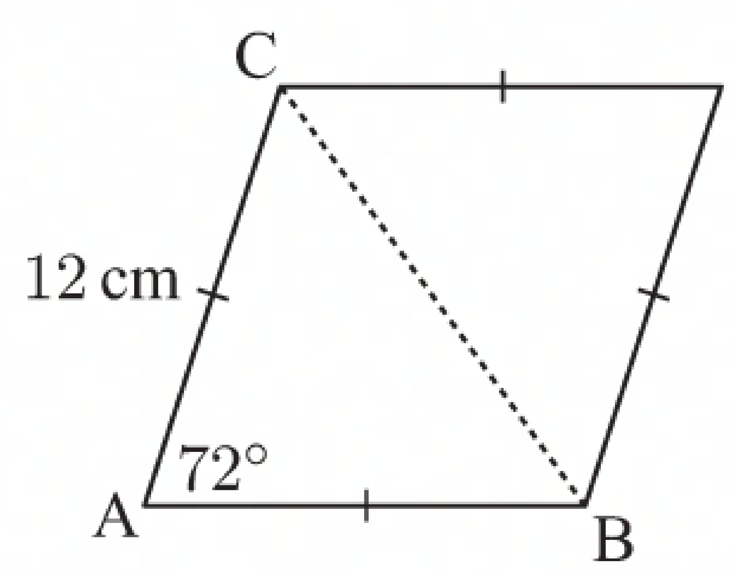
5

$$\text{Area} = 2 \times \text{area of } \triangle ABC$$
$$= 2 \times \tfrac{1}{2}bc\sin A$$
$$= 2 \times \tfrac{1}{2} \times 4 \times 6 \times \sin 52^\circ$$
$$\approx 18.9\ \text{cm}^2$$

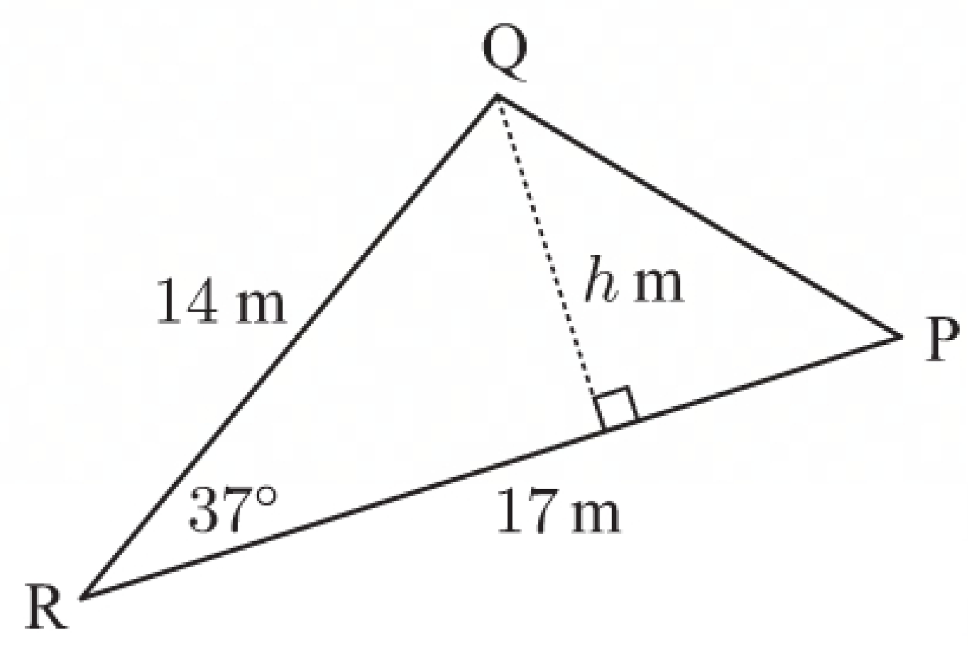


6

$$\text{Area} = 2 \times \text{area of } \triangle ABC$$
$$= 2 \times \tfrac{1}{2}bc\sin A$$
$$= 2 \times \tfrac{1}{2} \times 12 \times 12 \times \sin 72^\circ$$
$$\approx 137\ \text{cm}^2$$

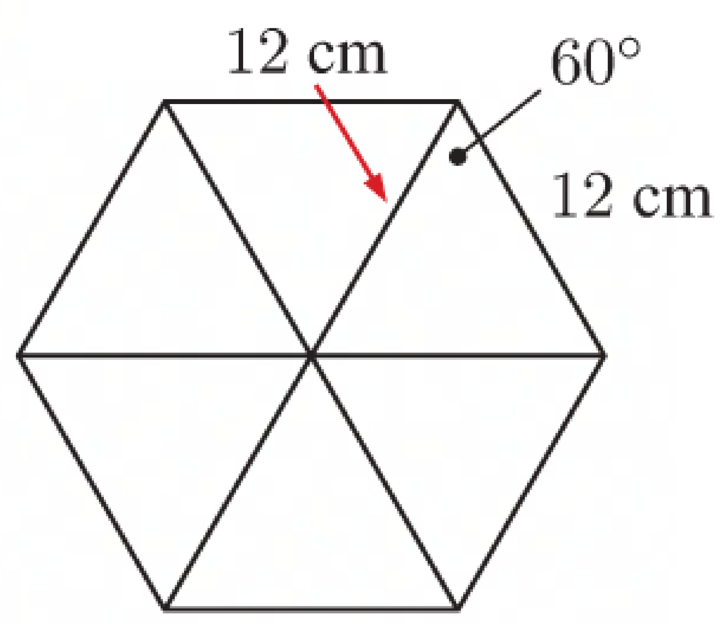


7 **a** Area of $\triangle PQR = \frac{1}{2}pq\sin R$
$= \frac{1}{2} \times 14 \times 17 \times \sin 37°$
$\approx 71.616 \text{ m}^2$

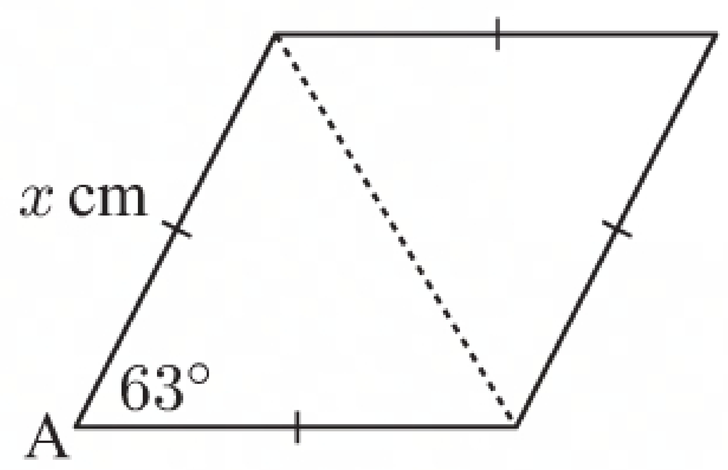


b Let the length from Q to [RP] be h m.
Area of $\triangle PQR = \frac{1}{2} \times \text{base} \times \text{height}$
$\therefore\ 71.616 \approx \frac{1}{2} \times 17 \times h$
$\therefore\ h \approx \dfrac{71.616}{\frac{1}{2} \times 17}$
$\therefore\ h \approx 8.43$
So the length of the altitude from Q to [RP] is about 8.43 m.

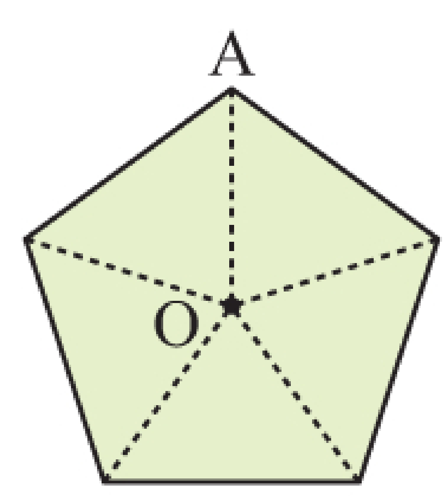
8 Area $= 6 \times$ area of one triangle
$= 6 \times \frac{1}{2} \times 12 \times 12 \times \sin 60°$
$\approx 374 \text{ cm}^2$

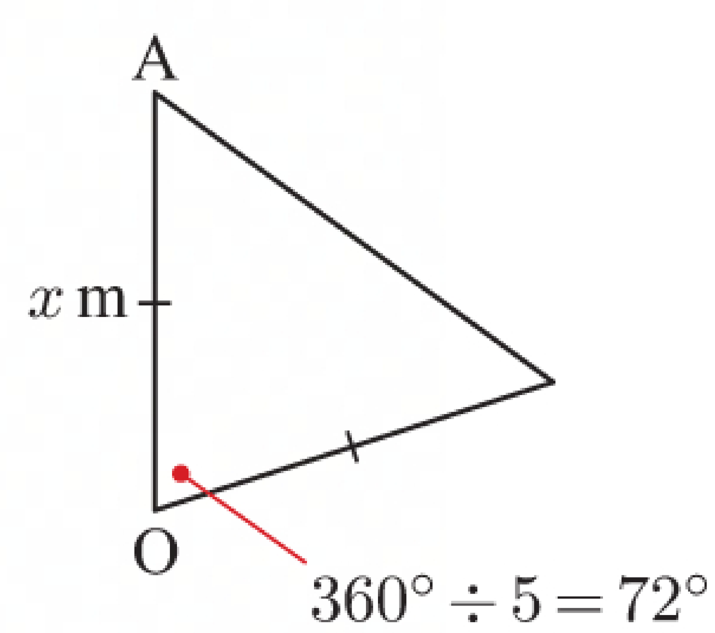


9 Let the side length be x cm.
Area $= 2 \times$ area of one triangle
$= 2 \times \frac{1}{2} \times x \times x \times \sin 63°$
$\therefore\ x^2 \sin 63° = 50$
$\therefore\ x^2 = \dfrac{50}{\sin 63°}$
$\therefore\ x = \sqrt{\dfrac{50}{\sin 63°}}$ $\{x > 0\}$
$\therefore\ x \approx 7.49$
So, the sides are approximately 7.49 cm long.



10 Area of one triangle $= \dfrac{338}{5} \text{ m}^2$
Let the side length of the triangle be x m.
$\therefore\ \frac{1}{2} \times x \times x \times \sin 72° = \dfrac{338}{5}$
$\therefore\ x^2 = \dfrac{2 \times 338}{5 \times \sin 72°}$
$\therefore\ x = \sqrt{\dfrac{2 \times 338}{5 \times \sin 72°}}$ $\{x > 0\}$
$\therefore\ x \approx 11.9$
So, OA ≈ 11.9 m



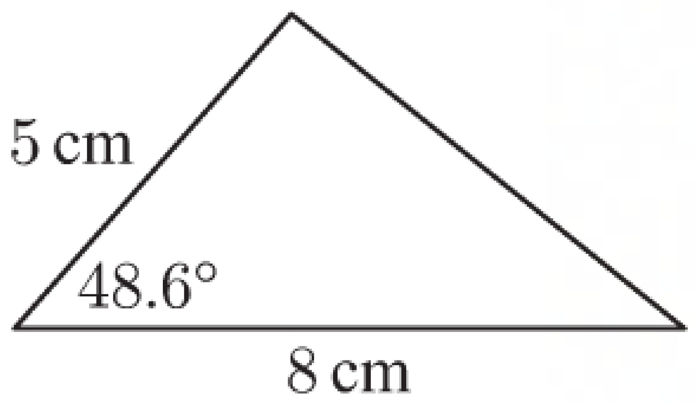


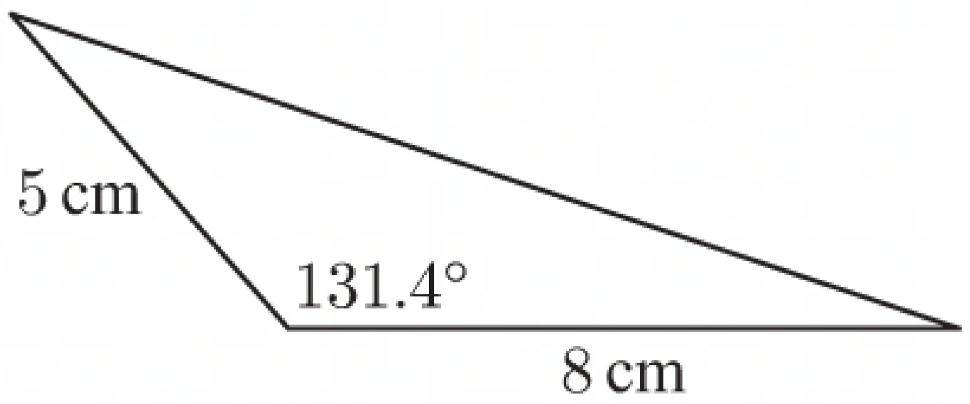
11 **a** If the included angle is θ, then
$$\begin{aligned}\tfrac{1}{2} \times 5 \times 8 \times \sin\theta &= 15\\ \therefore\quad 20\sin\theta &= 15\\ \therefore\quad \sin\theta &= \tfrac{15}{20} = \tfrac{3}{4}\end{aligned}$$

Now $\sin^{-1}\left(\tfrac{3}{4}\right) \approx 48.6^\circ$

$\therefore\quad \theta \approx 48.6^\circ$ or $(180 - 48.6)^\circ$

$\therefore\quad \theta \approx 48.6^\circ$ or 131.4°



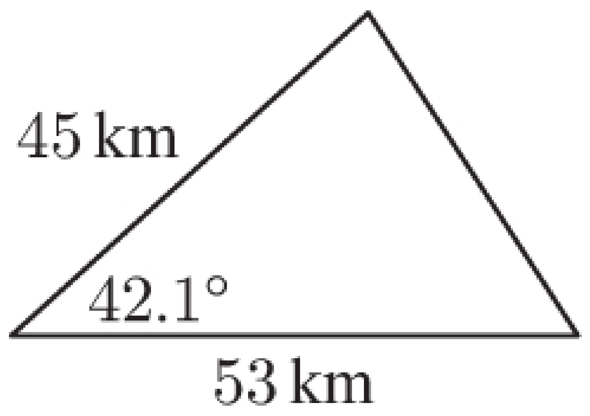


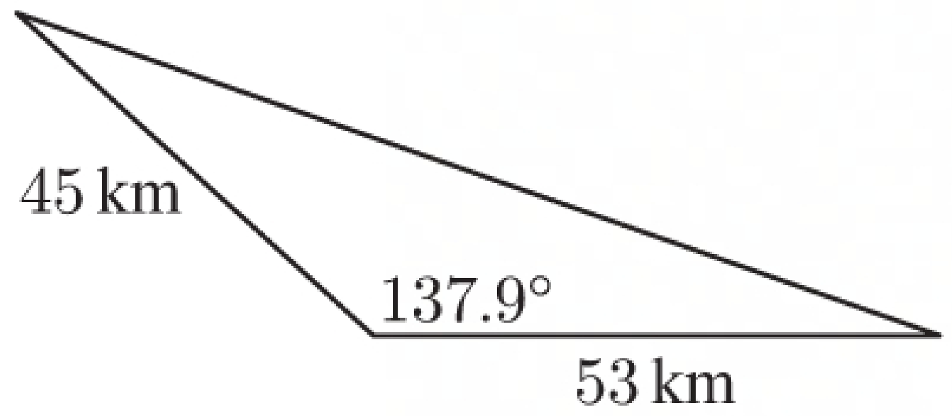
b If the included angle is θ, then
$$\begin{aligned}\tfrac{1}{2} \times 45 \times 53 \times \sin\theta &= 800\\ \therefore\quad \sin\theta &= \frac{800 \times 2}{45 \times 53}\end{aligned}$$

Now $\sin^{-1}\left(\tfrac{800\times 2}{45\times 53}\right) \approx 42.1^\circ$

$\therefore\quad \theta \approx 42.1^\circ$ or $(180 - 42.1)^\circ$

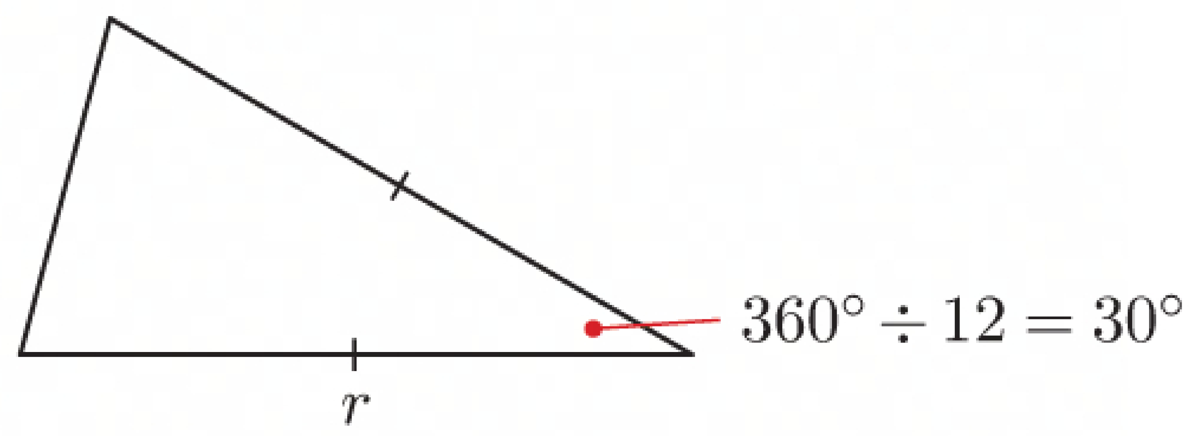
$\therefore\quad \theta \approx 42.1^\circ$ or 137.9°





12 Each coin is made up of 12 triangles.
Let half the length of a diagonal of a coin be r.

$$\begin{aligned}\text{Total area of 8 coins} &= 8 \times 12 \times \tfrac{1}{2} \times r \times r \times \sin 30^\circ\\ &= 48r^2\left(\tfrac{1}{2}\right)\\ &= 24r^2\end{aligned}$$



$$\begin{aligned}\text{Area of \$5 note} &= (4 \times 2r) \times (2 \times 2r)\\ &= 8r \times 4r\\ &= 32r^2\end{aligned}$$

$$\begin{aligned}\text{Fraction covered} &= \frac{24r^2}{32r^2}\\ &= \tfrac{3}{4}\end{aligned}$$

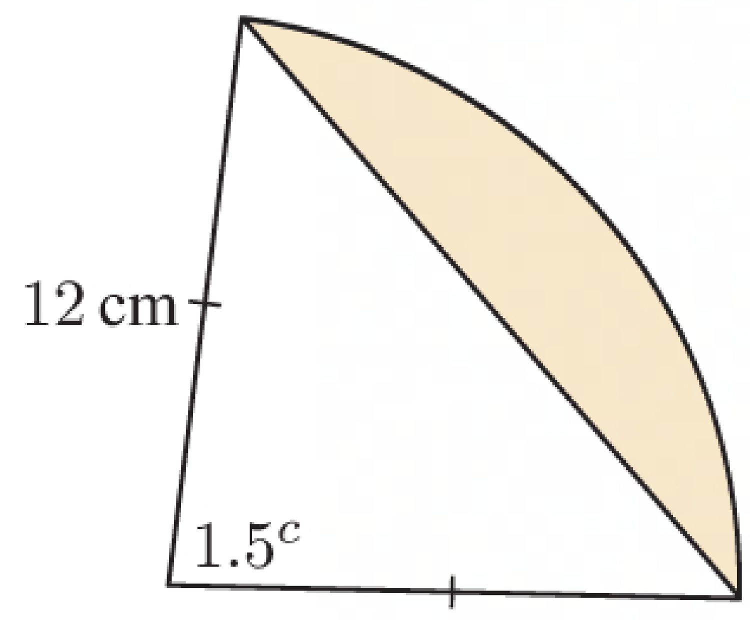
$\therefore\quad \tfrac{1}{4}$ is not covered.

13 **a** Shaded area

$= \text{area of sector} - \text{area of triangle}$

$= \tfrac{1}{2} \times 1.5 \times 12^2 - \tfrac{1}{2} \times 12 \times 12 \times \sin 1.5^c$

$\approx 36.2\ \text{cm}^2$

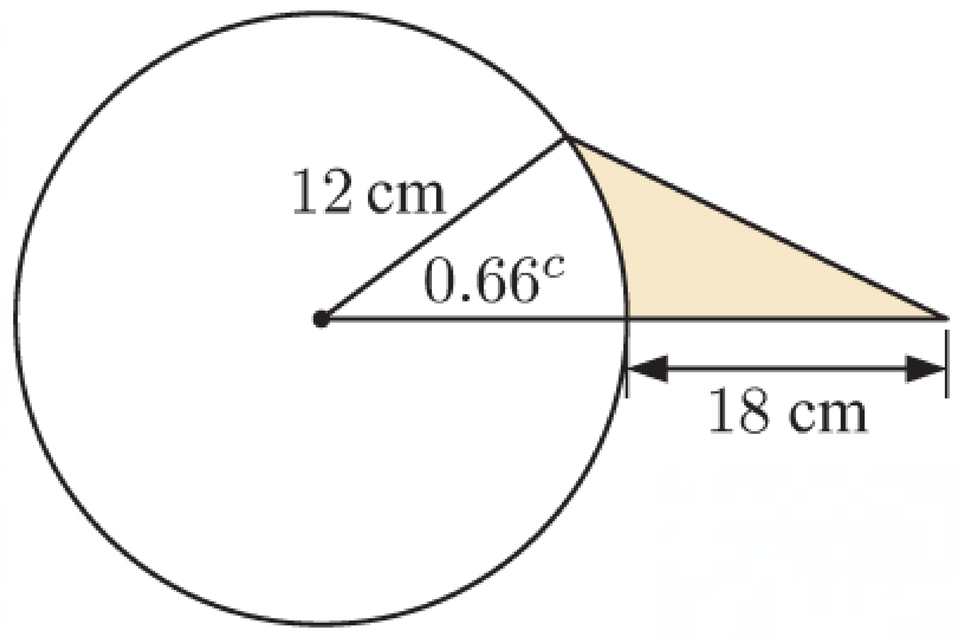


b Shaded area

$= \text{area of triangle} - \text{area of sector}$

$= \frac{1}{2} \times 12 \times (12 + 18) \times \sin 0.66^c - \frac{1}{2} \times 0.66 \times 12^2$

$\approx 62.8 \text{ cm}^2$

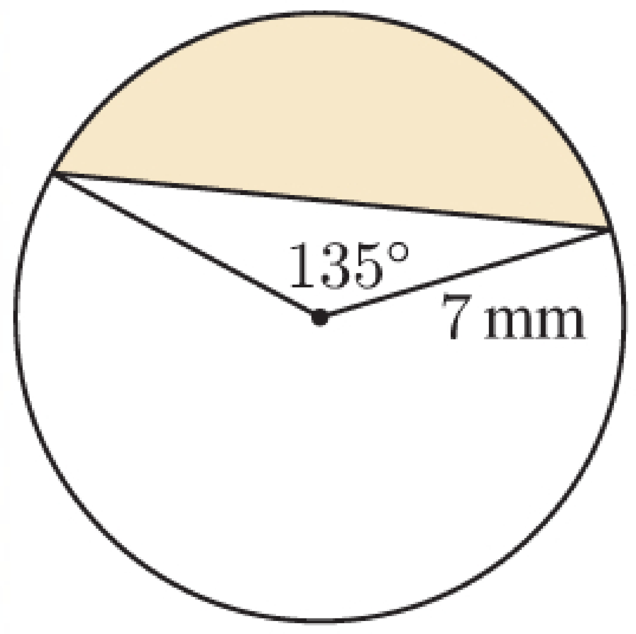


c Shaded area

$= \text{area of sector} - \text{area of triangle}$

$= \frac{135}{360} \times \pi \times 7^2 - \frac{1}{2} \times 7 \times 7 \times \sin 135°$

$\approx 40.4 \text{ mm}^2$



d $\triangle BCD$ is an equilateral triangle.

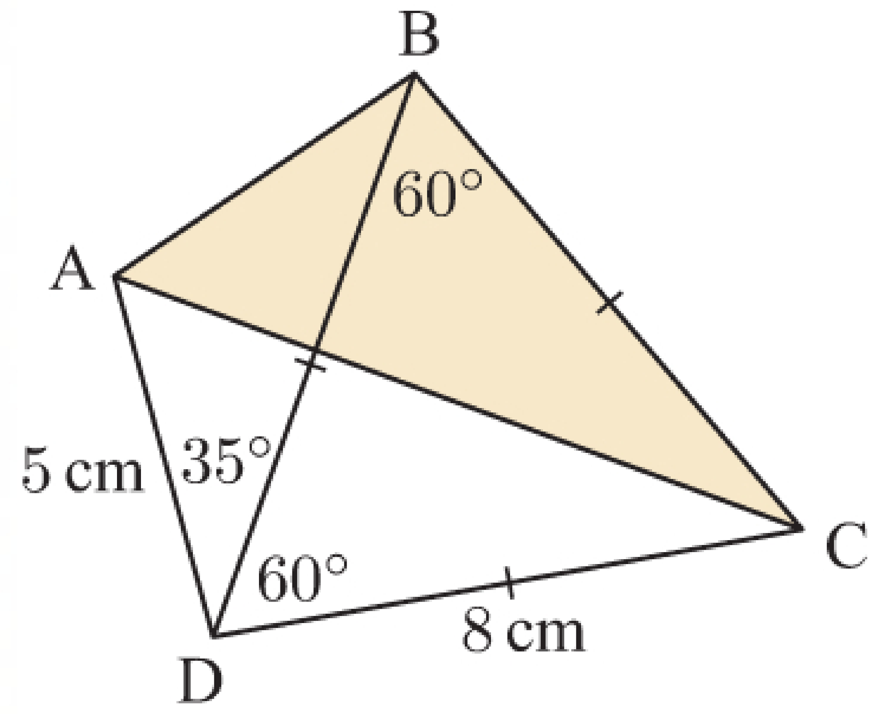
Shaded area

$= \text{area of } \triangle BCD + \text{area of } \triangle ABD - \text{area of } \triangle ACD$

$= \frac{1}{2} \times 8 \times 8 \times \sin 60° + \frac{1}{2} \times 5 \times 8 \times \sin 35°$

$\quad - \frac{1}{2} \times 5 \times 8 \times \sin 95°$

$\approx 19.3 \text{ cm}^2$



14 Area of segment AXBD

$= \text{area of sector ACBD} - \text{area of } \triangle ACB$

$= \frac{100}{360} \times \pi \times 7.3^2 - \frac{1}{2} \times 7.3 \times 7.3 \times \sin 100°$

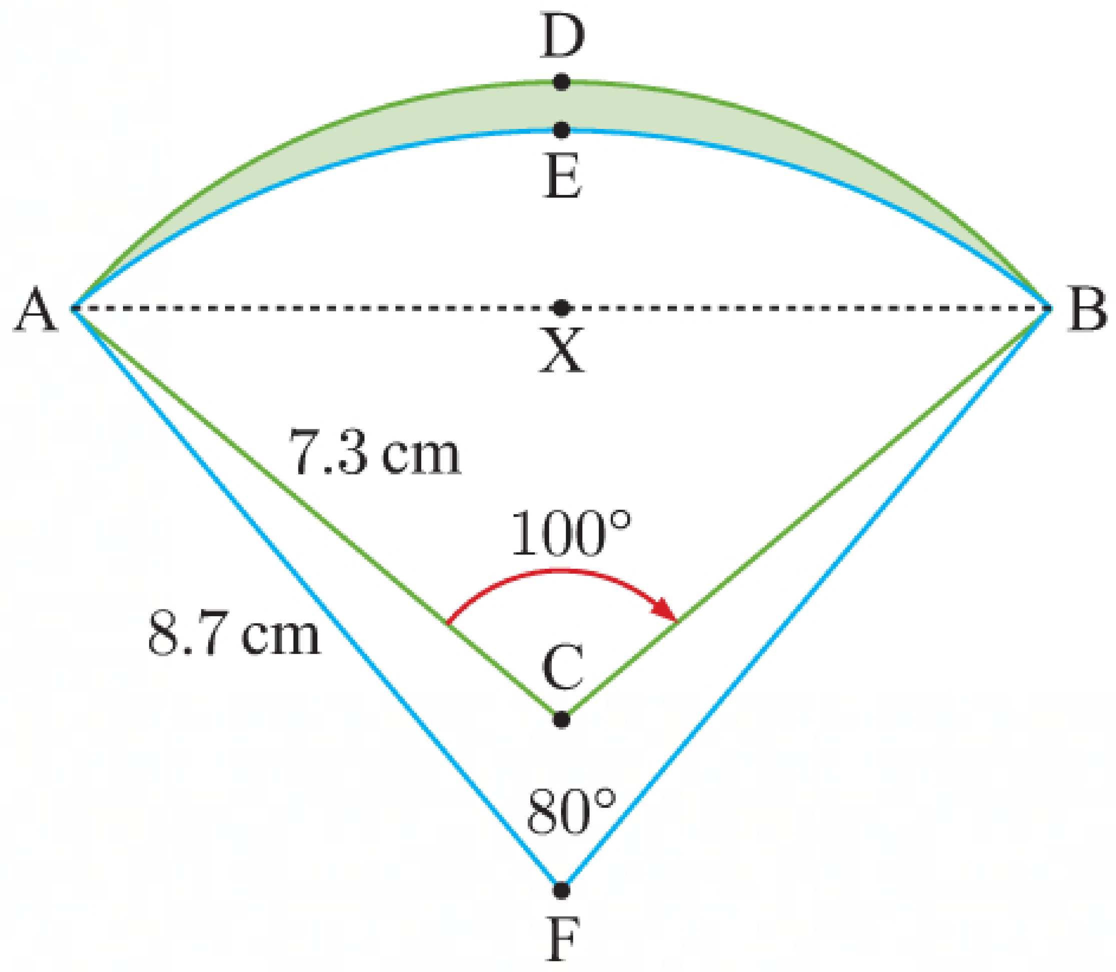
$\approx 20.264 \text{ cm}^2$

Area of segment AXBE

$= \text{area of sector AFBE} - \text{area of } \triangle AFB$

$= \frac{80}{360} \times \pi \times 8.7^2 - \frac{1}{2} \times 8.7 \times 8.7 \times \sin 80°$

$\approx 15.572 \text{ cm}^2$

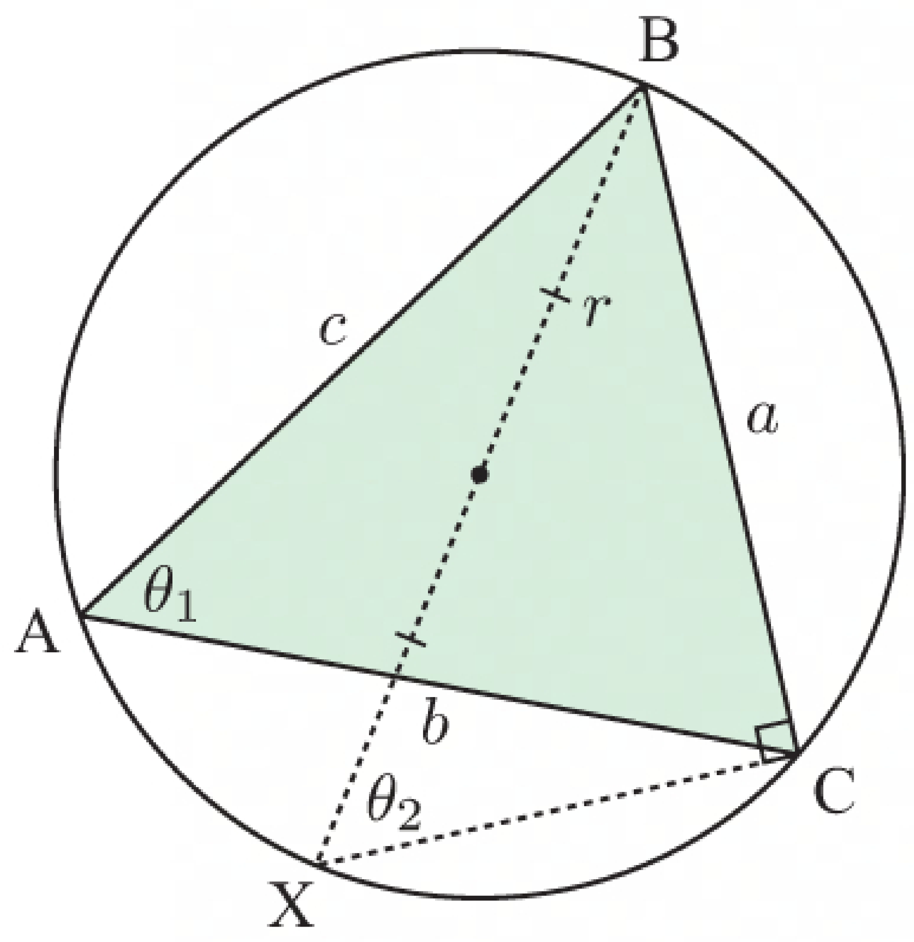


Shaded area $= \text{area of segment AXBD} - \text{area of segment AXBE}$

$\approx 20.264 - 15.572$

$\approx 4.69 \text{ cm}^2$

15



We draw diameter [BX] and line segment [CX].

Now $\theta_1 = \theta_2$ {angles subtended by the same arc}

and $\widehat{BCX} = 90°$ {angle in a semi-circle}

$$\begin{aligned}\text{Area of } \triangle ABC &= \tfrac{1}{2}bc\sin\theta_1\\ &= \tfrac{1}{2}bc\sin\theta_2\\ &= \tfrac{1}{2}bc \times \frac{a}{\text{BX}}\\ &= \tfrac{1}{2}bc \times \frac{a}{2r}\\ &= \frac{abc}{4r}\end{aligned}$$

EXERCISE 9B

1 **a** Let the remaining side have length x cm.

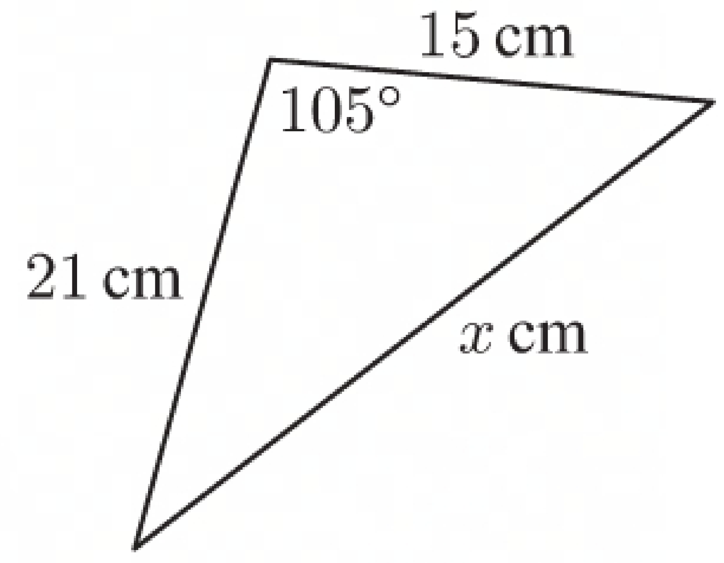
By the cosine rule:

$x^2 = 21^2 + 15^2 - 2 \times 21 \times 15 \times \cos 105°$

$\therefore\ x = \sqrt{21^2 + 15^2 - 2 \times 21 \times 15 \times \cos 105°}$ {as $x > 0$}

$\therefore\ x \approx 28.8$

The remaining side is about 28.8 cm in length.



b Let the remaining side have length x km.

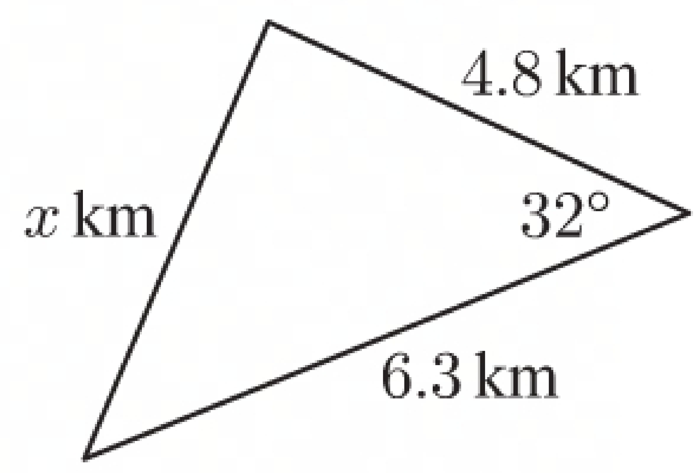
By the cosine rule:

$x^2 = 6.3^2 + 4.8^2 - 2 \times 6.3 \times 4.8 \times \cos 32°$

$\therefore\ x = \sqrt{6.3^2 + 4.8^2 - 2 \times 6.3 \times 4.8 \times \cos 32°}$ {as $x > 0$}

$\therefore\ x \approx 3.38$

The remaining side is about 3.38 km in length.



c Let the remaining side have length x m.

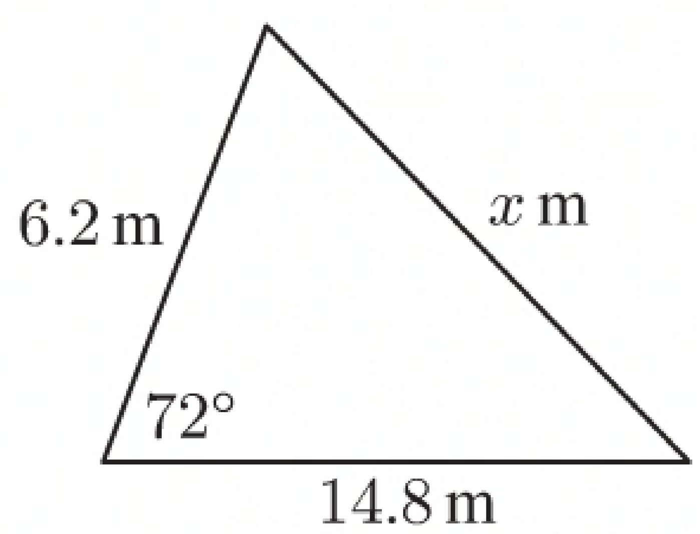
By the cosine rule:

$x^2 = 6.2^2 + 14.8^2 - 2 \times 6.2 \times 14.8 \times \cos 72°$

$\therefore\ x = \sqrt{6.2^2 + 14.8^2 - 2 \times 6.2 \times 14.8 \times \cos 72°}$ {as $x > 0$}

$\therefore\ x \approx 14.2$

The remaining side is about 14.2 m in length.

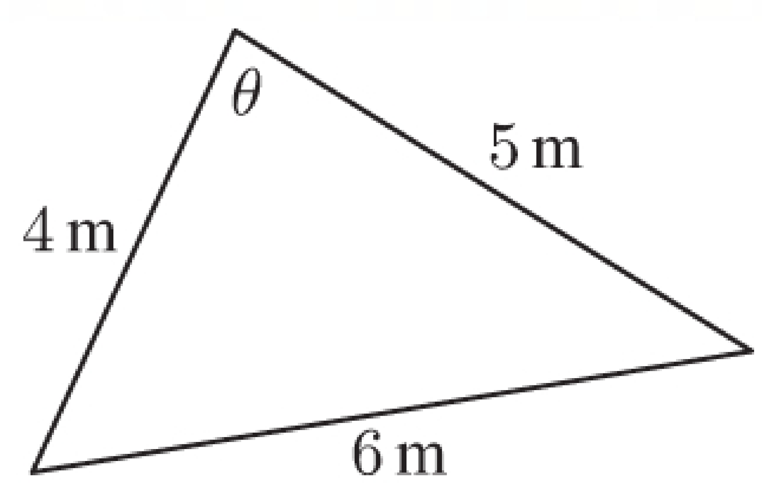


2 **a** By the cosine rule: $\cos\theta = \dfrac{4^2 + 5^2 - 6^2}{2 \times 4 \times 5}$

$\therefore\ \theta = \cos^{-1}\left(\dfrac{4^2 + 5^2 - 6^2}{2 \times 4 \times 5}\right)$

$\therefore\ \theta = \cos^{-1}\left(\frac{5}{40}\right)$

$\therefore\ \theta \approx 82.8°$



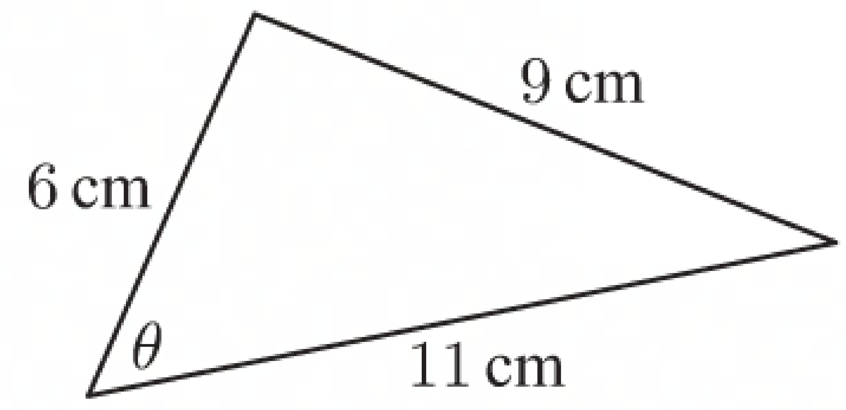
b By the cosine rule:

$$\cos\theta = \frac{6^2 + 11^2 - 9^2}{2 \times 6 \times 11}$$

$$\therefore \quad \theta = \cos^{-1}\left(\frac{6^2 + 11^2 - 9^2}{2 \times 6 \times 11}\right)$$

$$\therefore \quad \theta = \cos^{-1}\left(\tfrac{76}{132}\right)$$

$$\therefore \quad \theta \approx 54.8°$$



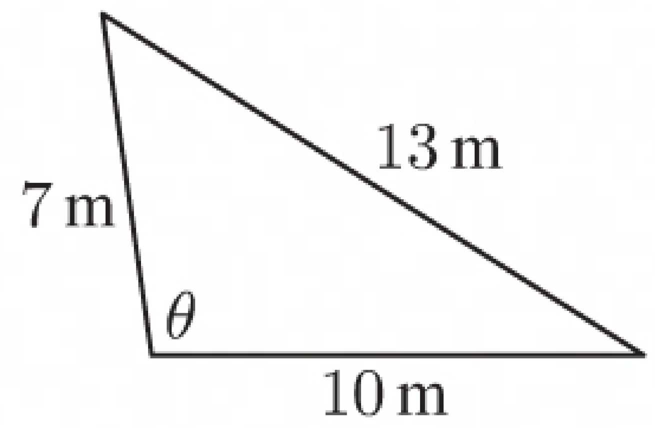
c By the cosine rule:

$$\cos\theta = \frac{7^2 + 10^2 - 13^2}{2 \times 7 \times 10}$$

$$\therefore \quad \theta = \cos^{-1}\left(\frac{7^2 + 10^2 - 13^2}{2 \times 7 \times 10}\right)$$

$$\therefore \quad \theta = \cos^{-1}\left(-\tfrac{20}{140}\right)$$

$$\therefore \quad \theta \approx 98.2°$$



3 By the cosine rule:

$$\cos \mathrm{B\widehat{A}C} = \frac{12^2 + 13^2 - 11^2}{2 \times 12 \times 13}$$

$$\therefore \quad \mathrm{B\widehat{A}C} = \cos^{-1}\left(\tfrac{192}{312}\right)$$

$$\therefore \quad \mathrm{B\widehat{A}C} \approx 52.0°$$

By the cosine rule:

$$\cos \mathrm{A\widehat{B}C} = \frac{13^2 + 11^2 - 12^2}{2 \times 13 \times 11}$$

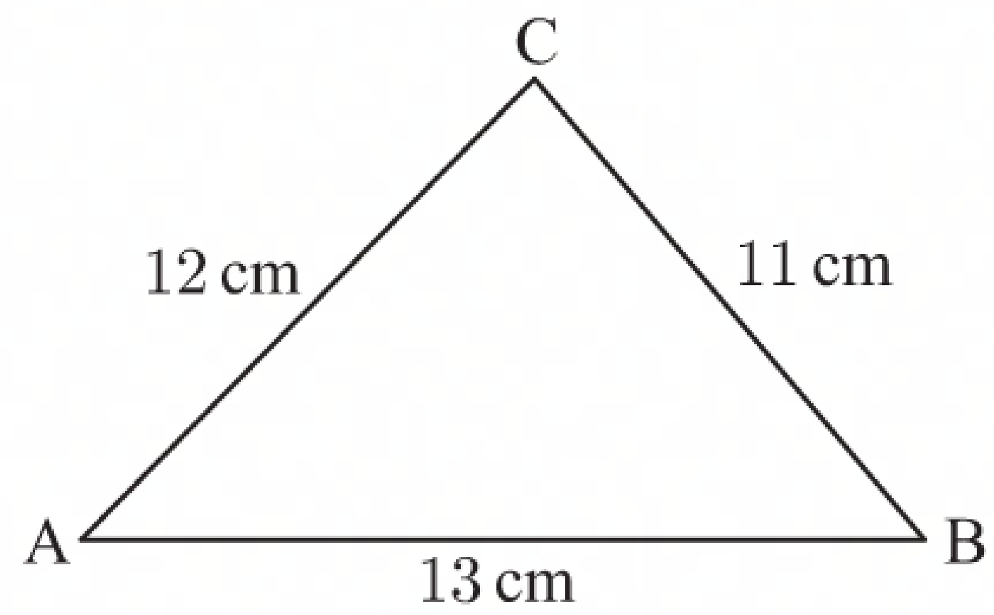
$$\therefore \quad \mathrm{A\widehat{B}C} = \cos^{-1}\left(\tfrac{146}{286}\right)$$

$$\therefore \quad \mathrm{A\widehat{B}C} \approx 59.3°$$

Also, $\mathrm{A\widehat{C}B} = 180° - \mathrm{B\widehat{A}C} - \mathrm{A\widehat{B}C}$

$$\approx 180° - 52.0° - 59.3°$$

$$\therefore \quad \mathrm{A\widehat{C}B} \approx 68.7°$$



4 **a**

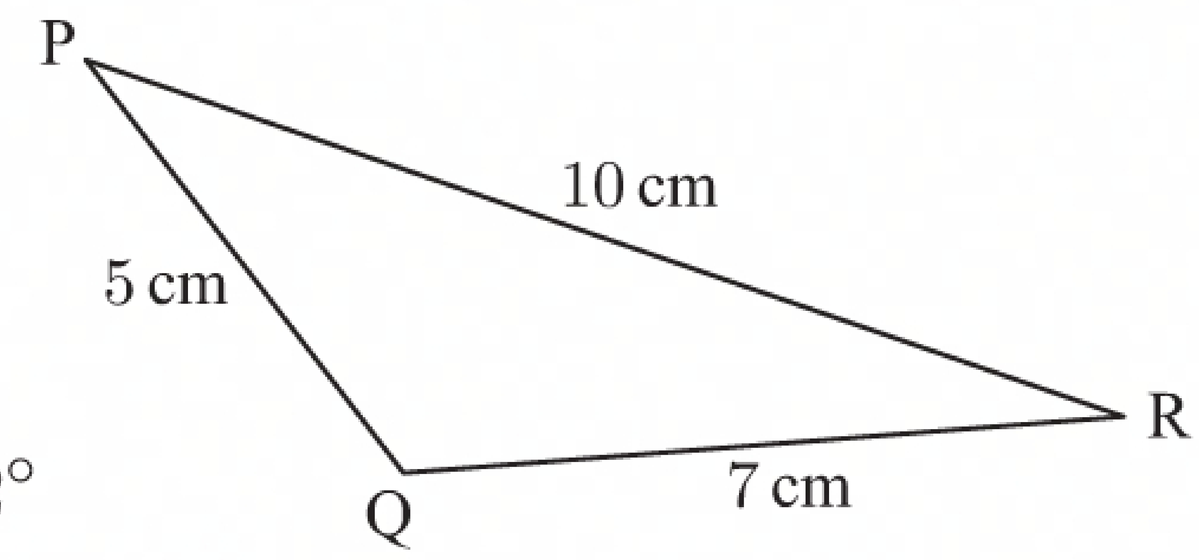
$$\cos \mathrm{P\widehat{Q}R} = \frac{5^2 + 7^2 - 10^2}{2 \times 5 \times 7}$$

$$\therefore \quad \mathrm{P\widehat{Q}R} = \cos^{-1}\left(-\tfrac{26}{70}\right)$$

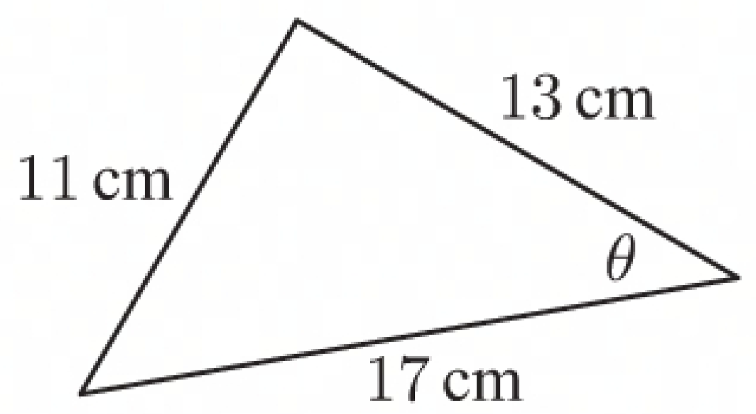
$$\therefore \quad \mathrm{P\widehat{Q}R} \approx 112°$$

b Area of $\triangle\mathrm{PQR} \approx \frac{1}{2} \times 5 \times 7 \times \sin 112°$

$$\approx 16.2 \text{ cm}^2$$



5 **a**



The smallest angle is opposite the shortest side.

By the cosine rule:

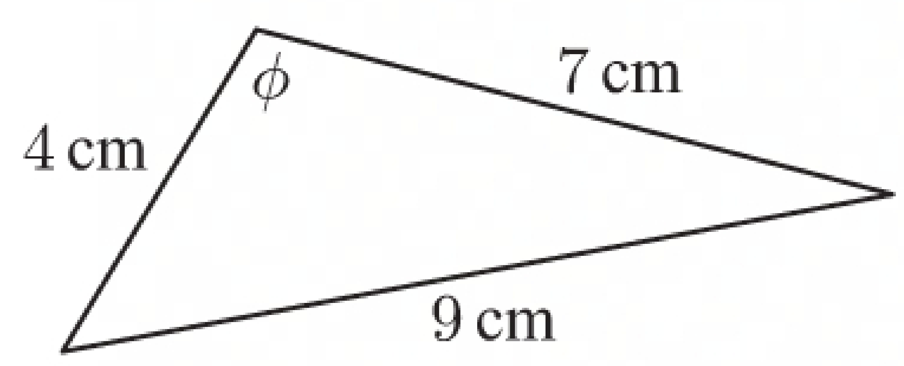
$$\cos\theta = \frac{13^2 + 17^2 - 11^2}{2 \times 13 \times 17}$$

$$\therefore \quad \theta = \cos^{-1}\left(\tfrac{337}{442}\right)$$

$$\approx 40.3°$$

The smallest angle measures about 40.3°.

b



The largest angle is opposite the longest side.

By the cosine rule:

$$\cos\phi = \frac{4^2 + 7^2 - 9^2}{2 \times 4 \times 7}$$

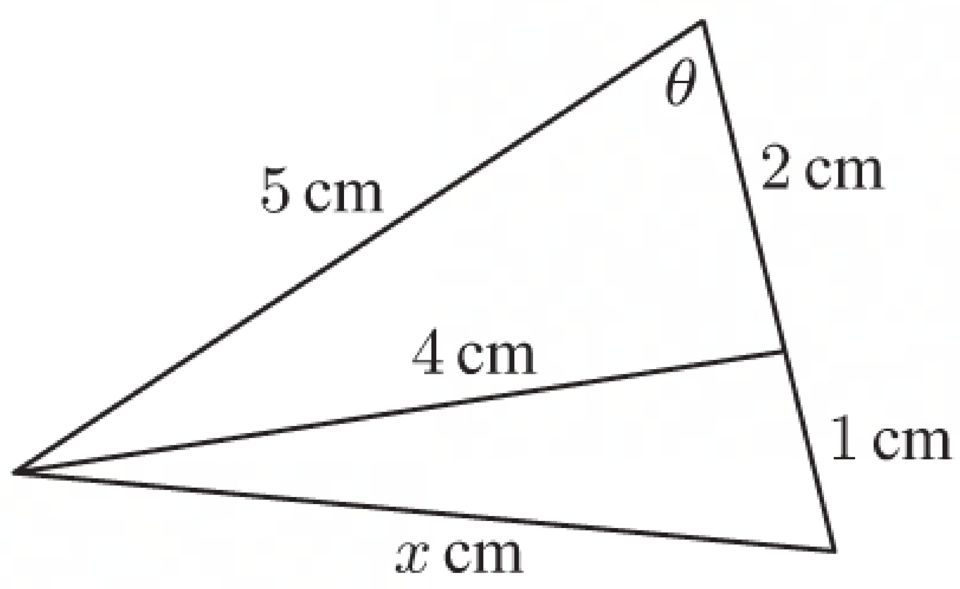
$$\therefore \quad \phi = \cos^{-1}\left(-\tfrac{16}{56}\right)$$

$$\approx 106.6°$$

The largest angle measures about 107°.

6 **a** By the cosine rule: $\cos\theta = \dfrac{2^2+5^2-4^2}{2\times 2\times 5}$

$\therefore \cos\theta = \dfrac{13}{20} = 0.65$



b By the cosine rule:

$x^2 = 5^2 + 3^2 - 2\times 5\times 3\times\cos\theta$

$\therefore\ x = \sqrt{5^2+3^2-2\times 5\times 3\times 0.65}$ {as $x > 0$}

$\therefore\ x \approx 3.81$

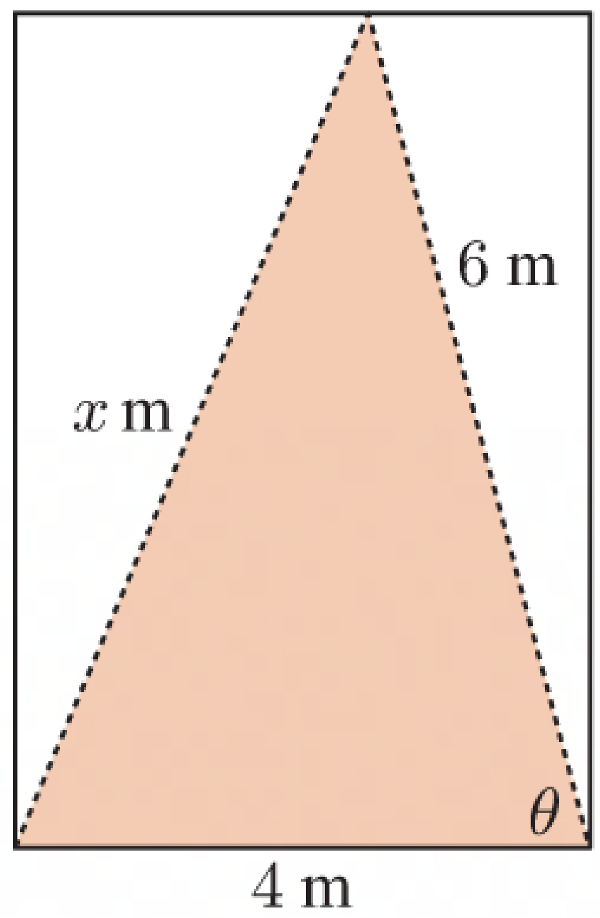
7 **a** Area $= 11.6\ \text{m}^2$

$\therefore\ \frac{1}{2}\times 6\times 4\times\sin\theta = 11.6$

$\therefore\ \sin\theta = \dfrac{11.6}{12}$

$\therefore\ \theta = \sin^{-1}\left(\frac{11.6}{12}\right)$

$\therefore\ \theta \approx 75.2^\circ$



b Let the third side have length x m.

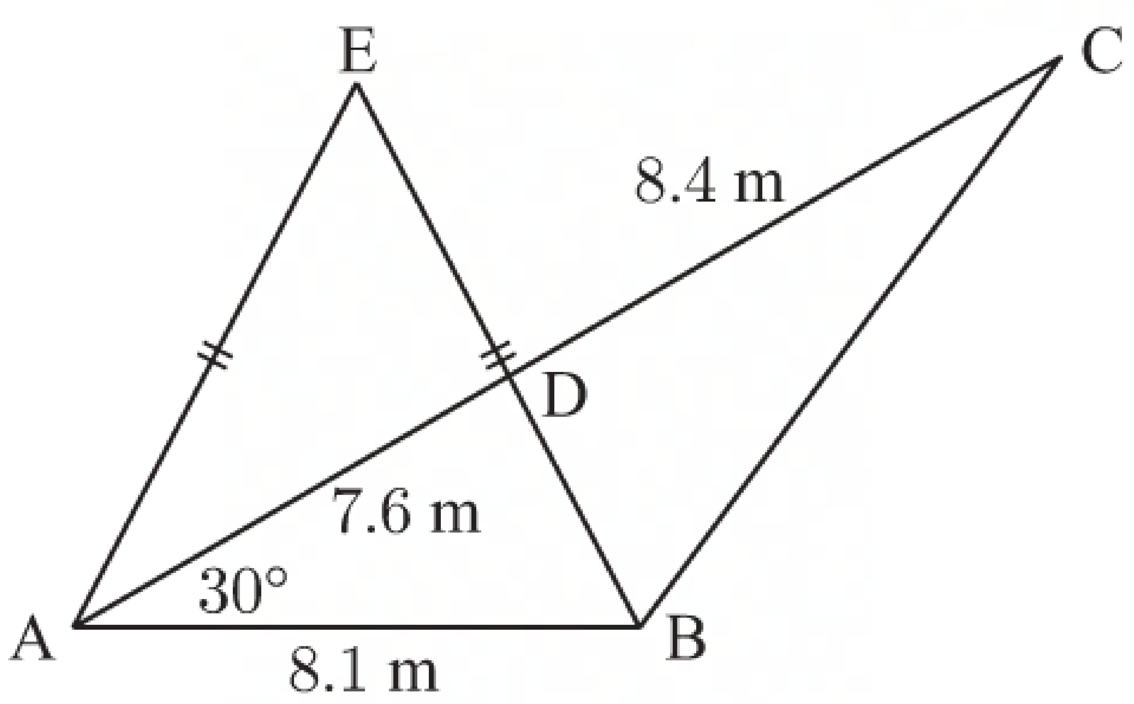
By the cosine rule: $x^2 = 6^2+4^2-2\times 6\times 4\times\cos\theta$

$\therefore\ x \approx \sqrt{6^2+4^2-2\times 6\times 4\times\cos 75.2^\circ}$

$\therefore\ x \approx 6.30$

The third side is about 6.30 m in length.

8



a In $\triangle$ABD, by the cosine rule:

$\text{DB}^2 = 7.6^2 + 8.1^2 - 2\times 7.6\times 8.1\times\cos 30^\circ$

$\therefore\ \text{DB} = \sqrt{7.6^2+8.1^2-2\times 7.6\times 8.1\times\cos 30^\circ}$ {as DB > 0}

$\therefore\ \text{DB} \approx 4.09$ m

Now AC = AD + DC $= 7.6 + 8.4 = 16$ m

In $\triangle$ABC, by the cosine rule:

$\text{BC}^2 = 8.1^2 + 16^2 - 2\times 8.1\times 16\times\cos 30^\circ$

$\therefore\ \text{BC} = \sqrt{8.1^2+16^2-2\times 8.1\times 16\times\cos 30^\circ}$ {as BC > 0}

$\therefore\ \text{BC} \approx 9.86$ m

b In $\triangle$ABD, $\cos \text{A}\widehat{\text{B}}\text{D} = \dfrac{8.1^2+\text{DB}^2-7.6^2}{2\times 8.1\times \text{DB}}$

$\therefore\ \text{A}\widehat{\text{B}}\text{D} \approx \cos^{-1}\left(\dfrac{8.1^2+4.09^2-7.6^2}{2\times 8.1\times 4.09}\right)$

$\therefore\ \text{A}\widehat{\text{B}}\text{D} \approx 68.2^\circ$

$\therefore\ \text{A}\widehat{\text{B}}\text{E} \approx 68.2^\circ$ {base angles in an isosceles triangle}

In $\triangle$DBC, $\cos D\widehat{B}C = \dfrac{DB^2 + BC^2 - 8.4^2}{2 \times DB \times BC}$

$\therefore\ D\widehat{B}C \approx \cos^{-1}\left(\dfrac{4.09^2 + 9.86^2 - 8.4^2}{2 \times 4.09 \times 9.86}\right)$

$\therefore\ D\widehat{B}C \approx 57.5°$

c Area of $\triangle$BCD $= \frac{1}{2} \times DB \times BC \times \sin D\widehat{B}C$

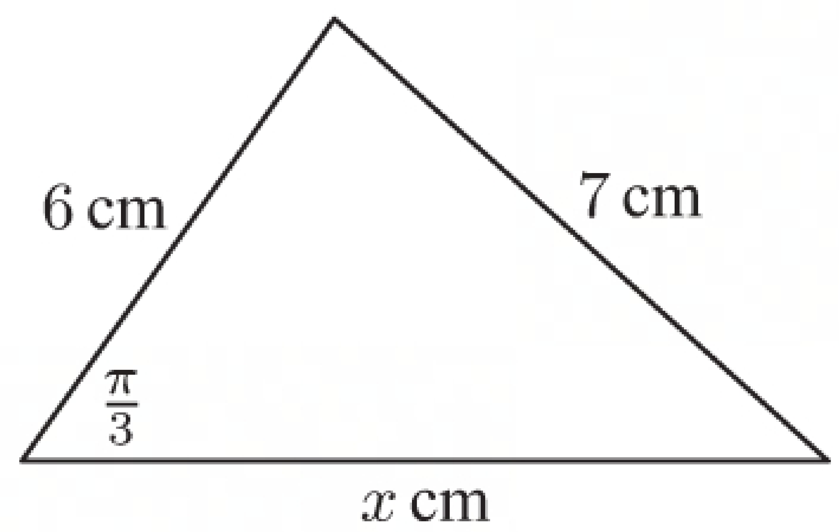
$\approx \frac{1}{2} \times 4.09 \times 9.86 \times \sin 57.5°$

$\approx 17.0\ m^2$

9 **a** By the cosine rule: $7^2 = x^2 + 6^2 - 2 \times x \times 6 \times \cos\frac{\pi}{3}$

$\therefore\ 49 = x^2 + 36 - 12x \times (\frac{1}{2})$

$\therefore\ x^2 - 6x - 13 = 0$



b $x = \dfrac{6 \pm \sqrt{6^2 - 4(1)(-13)}}{2}$

$= \dfrac{6 \pm \sqrt{88}}{2}$

$= 3 \pm \sqrt{22}$

But $x > 0$, so $x = 3 + \sqrt{22}$

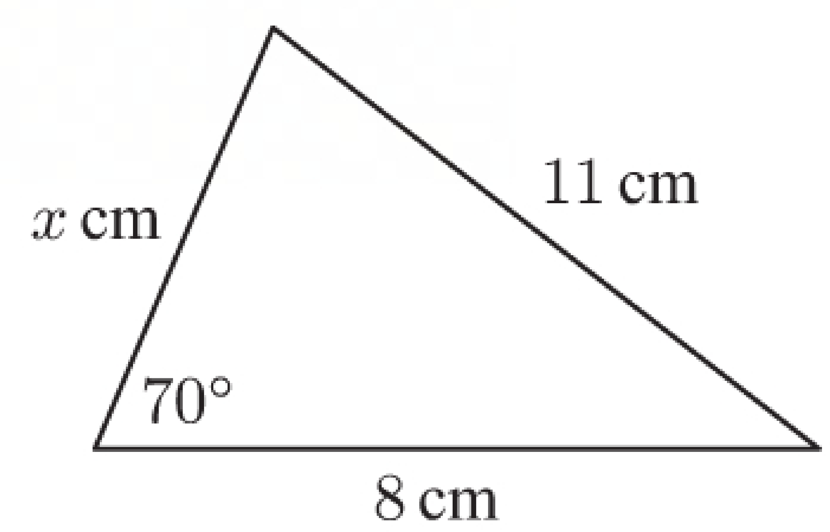
10 **a** By the cosine rule: $11^2 = x^2 + 8^2 - 2 \times x \times 8 \times \cos 70°$

$\therefore\ 121 = x^2 + 64 - 16x \times \cos 70°$

$\therefore\ x^2 - (16\cos 70°)x - 57 = 0$

Using technology, $x \approx -5.29$ or 10.8.

But $x > 0$, so $x \approx 10.8$.



b By the cosine rule: $5^2 = 3^2 + x^2 - 2 \times 3 \times x \times \cos\frac{2\pi}{3}$

$\therefore\ 25 = 9 + x^2 - 6x \times (-\frac{1}{2})$

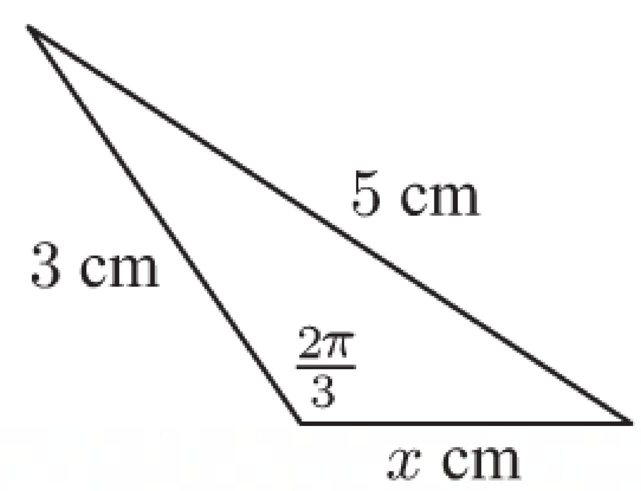
$\therefore\ x^2 + 3x - 16 = 0$

$x = \dfrac{-3 \pm \sqrt{3^2 - 4(1)(-16)}}{2}$

$= \dfrac{-3 \pm \sqrt{73}}{2}$

$= -\frac{3}{2} \pm \frac{\sqrt{73}}{2}$

But $x > 0$, so $x = -\frac{3}{2} + \frac{\sqrt{73}}{2} \approx 2.77$



c By the cosine rule:

$5^2 = x^2 + (2x)^2 - 2 \times x \times 2x \times \cos 60°$

$\therefore\ 25 = x^2 + 4x^2 - 4x^2 \times (\frac{1}{2})$

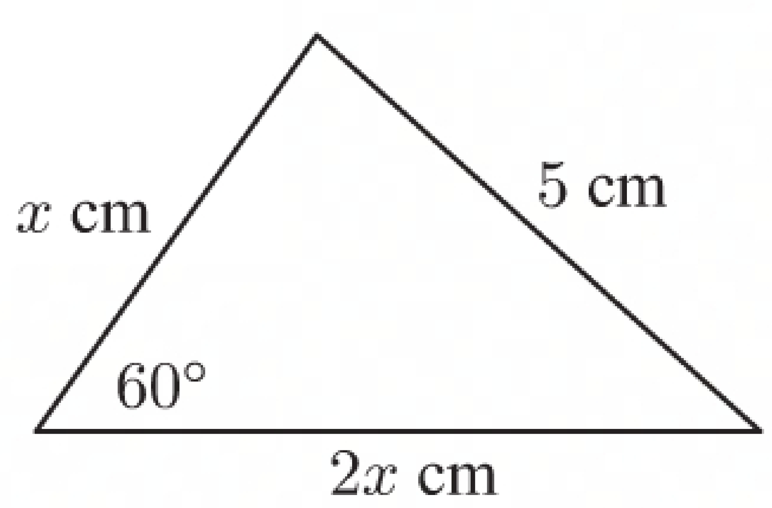
$= 5x^2 - 2x^2$

$= 3x^2$

$\therefore\ x^2 = \frac{25}{3}$

$\therefore\ x = \sqrt{\frac{25}{3}}$ {as $x > 0$}

$\therefore\ x \approx 2.89$

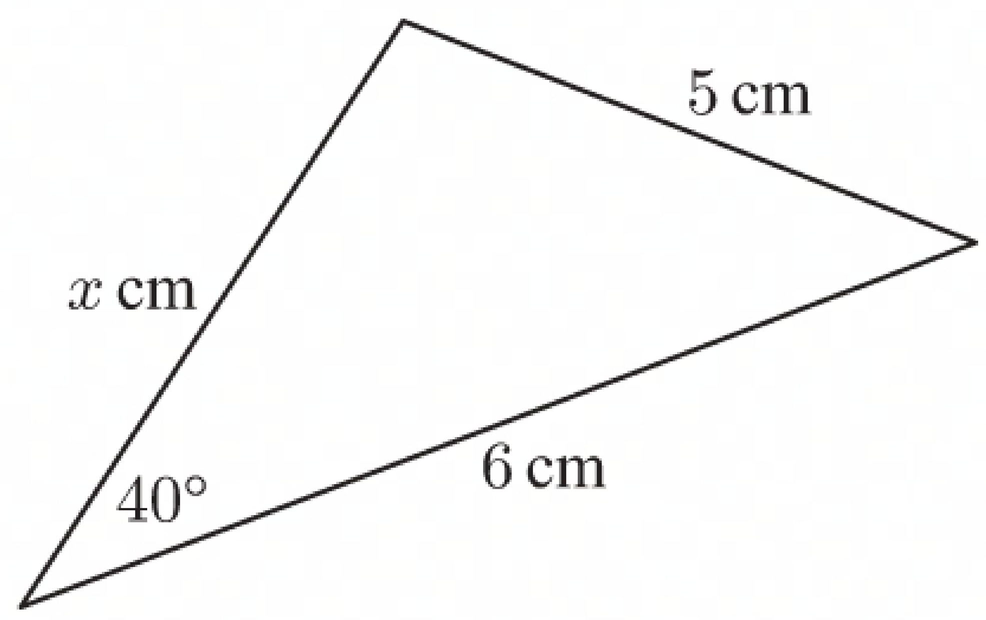


11 By the cosine rule: $5^2 = x^2 + 6^2 - 2 \times x \times 6 \times \cos 40°$

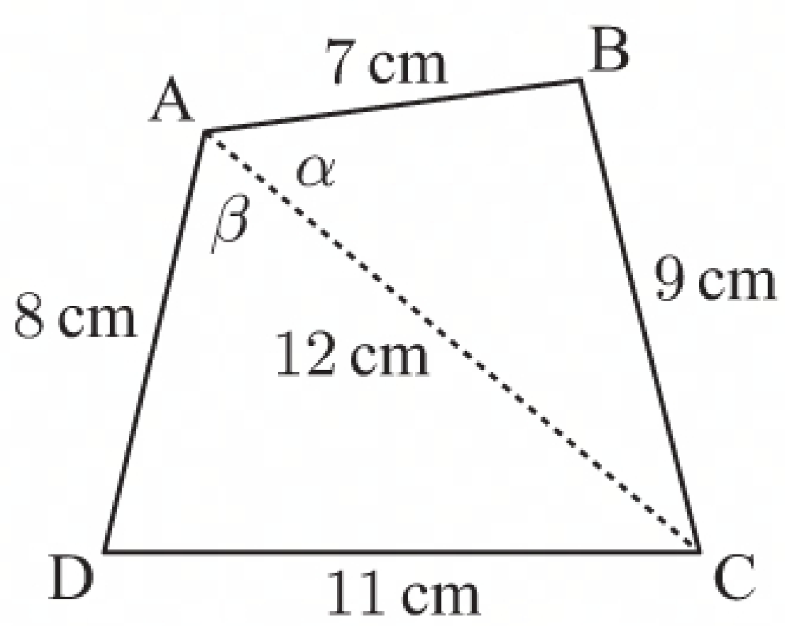
$\therefore\ 25 = x^2 + 36 - 12x \cos 40°$

$\therefore\ x^2 - (12 \cos 40°)x + 11 = 0$

Using technology, $x \approx 1.41$ or 7.78.



12



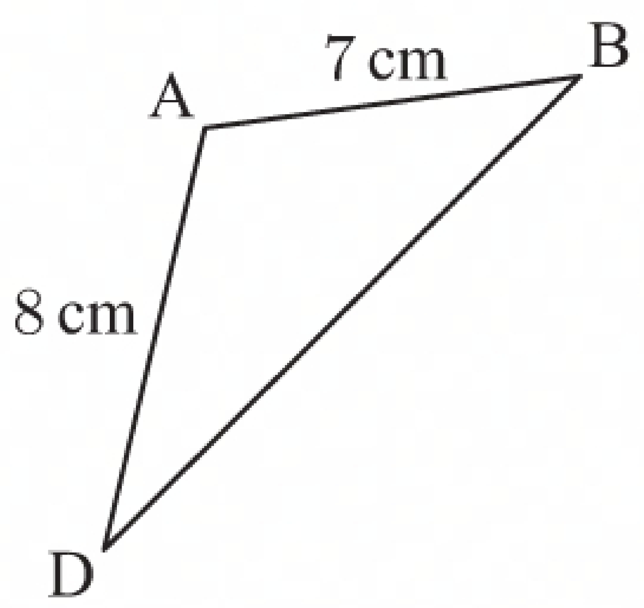
Let $\widehat{CAB}$ be α and $\widehat{DAC}$ be β.

In $\triangle$ABC, by the cosine rule: $\cos\alpha = \dfrac{7^2 + 12^2 - 9^2}{2 \times 7 \times 12}$

$\therefore\ \alpha = \cos^{-1}\left(\frac{112}{168}\right)$

In $\triangle$DAC, by the cosine rule: $\cos\beta = \dfrac{8^2 + 12^2 - 11^2}{2 \times 8 \times 12}$

$\therefore\ \beta = \cos^{-1}\left(\frac{87}{192}\right)$



Now in $\triangle$DAB, $\widehat{DAB} = \alpha + \beta$

$= \cos^{-1}\left(\frac{112}{168}\right) + \cos^{-1}\left(\frac{87}{192}\right)$

$\approx 111.2°$

By the cosine rule: $BD^2 \approx 8^2 + 7^2 - 2 \times 8 \times 7 \times \cos 111.2°$

$\therefore\ BD \approx \sqrt{8^2 + 7^2 - 2 \times 8 \times 7 \times \cos 111.2°}$

$\therefore\ BD \approx 12.4$

[BD] is about 12.4 cm long.

13 In $\triangle$BCD, by the cosine rule:

$BC^2 = 5^2 + 6^2 - 2 \times 5 \times 6 \times \cos 130°$

$\therefore\ BC = \sqrt{5^2 + 6^2 - 2 \times 5 \times 6 \times \cos 130°}$ {as $BC > 0$}

$\therefore\ BC \approx 9.98$

In $\triangle$ABC, by the cosine rule:

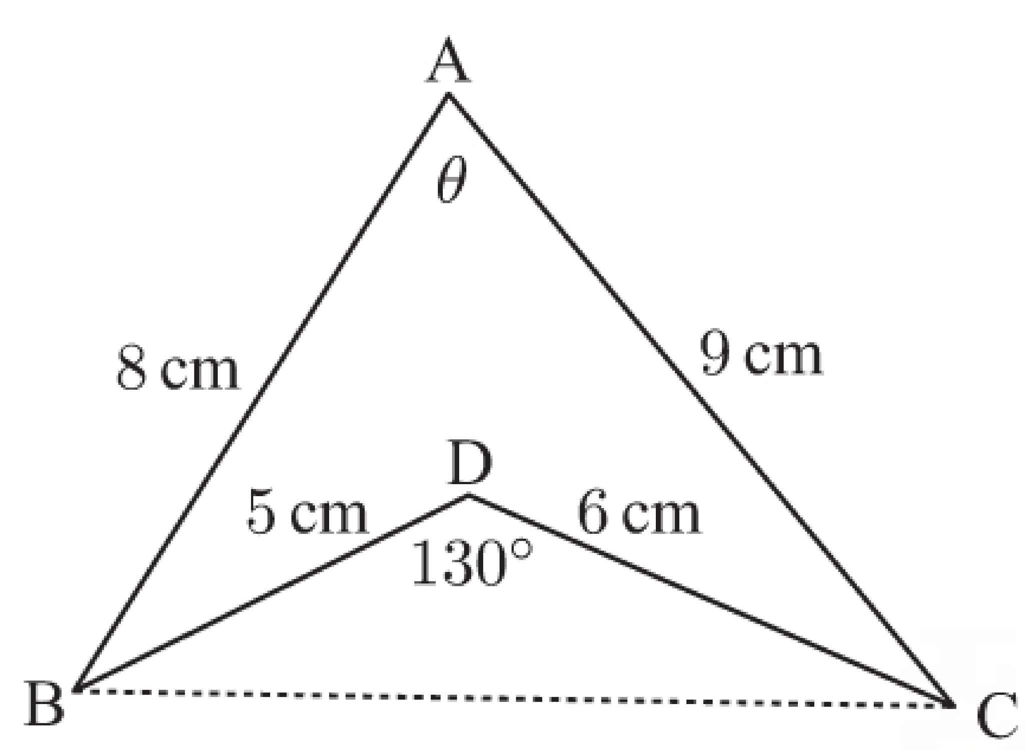
$BC^2 = 8^2 + 9^2 - 2 \times 8 \times 9 \times \cos\theta$

$\therefore\ 9.98^2 \approx 64 + 81 - 144 \cos\theta$

$\therefore\ \cos\theta \approx \dfrac{145 - 9.98^2}{144}$

$\therefore\ \theta \approx \cos^{-1}\left(\dfrac{145 - 9.98^2}{144}\right)$

$\therefore\ \theta \approx 71.6°$



14 Let the distance from P to C be x cm.

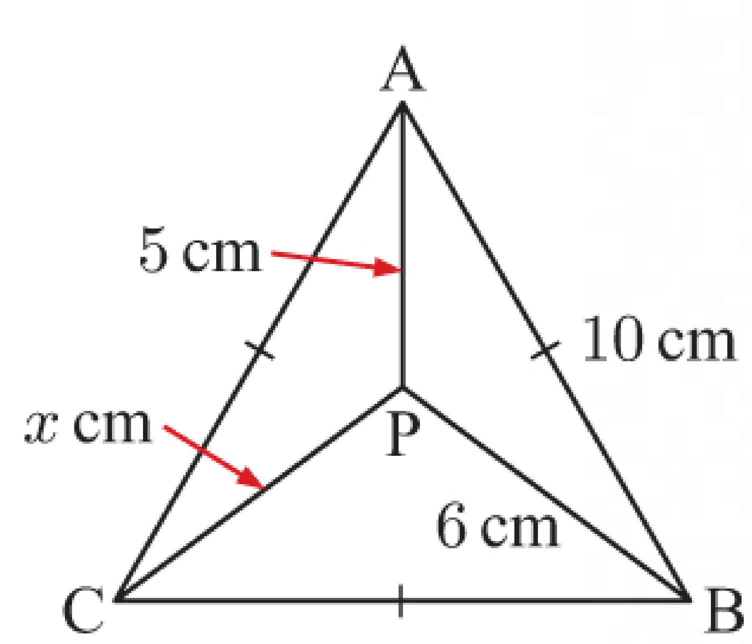
In $\triangle$ABP, $\cos\widehat{PAB} = \dfrac{5^2 + 10^2 - 6^2}{2 \times 5 \times 10}$

$= \frac{89}{100}$

$\therefore\ \widehat{PAB} = \cos^{-1}\left(\frac{89}{100}\right)$

$\therefore\ \widehat{PAC} = 60° - \cos^{-1}\left(\frac{89}{100}\right)$ {since $\triangle$ABC is equilateral}

$\therefore\ \widehat{PAC} \approx 32.87°$



Now, in $\triangle$APC, by the cosine rule:

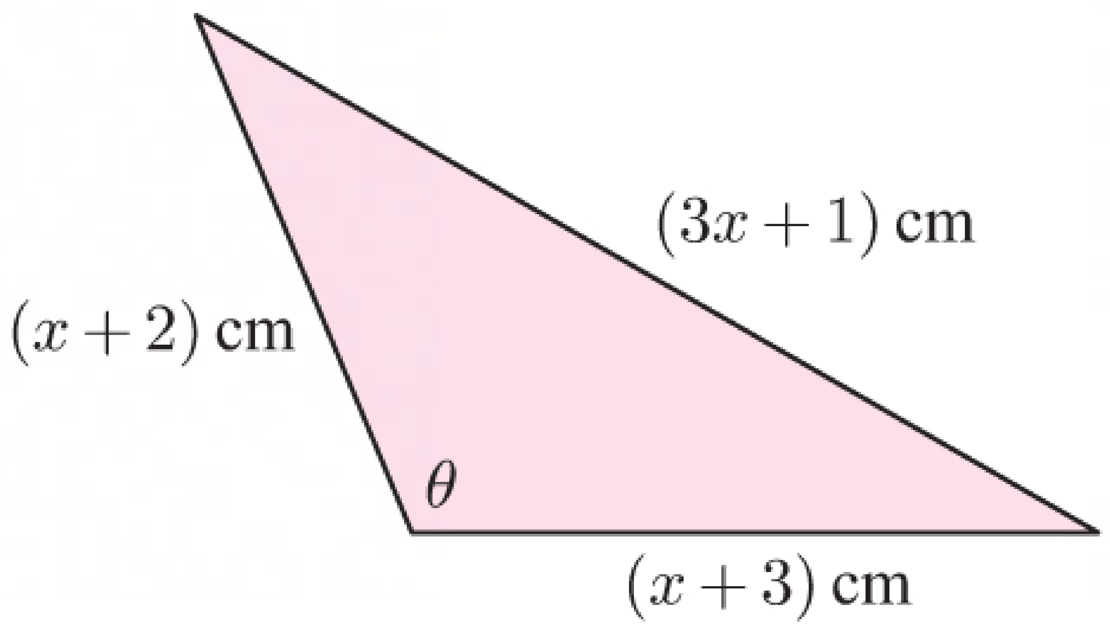
$$x^2 = 10^2 + 5^2 - 2 \times 10 \times 5 \times \cos \mathrm{P\widehat{A}C}$$

$$\therefore \quad x \approx \sqrt{10^2 + 5^2 - 2 \times 10 \times 5 \times \cos 32.87^\circ} \qquad \{\text{as } \ x > 0\}$$

$$\therefore \quad x \approx 6.40$$

So, P is about 6.40 cm from C.

15 a



By the cosine rule: $\quad (3x+1)^2 = (x+2)^2 + (x+3)^2 - 2(x+2)(x+3)\cos\theta$

$$\therefore \quad 9x^2 + 6x + 1 = x^2 + 4x + 4 + x^2 + 6x + 9 - 2(x^2 + 5x + 6)(-\tfrac{1}{5})$$

$$\therefore \quad 9x^2 + 6x + 1 = 2x^2 + 10x + 13 + \tfrac{2}{5}(x^2 + 5x + 6)$$

$$\therefore \quad 7x^2 - 4x - 12 = \tfrac{2}{5}x^2 + 2x + \tfrac{12}{5}$$

$$\therefore \quad \tfrac{33}{5}x^2 - 6x - \tfrac{72}{5} = 0$$

$$\therefore \quad 33x^2 - 30x - 72 = 0$$

$$\therefore \quad 3(11x^2 - 10x - 24) = 0$$

$$\therefore \quad 3(11x + 12)(x - 2) = 0$$

$$\therefore \quad (11x + 12)(x - 2) = 0$$

$$\therefore \quad x = -\tfrac{12}{11} \ \text{ or } \ 2$$

But $3x + 1 > 0 \quad \therefore \ x = 2$

b

$$\sin^2\theta + \cos^2\theta = 1$$

$$\therefore \quad \sin^2\theta + (-\tfrac{1}{5})^2 = 1$$

$$\therefore \quad \sin^2\theta + \tfrac{1}{25} = 1$$

$$\therefore \quad \sin^2\theta = \tfrac{24}{25}$$

$$\therefore \quad \sin\theta = \pm\tfrac{\sqrt{24}}{5}$$

But $0^\circ < \theta < 180^\circ$, so $\sin\theta > 0$

$$\therefore \quad \sin\theta = \tfrac{\sqrt{24}}{5}$$

Since $x = 2$, the sides have length $2 + 2 = 4$ cm, $2 + 3 = 5$ cm, and $3(2) + 1 = 7$ cm.

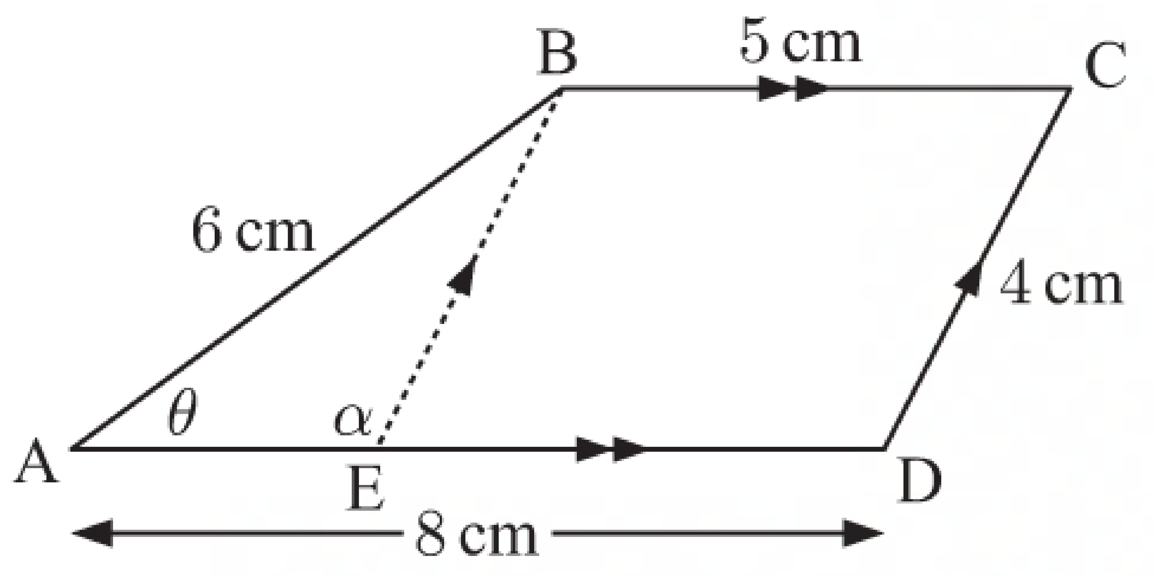
Area of the triangle $= \tfrac{1}{2} \times 4 \times 5 \times \sin\theta$

$$= 10 \times \tfrac{\sqrt{24}}{5}$$

$$= 2\sqrt{24}$$

$$= 4\sqrt{6} \text{ cm}^2$$

16



We draw [BE] to complete parallelogram BCDE.

$\therefore$ ED = BC = 5 cm and BE = CD = 4 cm {opposite sides of parallelogram}

$\therefore$ AE = 8 − ED
= 8 − 5 cm
= 3 cm

Using the cosine rule in $\triangle$ABE:

$\cos\theta = \dfrac{3^2+6^2-4^2}{2\times3\times6}$ and $\cos\alpha = \dfrac{3^2+4^2-6^2}{2\times3\times4}$

$\therefore\ \theta = \cos^{-1}\left(\dfrac{3^2+6^2-4^2}{2\times3\times6}\right)$ $\qquad\therefore\ \alpha = \cos^{-1}\left(\dfrac{3^2+4^2-6^2}{2\times3\times4}\right)$

$\therefore\ \theta = \cos^{-1}\left(\frac{29}{36}\right)$ $\qquad\therefore\ \alpha = \cos^{-1}\left(-\frac{11}{24}\right)$

$\therefore\ \theta \approx 36.3^\circ$ $\qquad\therefore\ \alpha \approx 117.3^\circ$

Now, $\mathrm{A\widehat{B}C} = 180^\circ - \theta$ {co-interior angles}
$\approx 143.7^\circ$

$\mathrm{B\widehat{E}D} = 180^\circ - \alpha$ {angles in a line}
$\approx 62.7^\circ$

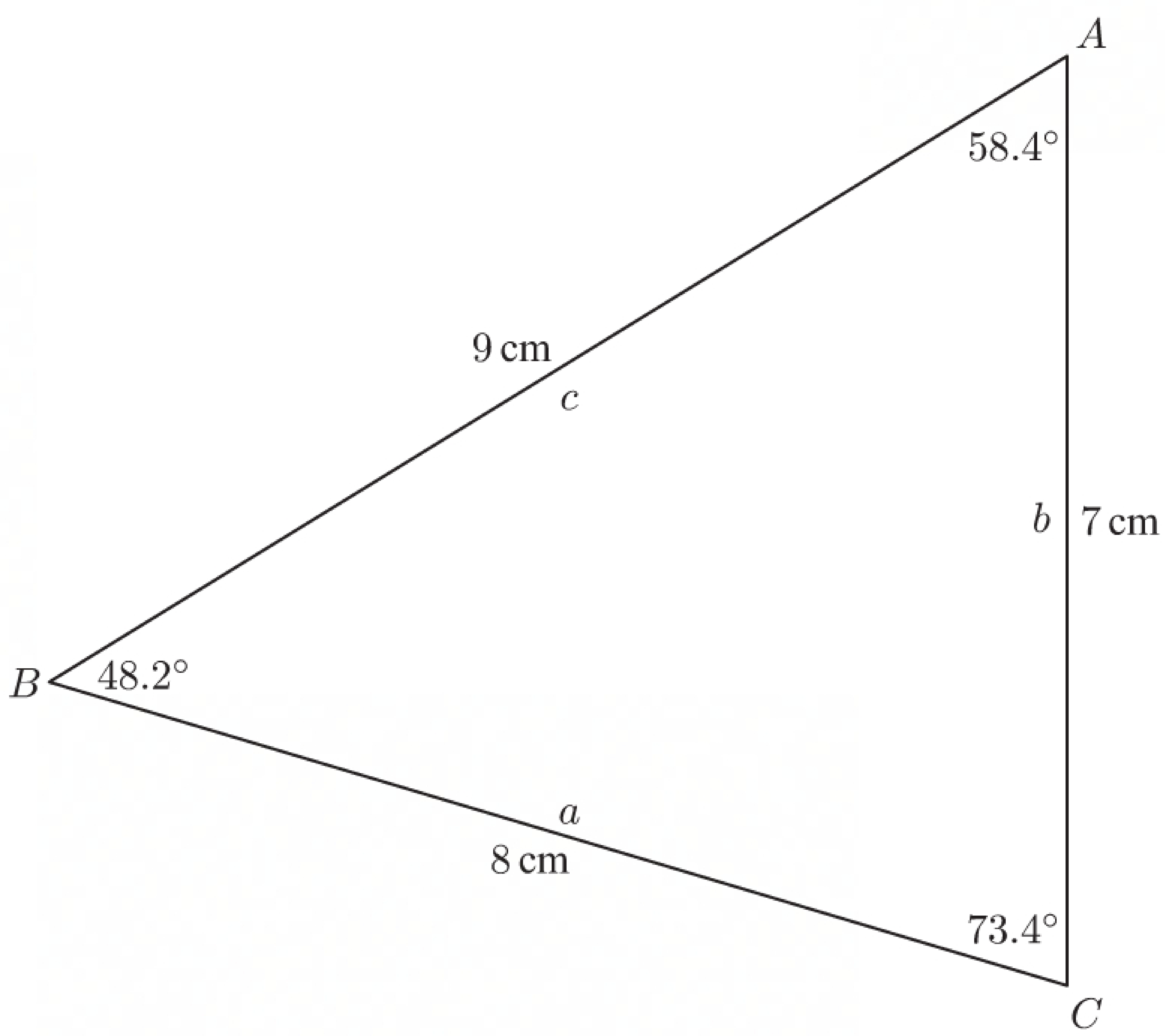
$\mathrm{B\widehat{C}D} = \mathrm{B\widehat{E}D}$ {opposite angles in parallelogram}
$\approx 62.7^\circ$

$\mathrm{C\widehat{D}E} = 180^\circ - \mathrm{B\widehat{E}D}$ {co-interior angles}
$\approx 117.3^\circ$

So, to the nearest degree, the angles of the trapezium are 36°, 63°, 117°, and 144°.

INVESTIGATION 1 THE SINE RULE

1, 2, 3



4

a	b	c	A	B	C	$\dfrac{\sin A}{a}$	$\dfrac{\sin B}{b}$	$\dfrac{\sin C}{c}$
8 cm	7 cm	9 cm	58.4°	48.2°	73.4°	0.1065	0.1065	0.1065

5 We notice that $\dfrac{\sin A}{a} = \dfrac{\sin B}{b} = \dfrac{\sin C}{c}$ or equivalently, $\dfrac{a}{\sin A} = \dfrac{b}{\sin B} = \dfrac{c}{\sin C}$.

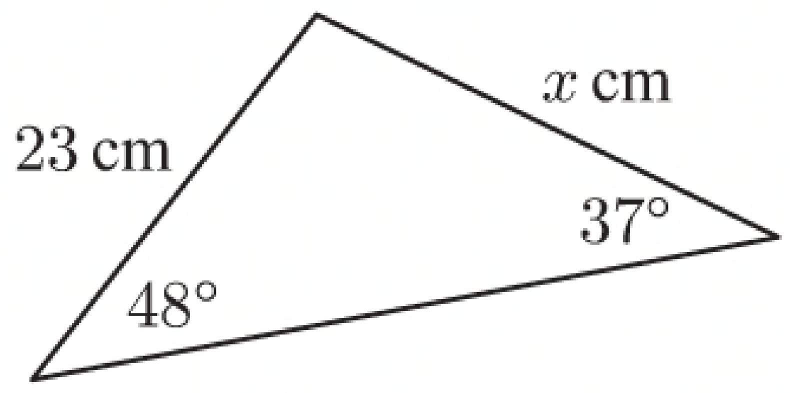
EXERCISE 9C.1

1 **a** Using the sine rule,

$$\frac{x}{\sin 48^\circ} = \frac{23}{\sin 37^\circ}$$

$$\therefore\ x = \frac{23 \times \sin 48^\circ}{\sin 37^\circ}$$

$$\therefore\ x \approx 28.4$$

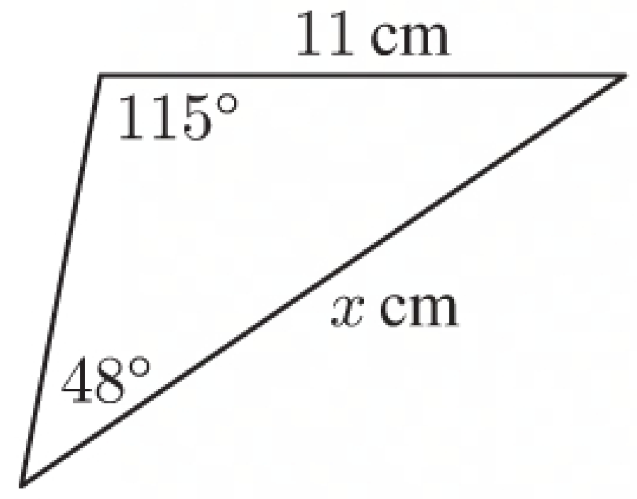


b Using the sine rule,

$$\frac{x}{\sin 115^\circ} = \frac{11}{\sin 48^\circ}$$

$$\therefore\ x = \frac{11 \times \sin 115^\circ}{\sin 48^\circ}$$

$$\therefore\ x \approx 13.4$$

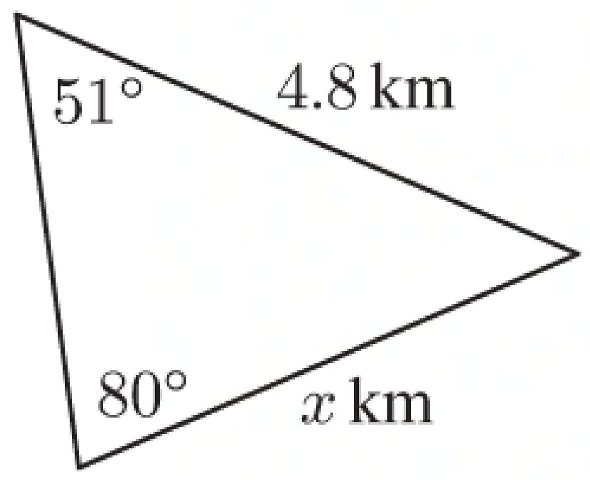


c Using the sine rule,

$$\frac{x}{\sin 51^\circ} = \frac{4.8}{\sin 80^\circ}$$

$$\therefore\ x = \frac{4.8 \times \sin 51^\circ}{\sin 80^\circ}$$

$$\therefore\ x \approx 3.79$$



d The unknown angle is $180^\circ - 100^\circ - 45^\circ$ {angles in a triangle}

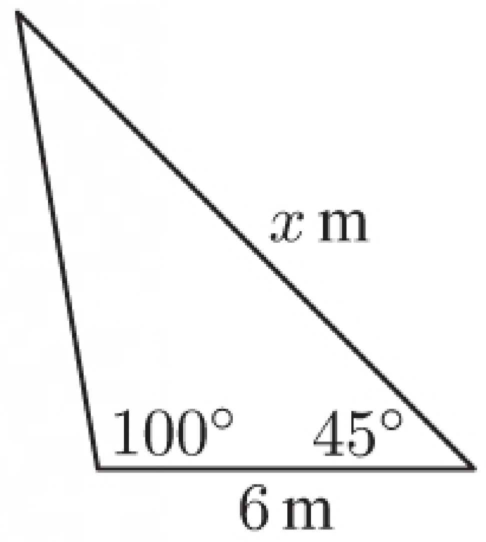
$= 35^\circ$

Using the sine rule,

$$\frac{x}{\sin 100^\circ} = \frac{6}{\sin 35^\circ}$$

$$\therefore\ x = \frac{6 \times \sin 100^\circ}{\sin 35^\circ}$$

$$\therefore\ x \approx 10.3$$



e The unknown angle is $180^\circ - 58^\circ - 55^\circ$ {angles in a triangle}

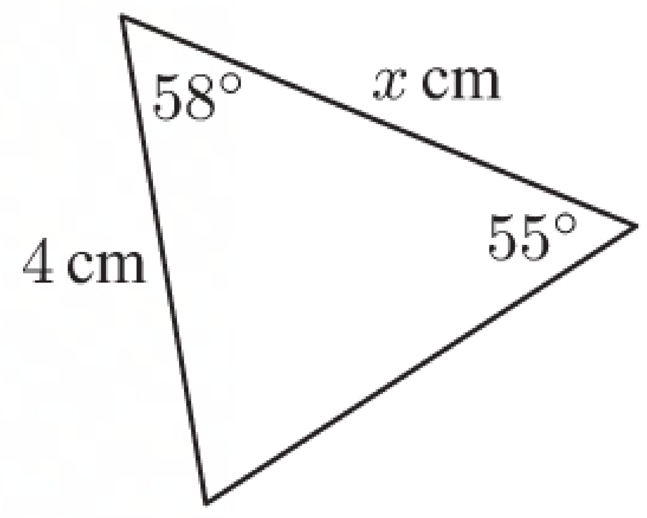
$= 67^\circ$

Using the sine rule,

$$\frac{x}{\sin 67^\circ} = \frac{4}{\sin 55^\circ}$$

$$\therefore\ x = \frac{4 \times \sin 67^\circ}{\sin 55^\circ}$$

$$\therefore\ x \approx 4.49$$



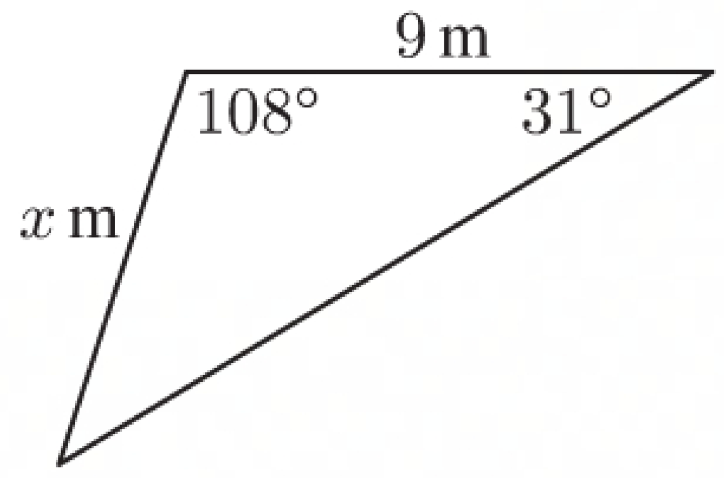
f The unknown angle is $180° - 108° - 31°$ {angles in a triangle}
$= 41°$

Using the sine rule,

$$\frac{x}{\sin 31°} = \frac{9}{\sin 41°}$$

$$\therefore \quad x = \frac{9 \times \sin 31°}{\sin 41°}$$

$$\therefore \quad x \approx 7.07$$

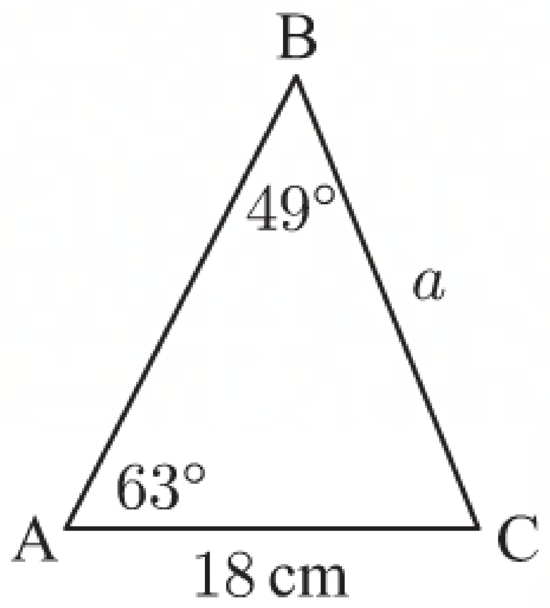


2 **a** Using the sine rule,

$$\frac{a}{\sin 63°} = \frac{18}{\sin 49°}$$

$$\therefore \quad a = \frac{18 \times \sin 63°}{\sin 49°}$$

$$\therefore \quad a \approx 21.3 \text{ cm}$$



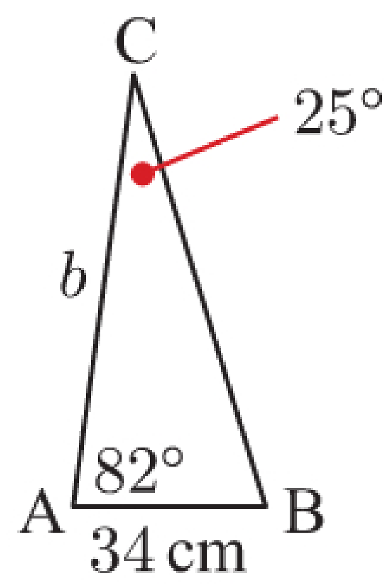
b The unknown angle is $180° - 25° - 82°$ {angles in a triangle}
$= 73°$

Using the sine rule,

$$\frac{b}{\sin 73°} = \frac{34}{\sin 25°}$$

$$\therefore \quad b = \frac{34 \times \sin 73°}{\sin 25°}$$

$$\therefore \quad b \approx 76.9 \text{ cm}$$



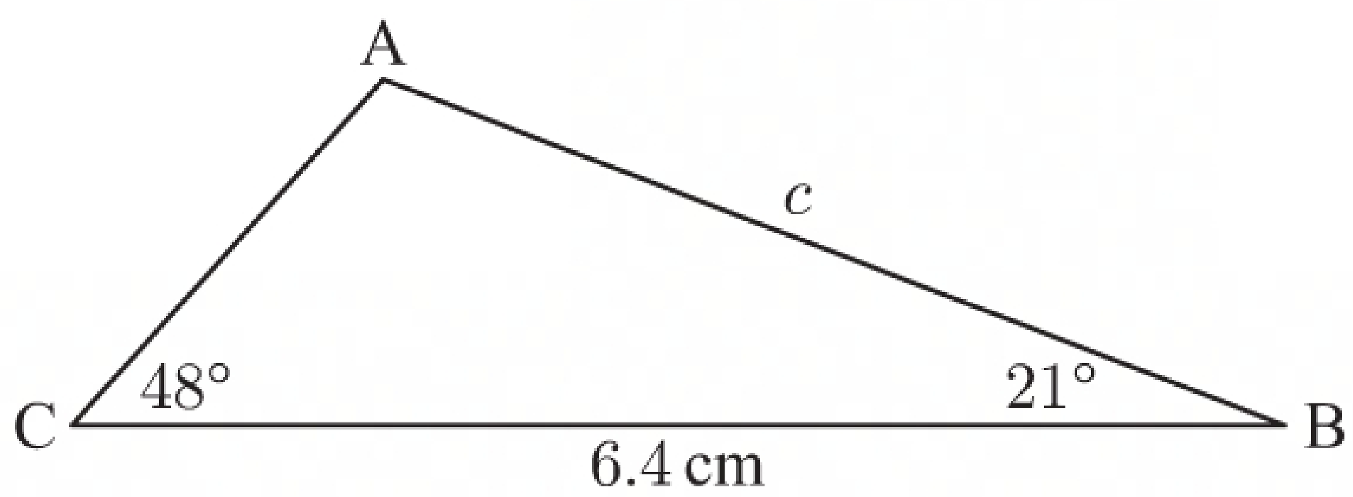
c The unknown angle is $180° - 48° - 21°$ {angles in a triangle}
$= 111°$

Using the sine rule,

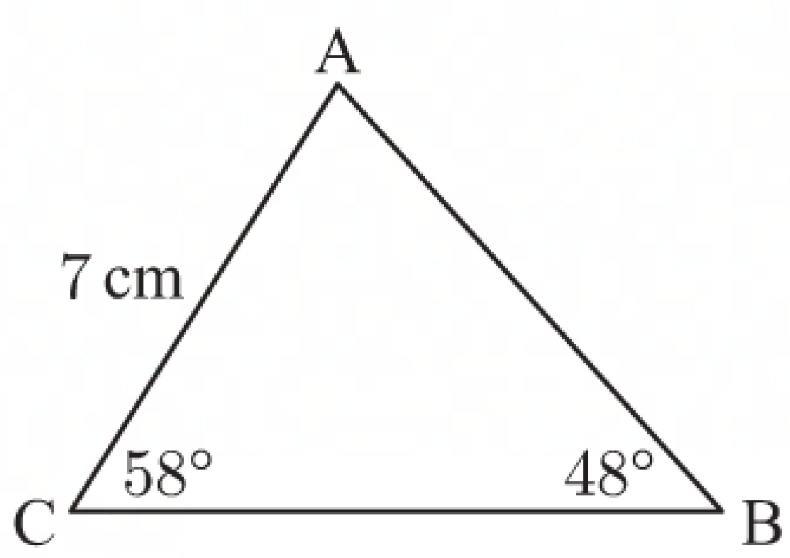
$$\frac{c}{\sin 48°} = \frac{6.4}{\sin 111°}$$

$$\therefore \quad c = \frac{6.4 \times \sin 48°}{\sin 111°}$$

$$\therefore \quad c \approx 5.09 \text{ cm}$$



3 **a** $B\widehat{A}C = 180° - 58° - 48°$ {angles in a triangle}
$= 74°$



Using the sine rule,

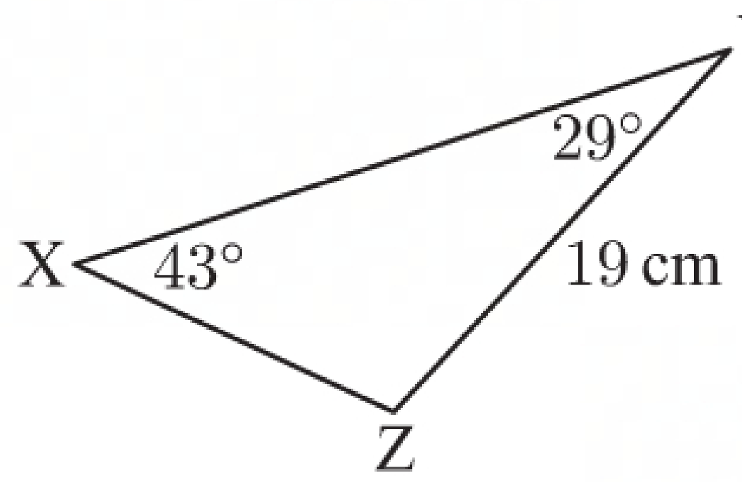
$$\frac{AB}{\sin 58°} = \frac{7}{\sin 48°} \quad \text{and} \quad \frac{BC}{\sin 74°} = \frac{7}{\sin 48°}$$

$$\therefore \quad AB = \frac{7 \times \sin 58°}{\sin 48°} \qquad \therefore \quad BC = \frac{7 \times \sin 74°}{\sin 48°}$$

$$\therefore \quad AB \approx 7.99 \qquad \therefore \quad BC \approx 9.05$$

So, $B\widehat{A}C = 74°$, $AB \approx 7.99$ cm, and $BC \approx 9.05$ cm.

b $X\widehat{Z}Y = 180^\circ - 43^\circ - 29^\circ$ {angles in a triangle}
$= 108^\circ$



Using the sine rule,

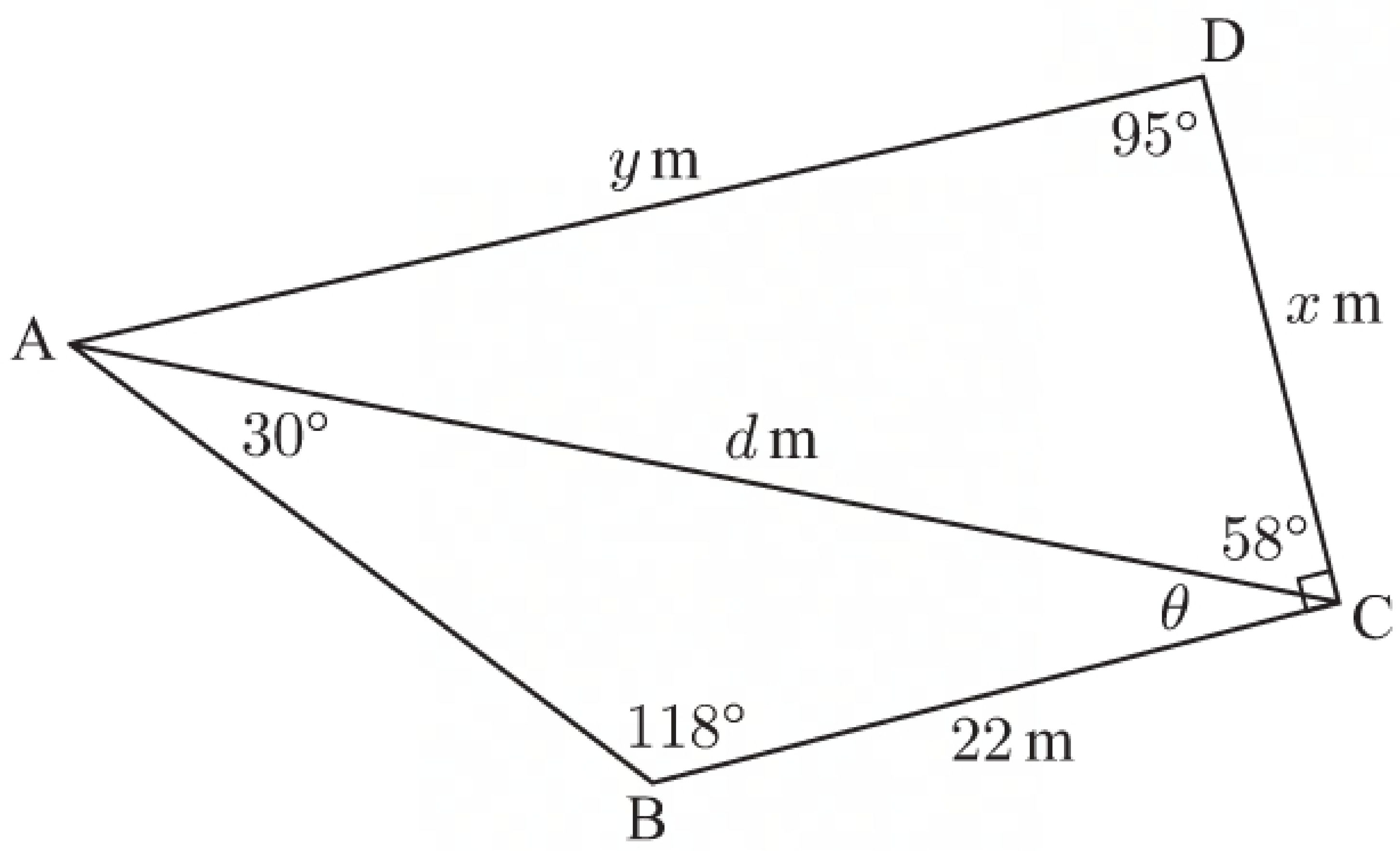
$$\frac{XZ}{\sin 29^\circ} = \frac{19}{\sin 43^\circ} \quad \text{and} \quad \frac{XY}{\sin 108^\circ} = \frac{19}{\sin 43^\circ}$$

$$\therefore \; XZ = \frac{19 \times \sin 29^\circ}{\sin 43^\circ} \qquad \therefore \; XY = \frac{19 \times \sin 108^\circ}{\sin 43^\circ}$$

$$\therefore \; XZ \approx 13.5 \qquad \therefore \; XY \approx 26.5$$

So, $X\widehat{Z}Y = 108^\circ$, $XZ \approx 13.5$ cm, and $XY \approx 26.5$ cm.

4 First we find the length of the diagonal, d m.
In $\triangle ABC$, using the sine rule,



$$\frac{d}{\sin 118^\circ} = \frac{22}{\sin 30^\circ}$$

$$\therefore \; d = \frac{22 \times \sin 118^\circ}{\sin 30^\circ}$$

$$\therefore \; d \approx 38.85$$

Now $\theta = 180^\circ - 30^\circ - 118^\circ = 32^\circ$

$$\therefore \; A\widehat{C}D = 90^\circ - 32^\circ = 58^\circ$$

$$\therefore \; D\widehat{A}C = 180^\circ - 95^\circ - 58^\circ = 27^\circ$$

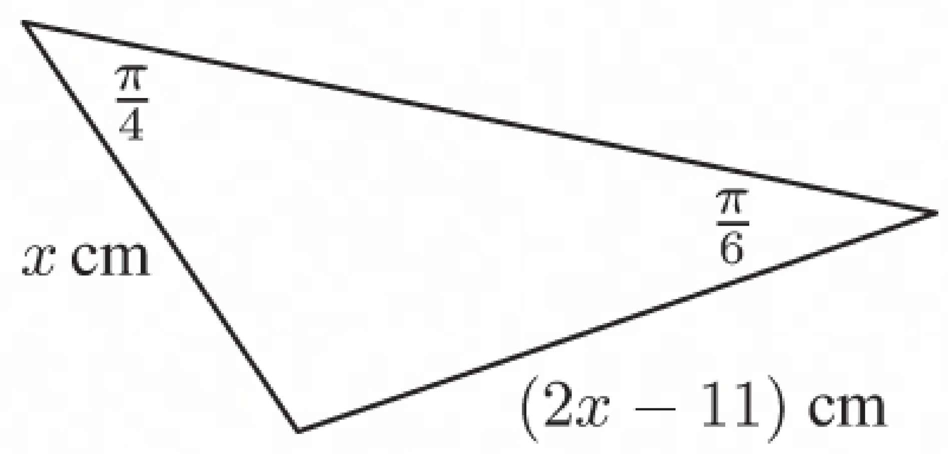
In $\triangle ACD$, using the sine rule,

$$\frac{x}{\sin 27^\circ} \approx \frac{38.85}{\sin 95^\circ} \quad \text{and} \quad \frac{y}{\sin 58^\circ} \approx \frac{38.85}{\sin 95^\circ}$$

$$\therefore \; x \approx \frac{38.85 \times \sin 27^\circ}{\sin 95^\circ} \qquad \therefore \; y \approx \frac{38.85 \times \sin 58^\circ}{\sin 95^\circ}$$

$$\therefore \; x \approx 17.7 \qquad \therefore \; y \approx 33.1$$

5 Using the sine rule,



$$\frac{x}{\sin\frac{\pi}{6}} = \frac{2x-11}{\sin\frac{\pi}{4}}$$

$$\therefore \frac{x}{\frac{1}{2}} = \frac{2x-11}{\frac{1}{\sqrt{2}}}$$

$$\therefore \frac{1}{\sqrt{2}}x = \tfrac{1}{2}(2x-11)$$

$$\therefore x = \tfrac{\sqrt{2}}{2}(2x-11)$$

$$= \sqrt{2}x - \frac{11\sqrt{2}}{2}$$

$$\therefore (1-\sqrt{2})x = -\frac{11\sqrt{2}}{2}$$

$$\therefore x = -\frac{11\sqrt{2}}{2(1-\sqrt{2})}$$

$$= \left(\frac{-11\sqrt{2}}{2-2\sqrt{2}}\right) \times \left(\frac{2+2\sqrt{2}}{2+2\sqrt{2}}\right)$$

$$= \frac{-22\sqrt{2}-22(2)}{2^2-(2\sqrt{2})^2}$$

$$= \frac{-22\sqrt{2}-44}{4-8}$$

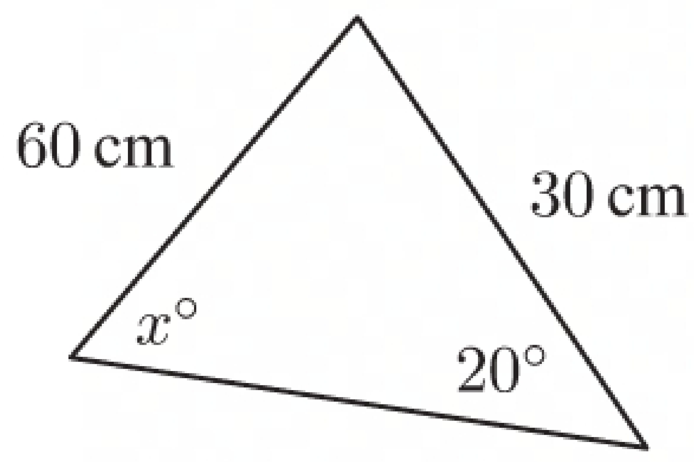
$$= \frac{-22\sqrt{2}-44}{-4}$$

$$= \tfrac{11}{2}\sqrt{2}+11$$

$$\therefore x = 11 + \tfrac{11}{2}\sqrt{2}$$

EXERCISE 9C.2

1 **a**



Using the sine rule,

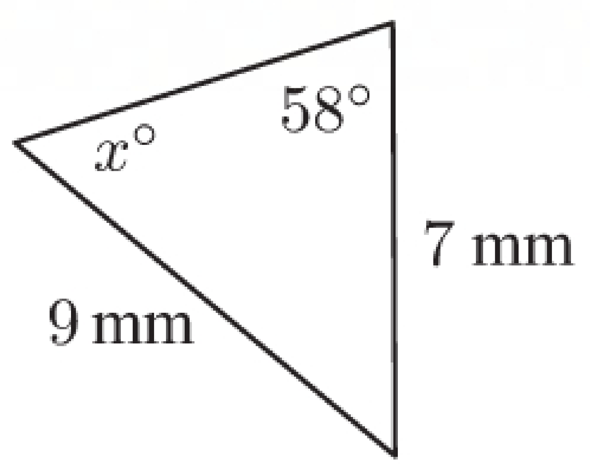
$$\frac{\sin x°}{30} = \frac{\sin 20°}{60}$$

$$\therefore \sin x° = \frac{30 \times \sin 20°}{60}$$

$$\therefore x = \sin^{-1}\left(\frac{30 \times \sin 20°}{60}\right)$$

$$\therefore x \approx 9.85$$

b



Using the sine rule,

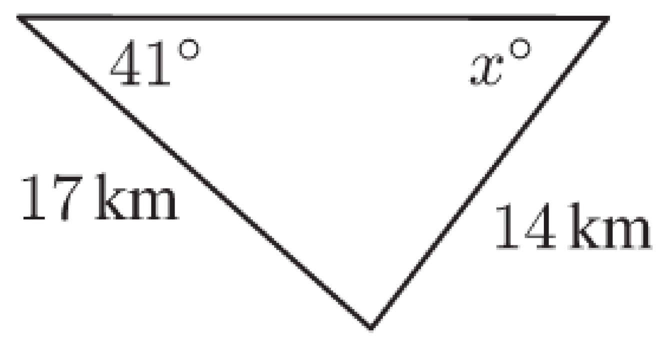
$$\frac{\sin x°}{7} = \frac{\sin 58°}{9}$$

$$\therefore \sin x° = \frac{7 \times \sin 58°}{9}$$

$$\therefore x = \sin^{-1}\left(\frac{7 \times \sin 58°}{9}\right)$$

$$\therefore x \approx 41.3$$

c



Using the sine rule,

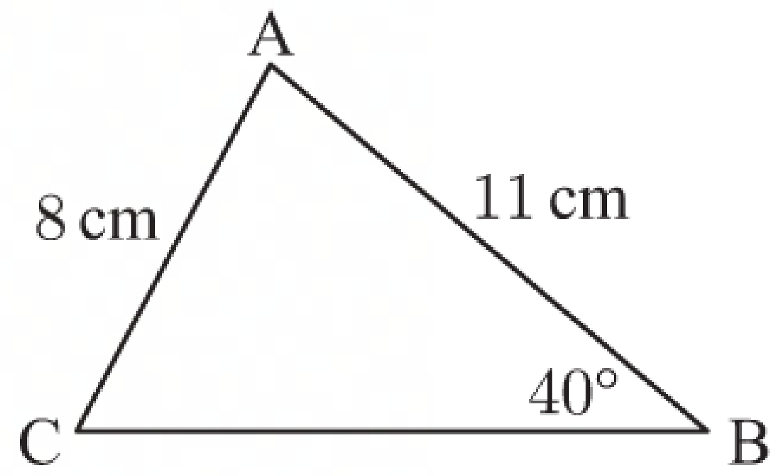
$$\frac{\sin x^\circ}{17} = \frac{\sin 41^\circ}{14}$$

$$\therefore \sin x^\circ = \frac{17 \times \sin 41^\circ}{14}$$

$$\therefore x = \sin^{-1}\left(\frac{17 \times \sin 41^\circ}{14}\right)$$

$$\therefore x \approx 52.8$$

2 Using the sine rule,



$$\frac{\sin C}{11} = \frac{\sin 40^\circ}{8}$$

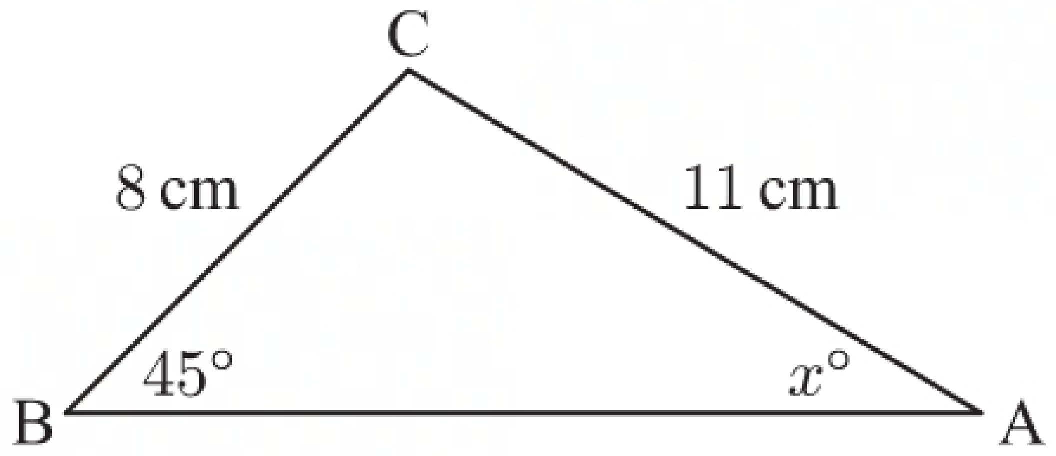
$$\therefore \sin C = \frac{11 \times \sin 40^\circ}{8}$$

$$\therefore C = \sin^{-1}\left(\frac{11 \times \sin 40^\circ}{8}\right) \text{ or its supplement}$$

$$\therefore C \approx 62.1^\circ \text{ or } (180 - 62.1)^\circ$$

$$\therefore C \approx 62.1^\circ \text{ or } 117.9^\circ$$

3 **a**



Let $\widehat{BAC}$ be x°.

Using the sine rule, $\frac{\sin x^\circ}{8} = \frac{\sin 45^\circ}{11}$

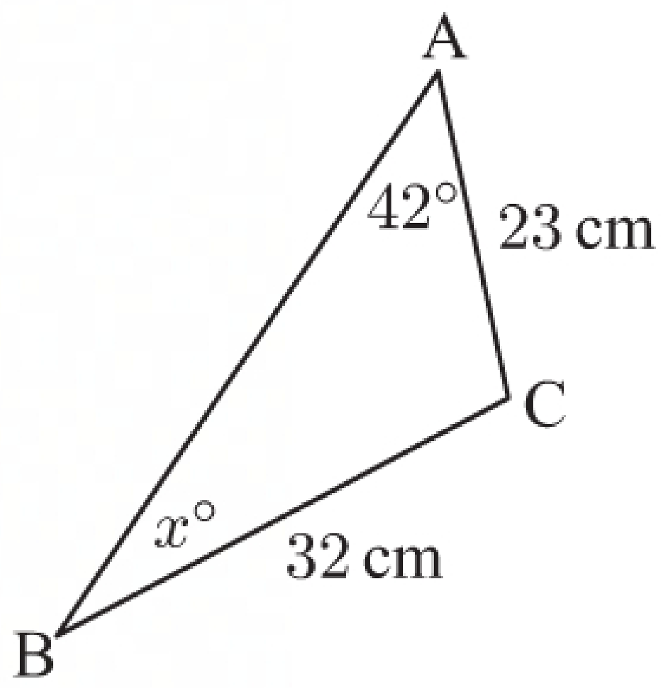
$$\therefore \sin x^\circ = \frac{8 \times \sin 45^\circ}{11}$$

$$\therefore x = \sin^{-1}\left(\frac{8 \times \sin 45^\circ}{11}\right)$$

$$\therefore x \approx 30.9$$

$\therefore$ $\widehat{BAC}$ is approximately 30.9°.

b



Let $\widehat{ABC}$ be x°.

Using the sine rule, $\frac{\sin x^\circ}{23} = \frac{\sin 42^\circ}{32}$

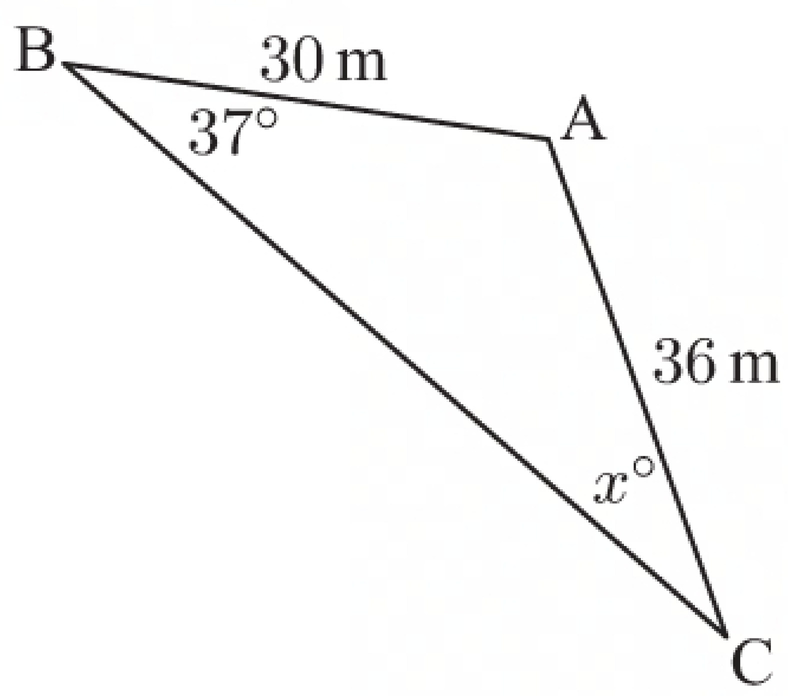
$$\therefore \sin x^\circ = \frac{23 \times \sin 42^\circ}{32}$$

$$\therefore x = \sin^{-1}\left(\frac{23 \times \sin 42^\circ}{32}\right)$$

$$\therefore x \approx 28.7$$

$\therefore$ $\widehat{ABC}$ is approximately 28.7°.

c



Let $\widehat{ACB}$ be x°.

Using the sine rule, $\frac{\sin x^\circ}{30} = \frac{\sin 37^\circ}{36}$

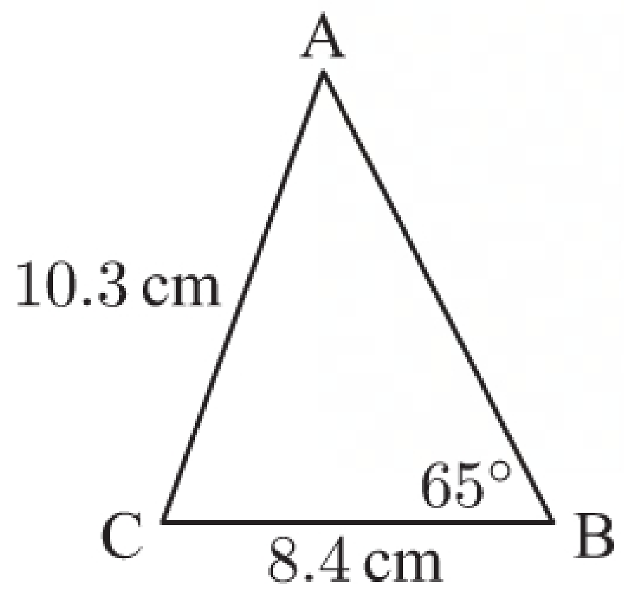
$$\therefore \sin x^\circ = \frac{30 \times \sin 37^\circ}{36}$$

$$\therefore x = \sin^{-1}\left(\frac{30 \times \sin 37^\circ}{36}\right)$$

$$\therefore x \approx 30.1$$

$\therefore$ $\widehat{ACB}$ is approximately 30.1°.

d $\dfrac{\sin \widehat{BAC}}{a} = \dfrac{\sin \widehat{ABC}}{b}$ {sine rule}



$\therefore \dfrac{\sin \widehat{BAC}}{8.4} = \dfrac{\sin 63°}{10.3}$

$\therefore \sin \widehat{BAC} = \dfrac{8.4 \times \sin 63°}{10.3}$

$\therefore \widehat{BAC} = \sin^{-1}\left(\dfrac{8.4 \times \sin 63°}{10.3}\right)$ or its supplement

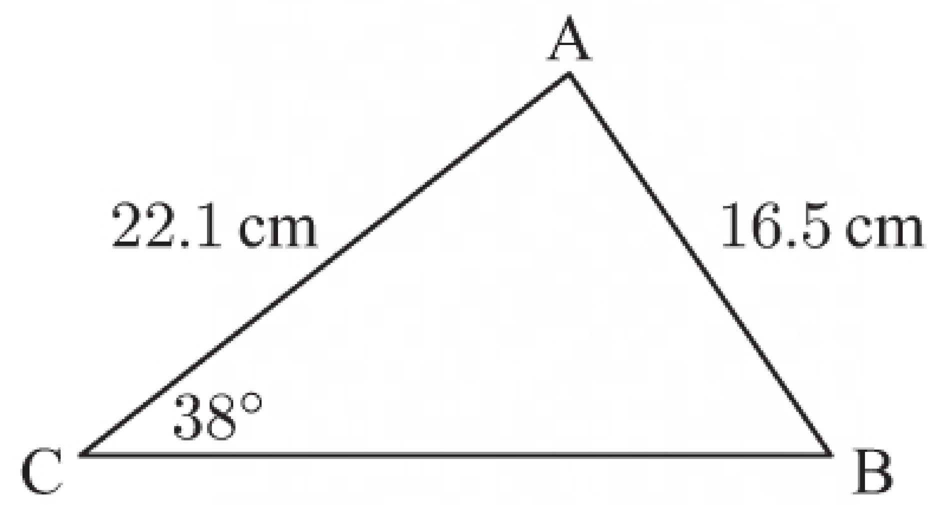
$\therefore \widehat{BAC} \approx 46.6°$ or $180° - 46.6°$

$\therefore \widehat{BAC} \approx 46.6°$ or $133.4°$

We reject $\widehat{BAC} = 133.4°$, since $133.4° + 63° > 180°$ which is impossible in a triangle.

$\therefore \widehat{BAC} \approx 46.6°$

e $\dfrac{\sin \widehat{ABC}}{22.1} = \dfrac{\sin 38°}{16.5}$ {sine rule}



$\therefore \sin \widehat{ABC} = \dfrac{22.1 \times \sin 38°}{16.5}$

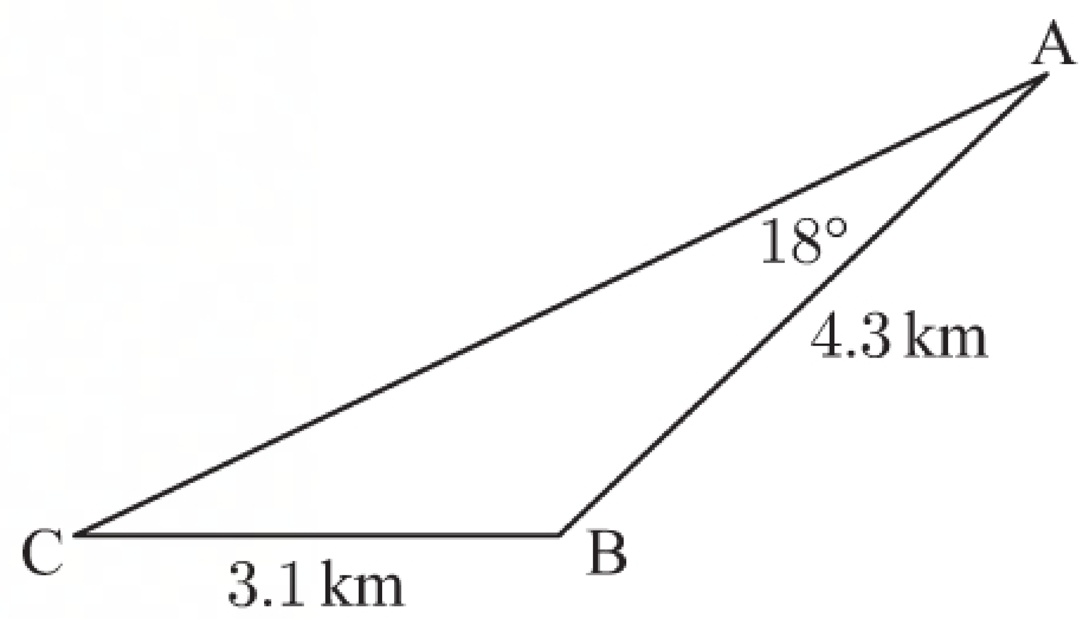
$\therefore \widehat{ABC} = \sin^{-1}\left(\dfrac{22.1 \times \sin 38°}{16.5}\right)$ or its supplement

$\therefore \widehat{ABC} \approx 55.5°$ or $180° - 55.5°$

$\therefore \widehat{ABC} \approx 55.5°$ or $124.5°$

both of which are possible as $124.5° + 38° = 162.5°$ which is $< 180°$.

f $\dfrac{\sin \widehat{ACB}}{4.3} = \dfrac{\sin 18°}{3.1}$ {sine rule}



$\therefore \sin \widehat{ACB} = \dfrac{4.3 \times \sin 18°}{3.1}$

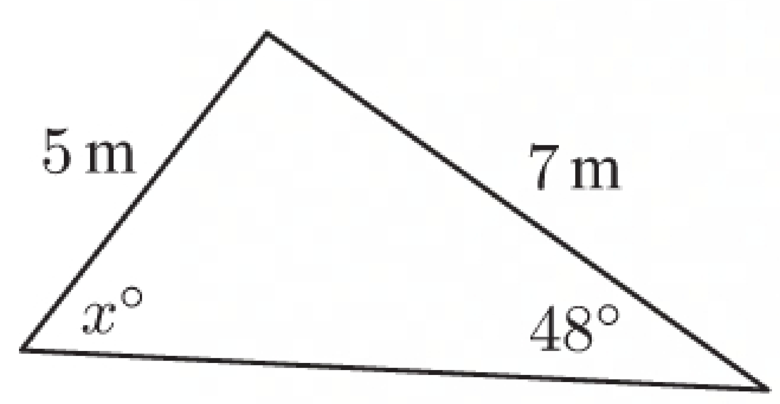
$\therefore \widehat{ACB} = \sin^{-1}\left(\dfrac{4.3 \times \sin 18°}{3.1}\right)$ or its supplement

$\therefore \widehat{ACB} \approx 25.4°$ or $180° - 25.4°$

$\therefore \widehat{ACB} \approx 25.4°$ or $154.6°$

both of which are possible as $154.6° + 18° = 172.6°$ which is $< 180°$.

4 **a** Using the sine rule, $\dfrac{\sin x°}{7} = \dfrac{\sin 48°}{5}$



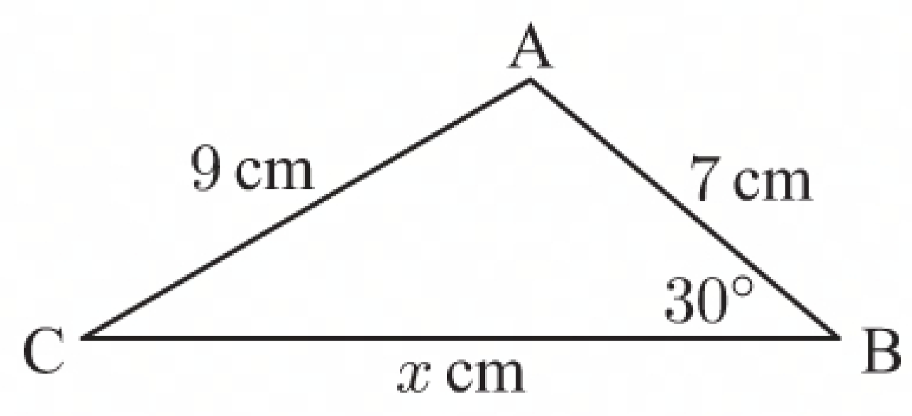
$\therefore \sin x° = \dfrac{7 \times \sin 48°}{5}$

$\therefore \sin x° \approx 1.04$

But $\sin x°$ is always between -1 and 1 (inclusive), so we cannot solve for x and the question cannot be solved.

b This means that it is impossible to draw a real diagram with the dimensions Mr Whiffen has given.

5 **a** **i** Using the sine rule,



$$\frac{\sin \widehat{ACB}}{7} = \frac{\sin 30°}{9}$$

$$\therefore \sin \widehat{ACB} = \frac{7 \times \sin 30°}{9}$$

$$\therefore \widehat{ACB} = \sin^{-1}\left(\frac{7 \times \sin 30°}{9}\right) \text{ or its supplement}$$

$$\therefore \widehat{ACB} \approx 22.9° \text{ or } 180° - 22.9°$$

$$\therefore \widehat{ACB} \approx 22.9° \text{ or } 157.1°$$

But $157.1 + 30 > 180$, so this case is impossible.

$$\therefore \widehat{ACB} \approx 22.9°$$

ii $\widehat{BAC} \approx 180° - 30° - 22.9°$ {angles in a triangle}

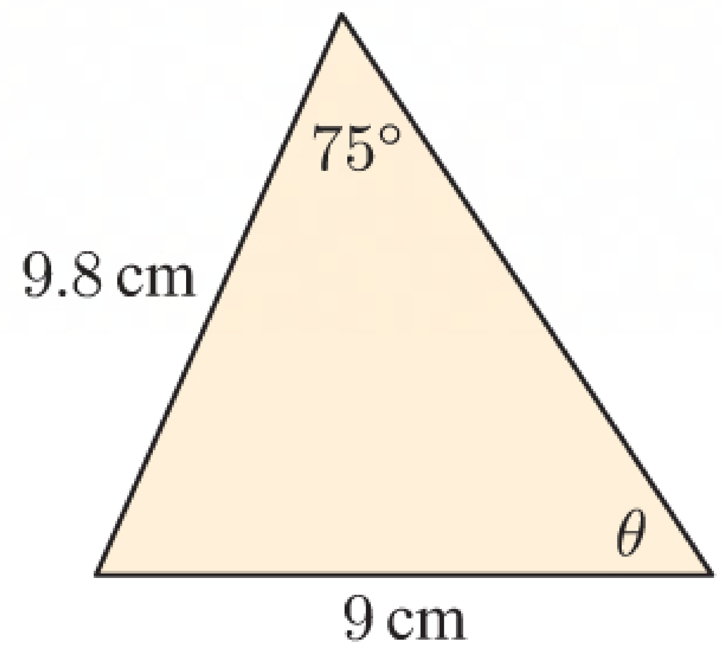
$\therefore \widehat{BAC} \approx 127°$

b Area of triangle ABC $= \frac{1}{2}bc\sin A$

$\approx \frac{1}{2} \times 9 \times 7 \times \sin 127°$

$\approx 25.1 \text{ cm}^2$

6 Let the angle opposite the 9.8 cm side be θ.



Using the sine rule, $\dfrac{\sin\theta}{9.8} = \dfrac{\sin 75°}{9}$

$$\therefore \sin\theta = \frac{9.8 \times \sin 75°}{9}$$

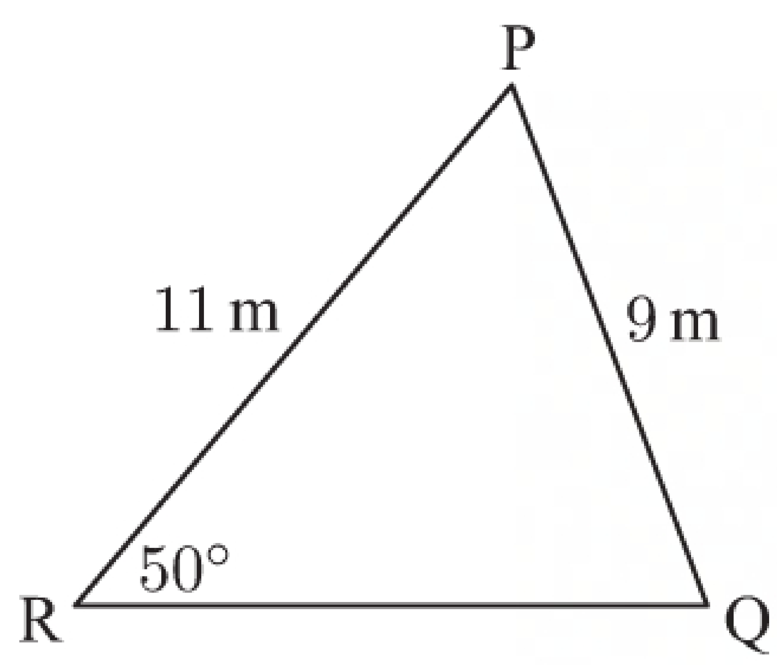
$$\therefore \sin\theta \approx 1.05$$

But $-1 \leqslant \sin\theta \leqslant 1$ for all θ

$\therefore \sin\theta \approx 1.05$ is impossible.

$\therefore$ it is not possible to have a triangle with the measurements shown.

7 **a** Using the sine rule,



$$\frac{\sin \widehat{PQR}}{11} = \frac{\sin 50°}{9}$$

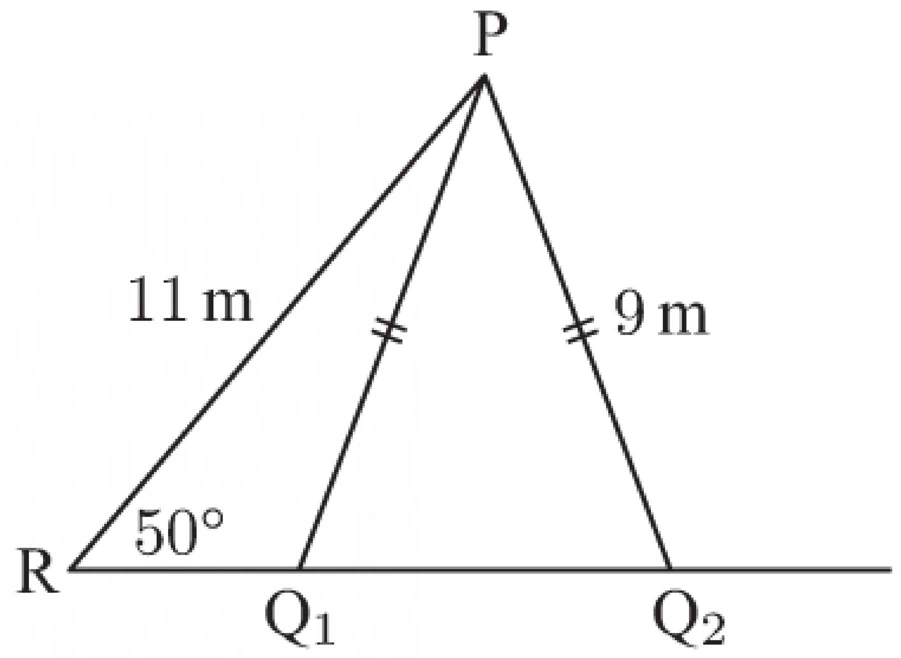
$$\therefore \sin \widehat{PQR} = \frac{11 \times \sin 50°}{9}$$

$$\therefore \widehat{PQR} = \sin^{-1}\left(\frac{11 \times \sin 50°}{9}\right) \text{ or its supplement}$$

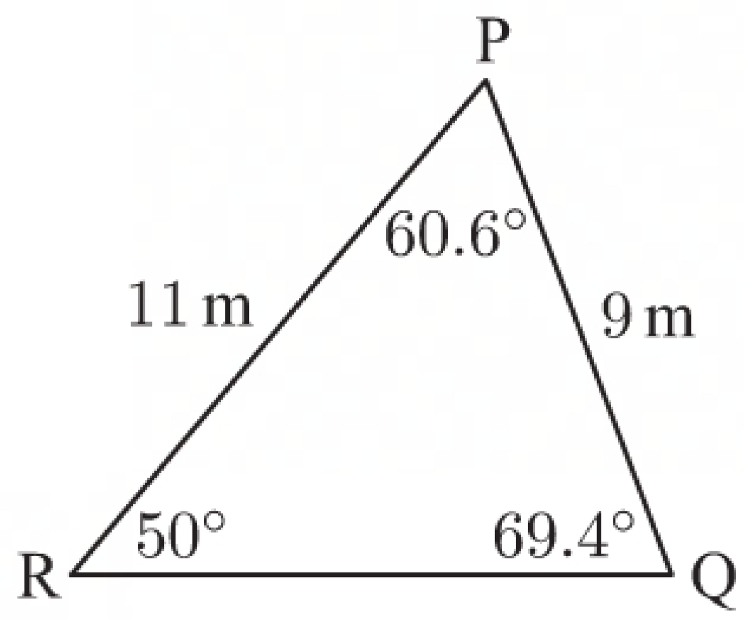
$$\therefore \widehat{PQR} \approx 69.4° \text{ or } 180° - 69.4°$$

$$\therefore \widehat{PQR} \approx 69.4° \text{ or } 110.6°$$

b



c For the case in which $\widehat{PQR} \approx 69.4°$:



i $\widehat{QPR} \approx 180° - 50° - 69.4°$ {angles in a triangle}

$\therefore\ \widehat{QPR} \approx 60.6°$

ii Area of triangle PQR $= \frac{1}{2} \times 9 \times 11 \times \sin\widehat{QPR}$

$\approx \frac{1}{2} \times 9 \times 11 \times \sin 60.6°$

$\approx 43.1\ \text{m}^2$

iii $\dfrac{QR}{\sin\widehat{QPR}} = \dfrac{PQ}{\sin\widehat{PRQ}}$ {sine rule}

$\therefore\ \dfrac{QR}{\sin 60.6°} \approx \dfrac{9}{\sin 50°}$

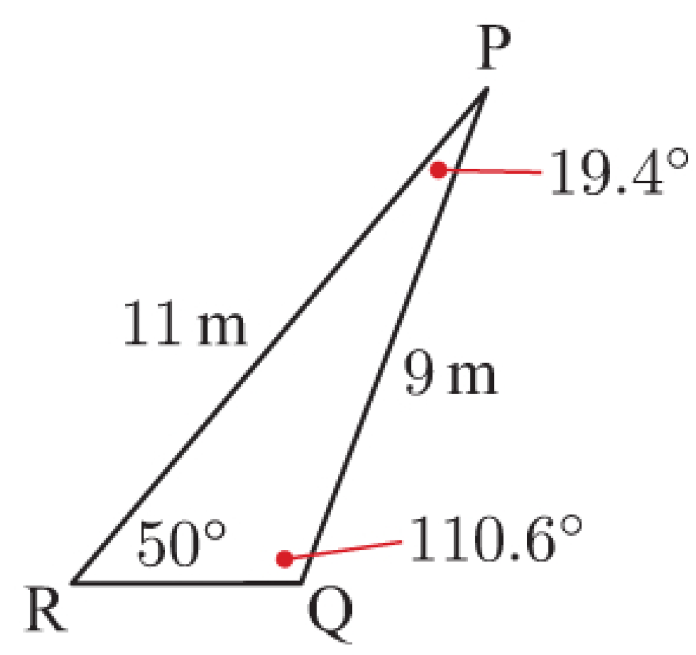
$\therefore\ QR \approx \dfrac{9 \times \sin 60.6°}{\sin 50°}$

$\therefore\ QR \approx 10.2$ m

So, perimeter of $\triangle PQR \approx (11 + 9 + 10.2)$ m

≈ 30.2 m

For the case in which $\widehat{PQR} \approx 110.6°$:



i $\widehat{QPR} \approx 180° - 50° - 110.6°$ {angles in a triangle}

$\therefore\ \widehat{QPR} \approx 19.4°$

ii Area of triangle PQR $= \frac{1}{2} \times 9 \times 11 \times \sin\widehat{QPR}$

$\approx \frac{1}{2} \times 9 \times 11 \times \sin 19.4°$

$\approx 16.5\ \text{m}^2$

iii $\dfrac{QR}{\sin\widehat{QPR}} = \dfrac{PQ}{\sin\widehat{PRQ}}$ {sine rule}

$\therefore\ \dfrac{QR}{\sin 19.4°} \approx \dfrac{9}{\sin 50°}$

$\therefore\ QR \approx \dfrac{9 \times \sin 19.4°}{\sin 50°}$

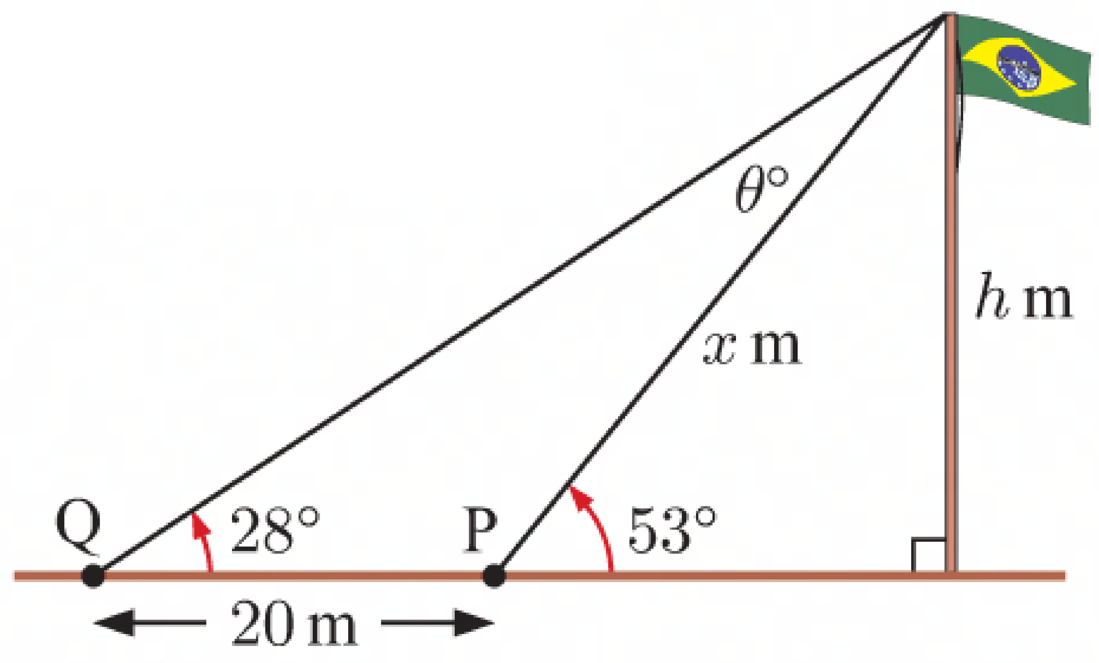
$\therefore\ QR \approx 3.91$ m

So, perimeter of $\triangle PQR \approx (11 + 9 + 3.91)$ m

≈ 23.9 m

EXERCISE 9D

1



Using the exterior angle of a triangle theorem,

$\theta° + 28° = 53°$

$\therefore \quad \theta = 25$

Let the flagpole be h m high.

Using the sine rule, $\dfrac{x}{\sin 28°} = \dfrac{20}{\sin 25°}$

$\therefore \quad x \approx \dfrac{20 \times \sin 28°}{\sin 25°}$

$\therefore \quad x \approx 22.22$

and $\quad \sin 53° = \dfrac{h}{x}$

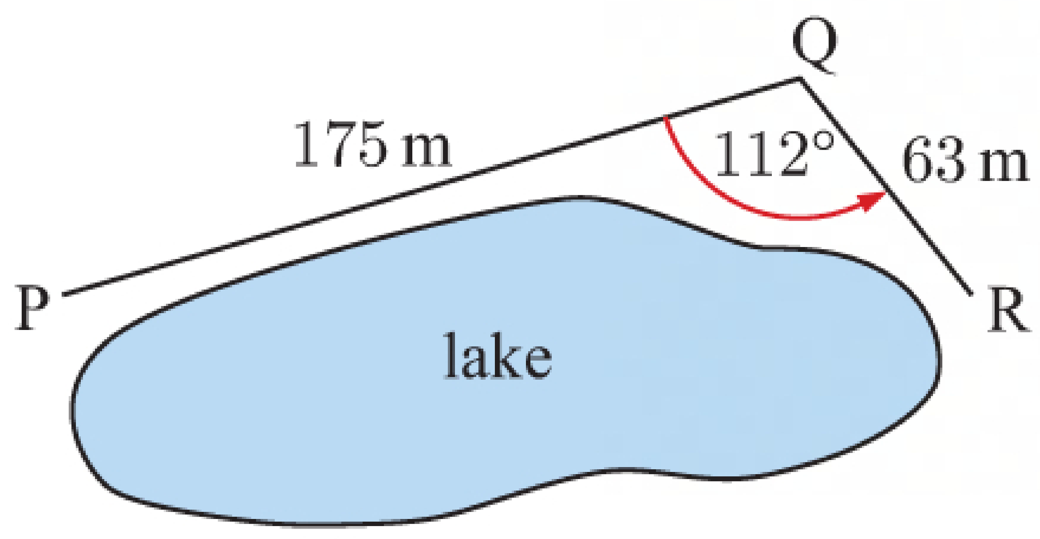
$\therefore \quad h = x \sin 53°$

$\approx 22.22 \times \sin 53°$

≈ 17.7 m

$\therefore$ the pole is approximately 17.7 m high.

2



By the cosine rule:

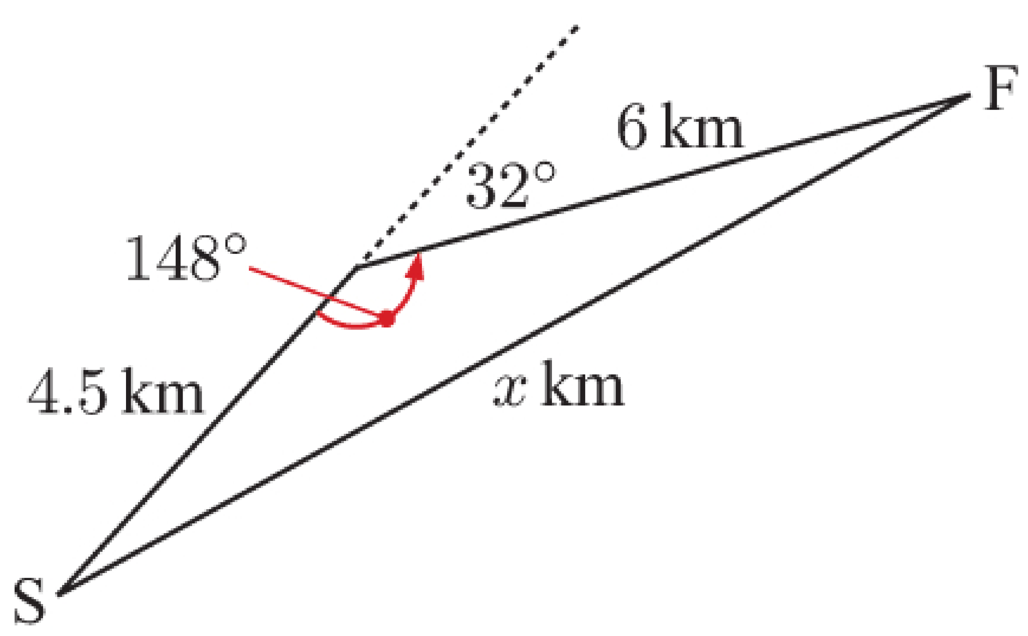
$\text{PR}^2 = 63^2 + 175^2 - 2 \times 63 \times 175 \times \cos 112°$

$\therefore \quad \text{PR} = \sqrt{63^2 + 175^2 - 2 \times 63 \times 175 \times \cos 112°}$

$\therefore \quad \text{PR} \approx 207$

So the distance from P to R is approximately 207 m.

3



By the cosine rule:

$x = \sqrt{6^2 + 4.5^2 - 2 \times 6 \times 4.5 \times \cos 148°}$

$\therefore \quad x \approx 10.1$

$\therefore$ the orienteer is about 10.1 km from her starting point.

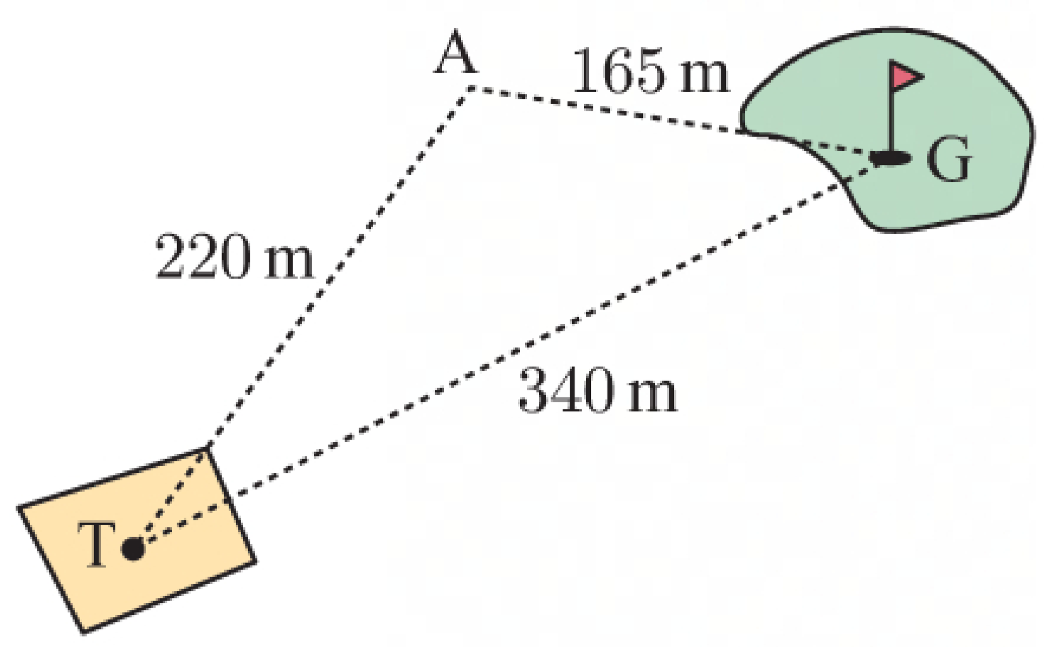
4 By the cosine rule:

$\cos T = \dfrac{220^2 + 340^2 - 165^2}{2 \times 220 \times 340}$

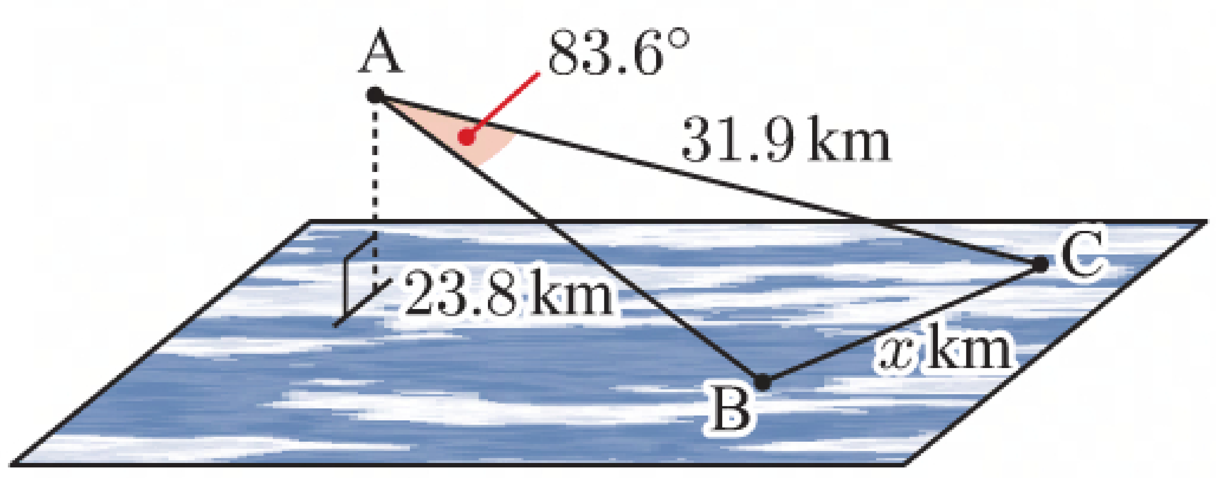
$\therefore \quad T = \cos^{-1}\left(\dfrac{136\,775}{149\,600}\right)$

$\therefore \quad T \approx 23.9$

$\therefore$ the tee shot was about 23.9° off line.



5



By the cosine rule:

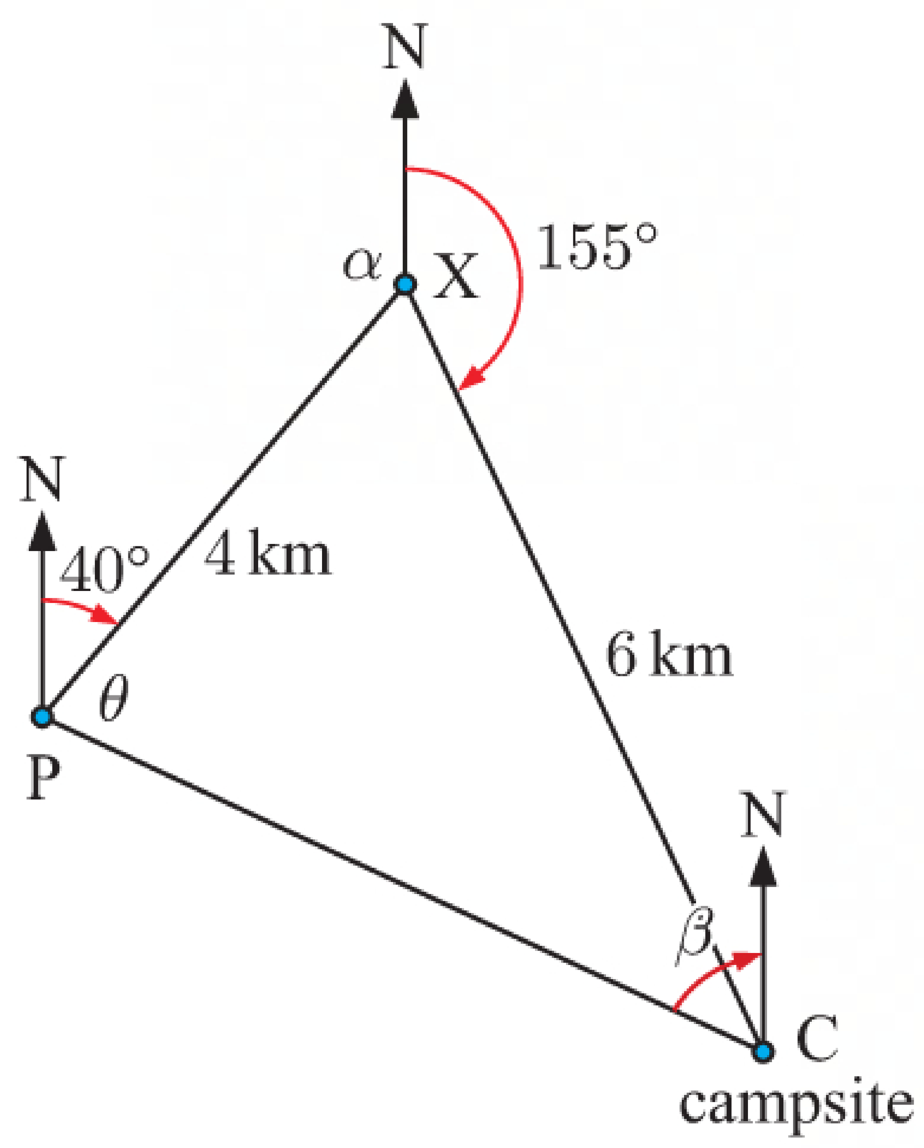
$x^2 = 23.8^2 + 31.9^2 - 2 \times 23.8 \times 31.9 \times \cos 83.6°$

$\therefore \quad x = \sqrt{23.8^2 + 31.9^2 - 2 \times 23.8 \times 31.9 \times \cos 83.6°}$

$\therefore \quad x \approx 37.6$

$\therefore$ ships B and C are about 37.6 km apart.

6



a $\alpha = 180° - 40°$ {co-interior angles}
$= 140°$

$\therefore$ $P\widehat{X}C = 360° - 155° - 140°$ {angles at a point}
$= 65°$

Using the cosine rule in $\triangle$PXC:

$PC^2 = 4^2 + 6^2 - 2 \times 4 \times 6 \times \cos 65°$

$\therefore$ $PC = \sqrt{4^2 + 6^2 - 2 \times 4 \times 6 \times \cos 65°}$ {as PC > 0}

$\therefore$ $PC \approx 5.6315$ km

So, Esko hikes about 5.63 km.

b By the cosine rule:

$$\cos\theta \approx \frac{4^2 + 5.6315^2 - 6^2}{2 \times 4 \times 5.6315}$$

$$\therefore \quad \theta \approx \cos^{-1}\left(\frac{4^2 + 5.6315^2 - 6^2}{2 \times 4 \times 5.6315}\right)$$

$\therefore$ $\theta \approx 74.9°$

So, Esko hikes on a bearing of $40° + 74.9° \approx 115°$.

c **i** Ritva travels a total distance of $4 + 6 = 10$ km.

$$\text{Ritva's time taken to reach the campsite} = \frac{\text{distance}}{\text{speed}} = \frac{10\text{ km}}{5\text{ km h}^{-1}} = 2\text{ hours}$$

$$\text{Esko's time taken to reach the campsite} = \frac{\text{distance}}{\text{speed}} \approx \frac{5.6315\text{ km}}{3\text{ km h}^{-1}} \approx 1.88\text{ hours}$$

So, Esko will arrive at the campsite first.

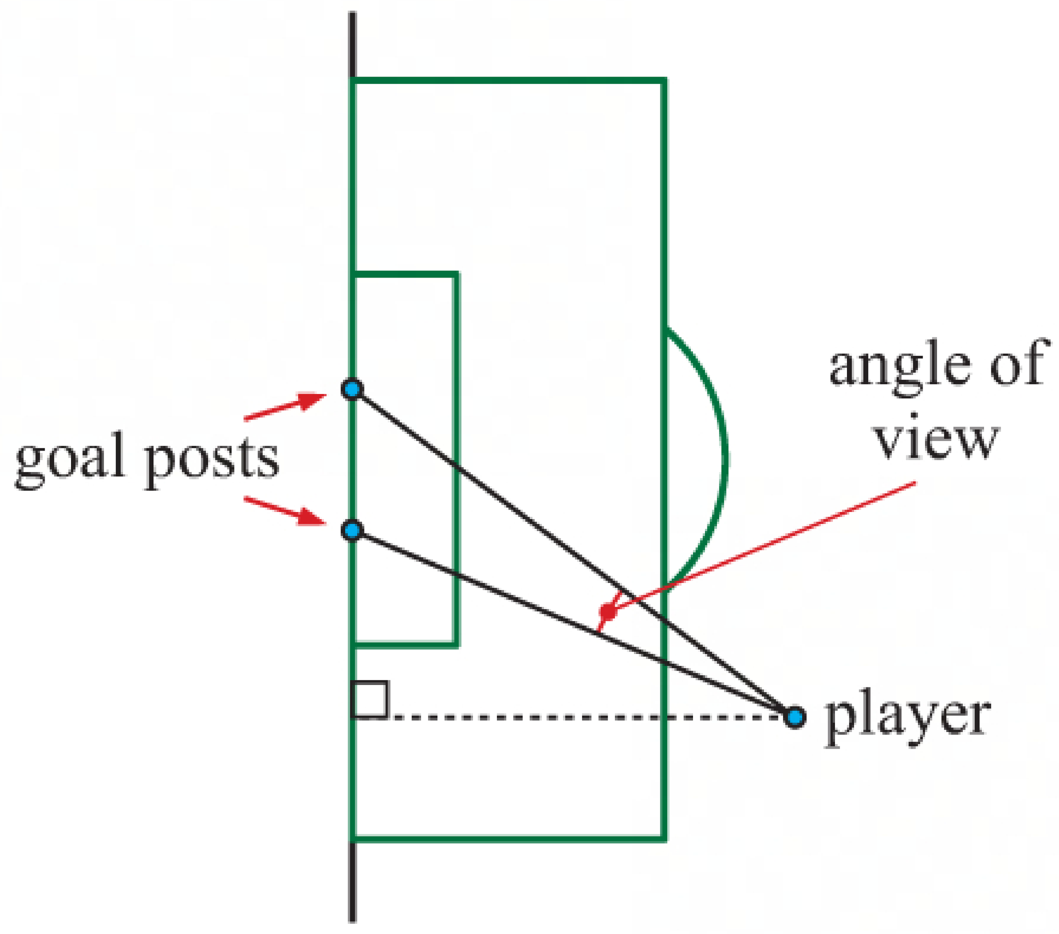
ii Difference in time taken to reach campsite $\approx$ 2 hours $-$ 1.88 hours
$\approx$ 0.123 hours
$\approx$ 7.37 minutes
$\approx$ 7 minutes 22 seconds

So, Esko will need to wait for about 7 minutes and 22 seconds before Ritva arrives.

d Now $\beta \approx 180° - 40° - 74.9°$ {co-interior angles}
$\approx 65.1°$

So, the hikers need to walk on a bearing of $360° - 65.1° \approx 295°$ to return directly to P.

7



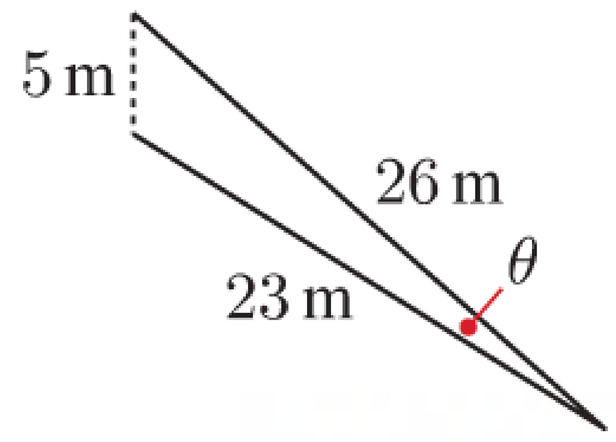
By the cosine rule:

$\cos\theta = \dfrac{23^2 + 26^2 - 5^2}{2 \times 23 \times 26}$

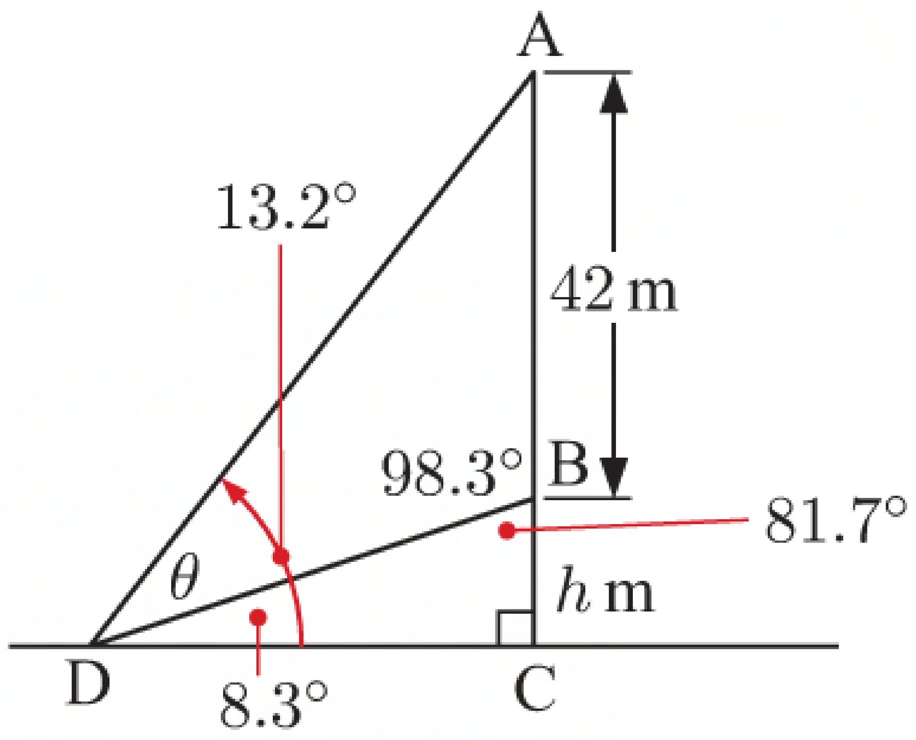
$\therefore \quad \theta = \cos^{-1}\left(\frac{1180}{1196}\right)$

$\therefore \quad \theta \approx 9.38°$

$\therefore$ the angle of view is about 9.38°.



8



$\theta = 13.2° - 8.3°$
$= 4.9°$

$D\widehat{B}C = 90° - 8.3°$ {angles in a triangle}
$= 81.7°$

$A\widehat{B}D = 180° - 81.7°$ {angles on a line}
$= 98.3°$

In $\triangle$ABD, by the sine rule, $\dfrac{AD}{\sin 98.3°} = \dfrac{42}{\sin 4.9°}$

$\therefore \quad AD = \dfrac{42 \times \sin 98.3°}{\sin 4.9°}$

$\therefore \quad AD \approx 486.56$ m

In $\triangle$ADC, $\sin 13.2° = \dfrac{h + 42}{AD}$

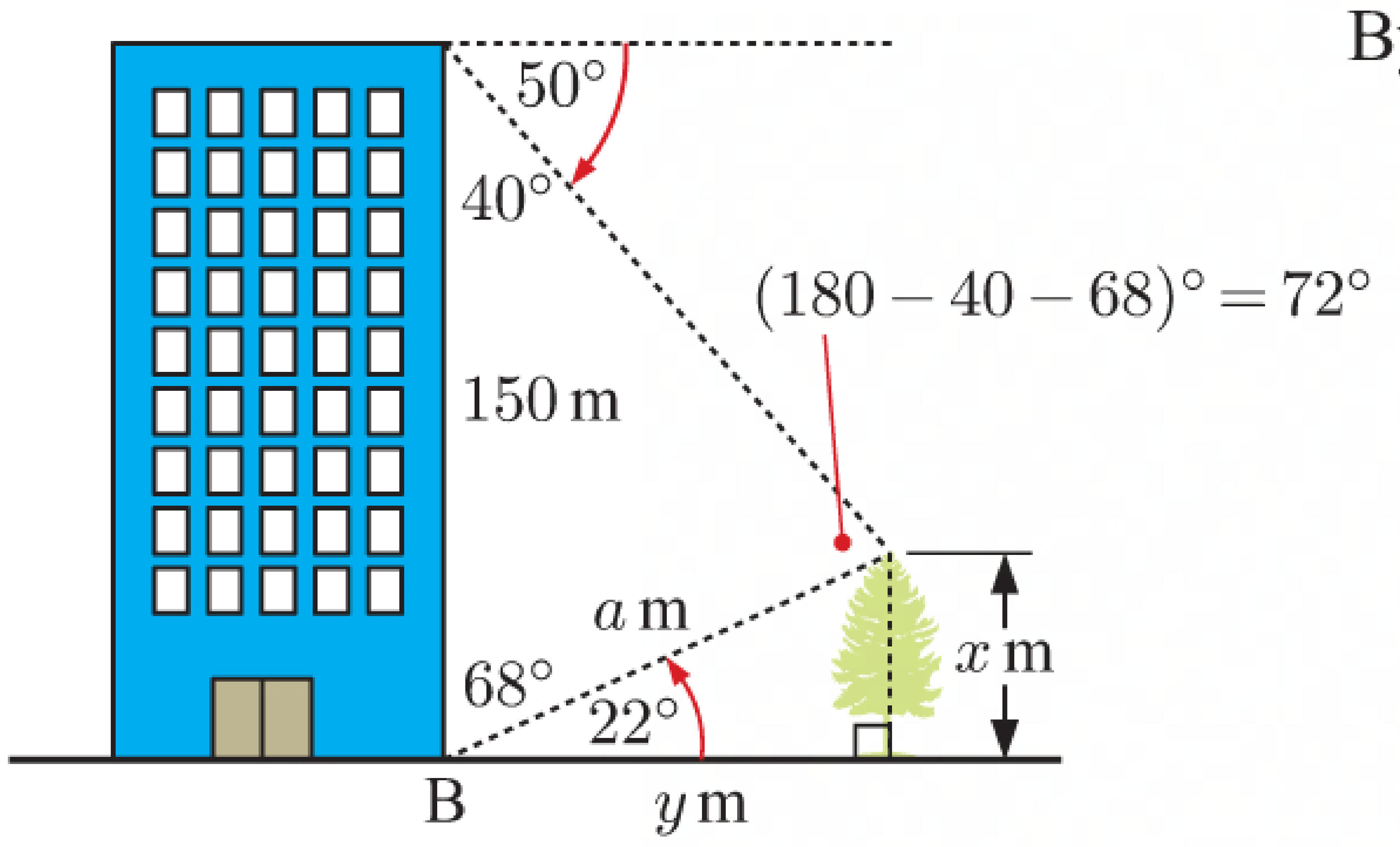
$\therefore \quad h + 42 \approx 486.56 \times \sin 13.2°$

$\therefore \quad h + 42 \approx 111.1$

$\therefore \quad h \approx 69.1$

$\therefore$ the hill is about 69.1 m high.

9



By the sine rule, $\dfrac{a}{\sin 40°} = \dfrac{150}{\sin 72°}$

$\therefore \quad a = \dfrac{150 \times \sin 40°}{\sin 72°}$

$\therefore \quad a \approx 101.38$

a $\sin 22° \approx \dfrac{x}{101.38}$

$\therefore \quad x \approx 101.38 \times \sin 22°$

$\therefore \quad x \approx 38.0$

$\therefore$ the tree is about 38.0 m high.

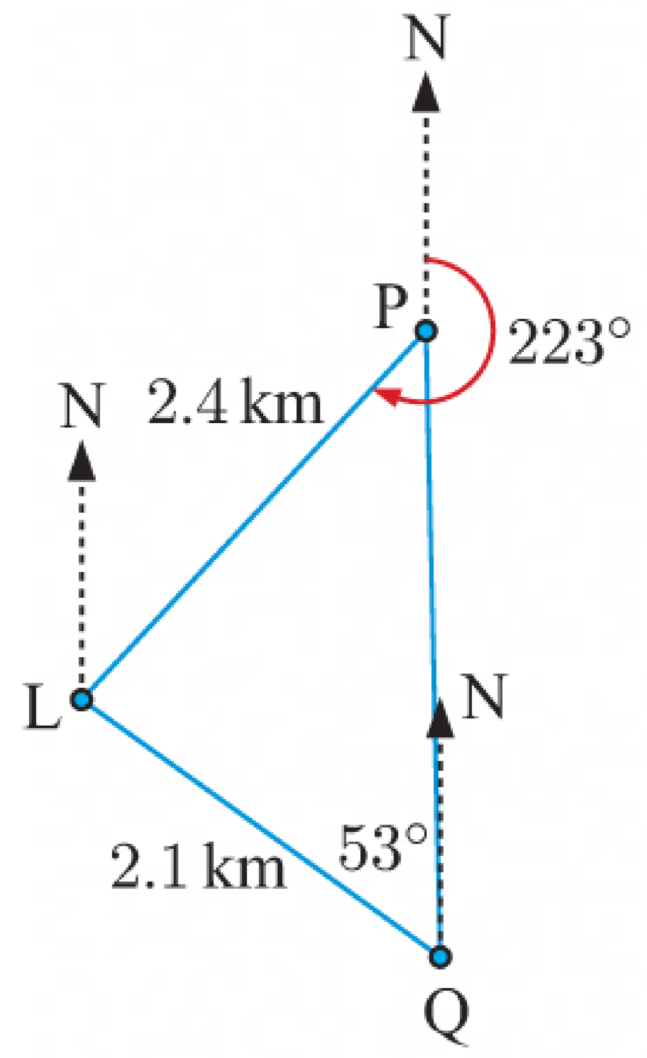
b $\cos 22° \approx \dfrac{y}{101.38}$

$\therefore \quad y \approx 101.38 \times \cos 22°$

$\therefore \quad y \approx 94.0$

$\therefore$ the tree is about 94.0 m from the building.

10 **a**



b Using the cosine rule in $\triangle$LPQ: $\quad 2.4^2 = 2.1^2 + \text{PQ}^2 - 2 \times 2.1 \times \text{PQ} \times \cos 53°$

$$\therefore \quad 5.76 = 4.41 + \text{PQ}^2 - 4.2 \times \text{PQ} \times \cos 53°$$

$$\therefore \quad \text{PQ}^2 - (4.2 \cos 53°)\,\text{PQ} - 1.35 = 0$$

Using technology, $\text{PQ} \approx 2.98$ or -0.453

But $\text{PQ} > 0$, so $\text{PQ} \approx 2.98$.

So, the yachts are about 2.98 km apart.

c Using the sine rule in $\triangle$LPQ, $\quad \dfrac{\sin \text{L}\widehat{\text{P}}\text{Q}}{2.1} = \dfrac{\sin 53°}{2.4}$

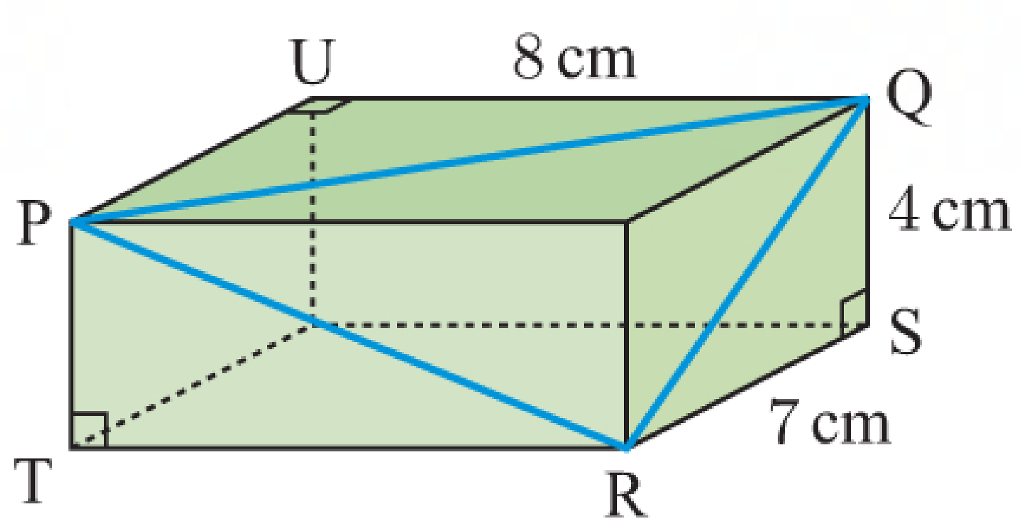
$$\therefore \quad \sin \text{L}\widehat{\text{P}}\text{Q} = \frac{2.1 \times \sin 53°}{2.4}$$

$$\therefore \quad \text{L}\widehat{\text{P}}\text{Q} = \sin^{-1}\left(\frac{2.1 \times \sin 53°}{2.4}\right)$$

$$\therefore \quad \text{L}\widehat{\text{P}}\text{Q} \approx 44.3°$$

The bearing of the *Queen Maria* from the *Porpoise* is $223° - 44.3° \approx 179°$.

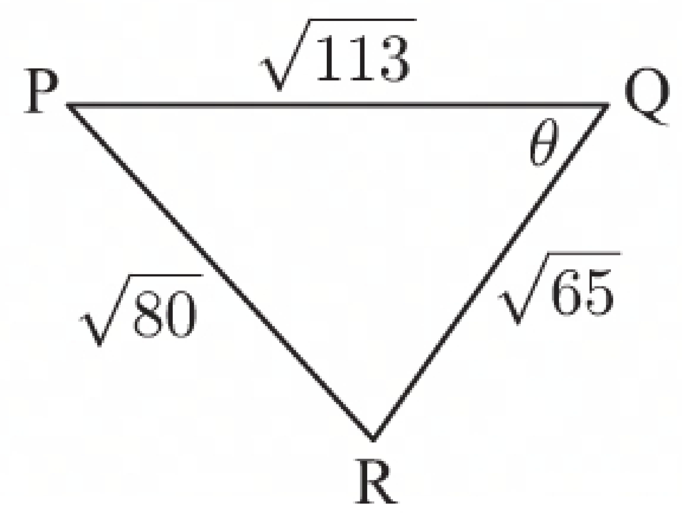
11 **a**



In $\triangle$QRS, $\quad \text{RQ} = \sqrt{4^2 + 7^2} = \sqrt{65}$ cm. $\quad$ {Pythagoras}

In $\triangle$PQU, $\quad \text{PQ} = \sqrt{8^2 + 7^2} = \sqrt{113}$ cm. $\quad$ {Pythagoras}

In $\triangle$PRT, $\quad \text{PR} = \sqrt{8^2 + 4^2} = \sqrt{80}$ cm. $\quad$ {Pythagoras}



By rearrangement of the cosine rule,

$$\cos\theta = \frac{(\sqrt{113})^2 + (\sqrt{65})^2 - (\sqrt{80})^2}{2\sqrt{113}\sqrt{65}}$$

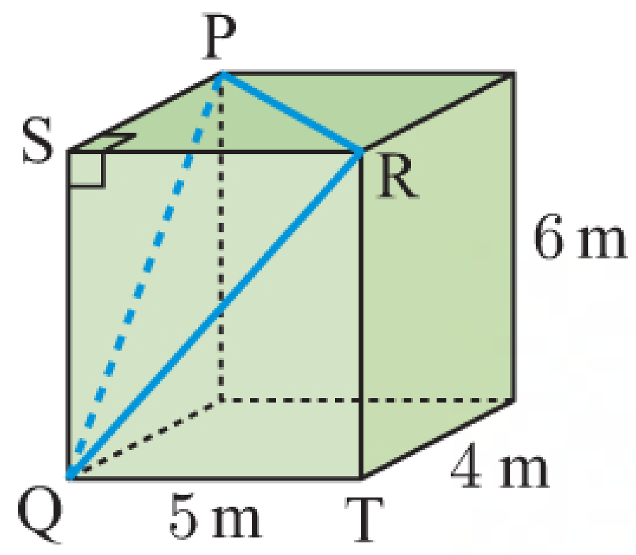
$$= \frac{113 + 65 - 80}{2\sqrt{113}\sqrt{65}}$$

$$= \frac{98}{2\sqrt{113}\sqrt{65}}$$

$$\therefore \quad \theta = \cos^{-1}\left(\frac{98}{2\sqrt{113}\sqrt{65}}\right) \approx 55.1^\circ$$

$\therefore$ $P\widehat{Q}R$ measures about 55.1°.

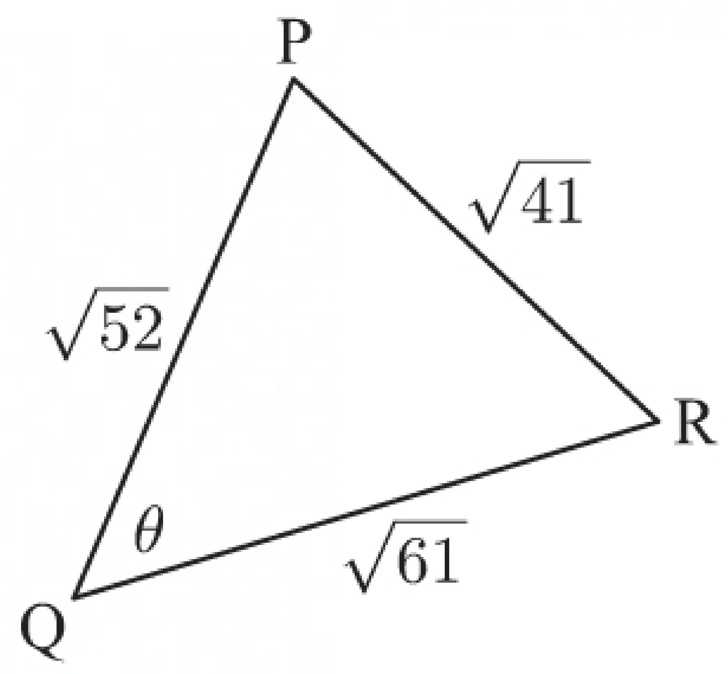
b



In $\triangle PRS$, $PR = \sqrt{5^2 + 4^2} = \sqrt{41}$ m. {Pythagoras}

In $\triangle PQS$, $PQ = \sqrt{6^2 + 4^2} = \sqrt{52}$ m. {Pythagoras}

In $\triangle QRT$, $QR = \sqrt{5^2 + 6^2} = \sqrt{61}$ m. {Pythagoras}



By rearrangement of the cosine rule,

$$\cos\theta = \frac{(\sqrt{52})^2 + (\sqrt{61})^2 - (\sqrt{41})^2}{2\sqrt{52}\sqrt{61}}$$

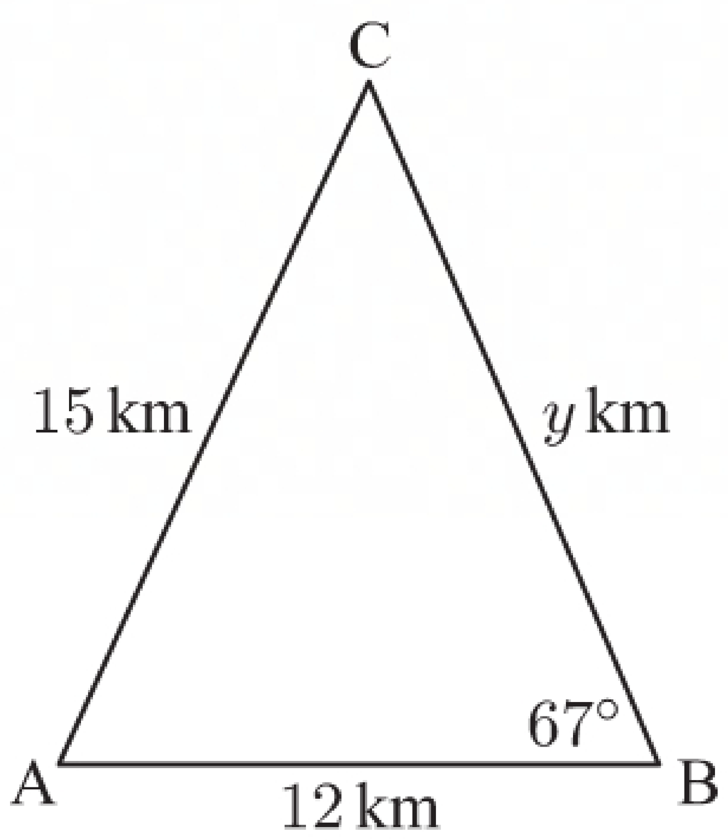
$$= \frac{52 + 61 - 41}{2\sqrt{52}\sqrt{61}}$$

$$= \frac{72}{2\sqrt{52}\sqrt{61}}$$

$$\therefore \quad \theta = \cos^{-1}\left(\frac{72}{2\sqrt{52}\sqrt{61}}\right) \approx 50.3^\circ$$

$\therefore$ $P\widehat{Q}R$ measures about 50.3°.

12



Using the sine rule, $\dfrac{\sin A\widehat{C}B}{12} = \dfrac{\sin 67^\circ}{15}$

$$\therefore \quad \sin A\widehat{C}B = \frac{12 \times \sin 67^\circ}{15}$$

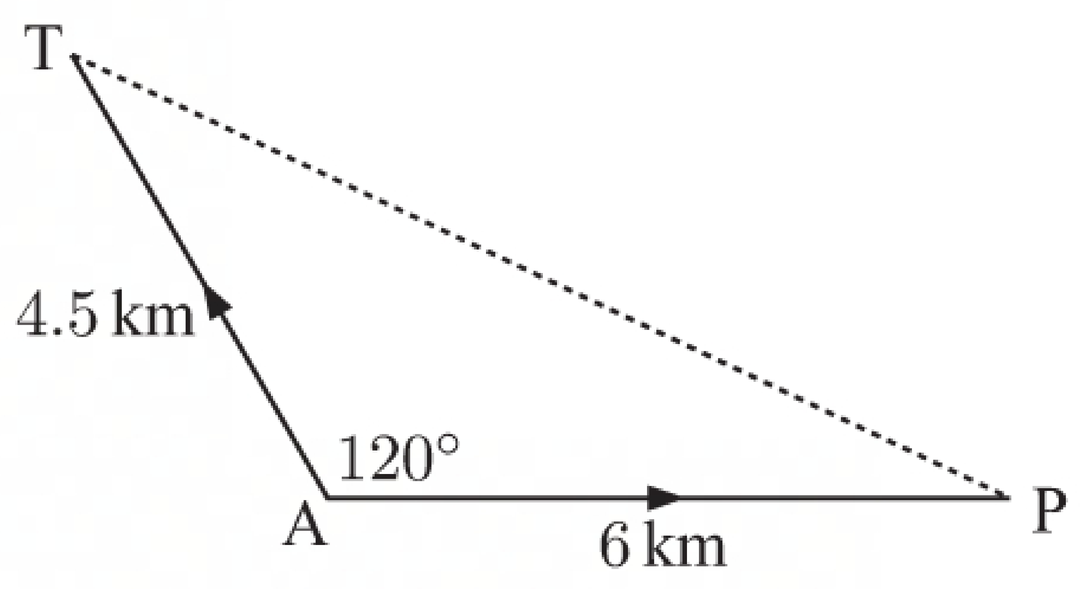
$$\therefore \quad A\widehat{C}B = \sin^{-1}\left(\frac{12 \times \sin 67^\circ}{15}\right)$$

$$\therefore \quad A\widehat{C}B \approx 47.4^\circ$$

Now, $C\widehat{A}B \approx 180^\circ - 67^\circ - 47.4^\circ$

$$\therefore \quad C\widehat{A}B \approx 65.6^\circ$$

13



$$\text{Distance} = \text{speed} \times \text{time}$$

So, after 45 min = 0.75 h,

$$\text{AT} = 6 \times 0.75 = 4.5 \text{ km}$$

and $\text{AP} = 8 \times 0.75 = 6$ km

By the cosine rule:

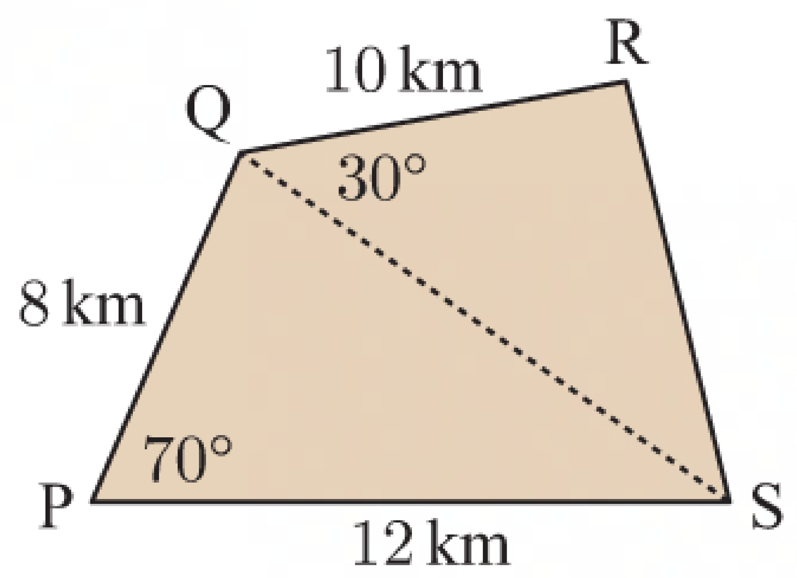
$$\text{PT}^2 = 4.5^2 + 6^2 - 2 \times 4.5 \times 6 \times \cos 120°$$

$$\therefore \ \text{PT} = \sqrt{4.5^2 + 6^2 - 2 \times 4.5 \times 6 \times \cos 120°}$$

$$\therefore \ \text{PT} \approx 9.12$$

So, after 45 minutes they are about 9.12 km apart.

14 **a** By the cosine rule:



$$\text{QS}^2 = 8^2 + 12^2 - 2 \times 8 \times 12 \times \cos 70°$$

$$\therefore \ \text{QS} = \sqrt{8^2 + 12^2 - 2 \times 8 \times 12 \times \cos 70°}$$

$$\approx 11.93$$

total area = area of $\triangle$PQS + area of $\triangle$QRS

$$\therefore \ \text{area} \approx \tfrac{1}{2} \times 8 \times 12 \times \sin 70° + \tfrac{1}{2} \times 10 \times 11.93 \times \sin 30°$$

$$\approx 74.9 \text{ km}^2$$

b 1 ha is $100 \text{ m} \times 100 \text{ m} = 0.1 \text{ km} \times 0.1 \text{ km}$

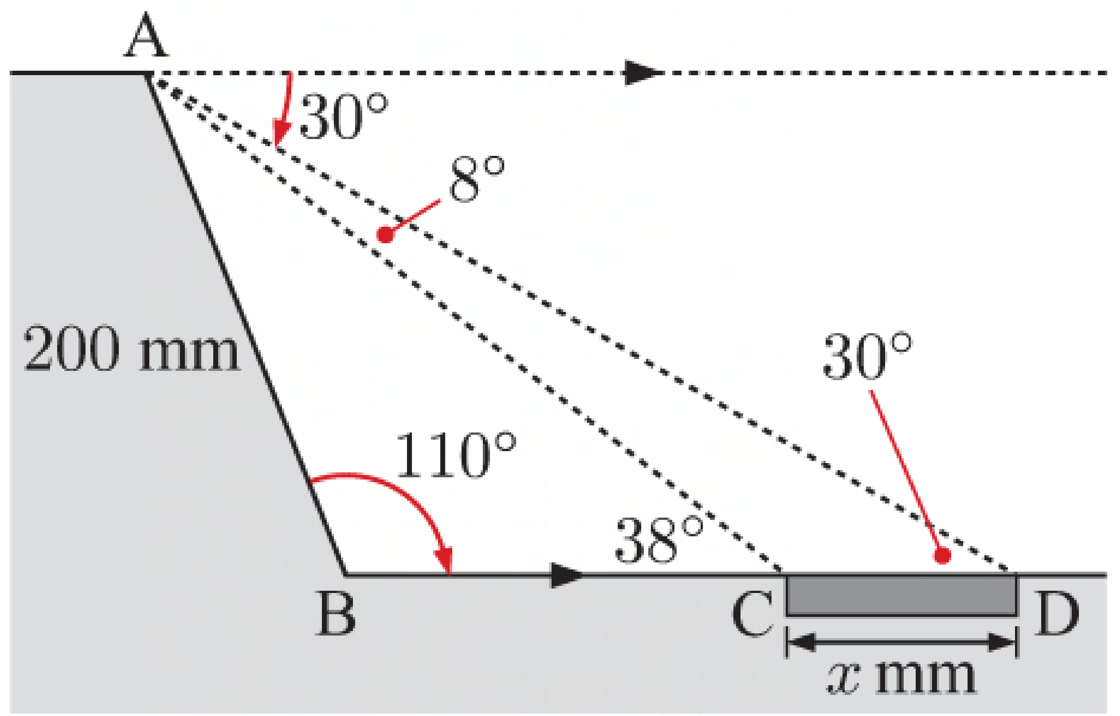
$$= 0.01 \text{ km}^2$$

$$\therefore \ 1 \text{ km}^2 = 100 \text{ ha}$$

$$\therefore \ \text{area} \approx (74.9 \times 100) \text{ ha}$$

$$\approx 7490 \text{ ha}$$

15



Using the sine rule in $\triangle$ABC,

$$\frac{\text{AC}}{\sin 110°} = \frac{200}{\sin 38°}$$

$$\therefore \ \text{AC} = \frac{200 \times \sin 110°}{\sin 38°}$$

$$\approx 305.26$$

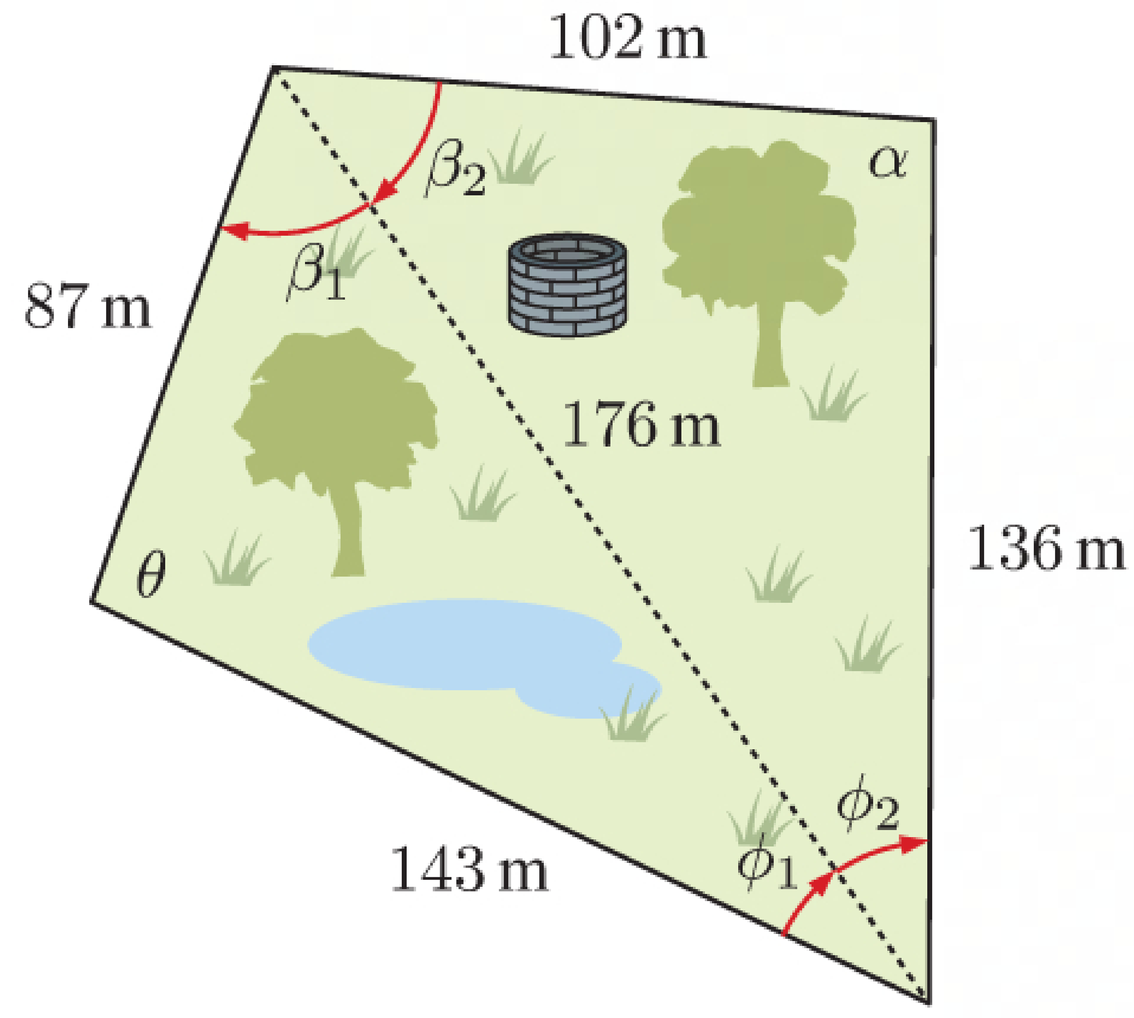
and in $\triangle$ACD, $\dfrac{x}{\sin 8°} \approx \dfrac{305.26}{\sin 30°}$

$$\therefore \ x \approx \frac{305.26 \times \sin 8°}{\sin 30°}$$

$$\approx 84.969$$

$\therefore$ the metal strip is about 85.0 mm wide.

16 By the cosine rule:



$$\cos\theta = \frac{87^2 + 143^2 - 176^2}{2 \times 87 \times 143}$$

$$\therefore \quad \theta = \cos^{-1}\left(\frac{-2958}{24\,882}\right)$$

$$\therefore \quad \theta \approx 96.8^\circ$$

Also by the cosine rule:

$$\cos\alpha = \frac{102^2 + 136^2 - 176^2}{2 \times 102 \times 136}$$

$$\therefore \quad \alpha = \cos^{-1}\left(\frac{-2076}{27\,744}\right)$$

$$\therefore \quad \alpha \approx 94.3^\circ$$

Now, by the sine rule, $\dfrac{\sin\beta_1}{143} = \dfrac{\sin\theta}{176}$

$$\therefore \quad \sin\beta_1 \approx \frac{143 \times \sin 96.8^\circ}{176}$$

$$\therefore \quad \beta_1 \approx \sin^{-1}\left(\frac{143 \times \sin 96.8^\circ}{176}\right)$$

$$\therefore \quad \beta_1 \approx 53.778^\circ$$

$$\phi_1 = 180^\circ - \beta_1 - \theta \qquad \{\text{angles in a triangle}\}$$

$$\approx 180^\circ - 53.778^\circ - 96.8^\circ$$

$$\therefore \quad \phi_1 \approx 29.394^\circ$$

Also by the sine rule, $\dfrac{\sin\beta_2}{136} = \dfrac{\sin\alpha}{176}$

$$\therefore \quad \sin\beta_2 \approx \frac{136 \times \sin 94.3^\circ}{176}$$

$$\therefore \quad \beta_2 \approx \sin^{-1}\left(\frac{136 \times \sin 94.3^\circ}{176}\right)$$

$$\therefore \quad \beta_2 \approx 50.404^\circ$$

$$\phi_2 = 180^\circ - \beta_2 - \alpha \qquad \{\text{angles in a triangle}\}$$

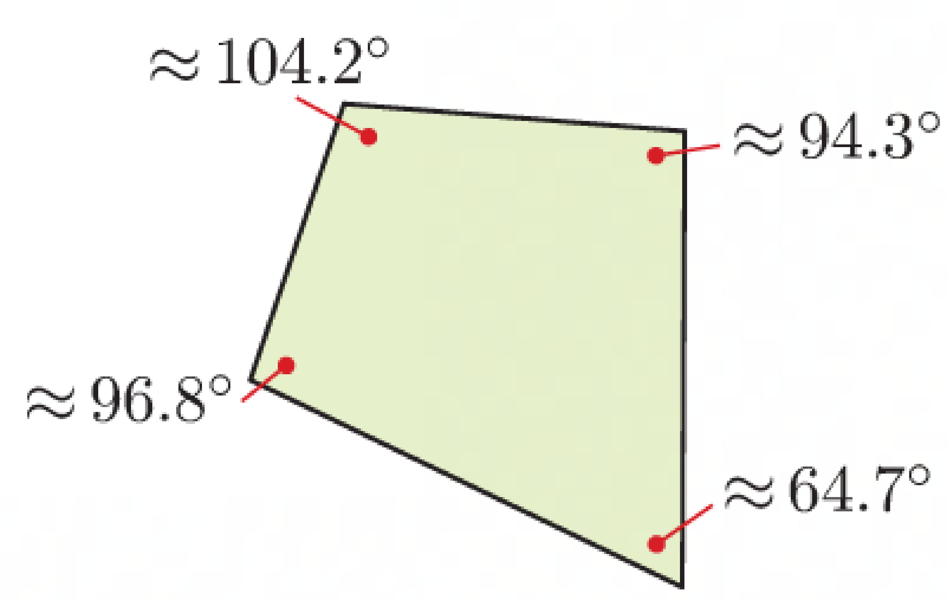
$$\approx 180^\circ - 50.404^\circ - 94.3^\circ$$

$$\therefore \quad \phi_2 \approx 35.304^\circ$$

So, $\beta_1 + \beta_2 \approx 53.778^\circ + 50.404^\circ \approx 104.2^\circ$

and $\phi_1 + \phi_2 \approx 29.394^\circ + 35.304^\circ \approx 64.7^\circ$

$\therefore$ the angles at each corner of the park are:



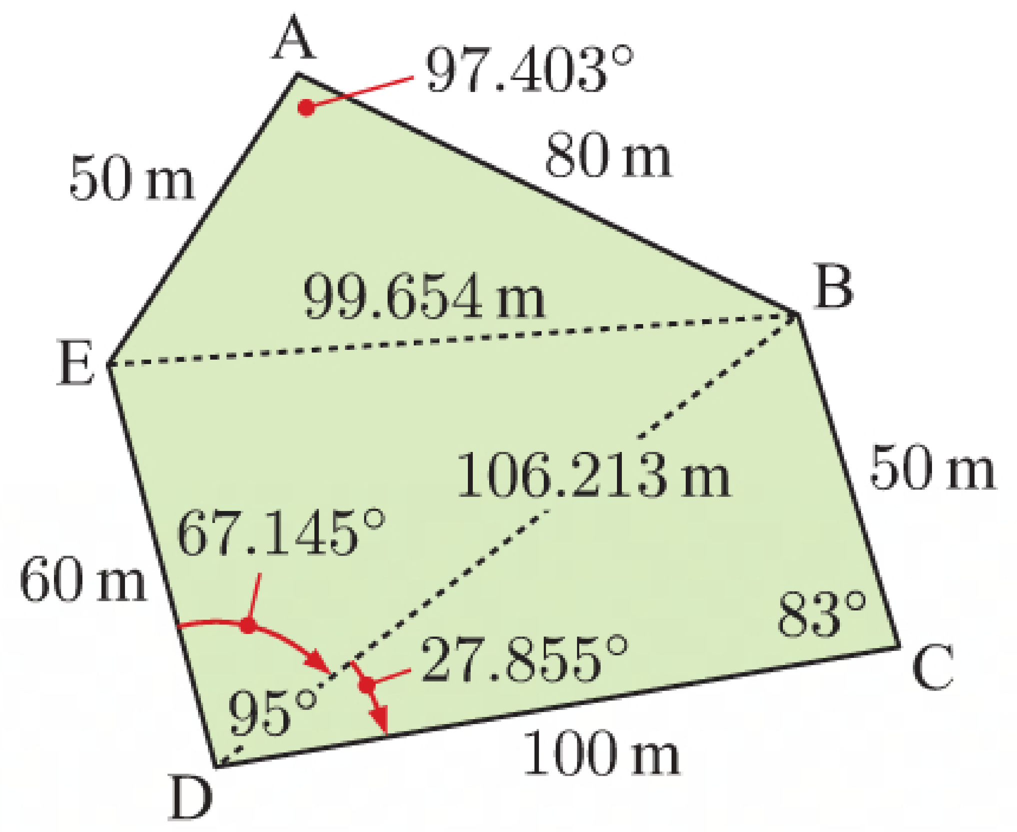
Area of the park = area of two triangles

$$= \tfrac{1}{2} \times 87 \times 143 \times \sin\theta \; + \; \tfrac{1}{2} \times 102 \times 136 \times \sin\alpha$$

$$\approx \tfrac{1}{2} \times 87 \times 143 \times \sin 96.8^\circ \; + \; \tfrac{1}{2} \times 102 \times 136 \times \sin 94.3^\circ$$

$$\approx 13\,100 \text{ m}^2$$

17



In $\triangle$BCD, using the cosine rule:

$$BD^2 = 50^2 + 100^2 - 2 \times 50 \times 100 \times \cos 83°$$

$$\therefore \quad BD = \sqrt{50^2 + 100^2 - 2 \times 50 \times 100 \times \cos 83°}$$

$$\approx 106.213 \text{ m}$$

In $\triangle$BCD, using the sine rule:

$$\frac{\sin B\hat{D}C}{50} \approx \frac{\sin 83°}{106.213}$$

$$\therefore \quad \sin B\hat{D}C \approx \frac{50 \times \sin 83°}{106.213}$$

$$\therefore \quad B\hat{D}C \approx \sin^{-1}\left(\frac{50 \times \sin 83°}{106.213}\right)$$

$$\therefore \quad B\hat{D}C \approx 27.855°$$

$$\therefore \quad B\hat{D}E \approx 95° - 27.855°$$

$$\approx 67.145°$$

In $\triangle$BED, using the cosine rule:

$$BE^2 \approx 60^2 + 106.213^2 - 2 \times 60 \times 106.213 \times \cos 67.145°$$

$$\therefore \quad BE \approx \sqrt{60^2 + 106.213^2 - 2 \times 60 \times 106.213 \times \cos 67.145°}$$

$$\approx 99.654 \text{ m}$$

In $\triangle$ABE, using the cosine rule:

$$\cos B\hat{A}E \approx \frac{50^2 + 80^2 - 99.654^2}{2 \times 50 \times 80}$$

$$\therefore \quad B\hat{A}E \approx \cos^{-1}\left(\frac{50^2 + 80^2 - 99.654^2}{8000}\right)$$

$$\approx 97.403°$$

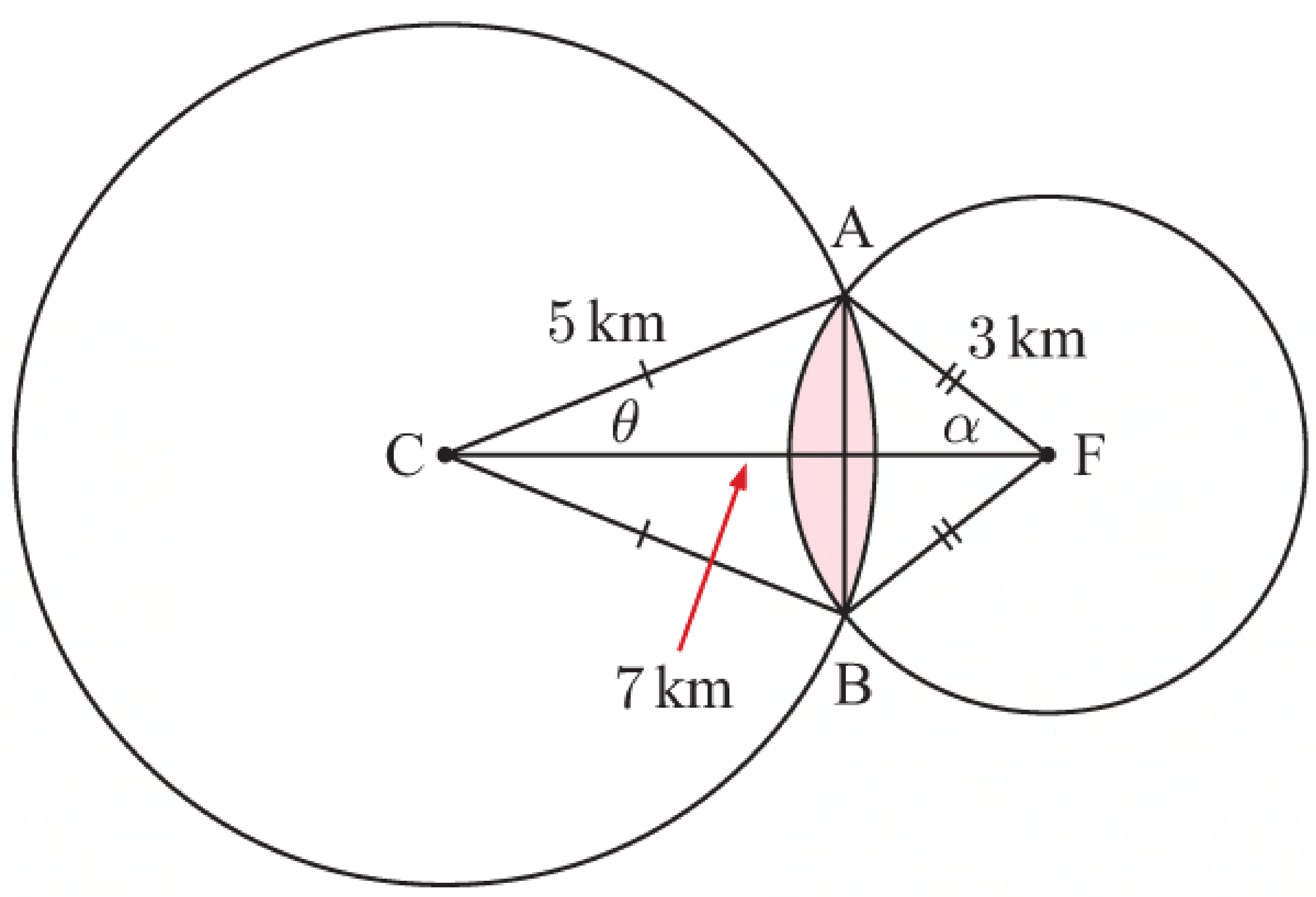
$\therefore$ area of the property

$= \text{area } \triangle BCD + \text{area } \triangle BED + \text{area } \triangle ABE$

$\approx \frac{1}{2} \times 50 \times 100 \times \sin 83° + \frac{1}{2} \times 60 \times 106.213 \times \sin 67.145° + \frac{1}{2} \times 50 \times 80 \times \sin 97.403°$

$\approx 7400 \text{ m}^2$

18 Let the location of the Chinese restaurant be C and the location of the fish and chip shop be F, such that $CF = 7$ km. The area enclosed by the larger circle shows the free delivery region for the Chinese restaurant and the area enclosed by the smaller circle shows the free delivery region for the fish and chip shop.



The region that receives free delivery from both locations is the intersection of these two regions. So, we need to find the shaded area.

Using the cosine rule in $\triangle$AFC: $\cos\theta = \dfrac{5^2+7^2-3^2}{2\times 5\times 7}$ and $\cos\alpha = \dfrac{3^2+7^2-5^2}{2\times 3\times 7}$

$\therefore\ \theta = \cos^{-1}\left(\frac{65}{70}\right)$ $\therefore\ \alpha = \cos^{-1}\left(\frac{33}{42}\right)$

Now $\triangle$AFC and $\triangle$BFC are congruent. {SSS}

$\therefore$ $A\widehat{C}B = 2\theta$ and $A\widehat{F}B = 2\alpha$

In the larger circle, area of $\triangle$ABC $= \frac{1}{2}\times 5\times 5\times \sin 2\theta$

$= \frac{25}{2}\times \sin\left(2\cos^{-1}\left(\frac{65}{70}\right)\right)$

$\approx 8.6161\ \text{km}^2$

Also, area of sector ABC $= \dfrac{2\theta}{360}\times\pi\times 5^2$

$= \dfrac{2\cos^{-1}\left(\frac{65}{70}\right)}{360}\times\pi\times 5^2$

$\approx 9.5063\ \text{km}^2$

Now, area of segment of larger circle = area of sector ABC − area of $\triangle$ABC

$\approx 9.5063 - 8.6161\ \text{km}^2$

$\approx 0.8902\ \text{km}^2$

In the smaller circle, area of $\triangle$ABF $= \frac{1}{2}\times 3\times 3\times\sin 2\alpha$

$= \frac{9}{2}\times\sin\left(2\cos^{-1}\left(\frac{33}{42}\right)\right)$

$\approx 4.3743\ \text{km}^2$

Also, area of sector ABF $= \dfrac{2\alpha}{360}\times\pi\times 3^2$

$= \dfrac{2\cos^{-1}\left(\frac{33}{42}\right)}{360}\times\pi\times 3^2$

$\approx 6.0025\ \text{km}^2$

Now, area of segment of smaller circle = area of sector ABF − area of $\triangle$ABF

$\approx 6.0025 - 4.3743\ \text{km}^2$

$\approx 1.6282\ \text{km}^2$

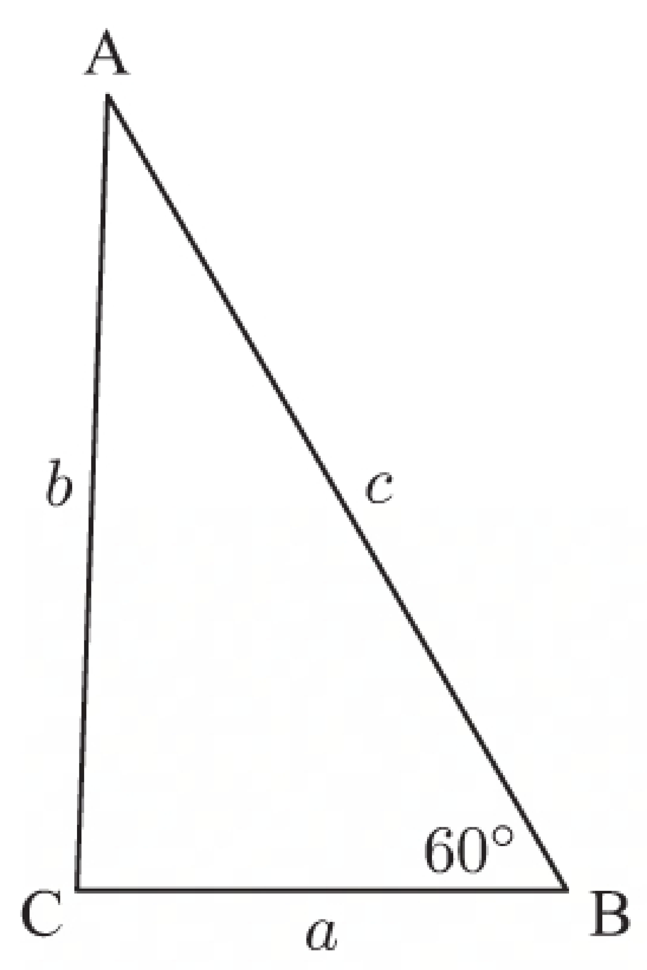
So, shaded area = area of segment of larger circle + area of segment of smaller circle

$\approx 0.8902 + 1.6282\ \text{km}^2$

$\approx 2.52\ \text{km}^2$

$\therefore$ the region that receives free delivery from both locations is about $2.52\ \text{km}^2$.

19



The perimeter is 36 m.

$\therefore \quad a + b + c = 36$

$\therefore \quad b = 36 - a - c \quad .... (1)$

The area is $30\sqrt{3}$ m^2.

$$\therefore \quad \tfrac{1}{2}ac\sin 60^\circ = 30\sqrt{3}$$

$$\therefore \quad \tfrac{1}{2}ac \times \left(\tfrac{\sqrt{3}}{2}\right) = 30\sqrt{3}$$

$$\therefore \quad \tfrac{\sqrt{3}}{4}ac = 30\sqrt{3}$$

$$\therefore \quad ac = 120$$

$$\therefore \quad a = \frac{120}{c} \quad .... (2)$$

Using the cosine rule: $b^2 = a^2 + c^2 - 2ac\cos 60^\circ$

$$\therefore \quad (36 - a - c)^2 = a^2 + c^2 - 2ac \times \left(\tfrac{1}{2}\right) \qquad \{\text{using (1)}\}$$

$$\therefore \quad (36 - a - c)(36 - a - c) = a^2 + c^2 - ac$$

$$\therefore \quad 1296 - 36a - 36c - 36a + \cancel{a^2} + ac - 36c + ac + \cancel{c^2} = \cancel{a^2} + \cancel{c^2} - ac$$

$$\therefore \quad 1296 - 72a - 72c + 3ac = 0$$

$$\therefore \quad 432 - 24a - 24c + ac = 0$$

$$\therefore \quad 432 - 24\left(\frac{120}{c}\right) - 24c + \left(\frac{120}{c}\right)c = 0 \qquad \{\text{using (2)}\}$$

$$\therefore \quad 432 - \frac{2880}{c} - 24c + 120 = 0$$

$$\therefore \quad 552 - \frac{2880}{c} - 24c = 0$$

$$\therefore \quad 24c^2 - 552c + 2880 = 0$$

$$\therefore \quad 24(c^2 - 23c + 120) = 0$$

$$\therefore \quad 24(c - 8)(c - 15) = 0$$

$$\therefore \quad c = 8 \text{ or } 15$$

Substituting $c = 8$ into (2) gives $a = \dfrac{120}{8} = 15$

Substituting $c = 15$ into (2) gives $a = \dfrac{120}{15} = 8$

When $c = 8$ and $a = 15$, $b = 36 - 15 - 8$ {using (1)}
$= 13$

When $c = 15$ and $a = 8$, $b = 36 - 8 - 15$ {using (1)}
$= 13$

Using the sine rule:

$$\frac{\sin B}{b} = \frac{\sin C}{c}$$

$$\therefore \frac{\sin 60°}{13} = \frac{\sin C}{8} \quad \{\text{when } c = 8\}$$

$$\therefore \frac{\frac{\sqrt{3}}{2}}{13} = \frac{\sin C}{8}$$

$$\therefore 4\sqrt{3} = 13\sin C$$

$$\therefore \sin C = \frac{4\sqrt{3}}{13}$$

$$\therefore C = \sin^{-1}\left(\tfrac{4\sqrt{3}}{13}\right)$$

$$\therefore C \approx 32.2°$$

and $A = 180° - B - C$ {angles in a triangle}

$$\therefore A \approx 180° - 60° - 32.2°$$

$$\therefore A \approx 87.8°$$

Also, using the sine rule:

$$\frac{\sin 60°}{13} = \frac{\sin C}{15}$$

$$\therefore \frac{\frac{\sqrt{3}}{2}}{13} = \frac{\sin C}{15} \quad \{\text{when } c = 15\}$$

$$\therefore \frac{15\sqrt{3}}{2} = 13\sin C$$

$$\therefore \sin C = \frac{15\sqrt{3}}{26}$$

$$\therefore C = \sin^{-1}\left(\tfrac{15\sqrt{3}}{26}\right)$$

$$\therefore C \approx 87.8°$$

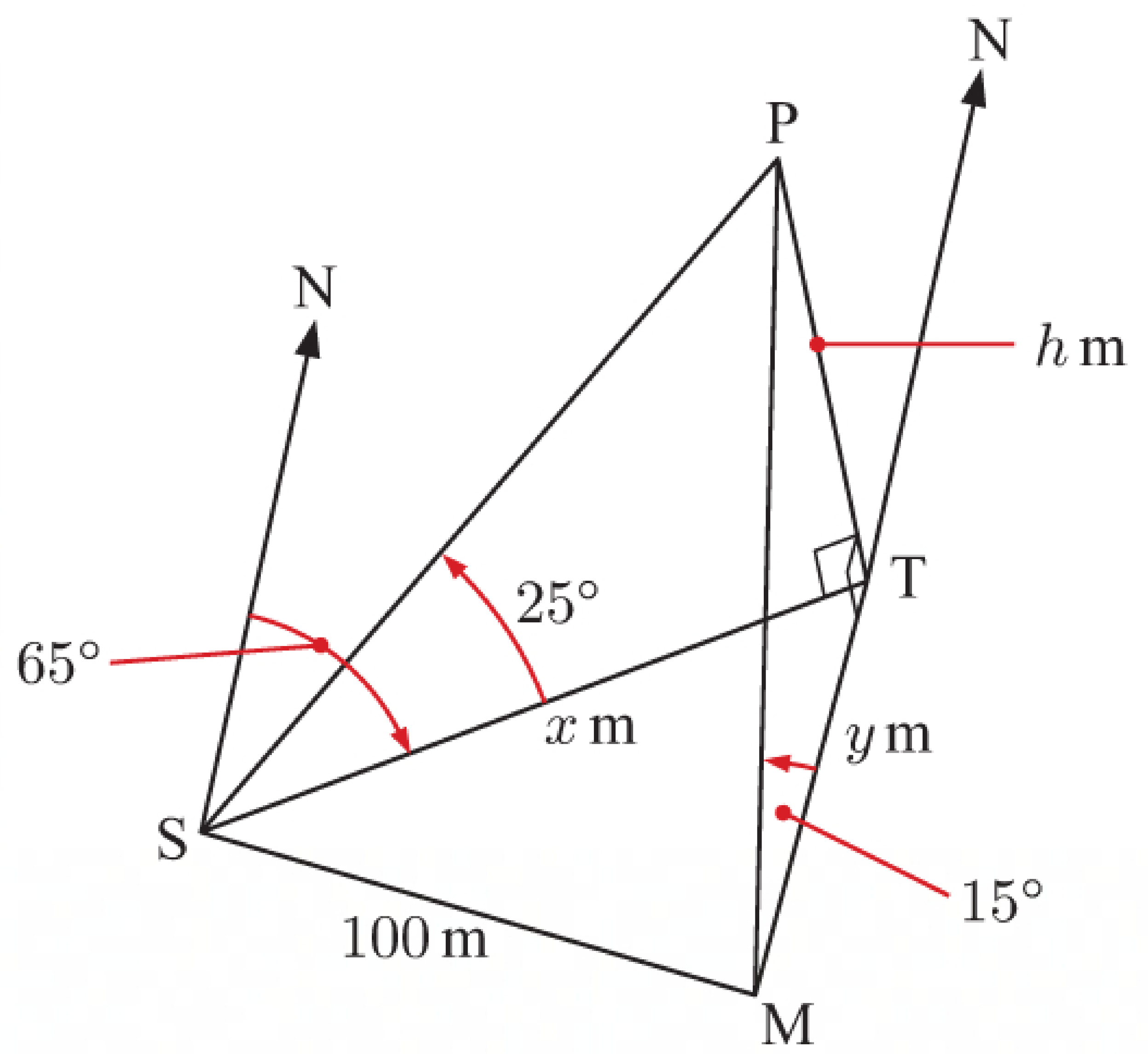
and $A = 180° - B - C$ {angles in a triangle}

$$\therefore A \approx 180° - 60° - 87.8°$$

$$\therefore A \approx 32.2°$$

So the remaining two angles of the garden are approximately 32.2° and 87.8°.

20 Suppose Sam and Markus are x m and y m from the tree respectively, and the tree is h m high.



In $\triangle$PST, $\tan 25° = \dfrac{h}{x}$

$$\therefore x = \frac{h}{\tan 25°} \approx 2.145h$$

In $\triangle$PMT, $\tan 15° = \dfrac{h}{y}$

$$\therefore y = \frac{h}{\tan 15°} \approx 3.732h$$

But $\widehat{STM} = 65°$ {equal alternate angles}

and $100^2 = x^2 + y^2 - 2xy\cos 65°$ {cosine rule}

$$\therefore 10\,000 \approx (2.145h)^2 + (3.732h)^2 - 2 \times (2.145)(3.732)h^2\cos 65°$$

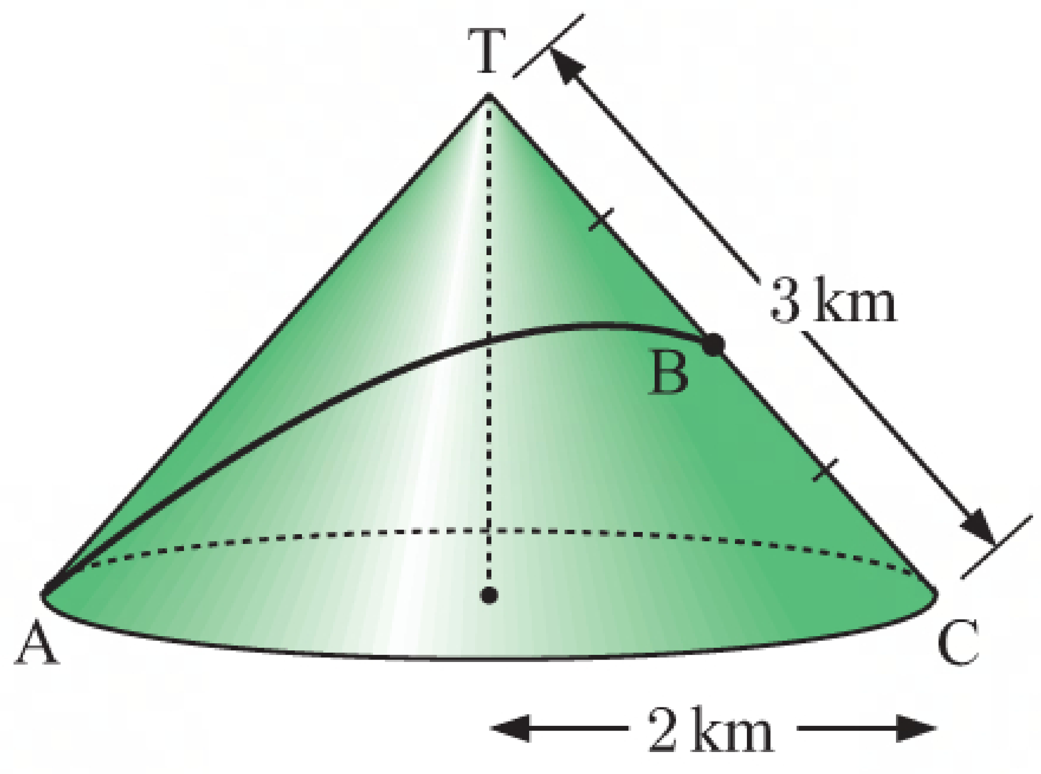
$$\therefore 10\,000 \approx 11.762\,h^2$$

$$\therefore h^2 \approx 850.17$$

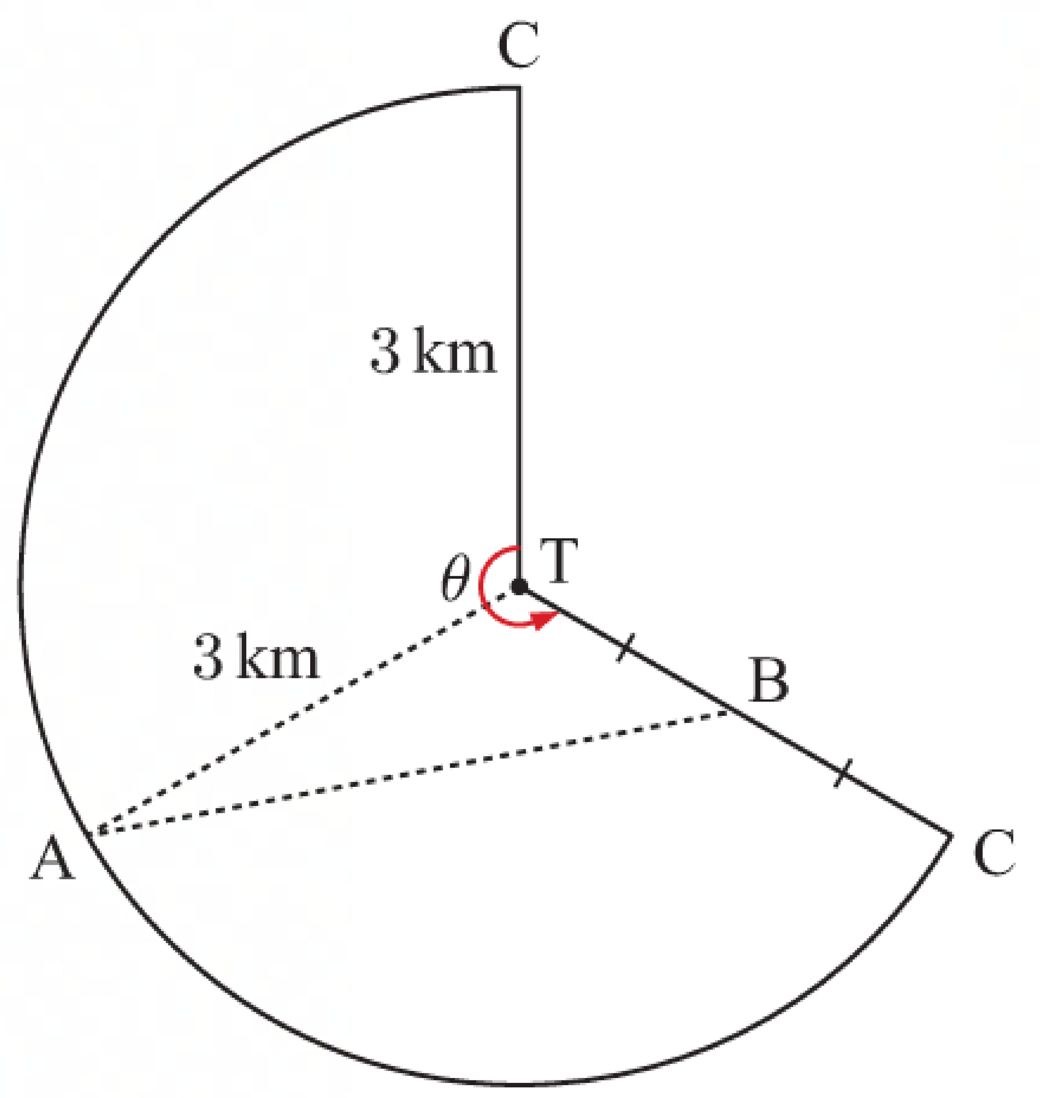
$$\therefore h \approx 29.2$$

So, the tree is about 29.2 m high.

21



a Consider the following net of the curved surface of the cone:



In order for AB to be as short as possible, [AB] must be a straight line.

The circumference of the cone's base is equal to the arc length of the sector.

$\therefore \ \theta \times 3 = 2\pi \times 2$

$\therefore \ 3\theta = 4\pi$

$\therefore \ \theta = \frac{4\pi}{3}$

Using the cosine rule in $\triangle$ABT,

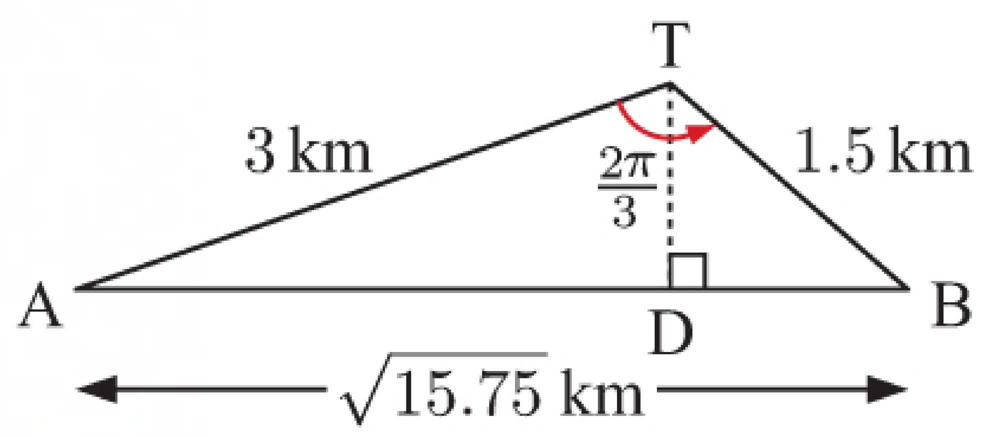
$$\text{AB}^2 = 3^2 + (\tfrac{3}{2})^2 - 2 \times 3 \times \tfrac{3}{2} \cos \tfrac{2\pi}{3}$$

$$\therefore \ \text{AB} = \sqrt{9 + \tfrac{9}{4} - 9(-\tfrac{1}{2})} \quad \{\text{as } \ \text{AB} > 0\}$$

$$\therefore \ \text{AB} = \sqrt{15.75} \approx 3.97 \text{ km}$$

The length of the path from A to B is approximately 3.97 km.

b The path from A to B is horizontal at the point D where [DT] is perpendicular to [AB].



Using the sine rule in $\triangle$TAB,

$$\frac{\sin \widehat{\text{ABT}}}{3} = \frac{\sin \frac{2\pi}{3}}{\sqrt{15.75}}$$

$$\therefore \ \widehat{\text{ABT}} = \sin^{-1}\left(\frac{3 \sin \frac{2\pi}{3}}{\sqrt{15.75}}\right)$$

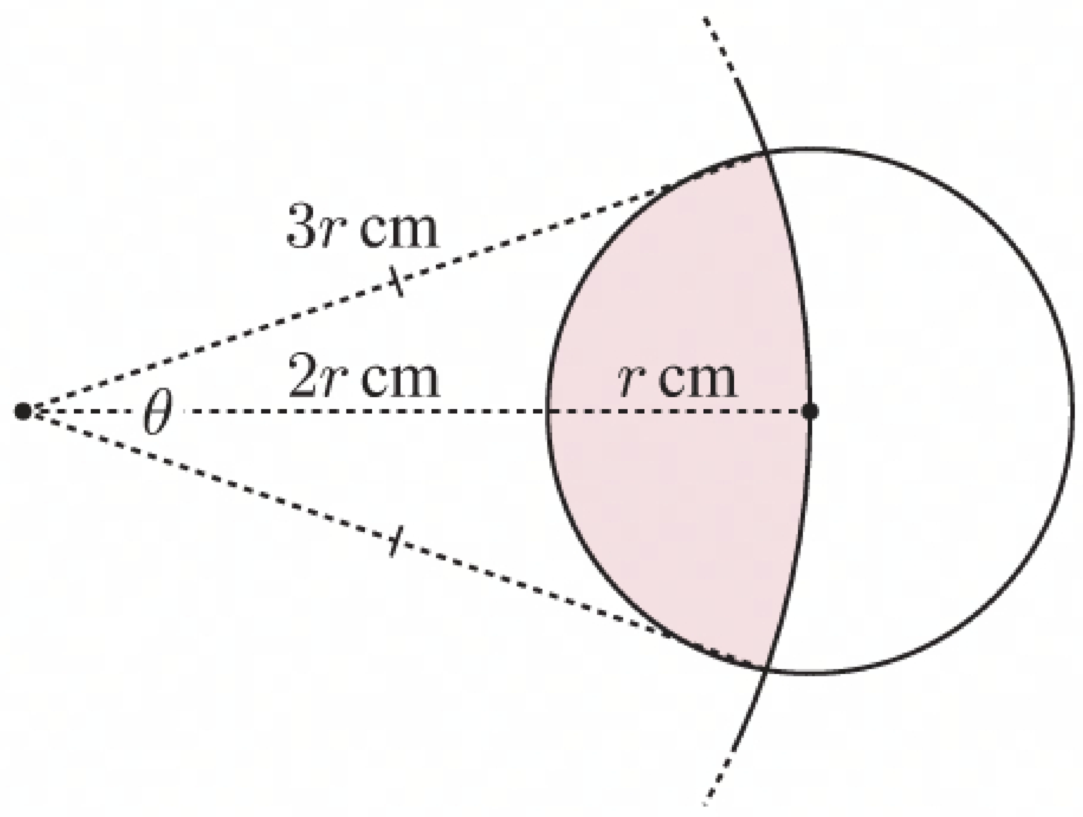
$$\therefore \ \widehat{\text{ABT}} \approx 40.89^\circ$$

Now in $\triangle$BDT, $\quad \cos 40.89^\circ \approx \dfrac{\text{BD}}{1.5}$

$$\therefore \ \text{BD} \approx 1.13 \text{ km}$$

The length of the part of the path from B to the point where the path is horizontal is approximately 1.13 km.

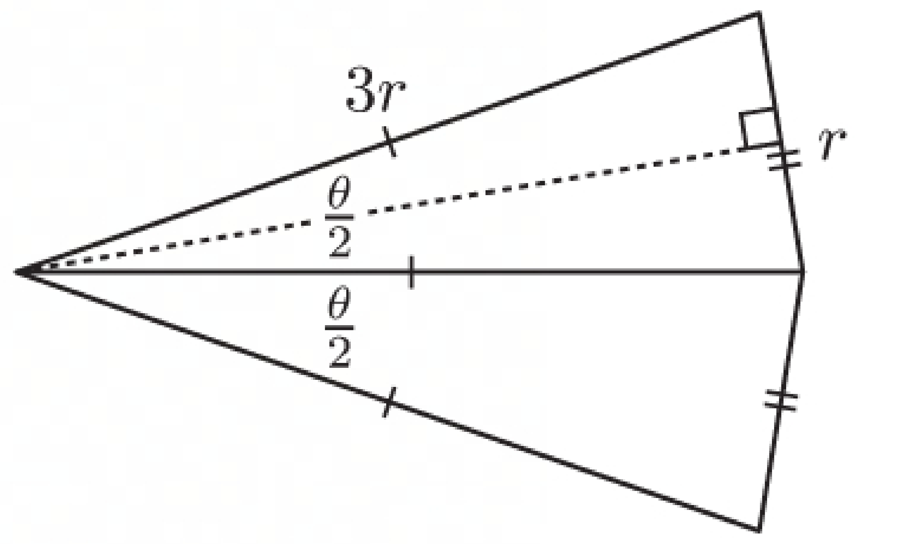
22 **a**



$$\sin \frac{\theta}{4} = \frac{\frac{r}{2}}{3r}$$

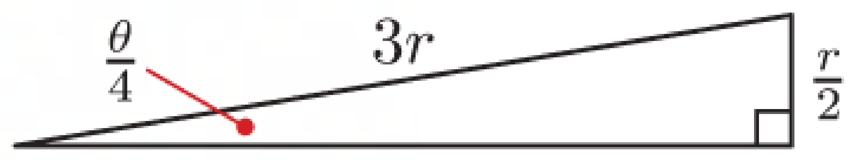
$$= \frac{r}{6r}$$

$$= \frac{1}{6}$$



$$\therefore \quad \frac{\theta}{4} = \sin^{-1}\left(\tfrac{1}{6}\right)$$

$$\therefore \quad \theta = 4\sin^{-1}\left(\tfrac{1}{6}\right)$$



b We can divide the figure into two parts, the sector of the larger circle, and the unshaded section of the smaller circle.

Area of sector of larger circle $= \frac{1}{2} \times \theta \times (3r)^2$ {radius $= 3r$ for larger circle}

$$= \tfrac{9}{2}\theta r^2$$

Area of unshaded section of smaller circle $= \pi r^2 -$ area of shaded section

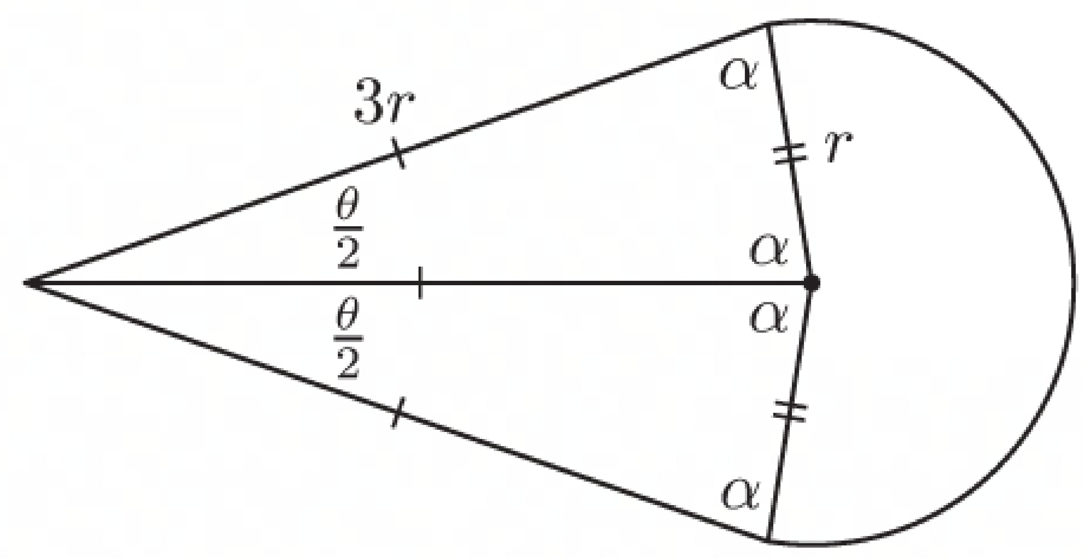
$$= \pi r^2 - \pi$$

Total area = area of sector of larger circle + area of unshaded section of smaller circle

$$= \tfrac{9}{2}\theta r^2 + \pi r^2 - \pi \quad \text{.... (1)}$$

The figure can also be divided into two isosceles triangles and a sector of the smaller circle as shown.

Let each of the base angles of the isosceles triangles be α.



$$\alpha + \alpha + \frac{\theta}{2} = \pi \quad \text{\{angles in a triangle\}}$$

$$\therefore \quad 2\alpha = \pi - \frac{\theta}{2}$$

Now, angle of sector of smaller circle $= 2\pi - 2\alpha$ {angles at a point}

$$= 2\pi - \left(\pi - \frac{\theta}{2}\right)$$

$$= \pi + \frac{\theta}{2}$$

Total area = area of two isosceles triangles + area of sector of smaller circle

$$= 2 \times \left(\tfrac{1}{2} \times 3r \times 3r \times \sin\tfrac{\theta}{2}\right) + \tfrac{1}{2} \times \left(\pi + \frac{\theta}{2}\right) \times r^2$$

$$= 9r^2 \sin\tfrac{\theta}{2} + \tfrac{1}{2}\pi r^2 + \tfrac{1}{4}\theta r^2 \quad \text{.... (2)}$$

Equating (1) and (2) gives $\frac{9}{2}\theta r^2 + \pi r^2 - \pi = 9r^2 \sin\frac{\theta}{2} + \frac{1}{2}\pi r^2 + \frac{1}{4}\theta r^2$

$$\therefore\ \tfrac{17}{4}\theta r^2 + \tfrac{1}{2}\pi r^2 - 9r^2\sin\tfrac{\theta}{2} = \pi$$

$$\therefore\ r^2\left(\tfrac{17}{4}\theta + \tfrac{\pi}{2} - 9\sin\tfrac{\theta}{2}\right) = \pi$$

$$\therefore\ r^2 = \frac{\pi}{\frac{17}{4}\theta + \frac{\pi}{2} - 9\sin\frac{\theta}{2}}$$

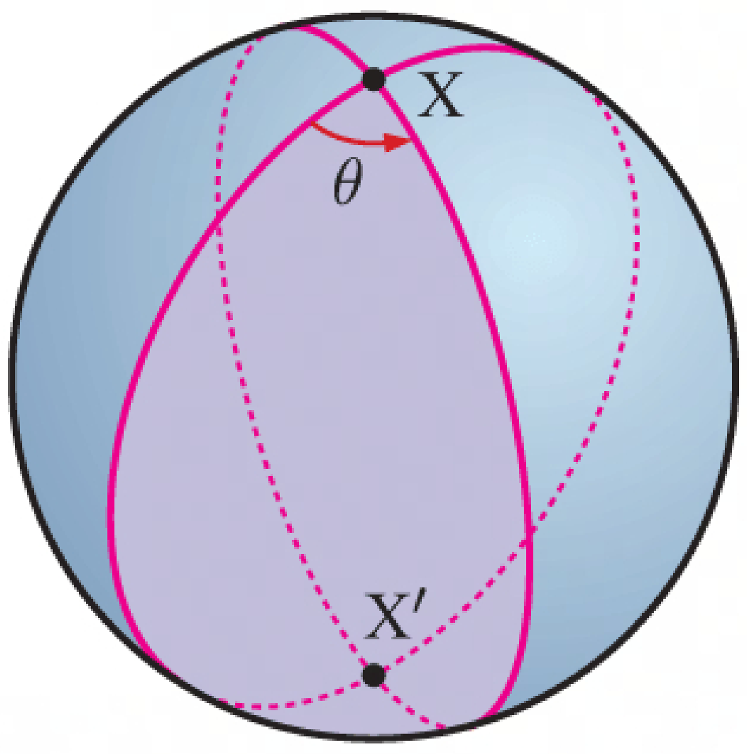
$$\therefore\ r = \sqrt{\frac{\pi}{\frac{17}{4}\theta + \frac{\pi}{2} - 9\sin\frac{\theta}{2}}} \qquad \{\text{as } r > 0\}$$

$$\therefore\ r \approx 1.467 \qquad \{\text{using } \theta = 4\sin^{-1}(\tfrac{1}{6}) \text{ from } \mathbf{a}\}$$

The radius of the smaller circle is approximately 1.467 cm.

ACTIVITY THE AREA OF A SPHERICAL TRIANGLE

1 The surface area of the lune is $\dfrac{\theta}{2\pi}$ of the total surface area of the sphere.

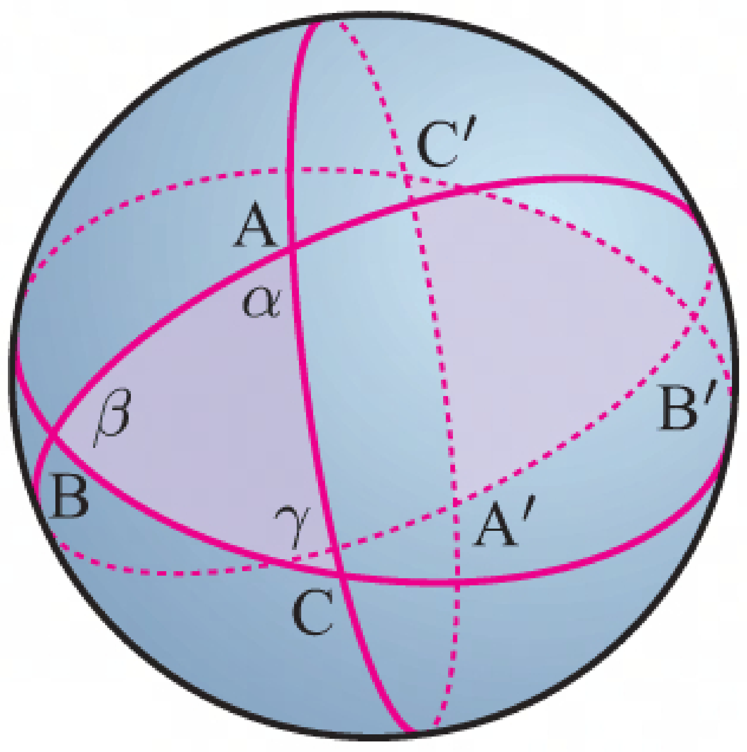


$$\therefore\ S_{X,\,\theta} = \frac{\theta}{2\pi} \times 4\pi r^2$$
$$= 2r^2\theta$$

2 **a** A′, B′, and C′ are diametrically opposite to A, B, and C respectively.

So, arc AB is diametrically opposite arc A′B′,
arc AC is diametrically opposite arc A′C′,
and arc BC is diametrically opposite arc B′C′.

Thus, AB = A′B′, AC = A′C′, and BC = B′C′ so that triangles ABC and A′B′C′ are congruent. {SSS}



b We can find the surface area of the sphere by adding the surface area of each lune twice, then subtracting the areas of the spherical triangles ABC and A′B′C′ that have been added more than once. Both triangles are counted 2 times more than necessary, so we subtract the area of $2 + 2 = 4$ spherical triangles.

$\therefore$ surface area of the sphere $= 4\pi r^2 = 2S_{A,\,\alpha} + 2S_{B,\,\beta} + 2S_{C,\,\gamma} - 4A$

c $S_{A,\,\alpha} = \dfrac{\alpha}{2\pi} \times 4\pi r^2 = 2\alpha r^2$ $\qquad S_{B,\,\beta} = \dfrac{\beta}{2\pi} \times 4\pi r^2 = 2\beta r^2$ $\qquad S_{C,\,\gamma} = \dfrac{\gamma}{2\pi} \times 4\pi r^2 = 2\gamma r^2$

$$\therefore\ 4\pi r^2 = 2S_{A,\,\alpha} + 2S_{B,\,\beta} + 2S_{C,\,\gamma} - 4A$$
$$\therefore\ 4\pi r^2 = 4\alpha r^2 + 4\beta r^2 + 4\gamma r^2 - 4A$$
$$\therefore\ 4A = 4\alpha r^2 + 4\beta r^2 + 4\gamma r^2 - 4\pi r^2$$
$$\therefore\ A = \alpha r^2 + \beta r^2 + \gamma r^2 - \pi r^2$$
$$\therefore\ A = (\alpha + \beta + \gamma - \pi)r^2$$

3 $A = (\alpha + \beta + \gamma - \pi)r^2 > 0$

$\therefore \quad \alpha + \beta + \gamma - \pi > 0 \quad \{A,\ r > 0\}$

$\therefore \quad \alpha + \beta + \gamma > \pi$

$\therefore$ the angle sum of a spherical triangle is greater than $180°$.

4 Suppose two spherical triangles on a given sphere are similar.

$\therefore$ the angles of the triangles must be the same.

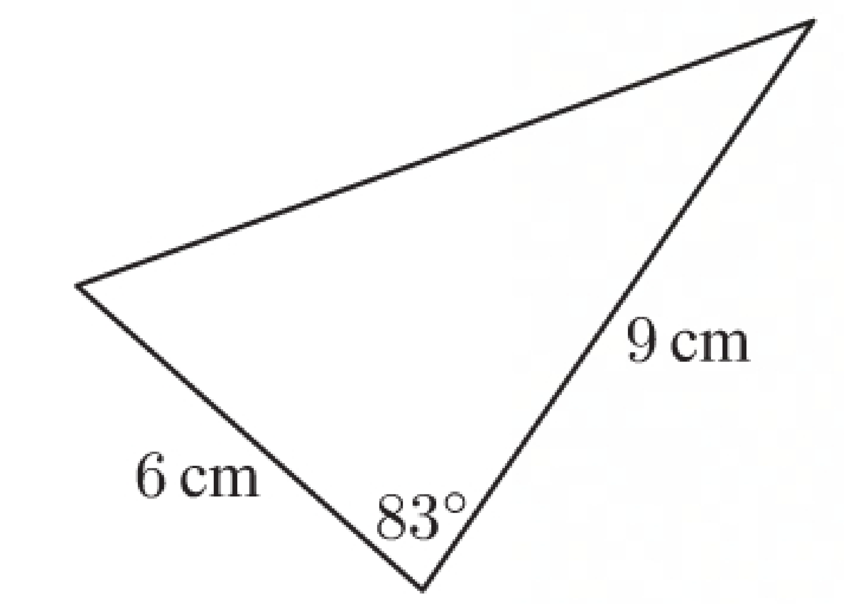
From **2 c**, two spherical triangles on a given sphere with the same angles must have the same area.

$\therefore$ the triangles have the same size and shape, and are therefore congruent.

It is therefore *not* possible for two spherical triangles on a given sphere to be similar but not congruent.

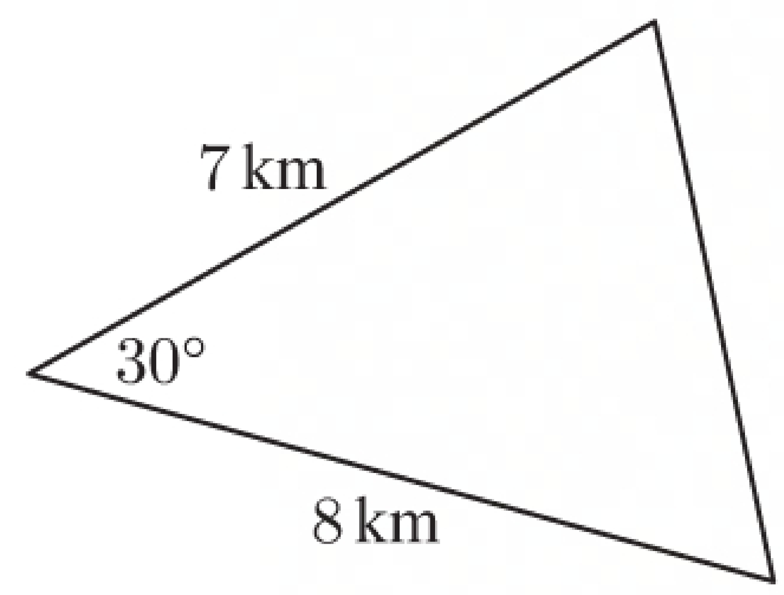
REVIEW SET 9A

1 **a**



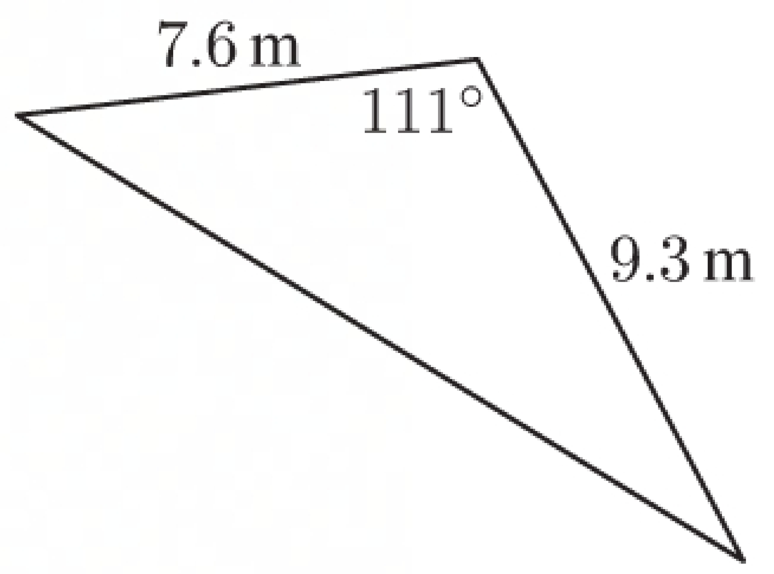
$$\text{Area} = \tfrac{1}{2} \times 6 \times 9 \times \sin 83°$$
$$\approx 26.8\ \text{cm}^2$$

b



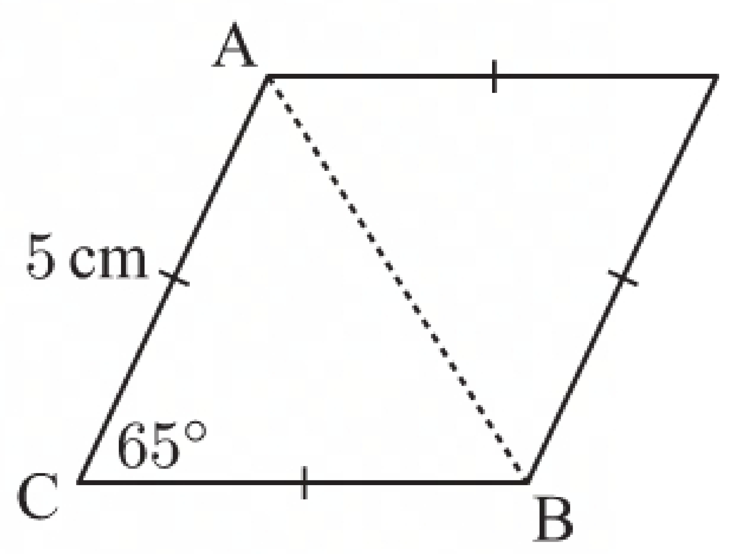
$$\text{Area} = \tfrac{1}{2} \times 7 \times 8 \times \sin 30°$$
$$= 28 \times \tfrac{1}{2}$$
$$= 14\ \text{km}^2$$

c



$$\text{Area} = \tfrac{1}{2} \times 7.6 \times 9.3 \times \sin 111°$$
$$\approx 33.0\ \text{m}^2$$

2



$$\text{Area} = 2 \times \text{area of } \triangle\text{ABC}$$
$$= 2 \times \tfrac{1}{2} \times 5 \times 5 \times \sin 65°$$
$$\approx 22.7\ \text{cm}^2$$

3 **a** Let the remaining side have length x cm.

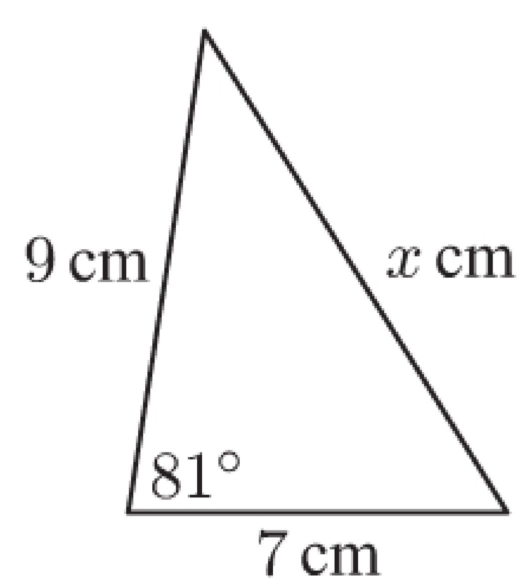
By the cosine rule:

$$x^2 = 9^2 + 7^2 - 2 \times 9 \times 7 \times \cos 81°$$

$\therefore \quad x = \sqrt{9^2 + 7^2 - 2 \times 9 \times 7 \times \cos 81°} \quad \{\text{as } x > 0\}$

$\therefore \quad x \approx 10.5$

The remaining side is about 10.5 cm in length.



b Let the remaining side have length x m.

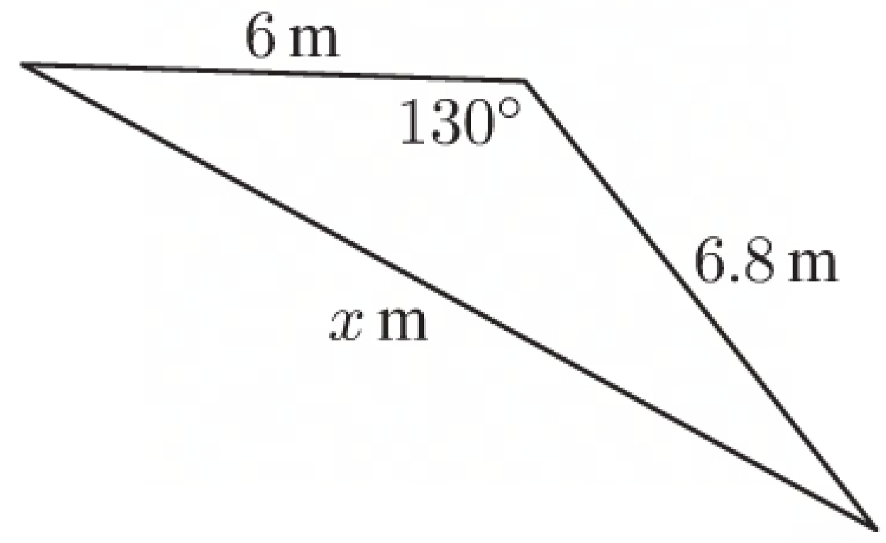
By the cosine rule:

$x^2 = 6^2 + 6.8^2 - 2 \times 6 \times 6.8 \times \cos 130°$

$\therefore \ x = \sqrt{6^2 + 6.8^2 - 2 \times 6 \times 6.8 \times \cos 130°}$ {as $x > 0$}

$\therefore \ x \approx 11.6$

The remaining side is about 11.6 m in length.

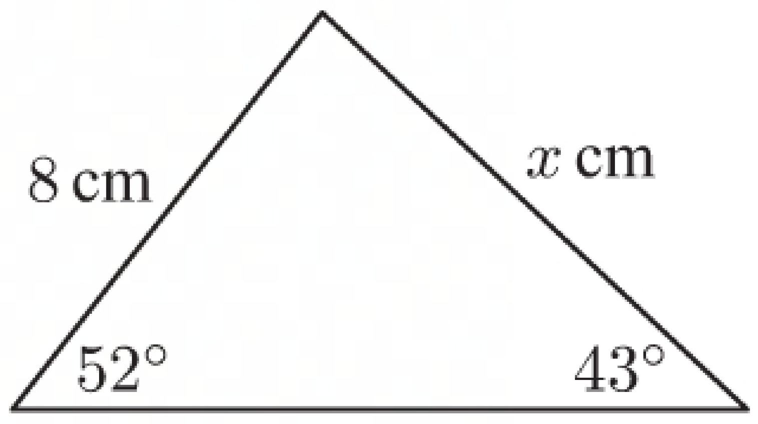


4 **a** Using the sine rule,

$$\frac{x}{\sin 52°} = \frac{8}{\sin 43°}$$

$$\therefore \ x = \frac{8 \times \sin 52°}{\sin 43°}$$

$$\therefore \ x \approx 9.24$$

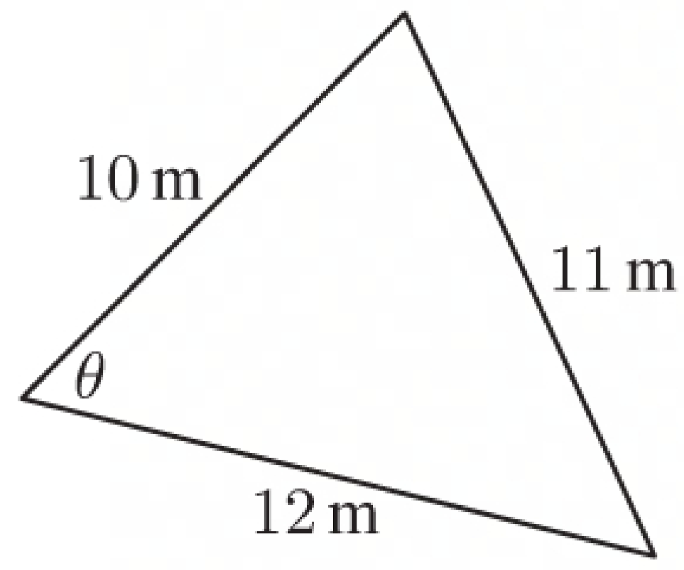


b Using the cosine rule, $\cos\theta = \dfrac{10^2 + 12^2 - 11^2}{2 \times 10 \times 12}$

$$\therefore \ \theta = \cos^{-1}\left(\frac{10^2 + 12^2 - 11^2}{2 \times 10 \times 12}\right)$$

$$\therefore \ \theta = \cos^{-1}\left(\tfrac{123}{240}\right)$$

$$\therefore \ \theta \approx 59.2°$$

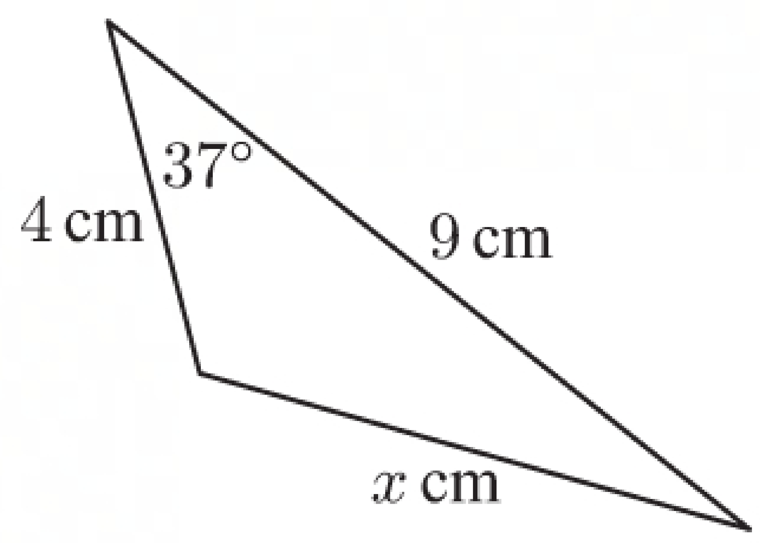


c Using the cosine rule,

$x^2 = 4^2 + 9^2 - 2 \times 4 \times 9 \times \cos 37°$

$\therefore \ x = \sqrt{4^2 + 9^2 - 2 \times 4 \times 9 \times \cos 37°}$ {as $x > 0$}

$\therefore \ x \approx 6.28$



5 By the cosine rule:

$DB^2 = 7^2 + 11^2 - 2 \times 7 \times 11 \times \cos 110°$

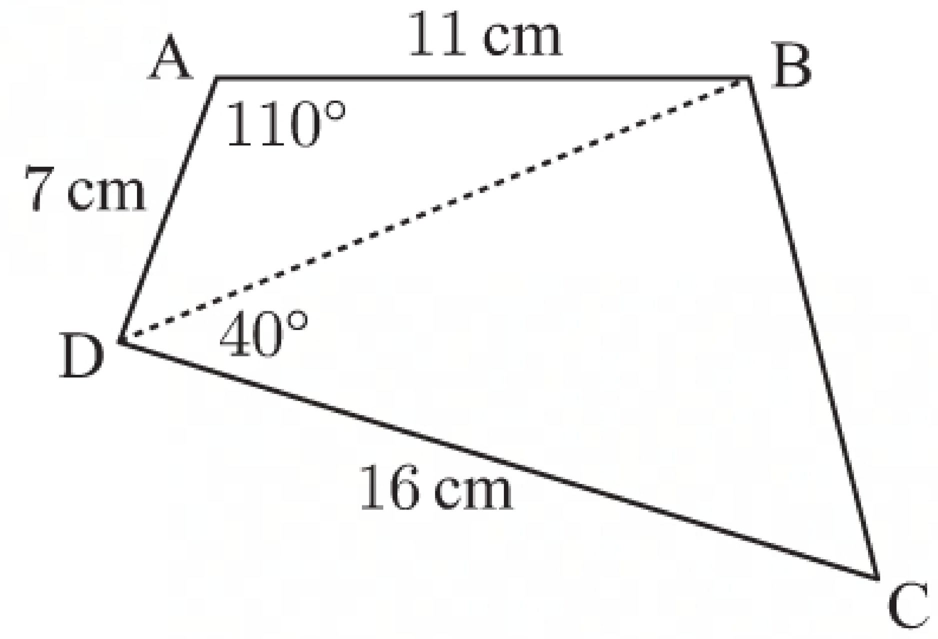
$\therefore \ DB = \sqrt{7^2 + 11^2 - 2 \times 7 \times 11 \times \cos 110°}$ {as $DB > 0$}

≈ 14.922 cm

Total area = area $\triangle$ABD + area $\triangle$BCD

$\approx \frac{1}{2} \times 7 \times 11 \times \sin 110° + \frac{1}{2} \times 16 \times 14.922 \times \sin 40°$

$\approx 113 \text{ cm}^2$

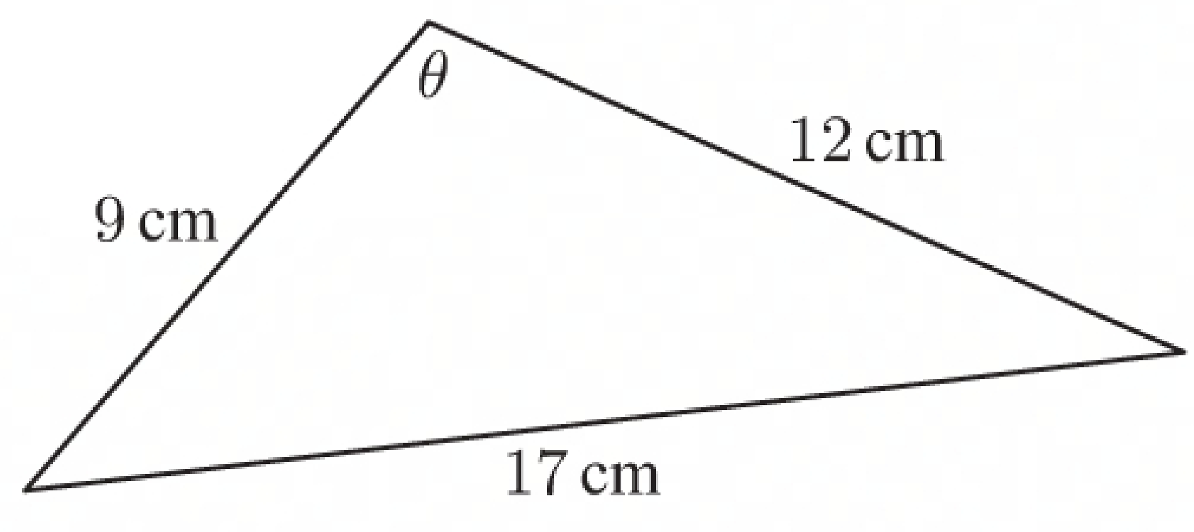


6 By the cosine rule: $\cos\theta = \dfrac{9^2 + 12^2 - 17^2}{2 \times 9 \times 12}$

$$\therefore \ \theta = \cos^{-1}\left(\frac{9^2 + 12^2 - 17^2}{2 \times 9 \times 12}\right)$$

$$\therefore \ \theta = \cos^{-1}\left(\tfrac{-64}{216}\right)$$

$$\therefore \ \theta \approx 107.2°$$



So, area of triangle $\approx \frac{1}{2} \times 9 \times 12 \times \sin 107.2°$

$\approx 51.6 \text{ cm}^2$

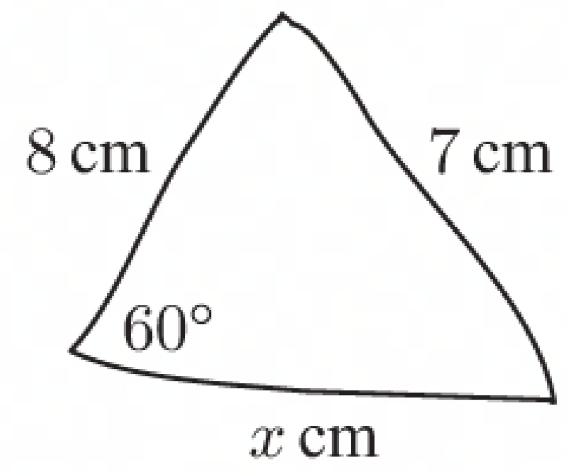
7 **a** By the cosine rule: $7^2 = x^2 + 8^2 - 2 \times x \times 8 \times \cos 60°$

$\therefore\ 49 = x^2 + 64 - 16x \times \frac{1}{2}$

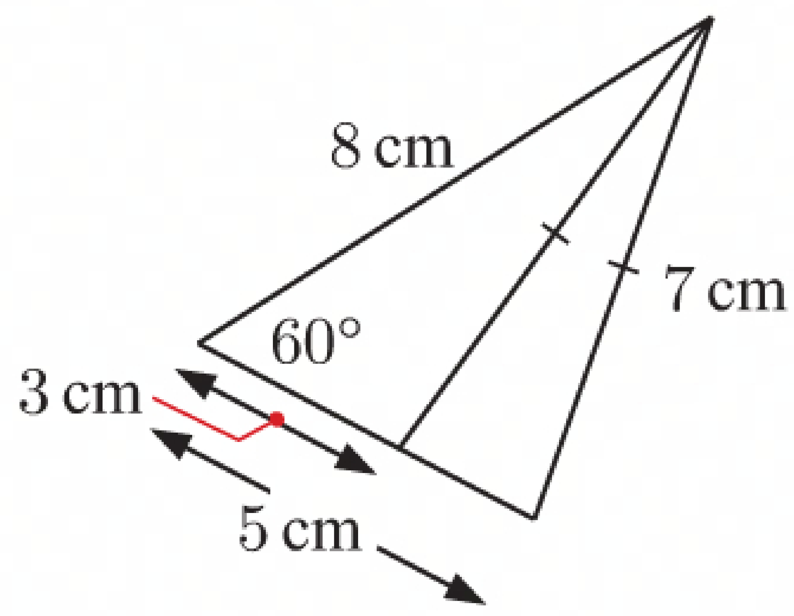
$\therefore\ x^2 - 8x + 15 = 0$

$\therefore\ (x-3)(x-5) = 0$

$\therefore\ x = 3$ or 5



b There are two possible values for x, so Kady can draw two triangles:



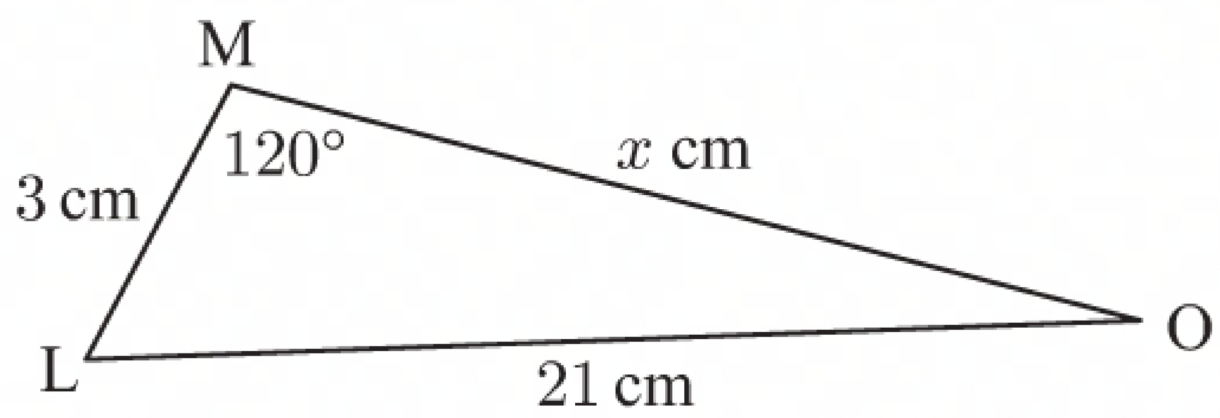
So, Kady's response should be that she needs more information to know which triangle to draw.

8 **a** By the cosine rule:

$21^2 = x^2 + 3^2 - 2 \times x \times 3 \times \cos 120°$

$\therefore\ 441 = x^2 + 9 - 6x \times (-\frac{1}{2})$

$\therefore\ x^2 + 3x - 432 = 0$



b $x = \dfrac{-3 \pm \sqrt{3^2 - 4(1)(-432)}}{2(1)}$

$= \dfrac{-3 \pm \sqrt{1737}}{2}$

$= -\frac{3}{2} \pm \frac{3\sqrt{193}}{2}$

But $x > 0$, so $x = -\frac{3}{2} + \frac{3\sqrt{193}}{2} \approx 19.3$

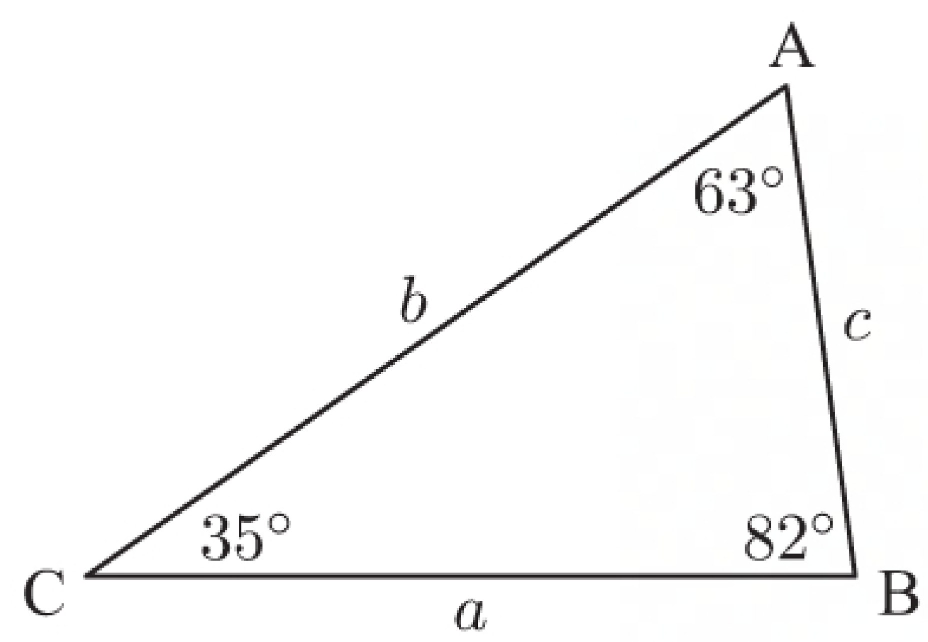
c Perimeter of triangle LMO $\approx 3 + 21 + 19.3$ cm

≈ 43.3 cm

9 Two of the angles are 35° and 82°.

$\therefore$ the third angle $= 180° - 35° - 82°$ {angles in a triangle}

$= 63°$



Area of triangle $= \frac{1}{2}ab\sin C$

$\therefore\ 40 = \frac{1}{2}ab\sin 35°$

$\therefore\ ab\sin 35° = 80$

$\therefore\ a = \dfrac{80}{b\sin 35°}$ .... (1)

Using the sine rule: $\dfrac{a}{\sin A} = \dfrac{b}{\sin B}$

$\therefore\ \dfrac{a}{\sin 63°} = \dfrac{b}{\sin 82°}$

$\therefore\ a = \dfrac{b\sin 63°}{\sin 82°}$ .... (2)

Equating (1) and (2) gives $\dfrac{80}{b\sin 35^\circ} = \dfrac{b\sin 63^\circ}{\sin 82^\circ}$

$$\therefore\ 80\sin 82^\circ = b^2 \sin 63^\circ \times \sin 35^\circ$$

$$\therefore\ b^2 = \frac{80\sin 82^\circ}{\sin 63^\circ \times \sin 35^\circ}$$

$$\therefore\ b = \sqrt{\frac{80\sin 82^\circ}{\sin 63^\circ \times \sin 35^\circ}} \qquad \{\text{as } b > 0\}$$

$$\therefore\ b \approx 12.5 \text{ cm}$$

$$\therefore\ a \approx \frac{80}{12.5\sin 35^\circ} \qquad \{\text{using (1)}\}$$

$$\therefore\ a \approx 11.2 \text{ cm}$$

Also, by the sine rule: $\dfrac{a}{\sin A} = \dfrac{c}{\sin C}$

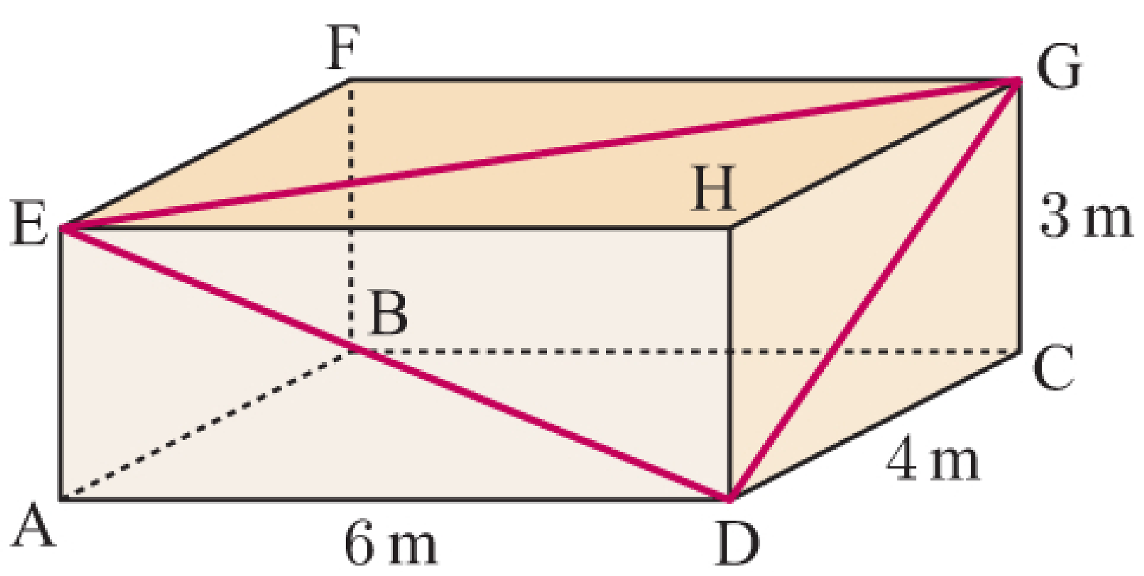
$$\therefore\ \frac{11.2}{\sin 63^\circ} \approx \frac{c}{\sin 35^\circ}$$

$$\therefore\ c \approx \frac{11.2\sin 35^\circ}{\sin 63^\circ}$$

$$\therefore\ c \approx 7.21 \text{ cm}$$

So, the triangle has sides of length ≈ 7.21 cm, ≈ 11.2 cm, and ≈ 12.5 cm.

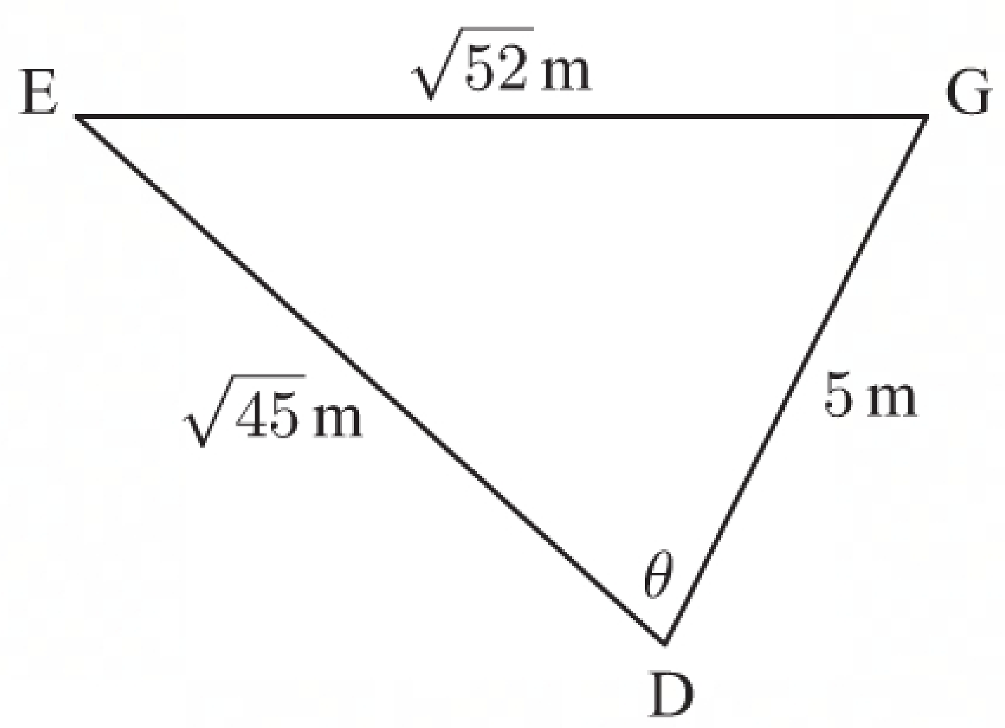
10



In $\triangle$ADE, $\text{DE} = \sqrt{6^2 + 3^2} = \sqrt{45}$ m. {Pythagoras}

In $\triangle$CDG, $\text{DG} = \sqrt{4^2 + 3^2} = \sqrt{25} = 5$ m. {Pythagoras}

In $\triangle$EGH, $\text{EG} = \sqrt{6^2 + 4^2} = \sqrt{52}$ m. {Pythagoras}



By rearrangement of the cosine rule,

$$\cos\theta = \frac{(\sqrt{45})^2 + 5^2 - (\sqrt{52})^2}{2\sqrt{45} \times 5}$$

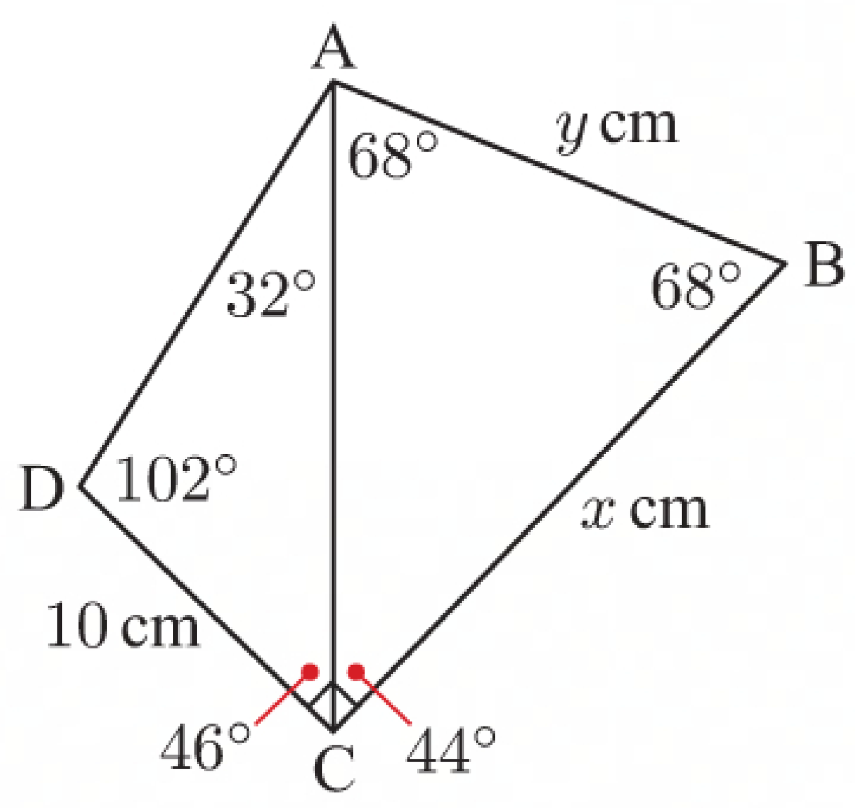
$$= \frac{45 + 25 - 52}{10\sqrt{45}}$$

$$= \frac{18}{10\sqrt{45}}$$

$$\therefore\ \theta = \cos^{-1}\left(\frac{18}{10\sqrt{45}}\right) \approx 74.4^\circ$$

$\therefore$ $\text{E}\widehat{\text{D}}\text{G}$ measures about 74.4°.

11



Using the sine rule in $\triangle ACD$, $\dfrac{AC}{\sin 102°} = \dfrac{10}{\sin 32°}$

$\therefore\ AC = \dfrac{10 \times \sin 102°}{\sin 32°}$

$\therefore\ AC \approx 18.5$ cm

$A\widehat{C}D = 180° - 32° - 102°$ {angles in a triangle}

$= 46°$

$\therefore\ B\widehat{C}A = 90° - 46°$

$= 44°$

$\therefore\ B\widehat{A}C = 180° - 44° - 68°$ {angles in a triangle}

$= 68°$

Now, $B\widehat{A}C = A\widehat{B}C = 68°$

$\therefore\ \triangle ABC$ is isosceles with $AC = BC$

$\therefore\ BC \approx 18.5$ cm

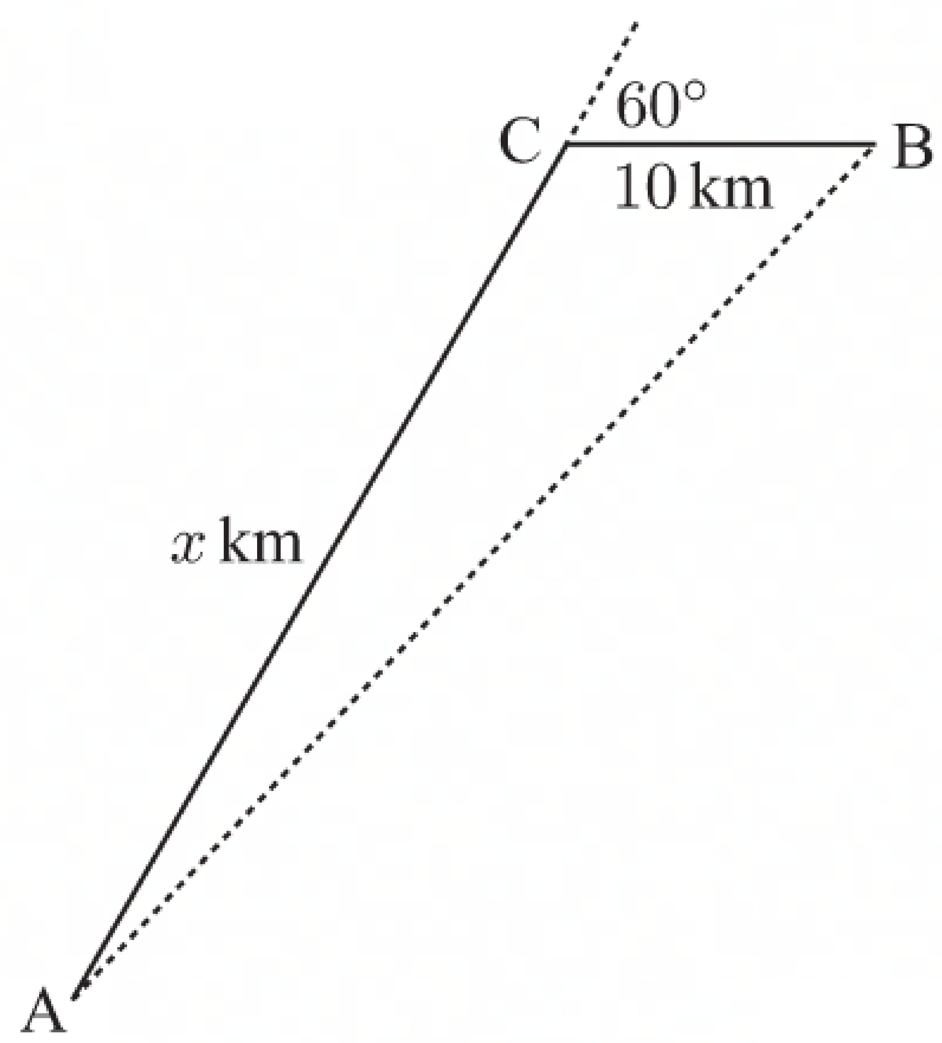
$\therefore\ x \approx 18.5$

Using the cosine rule in $\triangle ABC$, $y^2 = x^2 + x^2 - 2 \times x \times x \times \cos 44°$

$\therefore\ y \approx \sqrt{18.5^2 + 18.5^2 - 2 \times 18.5^2 \times \cos 44°}$ {as $y > 0$}

$\therefore\ y \approx 13.8$

12 Let AC be x km.



Now $AC + CB = x + 10$

and $AB = x + 10 - 4$

$= x + 6$ km

$A\widehat{C}B = 180° - 60°$ {angles on a line}

$= 120°$

By the cosine rule:

$(x+6)^2 = x^2 + 10^2 - 2 \times x \times 10 \times \cos 120°$

$\therefore\ x^2 + 12x + 36 = x^2 + 100 - 20x \times (-\frac{1}{2})$

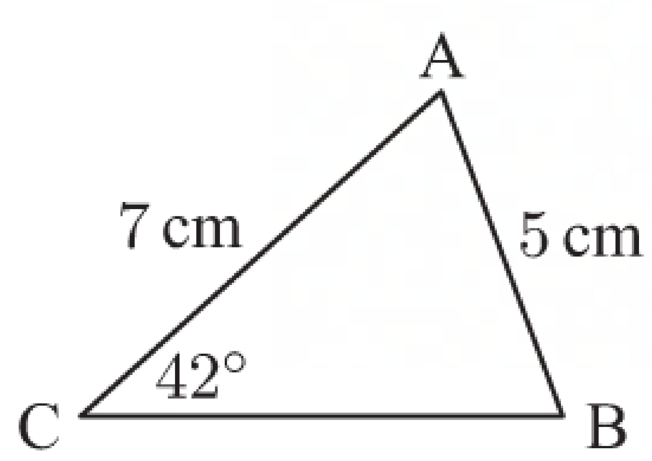
$\therefore\ 12x + 36 = 100 + 10x$

$\therefore\ 2x = 64$

$\therefore\ x = 32$

So, the boat travelled $32 + 10 = 42$ km.

13 **a** Using the sine rule,



$\dfrac{\sin A\widehat{B}C}{7} = \dfrac{\sin 42°}{5}$

$\therefore\ \sin A\widehat{B}C = \dfrac{7 \times \sin 42°}{5}$

$\therefore\ A\widehat{B}C = \sin^{-1}\left(\dfrac{7 \times \sin 42°}{5}\right)$ or its supplement

$\therefore\ A\widehat{B}C \approx 69.52°$ or $180° - 69.52°$

$\therefore\ A\widehat{B}C \approx 69.5°$ or $110.5°$

b For $\widehat{ABC} \approx 69.5°$, $\widehat{CAB} \approx 180° - 42° - 69.5°$ {angles in a triangle}

$\approx 68.5°$

area of $\triangle ABC \approx \frac{1}{2} \times 7 \times 5 \times \sin 68.5°$

$\approx 16.3 \text{ cm}^2$

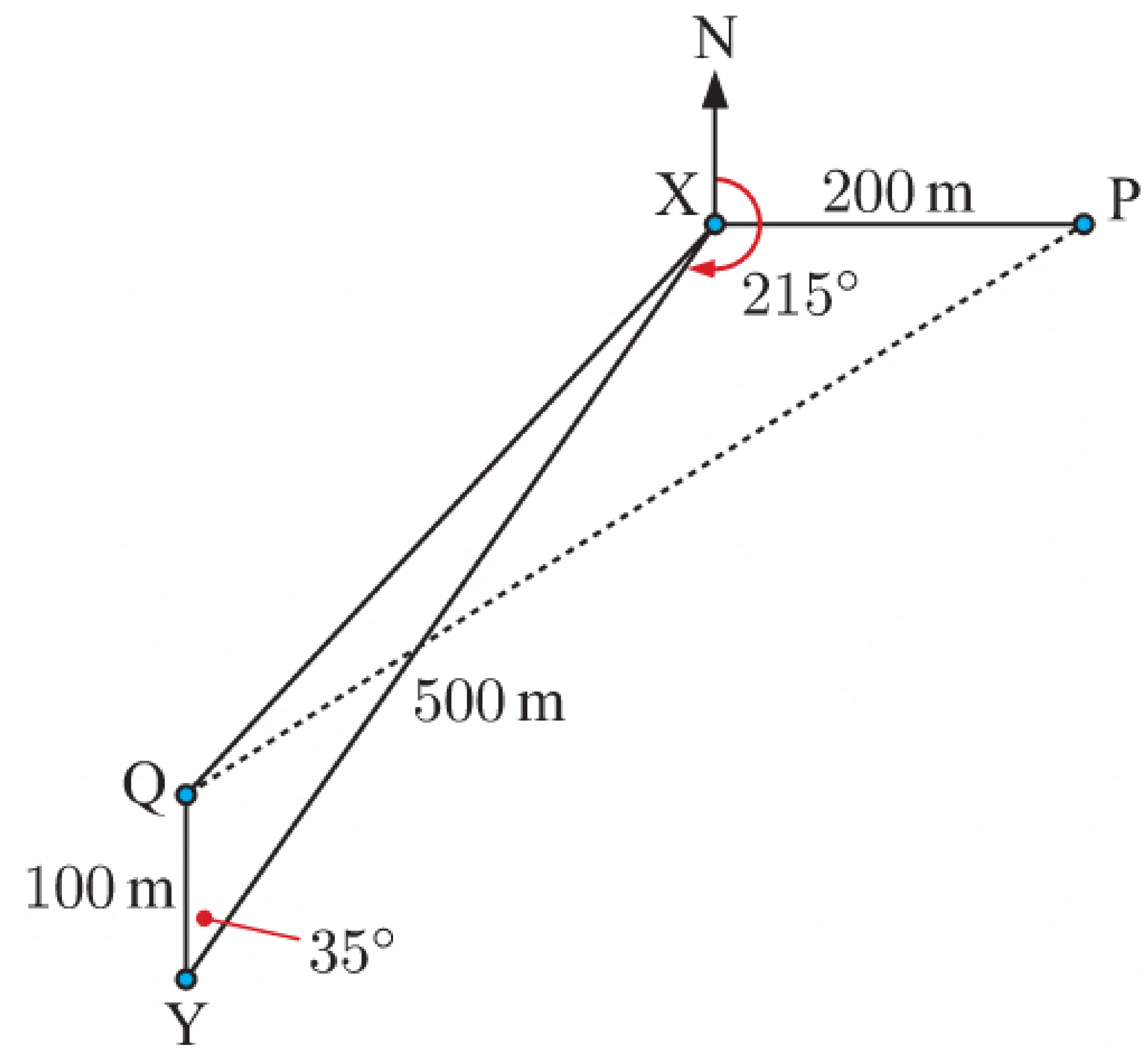
For $\widehat{ABC} \approx 110.5°$, $\widehat{CAB} \approx 180° - 42° - 110.5°$ {angles in a triangle}

$\approx 27.5°$

area of $\triangle ABC \approx \frac{1}{2} \times 7 \times 5 \times \sin 27.5°$

$\approx 8.09 \text{ cm}^2$

14 Suppose dune buggy X travels to P, and dune buggy Y travels to Q.



$\widehat{NXY} = 360° - 215°$ {angles at a point}

$= 145°$

$\widehat{QYX} = 180° - 145°$ {co-interior angles}

$= 35°$

Using the cosine rule in $\triangle QXY$,

$QX^2 = 100^2 + 500^2 - 2 \times 100 \times 500 \times \cos 35°$

$\therefore \quad QX = \sqrt{100^2 + 500^2 - 2 \times 100 \times 500 \times \cos 35°}$

$\therefore \quad QX \approx 422.0 \text{ m}$

Using the sine rule in $\triangle QXY$, $\dfrac{\sin \widehat{QXY}}{100} = \dfrac{\sin 35°}{QX}$

$\therefore \quad \sin \widehat{QXY} \approx \dfrac{100 \sin 35°}{422.0}$

$\therefore \quad \widehat{QXY} \approx \sin^{-1}\left(\dfrac{100 \sin 35°}{422.0}\right)$

$\therefore \quad \widehat{QXY} \approx 7.81°$

Now, $\widehat{PXQ} = 215° - 90° + \widehat{QXY}$

$\approx 215° - 90° + 7.81°$

$\approx 132.81°$

Using the cosine rule in $\triangle PXQ$,

$PQ^2 = 200^2 + QX^2 - 2 \times 200 \times QX \times \cos \widehat{PXQ}$

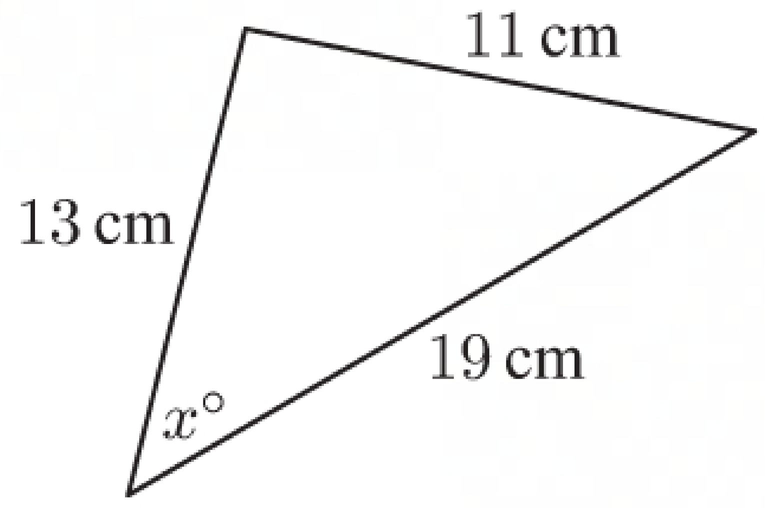
$\therefore \quad PQ \approx \sqrt{200^2 + (422.0)^2 - 2 \times 200 \times 422.0 \times \cos 132.81°}$

$\therefore \quad PQ \approx 576.9 \text{ m}$

The dune buggies X and Y are now approximately 577 m apart.

REVIEW SET 9B

1 **a**



By the cosine rule:

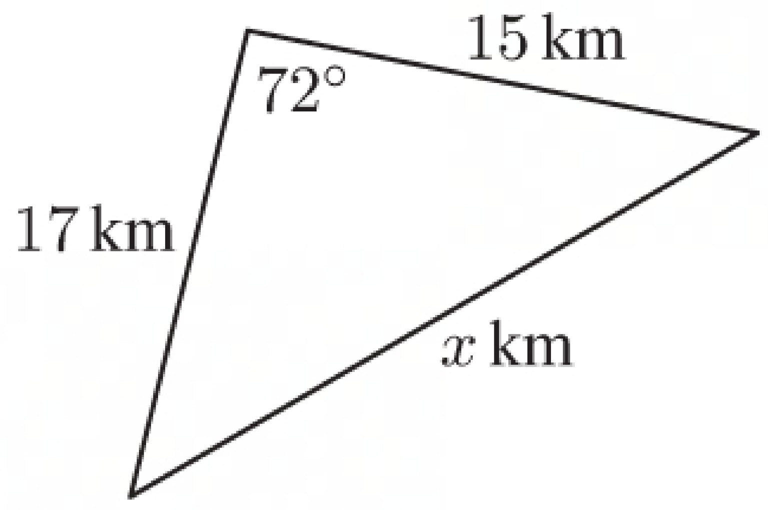
$$\cos x° = \frac{13^2 + 19^2 - 11^2}{2 \times 13 \times 19}$$

$$\therefore \ x° = \cos^{-1}\left(\frac{13^2 + 19^2 - 11^2}{2 \times 13 \times 19}\right)$$

$$\therefore \ x° = \cos^{-1}\left(\tfrac{409}{494}\right)$$

$$\therefore \ x \approx 34.1$$

b



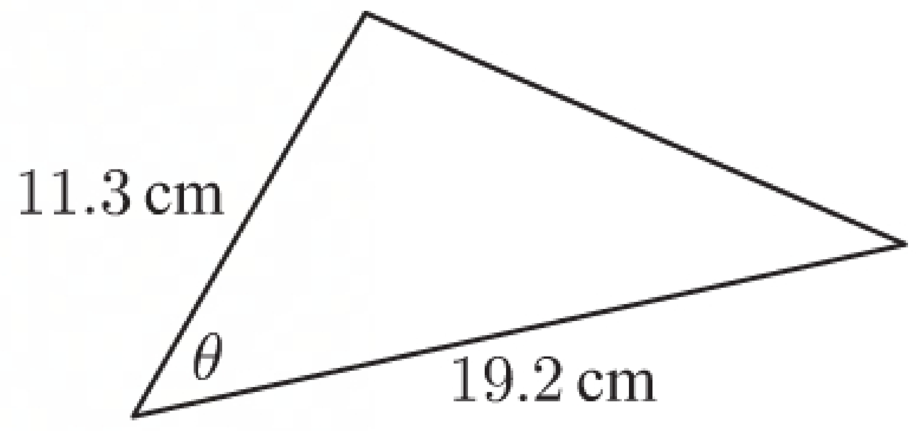
By the cosine rule:

$$x^2 = 15^2 + 17^2 - 2 \times 15 \times 17 \times \cos 72°$$

$$\therefore \ x = \sqrt{15^2 + 17^2 - 2 \times 15 \times 17 \times \cos 72°}$$

$$\therefore \ x \approx 18.9$$

2 Let the included angle be θ.



$$\text{area} = 80 \text{ cm}^2$$

$$\therefore \ \tfrac{1}{2} \times 11.3 \times 19.2 \times \sin\theta = 80$$

$$\therefore \ \sin\theta = \frac{160}{11.3 \times 19.2}$$

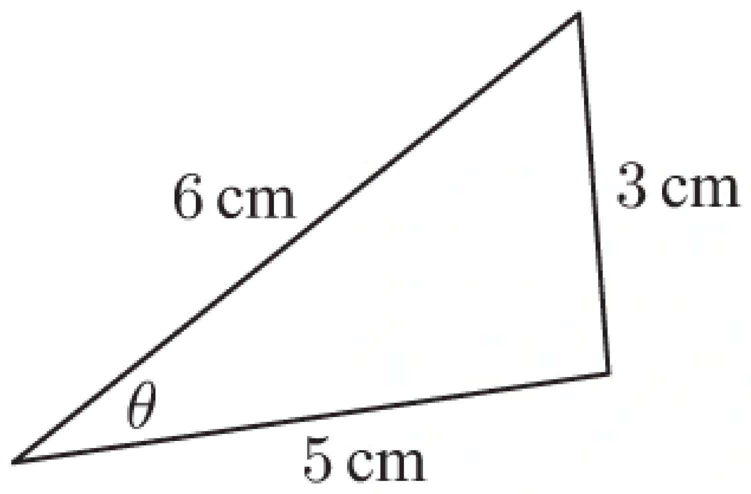
$$\therefore \ \theta = \sin^{-1}\left(\frac{160}{11.3 \times 19.2}\right) \text{ or its supplement}$$

$$\therefore \ \theta \approx 47.5° \text{ or } 180° - 47.5°$$

$$\therefore \ \theta \approx 47.5° \text{ or } 132.5°$$

So, the included angle is either $\approx 47.5°$ or $132.5°$.

3 **a**



Using the cosine rule:

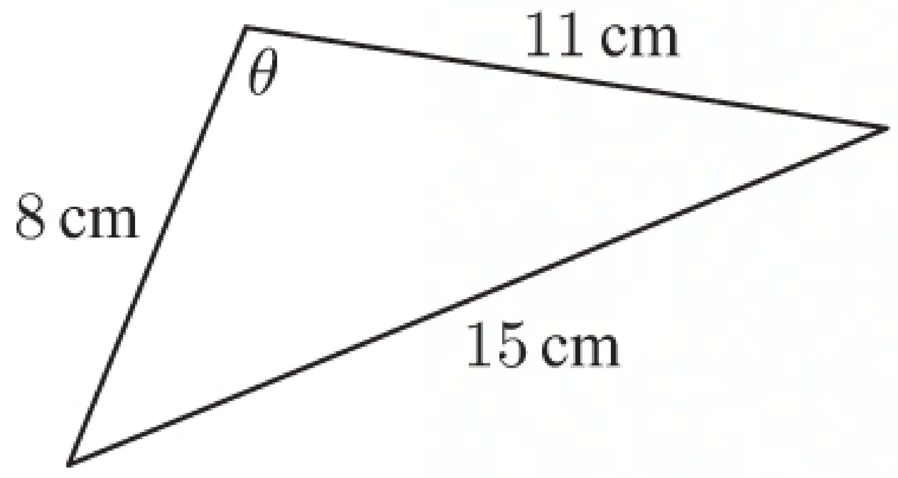
$$\cos\theta = \frac{5^2 + 6^2 - 3^2}{2 \times 5 \times 6}$$

$$\therefore \ \theta = \cos^{-1}\left(\frac{5^2 + 6^2 - 3^2}{2 \times 5 \times 6}\right)$$

$$\therefore \ \theta = \cos^{-1}\left(\tfrac{52}{60}\right)$$

$$\therefore \ \theta \approx 29.9°$$

b



Using the cosine rule:

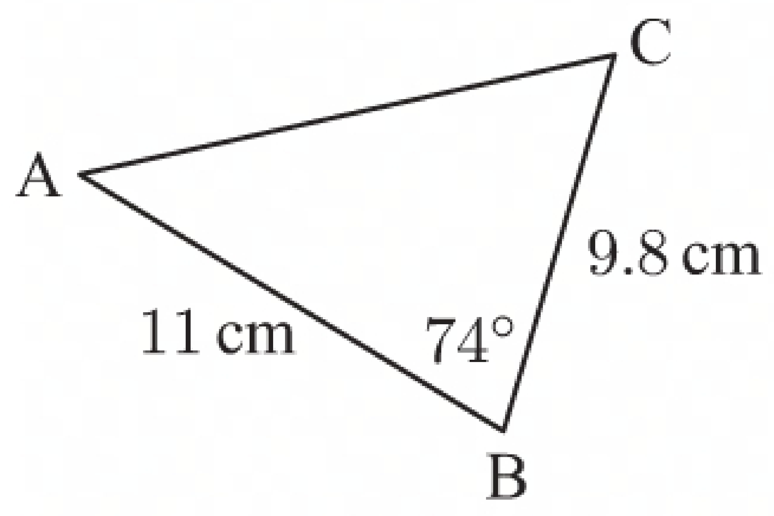
$$\cos\theta = \frac{8^2 + 11^2 - 15^2}{2 \times 8 \times 11}$$

$$\therefore \ \theta = \cos^{-1}\left(\frac{8^2 + 11^2 - 15^2}{2 \times 8 \times 11}\right)$$

$$\therefore \ \theta = \cos^{-1}\left(\tfrac{-40}{176}\right)$$

$$\therefore \ \theta \approx 103°$$

4 **a**



Using the cosine rule:

$$AC^2 = 11^2 + 9.8^2 - 2 \times 11 \times 9.8 \times \cos 74°$$

$$\therefore\ AC = \sqrt{11^2 + 9.8^2 - 2 \times 11 \times 9.8 \times \cos 74°}$$

$$\therefore\ AC \approx 12.554 \text{ cm}$$

$$\therefore\ AC \approx 12.6 \text{ cm}$$

Using the sine rule, $\dfrac{\sin A\widehat{C}B}{11} = \dfrac{\sin 74°}{AC}$

$$\therefore\ \sin A\widehat{C}B \approx \frac{11 \times \sin 74°}{12.554}$$

$$\therefore\ A\widehat{C}B \approx \sin^{-1}\left(\frac{11 \times \sin 74°}{12.554}\right) \text{ or its supplement}$$

$$\therefore\ A\widehat{C}B \approx 57.4° \text{ or } 180° - 57.4°$$

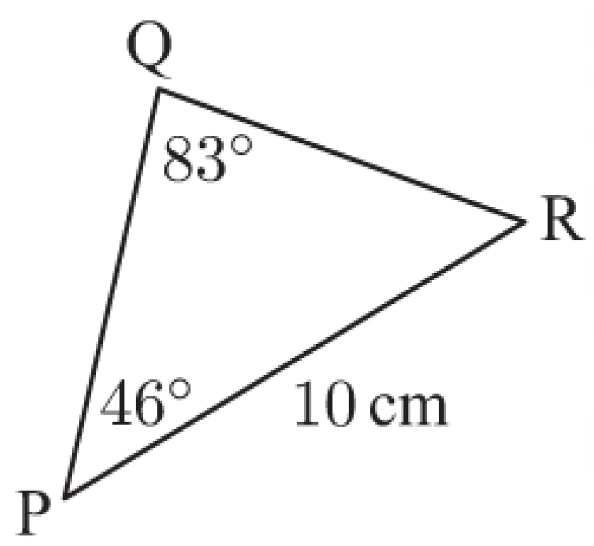
$$\therefore\ A\widehat{C}B \approx 57.4° \text{ or } 122.6°$$

impossible as $122.6° + 74° > 180°$

$\therefore$ $A\widehat{C}B$ measures about $57.4°$

and $B\widehat{A}C$ measures $180° - 74° - 57.4° \approx 48.6°$.

b



$P\widehat{R}Q = 180° - 46° - 83°$ {angles in a triangle}

$= 51°$

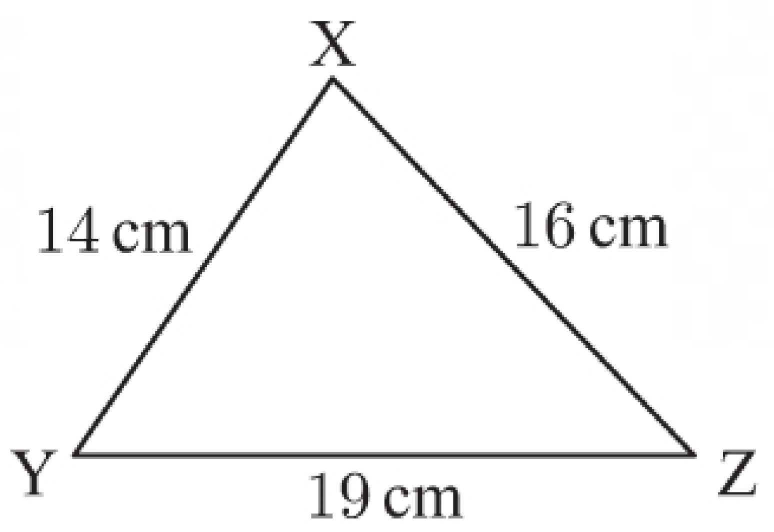
Using the sine rule,

$$\frac{PQ}{\sin 51°} = \frac{10}{\sin 83°} \quad \text{and} \quad \frac{QR}{\sin 46°} = \frac{10}{\sin 83°}$$

$$\therefore\ PQ = \frac{10 \times \sin 51°}{\sin 83°} \qquad \therefore\ QR = \frac{10 \times \sin 46°}{\sin 83°}$$

$$\therefore\ PQ \approx 7.83 \text{ cm} \qquad \therefore\ QR \approx 7.25 \text{ cm}$$

c



Using the cosine rule:

$$\cos Y\widehat{X}Z = \frac{14^2 + 16^2 - 19^2}{2 \times 14 \times 16} \qquad \cos X\widehat{Y}Z = \frac{14^2 + 19^2 - 16^2}{2 \times 14 \times 19}$$

$$\therefore\ Y\widehat{X}Z = \cos^{-1}\left(\frac{14^2 + 16^2 - 19^2}{2 \times 14 \times 16}\right) \qquad \therefore\ X\widehat{Y}Z = \cos^{-1}\left(\frac{14^2 + 19^2 - 16^2}{2 \times 14 \times 19}\right)$$

$$\therefore\ Y\widehat{X}Z = \cos^{-1}\left(\tfrac{91}{448}\right) \qquad \therefore\ X\widehat{Y}Z = \cos^{-1}\left(\tfrac{301}{532}\right)$$

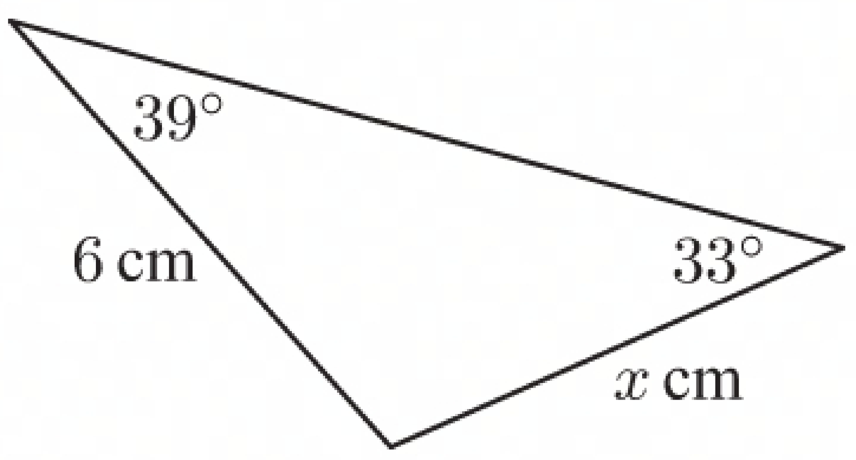
$$\therefore\ Y\widehat{X}Z \approx 78.3° \qquad \therefore\ X\widehat{Y}Z \approx 55.5°$$

$X\widehat{Z}Y = 180° - Y\widehat{X}Z - X\widehat{Y}Z$ {angles in a triangle}

$\approx 180° - 78.3° - 55.5°$

$\therefore\ X\widehat{Z}Y \approx 46.2°$

5 **a**



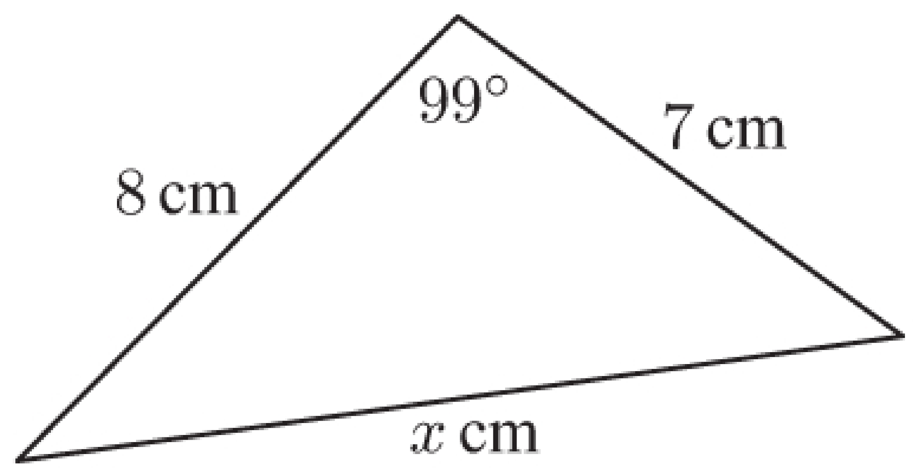
Using the sine rule,

$$\frac{x}{\sin 39°} = \frac{6}{\sin 33°}$$

$$\therefore \quad x = \frac{6 \times \sin 39°}{\sin 33°}$$

$$\therefore \quad x \approx 6.93$$

b



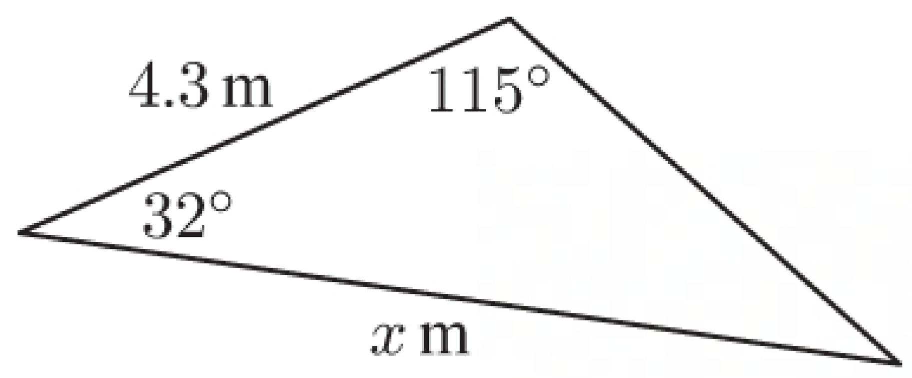
Using the cosine rule,

$$x^2 = 8^2 + 7^2 - 2 \times 8 \times 7 \times \cos 99°$$

$$\therefore \quad x = \sqrt{8^2 + 7^2 - 2 \times 8 \times 7 \times \cos 99°}$$

$$\therefore \quad x \approx 11.4$$

c



The unknown angle is

$180° - 32° - 115°$ {angles in a triangle}

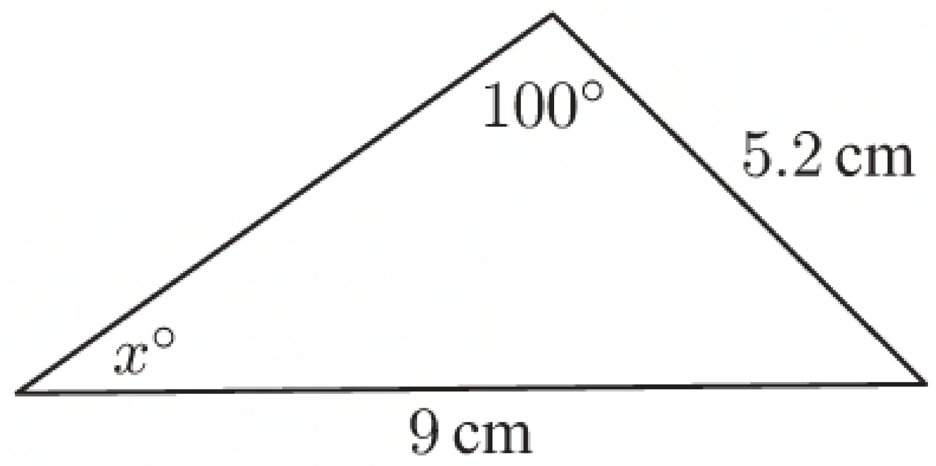
$= 33°$

Using the sine rule, $\dfrac{x}{\sin 115°} = \dfrac{4.3}{\sin 33°}$

$$\therefore \quad x = \frac{4.3 \times \sin 115°}{\sin 33°}$$

$$\therefore \quad x \approx 7.16$$

d



Using the sine rule,

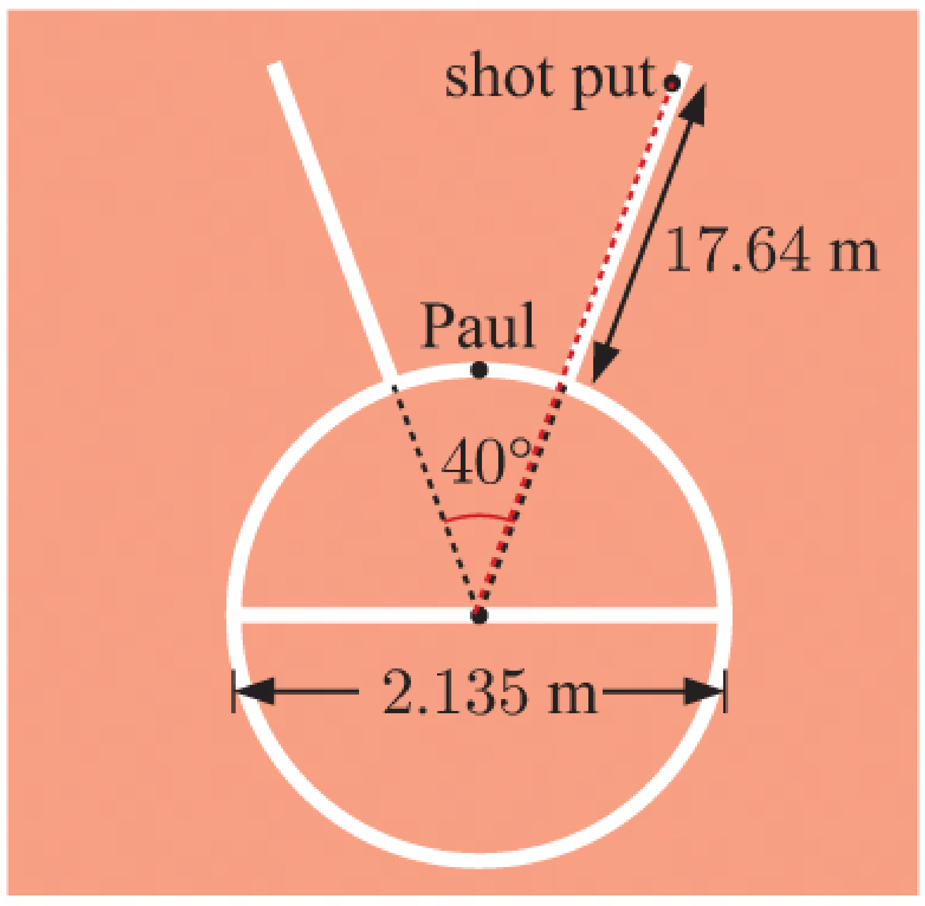
$$\frac{\sin x°}{5.2} = \frac{\sin 100°}{9}$$

$$\therefore \quad \sin x° = \frac{5.2 \times \sin 100°}{9}$$

$$\therefore \quad x = \sin^{-1}\left(\frac{5.2 \times \sin 100°}{9}\right)$$

$$\therefore \quad x \approx 34.7$$

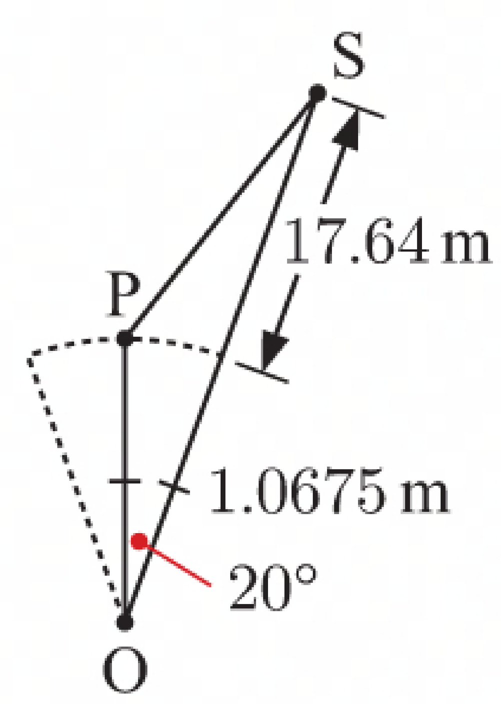
6



The throwing circle has radius $\dfrac{2.135}{2} = 1.0675$ m.

$OS = 1.0675 + 17.64$

$= 18.7075$ m



Using the cosine rule,

$$PS^2 = 1.0675^2 + 18.7075^2 - 2 \times 1.0675 \times 18.7075 \times \cos 20°$$

$$\therefore \quad PS = \sqrt{1.0675^2 + 18.7075^2 - 2 \times 1.0675 \times 18.7075 \times \cos 20°} \qquad \{\text{as } PS > 0\}$$

$$\therefore \quad PS \approx 17.7 \text{ m}$$

So, Paul actually put the shot approximately 17.7 m.

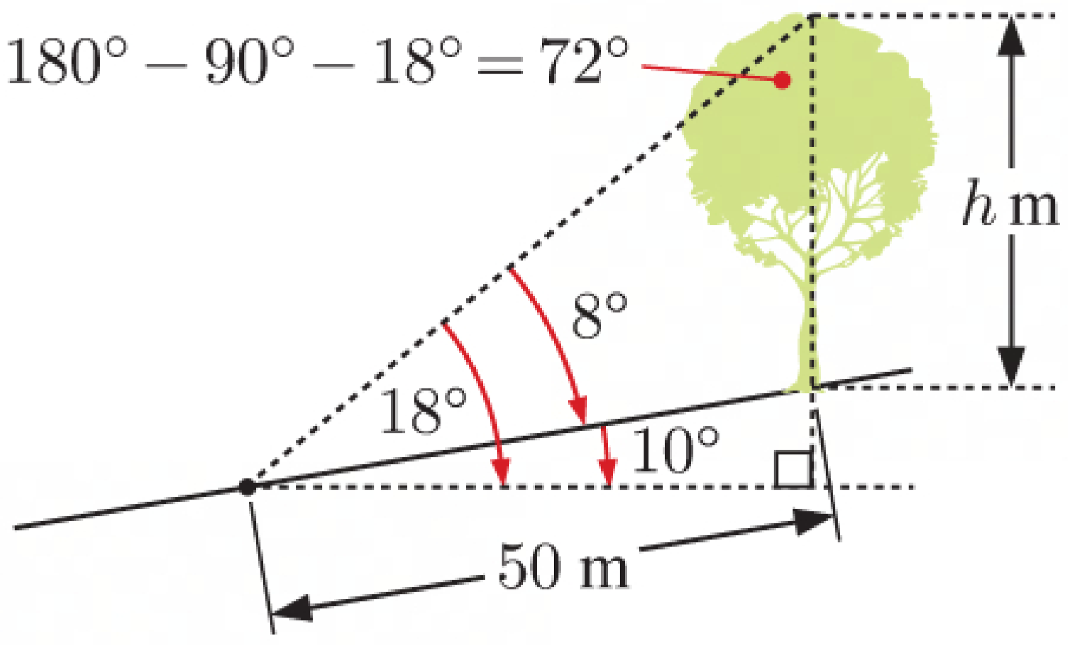
7 Let the height of the tree be h m.

Using the sine rule, $\dfrac{h}{\sin 8°} = \dfrac{50}{\sin 72°}$

$\therefore\ h = \dfrac{50 \times \sin 8°}{\sin 72°}$

$\therefore\ h \approx 7.32$

So, the tree is about 7.32 m high.

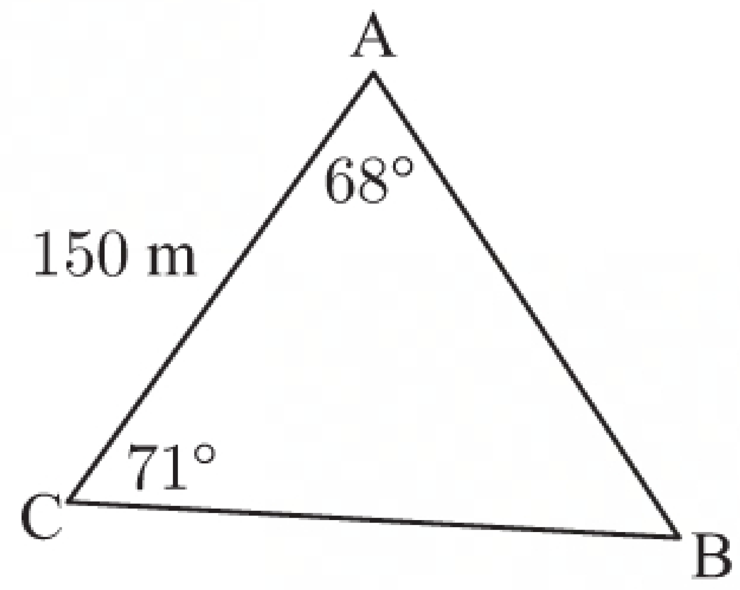


8 The unknown angle is $180° - 68° - 71°$ {angles in a triangle}
$= 41°$

Using the sine rule, $\dfrac{\text{AB}}{\sin 71°} = \dfrac{150}{\sin 41°}$

$\therefore\ \text{AB} = \dfrac{150 \times \sin 71°}{\sin 41°}$

$\therefore\ \text{AB} \approx 216.18$ m



Also using the sine rule, $\dfrac{\text{BC}}{\sin 68°} = \dfrac{150}{\sin 41°}$

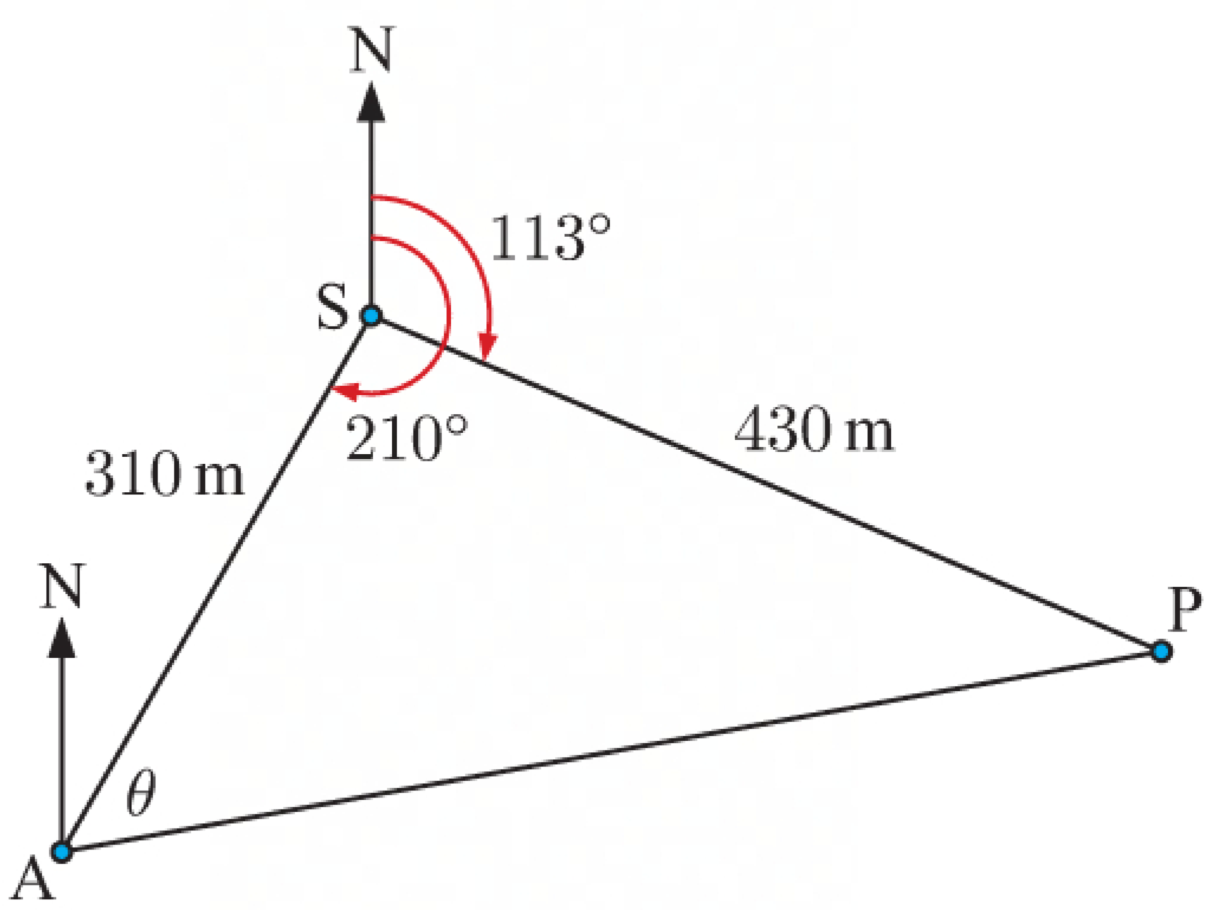
$\therefore\ \text{BC} = \dfrac{150 \times \sin 68°}{\sin 41°}$

$\therefore\ \text{BC} \approx 211.99$ m

So, the perimeter of the triangle $\approx 150 + 216.18 + 211.99$ m
≈ 578 m

Area of the triangle $= \frac{1}{2}bc\sin A$
$\approx \frac{1}{2} \times 150 \times 216.18 \times \sin 68°$
$\approx 15\,000$ m^2

9



$\text{A\hat{S}P} = 210° - 113° = 97°$

By the cosine rule:

$\text{AP}^2 = 310^2 + 430^2 - 2 \times 310 \times 430 \times \cos 97°$

$\therefore\ \text{AP} = \sqrt{310^2 + 430^2 - 2 \times 310 \times 430 \times \cos 97°}$ {as AP > 0}

$\therefore\ \text{AP} \approx 559.90$ m

Using the sine rule, $\dfrac{\sin\theta}{430} = \dfrac{\sin 97°}{\text{AP}}$

$\therefore\ \sin\theta \approx \dfrac{430 \times \sin 97°}{559.90}$

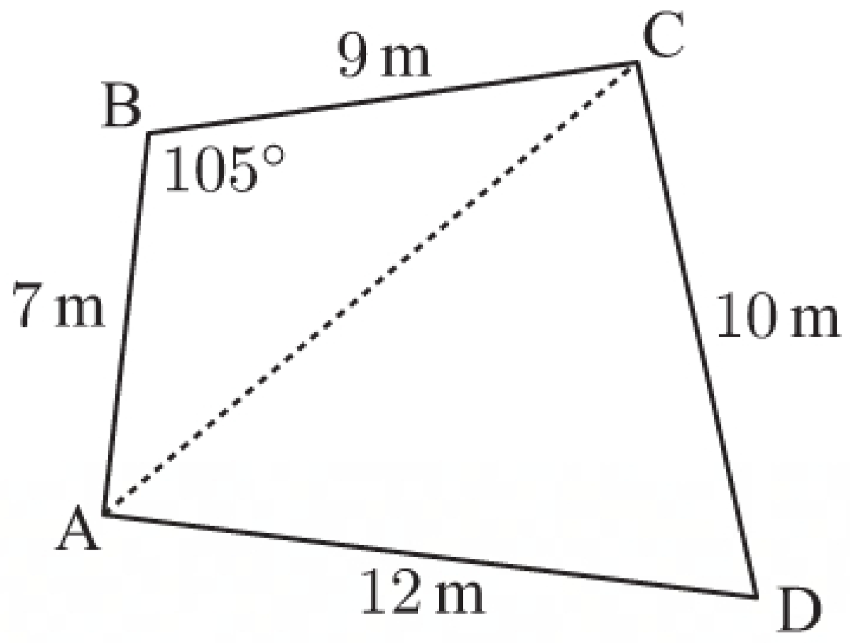
$\therefore\ \theta \approx \sin^{-1}\left(\dfrac{430 \times \sin 97°}{559.90}\right)$

$\therefore\ \theta \approx 49.664°$

Now $\text{A\hat{S}N} = 360° - 210°$ {angles at a point}
$= 150°$

So, Peter is about 560 m from Alix on a bearing of $(180° - 150°) + 49.664°$
$\approx 079.7°$

10



Using the cosine rule in $\triangle$ABC:

$$\text{AC}^2 = 7^2 + 9^2 - 2 \times 7 \times 9 \times \cos 105°$$

$$\therefore \ \text{AC} = \sqrt{7^2 + 9^2 - 2 \times 7 \times 9 \times \cos 105°} \quad \{\text{as AC} > 0\}$$

$$\therefore \ \text{AC} \approx 12.75 \text{ m}$$

Using the cosine rule in $\triangle$ACD:

$$\cos \text{A}\widehat{\text{D}}\text{C} \approx \frac{12^2 + 10^2 - 12.75^2}{2 \times 12 \times 10}$$

$$\therefore \ \text{A}\widehat{\text{D}}\text{C} \approx \cos^{-1}\left(\frac{12^2 + 10^2 - 12.75^2}{2 \times 12 \times 10}\right)$$

$$\therefore \ \text{A}\widehat{\text{D}}\text{C} \approx 70.2°$$

Using the sine rule in $\triangle$ABC:

$$\frac{\sin \text{B}\widehat{\text{A}}\text{C}}{9} = \frac{\sin 105°}{\text{AC}}$$

$$\therefore \ \sin \text{B}\widehat{\text{A}}\text{C} \approx \frac{9 \times \sin 105°}{12.75}$$

$$\therefore \ \text{B}\widehat{\text{A}}\text{C} \approx \sin^{-1}\left(\frac{9 \times \sin 105°}{12.75}\right)$$

$$\therefore \ \text{B}\widehat{\text{A}}\text{C} \approx 43.0°$$

Using the sine rule in $\triangle$ACD:

$$\frac{\sin \text{C}\widehat{\text{A}}\text{D}}{10} \approx \frac{\sin 70.2°}{\text{AC}}$$

$$\therefore \ \sin \text{C}\widehat{\text{A}}\text{D} \approx \frac{10 \times \sin 70.2°}{12.75}$$

$$\therefore \ \text{C}\widehat{\text{A}}\text{D} \approx \sin^{-1}\left(\frac{10 \times \sin 70.2°}{12.75}\right)$$

$$\therefore \ \text{C}\widehat{\text{A}}\text{D} \approx 47.5°$$

Now $\text{B}\widehat{\text{A}}\text{D} = \text{B}\widehat{\text{A}}\text{C} + \text{C}\widehat{\text{A}}\text{D}$

$$\therefore \ \text{B}\widehat{\text{A}}\text{D} \approx 43.0° + 47.5°$$

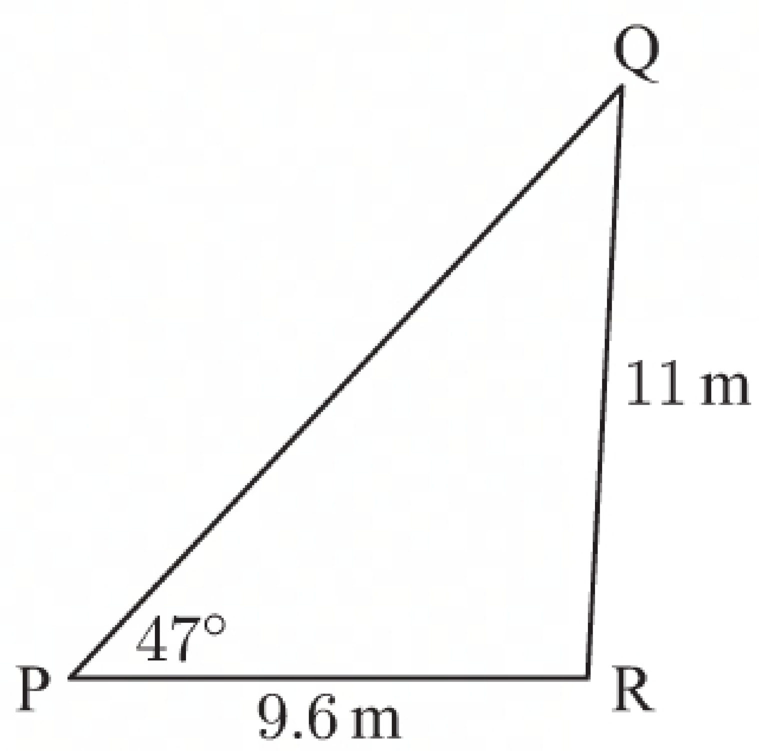
$$\therefore \ \text{B}\widehat{\text{A}}\text{D} \approx 90.5°$$

Also, $\text{B}\widehat{\text{C}}\text{D} = 360° - 105° - \text{B}\widehat{\text{A}}\text{D} - \text{A}\widehat{\text{D}}\text{C}$ {angles in a quadrilateral}

$$\approx 255° - 90.5° - 70.2°$$

$$\approx 94.3°$$

11



Using the sine rule,

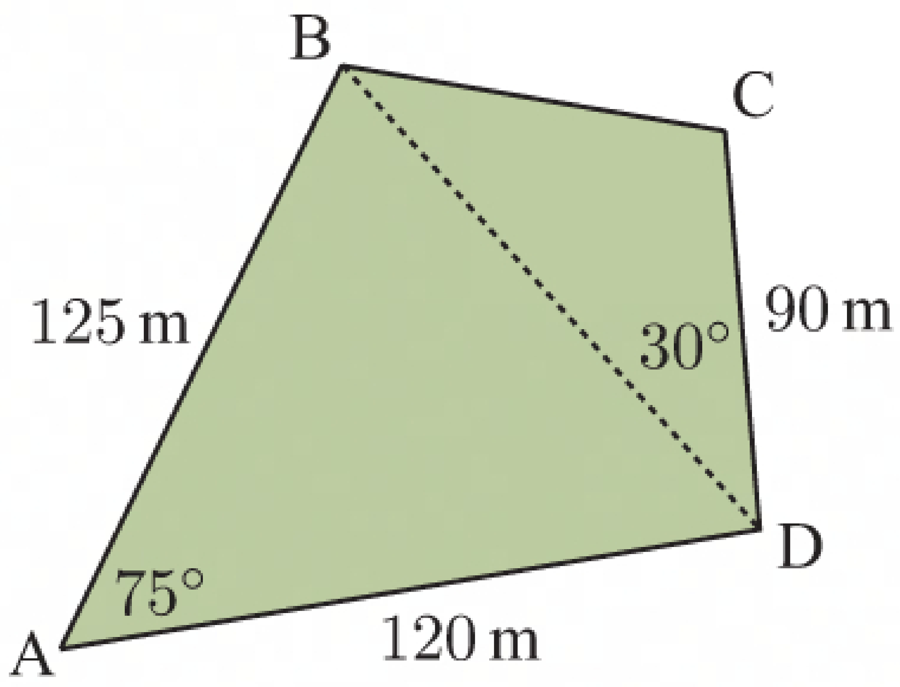
$$\frac{\sin Q}{9.6} = \frac{\sin 47°}{11}$$

$$\therefore \ \sin Q = \frac{9.6 \times \sin 47°}{11}$$

$$\therefore \ Q = \sin^{-1}\left(\frac{9.6 \times \sin 47°}{11}\right)$$

$$\therefore \ Q \approx 39.7°$$

12 **a**



By the cosine rule:

$$\text{BD}^2 = 120^2 + 125^2 - 2 \times 120 \times 125 \times \cos 75°$$

$$\therefore \ \text{BD} = \sqrt{120^2 + 125^2 - 2 \times 120 \times 125 \times \cos 75°} \quad \{\text{as BD} > 0\}$$

$$\therefore \ \text{BD} \approx 149.2 \text{ m}$$

The area of the block

$= \text{area of } \triangle\text{ABD} + \text{area of } \triangle\text{BCD}$

$\approx \frac{1}{2} \times 120 \times 125 \times \sin 75° + \frac{1}{2} \times 149.2 \times 90 \times \sin 30°$

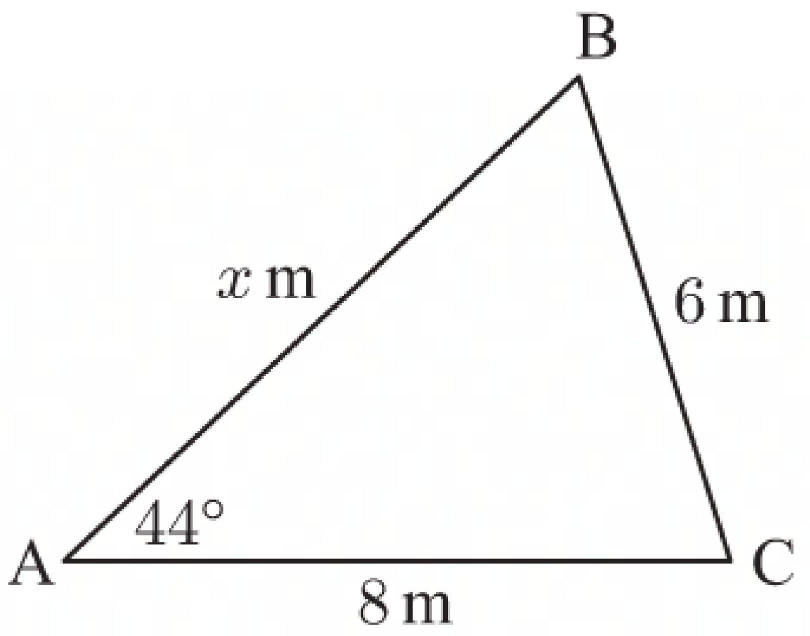
$\approx 10\,600 \text{ m}^2$

b Area $\approx 10\,600$ m^2

$\approx (10\,600 \div 10\,000)$ ha $\quad \{10\,000 \text{ m}^2 = 1 \text{ ha}\}$

≈ 1.06 ha

13 a



By the cosine rule, $6^2 = x^2 + 8^2 - 2 \times x \times 8 \times \cos 44°$

$\therefore\ 36 = x^2 + 64 - 16x \times \cos 44°$

$\therefore\ x^2 - 11.51x + 28 \approx 0$

$\therefore\ x \approx \dfrac{11.51 \pm \sqrt{11.51^2 - 4(1)(28)}}{2}$

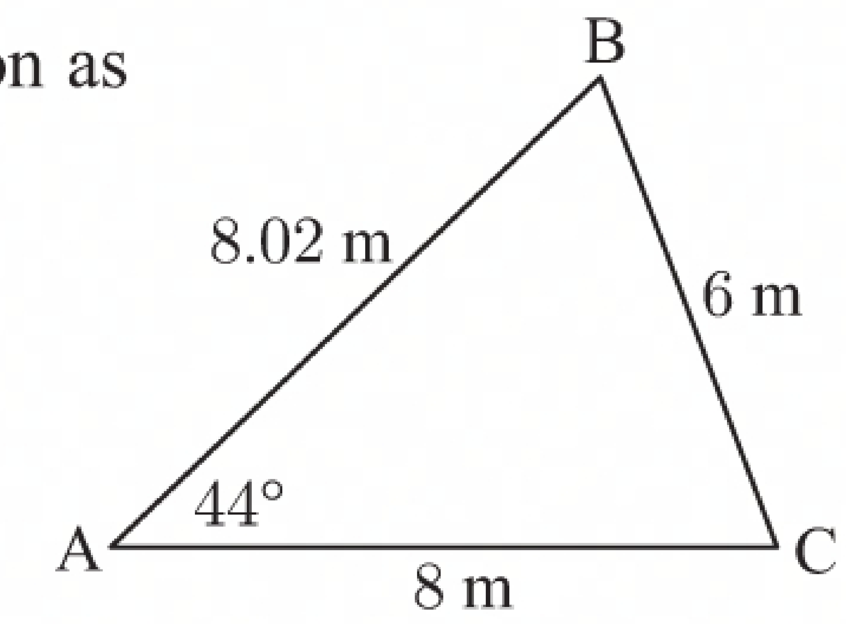
$\therefore\ x \approx \dfrac{11.51 \pm 4.524}{2}$

$\therefore\ x \approx 8.02$ or 3.49

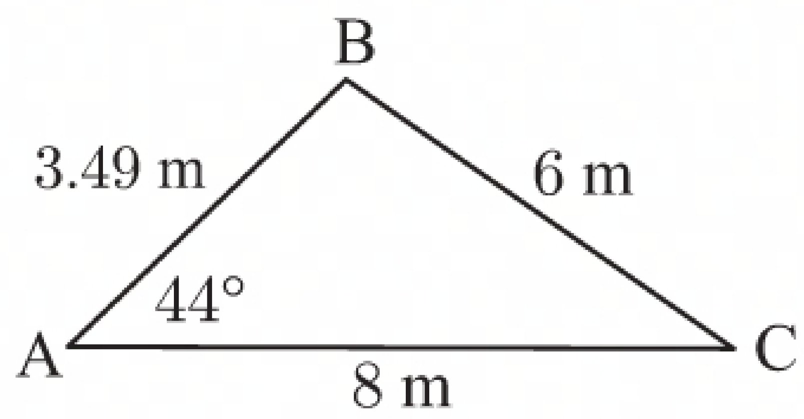
Frank needs additional information as there are two possible cases:

(1) when AB ≈ 8.02 m and

(2) when AB ≈ 3.49 m



Case (1)



Case (2)

b The area of the plot is a maximum when $x \approx 8.02$ m.

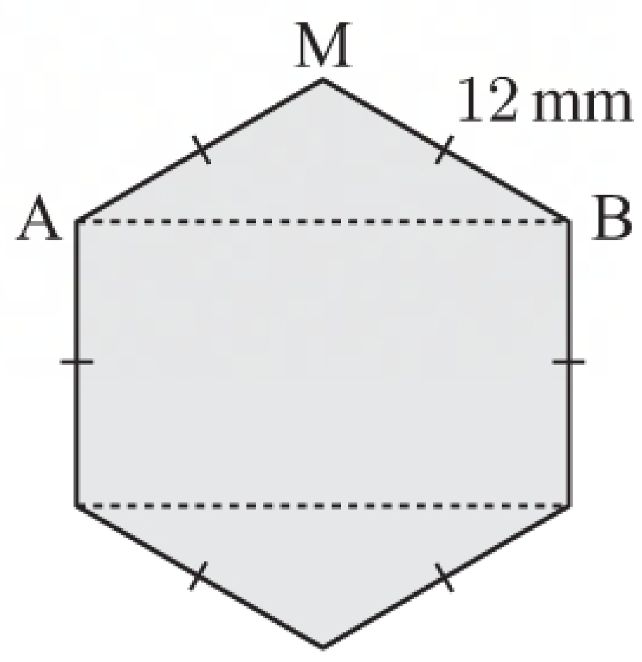
Volume = area × depth

$= \frac{1}{2} \times 8 \times x \times \sin 44° \times 0.1 \quad \{10 \text{ cm} \equiv 0.1 \text{ m}\}$

$\approx 4 \times 8.02 \times \sin 44° \times 0.1$

≈ 2.23 m^3

14 a i



The sum of the interior angles of a regular hexagon is $(6-2) \times 180° = 720°$.

$\therefore\ \widehat{\text{AMB}} = \dfrac{720°}{6} = 120°$

By the cosine rule in $\triangle$AMB:

$\text{AB}^2 = 12^2 + 12^2 - 2 \times 12 \times 12 \times \cos 120°$

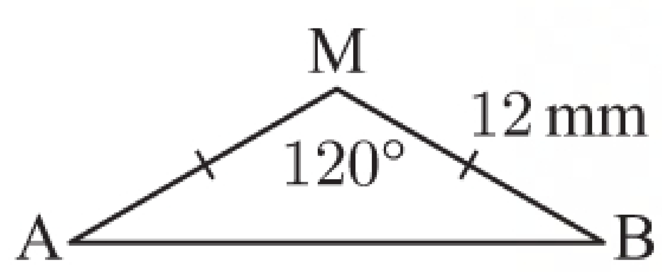
$\therefore\ \text{AB} = \sqrt{12^2 + 12^2 - 2 \times 12 \times 12 \times \cos 120°} \quad \{\text{as AB} > 0\}$

$\therefore\ \text{AB} \approx 20.8$ mm

ii Area of the hexagon = 2 × area of $\triangle$AMB + area of rectangle

$\approx 2 \times \frac{1}{2} \times 12 \times 12 \times \sin 120° + 20.8 \times 12$

≈ 374 mm^2



b Area of circular hole in nut $= \pi \times \left(\frac{8}{2}\right)^2$

$= 16\pi$ mm^2

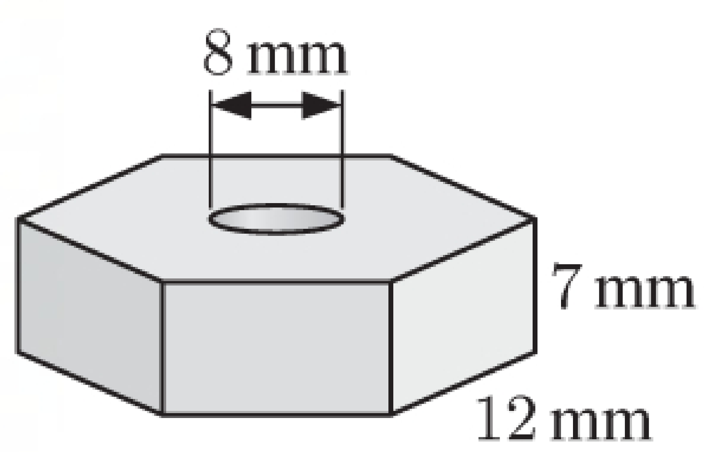
Volume of nut

= area of base × height

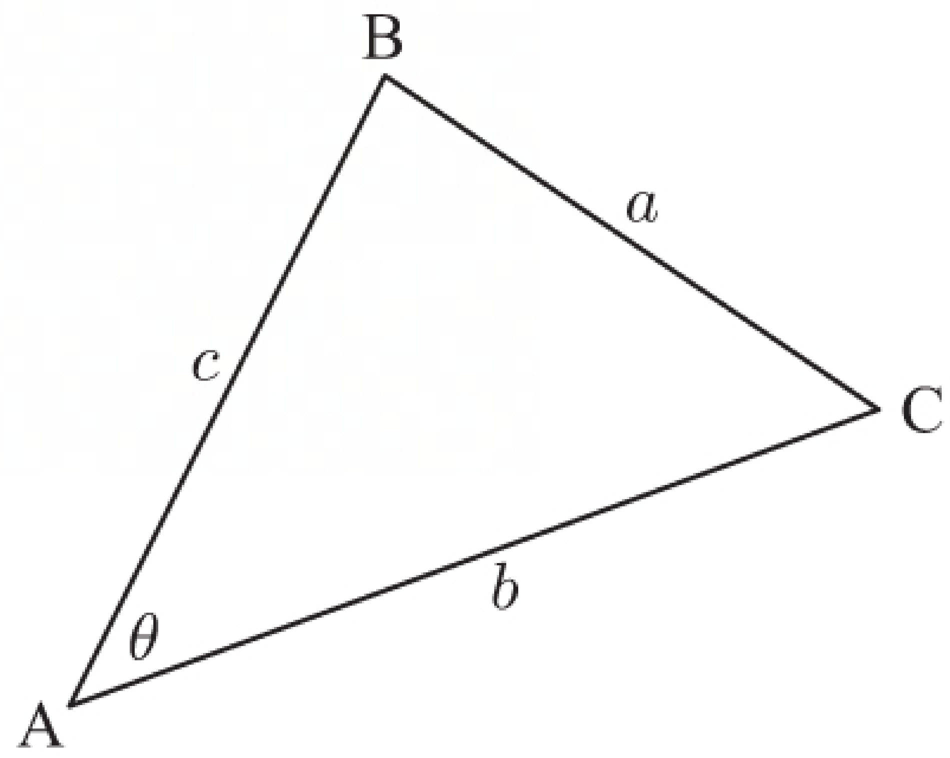
= (area of hexagon − area of circular hole) × height

$\approx (374 - 16\pi) \times 7$ mm^3

≈ 2270 mm^3



15 **a** Let $\mathrm{B\widehat{A}C} = \theta$, so $A = \frac{1}{2}bc\sin\theta$



$$\begin{aligned}
\therefore\ A^2 &= \left(\tfrac{1}{2}bc\sin\theta\right)^2 \\
&= \tfrac{1}{4}b^2c^2\sin^2\theta \\
&= \frac{b^2c^2}{4}\times(1-\cos^2\theta) \\
&= \frac{b^2c^2}{4}(1+\cos\theta)(1-\cos\theta)
\end{aligned}$$

Using the cosine rule, $\cos\theta = \dfrac{b^2+c^2-a^2}{2bc}$

$$\therefore\ A^2 = \frac{b^2c^2}{4}\left(1+\frac{b^2+c^2-a^2}{2bc}\right)\left(1-\frac{b^2+c^2-a^2}{2bc}\right)$$

b

$$\begin{aligned}
A^2 &= \frac{b^2c^2}{4}\left(1+\frac{b^2+c^2-a^2}{2bc}\right)\left(1-\frac{b^2+c^2-a^2}{2bc}\right) \\
&= \frac{b^2c^2}{4}\left(1+\frac{b^2+c^2-a^2}{2bc}\right)\left(1-\frac{b^2+c^2-a^2}{2bc}\right) \\
&= \frac{b^2c^2}{4}\left(\frac{2bc+b^2+c^2-a^2}{2bc}\right)\left(\frac{2bc-b^2-c^2+a^2}{2bc}\right) \\
&= \tfrac{1}{16}(b^2+2bc+c^2-a^2)(a^2-(b^2-2bc+c^2)) \\
&= \tfrac{1}{16}((b+c)^2-a^2)(a^2-(b-c)^2) \\
&= \tfrac{1}{16}(b+c+a)(b+c-a)(a-b+c)(a+b-c) \\
&= \left(\frac{a+b+c}{2}\right)\left(\frac{b+c-a}{2}\right)\left(\frac{a-b+c}{2}\right)\left(\frac{a+b-c}{2}\right) \\
&= \left(\frac{a+b+c}{2}\right)\left(\frac{a+b+c}{2}-a\right)\left(\frac{a+b+c}{2}-b\right)\left(\frac{a+b+c}{2}-c\right) \\
&= s(s-a)(s-b)(s-c) \quad \text{where } s = \frac{a+b+c}{2}
\end{aligned}$$

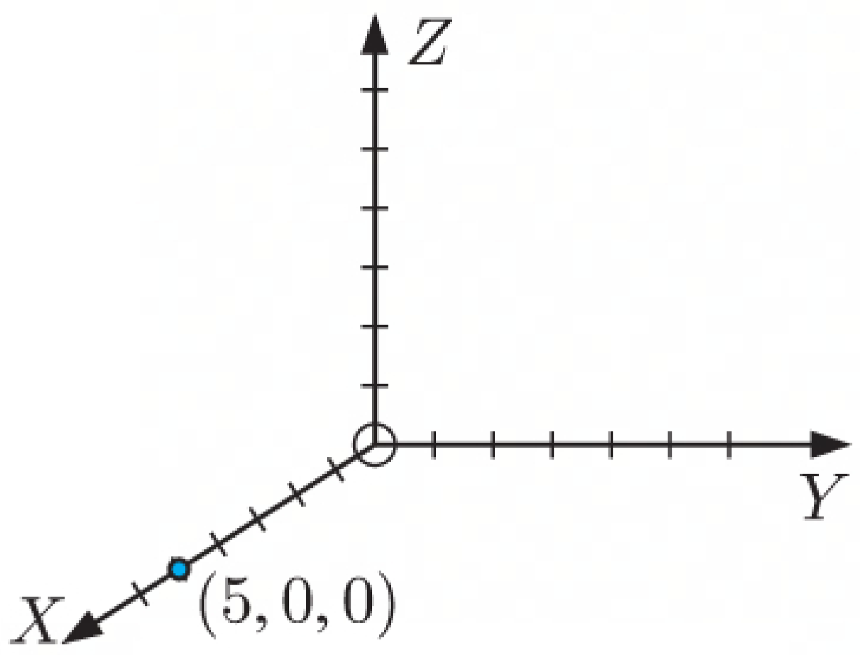
$$\therefore\ A = \sqrt{s(s-a)(s-b)(s-c)} \qquad \{\text{as } A > 0\}$$

Chapter 10

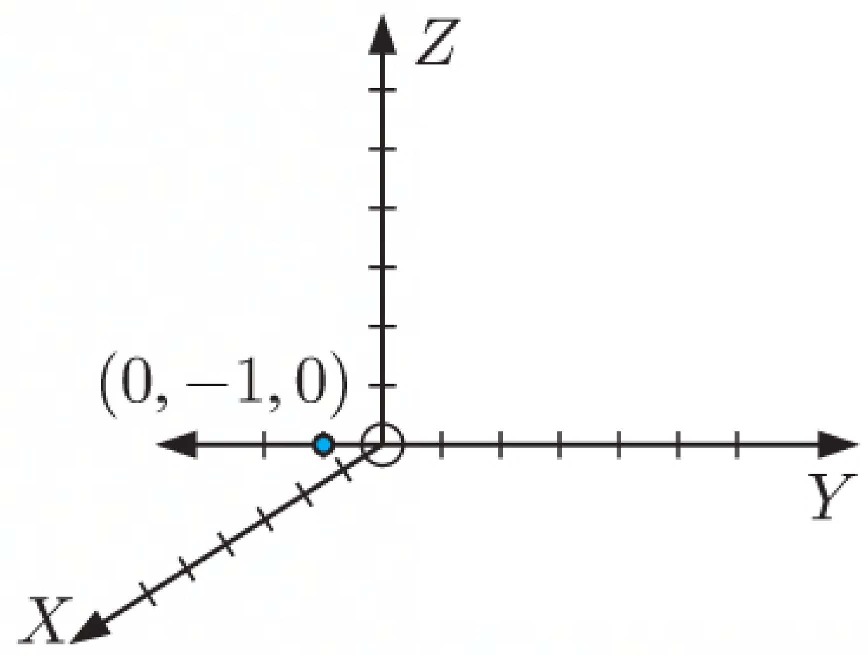
POINTS IN SPACE

EXERCISE 10A

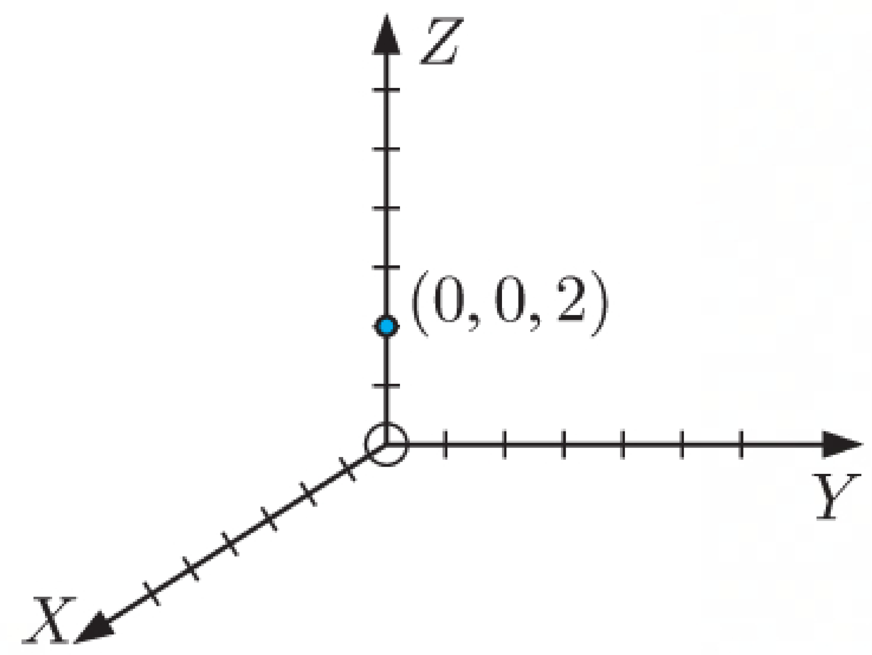
1 **a**



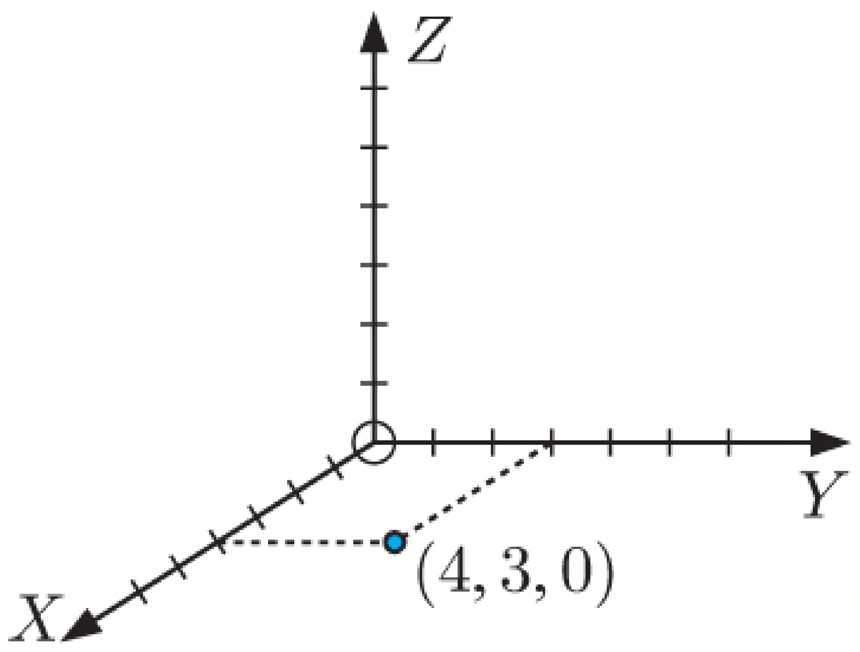
b



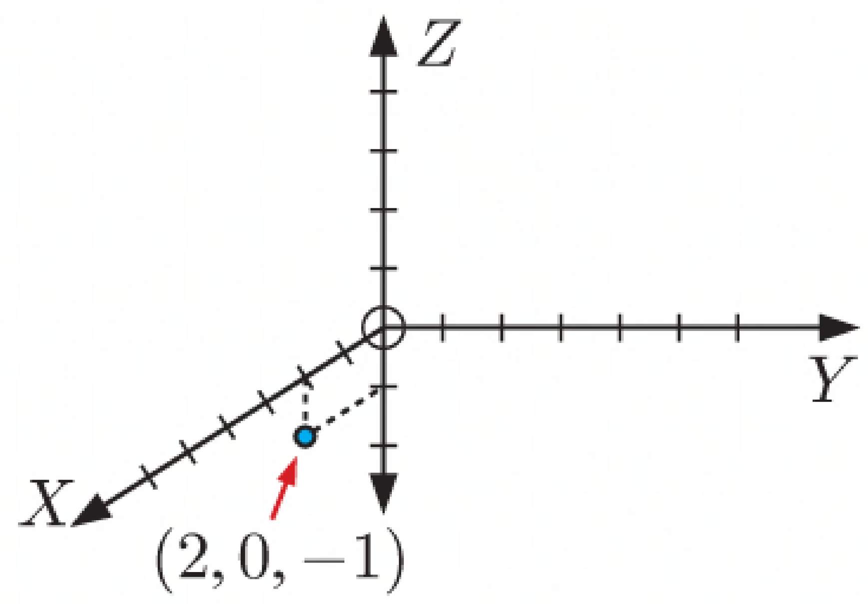
c



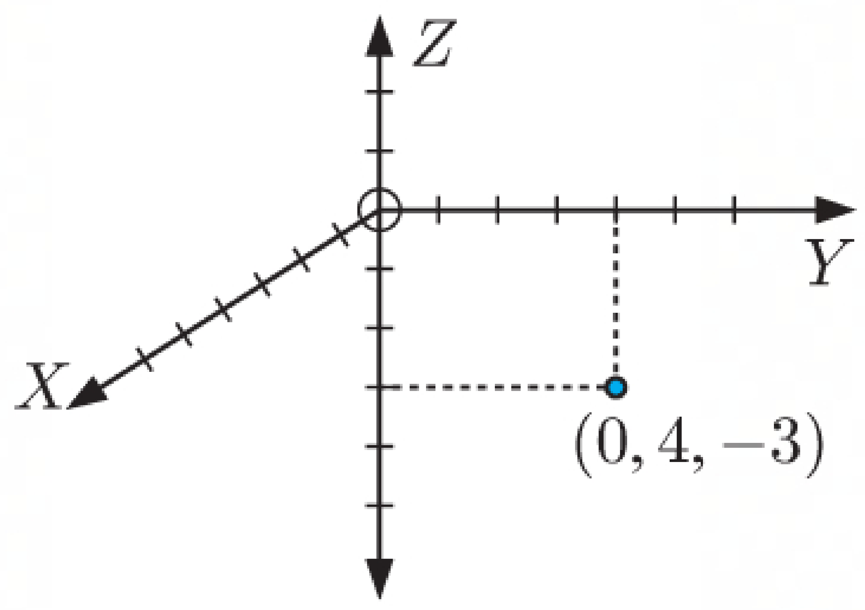
d



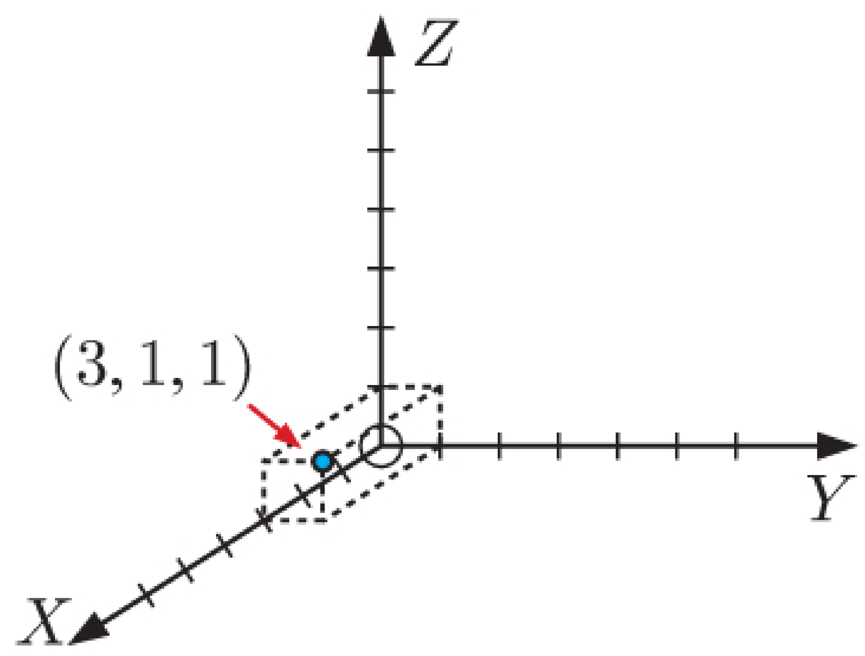
e



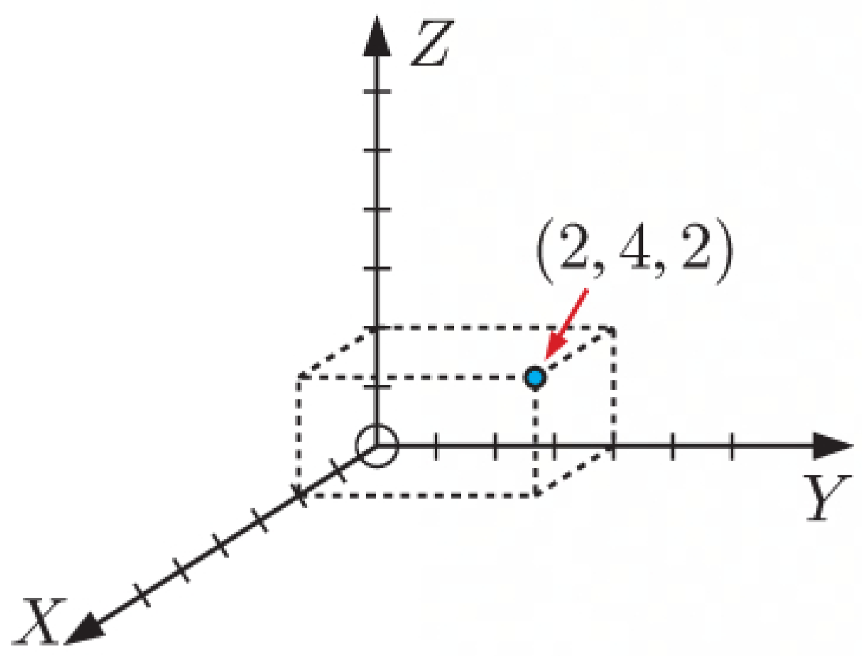
f



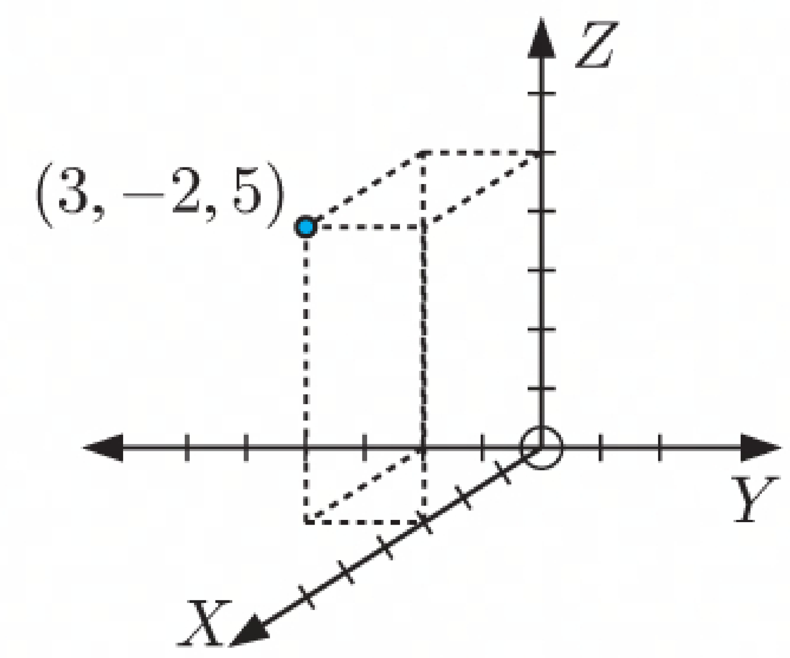
g



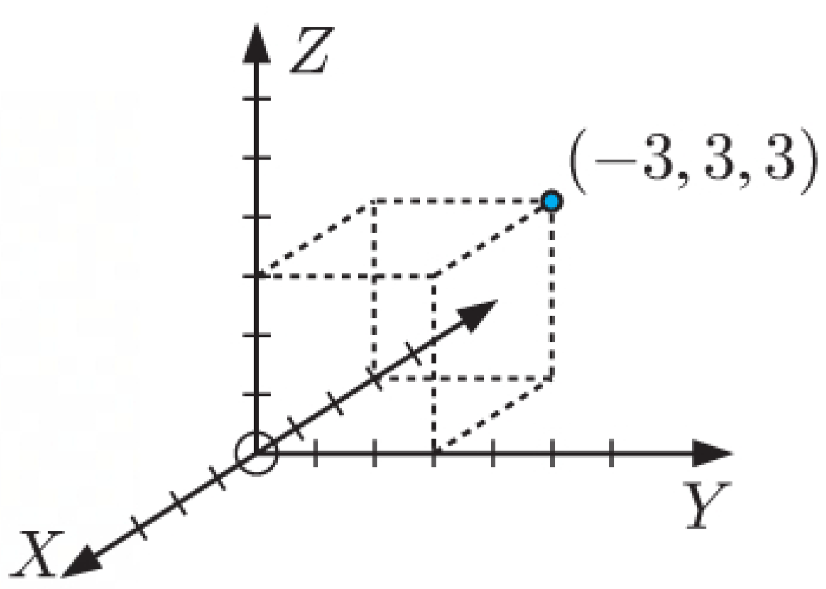
h



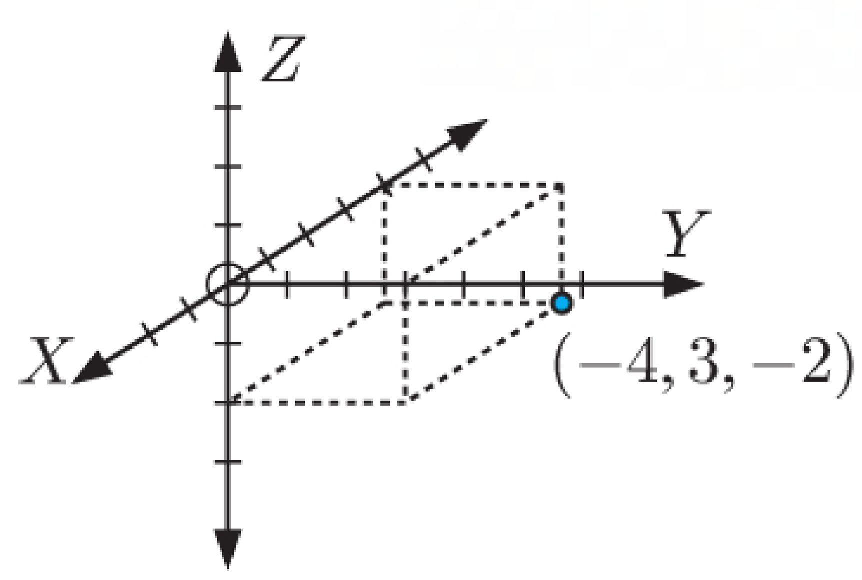
i



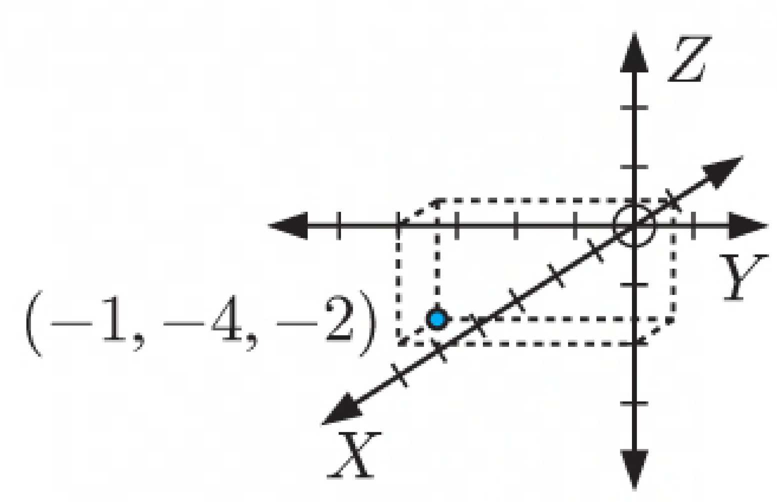
j



k



l



2 **a** **i** $\text{AB} = \sqrt{(6-0)^2 + (-4-0)^2 + (2-0)^2}$

$= \sqrt{6^2 + (-4)^2 + 2^2}$

$= \sqrt{36 + 16 + 4}$

$= \sqrt{56}$

$= 2\sqrt{14}$ units

ii The midpoint is

$\left(\dfrac{0+6}{2}, \dfrac{0+-4}{2}, \dfrac{0+2}{2}\right)$,

which is $(3, -2, 1)$.

b **i** $AB = \sqrt{(0-4)^2 + (1-1)^2 + (-2-0)^2}$
$= \sqrt{(-4)^2 + 0^2 + (-2)^2}$
$= \sqrt{16+0+4}$
$= \sqrt{20}$
$= 2\sqrt{5}$ units

ii The midpoint is $\left(\frac{4+0}{2}, \frac{1+1}{2}, \frac{0+-2}{2}\right)$, which is $(2, 1, -1)$.

c **i** $AB = \sqrt{(5-1)^2 + (-3--1)^2 + (0-2)^2}$
$= \sqrt{4^2 + (-2)^2 + (-2)^2}$
$= \sqrt{16+4+4}$
$= \sqrt{24}$
$= 2\sqrt{6}$ units

ii The midpoint is $\left(\frac{1+5}{2}, \frac{-1+-3}{2}, \frac{2+0}{2}\right)$, which is $(3, -2, 1)$.

d **i** $AB = \sqrt{(-6--2)^2 + (7-0)^2 + (3-5)^2}$
$= \sqrt{(-4)^2 + 7^2 + (-2)^2}$
$= \sqrt{16+49+4}$
$= \sqrt{69}$ units

ii The midpoint is $\left(\frac{-2+-6}{2}, \frac{0+7}{2}, \frac{5+3}{2}\right)$, which is $\left(-4, \frac{7}{2}, 4\right)$.

e **i** $AB = \sqrt{(4--1)^2 + (1-5)^2 + (-1-2)^2}$
$= \sqrt{5^2 + (-4)^2 + (-3)^2}$
$= \sqrt{25+16+9}$
$= \sqrt{50}$
$= 5\sqrt{2}$ units

ii The midpoint is $\left(\frac{-1+4}{2}, \frac{5+1}{2}, \frac{2+-1}{2}\right)$, which is $\left(\frac{3}{2}, 3, \frac{1}{2}\right)$.

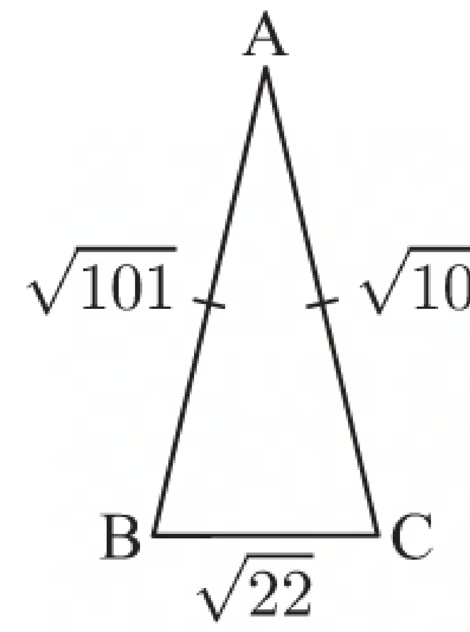
f **i** $AB = \sqrt{(-5-2)^2 + (3-6)^2 + (2--3)^2}$
$= \sqrt{(-7)^2 + (-3)^2 + 5^2}$
$= \sqrt{49+9+25}$
$= \sqrt{83}$ units

ii The midpoint is $\left(\frac{2+-5}{2}, \frac{6+3}{2}, \frac{-3+2}{2}\right)$, which is $\left(-\frac{3}{2}, \frac{9}{2}, -\frac{1}{2}\right)$.

3 **a** $AB = \sqrt{(-5-2)^2 + (5-1)^2 + (3--3)^2}$
$= \sqrt{(-7)^2 + 4^2 + 6^2}$
$= \sqrt{49+16+36}$
$= \sqrt{101}$ units

$AC = \sqrt{(-2-2)^2 + (3-1)^2 + (6--3)^2}$
$= \sqrt{(-4)^2 + 2^2 + 9^2}$
$= \sqrt{16+4+81}$
$= \sqrt{101}$ units

$BC = \sqrt{(-2--5)^2 + (3-5)^2 + (6-3)^2}$
$= \sqrt{3^2 + (-2)^2 + 3^2}$
$= \sqrt{9+4+9}$
$= \sqrt{22}$ units



$AB = AC = \sqrt{101}$ units and $BC \neq AB$,
so $\triangle ABC$ is isosceles.

b $\text{AB} = \sqrt{(-1-3)^2 + (-4 - -1)^2 + (0-5)^2}$

$= \sqrt{(-4)^2 + (-3)^2 + (-5)^2}$

$= \sqrt{16+9+25}$

$= \sqrt{50}$

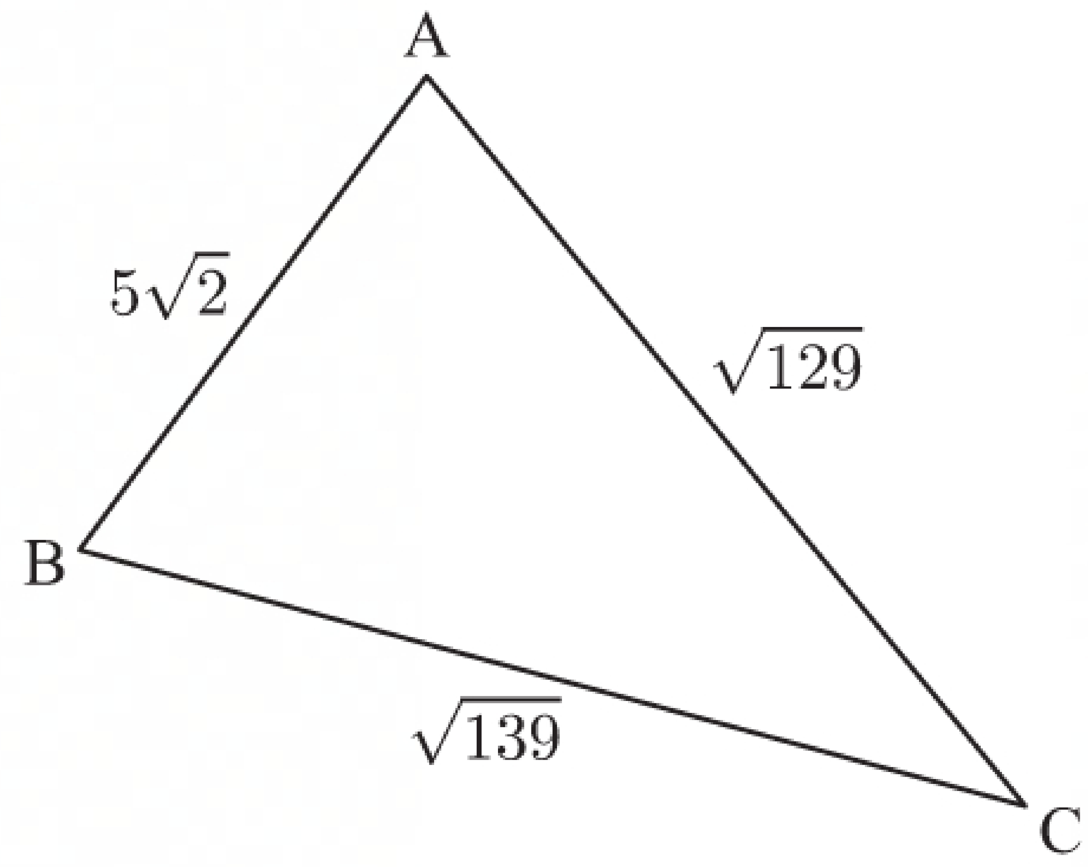
$= 5\sqrt{2}$ units

$\text{AC} = \sqrt{(2-3)^2 + (7 - -1)^2 + (-3-5)^2}$

$= \sqrt{(-1)^2 + 8^2 + (-8)^2}$

$= \sqrt{1+64+64}$

$= \sqrt{129}$ units



$\text{BC} = \sqrt{(2 - -1)^2 + (7 - -4)^2 + (-3-0)^2}$

$= \sqrt{3^2 + 11^2 + (-3)^2}$

$= \sqrt{9+121+9}$

$= \sqrt{139}$ units

AB $\neq$ AC $\neq$ BC, so $\triangle$ABC is scalene.

4 $\text{AB} = \sqrt{(-6-3)^2 + (7-1)^2 + (13 - -2)^2}$

$= \sqrt{(-9)^2 + 6^2 + 15^2}$

$= \sqrt{81+36+225}$

$= \sqrt{342}$ units

$\text{AC} = \sqrt{(5-3)^2 + (9-1)^2 + (-4 - -2)^2}$

$= \sqrt{2^2 + 8^2 + (-2)^2}$

$= \sqrt{4+64+4}$

$= \sqrt{72}$ units

$\text{BC} = \sqrt{(5 - -6)^2 + (9-7)^2 + (-4-13)^2}$

$= \sqrt{11^2 + 2^2 + (-17)^2}$

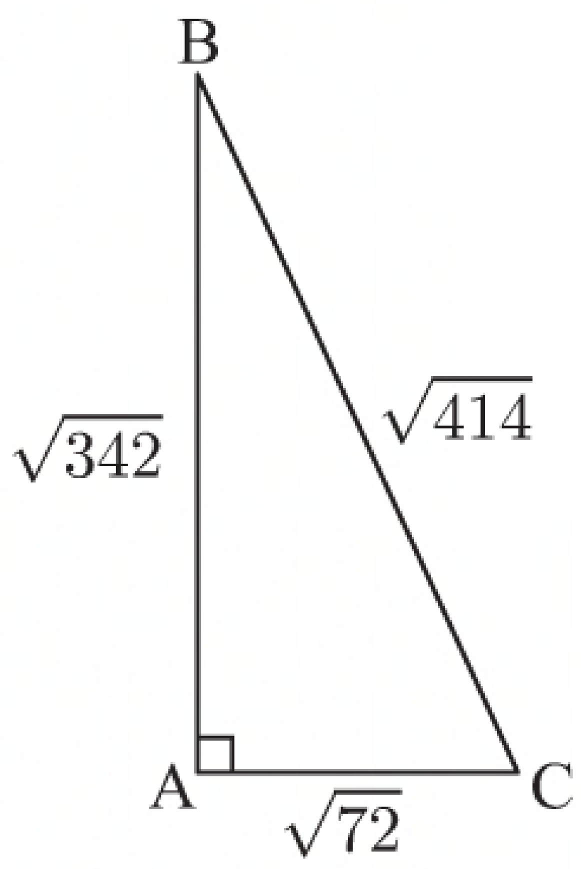
$= \sqrt{121+4+289}$

$= \sqrt{414}$ units

$\text{AB}^2 + \text{AC}^2 = (\sqrt{342})^2 + (\sqrt{72})^2 = 414$

and $\text{BC}^2 = (\sqrt{414})^2 = 414$

$\therefore$ triangle ABC is right angled at A.



5 **a** The midpoint of [PQ] is $\left(\frac{1+6}{2}, \frac{4+-8}{2}, \frac{-1+7}{2}\right)$ which is $\left(\frac{7}{2}, -2, 3\right)$.

The midpoint of [QR] is $\left(\frac{6+-5}{2}, \frac{-8+-2}{2}, \frac{7+-9}{2}\right)$ which is $\left(\frac{1}{2}, -5, -1\right)$.

So, M is $\left(\frac{7}{2}, -2, 3\right)$ and N is $\left(\frac{1}{2}, -5, -1\right)$.

b $\text{PR} = \sqrt{(-5-1)^2 + (-2-4)^2 + (-9--1)^2}$
$= \sqrt{(-6)^2 + (-6)^2 + (-8)^2}$
$= \sqrt{36+36+64}$
$= \sqrt{136}$
$= 2\sqrt{34}$ units

$\text{MN} = \sqrt{\left(\frac{1}{2}-\frac{7}{2}\right)^2 + (-5--2)^2 + (-1-3)^2}$
$= \sqrt{(-3)^2 + (-3)^2 + (-4)^2}$
$= \sqrt{9+9+16}$
$= \sqrt{34}$ units
$= \frac{1}{2} \times \text{PR}$

So, [MN] is half the length of [PR].

6 Let B have coordinates (x, y, z).

The midpoint M of [AB] is $\left(\frac{3+x}{2}, \frac{5+y}{2}, \frac{-2+z}{2}\right)$, which is $(-2, 8, \frac{1}{2})$.

Equating coordinates:

$\frac{3+x}{2} = -2$ $\quad$ $\frac{5+y}{2} = 8$ $\quad$ $\frac{-2+z}{2} = \frac{1}{2}$

$\therefore\ 3+x = -4$ $\quad$ $\therefore\ 5+y = 16$ $\quad$ $\therefore\ -2+z = 1$

$\therefore\ x = -7$ $\quad$ $\therefore\ y = 11$ $\quad$ $\therefore\ z = 3$

So, B is $(-7, 11, 3)$.

7 Let P have coordinates (p_1, p_2, p_3), Q have coordinates (q_1, q_2, q_3), and R have coordinates (r_1, r_2, r_3).

The midpoint of [PQ] is $\left(\frac{p_1+q_1}{2}, \frac{p_2+q_2}{2}, \frac{p_3+q_3}{2}\right)$, which is $(1, 2, \frac{3}{2})$.

The midpoint of [QR] is $\left(\frac{q_1+r_1}{2}, \frac{q_2+r_2}{2}, \frac{q_3+r_3}{2}\right)$, which is $(\frac{5}{2}, 3, \frac{9}{2})$.

The midpoint of [PR] is $\left(\frac{p_1+r_1}{2}, \frac{p_2+r_2}{2}, \frac{p_3+r_3}{2}\right)$, which is $(-\frac{1}{2}, 6, 3)$.

Equating coordinates for the midpoint of [PQ]:

$\frac{p_1+q_1}{2} = 1$ $\quad$ $\frac{p_2+q_2}{2} = 2$ $\quad$ $\frac{p_3+q_3}{2} = \frac{3}{2}$

$\therefore\ p_1+q_1 = 2$ $\quad$ $\therefore\ p_2+q_2 = 4$ $\quad$ $\therefore\ p_3+q_3 = 3$

$\therefore\ p_1 = 2-q_1$ $\quad$ $\therefore\ p_2 = 4-q_2$ $\quad$ $\therefore\ p_3 = 3-q_3$

Equating coordinates for the midpoint of [QR]:

$\frac{q_1+r_1}{2} = \frac{5}{2}$ $\quad$ $\frac{q_2+r_2}{2} = 3$ $\quad$ $\frac{q_3+r_3}{2} = \frac{9}{2}$

$\therefore\ q_1+r_1 = 5$ $\quad$ $\therefore\ q_2+r_2 = 6$ $\quad$ $\therefore\ q_3+r_3 = 9$

$\therefore\ q_1 = 5-r_1$ $\quad$ $\therefore\ q_2 = 6-r_2$ $\quad$ $\therefore\ q_3 = 9-r_3$

Equating coordinates for the midpoint of [PR]:

$$\frac{p_1+r_1}{2} = -\tfrac{1}{2} \qquad \frac{p_2+r_2}{2} = 6 \qquad \frac{p_3+r_3}{2} = 3$$

$$\therefore\ p_1+r_1=-1 \qquad \therefore\ p_2+r_2=12 \qquad \therefore\ p_3+r_3=6$$

$$\therefore\ r_1=-1-p_1 \qquad \therefore\ r_2=12-p_2 \qquad \therefore\ r_3=6-p_3$$

So, we have $p_1 = 2-q_1$
$q_1 = 5-r_1$ .... (1)
$r_1 = -1-p_1$ .... (2)

$$\begin{aligned}
\therefore\ p_1 &= 2-(5-r_1) && \{\text{using (1)}\}\\
&= -3+r_1\\
&= -3+(-1-p_1) && \{\text{using (2)}\}\\
&= -4-p_1\\
\therefore\ 2p_1 &= -4\\
\therefore\ p_1 &= -2\\
\therefore\ r_1 &= -1-(-2)=1 && \{\text{using (2) again}\}\\
\text{and}\quad q_1 &= 5-1=4 && \{\text{using (1) again}\}
\end{aligned}$$

Also, $p_2 = 4-q_2$
$q_2 = 6-r_2$ .... (3)
$r_2 = 12-p_2$ .... (4)

$$\begin{aligned}
\therefore\ p_2 &= 4-(6-r_2) && \{\text{using (3)}\}\\
&= -2+r_2\\
&= -2+(12-p_2) && \{\text{using (4)}\}\\
&= 10-p_2\\
\therefore\ 2p_2 &= 10\\
\therefore\ p_2 &= 5\\
\therefore\ r_2 &= 12-5=7 && \{\text{using (4) again}\}\\
\text{and}\quad q_2 &= 6-7=-1 && \{\text{using (3) again}\}
\end{aligned}$$

Also, $p_3 = 3-q_3$
$q_3 = 9-r_3$ .... (5)
$r_3 = 6-p_3$ .... (6)

$$\begin{aligned}
\therefore\ p_3 &= 3-(9-r_3) && \{\text{using (5)}\}\\
&= -6+r_3\\
&= -6+(6-p_3) && \{\text{using (6)}\}\\
&= -p_3\\
\therefore\ 2p_3 &= 0\\
\therefore\ p_3 &= 0\\
\therefore\ r_3 &= 6-0=6 && \{\text{using (6) again}\}\\
\text{and}\quad q_3 &= 9-6=3 && \{\text{using (5) again}\}
\end{aligned}$$

So, the coordinates are P(−2, 5, 0), Q(4, −1, 3), and R(1, 7, 6).

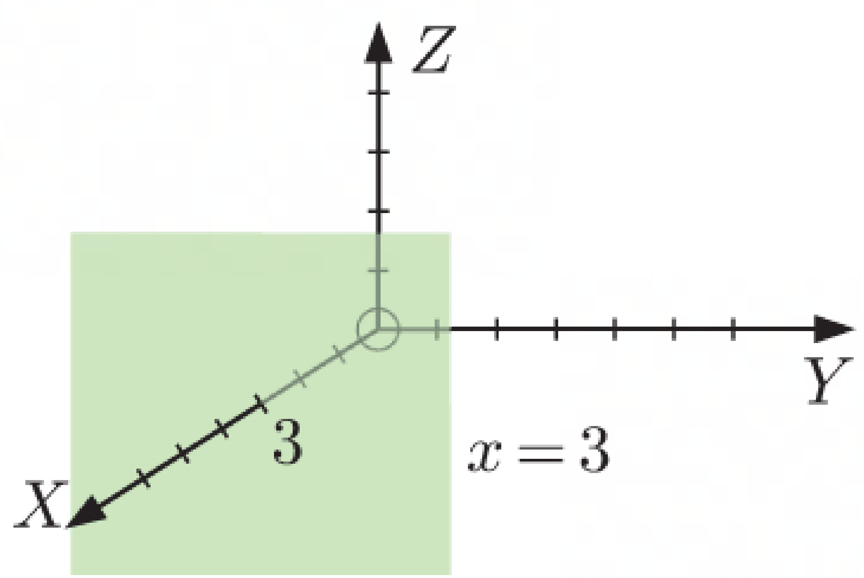
8 $\text{PQ} = \sqrt{(k-2)^2+(-1-4)^2+(-2--3)^2} = 7$

$$\begin{aligned}
\therefore\ \sqrt{k^2-4k+4+(-5)^2+1^2} &= 7\\
\therefore\ \sqrt{k^2-4k+4+25+1} &= 7\\
\therefore\ \sqrt{k^2-4k+30} &= 7\\
\therefore\ k^2-4k+30 &= 49 && \{\text{squaring both sides}\}\\
\therefore\ k^2-4k &= 19\\
\therefore\ k^2-4k+(-2)^2 &= 19+(-2)^2 && \{\text{completing the square}\}\\
\therefore\ (k-2)^2 &= 23\\
\therefore\ k-2 &= \pm\sqrt{23}\\
\therefore\ k &= 2\pm\sqrt{23}
\end{aligned}$$

9 **a**
$$OP = \sqrt{(x-0)^2 + (y-0)^2 + (z-0)^2}$$
$$\therefore\ 3 = \sqrt{x^2 + y^2 + z^2}$$
$$\therefore\ x^2 + y^2 + z^2 = 9$$
So, P lies on a sphere with centre $O(0, 0, 0)$ and radius 3 units.

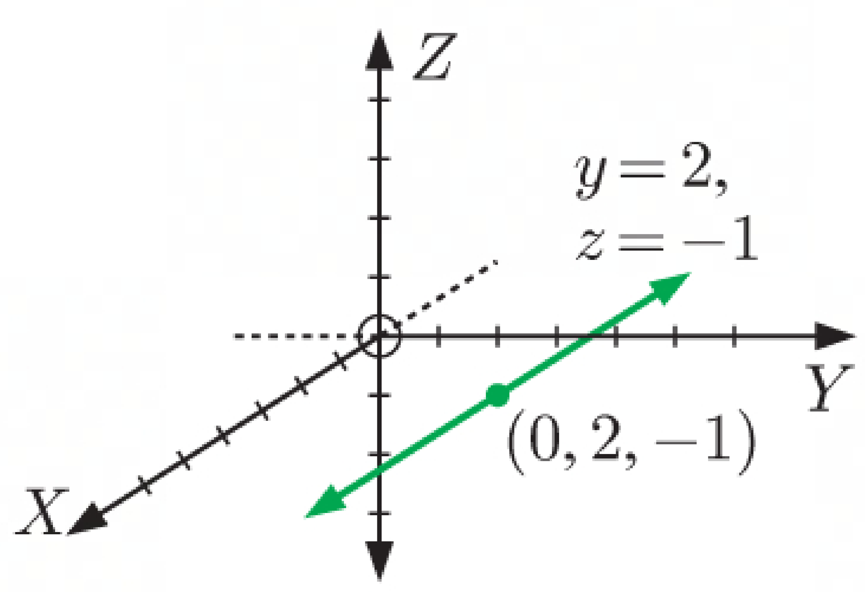
b
$$AP = \sqrt{(x-2)^2 + (y-5)^2 + (z-4)^2}$$
$$\therefore\ 1 = \sqrt{(x-2)^2 + (y-5)^2 + (z-4)^2}$$
$$\therefore\ (x-2)^2 + (y-5)^2 + (z-4)^2 = 1$$
So, P lies on a sphere with centre $(2, 5, 4)$ and radius 1 unit.

10 **a**



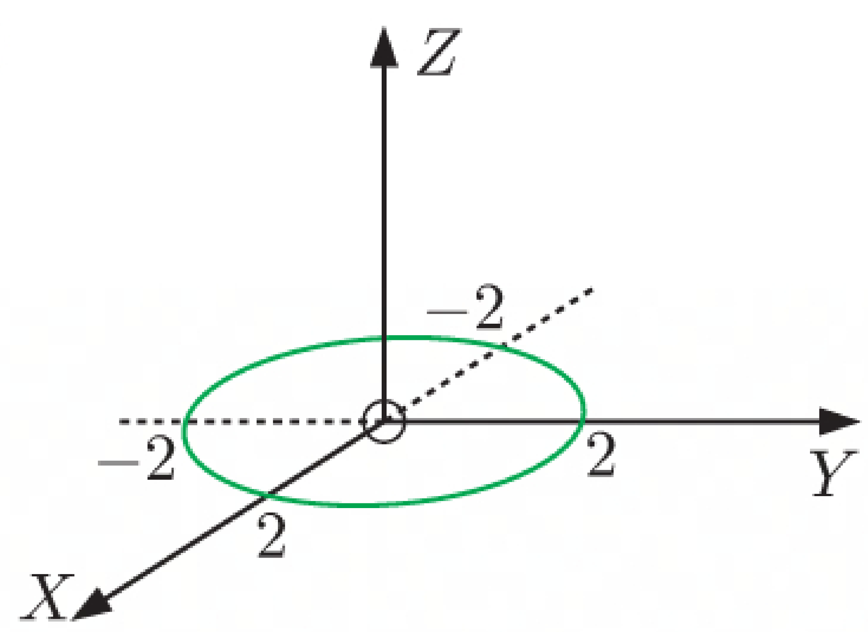
A plane parallel to the Y-Z plane, passing through $(3, 0, 0)$.

b



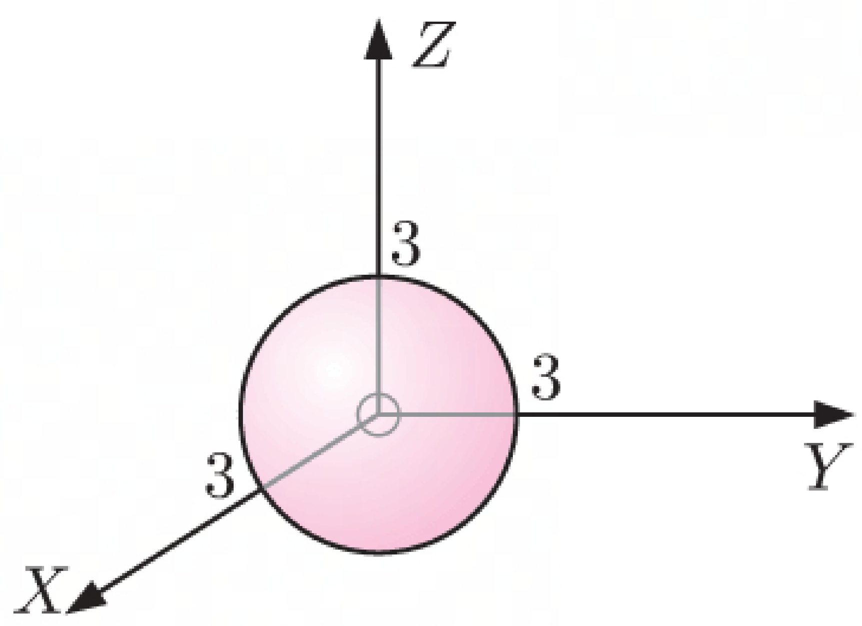
A line parallel to the X-axis, passing through $(0, 2, -1)$.

c



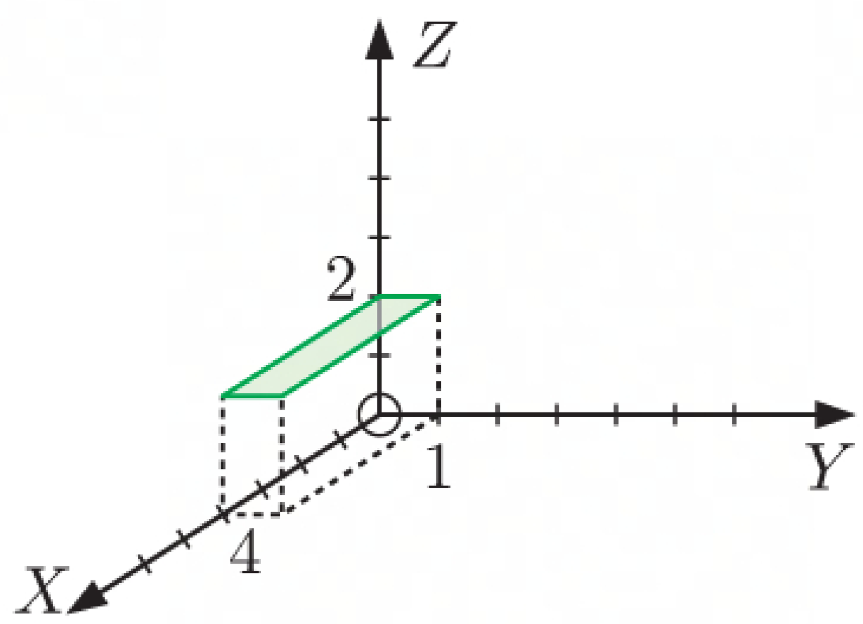
A circle in the X-Y plane, centre $(0, 0, 0)$, radius 2 units.

d



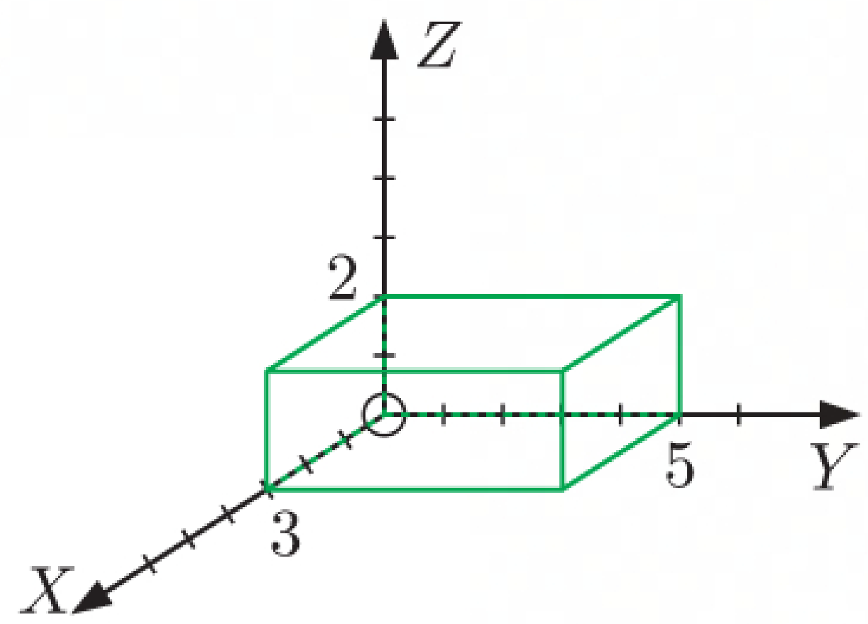
A sphere, centre $(0, 0, 0)$, radius 3 units.

e



A 4 by 1 rectangular plane 2 units above the X-Y plane (as shown).

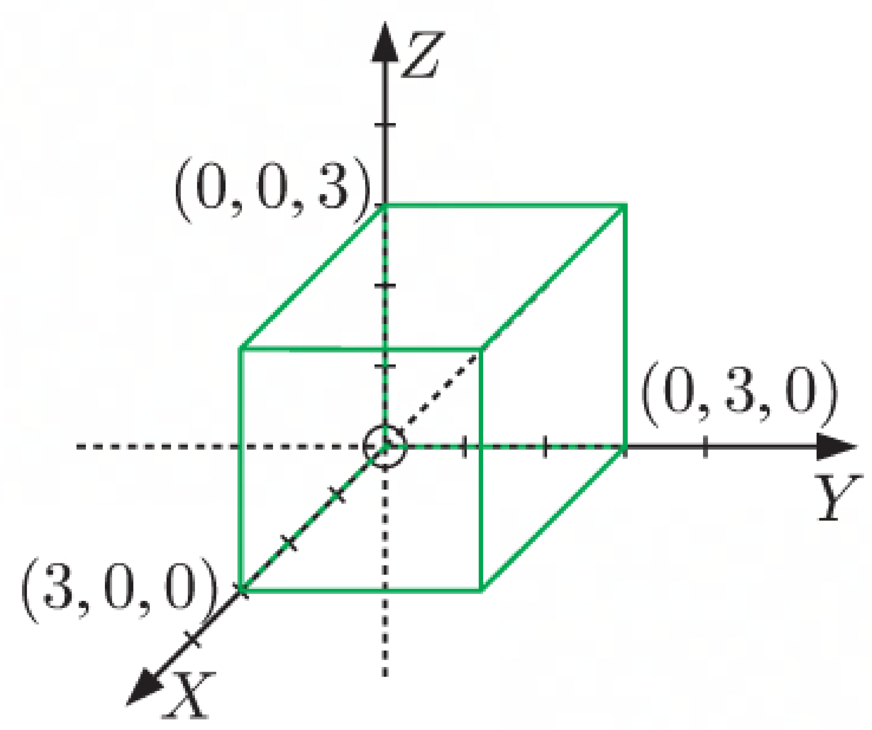
f



All points on and within a $3 \times 5 \times 2$ rectangular prism (as shown).

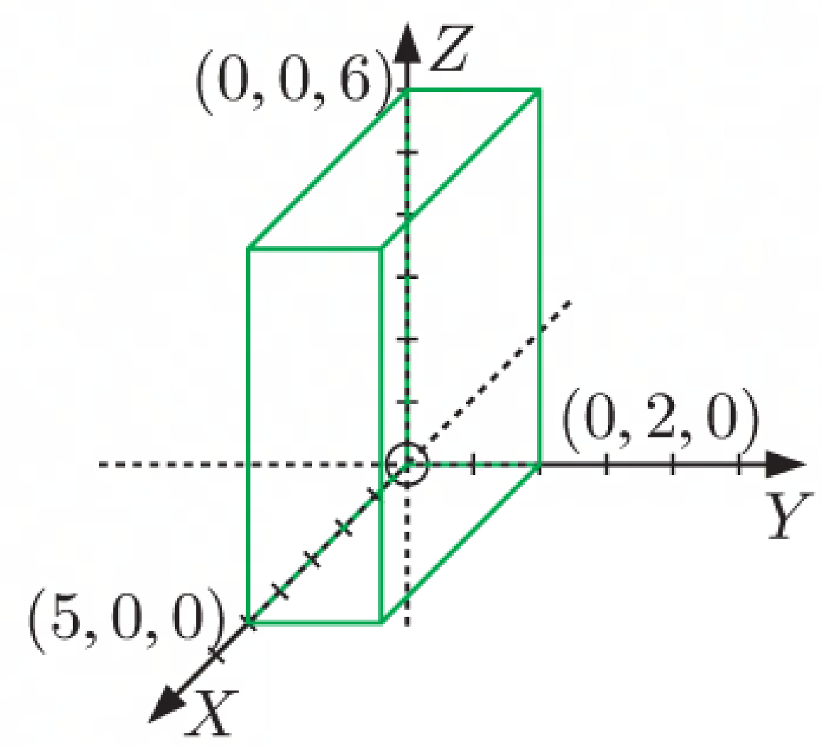
EXERCISE 10B

1 **a**



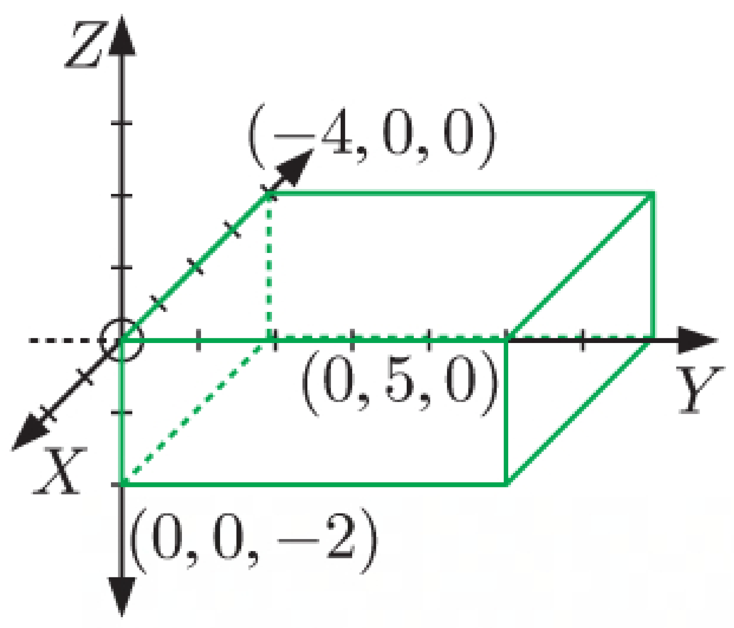
Volume = length × width × height
$= 3 \times 3 \times 3$
$= 27$ units^3

b



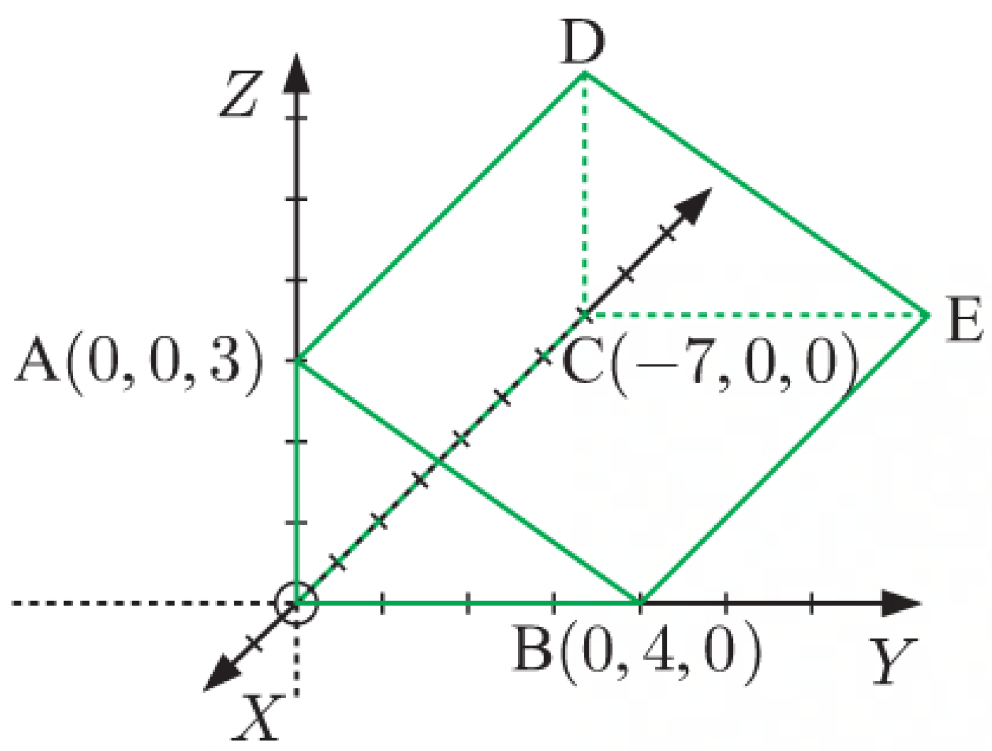
Volume = length × width × height
$= 5 \times 2 \times 6$
$= 60$ units^3

c



Volume = length × width × height
$= 5 \times 4 \times 2$
$= 40$ units^3

2 **a** D is $(-7, 0, 3)$ and E is $(-7, 4, 0)$.



b Volume = area of end × length
$= \frac{1}{2} \times$ base × height × length
$= \frac{1}{2} \times 4 \times 3 \times 7$
$= 42$ units^3

c $AB = \sqrt{(0-0)^2 + (4-0)^2 + (0-3)^2}$
$= \sqrt{0^2 + 4^2 + (-3)^2}$
$= \sqrt{0 + 16 + 9}$
$= \sqrt{25}$
$= 5$ units

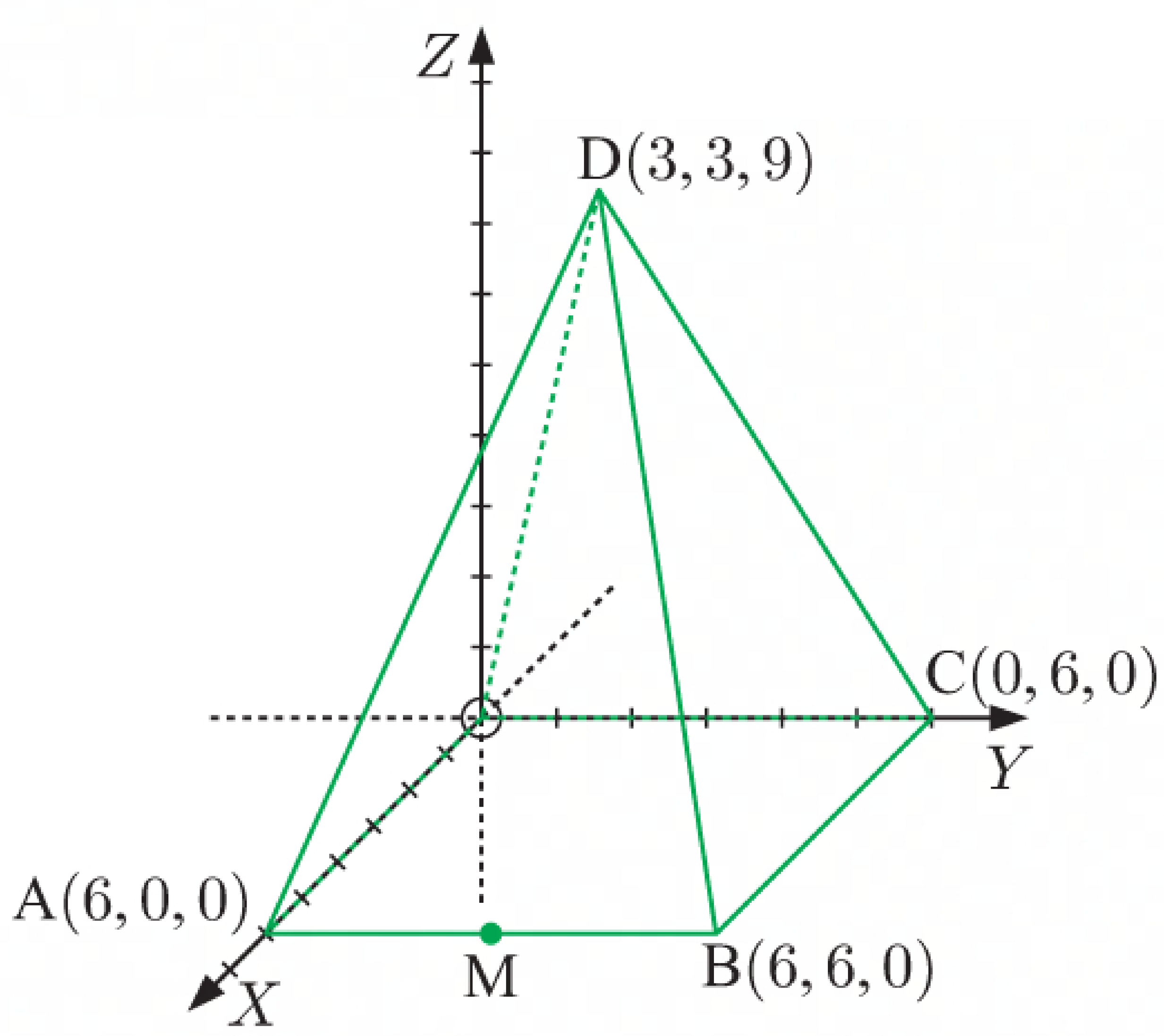
d Surface area of prism
= area of base + area of 2 triangular faces + area of 2 rectangular faces
$= 7 \times 4 + 2 \times$ area of $\triangle$OAB + area of quadrilateral OADC + area of quadrilateral ABED
$= 28 + 2 \times \frac{1}{2} \times 4 \times 3 + 7 \times 3 + 7 \times 5$
$= 96$ units^2

3 **a** To find the centre of the base, we locate the midpoints of the diagonals.

The midpoint of [OB] is $\left(\frac{0+6}{2}, \frac{0+6}{2}, \frac{0+0}{2}\right)$
which is $(3, 3, 0)$.

The midpoint of [AC] is $\left(\frac{6+0}{2}, \frac{0+6}{2}, \frac{0+0}{2}\right)$
which is $(3, 3, 0)$.

$\therefore$ the centre of the base is $(3, 3, 0)$.
$\therefore$ the apex $(3, 3, 9)$ lies directly above the centre of the base.

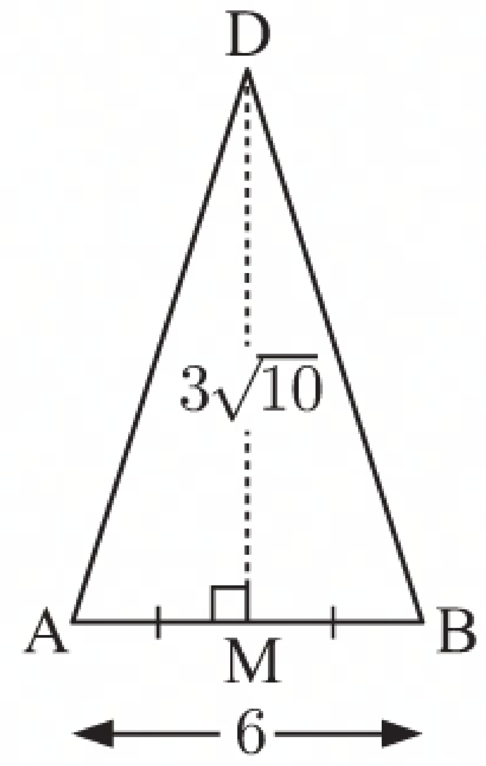


b Volume $= \frac{1}{3}$(area of base × height)
$= \frac{1}{3} \times 6 \times 6 \times 9$
$= 108$ units^3

c **i** M is $\left(\frac{6+6}{2}, \frac{0+6}{2}, \frac{0+0}{2}\right)$ which is $(6, 3, 0)$.

ii $\text{MD} = \sqrt{(3-6)^2 + (3-3)^2 + (9-0)^2}$
$= \sqrt{(-3)^2 + 0^2 + 9^2}$
$= \sqrt{9+0+81}$
$= \sqrt{90}$
$= 3\sqrt{10}$ units

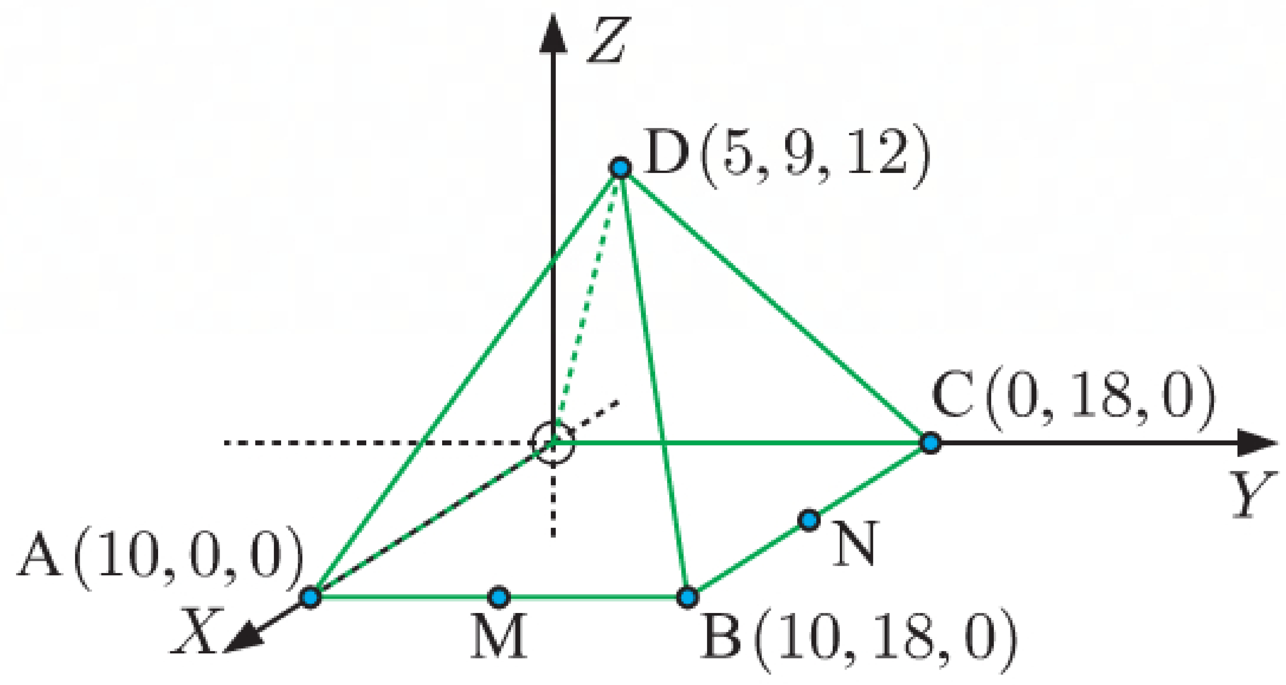
iii Area of triangle ABD $= \frac{1}{2} \times 6 \times 3\sqrt{10}$
$= 9\sqrt{10}$ units^2



Surface area of pyramid
$=$ area of base $+$ area of 4 triangular faces
$= 6 \times 6 \ + \ 4 \times 9\sqrt{10}$
$= 36 + 36\sqrt{10}$
$= 36(1+\sqrt{10})$ units^2

4 The midpoint of [OB] is $\left(\frac{0+10}{2}, \frac{0+18}{2}, \frac{0+0}{2}\right)$ which is $(5, 9, 0)$.

The midpoint of [AC] is $\left(\frac{10+0}{2}, \frac{0+18}{2}, \frac{0+0}{2}\right)$ which is $(5, 9, 0)$.



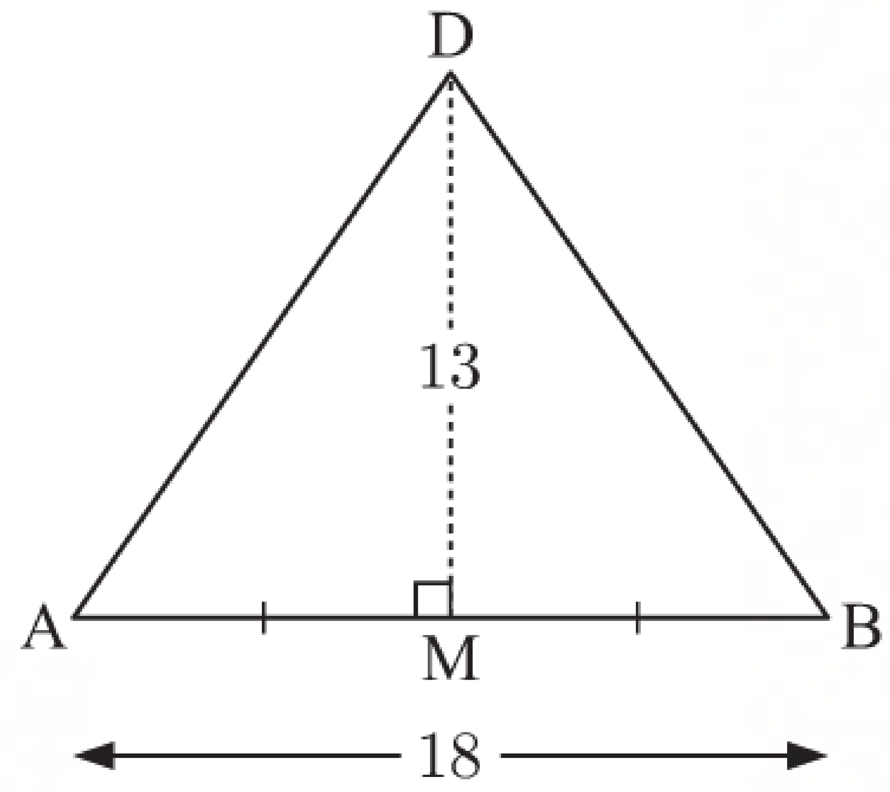
$\therefore$ the centre of the base is $(5, 9, 0)$ which lies directly below the apex $(5, 9, 12)$.

Volume $= \frac{1}{3}$(area of base $\times$ height)
$= \frac{1}{3} \times 18 \times 10 \times 12$
$= 720$ units^3

Let the midpoint of [AB] be M.

M is $\left(\frac{10+10}{2}, \frac{0+18}{2}, \frac{0+0}{2}\right)$ which is $(10, 9, 0)$.

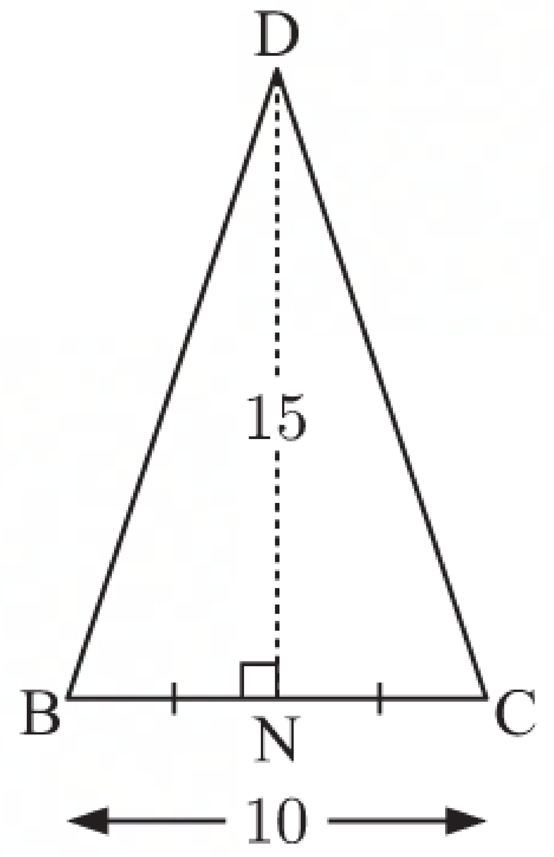
$\text{MD} = \sqrt{(5-10)^2 + (9-9)^2 + (12-0)^2}$
$= \sqrt{(-5)^2 + 0^2 + 12^2}$
$= \sqrt{25+0+144}$
$= \sqrt{169}$
$= 13$ units



Area of triangle ABD $= \frac{1}{2} \times 18 \times 13$
$= 117$ units^2

Let the midpoint of [BC] be N.

N is $\left(\frac{10+0}{2}, \frac{18+18}{2}, \frac{0+0}{2}\right)$ which is $(5, 18, 0)$.



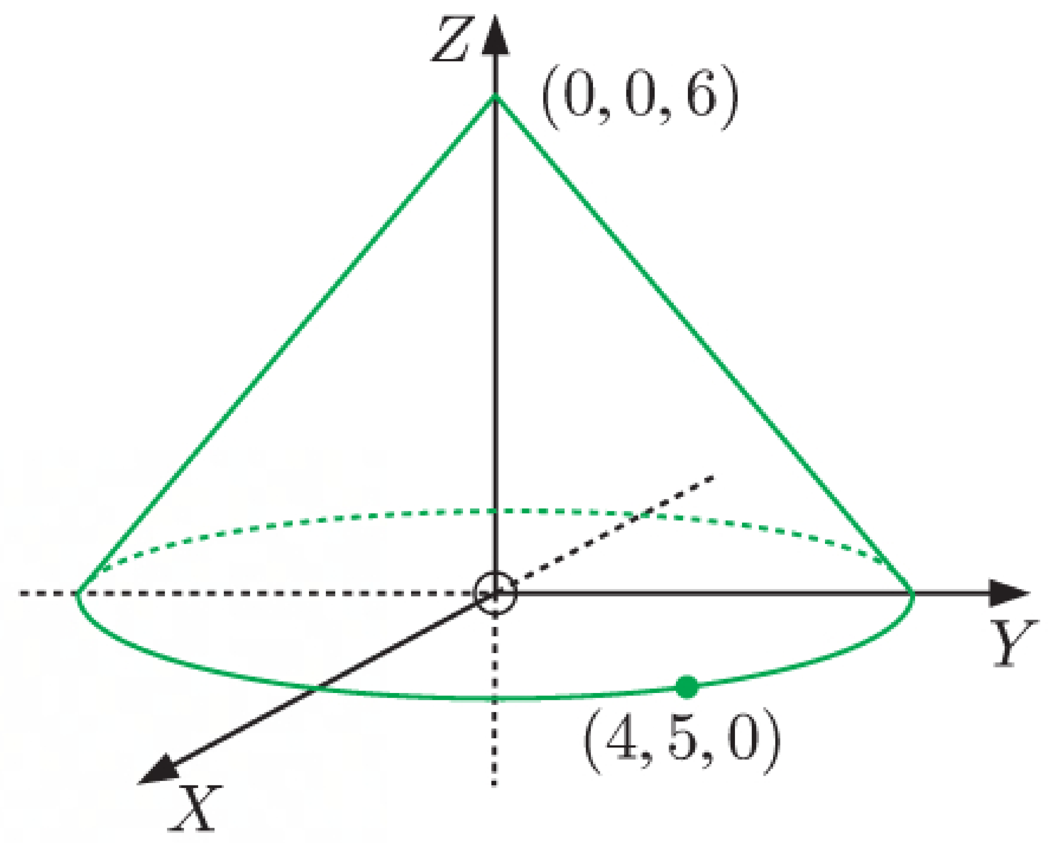
$$\begin{aligned} \text{ND} &= \sqrt{(5-5)^2+(9-18)^2+(12-0)^2} \\ &= \sqrt{0^2+(-9)^2+12^2} \\ &= \sqrt{0+81+144} \\ &= \sqrt{225} \\ &= 15 \text{ units} \end{aligned}$$

$$\begin{aligned} \text{Area of triangle BCD} &= \tfrac{1}{2} \times 10 \times 15 \\ &= 75 \text{ units}^2 \end{aligned}$$

$$\begin{aligned} \text{Surface area of pyramid} &= \text{area of base} + \text{area of 4 triangular faces} \\ &= 18 \times 10 + 2 \times \text{area of } \triangle\text{ABD} + 2 \times \text{area of } \triangle\text{BCD} \\ &= 180 + 2 \times 117 + 2 \times 75 \\ &= 564 \text{ units}^2 \end{aligned}$$

5 **a** Base radius of cone

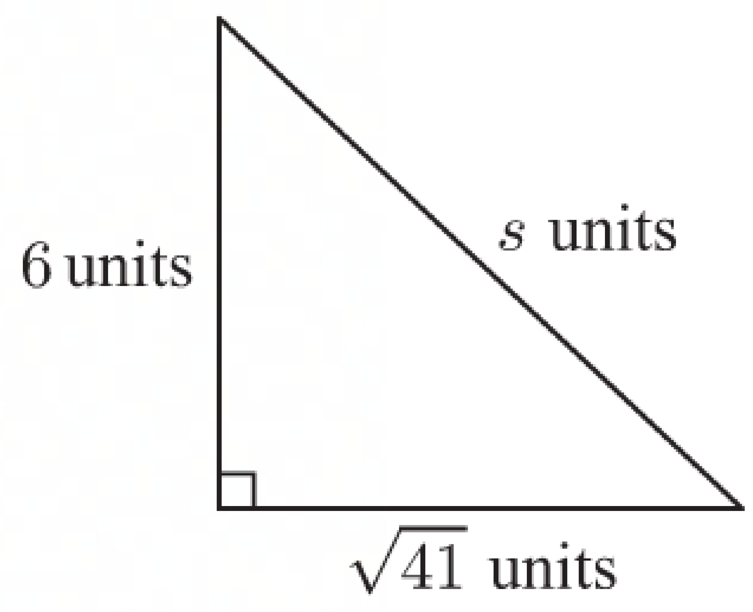
$$\begin{aligned} &= \text{distance from centre } (0, 0, 0) \text{ to point } (4, 5, 0) \\ &= \sqrt{(4-0)^2+(5-0)^2+(0-0)^2} \\ &= \sqrt{4^2+5^2+0^2} \\ &= \sqrt{16+25+0} \\ &= \sqrt{41} \text{ units} \end{aligned}$$



b
$$\begin{aligned} \text{Volume of cone} &= \tfrac{1}{3}(\text{area of base} \times \text{height}) \\ &= \tfrac{1}{3} \times \pi \times (\sqrt{41})^2 \times 6 \\ &= 82\pi \text{ units}^3 \end{aligned}$$

c Let the slant height be s units.

$$\begin{aligned} s^2 &= 6^2 + (\sqrt{41})^2 \quad \{\text{Pythagoras}\} \\ \therefore\ s^2 &= 36 + 41 \\ \therefore\ s^2 &= 77 \\ \therefore\ s &= \sqrt{77} \quad \{\text{as } s > 0\} \end{aligned}$$



So, the slant height of the cone is $\sqrt{77}$ units.

d
$$\begin{aligned} \text{Surface area of cone} &= \pi r s + \pi r^2 \\ &= \pi \times \sqrt{41} \times \sqrt{77} + \pi \times (\sqrt{41})^2 \\ &\approx 305 \text{ units}^2 \end{aligned}$$

6 **a** Radius of sphere

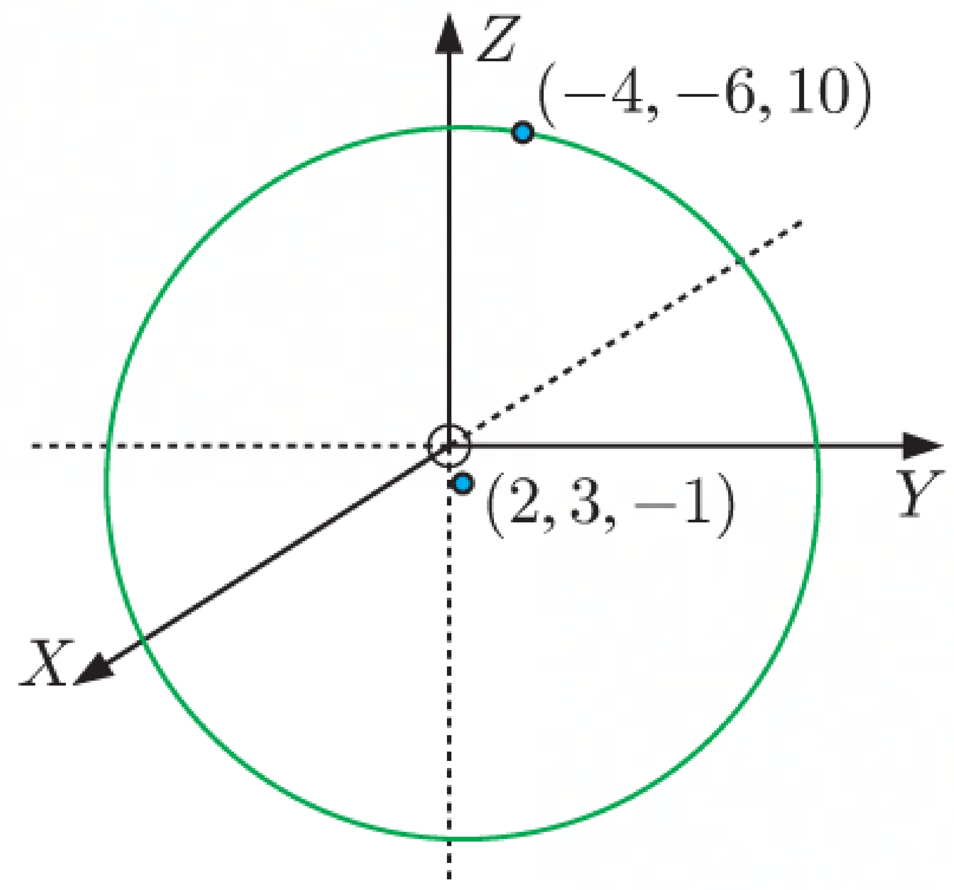
$= \text{distance from centre } (2, 3, -1) \text{ to point } (-4, -6, 10)$

$= \sqrt{(-4-2)^2 + (-6-3)^2 + (10 - -1)^2}$

$= \sqrt{(-6)^2 + (-9)^2 + 11^2}$

$= \sqrt{36 + 81 + 121}$

$= \sqrt{238}$ units



b Volume of sphere $= \frac{4}{3}\pi r^3$

$= \frac{4}{3} \times \pi \times (\sqrt{238})^3$

$\approx 15\,400$ units^3

7 **a** Centre of sphere = midpoint of [PQ]

$= \left(\frac{-1 + -5}{2}, \frac{1+7}{2}, \frac{2 + -8}{2}\right)$

$= (-3, 4, -3)$

b Radius of sphere = distance from centre $(-3, 4, -3)$ to point P$(-1, 1, 2)$

$= \sqrt{(-1 - -3)^2 + (1-4)^2 + (2 - -3)^2}$

$= \sqrt{2^2 + (-3)^2 + 5^2}$

$= \sqrt{4 + 9 + 25}$

$= \sqrt{38}$ units

c Volume of sphere $= \frac{4}{3}\pi r^3$

$= \frac{4}{3} \times \pi \times (\sqrt{38})^3$

≈ 981 units^3

Surface area of sphere $= 4\pi r^2$

$= 4 \times \pi \times (\sqrt{38})^2$

≈ 478 units^2

8 **a** Radius of cylinder = distance from centre of base $(1, -3, 0)$ to point $(-2, -2, 0)$

$= \sqrt{(-2-1)^2 + (-2 - -3)^2 + (0-0)^2}$

$= \sqrt{(-3)^2 + 1^2 + 0^2}$

$= \sqrt{9+1}$

$= \sqrt{10}$ units

Volume of cylinder $= 40\pi$ units^3

$\therefore \quad \pi r^2 h = 40\pi$

$\therefore \quad \pi \times (\sqrt{10})^2 \times h = 40\pi$

$\therefore \quad 10h = 40$

$\therefore \quad h = 4$

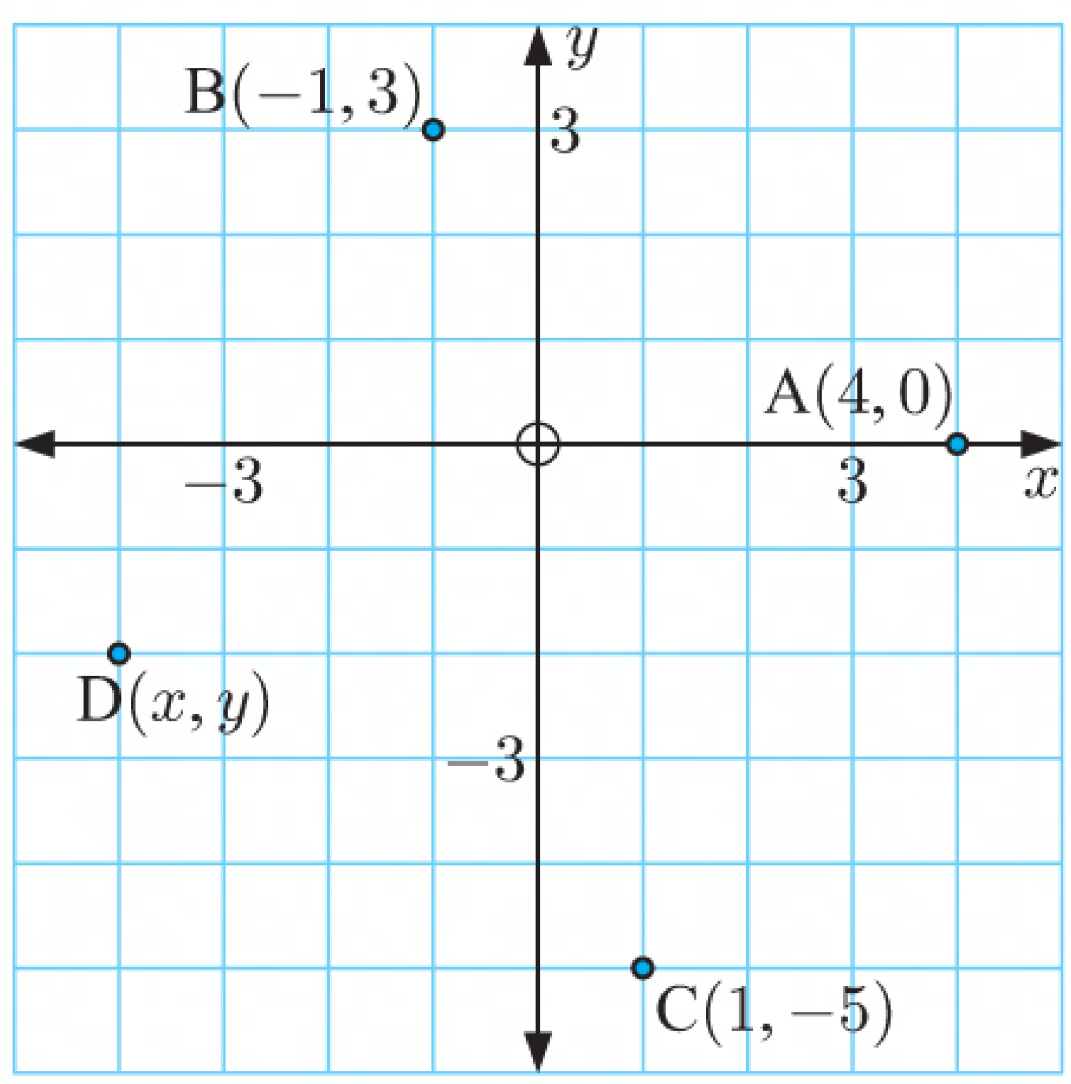
So, the height of the cylinder is 4 units.

b From **a**, the radius of the cylinder is $\sqrt{10}$ units, and the height of the cylinder is 4 units.
Since the base of the cylinder lies in the X-Y plane, a point P(x, y, z) lies on the curved surface of the cylinder if:
$(x-1)^2 + (y+3)^2 = (\sqrt{10})^2 = 10$ and $0 \leqslant z \leqslant 4$.
The point $(3, k, 2)$ lies on the curved surface of the cylinder.

$$\therefore\ (3-1)^2 + (k+3)^2 = 10$$
$$\therefore\ 2^2 + (k+3)^2 = 10$$
$$\therefore\ (k+3)^2 = 6$$
$$\therefore\ k+3 = \pm\sqrt{6}$$
$$\therefore\ k = -3 \pm \sqrt{6}$$

c Surface area of cylinder $= 2\pi rh + 2\pi r^2$
$$= 2 \times \pi \times \sqrt{10} \times 4\ +\ 2 \times \pi \times (\sqrt{10})^2$$
$$\approx 142 \text{ units}^2$$

9 **a** Let D have coordinates (x, y, z).
Since A$(4, 0, 2)$, B$(-1, 3, 2)$, and C$(1, -5, 2)$ lie in the plane $Z = 2$, D must also lie in this plane. $\therefore\ z = 2$
Consider the square base of the pyramid in two dimensions.



Now, gradient of [AB] = gradient of [CD]

$$\therefore\ \frac{3-0}{-1-4} = \frac{y-(-5)}{x-1}$$
$$\therefore\ \frac{3}{-5} = \frac{y+5}{x-1}$$
$$\therefore\ 3(x-1) = -5(y+5)$$
$$\therefore\ 3x-3 = -5y-25$$
$$\therefore\ 3x+5y = -22$$
$$\therefore\ 5y = -22-3x$$
$$\therefore\ y = \frac{-22-3x}{5} \quad \text{.... (1)}$$

and gradient of [AC] = gradient of [BD]

$$\therefore\ \frac{-5-0}{1-4} = \frac{y-3}{x--1}$$
$$\therefore\ \frac{-5}{-3} = \frac{y-3}{x+1}$$
$$\therefore\ -5(x+1) = -3(y-3)$$
$$\therefore\ -5x-5 = -3y+9$$
$$\therefore\ 5x-3y = -14 \quad \text{.... (2)}$$

Substituting (1) into (2) gives $\quad 5x - 3\left(\frac{-22-3x}{5}\right) = -14$

$$\therefore\ 5x + \frac{66+9x}{5} = -14$$
$$\therefore\ 25x + 66 + 9x = -70$$
$$\therefore\ 34x = -136$$
$$\therefore\ x = -4$$

Substituting $x = -4$ into (2) gives

$$\begin{aligned} 5(-4) - 3y &= -14 \\ \therefore\ -20 - 3y &= -14 \\ \therefore\ -3y &= 6 \\ \therefore\ y &= -2 \end{aligned}$$

So, D has x-coordinate -4, y-coordinate -2, and z-coordinate 2 (since it lies on the same plane as A, B, and C).

$\therefore$ D is $(-4, -2, 2)$.

b We first find the centre of the square base ABCD.

The midpoint of [BC] is $\left(\dfrac{-1+1}{2}, \dfrac{3+-5}{2}, \dfrac{2+2}{2}\right) = (0, -1, 2)$

The midpoint of [AD] is $\left(\dfrac{4+-4}{2}, \dfrac{0+-2}{2}, \dfrac{2+2}{2}\right) = (0, -1, 2)$

So, the centre of the square base ABCD is $(0, -1, 2)$.

$\therefore$ the apex V of the pyramid has coordinates $(0, -1, z)$ where $z > 2$ since the apex lies directly above the centre of the base.

Now, surface area of a square-based pyramid $=$ area of 4 identical sides $+$ area of square base $= 210$ units^2

$$\begin{aligned} AB &= \sqrt{(-1-4)^2 + (3-0)^2} \\ &= \sqrt{(-5)^2 + 3^2} \\ &= \sqrt{25+9} \\ &= \sqrt{34} \end{aligned}$$

$\therefore$ area of square base $= (\sqrt{34})^2 = 34$ units^2

$\therefore$ area of 4 identical sides $+ 34$ units$^2 = 210$ units^2

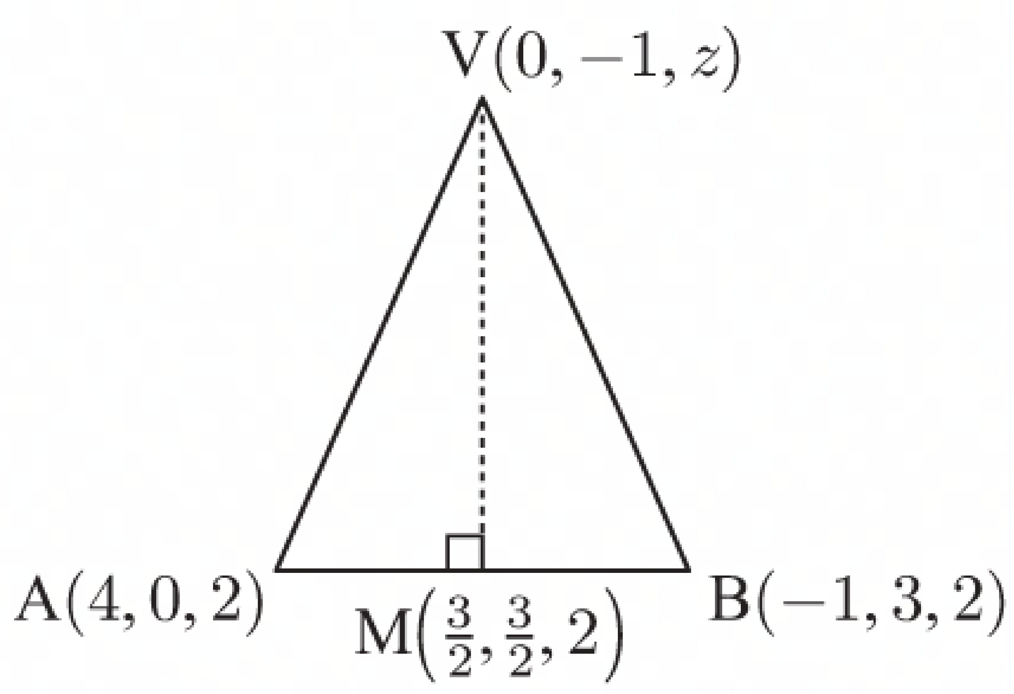
$\therefore$ area of 4 identical sides $= 176$ units^2

$\therefore$ area of triangular face $= 44$ units^2

[AB] has midpoint $M\left(\dfrac{4+-1}{2}, \dfrac{0+3}{2}, \dfrac{2+2}{2}\right)$ which is $M\left(\frac{3}{2}, \frac{3}{2}, 2\right)$.

$$\begin{aligned} VM &= \sqrt{(\tfrac{3}{2}-0)^2 + (\tfrac{3}{2}--1)^2 + (2-z)^2} \\ &= \sqrt{(\tfrac{3}{2})^2 + (\tfrac{5}{2})^2 + (2-z)^2} \\ &= \sqrt{\tfrac{17}{2} + (2-z)^2} \end{aligned}$$

$AB = \sqrt{34}$ {from above}



$$\text{Area of triangle ABV} = 44 \text{ units}^2$$
$$\therefore\ \tfrac{1}{2} \times \text{AB} \times \text{VM} = 44$$
$$\therefore\ \tfrac{1}{2} \times \sqrt{34} \times \sqrt{\tfrac{17}{2} + (2-z)^2} = 44$$
$$\therefore\ \sqrt{\tfrac{17}{2} + (2-z)^2} = \frac{88}{\sqrt{34}}$$
$$\therefore\ \tfrac{17}{2} + (2-z)^2 = \frac{3872}{17}$$
$$\therefore\ (2-z)^2 = \frac{7455}{34}$$
$$\therefore\ 2 - z = -\sqrt{\frac{7455}{34}} \quad \{\text{as } z > 2\}$$
$$\therefore\ z = 2 + \sqrt{\frac{7455}{34}}$$
$$\therefore\ z \approx 16.8$$

Since the base of the pyramid is in the plane $Z = 2$, the height of the pyramid, to 1 decimal place, is $16.8 - 2 = 14.8$ units.

c Volume of pyramid $= \frac{1}{3} \times \text{base} \times \text{height}$

$$\approx \tfrac{1}{3} \times 34 \times 14.8$$
$$\approx 168 \text{ units}^3$$

10 **a** **i** A has coordinates $(10, 40)$ on the 2-dimensional plane.

$\therefore$ A is $(10, 40, 0)$ on the 3-dimensional plane.

B has coordinates $(50, 160)$ on the 2-dimensional plane.

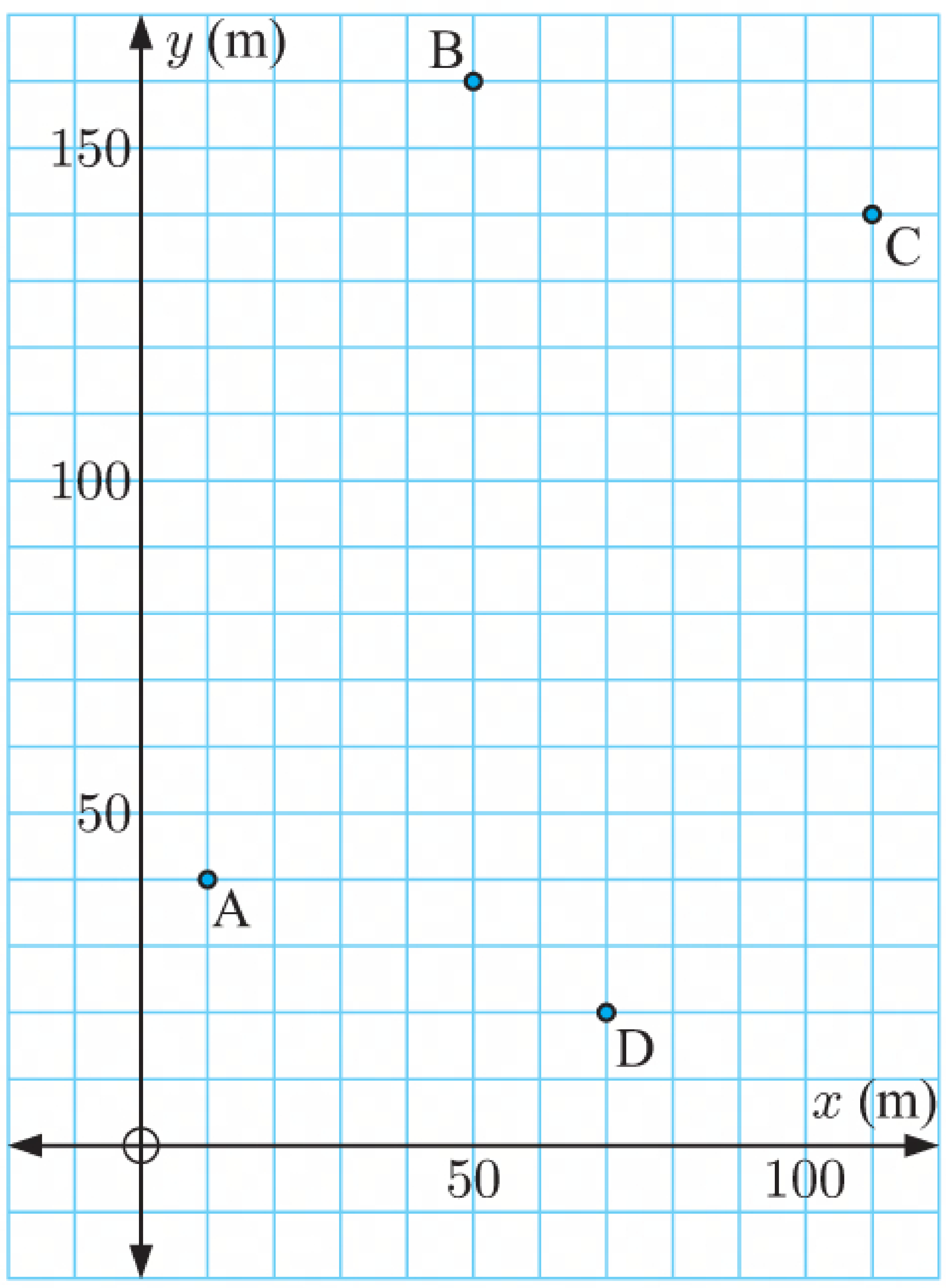
$\therefore$ B is $(50, 160, 0)$ on the 3-dimensional plane.

C has coordinates $(110, 140)$ on the 2-dimensional plane.

$\therefore$ C is $(110, 140, 0)$ on the 3-dimensional plane.

D has coordinates $(70, 20)$ on the 2-dimensional plane.

$\therefore$ D is $(70, 20, 0)$ on the 3-dimensional plane.



ii The apex is 15 m above the centre of the base.

To find the centre of the base, we locate the midpoints of the diagonals.

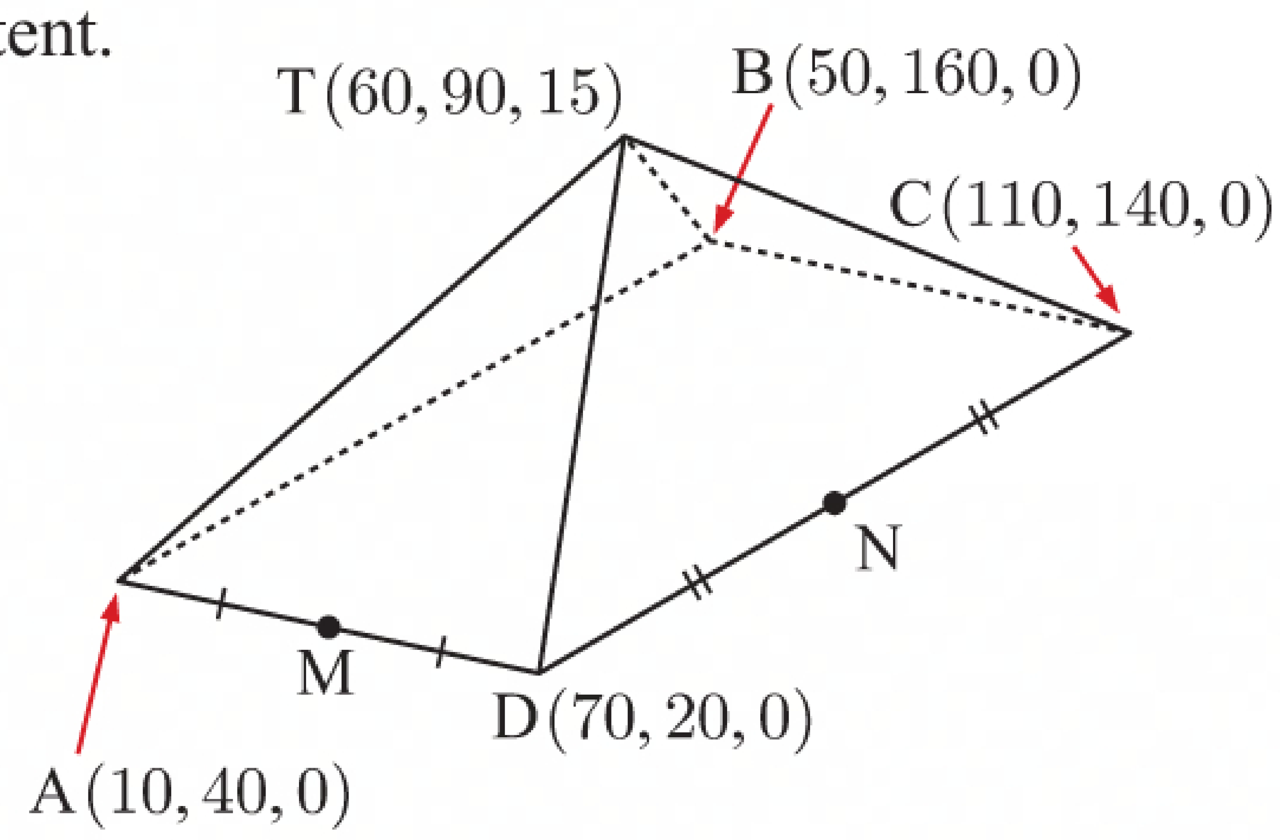
The midpoint of [AC] is $\left(\frac{10+110}{2}, \frac{40+140}{2}, \frac{0+0}{2}\right)$ which is $(60, 90, 0)$.

The midpoint of [BD] is $\left(\frac{50+70}{2}, \frac{160+20}{2}, \frac{0+0}{2}\right)$ which is $(60, 90, 0)$.

$\therefore$ the centre of the base is $(60, 90, 0)$.

$\therefore$ the apex is $(60, 90, 15)$.

b We need to find the side lengths of the base of the tent.



$$\begin{aligned} AD &= \sqrt{(70-10)^2 + (20-40)^2 + (0-0)^2} \\ &= \sqrt{60^2 + (-20)^2 + 0^2} \\ &= \sqrt{4000} \text{ m} \end{aligned}$$

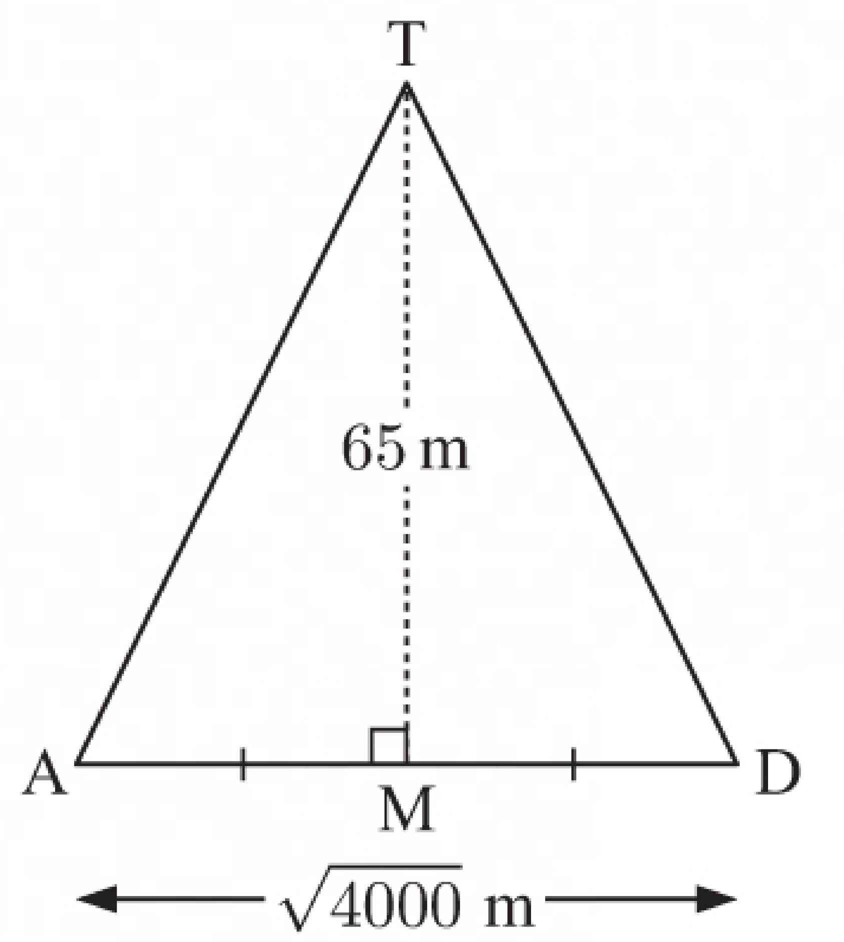
$$\begin{aligned} CD &= \sqrt{(70-110)^2 + (20-140)^2 + (0-0)^2} \\ &= \sqrt{(-40)^2 + (-120)^2 + 0^2} \\ &= \sqrt{16\,000} \text{ m} \end{aligned}$$

$$\begin{aligned} \text{Volume of air inside tent} &= \tfrac{1}{3}(\text{area of base} \times \text{height}) \\ &= \tfrac{1}{3} \times \sqrt{16\,000} \times \sqrt{4000} \times 15 \\ &= 40\,000 \text{ m}^3 \end{aligned}$$

c Let the apex $(60, 90, 15)$ be T.

Let the midpoint of side [AD] be M.

M is $\left(\dfrac{10+70}{2}, \dfrac{40+20}{2}, \dfrac{0+0}{2}\right)$ which is $(40, 30, 0)$.

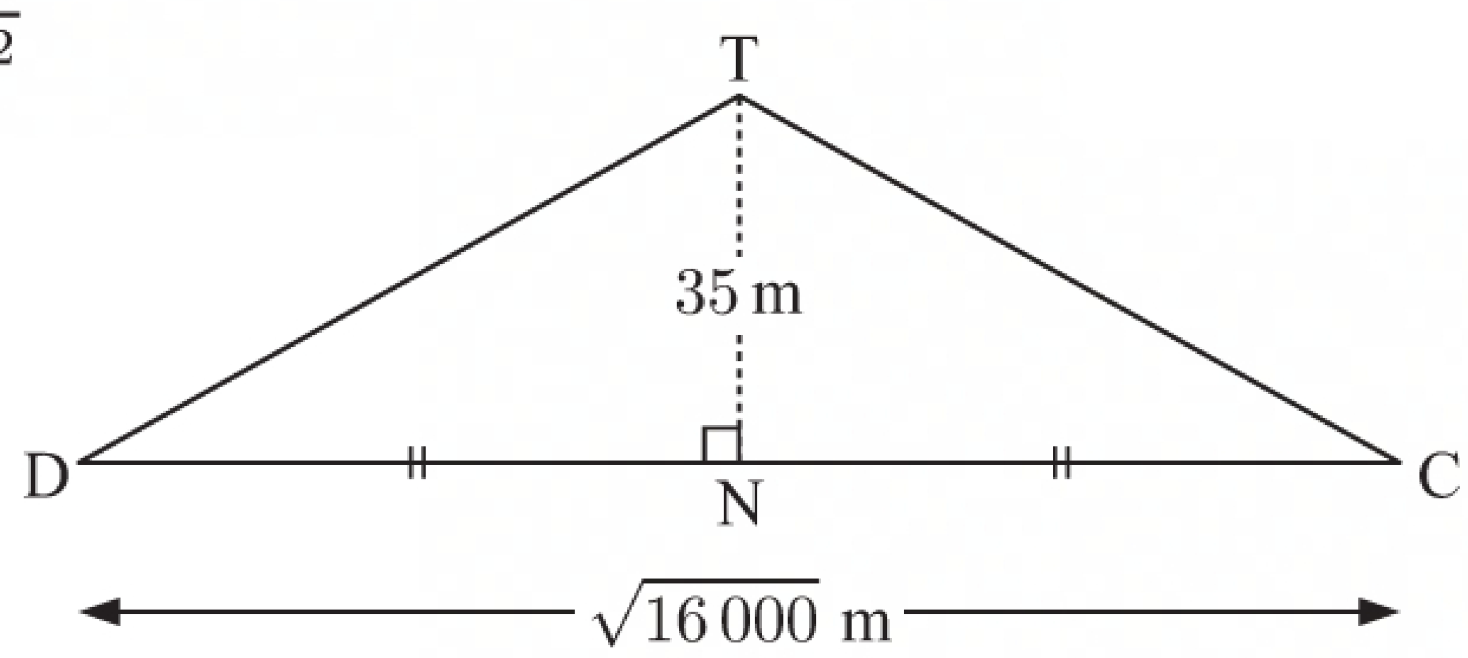


$$\begin{aligned} MT &= \sqrt{(60-40)^2 + (90-30)^2 + (15-0)^2} \\ &= \sqrt{20^2 + 60^2 + 15^2} \\ &= \sqrt{4225} \\ &= 65 \text{ m} \end{aligned}$$

$$\begin{aligned} \text{Area of triangle ADT} &= \tfrac{1}{2} \times \sqrt{4000} \times 65 \\ &= \tfrac{65}{2}\sqrt{4000} \text{ m}^2 \end{aligned}$$

Let the midpoint of side [CD] be N.

N is $\left(\dfrac{110+70}{2}, \dfrac{140+20}{2}, \dfrac{0+0}{2}\right)$ which is $(90, 80, 0)$.



$$\begin{aligned} NT &= \sqrt{(60-90)^2 + (90-80)^2 + (15-0)^2} \\ &= \sqrt{(-30)^2 + 10^2 + 15^2} \\ &= \sqrt{1225} \\ &= 35 \text{ m} \end{aligned}$$

$$\begin{aligned} \text{Area of triangle CDT} &= \tfrac{1}{2} \times \sqrt{16\,000} \times 35 \\ &= \tfrac{35}{2}\sqrt{16\,000} \text{ m}^2 \end{aligned}$$

$$\begin{aligned} \text{Area of material needed for the tent} &= \text{area of 4 triangular faces} \\ &= 2 \times \text{area of } \triangle ADT + 2 \times \text{area of } \triangle CDT \\ &= 2 \times \tfrac{65}{2}\sqrt{4000} + 2 \times \tfrac{35}{2}\sqrt{16\,000} \\ &\approx 8540 \text{ m}^2 \end{aligned}$$

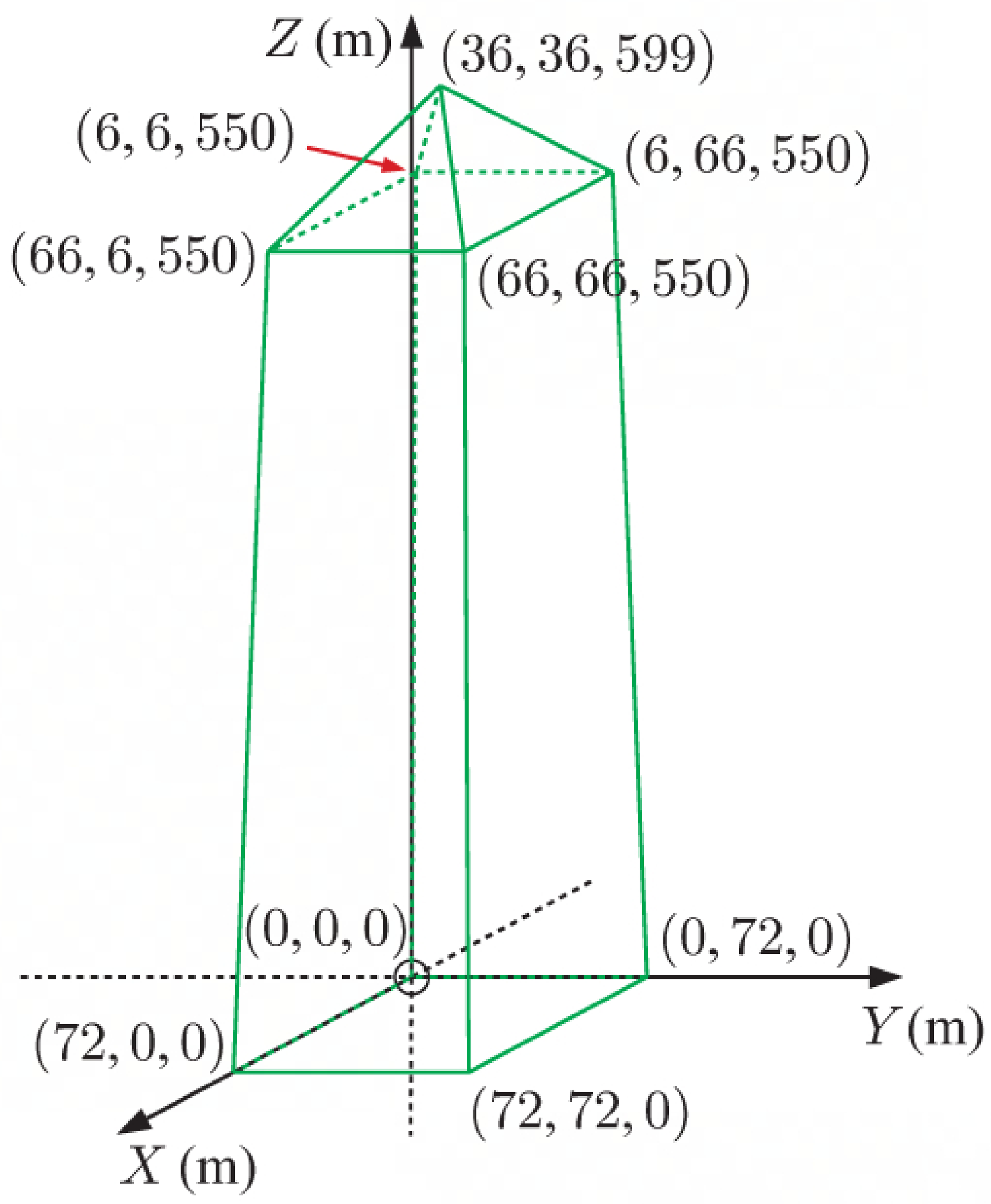
11 Let the pyramid at the top of the structure have volume V_{top}, and the rest of the structure have volume V_{base}.

The pyramid at the top has

base side lengths $66 - 6 = 60$ m

and height $599 - 550 = 49$ m

$$\therefore \quad V_{\text{top}} = \tfrac{1}{3} \times 60 \times 60 \times 49$$
$$= 58\,800 \text{ m}^3$$

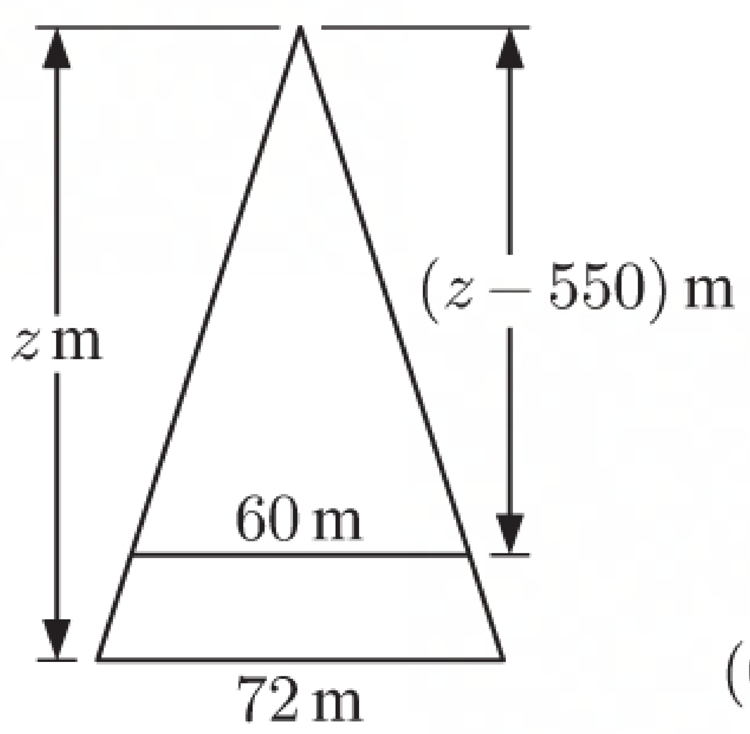


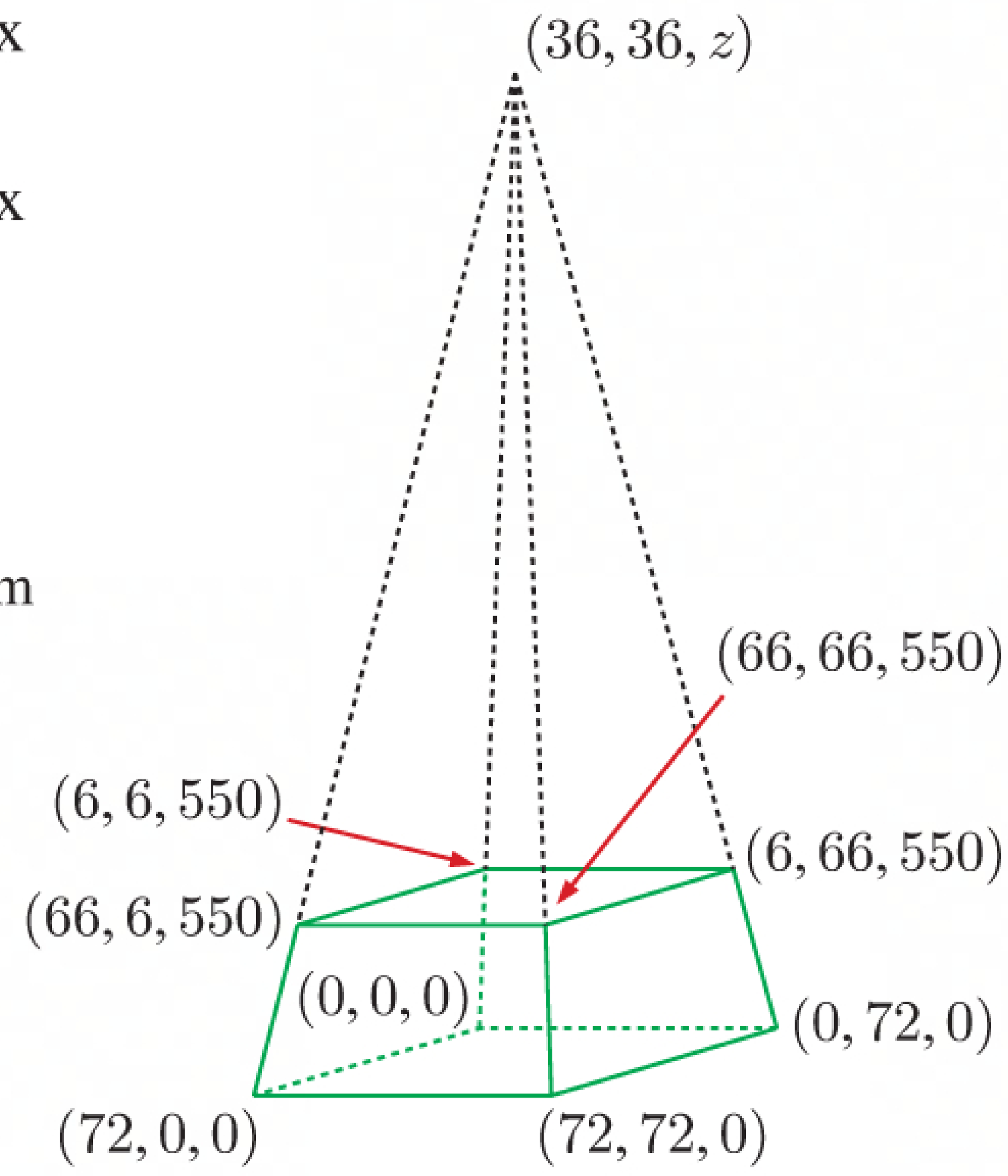
Now, the rest of the structure is *part* of a pyramid with apex $(36,\ 36,\ z)$, $\ z > 0$.

Consider the cross-section of this pyramid through the apex and parallel to the Y-Z plane.

Using similar triangles,

$$\frac{z}{z - 550} = \frac{72}{60}$$
$$\therefore \quad 60z = 72z - 39\,600$$
$$\therefore \quad 12z = 39\,600$$
$$\therefore \quad z = 3300$$





$\therefore \quad V_{\text{base}}$

$=$ volume of the whole pyramid

$\quad -$ volume of the top pyramid

$$= \tfrac{1}{3} \times 72 \times 72 \times 3300 \ - \ \tfrac{1}{3} \times 60 \times 60 \times (3300 - 550)$$
$$= 2\,402\,400 \text{ m}^3$$

$$\therefore \quad \text{volume of the whole building} = V_{\text{top}} + V_{\text{base}}$$
$$= 58\,800 + 2\,402\,400$$
$$= 2\,461\,200 \text{ m}^3$$

EXERCISE 10C

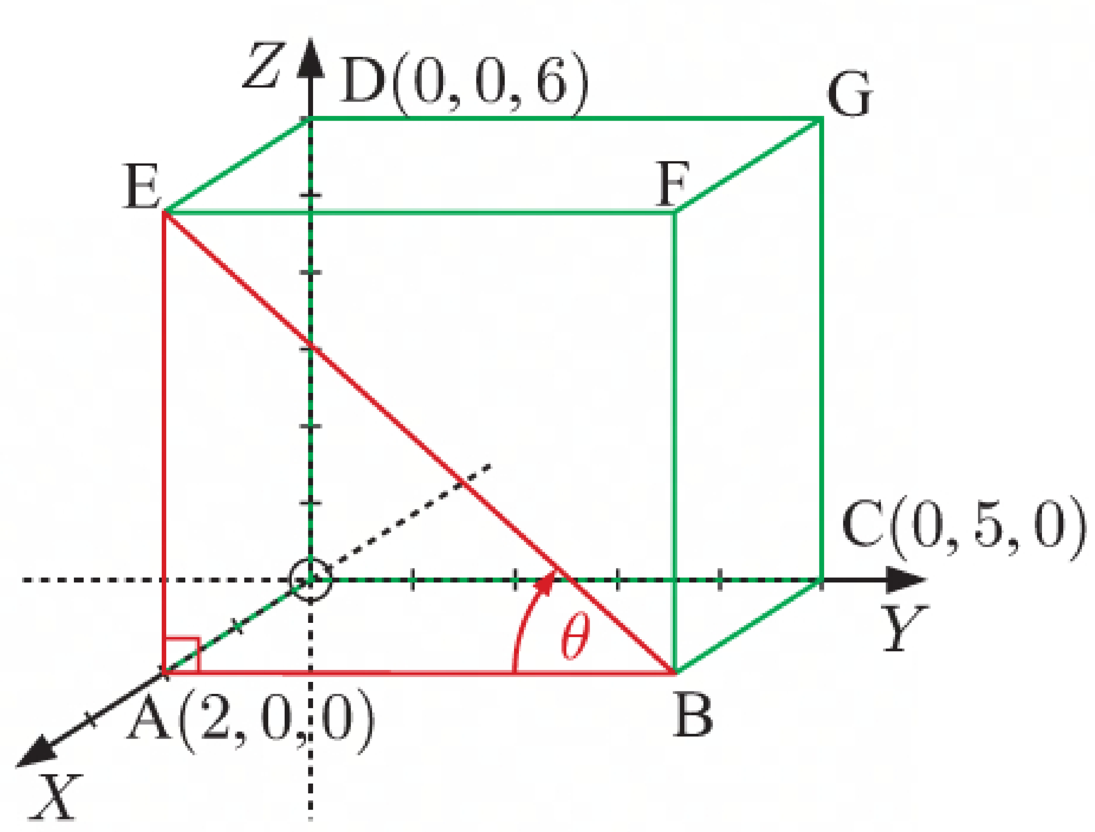
1 **a** The required angle is $\text{A}\widehat{\text{B}}\text{E}$.

Now $\quad \text{AE} = 6$ units

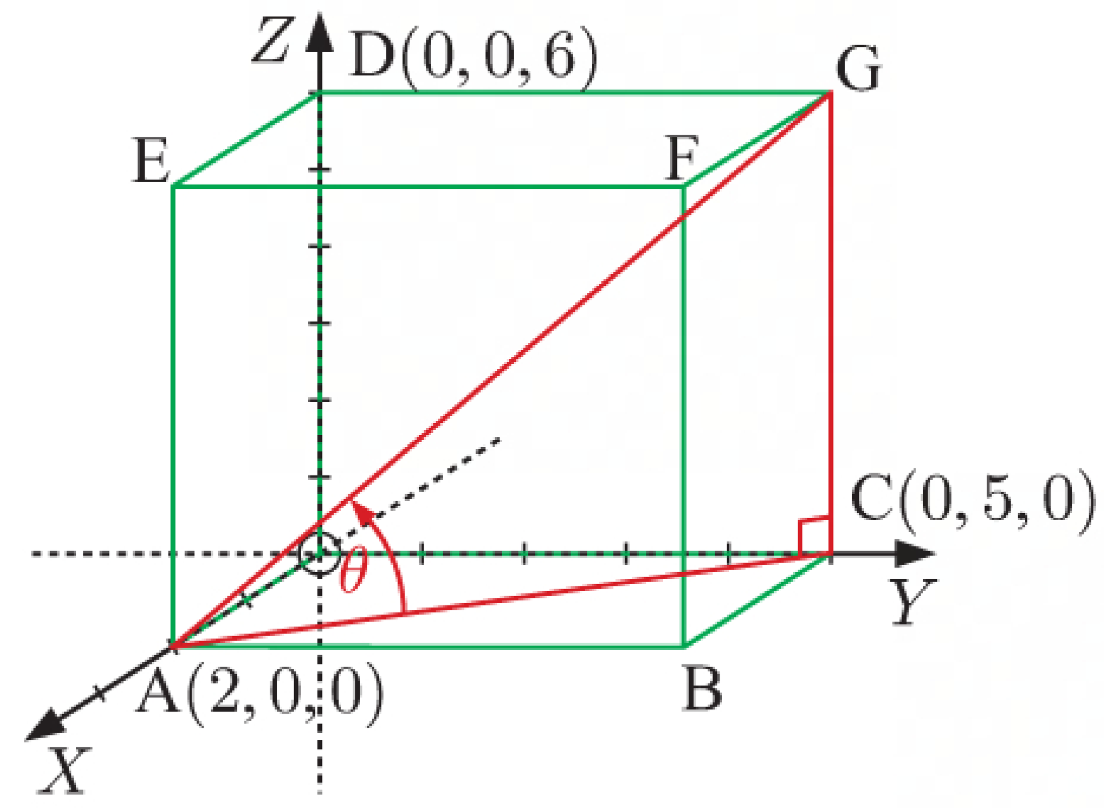
and $\quad \text{AB} = 5$ units

$$\therefore \quad \tan\theta = \frac{6}{5}$$
$$\therefore \quad \theta = \tan^{-1}\left(\tfrac{6}{5}\right)$$
$$\approx 50.2^\circ$$

The angle is about 50.2°.



b The required angle is $C\widehat{A}G$.



Now $\quad CG = 6$ units

and $\quad AC = \sqrt{(0-2)^2 + (5-0)^2 + (0-0)^2}$

$= \sqrt{(-2)^2 + 5^2 + 0^2}$

$= \sqrt{4 + 25 + 0}$

$= \sqrt{29}$ units

$\therefore \quad \tan\theta = \dfrac{6}{\sqrt{29}}$

$\therefore \quad \theta = \tan^{-1}\left(\frac{6}{\sqrt{29}}\right)$

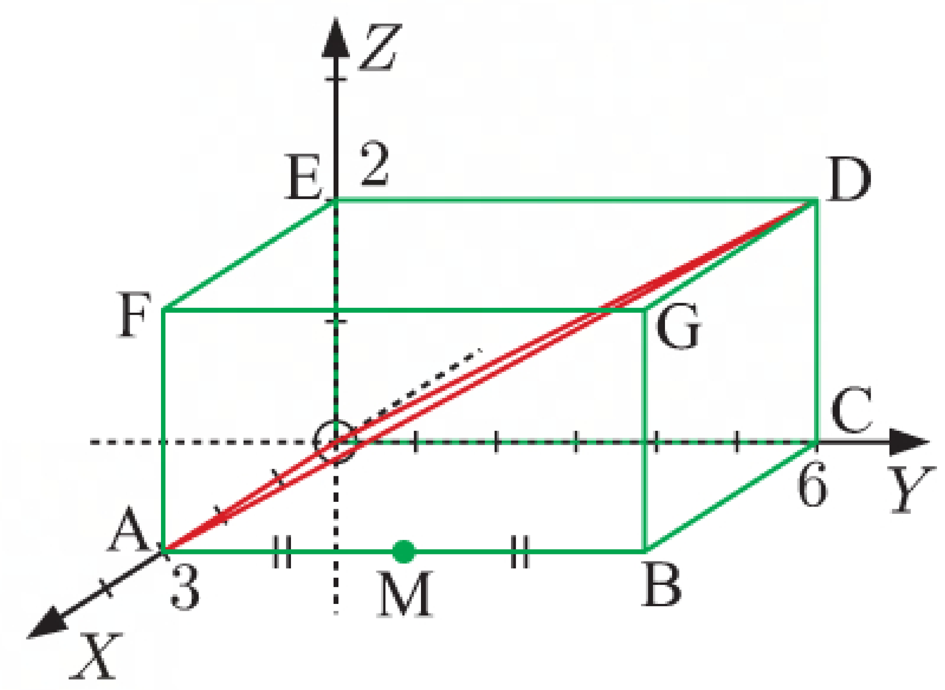
$\approx 48.1°$

The angle is about $48.1°$.

2 **a** A is $(3, 0, 0)$ and B is $(3, 6, 0)$.

The midpoint M of [AB] is $\left(\dfrac{3+3}{2}, \dfrac{0+6}{2}, \dfrac{0+0}{2}\right)$ which is $(3, 3, 0)$.

b D is $(0, 6, 2)$, and $\triangle ADO$ is right angled at O.



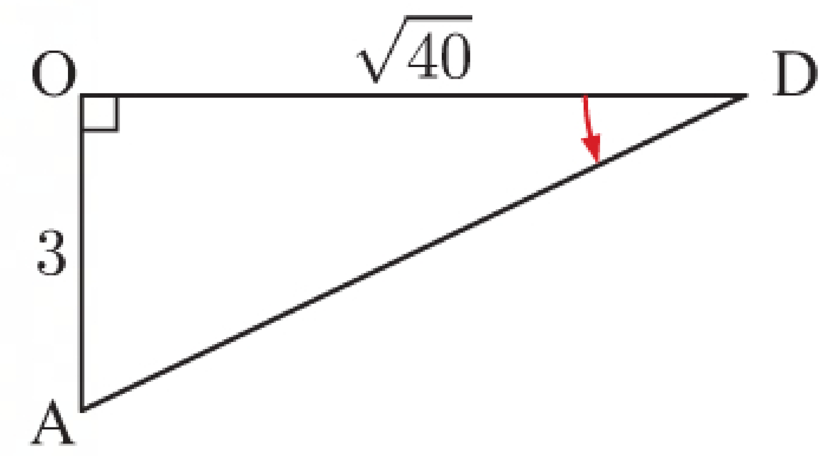
Now $\quad OA = 3$ units

and $\quad OD = \sqrt{(0-0)^2 + (6-0)^2 + (2-0)^2}$

$= \sqrt{0^2 + 6^2 + 2^2}$

$= \sqrt{0 + 36 + 4}$

$= \sqrt{40}$ units

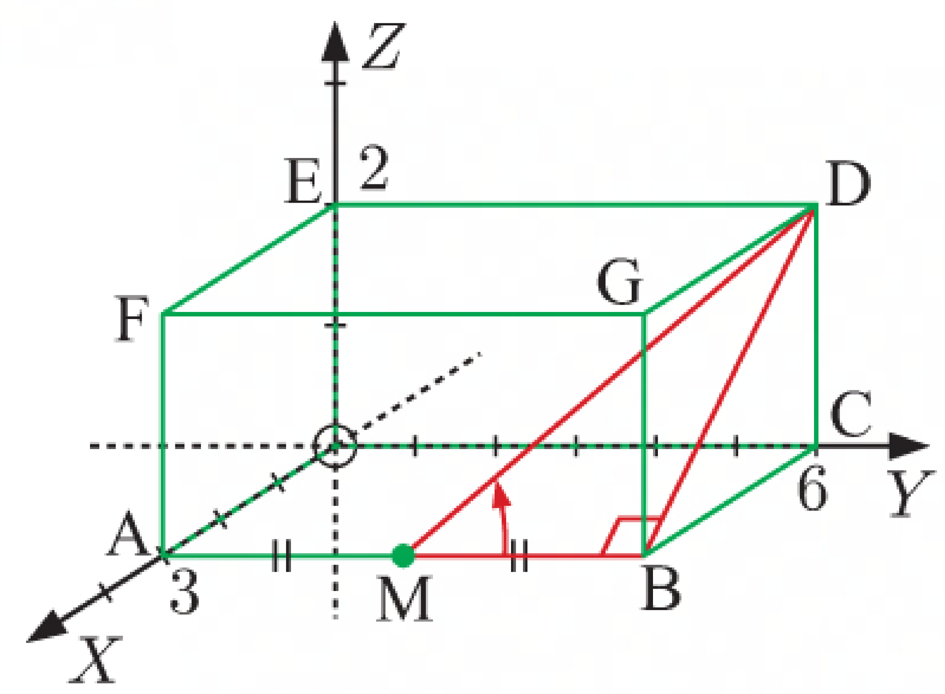


$\therefore \quad \tan A\widehat{D}O = \dfrac{3}{\sqrt{40}}$

$\therefore \quad A\widehat{D}O = \tan^{-1}\left(\frac{3}{\sqrt{40}}\right)$

$\therefore \quad A\widehat{D}O \approx 25.4°$

c Now $\quad BM = 3$ units



and $\quad BD = \sqrt{(0-3)^2 + (6-6)^2 + (2-0)^2}$

$= \sqrt{(-3)^2 + 0^2 + 2^2}$

$= \sqrt{9 + 0 + 4}$

$= \sqrt{13}$ units

$\therefore \quad \tan B\widehat{M}D = \dfrac{\sqrt{13}}{3}$

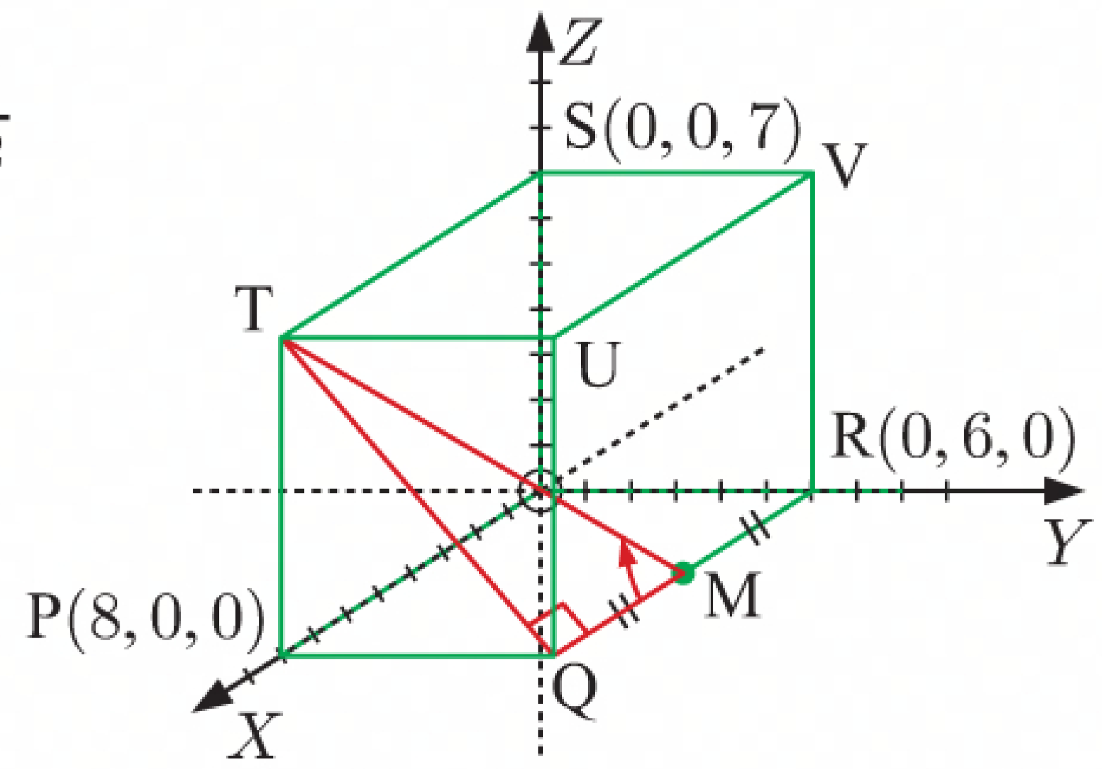
$\therefore \quad B\widehat{M}D = \tan^{-1}\left(\frac{\sqrt{13}}{3}\right)$

$\therefore \quad B\widehat{M}D \approx 50.2°$

3 **a** Q is $(8, 6, 0)$.

The midpoint M of [QR] is $\left(\dfrac{8+0}{2}, \dfrac{6+6}{2}, \dfrac{0+0}{2}\right)$ which is $(4, 6, 0)$.

b T is (8, 0, 7).



Now QM = 4 units

and $\text{QT} = \sqrt{(8-8)^2 + (0-6)^2 + (7-0)^2}$

$= \sqrt{0^2 + (-6)^2 + 7^2}$

$= \sqrt{0 + 36 + 49}$

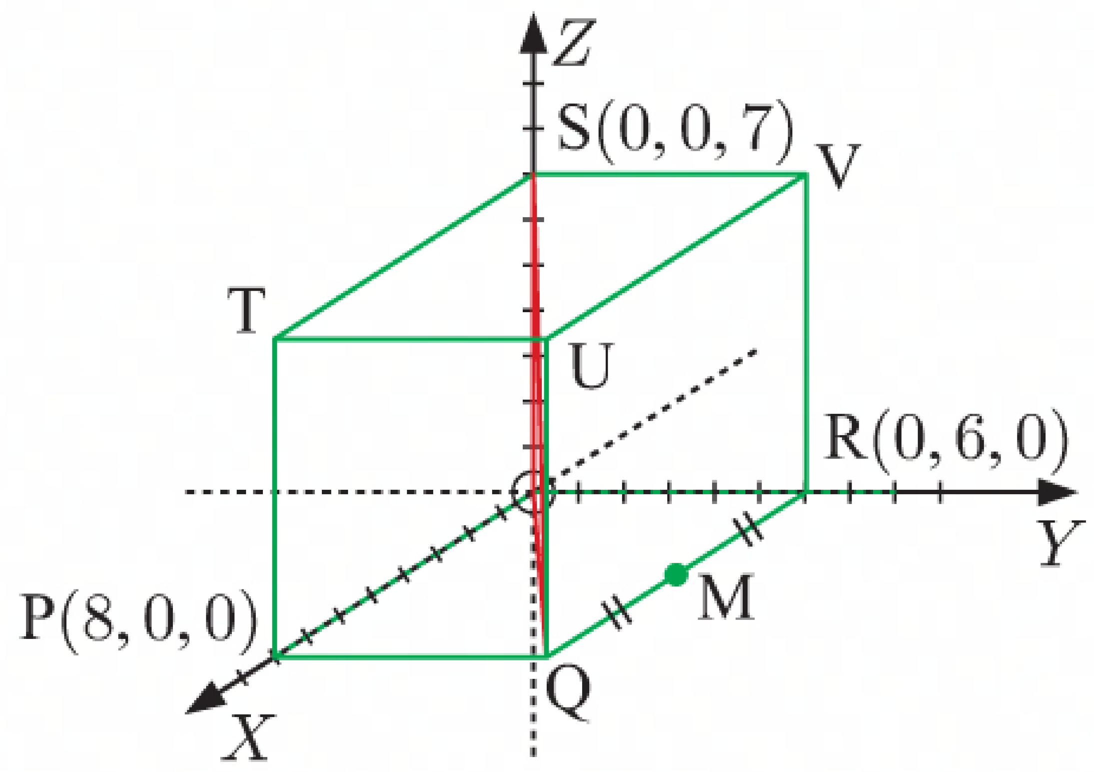
$= \sqrt{85}$ units

$\therefore \tan \text{Q}\widehat{\text{M}}\text{T} = \dfrac{\sqrt{85}}{4}$

$\therefore \text{Q}\widehat{\text{M}}\text{T} = \tan^{-1}\left(\frac{\sqrt{85}}{4}\right)$

$\therefore \text{Q}\widehat{\text{M}}\text{T} \approx 66.5°$

c **i** The required angle is $\text{O}\widehat{\text{Q}}\text{S}$.



Now OS = 7 units

and $\text{OQ} = \sqrt{(8-0)^2 + (6-0)^2 + (0-0)^2}$

$= \sqrt{8^2 + 6^2 + 0^2}$

$= \sqrt{64 + 36}$

$= \sqrt{100}$

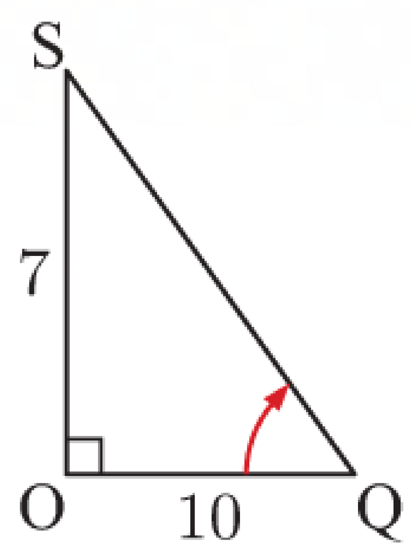
$= 10$ units

$\therefore \tan \text{O}\widehat{\text{Q}}\text{S} = \dfrac{7}{10}$

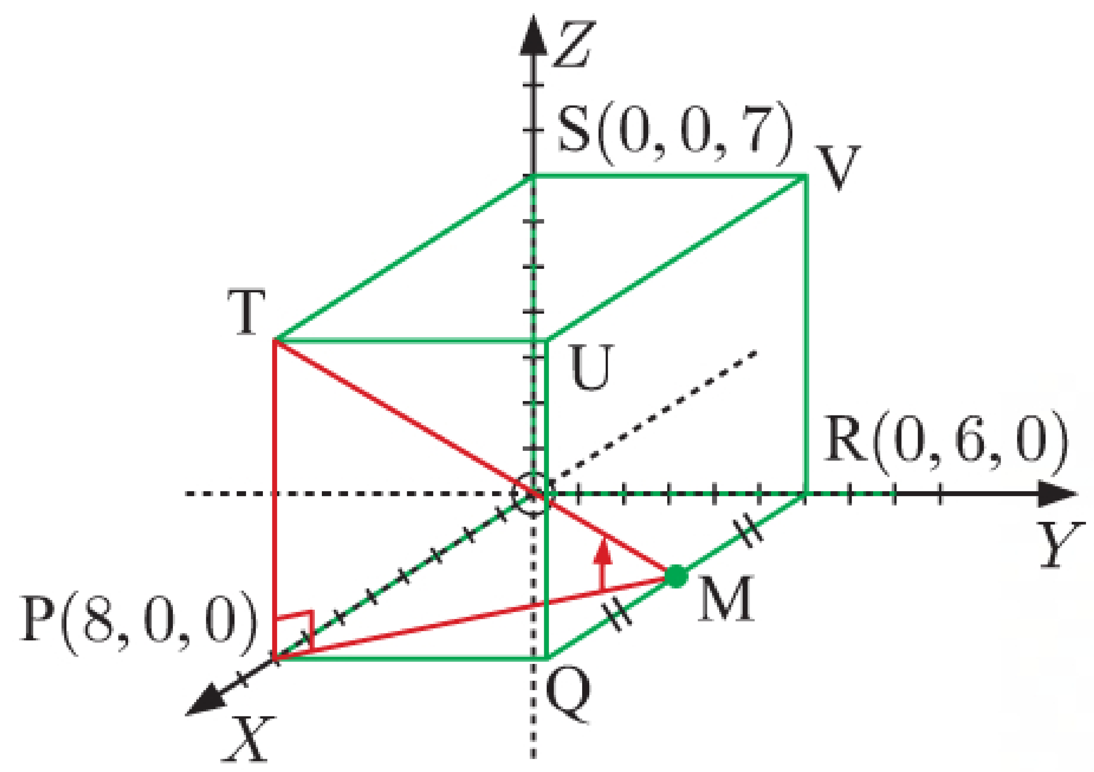
$\therefore \text{O}\widehat{\text{Q}}\text{S} = \tan^{-1}\left(\frac{7}{10}\right)$

$\therefore \text{O}\widehat{\text{Q}}\text{S} \approx 35.0°$

The angle is about 35.0°.



ii The required angle is $\text{T}\widehat{\text{M}}\text{P}$.



Now PT = 7 units

and $\text{MP} = \sqrt{(8-4)^2 + (0-6)^2 + (0-0)^2}$

$= \sqrt{4^2 + (-6)^2 + 0^2}$

$= \sqrt{16 + 36 + 0}$

$= \sqrt{52}$ units

$\therefore \tan \text{T}\widehat{\text{M}}\text{P} = \dfrac{7}{\sqrt{52}}$

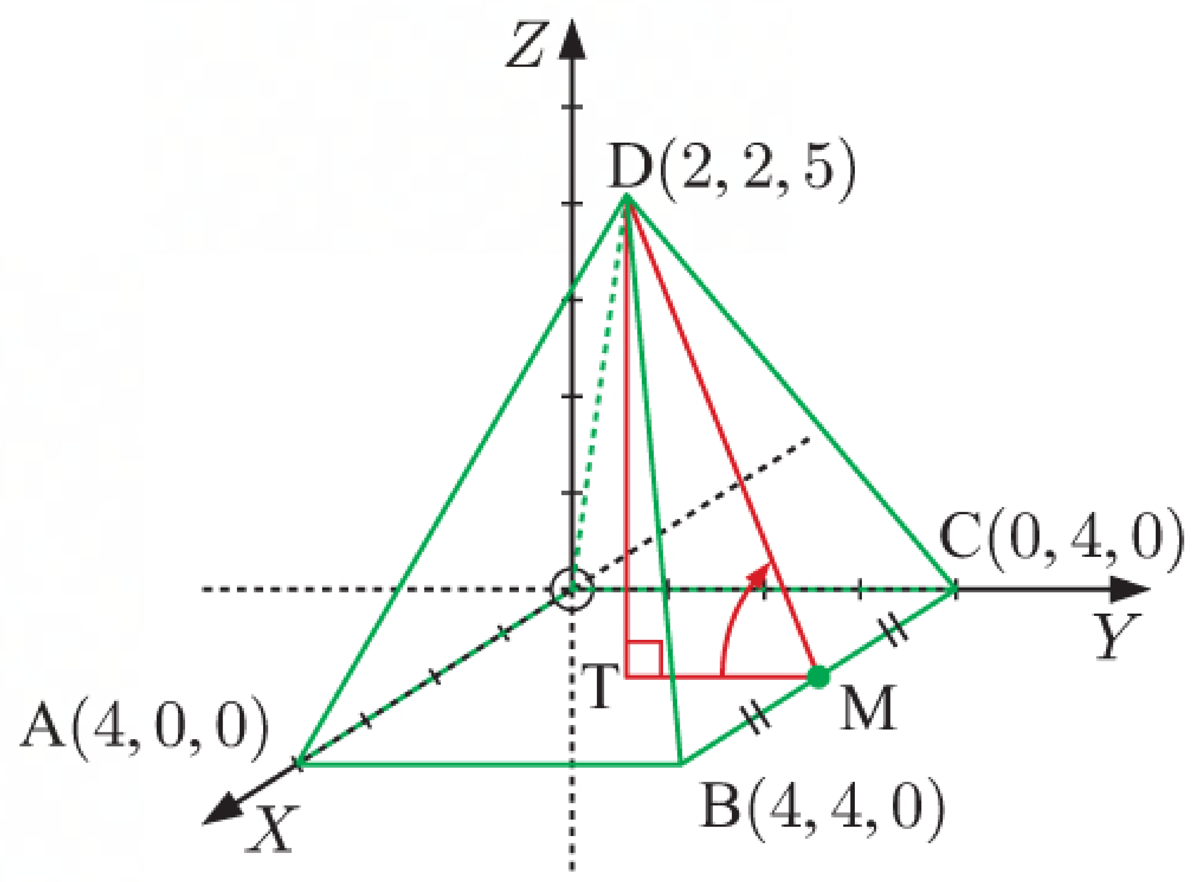
$\therefore \text{T}\widehat{\text{M}}\text{P} = \tan^{-1}\left(\frac{7}{\sqrt{52}}\right)$

$\therefore \text{T}\widehat{\text{M}}\text{P} \approx 44.1°$

The angle is about 44.1°.

4 **a** The midpoint M of [BC] is $\left(\frac{4+0}{2}, \frac{4+4}{2}, \frac{0+0}{2}\right)$ which is $(2, 4, 0)$.

b **i** The required angle is $\widehat{DMT}$, where T is the centre of the base.
To find the centre of the base, we locate the midpoints of the diagonals.



The midpoint of [AC] is
$\left(\frac{4+0}{2}, \frac{0+4}{2}, \frac{0+0}{2}\right)$ which is $(2, 2, 0)$.

The midpoint of [BO] is
$\left(\frac{4+0}{2}, \frac{4+0}{2}, \frac{0+0}{2}\right)$ which is $(2, 2, 0)$.

$\therefore$ the centre of the base is T(2, 2, 0).

Now DT = 5 units and MT = 2 units

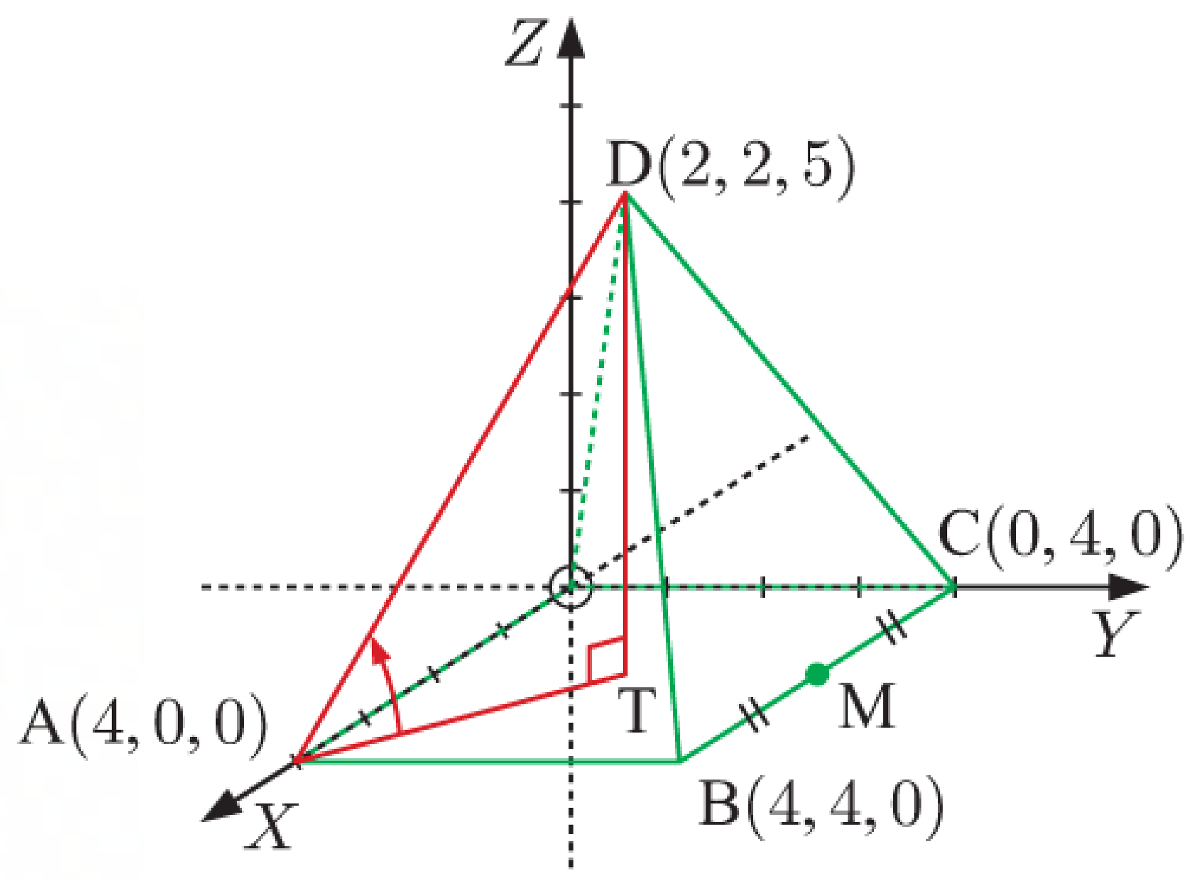
$\therefore \tan \widehat{DMT} = \frac{5}{2}$

$\therefore \widehat{DMT} = \tan^{-1}\left(\frac{5}{2}\right)$

$\therefore \widehat{DMT} \approx 68.2°$

The angle is about 68.2°.

ii The required angle is $\widehat{DAT}$.



Now DT = 5 units

$$\text{and} \quad AT = \sqrt{(2-4)^2 + (2-0)^2 + (0-0)^2}$$
$$= \sqrt{(-2)^2 + 2^2 + 0^2}$$
$$= \sqrt{4+4+0}$$
$$= \sqrt{8} \text{ units}$$

$\therefore \tan \widehat{DAT} = \frac{5}{\sqrt{8}}$

$\therefore \widehat{DAT} = \tan^{-1}\left(\frac{5}{\sqrt{8}}\right)$

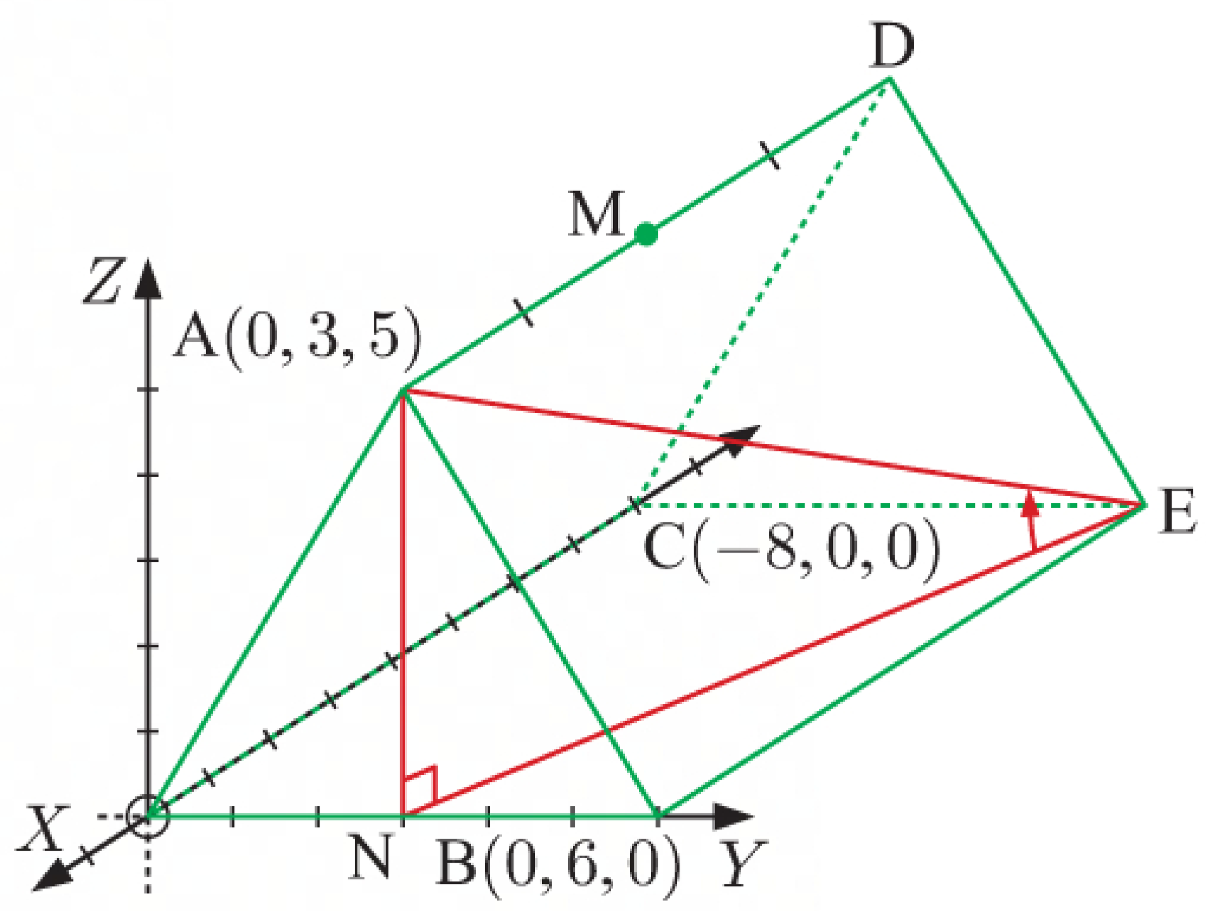
$\therefore \widehat{DAT} \approx 60.5$

The angle is about 60.5°.

5 **a** D is $(-8, 3, 5)$.

The midpoint M of [AD] is $\left(\frac{0+-8}{2}, \frac{3+3}{2}, \frac{5+5}{2}\right)$ which is $(-4, 3, 5)$.

b **i** The required angle is $\widehat{AEN}$, where N is the midpoint of [OB].



N is $\left(\frac{0+0}{2}, \frac{0+6}{2}, \frac{0+0}{2}\right)$ which is

$(0, 3, 0)$, and E is $(-8, 6, 0)$.

$$\begin{aligned}\text{Now}\quad AN &= 5 \text{ units}\\ \text{and}\quad EN &= \sqrt{(0--8)^2+(3-6)^2+(0-0)^2}\\ &= \sqrt{8^2+(-3)^2+0^2}\\ &= \sqrt{64+9+0}\\ &= \sqrt{73}\text{ units}\end{aligned}$$

$$\therefore\ \tan A\widehat{E}N = \frac{5}{\sqrt{73}}$$

$$\therefore\ A\widehat{E}N = \tan^{-1}\left(\tfrac{5}{\sqrt{73}}\right)$$

$$\therefore\ A\widehat{E}N \approx 30.3^\circ$$

The angle is about 30.3°.

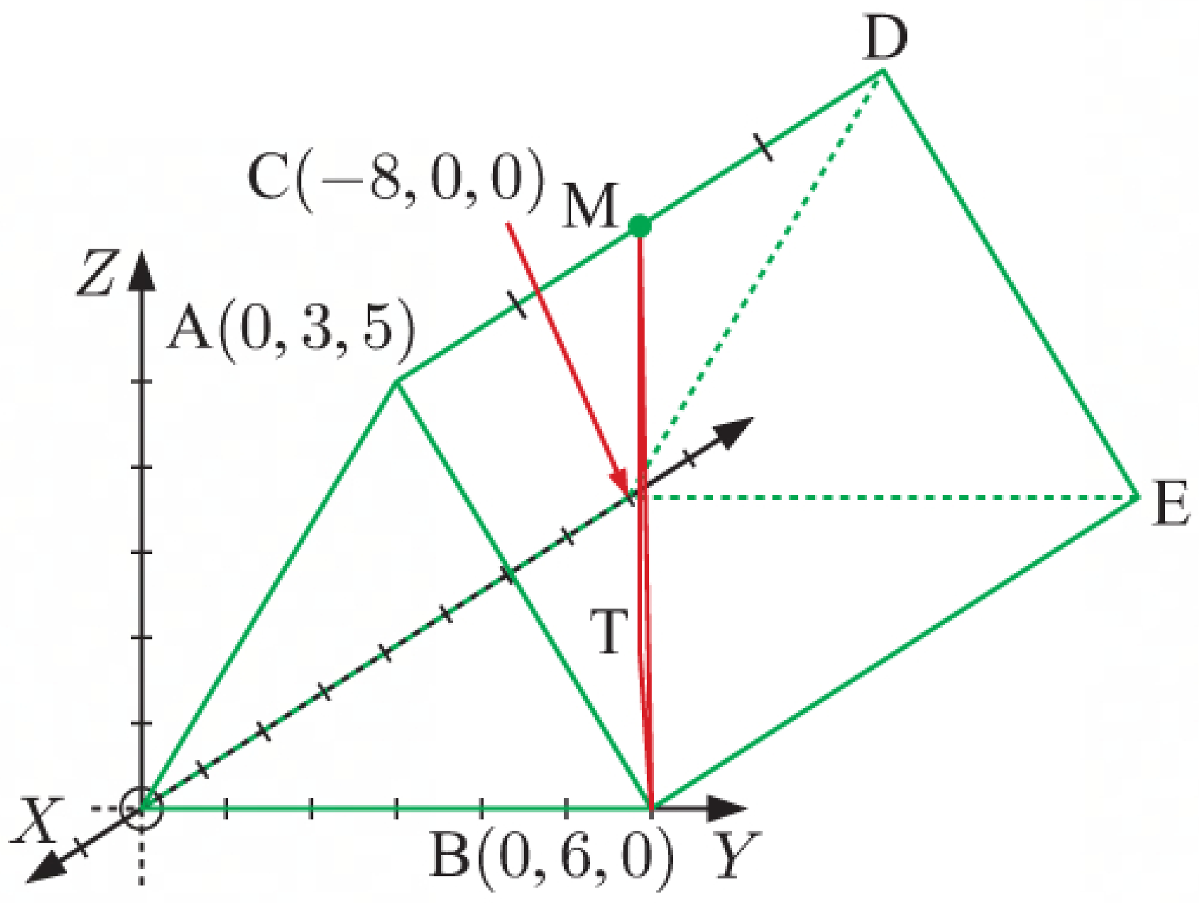
ii The required angle is $M\widehat{B}T$, where T is the centre of the base.
To find the centre of the base, we locate the midpoints of the diagonals.
The midpoint of [OE] is

$$\left(\frac{0+-8}{2}, \frac{0+6}{2}, \frac{0+0}{2}\right) \text{ which is } (-4, 3, 0).$$

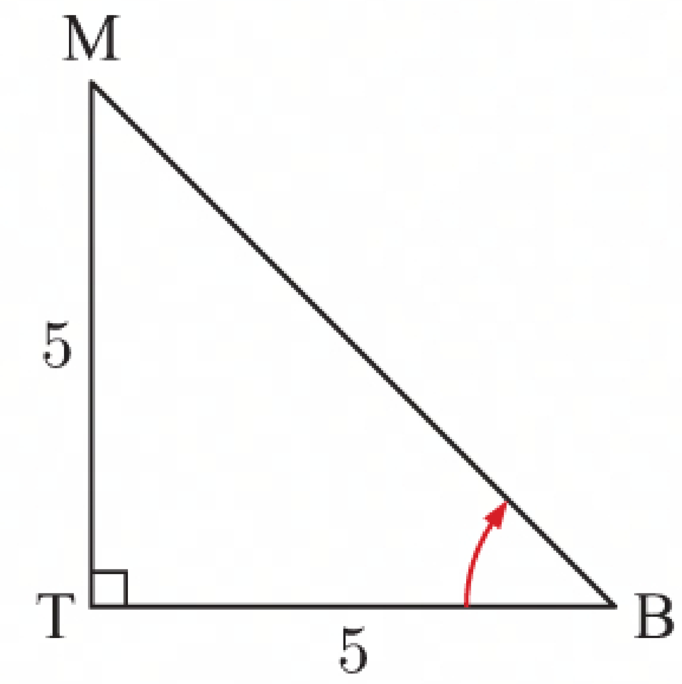
The midpoint of [BC] is

$$\left(\frac{0+-8}{2}, \frac{6+0}{2}, \frac{0+0}{2}\right) \text{ which is } (-4, 3, 0).$$

$\therefore$ the centre of the base is T$(-4, 3, 0)$.



$$\begin{aligned}\text{Now}\quad MT &= 5 \text{ units}\\ \text{and}\quad BT &= \sqrt{(-4-0)^2+(3-6)^2+(0-0)^2}\\ &= \sqrt{(-4)^2+(-3)^2+0^2}\\ &= \sqrt{16+9+0}\\ &= \sqrt{25}\\ &= 5\text{ units}\end{aligned}$$



$$\therefore\ \tan M\widehat{B}T = \frac{5}{5} = 1$$

$$\therefore\ M\widehat{B}T = \tan^{-1}(1)$$

$$\therefore\ M\widehat{B}T = 45^\circ$$

The angle is 45°.

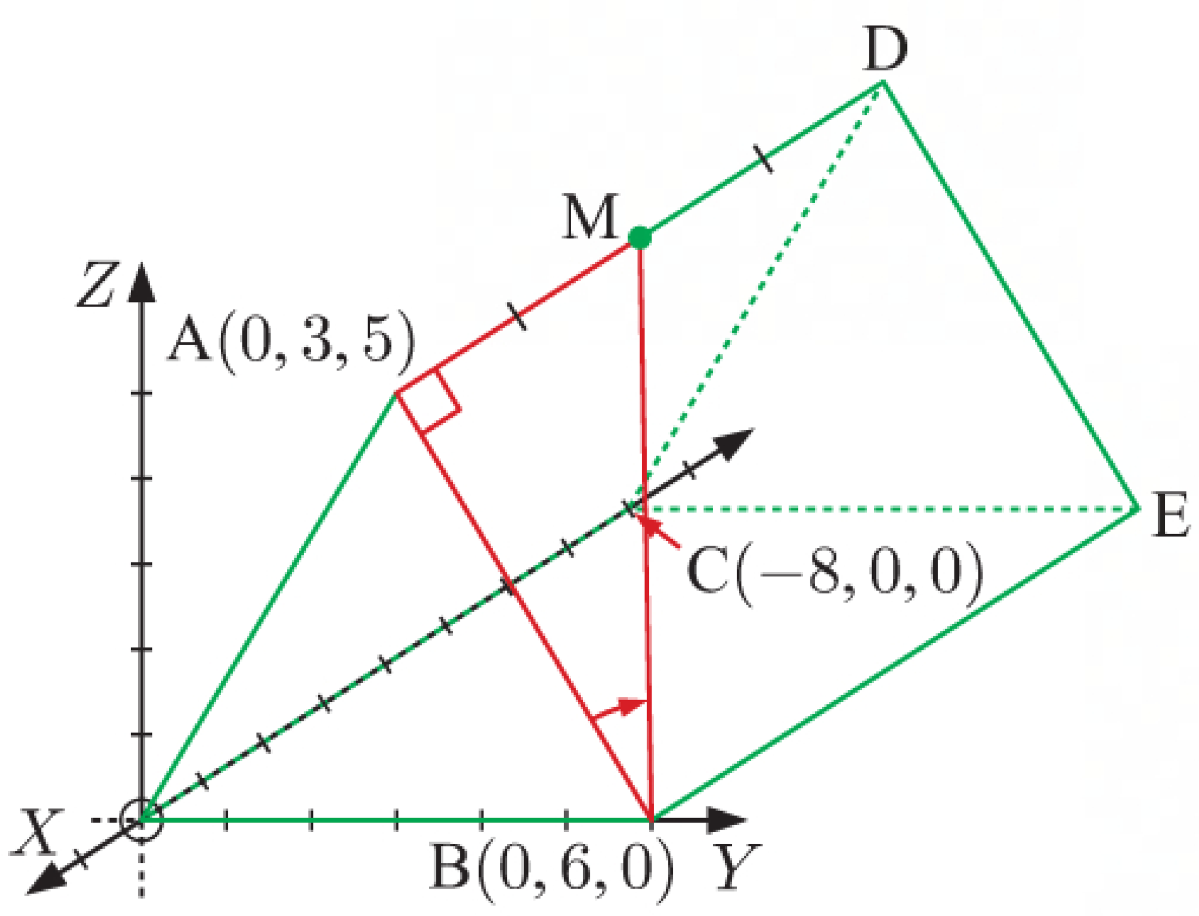
c

$$\begin{aligned}AB &= \sqrt{(0-0)^2+(6-3)^2+(0-5)^2}\\ &= \sqrt{0^2+3^2+(-5)^2}\\ &= \sqrt{0+9+25}\\ &= \sqrt{34}\text{ units}\end{aligned}$$

$$\therefore\ \tan A\widehat{B}M = \frac{4}{\sqrt{34}}$$

$$\therefore\ A\widehat{B}M = \tan^{-1}\left(\tfrac{4}{\sqrt{34}}\right)$$

$$\therefore\ A\widehat{B}M \approx 34.4^\circ$$



6 **a** **i** $AB = \sqrt{(4 - -1)^2 + (1-0)^2 + (1-2)^2}$

$= \sqrt{5^2 + 1^2 + (-1)^2}$

$= \sqrt{27}$

$= 3\sqrt{3}$ units

ii $AC = \sqrt{(-2 - -1)^2 + (2-0)^2 + (0-2)^2}$

$= \sqrt{(-1)^2 + 2^2 + (-2)^2}$

$= \sqrt{9}$

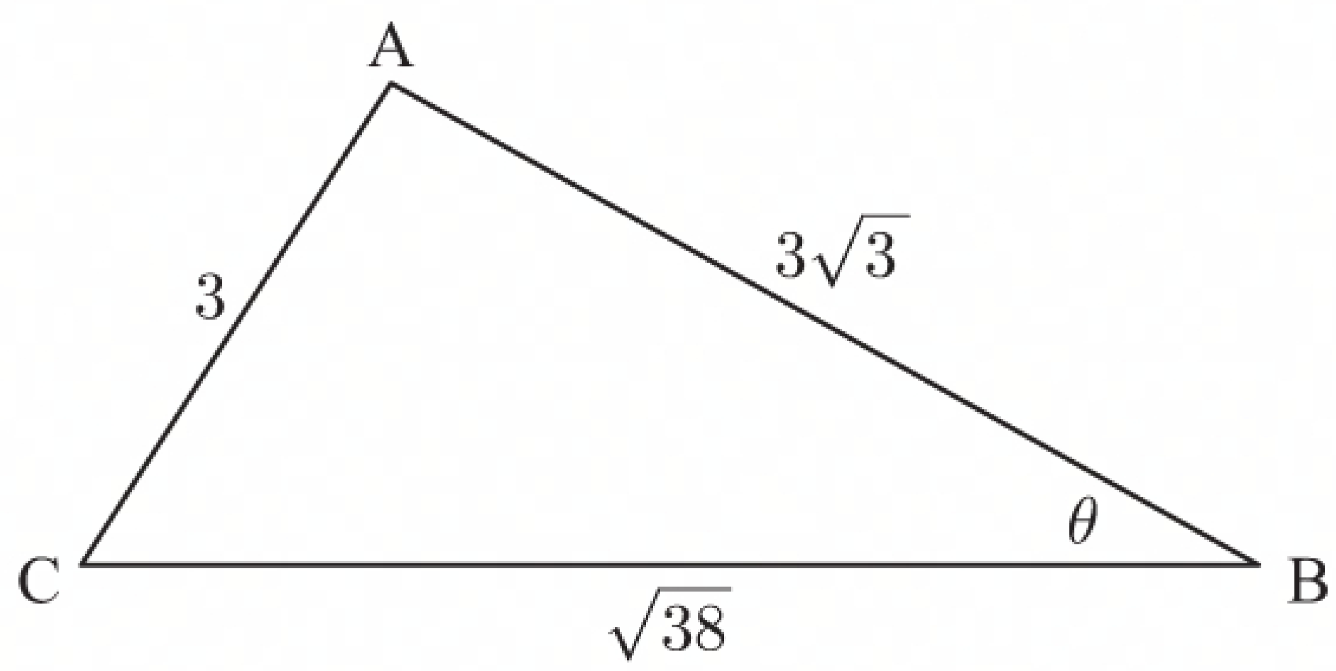
$= 3$ units

iii $BC = \sqrt{(-2-4)^2 + (2-1)^2 + (0-1)^2}$

$= \sqrt{(-6)^2 + 1^2 + (-1)^2}$

$= \sqrt{38}$ units

b By the cosine rule,



$$\cos\theta = \frac{(3\sqrt{3})^2 + (\sqrt{38})^2 - 3^2}{2 \times 3\sqrt{3} \times \sqrt{38}}$$

$$\therefore \quad \cos\theta = \frac{27 + 38 - 9}{6\sqrt{3} \times \sqrt{38}}$$

$$\therefore \quad \theta = \cos^{-1}\left(\frac{56}{6\sqrt{3} \times \sqrt{38}}\right) \approx 29.1°$$

So, $A\widehat{B}C$ is about 29.1°.

7 **a** $PQ = \sqrt{(3-0)^2 + (-1-2)^2 + (-4 - -2)^2}$

$= \sqrt{3^2 + (-3)^2 + (-2)^2}$

$= \sqrt{22}$ units

$PR = \sqrt{(4-0)^2 + (0-2)^2 + (-1 - -2)^2}$

$= \sqrt{4^2 + (-2)^2 + 1^2}$

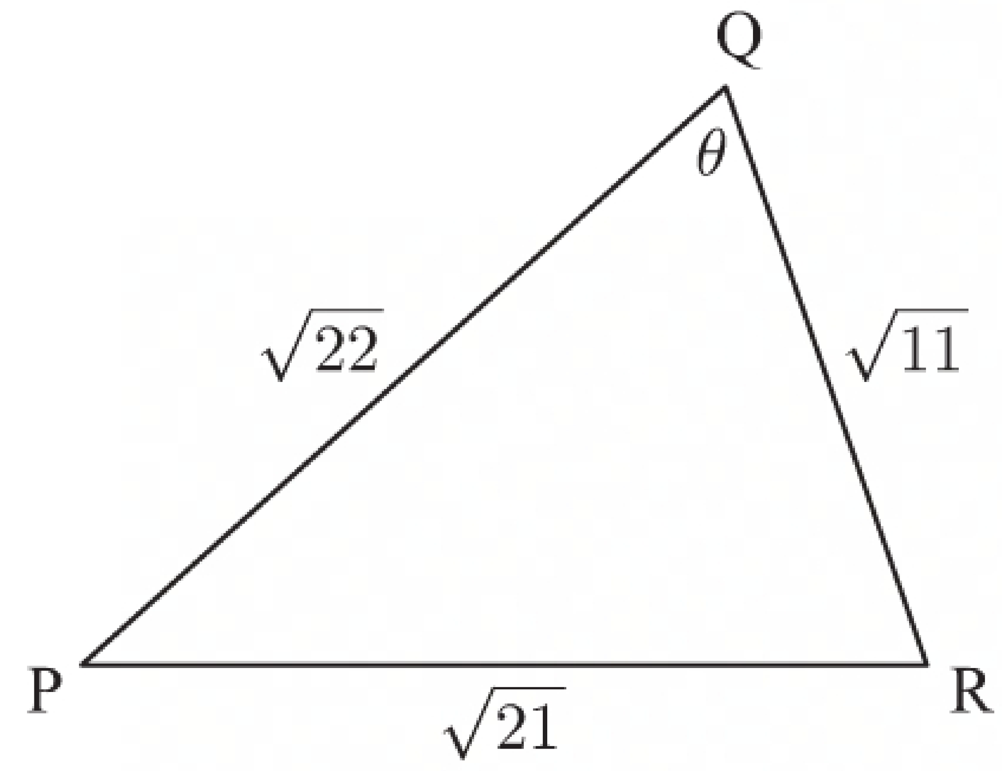
$= \sqrt{21}$ units

$QR = \sqrt{(4-3)^2 + (0 - -1)^2 + (-1 - -4)^2}$

$= \sqrt{1^2 + 1^2 + 3^2}$

$= \sqrt{11}$ units

By the cosine rule,



$$\cos\theta = \frac{(\sqrt{22})^2 + (\sqrt{11})^2 - (\sqrt{21})^2}{2 \times \sqrt{22} \times \sqrt{11}}$$

$$\therefore \quad \cos\theta = \frac{22 + 11 - 21}{2 \times \sqrt{22} \times \sqrt{11}}$$

$$\therefore \quad \theta = \cos^{-1}\left(\frac{12}{2 \times \sqrt{22} \times \sqrt{11}}\right) \approx 67.3°$$

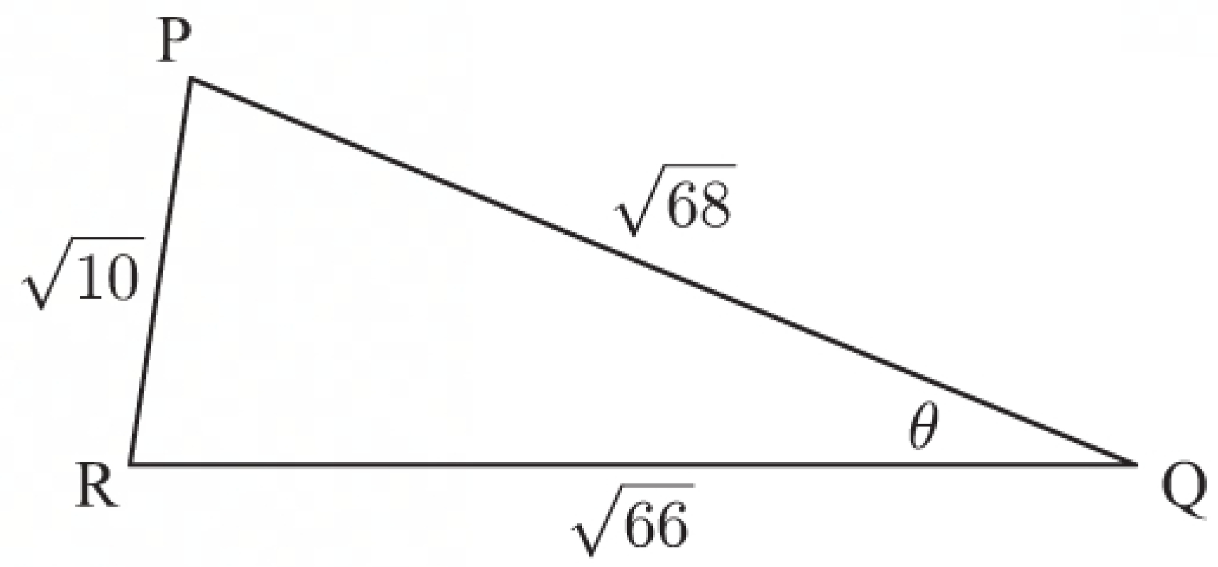
So, $P\widehat{Q}R$ is about 67.3°.

b $\text{PQ} = \sqrt{(5 - -3)^2 + (0 - 2)^2 + (1 - 1)^2}$
$= \sqrt{8^2 + (-2)^2 + 0^2}$
$= \sqrt{68}$ units

$\text{PR} = \sqrt{(-3 - -3)^2 + (-1 - 2)^2 + (2 - 1)^2}$
$= \sqrt{0^2 + (-3)^2 + 1^2}$
$= \sqrt{10}$ units

$\text{QR} = \sqrt{(-3 - 5)^2 + (-1 - 0)^2 + (2 - 1)^2}$
$= \sqrt{(-8)^2 + (-1)^2 + 1^2}$
$= \sqrt{66}$ units

By the cosine rule,



$$\cos\theta = \frac{(\sqrt{68})^2 + (\sqrt{66})^2 - (\sqrt{10})^2}{2 \times \sqrt{68} \times \sqrt{66}}$$

$$\therefore \quad \cos\theta = \frac{68 + 66 - 10}{2 \times \sqrt{68} \times \sqrt{66}}$$

$$\therefore \quad \theta = \cos^{-1}\left(\frac{124}{2 \times \sqrt{68} \times \sqrt{66}}\right) \approx 22.3°$$

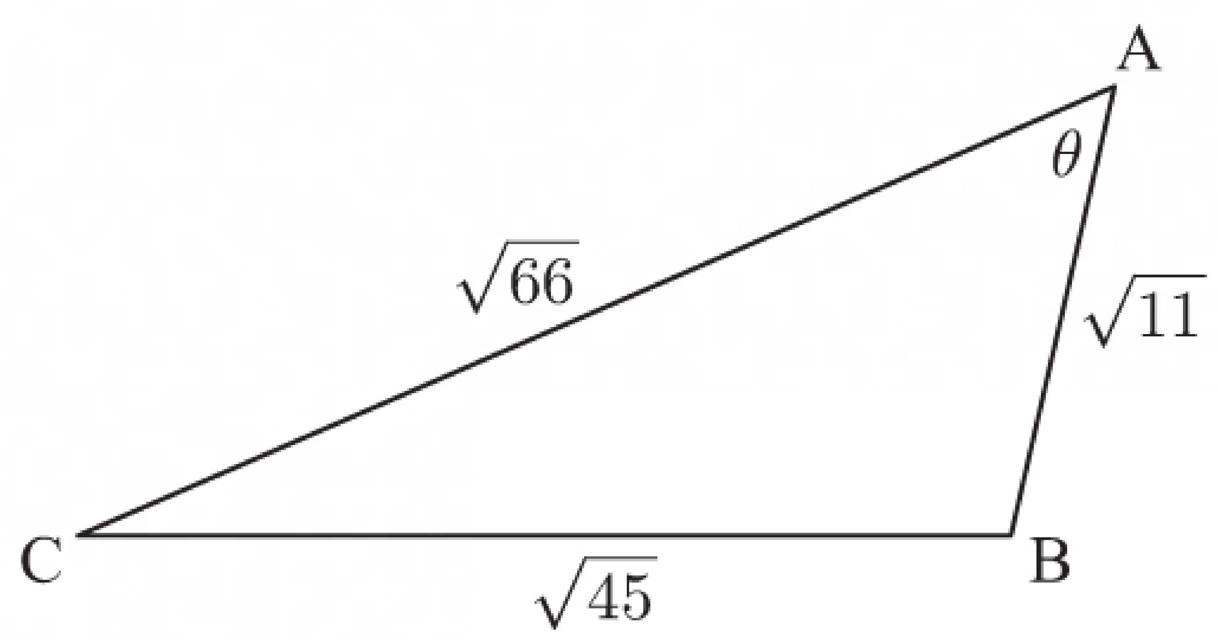
So, $\text{P}\widehat{\text{Q}}\text{R}$ is about 22.3°.

8 **a** $\text{AB} = \sqrt{(0 - -1)^2 + (-2 - -5)^2 + (3 - 2)^2}$
$= \sqrt{1^2 + 3^2 + 1^2}$
$= \sqrt{11}$ units

$\text{AC} = \sqrt{(4 - -1)^2 + (0 - -5)^2 + (-2 - 2)^2}$
$= \sqrt{5^2 + 5^2 + (-4)^2}$
$= \sqrt{66}$ units

$\text{BC} = \sqrt{(4 - 0)^2 + (0 - -2)^2 + (-2 - 3)^2}$
$= \sqrt{4^2 + 2^2 + (-5)^2}$
$= \sqrt{45}$ units

By the cosine rule,



$$\cos\theta = \frac{(\sqrt{11})^2 + (\sqrt{66})^2 - (\sqrt{45})^2}{2 \times \sqrt{11} \times \sqrt{66}}$$

$$\therefore \quad \cos\theta = \frac{11 + 66 - 45}{2 \times \sqrt{11} \times \sqrt{66}}$$

$$\therefore \quad \theta = \cos^{-1}\left(\frac{32}{2 \times \sqrt{11} \times \sqrt{66}}\right) \approx 53.6°$$

So, $\text{B}\widehat{\text{A}}\text{C}$ is about 53.6°.

b Area of triangle ABC $= \frac{1}{2} \times \sqrt{11} \times \sqrt{66} \times \sin\theta$

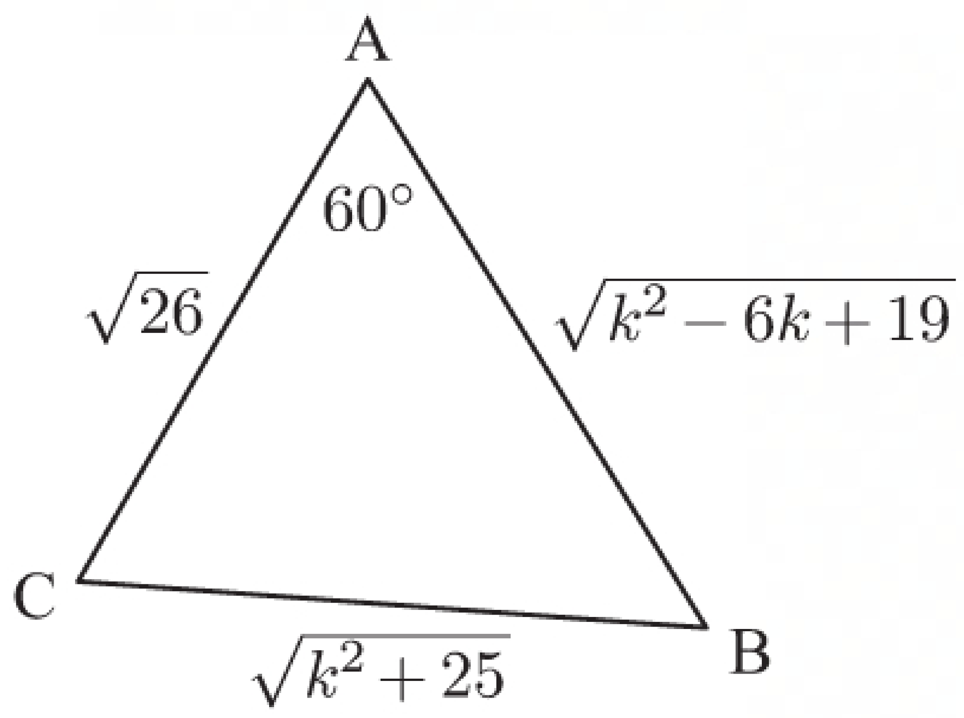
$\approx \frac{1}{2} \times \sqrt{11} \times \sqrt{66} \times \sin 53.6°$

≈ 10.8 units^2

9 $AB = \sqrt{(2-1)^2+(5-2)^2+(k-3)^2}$

$= \sqrt{1^2+3^2+(k^2-6k+9)}$

$= \sqrt{k^2-6k+19}$ units



$AC = \sqrt{(5-1)^2+(1-2)^2+(0-3)^2}$

$= \sqrt{4^2+(-1)^2+(-3)^2}$

$= \sqrt{26}$ units

$BC = \sqrt{(5-2)^2+(1-5)^2+(0-k)^2}$

$= \sqrt{3^2+(-4)^2+k^2}$

$= \sqrt{k^2+25}$ units

By the cosine rule,

$$(\sqrt{k^2+25})^2 = (\sqrt{k^2-6k+19})^2 + (\sqrt{26})^2 - 2 \times \sqrt{k^2-6k+19} \times \sqrt{26} \times \cos 60°$$

$$\therefore\ k^2+25 = k^2-6k+19+26-2\times\sqrt{k^2-6k+19}\times\sqrt{26}\times\tfrac{1}{2}$$

$$\therefore\ \cancel{k^2}+25 = \cancel{k^2}-6k+45-\sqrt{26}\times\sqrt{k^2-6k+19}$$

$$\therefore\ 6k-20 = -\sqrt{26}\times\sqrt{k^2-6k+19}$$

$$\therefore\ \frac{20-6k}{\sqrt{26}} = \sqrt{k^2-6k+19}$$

$$\therefore\ \left(\frac{20-6k}{\sqrt{26}}\right)^2 = (\sqrt{k^2-6k+19})^2 \quad \text{provided } 20-6k \geqslant 0$$

$$\therefore\ \frac{400-240k+36k^2}{26} = k^2-6k+19$$

$$\therefore\ 400-240k+36k^2 = 26(k^2-6k+19)$$

$$\therefore\ 400-240k+36k^2 = 26k^2-156k+494$$

$$\therefore\ 10k^2-84k-94=0$$

$$\therefore\ 2(5k^2-42k-47)=0$$

$$\therefore\ 5k^2-42k-47=0$$

$$\therefore\ (5k-47)(k+1)=0$$

$$\therefore\ k=\tfrac{47}{5} \text{ or } -1$$

The reasoning above is only true provided $20-6k \geqslant 0$.

When $k=\frac{47}{5}$, $20-6k = 20-6(\frac{47}{5})$

$= -\frac{182}{5}$ which is < 0

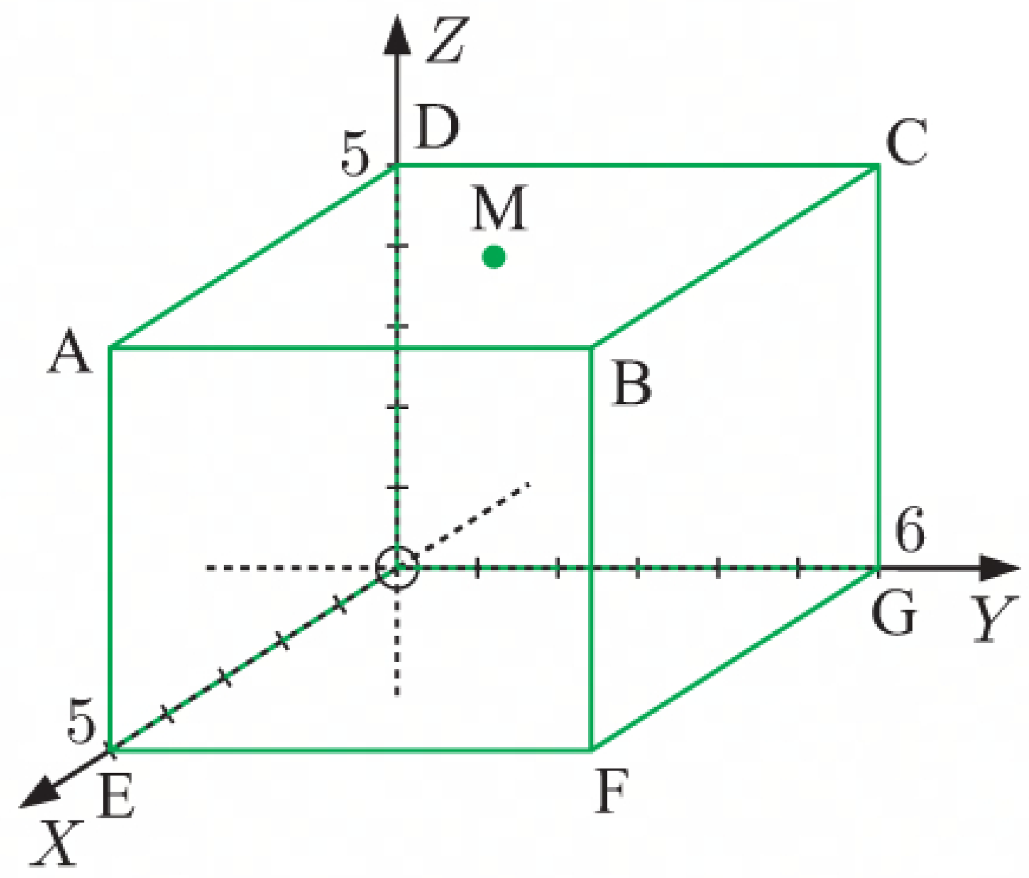
$\therefore$ $k=\frac{47}{5}$ is not a valid solution.

When $k=-1$, $20-6k = 20-6(-1)$

$= 26$ which is > 0

$\therefore$ $k=-1$ is the only solution.

10 **a** To find the centre of the face ABCD, we locate the midpoints of the diagonals.



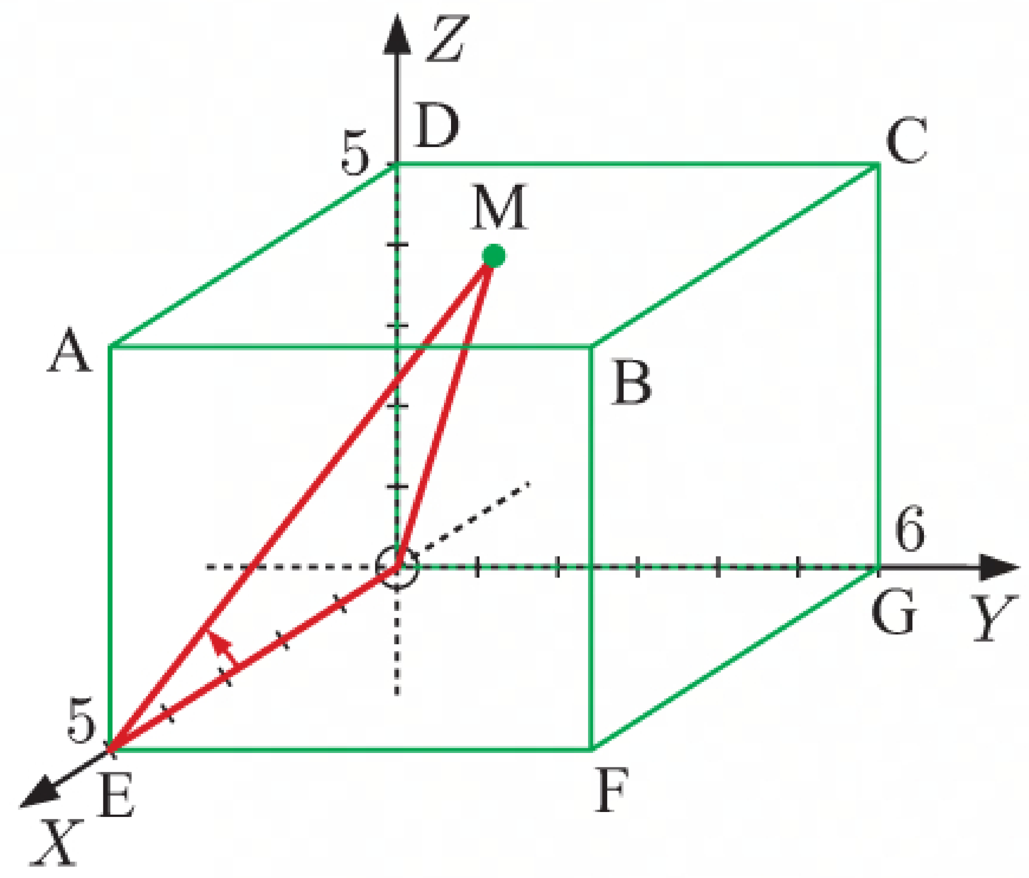
A is $(5, 0, 5)$, B is $(5, 6, 5)$, C is $(0, 6, 5)$, and D is $(0, 0, 5)$.

The midpoint of [AC] is $\left(\frac{5+0}{2}, \frac{0+6}{2}, \frac{5+5}{2}\right)$ which is $\left(\frac{5}{2}, 3, 5\right)$.

The midpoint of [BD] is $\left(\frac{5+0}{2}, \frac{6+0}{2}, \frac{5+5}{2}\right)$ which is $\left(\frac{5}{2}, 3, 5\right)$.

$\therefore$ the centre of the face ABCD is $M\left(\frac{5}{2}, 3, 5\right)$.

b **i** E is $(5, 0, 0)$ and M is $\left(\frac{5}{2}, 3, 5\right)$.

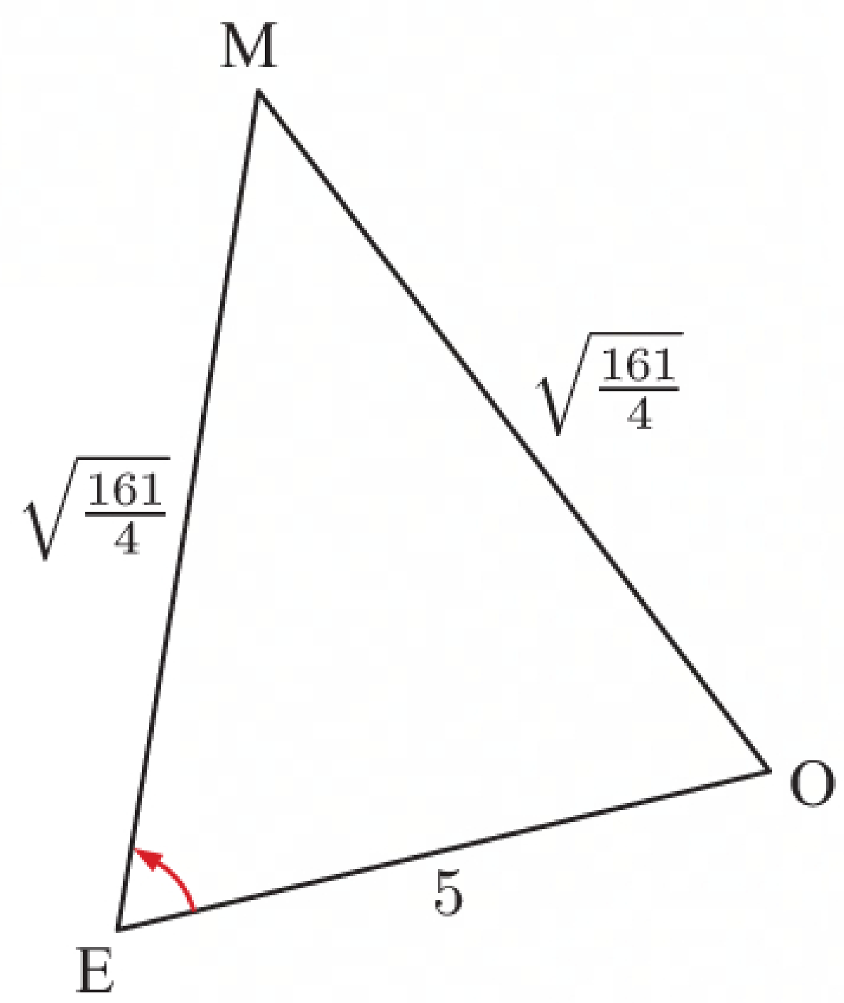


Now $\text{OE} = 5$ units

$$\text{and} \quad \text{EM} = \sqrt{(\tfrac{5}{2}-5)^2+(3-0)^2+(5-0)^2}$$
$$= \sqrt{(-\tfrac{5}{2})^2+3^2+5^2}$$
$$= \sqrt{\tfrac{25}{4}+9+25}$$
$$= \sqrt{\tfrac{161}{4}} \text{ units}$$

$$\text{also} \quad \text{OM} = \sqrt{(\tfrac{5}{2}-0)^2+(3-0)^2+(5-0)^2}$$
$$= \sqrt{(\tfrac{5}{2})^2+3^2+5^2}$$
$$= \sqrt{\tfrac{25}{4}+9+25}$$
$$= \sqrt{\tfrac{161}{4}} \text{ units}$$

By the cosine rule,



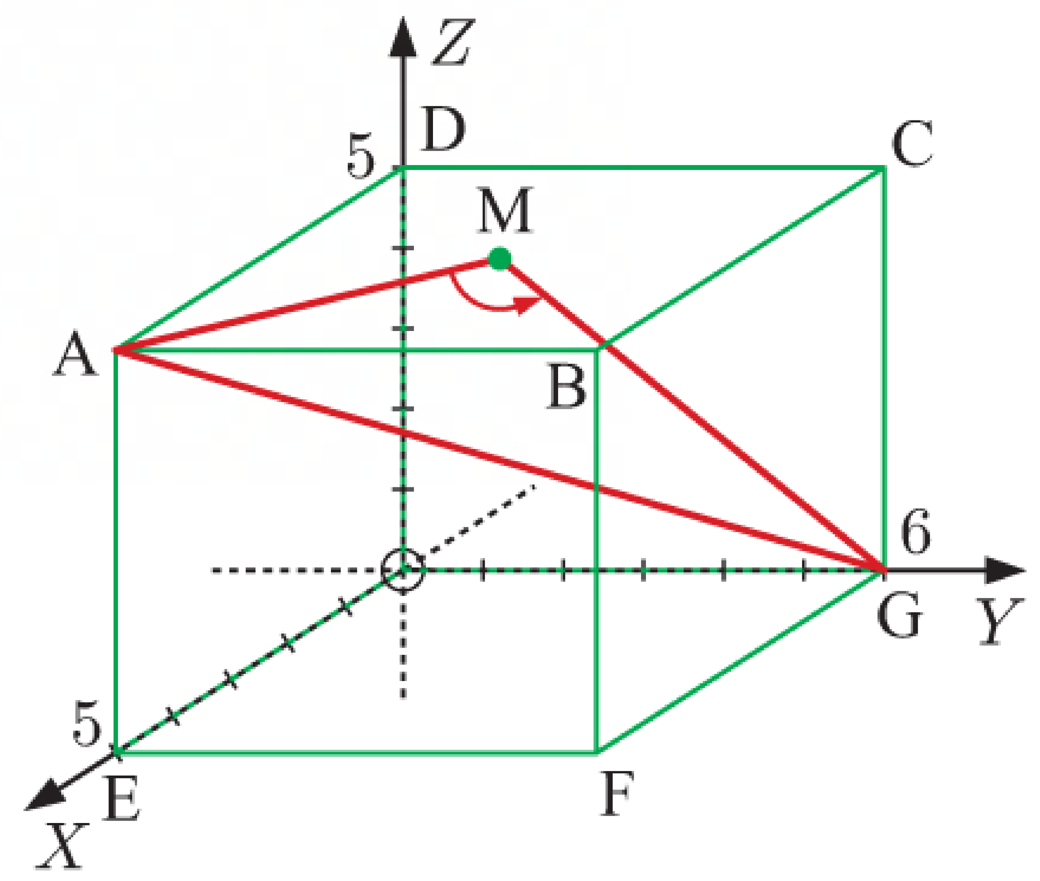
$$\cos \widehat{\text{OEM}} = \frac{5^2 + \left(\sqrt{\frac{161}{4}}\right)^2 - \left(\sqrt{\frac{161}{4}}\right)^2}{2 \times 5 \times \sqrt{\frac{161}{4}}}$$

$$\therefore \quad \cos \widehat{\text{OEM}} = \frac{25 + \frac{161}{4} - \frac{161}{4}}{10 \times \frac{\sqrt{161}}{2}}$$

$$\therefore \quad \cos \widehat{\text{OEM}} = \frac{25}{5\sqrt{161}}$$

$$\therefore \quad \widehat{\text{OEM}} = \cos^{-1}\left(\frac{5}{\sqrt{161}}\right) \approx 66.8^\circ$$

ii A is $(5, 0, 5)$, M is $\left(\frac{5}{2}, 3, 5\right)$, and G is $(0, 6, 0)$.

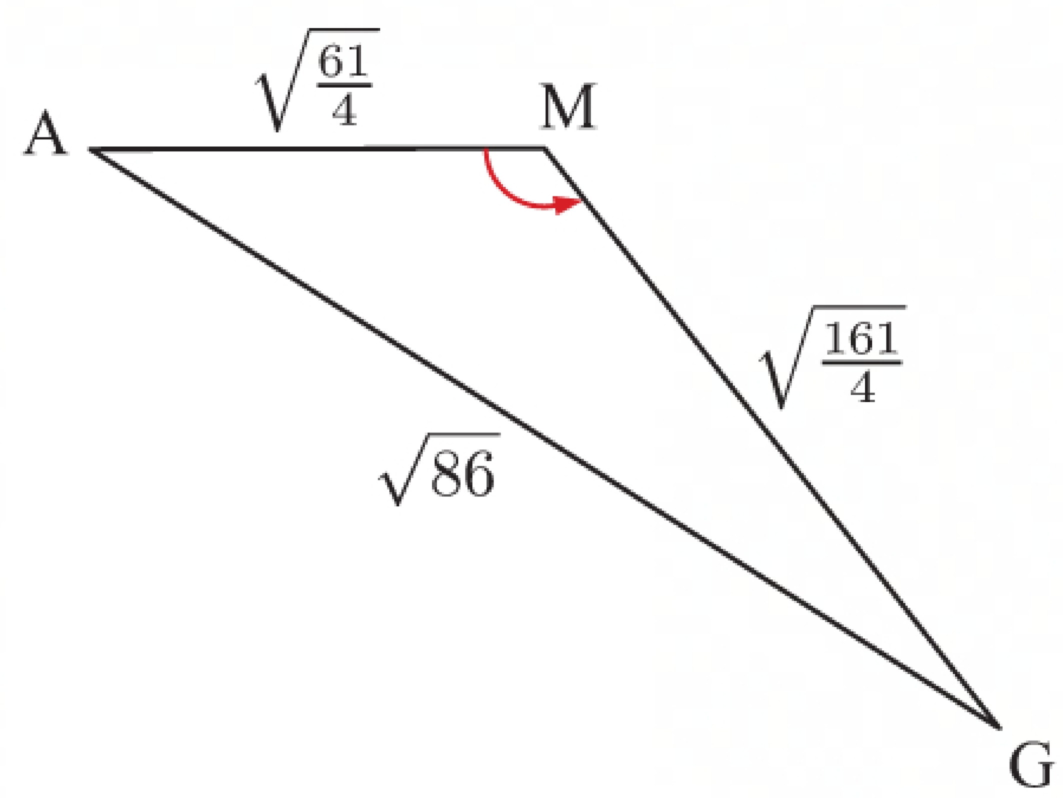


$$\begin{aligned} AM &= \sqrt{(\tfrac{5}{2}-5)^2+(3-0)^2+(5-5)^2} \\ &= \sqrt{(-\tfrac{5}{2})^2+3^2+0^2} \\ &= \sqrt{\tfrac{25}{4}+9} \\ &= \sqrt{\tfrac{61}{4}} \text{ units} \end{aligned}$$

$$\begin{aligned} AG &= \sqrt{(0-5)^2+(6-0)^2+(0-5)^2} \\ &= \sqrt{(-5)^2+6^2+(-5)^2} \\ &= \sqrt{25+36+25} \\ &= \sqrt{86} \text{ units} \end{aligned}$$

$$\begin{aligned} \text{and} \quad GM &= \sqrt{(\tfrac{5}{2}-0)^2+(3-6)^2+(5-0)^2} \\ &= \sqrt{(\tfrac{5}{2})^2+(-3)^2+5^2} \\ &= \sqrt{\tfrac{25}{4}+9+25} \\ &= \sqrt{\tfrac{161}{4}} \text{ units} \end{aligned}$$

By the cosine rule,



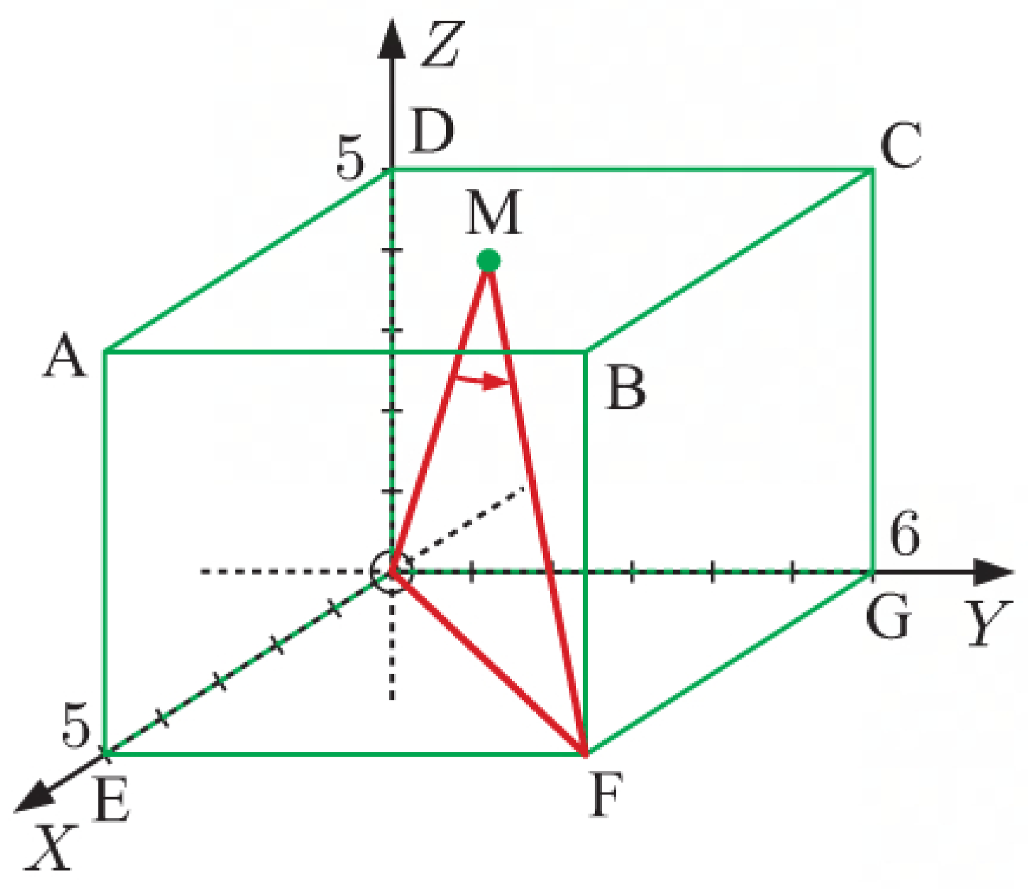
$$\cos A\widehat{M}G = \frac{\left(\sqrt{\frac{61}{4}}\right)^2+\left(\sqrt{\frac{161}{4}}\right)^2-(\sqrt{86})^2}{2\times\sqrt{\frac{61}{4}}\times\sqrt{\frac{161}{4}}}$$

$$\therefore \quad \cos A\widehat{M}G = \frac{\frac{61}{4}+\frac{161}{4}-86}{2\times\frac{\sqrt{61}}{2}\times\frac{\sqrt{161}}{2}}$$

$$\therefore \quad \cos A\widehat{M}G = \frac{-\frac{61}{2}}{\frac{\sqrt{61}\times\sqrt{161}}{2}}$$

$$\therefore \quad A\widehat{M}G = \cos^{-1}\left(\frac{-61}{\sqrt{61}\times\sqrt{161}}\right) \approx 128^\circ$$

iii M is $\left(\frac{5}{2}, 3, 5\right)$ and F is $(5, 6, 0)$.

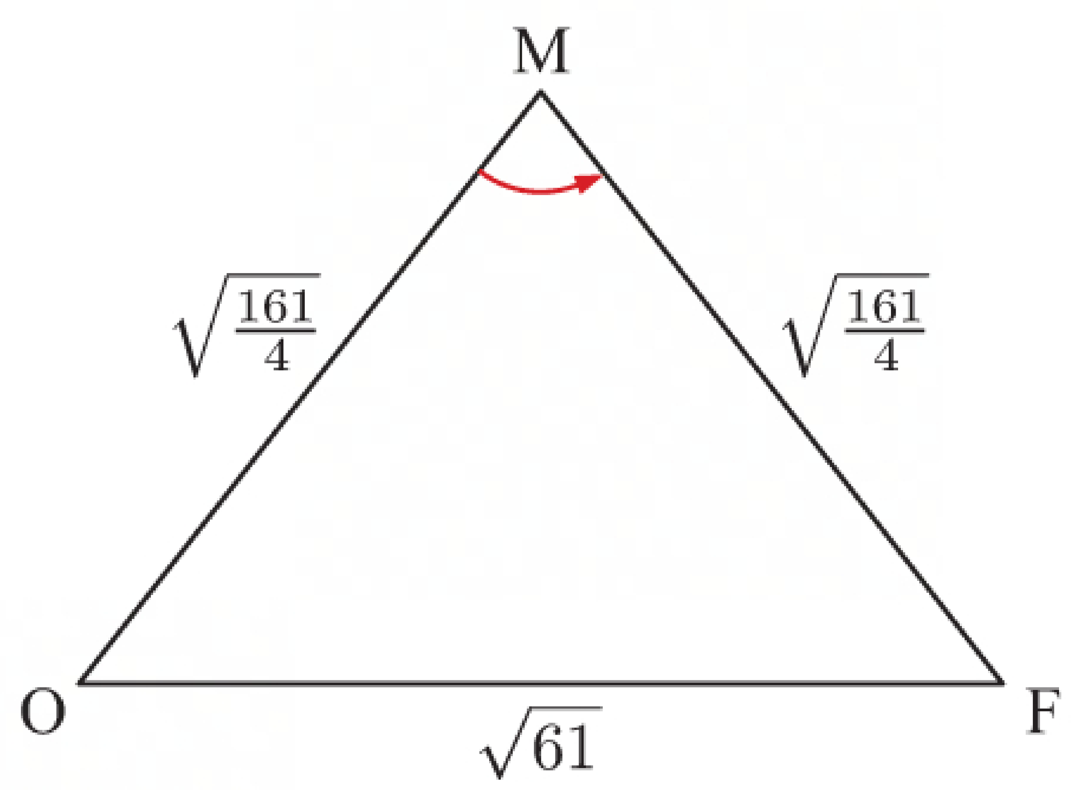


$$\text{OM} = \sqrt{\tfrac{161}{4}} \text{ units} \qquad \{\text{from } \mathbf{i}\}$$

$$\begin{aligned}\text{OF} &= \sqrt{(5-0)^2 + (6-0)^2 + (0-0)^2} \\ &= \sqrt{5^2 + 6^2 + 0^2} \\ &= \sqrt{25 + 36} \\ &= \sqrt{61} \text{ units}\end{aligned}$$

and
$$\begin{aligned}\text{FM} &= \sqrt{\left(\tfrac{5}{2} - 5\right)^2 + (3-6)^2 + (5-0)^2} \\ &= \sqrt{\left(-\tfrac{5}{2}\right)^2 + (-3)^2 + 5^2} \\ &= \sqrt{\tfrac{25}{4} + 9 + 25} \\ &= \sqrt{\tfrac{161}{4}} \text{ units}\end{aligned}$$

By the cosine rule,



$$\cos \text{O}\widehat{\text{M}}\text{F} = \frac{\left(\sqrt{\frac{161}{4}}\right)^2 + \left(\sqrt{\frac{161}{4}}\right)^2 - (\sqrt{61})^2}{2 \times \sqrt{\frac{161}{4}} \times \sqrt{\frac{161}{4}}}$$

$$\therefore \quad \cos \text{O}\widehat{\text{M}}\text{F} = \frac{\frac{161}{4} + \frac{161}{4} - 61}{2 \times \frac{161}{4}}$$

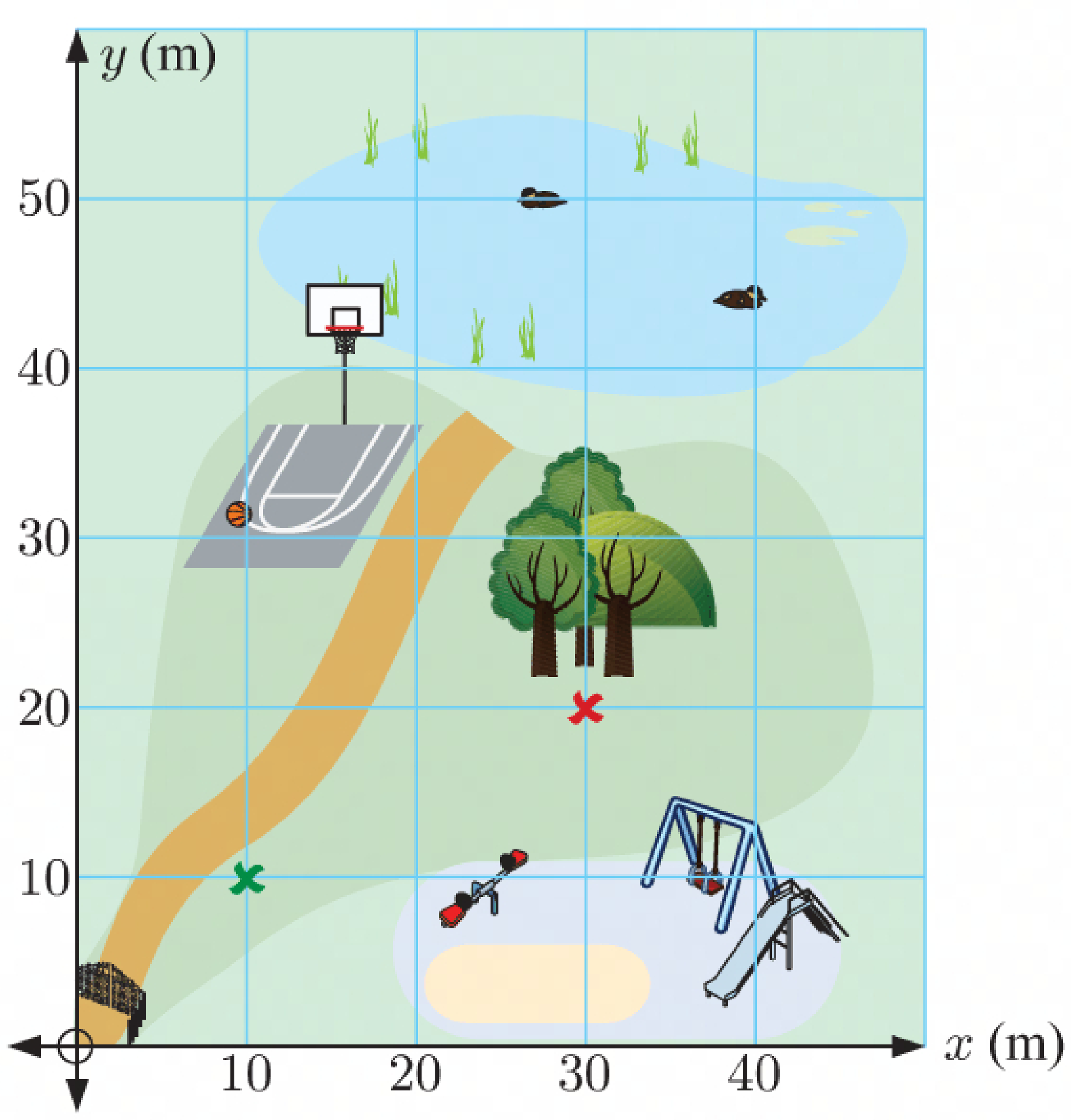
$$\therefore \quad \cos \text{O}\widehat{\text{M}}\text{F} = \frac{\frac{39}{2}}{\frac{161}{2}}$$

$$\therefore \quad \text{O}\widehat{\text{M}}\text{F} = \cos^{-1}\left(\tfrac{39}{161}\right) \approx 76.0^\circ$$

11 **a** The bird is sitting in a tree, 10 m directly above Ayla. We describe the location of the bird using 3-dimensional coordinates.

$\therefore$ the bird is located at $(30, 20, 10)$.

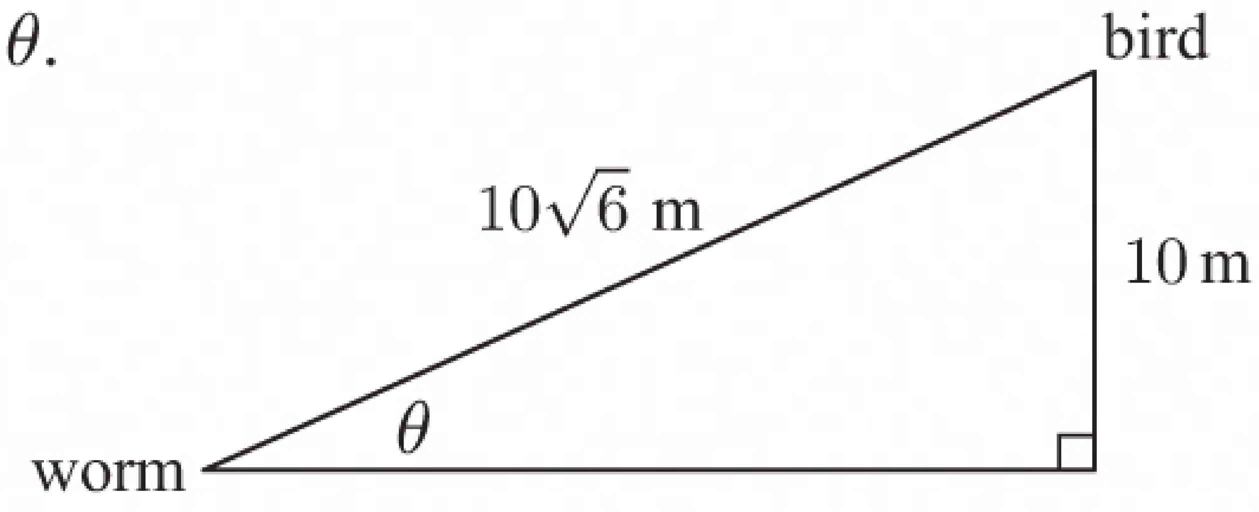
b **i** The worm is located at ×, which is at $(10, 10, 0)$.



The distance from the worm to the bird

$$\begin{aligned} &= \sqrt{(30-10)^2 + (20-10)^2 + (10-0)^2} \\ &= \sqrt{20^2 + 10^2 + 10^2} \\ &= \sqrt{400 + 100 + 100} \\ &= \sqrt{600} \\ &= 10\sqrt{6} \text{ m} \\ &\approx 24.5 \text{ m}\end{aligned}$$

ii Let the angle that the bird flies to the ground be θ.



$$\sin\theta = \frac{10}{10\sqrt{6}}$$

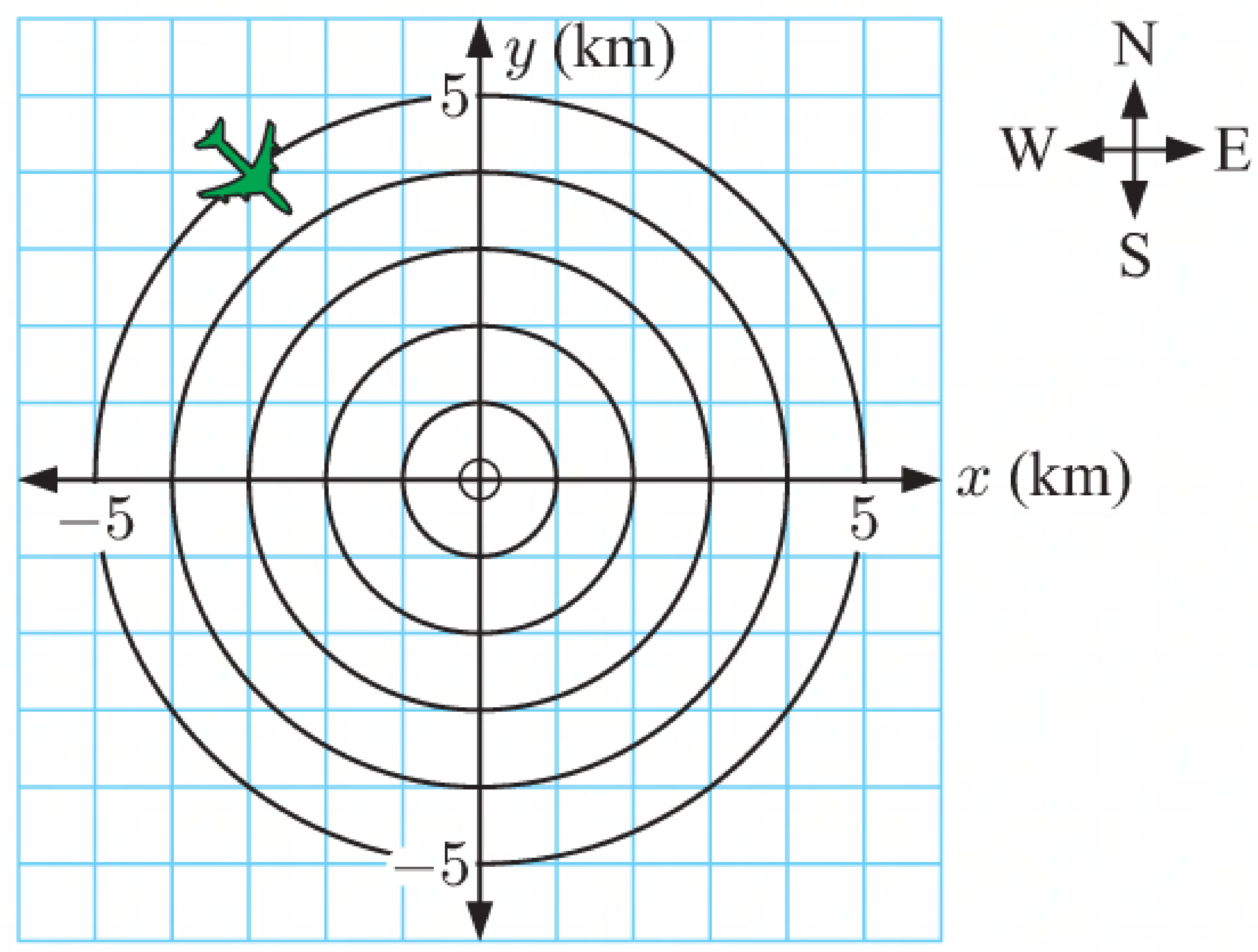
$$\therefore \quad \theta = \sin^{-1}\left(\frac{10}{10\sqrt{6}}\right)$$

$$\therefore \quad \theta \approx 24.1^\circ$$

So the bird flies at an angle of about 24.1° to the ground.

12 **a** The plane is 500 m, or $\frac{1}{2}$ km above $(-3, 4)$.

$\therefore$ the plane is at $\left(-3, 4, \frac{1}{2}\right)$.



b Distance of plane from control centre

$$= \sqrt{(0 - -3)^2 + (0-4)^2 + (0 - \tfrac{1}{2})^2}$$

$$= \sqrt{3^2 + (-4)^2 + (-\tfrac{1}{2})^2}$$

$$= \sqrt{9 + 16 + \tfrac{1}{4}}$$

$$= \sqrt{\tfrac{101}{4}}$$

$$= \frac{\sqrt{101}}{2}$$

$$\approx 5.02 \text{ km}$$

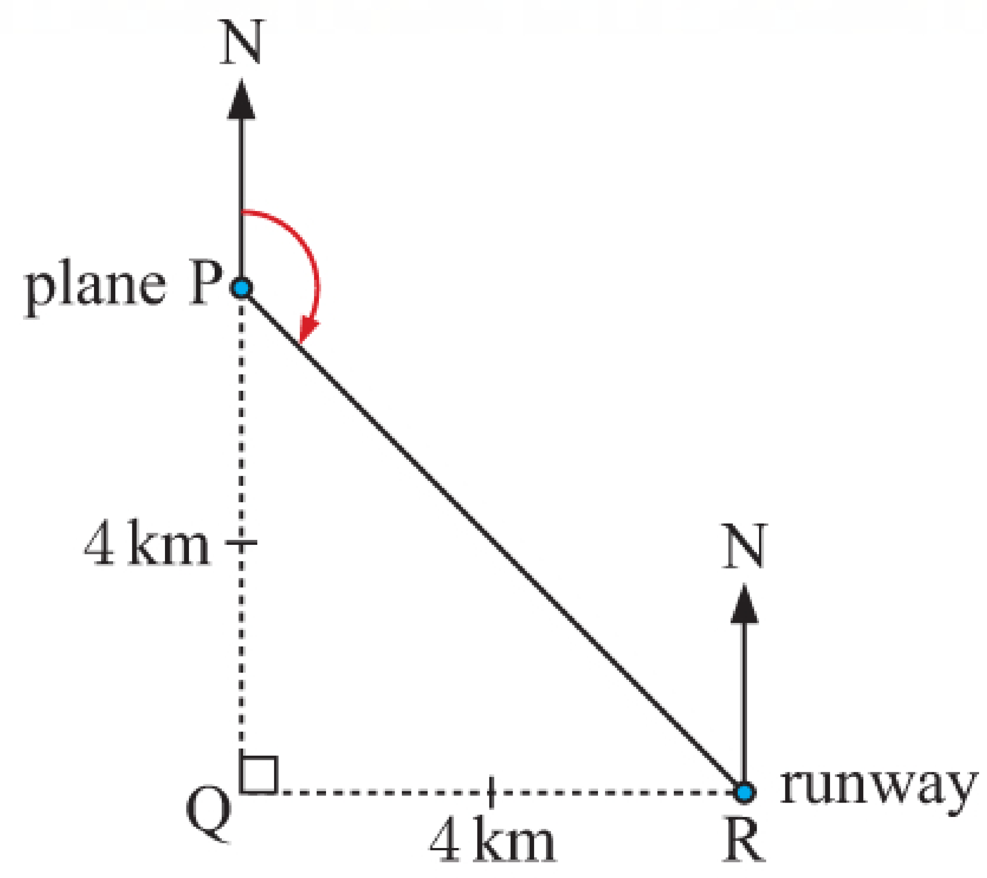
c The aeroplane is at $\left(-3, 4, \frac{1}{2}\right)$, so it is 3 km west and 4 km north of the control centre.

The runway is 1 km east of the control centre, so it is $3 + 1 = 4$ km east and 4 km south of the plane.

$\therefore$ $\triangle$PQR is right angled isosceles, with QP = QR = 4 km.

$\therefore$ $Q\widehat{P}R = Q\widehat{R}P = 45^\circ$ {equal base angles}

$\therefore$ the runway is at a bearing of $90^\circ + 45^\circ = 135^\circ$ from the plane.



d P is at $\left(-3, 4, \frac{1}{2}\right)$ and R is at $(1, 0, 0)$.

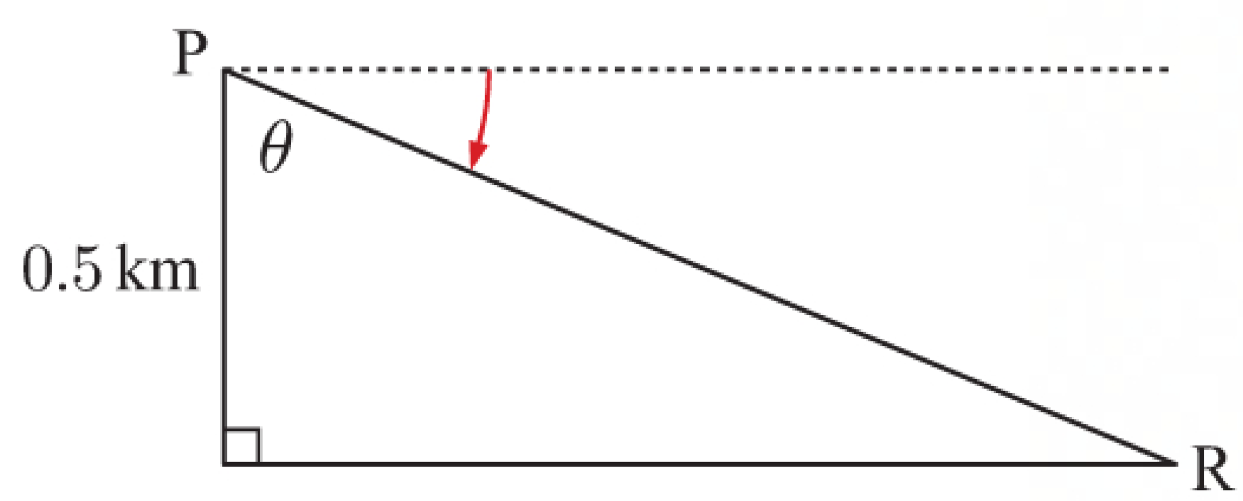
$$\text{PR} = \sqrt{(1 - -3)^2 + (0-4)^2 + (0 - \tfrac{1}{2})^2}$$

$$= \sqrt{4^2 + (-4)^2 + (-\tfrac{1}{2})^2}$$

$$= \sqrt{16 + 16 + \tfrac{1}{4}}$$

$$= \sqrt{\tfrac{129}{4}}$$

$$= \frac{\sqrt{129}}{2} \text{ km}$$



Now $\cos\theta = \dfrac{0.5}{\text{PR}}$

$\therefore\ \cos\theta = \dfrac{0.5}{\left(\frac{\sqrt{129}}{2}\right)} = \frac{1}{\sqrt{129}}$

$\therefore\ \theta = \cos^{-1}\left(\frac{1}{\sqrt{129}}\right)$

$\therefore\ \theta \approx 84.95°$

So, the angle of the plane's descent is $90° - 84.95° \approx 5.05°$.

13 **a** Jack is at $(-1, -3, 1.2)$ and Malina is at $(0, 4, 0.7)$.

$$\begin{aligned}\text{Distance between Jack and Malina} &= \sqrt{(0--1)^2+(4--3)^2+(0.7-1.2)^2}\\ &= \sqrt{1^2+7^2+(-0.5)^2}\\ &= \sqrt{1+49+0.25}\\ &= \sqrt{50.25}\\ &\approx 7.09\end{aligned}$$

So, Jack and Malina are approximately 7.09 km apart.

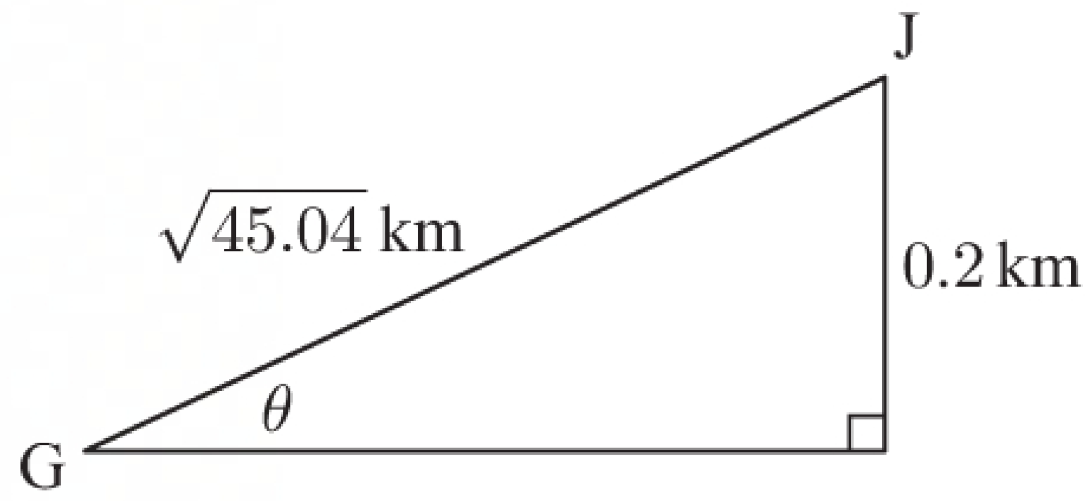
b $Z = 0$ represents sea level, so Gabriel is 1 km above sea level, Jack is 1.2 km above sea level, and Malina is 0.7 km above sea level.

$\therefore$ the explorers, from highest altitude to lowest altitude, are Jack, Gabriel, and Malina.

c **i** Gabriel is at $(2, 3, 1)$ and Jack is at $(-1, -3, 1.2)$.

$$\begin{aligned}\text{Distance between Gabriel and Jack} &= \sqrt{(-1-2)^2+(-3-3)^2+(1.2-1)^2}\\ &= \sqrt{(-3)^2+(-6)^2+(0.2)^2}\\ &= \sqrt{9+36+0.04}\\ &= \sqrt{45.04}\text{ km}\end{aligned}$$

Let the angle of elevation from Gabriel to Jack be θ.



$\sin\theta = \dfrac{0.2}{\sqrt{45.04}}$

$\therefore\ \theta = \sin^{-1}\left(\dfrac{0.2}{\sqrt{45.04}}\right)$

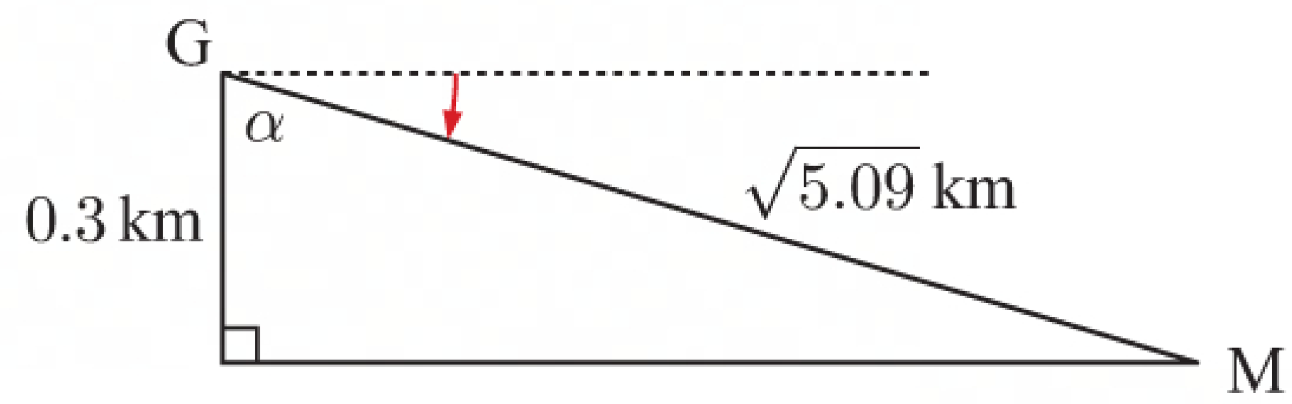
$\therefore\ \theta \approx 1.71°$

So, the angle of elevation from Gabriel to Jack is approximately 1.71°.

ii Gabriel is at $(2, 3, 1)$ and Malina is at $(0, 4, 0.7)$.

$$\begin{aligned}\text{Distance between Gabriel and Malina} &= \sqrt{(0-2)^2+(4-3)^2+(0.7-1)^2}\\ &= \sqrt{(-2)^2+1^2+(-0.3)^2}\\ &= \sqrt{4+1+0.09}\\ &= \sqrt{5.09}\text{ km}\end{aligned}$$

Let the angle of depression from Gabriel to Malina be $90° - \alpha$.



$$\cos\alpha = \frac{0.3}{\sqrt{5.09}}$$

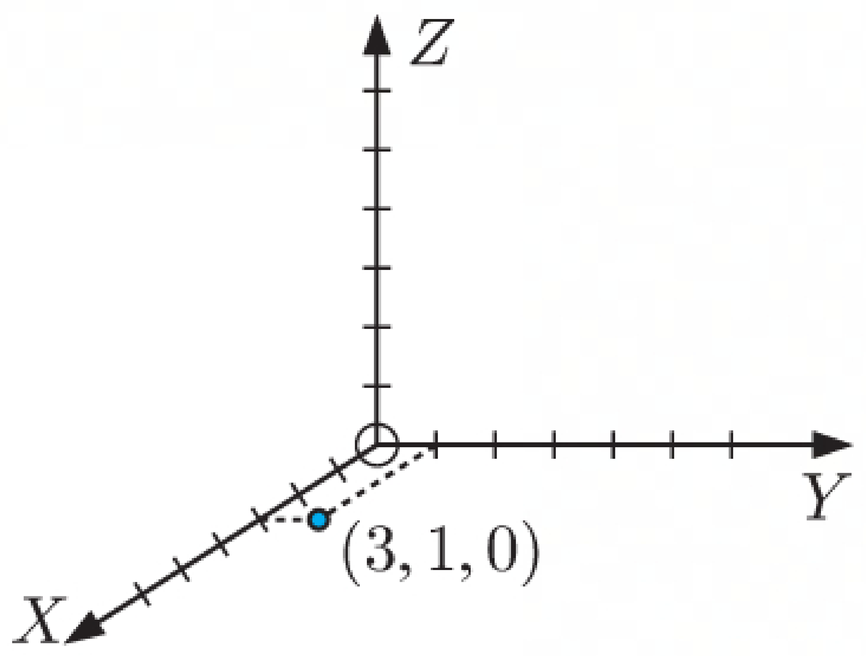
$$\therefore\ \alpha = \cos^{-1}\left(\frac{0.3}{\sqrt{5.09}}\right)$$

$$\therefore\ \alpha \approx 82.36°$$

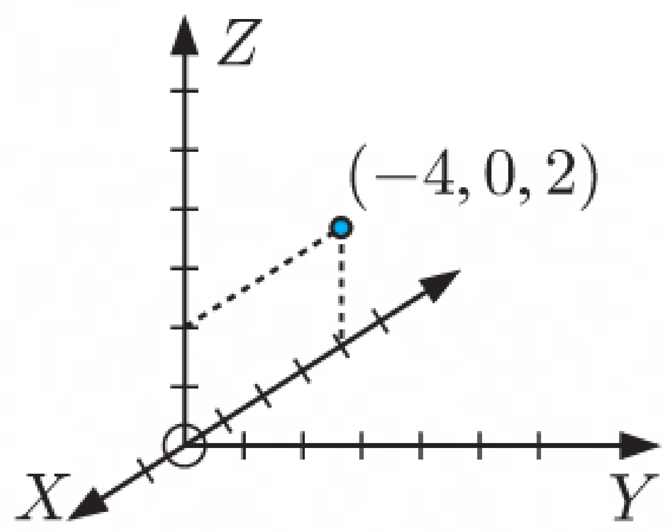
So, the angle of depression from Gabriel to Malina is approximately $90° - 82.36° \approx 7.64°$.

REVIEW SET 10A

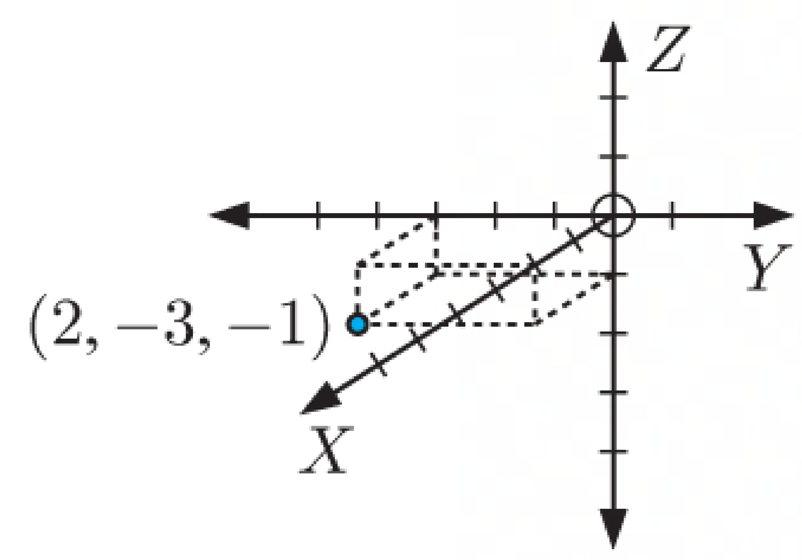
1 **a**



b



c



2 **a** **i**

$$\begin{aligned} PQ &= \sqrt{(-3-1)^2 + (-6 - -2)^2 + (2-0)^2} \\ &= \sqrt{(-4)^2 + (-4)^2 + 2^2} \\ &= \sqrt{16+16+4} \\ &= \sqrt{36} \\ &= 6 \text{ units} \end{aligned}$$

ii The midpoint is

$$\left(\frac{1+-3}{2}, \frac{-2+-6}{2}, \frac{0+2}{2}\right)$$

which is $(-1, -4, 1)$.

b **i**

$$\begin{aligned} PQ &= \sqrt{(-2 - -3)^2 + (-7-1)^2 + (1-6)^2} \\ &= \sqrt{1^2 + (-8)^2 + (-5)^2} \\ &= \sqrt{1+64+25} \\ &= \sqrt{90} \\ &= 3\sqrt{10} \text{ units} \end{aligned}$$

ii The midpoint is

$$\left(\frac{-3+-2}{2}, \frac{1+-7}{2}, \frac{6+1}{2}\right)$$

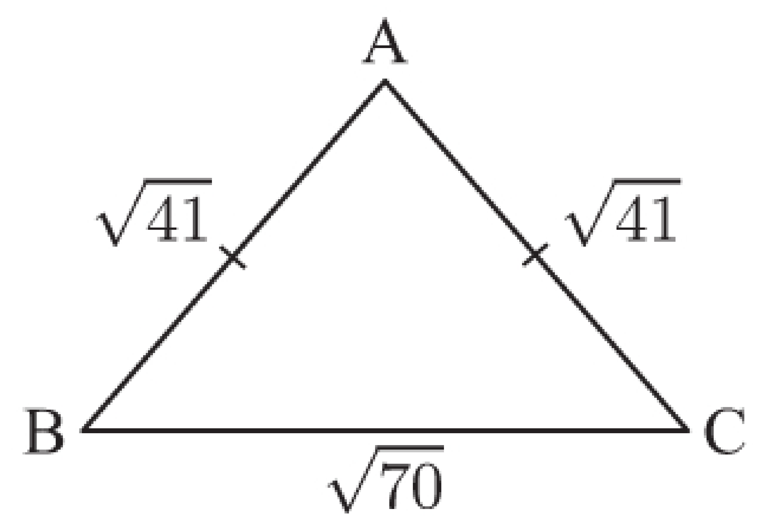
which is $(-\frac{5}{2}, -3, \frac{7}{2})$.

3

$$\begin{aligned} AB &= \sqrt{(3 - -2)^2 + (5-5)^2 + (-3-1)^2} \\ &= \sqrt{5^2 + 0^2 + (-4)^2} \\ &= \sqrt{25+0+16} \\ &= \sqrt{41} \text{ units} \end{aligned}$$

$$\begin{aligned} AC &= \sqrt{(0 - -2)^2 + (-1-5)^2 + (2-1)^2} \\ &= \sqrt{2^2 + (-6)^2 + 1^2} \\ &= \sqrt{4+36+1} \\ &= \sqrt{41} \text{ units} \end{aligned}$$

$$\begin{aligned} BC &= \sqrt{(0-3)^2 + (-1-5)^2 + (2 - -3)^2} \\ &= \sqrt{(-3)^2 + (-6)^2 + 5^2} \\ &= \sqrt{9+36+25} \\ &= \sqrt{70} \text{ units} \end{aligned}$$

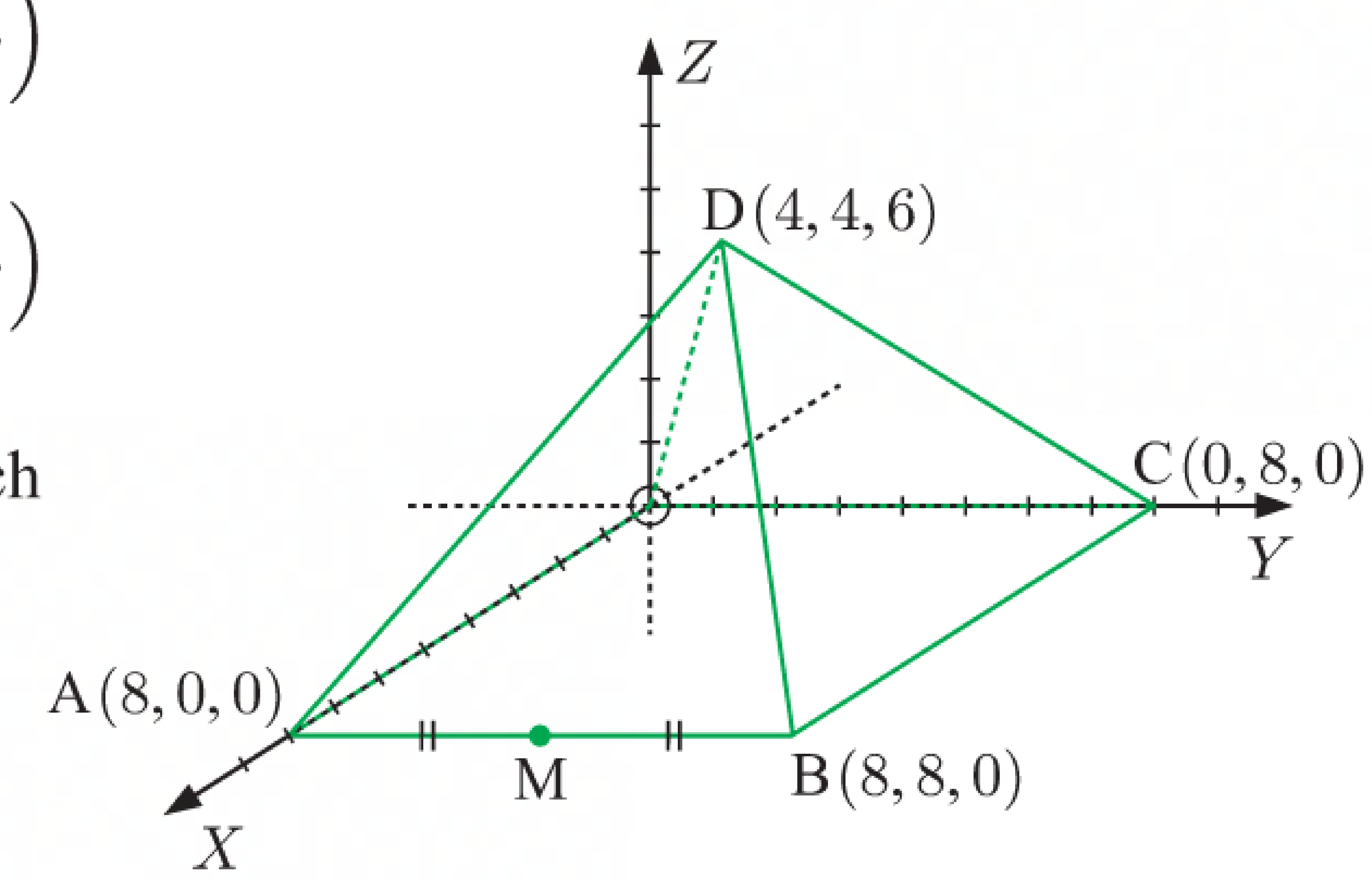


$AB = AC = \sqrt{41}$ units and $BC \neq AB$,
so $\triangle ABC$ is isosceles.

4 **a** The midpoint of [OB] is $\left(\dfrac{0+8}{2}, \dfrac{0+8}{2}, \dfrac{0+0}{2}\right)$ which is $(4, 4, 0)$.

The midpoint of [AC] is $\left(\dfrac{8+0}{2}, \dfrac{0+8}{2}, \dfrac{0+0}{2}\right)$ which is $(4, 4, 0)$.

$\therefore$ the centre of the base is $(4, 4, 0)$ which lies directly below the apex $D(4, 4, 6)$.



Volume of pyramid

$= \frac{1}{3}(\text{area of base} \times \text{height})$

$= \frac{1}{3} \times 8 \times 8 \times 6$

$= 128 \text{ units}^3$

b The midpoint M of [AB] is $\left(\dfrac{8+8}{2}, \dfrac{0+8}{2}, \dfrac{0+0}{2}\right)$ which is $(8, 4, 0)$.

c $\text{MD} = \sqrt{(4-8)^2 + (4-4)^2 + (6-0)^2}$

$= \sqrt{(-4)^2 + 0^2 + 6^2}$

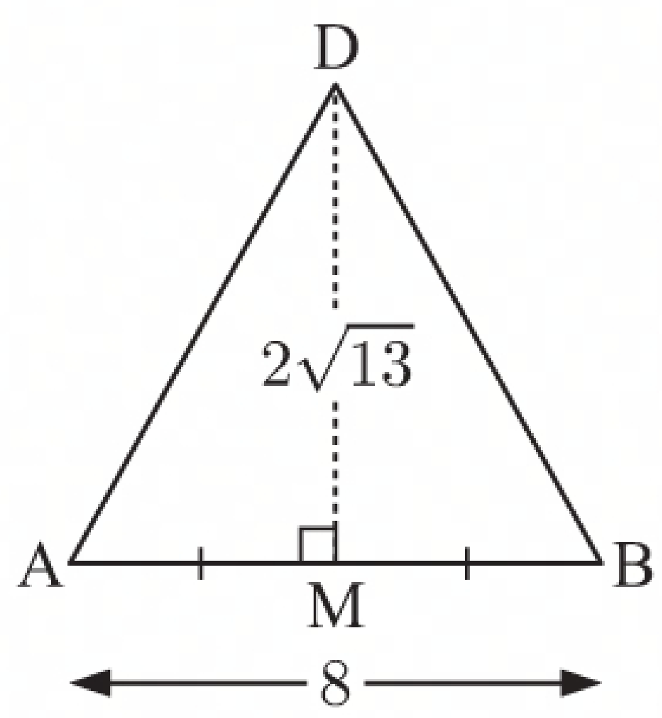
$= \sqrt{16 + 0 + 36}$

$= \sqrt{52}$

$= 2\sqrt{13} \text{ units}$

d Area of triangle ABD $= \frac{1}{2} \times 8 \times 2\sqrt{13}$

$= 8\sqrt{13} \text{ units}^2$



Surface area of pyramid

$=$ area of base $+$ area of 4 triangular faces

$= 8 \times 8 \ + \ 4 \times 8\sqrt{13}$

$= 64 + 32\sqrt{13}$

$= 32(2 + \sqrt{13}) \text{ units}^2$

$\approx 179 \text{ units}^2$

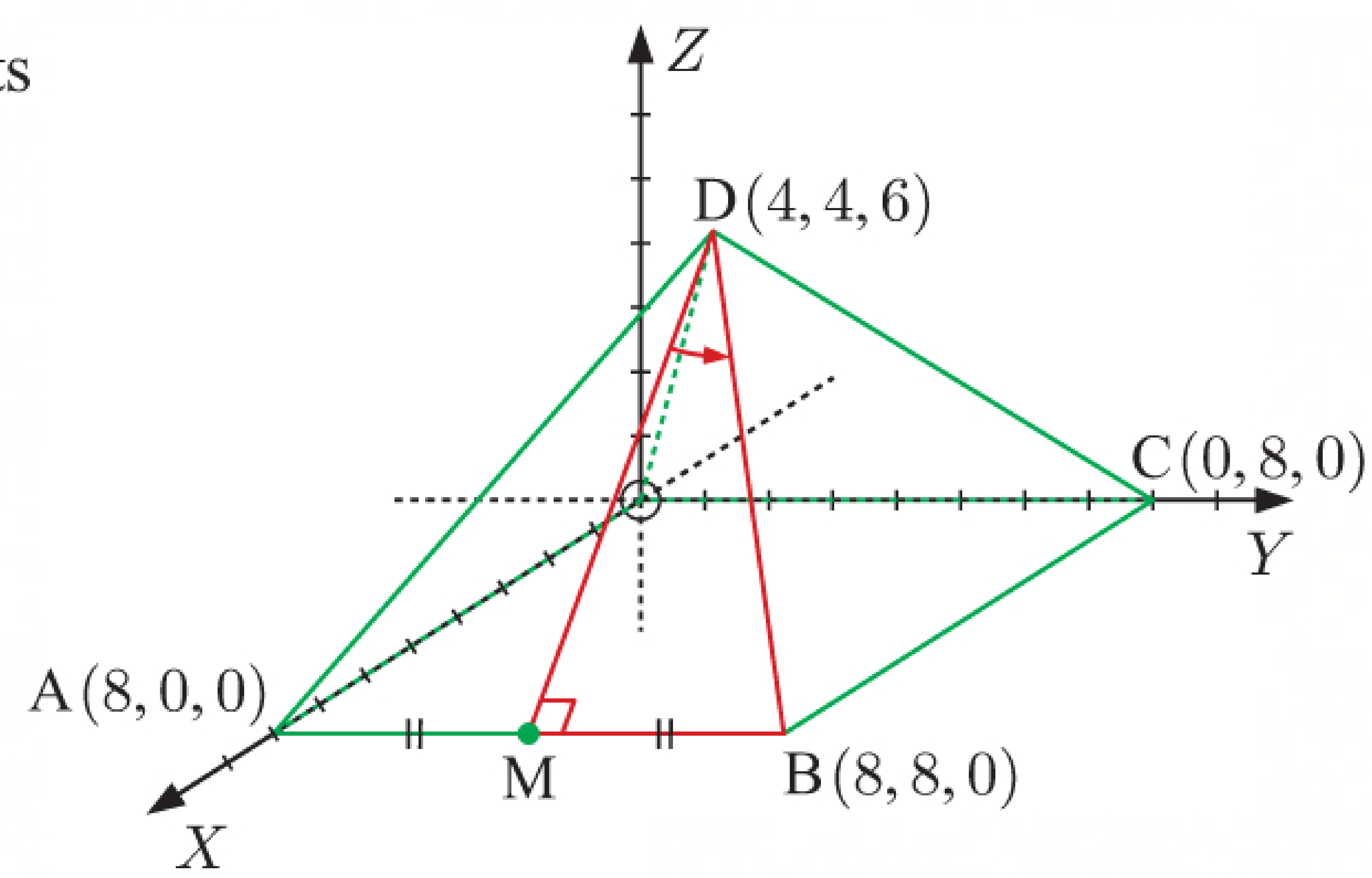
e Now $\text{MB} = 4$ units and $\text{MD} = 2\sqrt{13}$ units

$\therefore \tan M\widehat{D}B = \dfrac{\text{MB}}{\text{MD}}$

$= \dfrac{4}{2\sqrt{13}}$

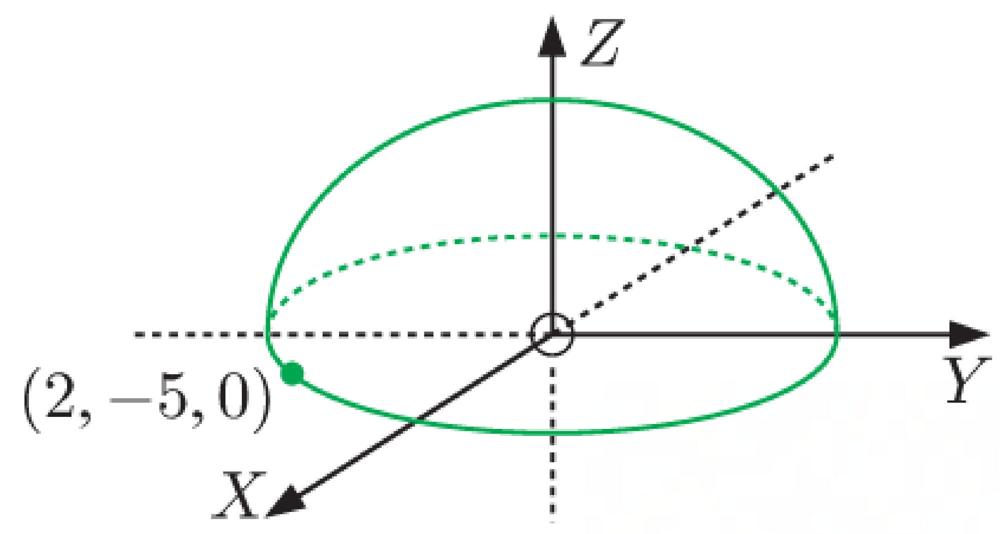
$\therefore M\widehat{D}B = \tan^{-1}\left(\frac{4}{2\sqrt{13}}\right)$

$\therefore M\widehat{D}B \approx 29.0^\circ$



5 **a** Radius of hemisphere = distance from centre $(0, 0, 0)$ to point $(2, -5, 0)$

$$= \sqrt{(2-0)^2 + (-5-0)^2 + (0-0)^2}$$
$$= \sqrt{2^2 + (-5)^2 + 0^2}$$
$$= \sqrt{4 + 25 + 0}$$
$$= \sqrt{29} \text{ units}$$



b Volume of hemisphere $= \frac{1}{2} \times \frac{4}{3}\pi r^3$

$$= \tfrac{2}{3} \times \pi \times (\sqrt{29})^3$$
$$\approx 327 \text{ units}^3$$

Surface area of hemisphere $= \frac{1}{2} \times 4\pi r^2 + \pi r^2$

$$= 3\pi r^2$$
$$= 3 \times \pi \times (\sqrt{29})^2$$
$$\approx 273 \text{ units}^2$$

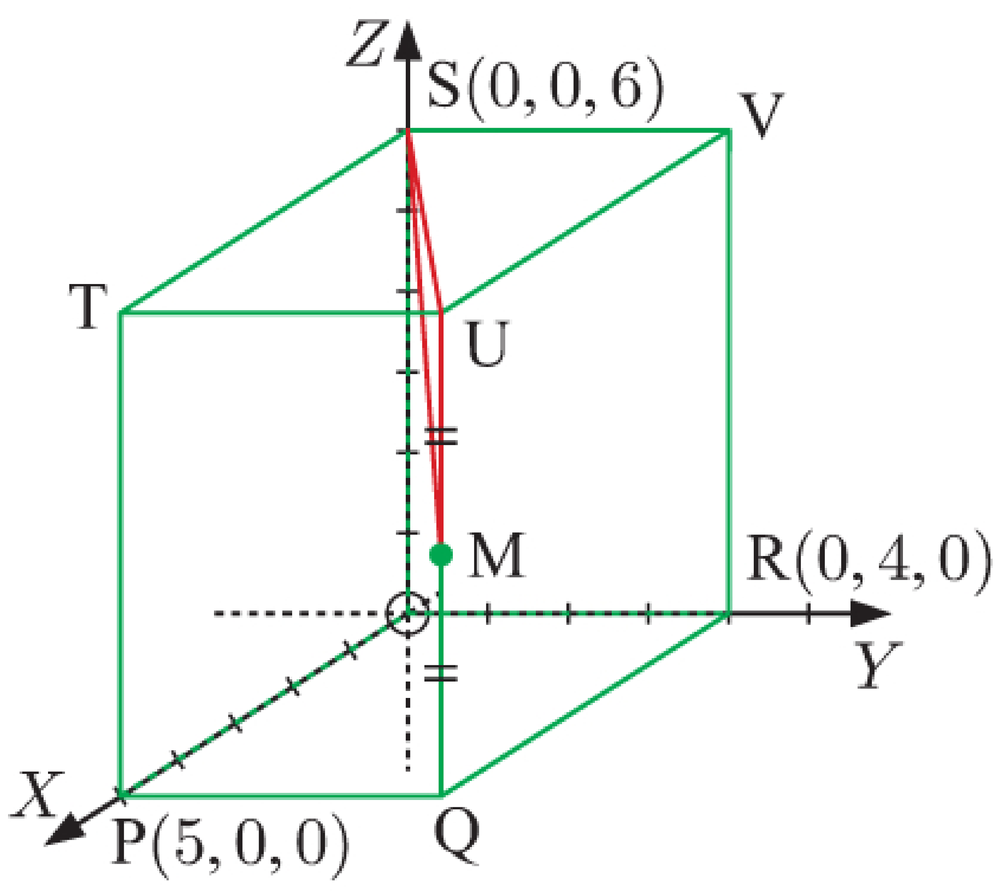
6 **a** Q is at $(5, 4, 0)$ and U is at $(5, 4, 6)$.

The midpoint M of [UQ] is $\left(\frac{5+5}{2}, \frac{4+4}{2}, \frac{6+0}{2}\right)$ which is $(5, 4, 3)$.

b Now $\text{UM} = 3$ units

and $\text{SU} = \sqrt{(5-0)^2 + (4-0)^2 + (6-6)^2}$

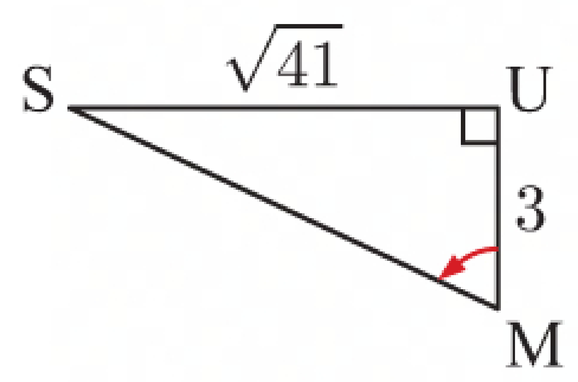
$$= \sqrt{5^2 + 4^2 + 0^2}$$
$$= \sqrt{25 + 16 + 0}$$
$$= \sqrt{41} \text{ units}$$



$\therefore \tan \widehat{\text{UMS}} = \dfrac{\sqrt{41}}{3}$

$\therefore \widehat{\text{UMS}} = \tan^{-1}\left(\frac{\sqrt{41}}{3}\right)$

$\therefore \widehat{\text{UMS}} \approx 64.9°$

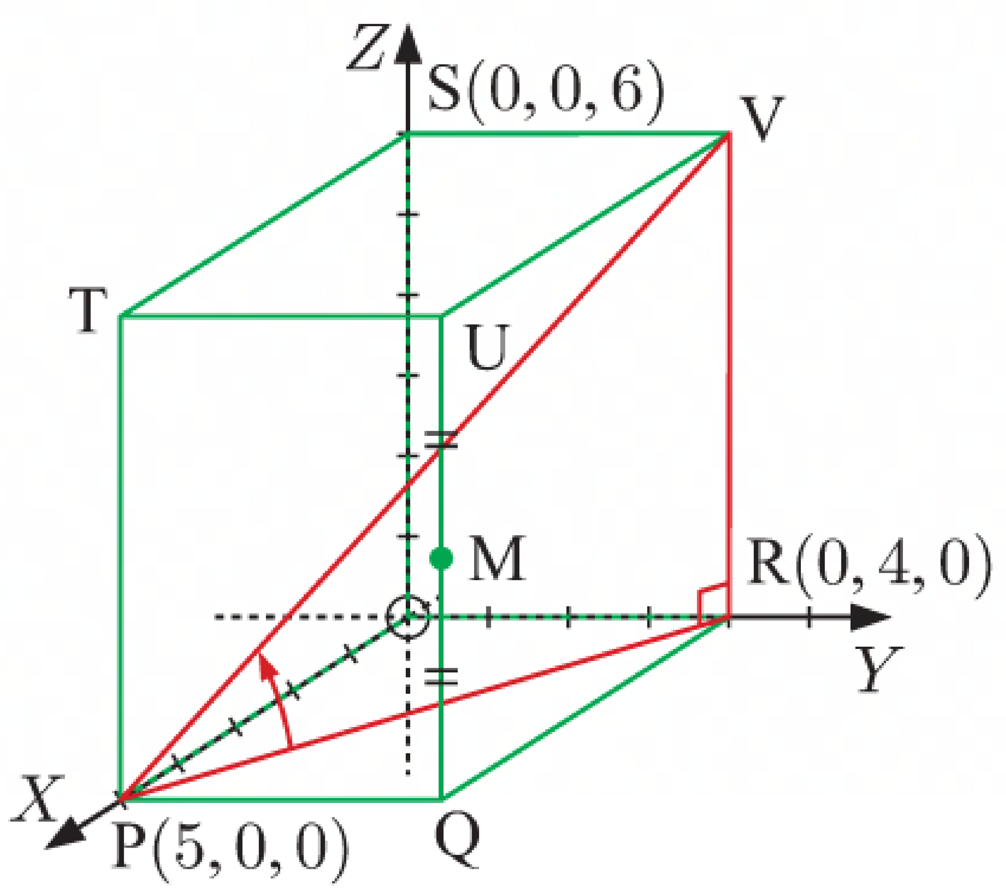


c **i** The required angle is $\widehat{\text{RPV}}$.

Now $\text{RV} = 6$ units

and $\text{PR} = \sqrt{(0-5)^2 + (4-0)^2 + (0-0)^2}$

$$= \sqrt{(-5)^2 + 4^2 + 0^2}$$
$$= \sqrt{25 + 16 + 0}$$
$$= \sqrt{41} \text{ units}$$



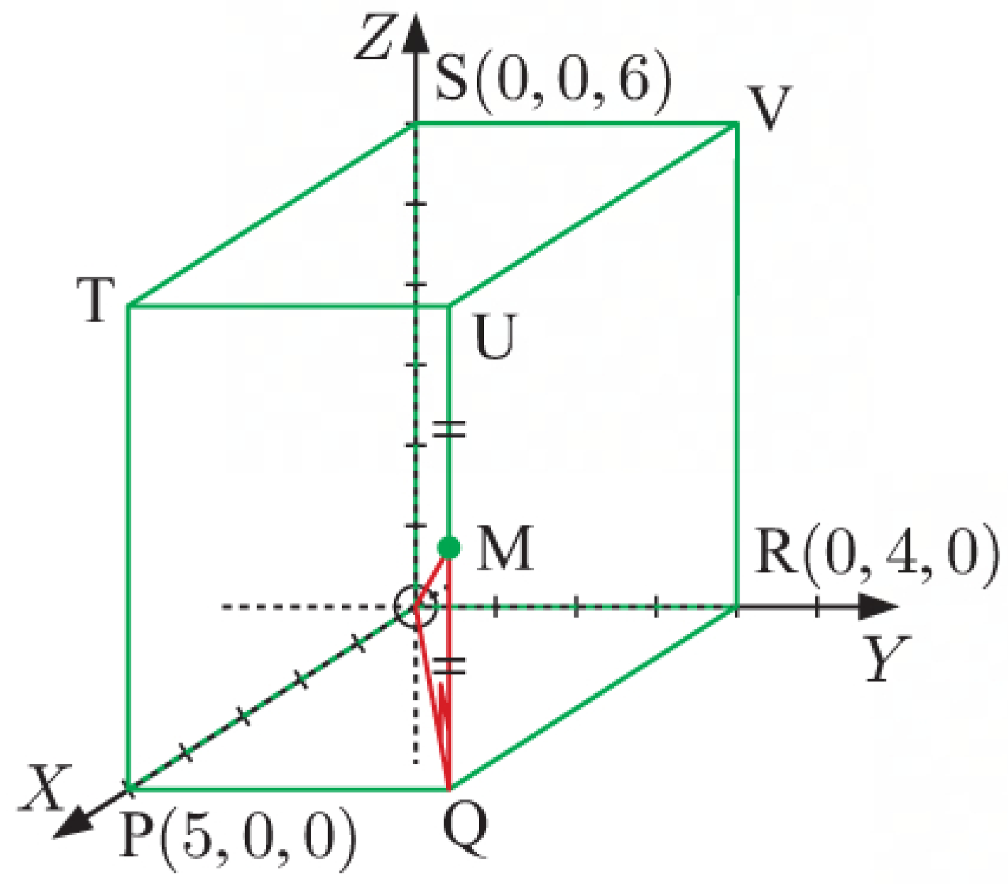
$\therefore \tan \widehat{\text{RPV}} = \dfrac{6}{\sqrt{41}}$

$\therefore \widehat{\text{RPV}} = \tan^{-1}\left(\frac{6}{\sqrt{41}}\right)$

$\therefore \widehat{\text{RPV}} \approx 43.1°$

The angle is about $43.1°$.

ii The required angle is $\widehat{MOQ}$.

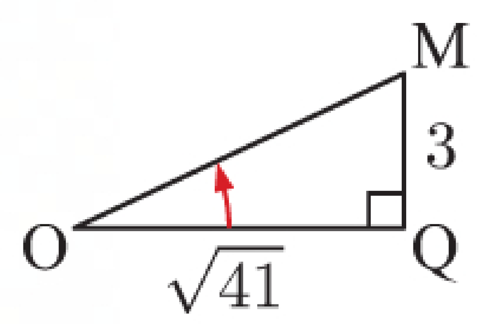


Now $MQ = 3$ units

and $OQ = PR = \sqrt{41}$ units

{diagonals of rectangle are equal in length}

$\therefore \tan \widehat{MOQ} = \dfrac{3}{\sqrt{41}}$



$\therefore \widehat{MOQ} = \tan^{-1}\left(\frac{3}{\sqrt{41}}\right)$

$\therefore \widehat{MOQ} \approx 25.1°$

The angle is about $25.1°$.

7 $AB = \sqrt{(0 - -1)^2 + (3-2)^2 + (1-5)^2}$

$= \sqrt{1^2 + 1^2 + (-4)^2}$

$= \sqrt{18}$ units

$AC = \sqrt{(2 - -1)^2 + (-4-2)^2 + (0-5)^2}$

$= \sqrt{3^2 + (-6)^2 + (-5)^2}$

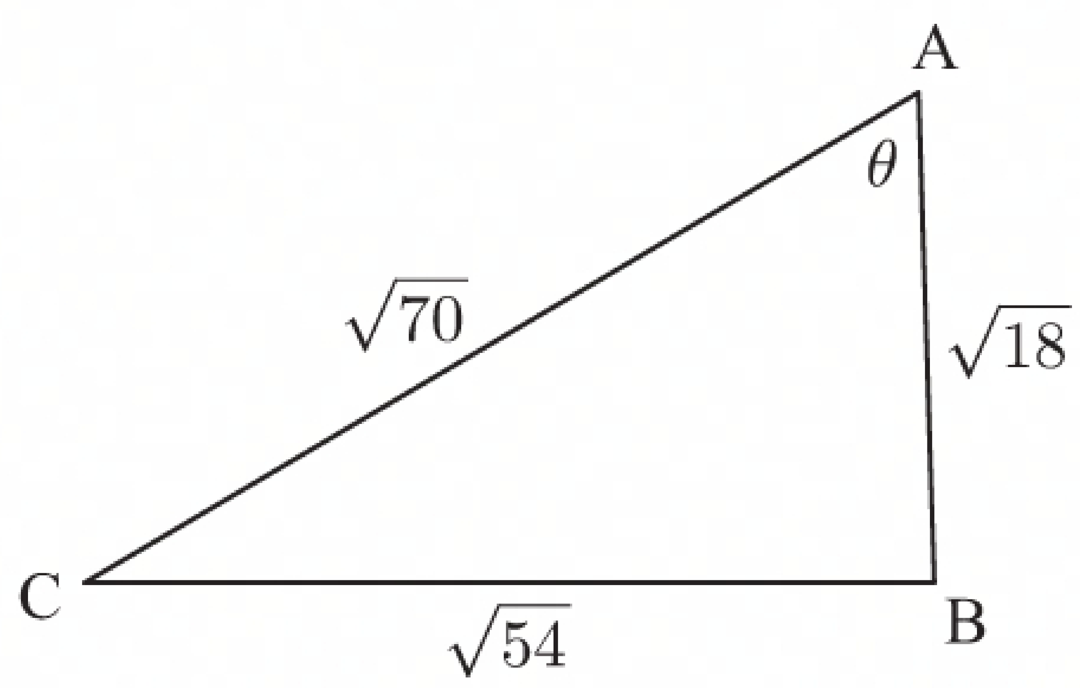
$= \sqrt{70}$ units

$BC = \sqrt{(2-0)^2 + (-4-3)^2 + (0-1)^2}$

$= \sqrt{2^2 + (-7)^2 + (-1)^2}$

$= \sqrt{54}$ units

By the cosine rule,



$$\cos\theta = \frac{(\sqrt{18})^2 + (\sqrt{70})^2 - (\sqrt{54})^2}{2 \times \sqrt{18} \times \sqrt{70}}$$

$$\therefore \cos\theta = \frac{18 + 70 - 54}{2 \times \sqrt{18} \times \sqrt{70}}$$

$$\therefore \theta = \cos^{-1}\left(\frac{34}{2 \times \sqrt{18} \times \sqrt{70}}\right) \approx 61.4°$$

So, $\widehat{BAC}$ is about $61.4°$.

8 **a** $CA = \sqrt{(8-1)^2 + (-7 - -3)^2 + (2-2)^2}$

$= \sqrt{7^2 + (-4)^2 + 0^2}$

$= \sqrt{65}$ units

$CB = \sqrt{(-1-1)^2 + (k - -3)^2 + (8-2)^2}$

$= \sqrt{(-2)^2 + (k+3)^2 + 6^2}$

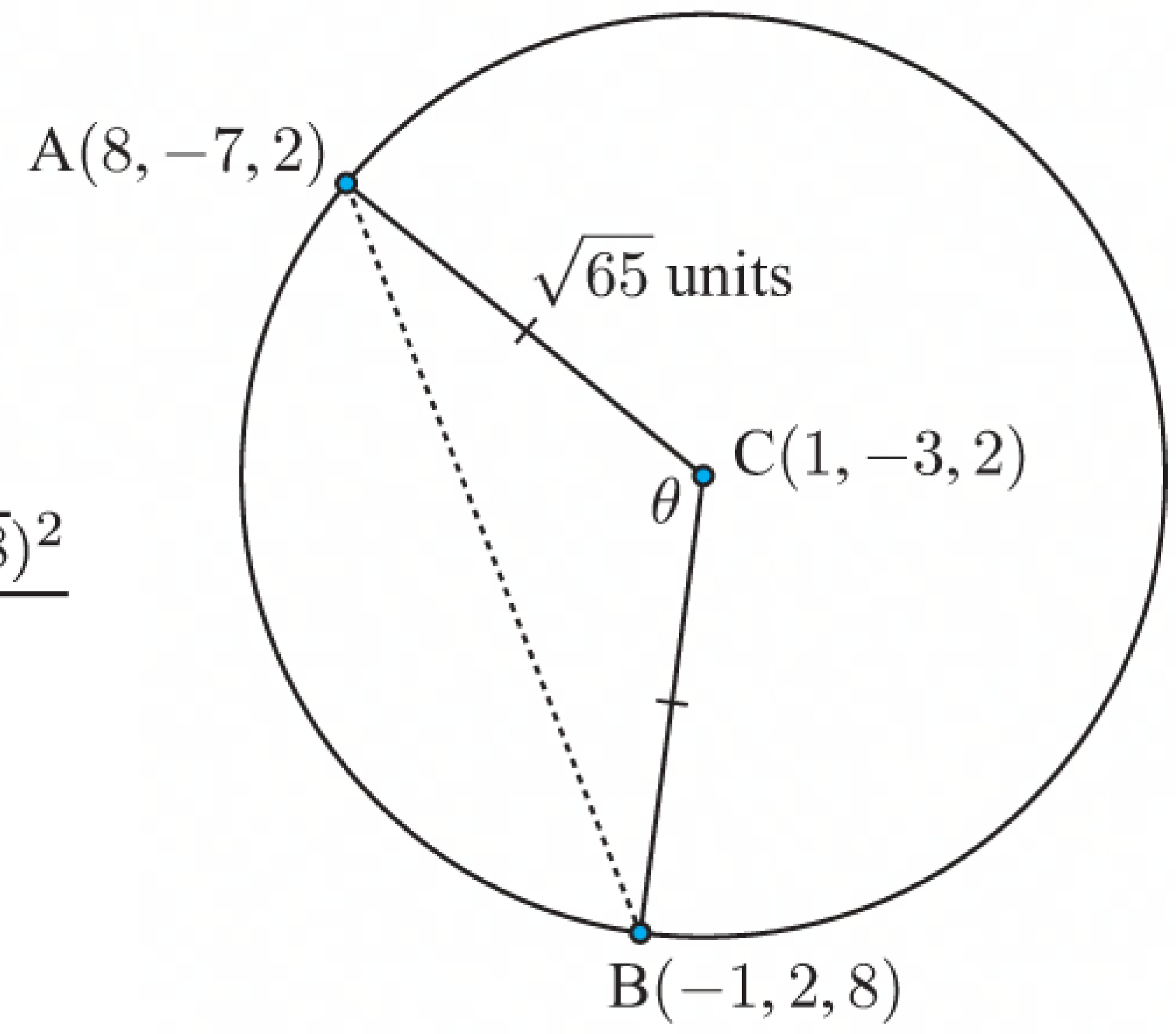
$= \sqrt{4 + k^2 + 6k + 9 + 36}$

$= \sqrt{k^2 + 6k + 49}$ units

$$\begin{aligned}
\text{Now} \quad CA &= CB \\
\therefore \quad \sqrt{65} &= \sqrt{k^2+6k+49} \\
\therefore \quad 65 &= k^2+6k+49 \\
\therefore \quad k^2+6k-16 &= 0 \\
\therefore \quad (k+8)(k-2) &= 0 \\
\therefore \quad k &= -8 \text{ or } 2
\end{aligned}$$

but $k > 0$ $\quad \therefore \quad k = 2$

b Since $k = 2$, B is $(-1, 2, 8)$.



$$\begin{aligned}
AB &= \sqrt{(-1-8)^2+(2--7)^2+(8-2)^2} \\
&= \sqrt{(-9)^2+9^2+6^2} \\
&= \sqrt{198} \text{ units}
\end{aligned}$$

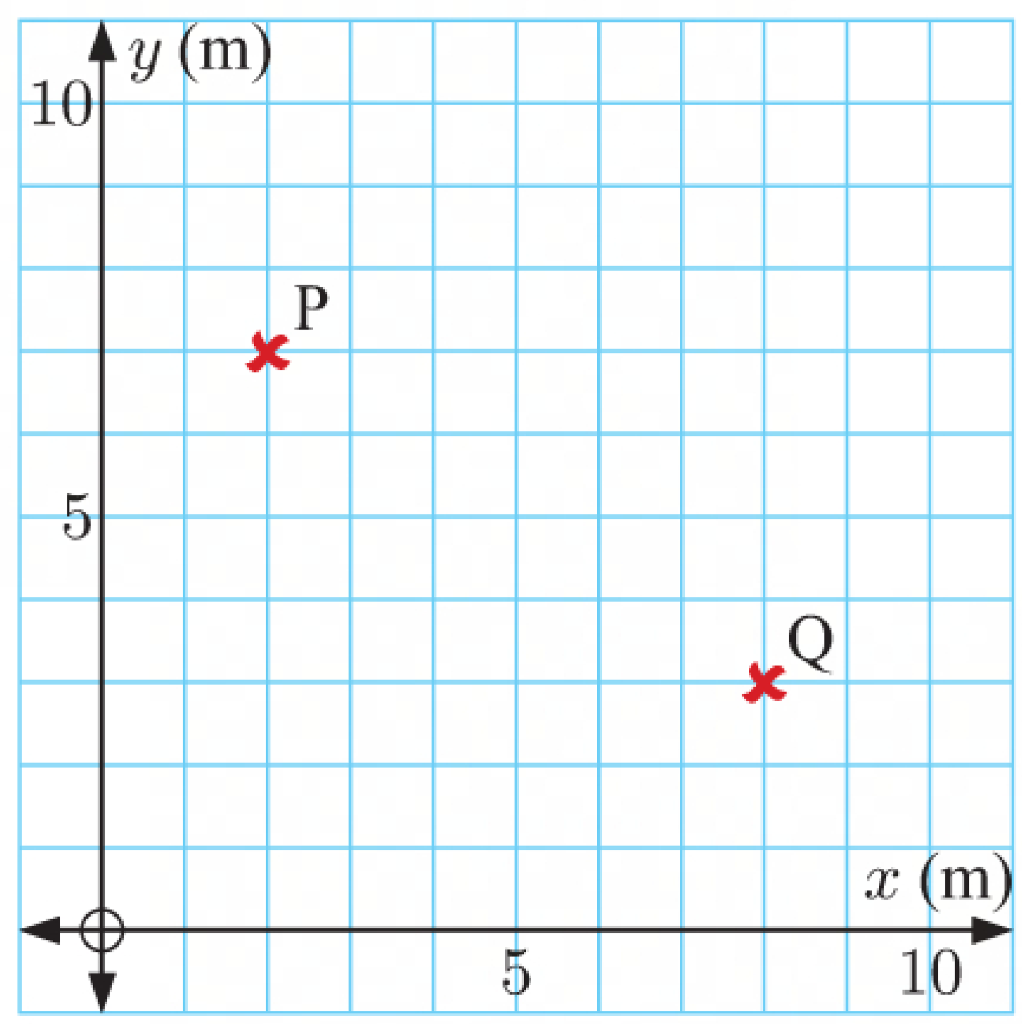
By the cosine rule, $\cos\theta = \dfrac{(\sqrt{65})^2+(\sqrt{65})^2-(\sqrt{198})^2}{2\times\sqrt{65}\times\sqrt{65}}$

$$\therefore \quad \cos\theta = \frac{65+65-198}{2\times 65}$$

$$\therefore \quad \theta = \cos^{-1}\left(-\tfrac{68}{130}\right) \approx 121.5^\circ$$

$$\begin{aligned}
\text{Now, area of minor sector CAB} &= \frac{\theta}{360}\times\pi r^2 \\
&\approx \tfrac{121.5}{360}\times\pi\times(\sqrt{65})^2 \\
&\approx 68.9 \text{ units}^2
\end{aligned}$$

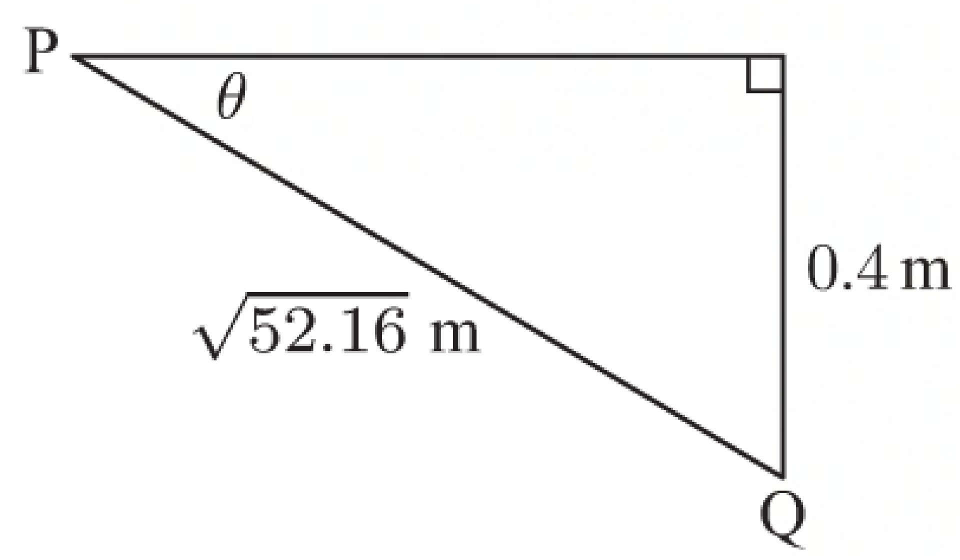
9 **a** The fossil at P has coordinates $(2, 7, -2.5)$.
The fossil at Q has coordinates $(8, 3, -2.9)$.



b
$$\begin{aligned}
PQ &= \sqrt{(8-2)^2+(3-7)^2+(-2.9--2.5)^2} \\
&= \sqrt{6^2+(-4)^2+(-0.4)^2} \\
&= \sqrt{36+16+0.16} \\
&= \sqrt{52.16} \\
&\approx 7.22 \text{ m}
\end{aligned}$$

The distance between the fossils is about 7.22 m.

c Fossil Q is $2.9 - 2.5 = 0.4$ m deeper underground than fossil P.



$$\sin\theta = \frac{0.4}{\sqrt{52.16}}$$

$$\therefore \quad \theta = \sin^{-1}\left(\frac{0.4}{\sqrt{52.16}}\right)$$

$$\therefore \quad \theta \approx 3.17^\circ$$

The angle of depression from P to Q is about 3.17°.

REVIEW SET 10B

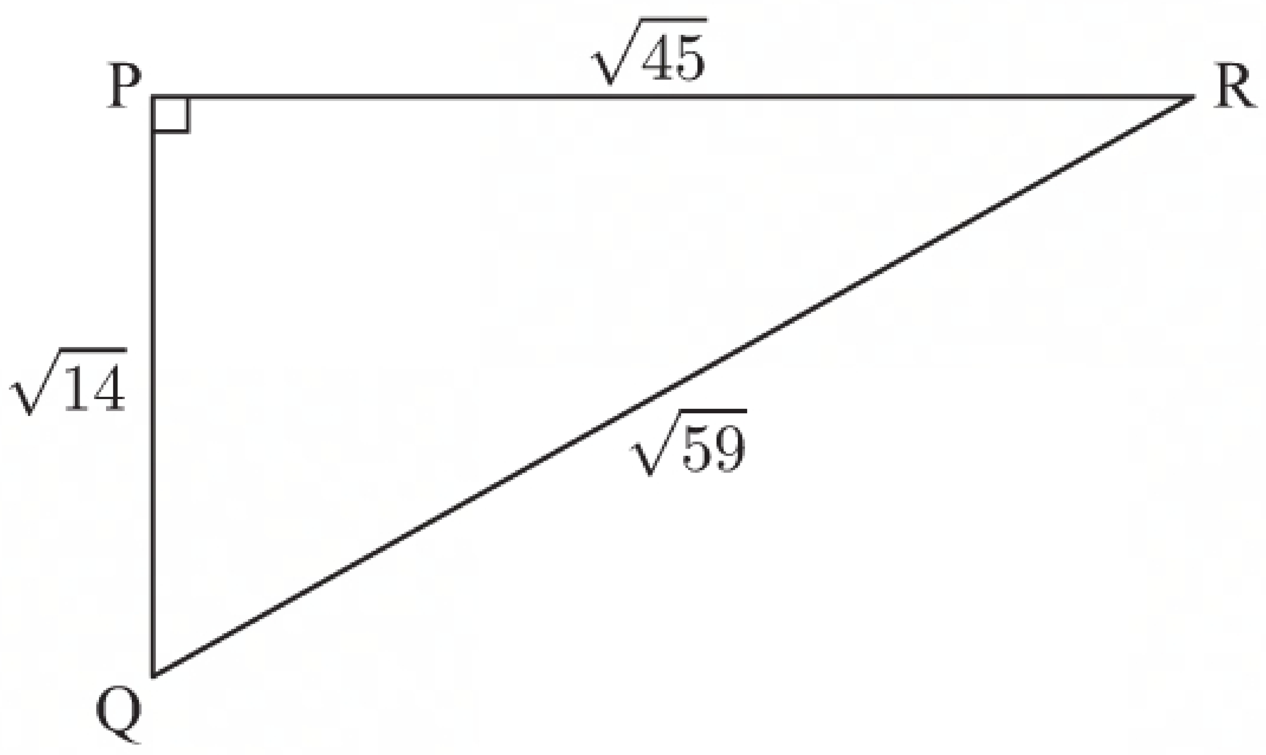
1 **a** **i** $\text{AB} = \sqrt{(-1 - -3)^2 + (6-0)^2 + (4-5)^2}$
$= \sqrt{2^2 + 6^2 + (-1)^2}$
$= \sqrt{4 + 36 + 1}$
$= \sqrt{41}$ units

ii The midpoint is
$\left(\frac{-3+-1}{2}, \frac{0+6}{2}, \frac{5+4}{2}\right)$
which is $\left(-2, 3, \frac{9}{2}\right)$.

b **i** $\text{AB} = \sqrt{(-2 - -7)^2 + (1-4)^2 + (-1-6)^2}$
$= \sqrt{5^2 + (-3)^2 + (-7)^2}$
$= \sqrt{25 + 9 + 49}$
$= \sqrt{83}$ units

ii The midpoint is
$\left(\frac{-7+-2}{2}, \frac{4+1}{2}, \frac{6+-1}{2}\right)$
which is $\left(-\frac{9}{2}, \frac{5}{2}, \frac{5}{2}\right)$.

2 **a** $\text{PQ} = \sqrt{(-2 - -5)^2 + (-2-0)^2 + (2-1)^2}$
$= \sqrt{3^2 + (-2)^2 + 1^2}$
$= \sqrt{9 + 4 + 1}$
$= \sqrt{14}$ units

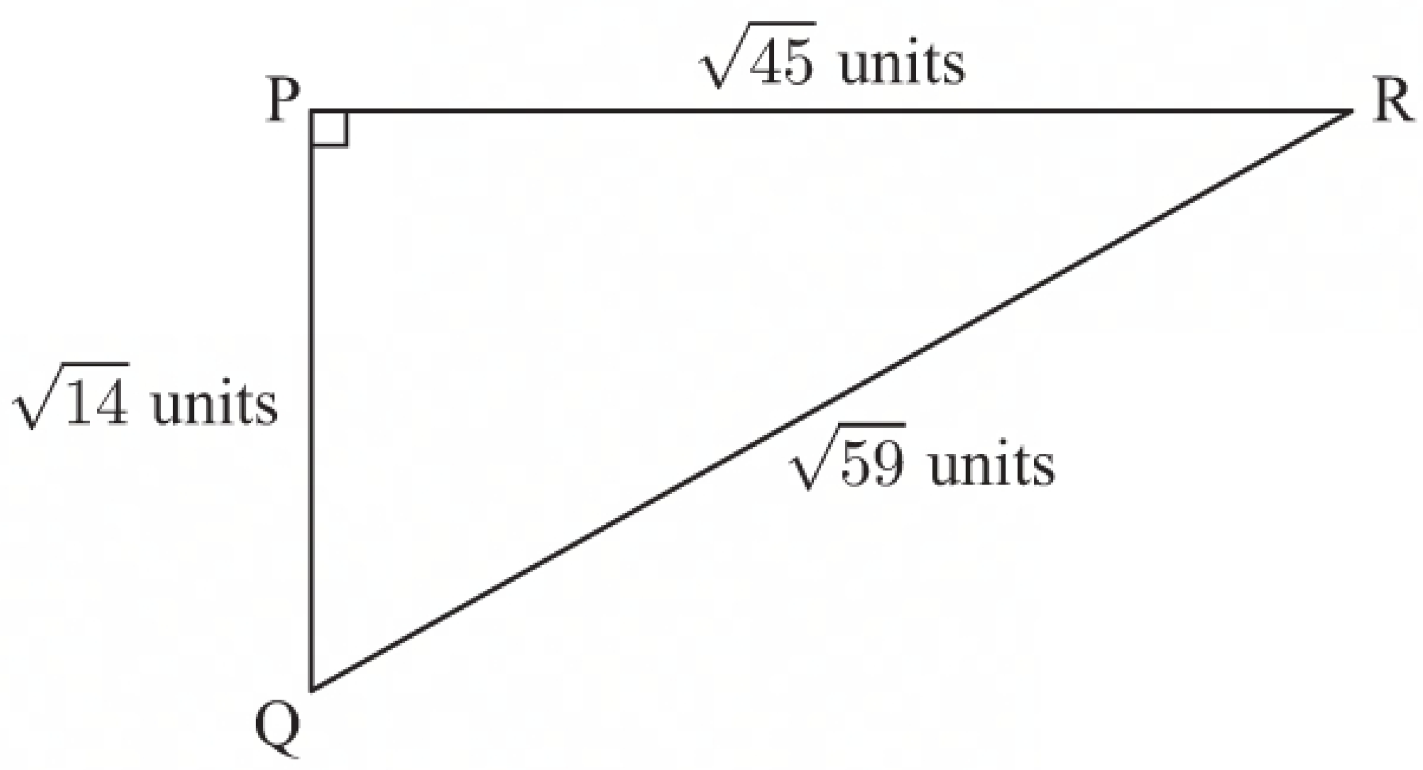


$\text{PR} = \sqrt{(-1 - -5)^2 + (5-0)^2 + (-1-1)^2}$
$= \sqrt{4^2 + 5^2 + (-2)^2}$
$= \sqrt{16 + 25 + 4}$
$= \sqrt{45}$ units

$\text{QR} = \sqrt{(-1 - -2)^2 + (5 - -2)^2 + (-1-2)^2}$
$= \sqrt{1^2 + 7^2 + (-3)^2}$
$= \sqrt{1 + 49 + 9}$
$= \sqrt{59}$ units

$\text{PQ}^2 + \text{PR}^2 = (\sqrt{14})^2 + (\sqrt{45})^2 = 59$
and $\text{QR}^2 = (\sqrt{59})^2 = 59$
$\therefore$ triangle PQR is right angled at P.

b $\tan P\widehat{Q}R = \dfrac{\sqrt{45}}{\sqrt{14}}$
$\therefore$ $P\widehat{Q}R = \tan^{-1}\left(\frac{\sqrt{45}}{\sqrt{14}}\right)$
$\therefore$ $P\widehat{Q}R \approx 60.8°$



3 The distance from $(4, -2, 1)$ to $(1, 3, k)$

is $\sqrt{(1-4)^2+(3--2)^2+(k-1)^2}=8$

$\therefore \sqrt{(-3)^2+5^2+k^2-2k+1}=8$

$\therefore \sqrt{9+25+k^2-2k+1}=8$

$\therefore \sqrt{k^2-2k+35}=8$

$\therefore k^2-2k+35=64$ {squaring both sides}

$\therefore k^2-2k=29$

$\therefore k^2-2k+(-1)^2=29+(-1)^2$ {completing the square}

$\therefore (k-1)^2=30$

$\therefore k-1=\pm\sqrt{30}$

$\therefore k=1\pm\sqrt{30}$

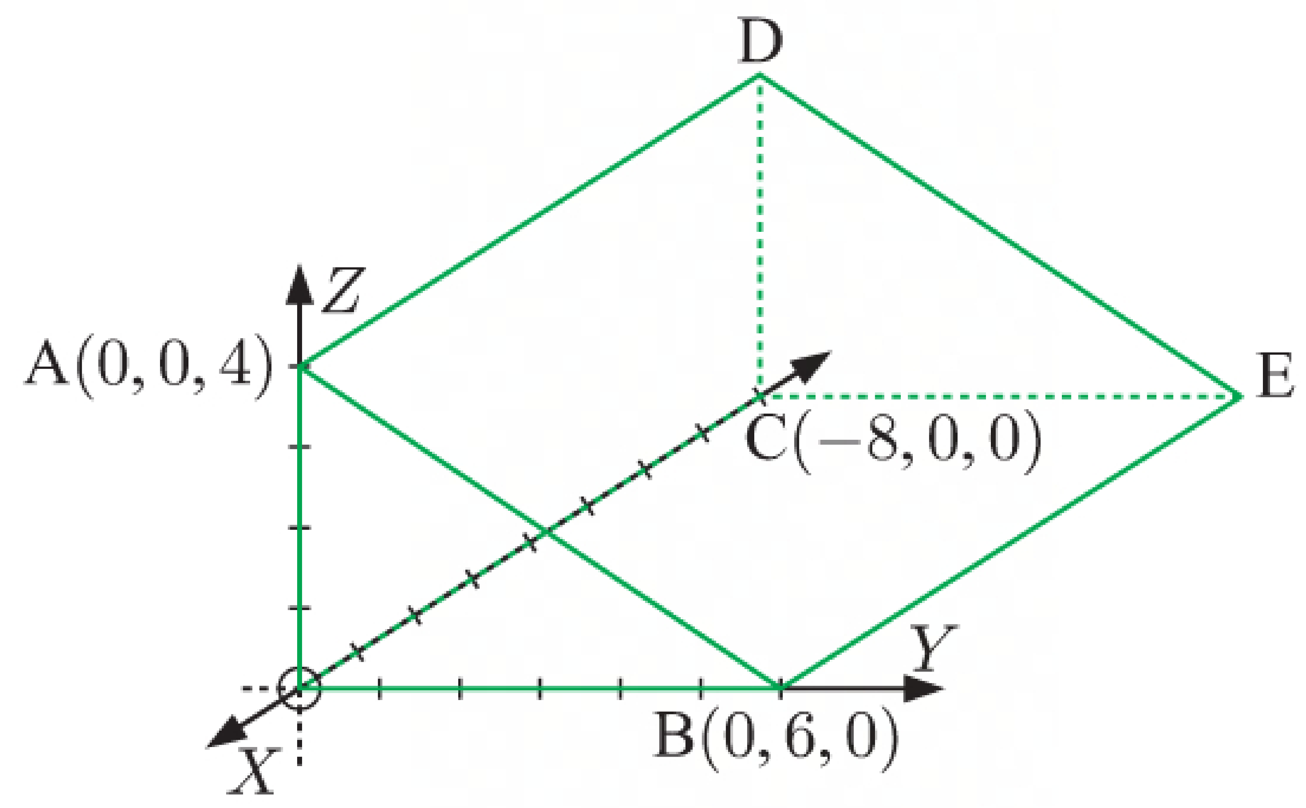
4 **a** Volume of prism

$=$ area of end $\times$ length

$=\frac{1}{2}\times$ base $\times$ height $\times$ length

$=\frac{1}{2}\times 6\times 4\times 8$

$= 96$ units^3



b $\text{AB}=\sqrt{(0-0)^2+(6-0)^2+(0-4)^2}$

$=\sqrt{0^2+6^2+(-4)^2}$

$=\sqrt{0+36+16}$

$=\sqrt{52}$

$=2\sqrt{13}$ units

c D is $(-8, 0, 4)$ and E is $(-8, 6, 0)$.

Surface area of prism

$=$ area of base $+$ area of 2 triangular faces $+$ area of 2 rectangular faces

$=8\times 6 + 2\times$ area of $\triangle$OAB $+$ area of quadrilateral OADC $+$ area of quadrilateral ABED

$=48 + 2\times\frac{1}{2}\times 6\times 4 + 8\times 4 + 8\times 2\sqrt{13}$

$=104+16\sqrt{13}$

≈ 162 units^2

5 **a** Centre of sphere $=$ midpoint of [PQ]

$=\left(\frac{4+-6}{2}, \frac{-2+2}{2}, \frac{3+-5}{2}\right)$

$=(-1, 0, -1)$

b Radius of sphere $=$ distance from centre $(-1, 0, -1)$ to P$(4, -2, 3)$

$=\sqrt{(4--1)^2+(-2-0)^2+(3--1)^2}$

$=\sqrt{5^2+(-2)^2+4^2}$

$=\sqrt{25+4+16}$

$=\sqrt{45}$

$=3\sqrt{5}$ units

c Volume of sphere $= \frac{4}{3}\pi r^3$
$= \frac{4}{3} \times \pi \times (3\sqrt{5})^3$
≈ 1260 units^3

Surface area of sphere $= 4\pi r^2$
$= 4 \times \pi \times (3\sqrt{5})^2$
≈ 565 units^2

6 **a** $PQ = \sqrt{(0 - -2)^2 + (-1-1)^2 + (4-3)^2}$
$= \sqrt{2^2 + (-2)^2 + 1^2}$
$= \sqrt{9} = 3$ units

$PR = \sqrt{(-3 - -2)^2 + (2-1)^2 + (0-3)^2}$
$= \sqrt{(-1)^2 + 1^2 + (-3)^2}$
$= \sqrt{11}$ units

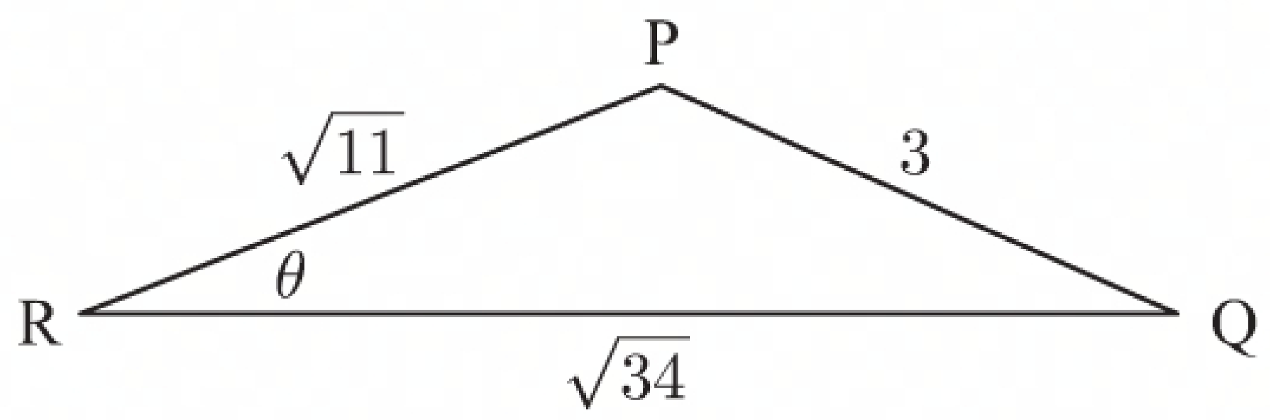
$QR = \sqrt{(-3-0)^2 + (2 - -1)^2 + (0-4)^2}$
$= \sqrt{(-3)^2 + 3^2 + (-4)^2}$
$= \sqrt{34}$ units

By the cosine rule,

$$\cos\theta = \frac{(\sqrt{11})^2 + (\sqrt{34})^2 - 3^2}{2 \times \sqrt{11} \times \sqrt{34}}$$

$$\therefore \cos\theta = \frac{11 + 34 - 9}{2 \times \sqrt{11} \times \sqrt{34}}$$

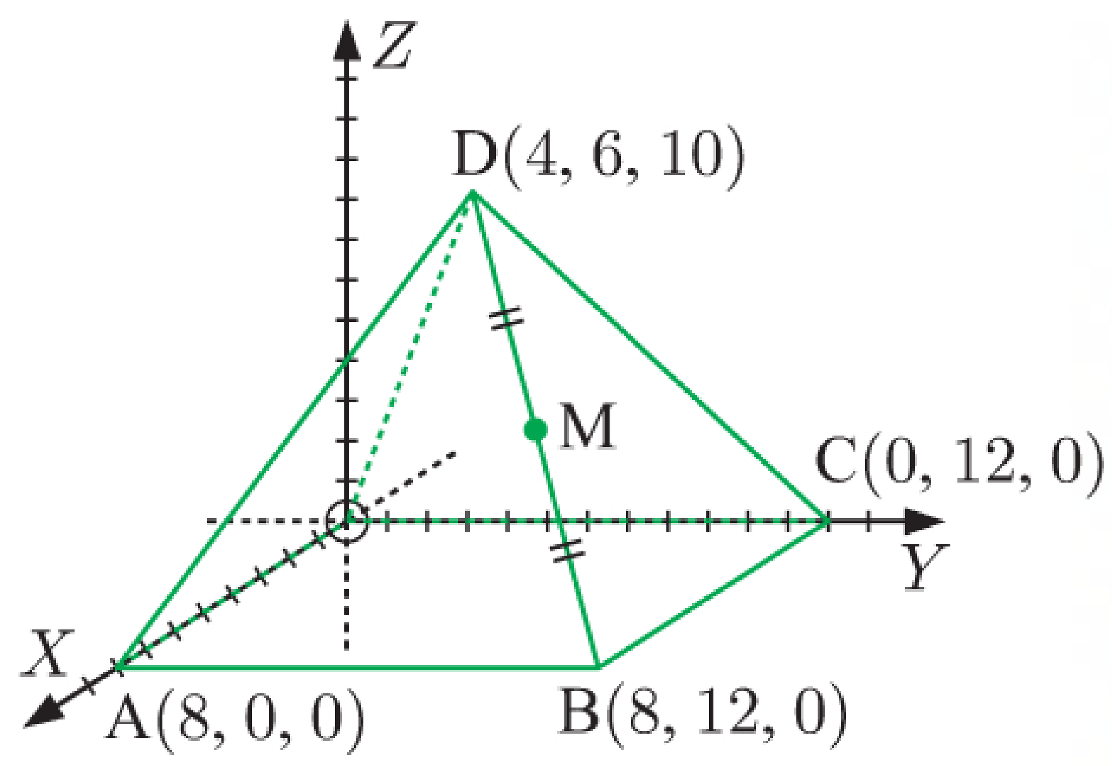
$$\therefore \theta = \cos^{-1}\left(\frac{36}{2 \times \sqrt{11} \times \sqrt{34}}\right) \approx 21.4^\circ$$



So, $P\widehat{R}Q$ is about 21.4°.

b Area of triangle PQR $= \frac{1}{2} \times \sqrt{11} \times \sqrt{34} \times \sin\theta$
$\approx \frac{1}{2} \times \sqrt{11} \times \sqrt{34} \times \sin 21.4^\circ$
≈ 3.54 units^2

7 **a** The midpoint M of [BD] is
$\left(\frac{8+4}{2}, \frac{12+6}{2}, \frac{0+10}{2}\right)$ which is $(6, 9, 5)$.



b $AC = \sqrt{(0-8)^2 + (12-0)^2 + (0-0)^2}$
$= \sqrt{(-8)^2 + 12^2 + 0^2}$
$= \sqrt{208}$ units

$AD = \sqrt{(4-8)^2 + (6-0)^2 + (10-0)^2}$
$= \sqrt{(-4)^2 + 6^2 + 10^2}$
$= \sqrt{152}$ units

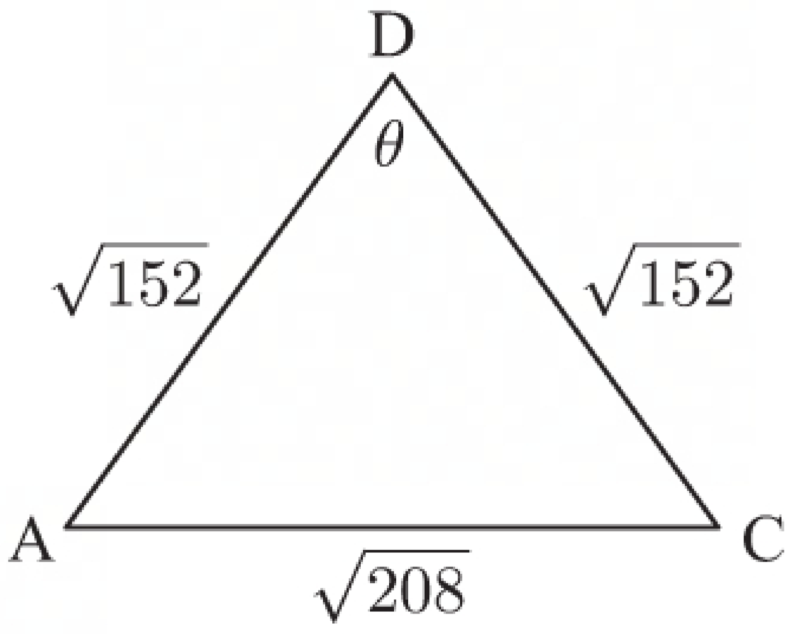
$CD = \sqrt{(4-0)^2 + (6-12)^2 + (10-0)^2}$
$= \sqrt{4^2 + (-6)^2 + 10^2}$
$= \sqrt{152}$ units

By the cosine rule,

$$\cos\theta = \frac{(\sqrt{152})^2 + (\sqrt{152})^2 - (\sqrt{208})^2}{2 \times \sqrt{152} \times \sqrt{152}}$$

$$\therefore \quad \cos\theta = \frac{152 + 152 - 208}{2 \times 152}$$

$$\therefore \quad \theta = \cos^{-1}\left(\tfrac{96}{304}\right) \approx 71.6°$$



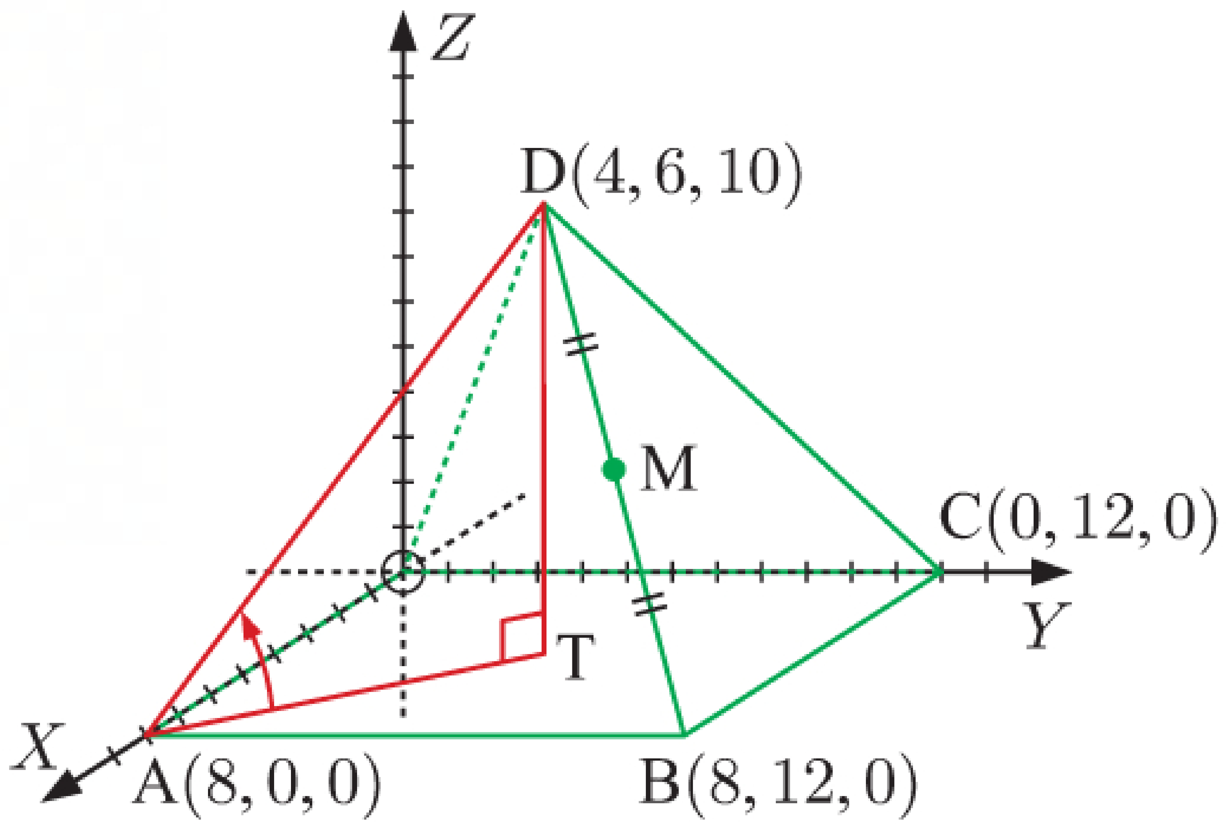
So, $\widehat{ADC}$ is about $71.6°$.

c **i** The required angle is $\widehat{DAT}$, where T is the centre of the base.

To find the centre of the base, we locate the midpoints of the diagonals.

The midpoint of [AC] is

$$\left(\frac{8+0}{2}, \frac{0+12}{2}, \frac{0+0}{2}\right) \text{ which is } (4, 6, 0).$$



The midpoint of [BO] is

$$\left(\frac{8+0}{2}, \frac{12+0}{2}, \frac{0+0}{2}\right) \text{ which is } (4, 6, 0).$$

$\therefore$ the centre of the base is $T(4, 6, 0)$.

Now $\quad DT = 10$ units

$$\text{and} \quad AT = \sqrt{(4-8)^2 + (6-0)^2 + (0-0)^2}$$
$$= \sqrt{(-4)^2 + 6^2 + 0^2}$$
$$= \sqrt{16 + 36 + 0}$$
$$= \sqrt{52} \text{ units}$$

$$\therefore \quad \tan\widehat{DAT} = \frac{10}{\sqrt{52}}$$

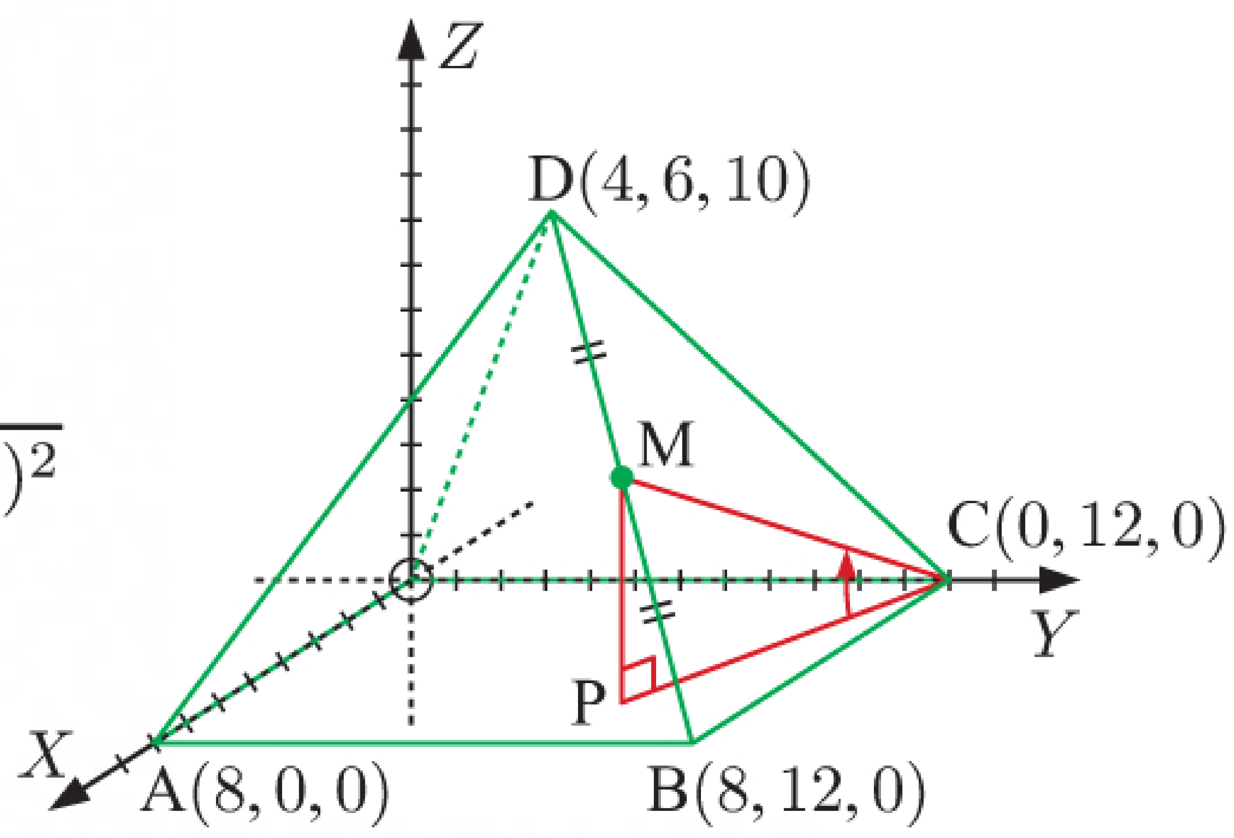
$$\therefore \quad \widehat{DAT} = \tan^{-1}\left(\tfrac{10}{\sqrt{52}}\right)$$

$$\therefore \quad \widehat{DAT} \approx 54.2°$$

The angle is about $54.2°$.

ii The required angle is $\widehat{MCP}$, where P is the point on the base plane which is directly below M.

M is $(6, 9, 5)$, so P is $(6, 9, 0)$.



Now $\quad MP = 5$ units

$$\text{and} \quad PC = \sqrt{(0-6)^2 + (12-9)^2 + (0-0)^2}$$
$$= \sqrt{(-6)^2 + 3^2 + 0^2}$$
$$= \sqrt{36 + 9 + 0}$$
$$= \sqrt{45} \text{ units}$$

$$\therefore \quad \tan\widehat{MCP} = \frac{5}{\sqrt{45}}$$

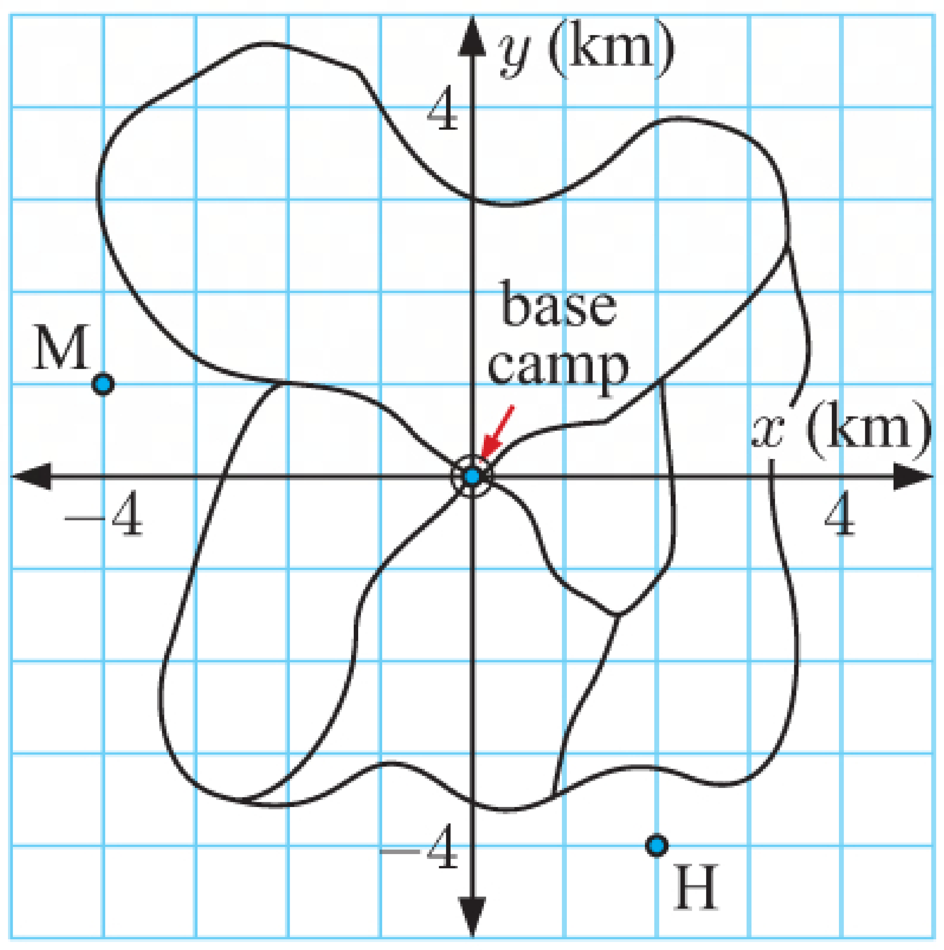
$$\therefore \quad \widehat{MCP} = \tan^{-1}\left(\tfrac{5}{\sqrt{45}}\right)$$

$$\therefore \quad \widehat{MCP} \approx 36.7°$$

The angle is about $36.7°$.

8 **a** $200\text{ m} \equiv 0.2\text{ km} \equiv \frac{1}{5}\text{ km}$

$\therefore$ the hiker is located at $\left(2,\ -4,\ \frac{1}{5}\right)$.



b Distance of hiker from base camp $(0,\ 0,\ 0)$

$$= \sqrt{(0-2)^2 + (0 - -4)^2 + (0 - \tfrac{1}{5})^2}$$
$$= \sqrt{(-2)^2 + 4^2 + (-\tfrac{1}{5})^2}$$
$$= \sqrt{4 + 16 + \tfrac{1}{25}}$$
$$= \sqrt{\tfrac{501}{25}}$$
$$= \frac{\sqrt{501}}{5}$$
$$\approx 4.48\text{ km}$$

c **i** $500\text{ m} \equiv 0.5\text{ km} \equiv \frac{1}{2}\text{ km}$

$\therefore$ the mountain top is located at $\left(-4,\ 1,\ \frac{1}{2}\right)$.

ii Distance between the hiker and the mountain top

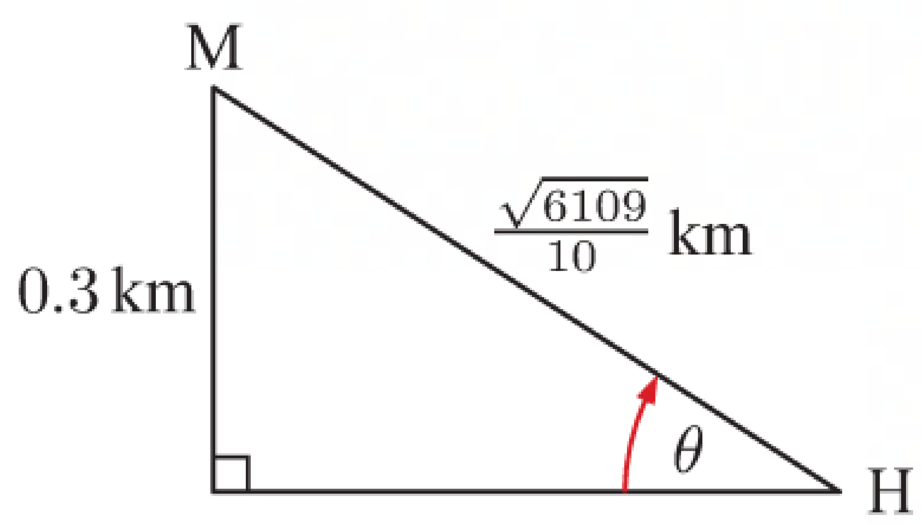
$$= \sqrt{(-4-2)^2 + (1 - -4)^2 + (\tfrac{1}{2} - \tfrac{1}{5})^2}$$
$$= \sqrt{(-6)^2 + 5^2 + (\tfrac{3}{10})^2}$$
$$= \sqrt{36 + 25 + \tfrac{9}{100}}$$
$$= \sqrt{\tfrac{6109}{100}}$$
$$= \frac{\sqrt{6109}}{10}$$
$$\approx 7.82\text{ km}$$

iii The mountain top is $0.5 - 0.2 = 0.3$ km above the hiker.

$$\sin\theta = \frac{0.3}{\left(\frac{\sqrt{6109}}{10}\right)} = \frac{3}{\sqrt{6109}}$$

$$\therefore\ \theta = \sin^{-1}\left(\frac{3}{\sqrt{6109}}\right)$$

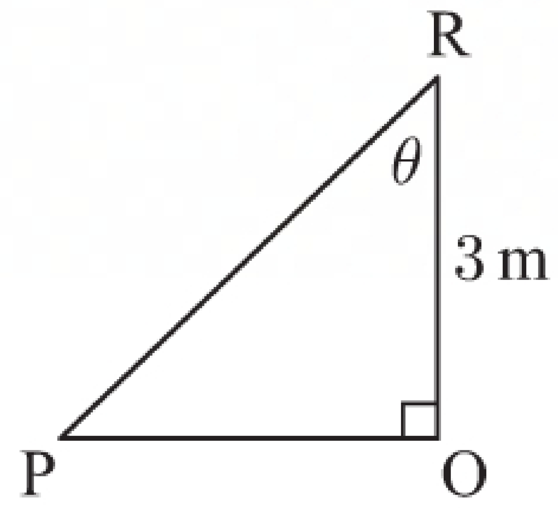
$$\therefore\ \theta \approx 2.20^\circ$$



The angle of elevation from the hiker to the mountain top is about 2.20°.

9 **a** R is 3 m up the flagpole, so its coordinates are $(0,\ 0,\ 3)$.

b Let the angle between the wire from P to R and the flagpole be θ.



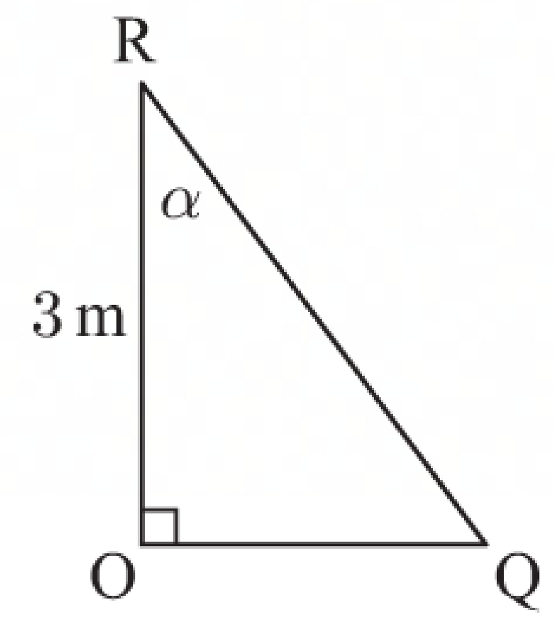
$$\text{OP} = \sqrt{(3-0)^2 + (1-0)^2 + (0-0)^2}$$
$$= \sqrt{3^2 + 1^2 + 0^2}$$
$$= \sqrt{10}\text{ m}$$

$$\tan\theta = \frac{\text{OP}}{\text{OR}} = \frac{\sqrt{10}}{3}$$

$$\therefore\ \theta = \tan^{-1}\left(\tfrac{\sqrt{10}}{3}\right) \approx 46.5^\circ$$

So, the wire from P to R makes an angle of approximately 46.5° with the flagpole.

Let the angle between the wire from Q to R and the flagpole be α.



$$\begin{aligned}OQ &= \sqrt{(-1-0)^2+(2-0)^2+(0-0)^2}\\ &= \sqrt{(-1)^2+2^2+0^2}\\ &= \sqrt{5}\text{ m}\end{aligned}$$

$$\tan\alpha = \frac{OQ}{OR} = \frac{\sqrt{5}}{3}$$

$$\therefore\ \alpha = \tan^{-1}\left(\frac{\sqrt{5}}{3}\right) \approx 36.7^\circ$$

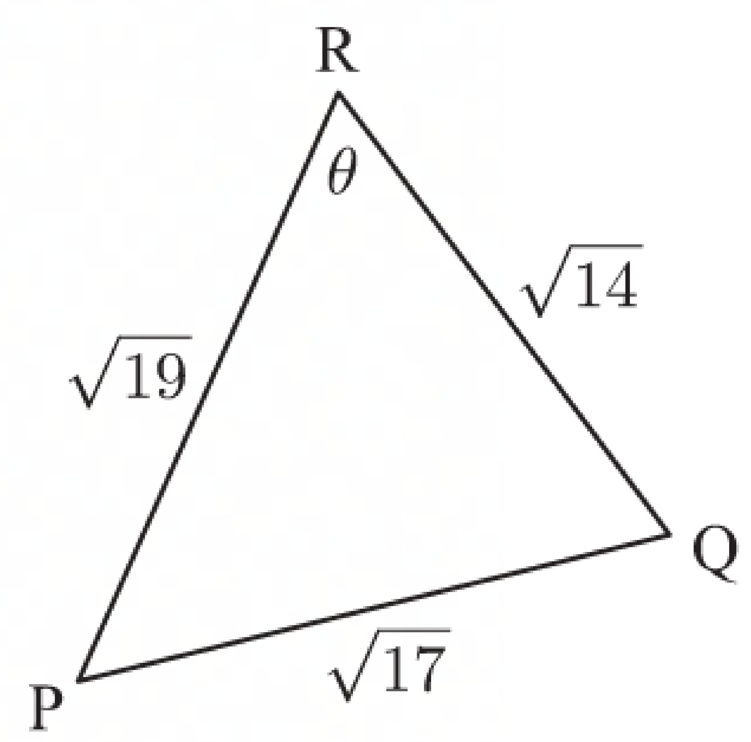
So, the wire from Q to R makes an angle of approximately 36.7° with the flagpole.

c $$\begin{aligned}PQ &= \sqrt{(-1-3)^2+(2-1)^2+(0-0)^2}\\ &= \sqrt{(-4)^2+1^2+0^2}\\ &= \sqrt{17}\text{ m}\end{aligned}$$

$$\begin{aligned}PR &= \sqrt{(0-3)^2+(0-1)^2+(3-0)^2}\\ &= \sqrt{(-3)^2+(-1)^2+3^2}\\ &= \sqrt{19}\text{ m}\end{aligned}$$

$$\begin{aligned}QR &= \sqrt{(0--1)^2+(0-2)^2+(3-0)^2}\\ &= \sqrt{1^2+(-2)^2+3^2}\\ &= \sqrt{14}\text{ m}\end{aligned}$$

By the cosine rule,



$$\cos\theta = \frac{(\sqrt{19})^2+(\sqrt{14})^2-(\sqrt{17})^2}{2\times\sqrt{19}\times\sqrt{14}}$$

$$\therefore\ \cos\theta = \frac{19+14-17}{2\times\sqrt{19}\times\sqrt{14}}$$

$$\therefore\ \theta = \cos^{-1}\left(\frac{16}{2\times\sqrt{19}\times\sqrt{14}}\right) \approx 60.6^\circ$$

So, the wires meet at an angle of approximately 60.6°.

Chapter 11

PROBABILITY

INVESTIGATION 1 — DICE ROLLING EXPERIMENT

1 The possible outcomes for the uppermost face when the die is rolled are 1, 2, 3, 4, 5, and 6.

2 Each outcome is equally likely to occur, and there are 6 possible outcomes. We expect the relative frequency of rolling a 2 to be $\frac{1}{6}$.

3 **Note:** The answers below are only one of many possible results from the experiment. Your results will differ.

Outcome	*Tally*	*Frequency*
1	$\vert\vert$	2
2	$\vert\vert\vert$	3
3	$\cancel{\vert\vert\vert\vert}$	5
4	$\vert\vert\vert$	3
5	$\vert\vert\vert$	3
6	$\vert\vert\vert\vert$	4

relative frequency of rolling a 2 $= \frac{3}{20} = 0.15$

4

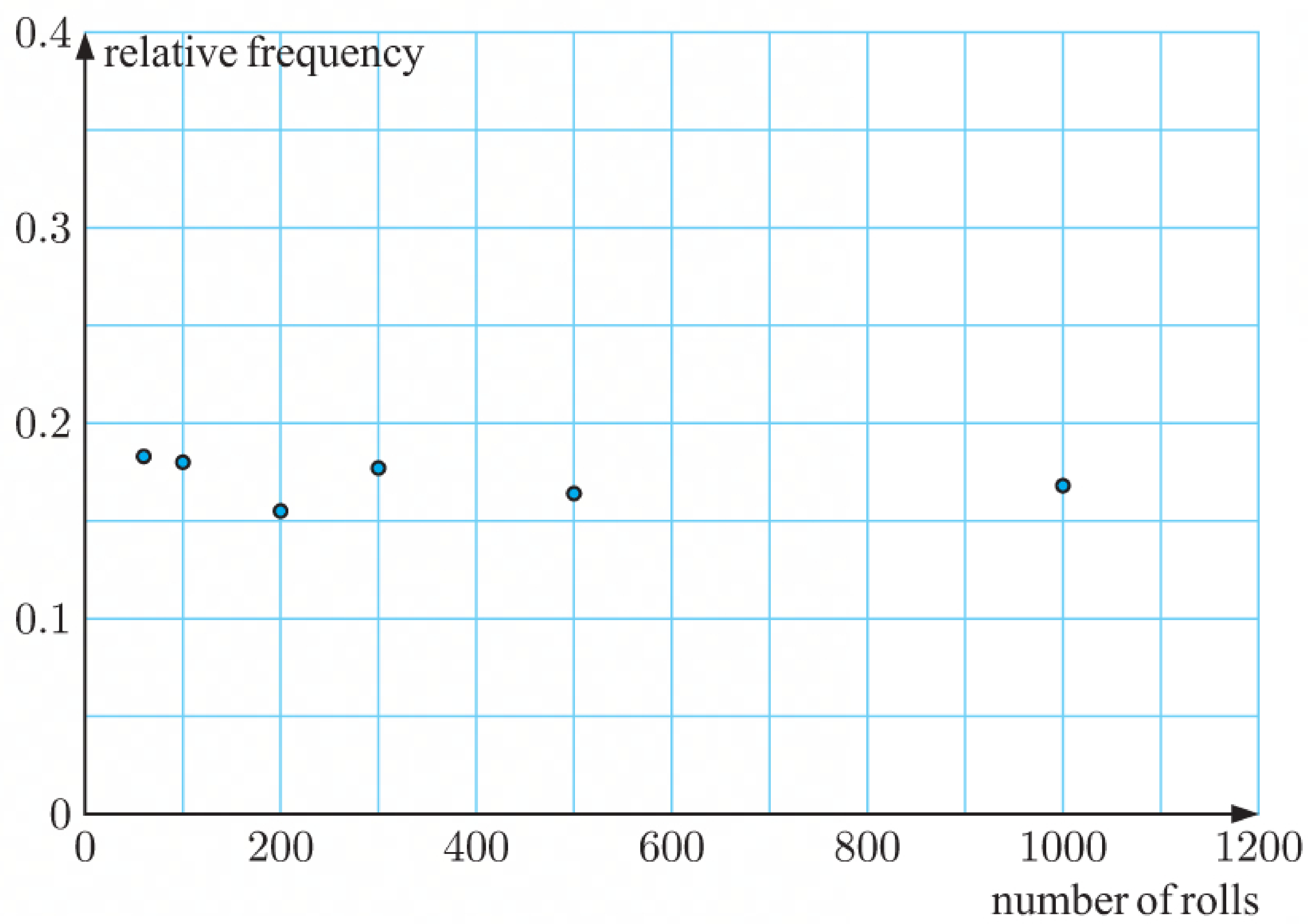
Outcome	*Tally*	*Frequency*
1	$\cancel{\vert\vert\vert\vert}$	5
2	$\cancel{\vert\vert\vert\vert}\ \vert\vert$	7
3	$\cancel{\vert\vert\vert\vert}\ \vert\vert\vert$	8
4	$\cancel{\vert\vert\vert\vert}$	5
5	$\cancel{\vert\vert\vert\vert}\ \vert$	6
6	$\cancel{\vert\vert\vert\vert}\ \vert\vert\vert\vert$	9

relative frequency of rolling a 2 $= \frac{7}{40} = 0.175$

5

Number of rolls	*Frequency of rolling a* 2	*Relative frequency of rolling a* 2
60	11	$\frac{11}{60} \approx 0.183$
100	18	$\frac{18}{100} = 0.18$
200	31	$\frac{31}{200} = 0.155$
300	53	$\frac{53}{300} \approx 0.177$
500	82	$\frac{82}{500} = 0.164$
1000	168	$\frac{168}{1000} = 0.168$

6



The relative frequencies become more consistent and approach the value $\frac{1}{6} \approx 0.167$ as the number of rolls increases.

7 As the number of rolls increases, the relative frequency of rolling a 2 will approach $\frac{1}{6}$.

INVESTIGATION 2 TOSSING DRAWING PINS

Note: These are example results only, your results will differ.

1, 2

Outcome	*Frequency*	*Relative frequency*
Two backs	8	$\frac{8}{80} = 0.1$
Back and side	38	$\frac{38}{80} = 0.475$
Two sides	34	$\frac{34}{80} = 0.425$

3

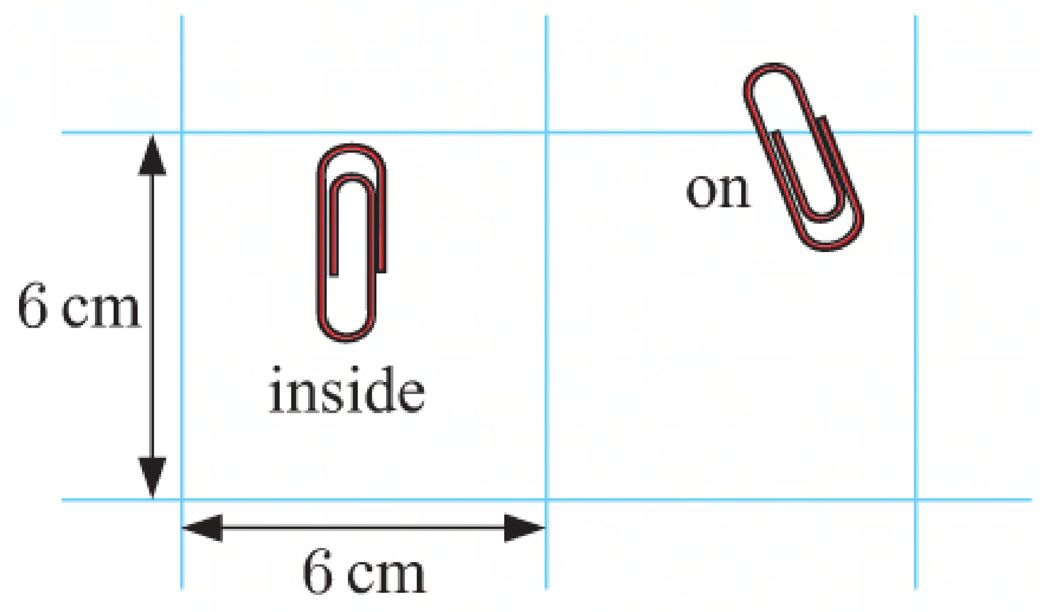
Outcome	*Frequency*	*Relative frequency*
Two backs	45	$\frac{45}{400} = 0.1125$
Back and side	175	$\frac{175}{400} = 0.4375$
Two sides	180	$\frac{180}{400} = 0.45$

4 The whole group's data has a larger sample size and hence will provide more reliable probability estimates.

EXERCISE 11A

1 **a** $P(\text{inside a square}) = \frac{113}{145}$
≈ 0.78

b $P(\text{on a line}) = \frac{32}{145}$
≈ 0.22



2

Length	*Frequency*
0 - 19	17
20 - 39	38
40 - 59	19
60+	4

Total frequency $= 17 + 38 + 19 + 4 = 78$

a $P(\text{20 to 39 seconds}) = \frac{38}{78}$
≈ 0.487

b $P(\text{at least one minute}) = P(\geqslant 60 \text{ seconds})$
$= \frac{4}{78}$
≈ 0.051

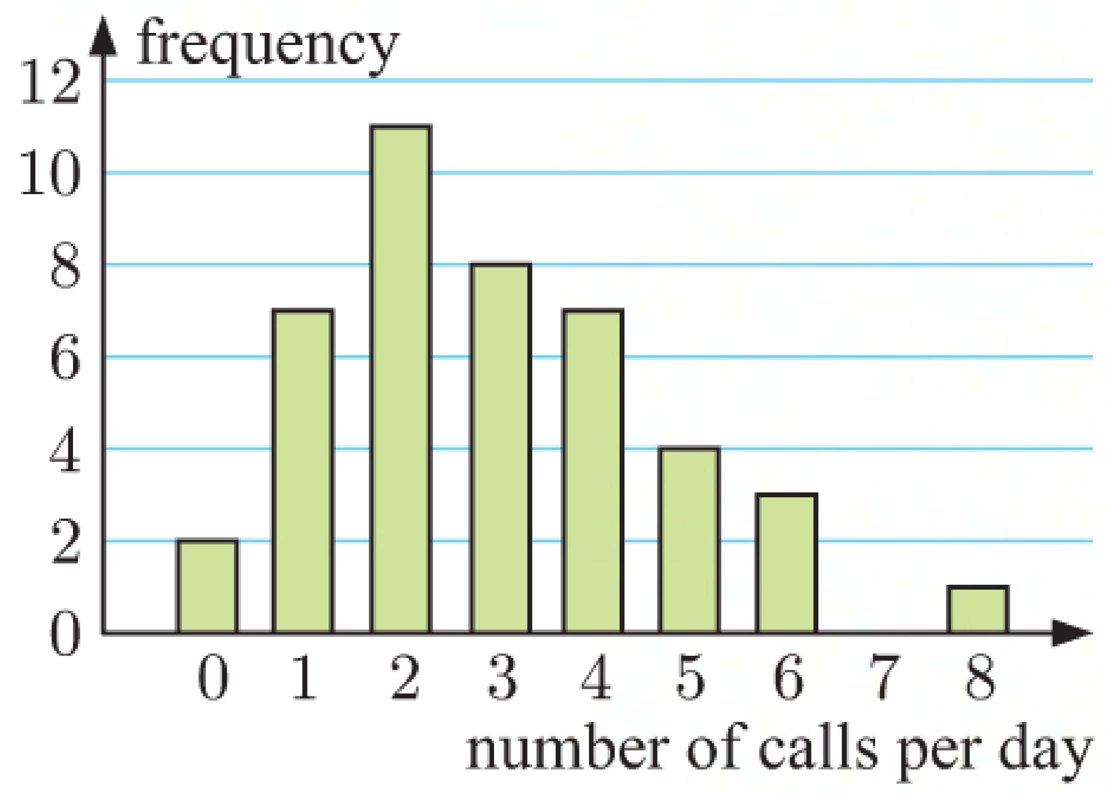
c $P(\text{between 20 and 59 seconds inclusive}) = \frac{38+19}{78}$
≈ 0.731

3 **a** The survey lasted $2 + 7 + 11 + 8 + 7 + 4 + 3 + 0 + 1$
$= 43$ days

b **i** $P(0 \text{ calls}) \approx \frac{2}{43}$
≈ 0.0465

ii $P(\geqslant 5 \text{ calls}) \approx \frac{4+3+0+1}{43}$
≈ 0.186

iii $P(< 3 \text{ calls}) \approx \frac{2+7+11}{43}$
≈ 0.465



4

Days between refills	*Frequency*
1	37
2	81
3	48
4	17
5	6
6	1

$$\begin{aligned}\text{Total frequency} &= 37 + 81 + 48 + 17 + 6 + 1 \\ &= 190\end{aligned}$$

a $\begin{aligned}\text{P(4 days gap)} &\approx \frac{17}{190} \\ &\approx 0.0895\end{aligned}$

b $\begin{aligned}\text{P(at least 4 days gap)} &\approx \frac{17+6+1}{190} \\ &\approx 0.126\end{aligned}$

5

School	*Number of 15 year olds*		*Number of smokers*	
	Male	*Female*	*Male*	*Female*
A	45	51	10	11
B	36	42	9	6
C	52	49	13	13
D	28	33	9	10
E	40	39	7	4
Total	201	214	48	44

a $\text{P(female 15 year old at school } \mathbf{C} \text{ is a smoker)} \approx \dfrac{13}{49}$ ← number of female smokers at school **C** / ← total number of females aged 15 at school **C**

≈ 0.265

b At school **E**, there are $40 + 39 = 79$ 15 year olds.
$7 + 4 = 11$ of these students are smokers, so $79 - 11 = 68$ are non-smokers.

$\therefore$ P(15 year old from school **E** is *not* a smoker) $= \dfrac{68}{79} \approx 0.861$

c There are $48 + 44 = 92$ smokers in total,
and $201 + 214 = 415$ 15 year olds.

$\therefore$ $\begin{aligned}\text{P(15 year old from any of the five schools is a smoker)} &\approx \frac{92}{415} \\ &\approx 0.222\end{aligned}$

6

Reason	2014/15	2015/16	2016/17	2017/18
Access	585	1127	2545	1612
Billing	1822	2102	3136	3582
Contracts	242	440	719	836
Credit control	3	44	118	136
Customer Service	12	282	1181	1940
Disconnection	n/a	n/a	n/a	248
Faults	86	79	120	384
Privacy	93	86	57	60
Provision	172	122	209	311
Total	3015	4282	8085	9109

a P(complaint received in 2016/17 was about customer service)

$$\approx \frac{1181}{8085} \begin{array}{l} \leftarrow \text{number of customer service complaints in 2016/17} \\ \leftarrow \text{total number of complaints in 2016/17} \end{array}$$

≈ 0.146

b There were $1822 + 2102 + 3136 + 3582 = 10\,642$ billing complaints in total,
and $3015 + 4282 + 8085 + 9109 = 24\,491$ complaints in total.

$\therefore$ P(complaint received at any time was related to billing) $\approx \dfrac{10\,642}{24\,491} \approx 0.435$

c In 2017/18, 3582 complaints were related to billing and 384 complaints were related to faults.
So, $9109 - 3582 - 384 = 5143$ complaints in 2017/18 did *not* relate to either billing or faults.

$\therefore$ P(complaint received in 2017/18 did *not* relate to either billing or faults) $\approx \dfrac{5143}{9109}$

≈ 0.565

7

Summer Temperatures in Barcelona	*Month*		
	June	*July*	*Aug*
Mean days max. $\geqslant 40°C$	0.3	1.2	0.7
Mean days max. $\geqslant 35°C$	3.0	5.8	5.3
Mean days max. $\geqslant 30°C$	9.4	12.3	12.0

a **i** P(August day $\geqslant 35°C$) $\approx \dfrac{5.3}{31} \begin{array}{l} \leftarrow \text{number of August days} \geqslant 35°\text{C} \\ \leftarrow \text{number of days in August} \end{array}$

≈ 0.171

ii 12.0 days in August are $\geqslant 30°C$, so $31 - 12.0 = 19$ days in August are $< 30°C$.

$\therefore$ P(August day $< 30°C$) $\approx \dfrac{19}{31} \approx 0.613$

b There are $9.4 + 12.3 + 12.0 = 33.7$ days in total during summer which are $\geqslant 30°C$,
and $30 + 31 + 31 = 92$ days in total during summer.

$\therefore$ P(any summer day will be $\geqslant 30°C$) $\approx \dfrac{33.7}{92} \approx 0.366$

c There are $0.3 + 1.2 + 0.7 = 2.2$ days in total during summer which are $\geqslant 40°C$, and 1.2 days in July which are $\geqslant 40°C$.

$\therefore$ P(a summer day $\geqslant 40°C$ is in July) $\approx \frac{1.2}{2.2} \approx 0.545$

EXERCISE 11B

1 We extend the table to include totals for each row and column.

	Adult	*Child*	*Total*
Season ticket holder	1824	779	2603
Not a season ticket holder	3247	1660	4907
Total	5071	2439	7510

a The total attendance for the match was 7510 people.

b **i** 2439 out of the 7510 people at the match were children.

$\therefore$ P(a child is selected) $\approx \frac{2439}{7510} \approx 0.325$

ii 4907 out of the 7510 people at the match were not season ticket holders.

$\therefore$ P(a non-season ticket holder is selected) $\approx \frac{4907}{7510} \approx 0.653$

iii 1824 out of the 7510 people at the match were adult season ticket holders.

$\therefore$ P(an adult season ticket holder is selected) $\approx \frac{1824}{7510} \approx 0.243$

2 **a**

	Junior	*Middle*	*Senior*	*Total*
Sport	131	164	141	436
No sport	28	81	176	285
Total	159	245	317	721

b **i** 436 out of the 721 students surveyed play sport.

$\therefore$ P(plays sport) $\approx \frac{436}{721} \approx 0.605$

ii 131 out of the 721 students surveyed play sport and are in the junior school.

$\therefore$ P(plays sport and is in junior school) $\approx \frac{131}{721} \approx 0.182$

iii $81 + 176 = 257$ students out of the 721 surveyed do not play sport and are in the middle or senior school.

$\therefore$ P(does not play sport and is in middle school or higher) $\approx \frac{257}{721} \approx 0.356$

3 We extend the table to include totals for each row and column.

	Single	*Double*	*Family*	*Total*
Peak season	225	420	98	743
Off-peak season	148	292	52	492
Total	373	712	150	1235

a **i** 743 out of the 1235 bookings made were in the peak season.

$\therefore$ P(in the peak season) $\approx \dfrac{743}{1235} \approx 0.602$

ii 148 out of the 1235 bookings made were for a single room in the off-peak season.

$\therefore$ P(a single room in the off-peak season) $\approx \dfrac{148}{1235} \approx 0.120$

iii $373 + 712 = 1085$ bookings out of the 1235 were for a single or a double room.

$\therefore$ P(a single or a double room) $\approx \dfrac{1085}{1235} \approx 0.879$

iv $225 + 420 + 98 + 52 = 795$ bookings out of the 1235 were made during the peak season or were for a family room.

$\therefore$ P(during the peak season or a family room) $\approx \dfrac{795}{1235} \approx 0.644$

b 52 out of the 492 bookings made during the off-peak season were for a family room.

$\therefore$ P(booking made in off-peak season is for family room) $\approx \dfrac{52}{492} \approx 0.106$

c $420 + 98 = 518$ bookings out of $712 + 150 = 862$ were made in the peak season for double or family rooms.

$\therefore$ P(booking made for a double or family room was in peak season) $\approx \dfrac{518}{862} \approx 0.601$

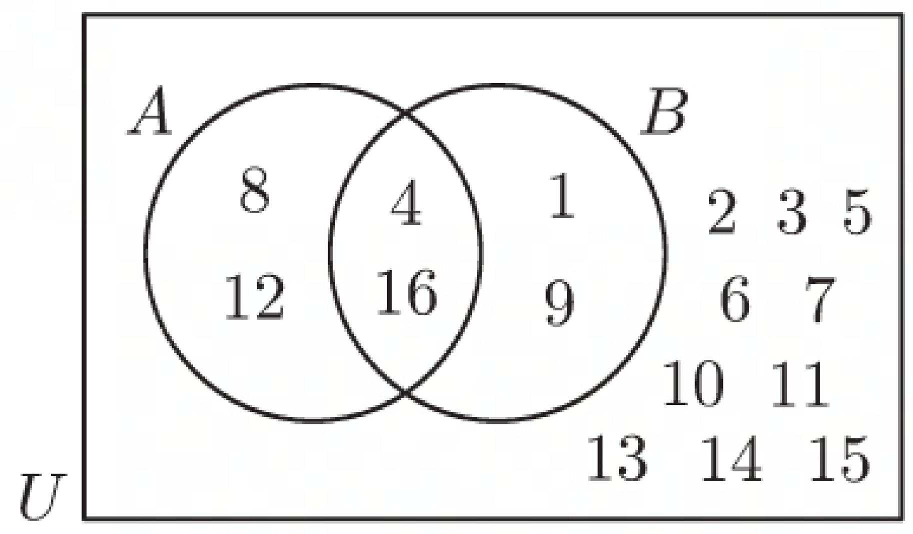
EXERCISE 11C

1 **a** {A, B, C, D} **b** {1, 2, 3, 4, 5, 6, 7, 8} **c** {MM, MF, FM, FF}

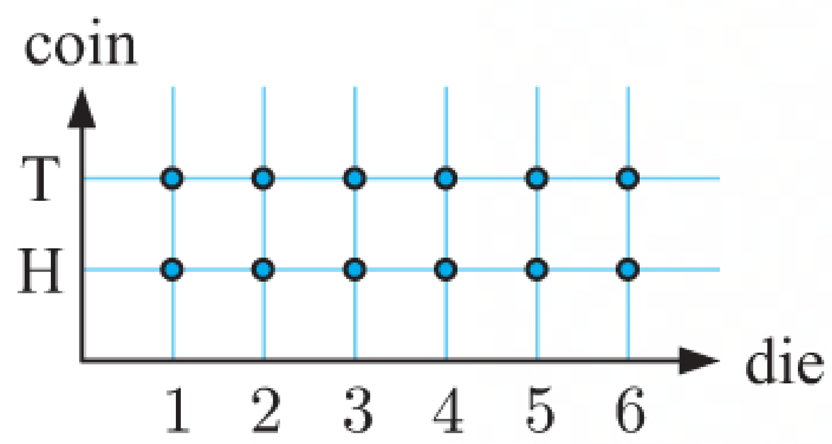
2 **a** $U = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16\}$

b **i** $A = \{4, 8, 12, 16\}$ **ii** $B = \{1, 4, 9, 16\}$

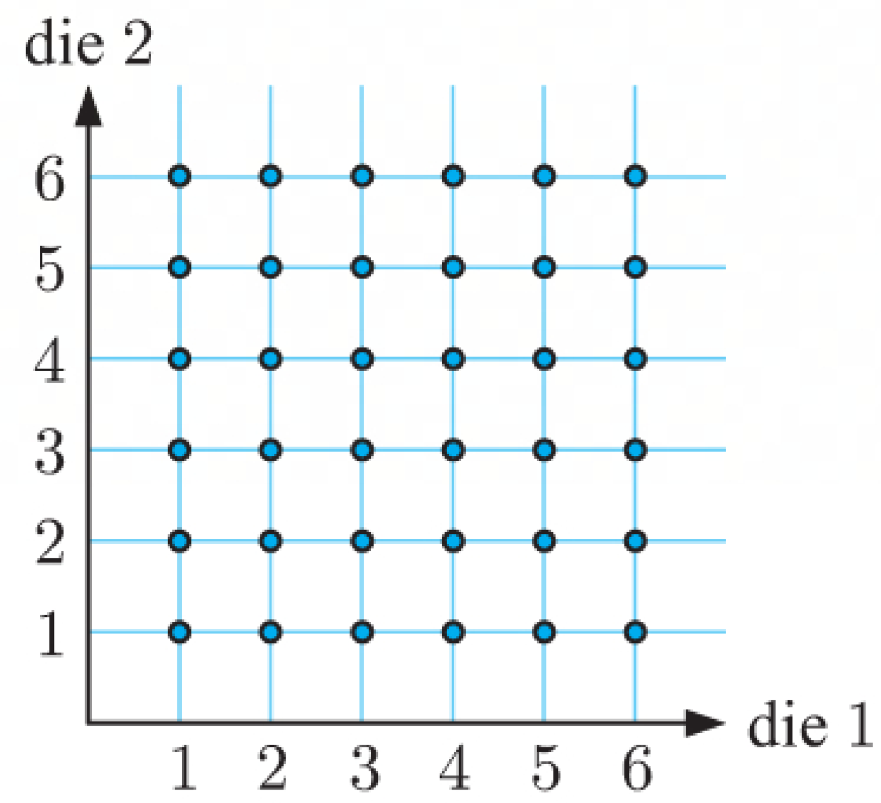
c



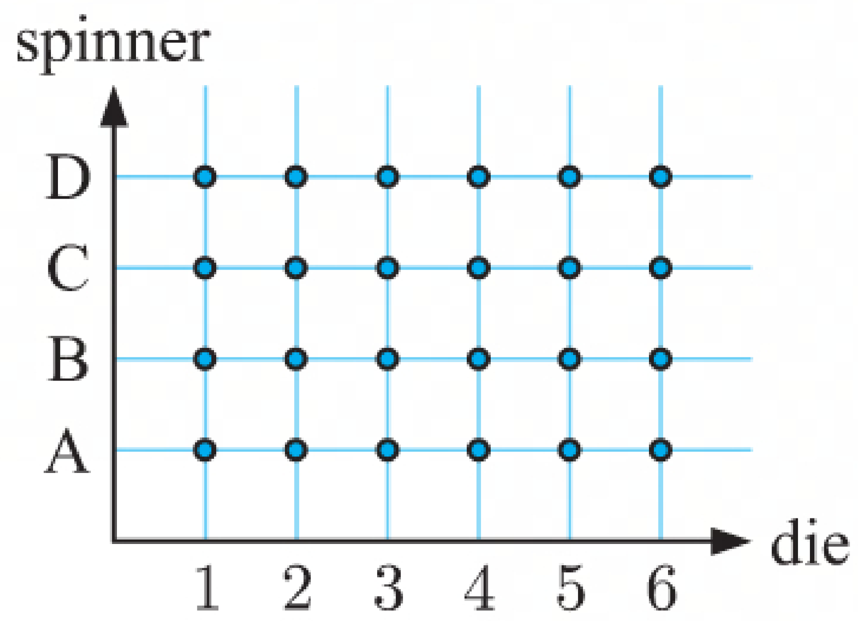
3 **a**



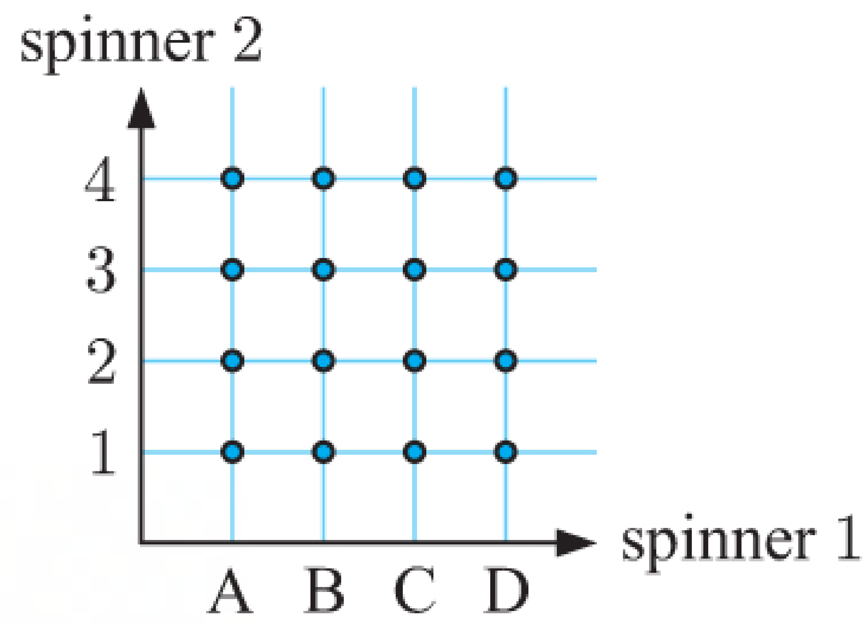
b



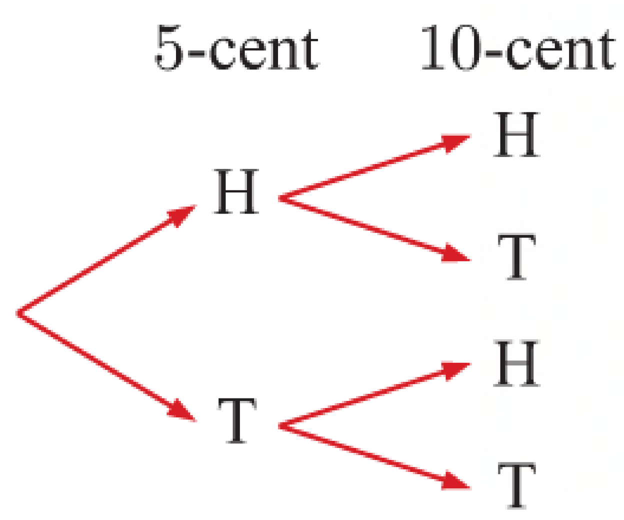
c



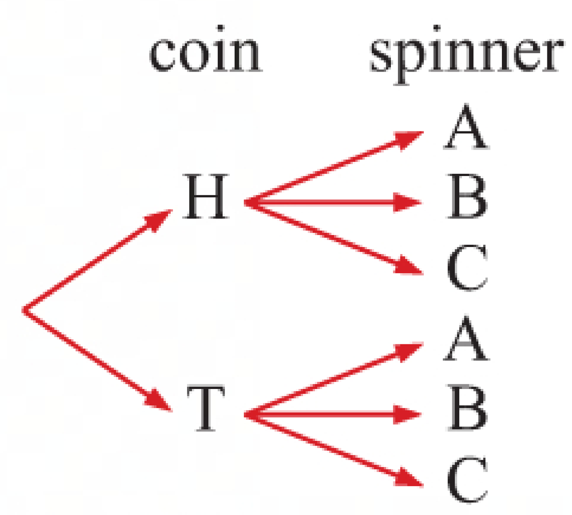
d



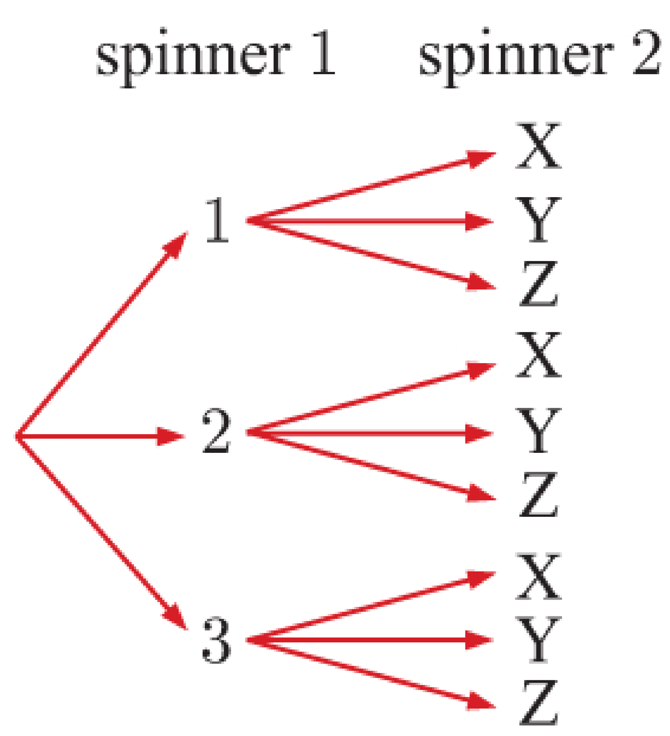
4 **a**



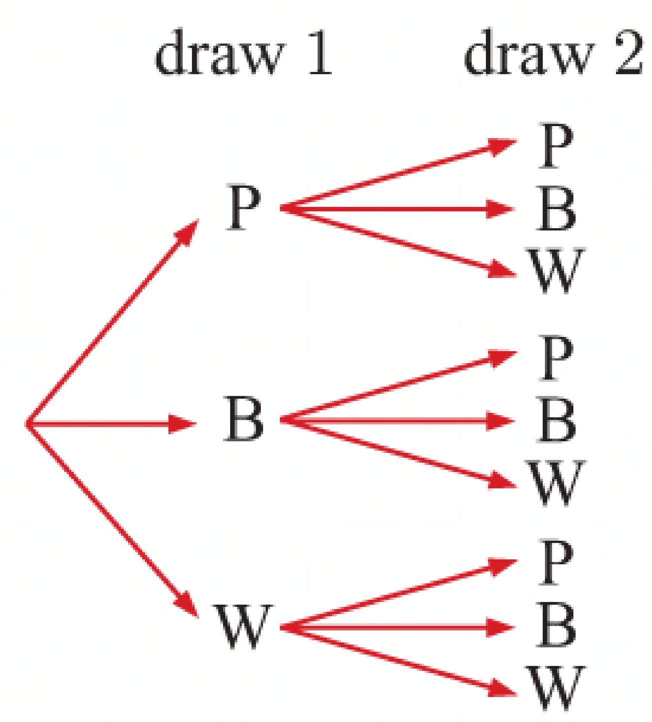
b



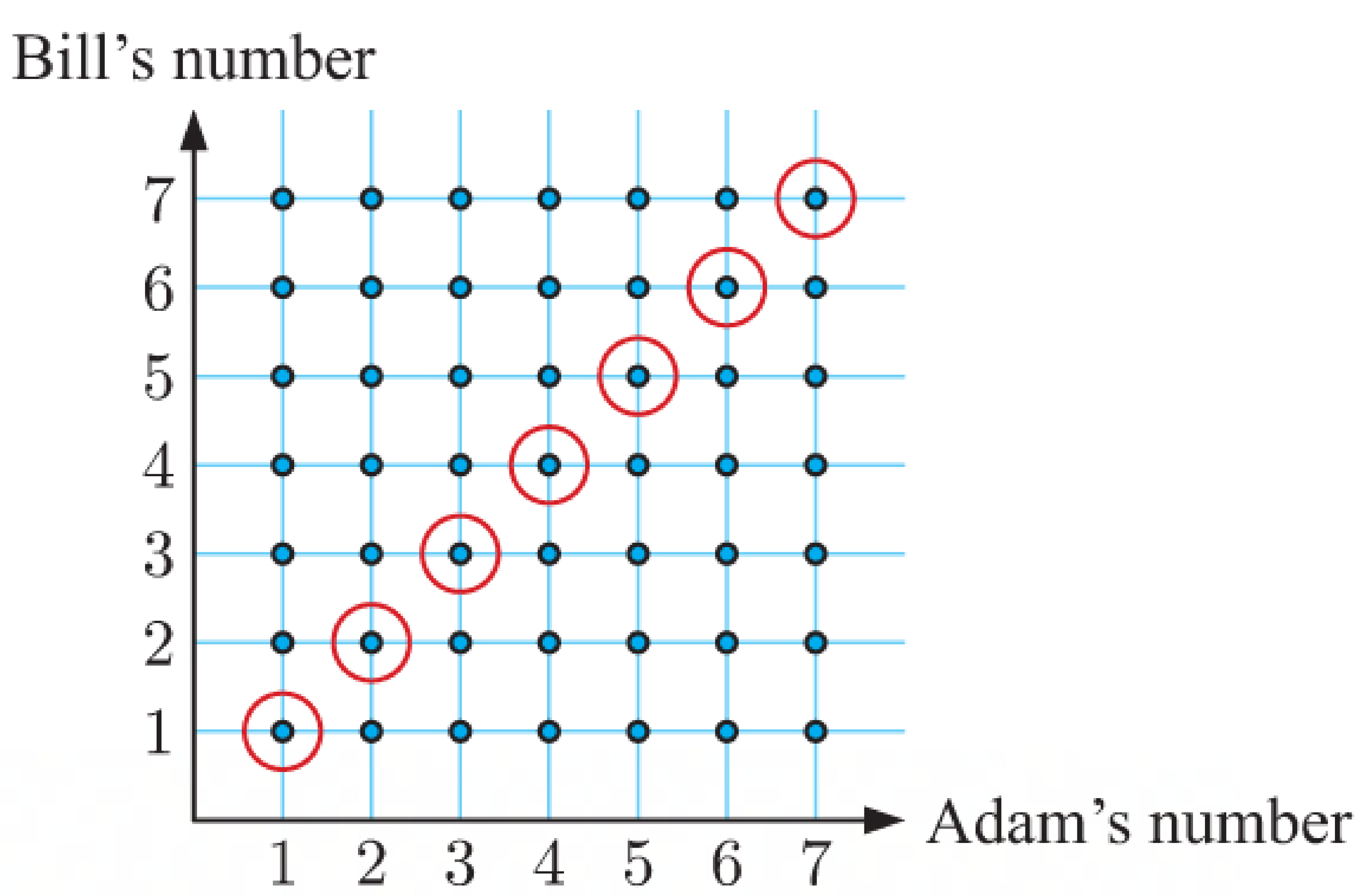
c



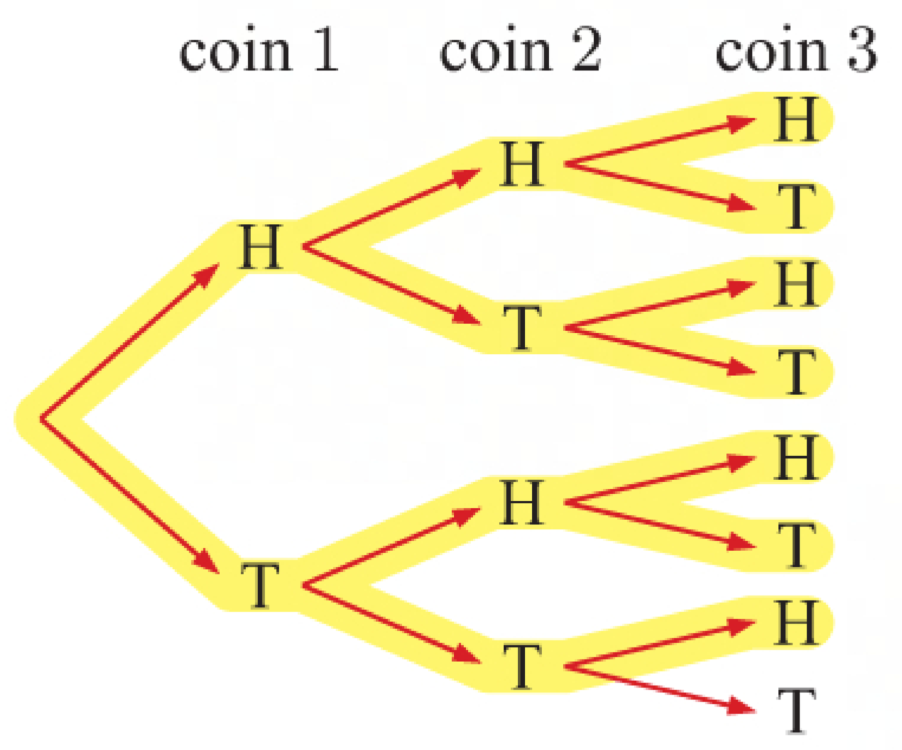
d



5



6



EXERCISE 11D

1 Total number of marbles $= 5 + 3 + 7 = 15$

a $P(\text{red}) = \dfrac{3}{15} = \frac{1}{5}$

b $P(\text{green}) = \dfrac{5}{15} = \frac{1}{3}$

c $P(\text{blue}) = \frac{7}{15}$

d $P(\text{not red}) = P(\text{green or blue})$

$= \dfrac{5+7}{15}$

$= \dfrac{12}{15}$

$= \frac{4}{5}$

e $P(\text{neither green nor blue}) = P(\text{red})$

$= \frac{1}{5}$

f $P(\text{green or red}) = \dfrac{5+3}{15}$

$= \frac{8}{15}$

2 There are 200 tickets which could be selected with equal chance.

a $P(\text{even number}) = P(\text{number is a multiple of } 2)$

$= \dfrac{100}{200}$

$= \frac{1}{2}$

b $P(\text{multiple of } 12)$

$= P(12, 24, 36, 48, 60, 72, 84, 96, 108, 120, 132, 144, 156, 168, 180, \text{ or } 192)$

$= \dfrac{16}{200}$

$= \frac{2}{25}$

c The numbers from 1 to 200 which contain a 2 are:

2, 12, 20, 21, ..., 28, 29, 32, 42, 52, 62, 72, 82, 92,
102, 112, 120, 121, ..., 128, 129, 132, 142, 152, 162, 172, 182, 192, and 200.

$\therefore$ $P(\text{number contains a } 2) = \dfrac{39}{200}$

3 **a** P(multiple of 4)
$= \text{P}(4, 8, 12, 16, 20, 24, 28, 32, \text{ or } 36)$
$= \dfrac{9}{36}$
$= \frac{1}{4}$

b P(between 6 and 9 inclusive)
$= \text{P}(6, 7, 8, \text{ or } 9)$
$= \dfrac{4}{36}$
$= \frac{1}{9}$

c $\text{P}(> 20)$
$= \text{P}(21, 22, 23, 24, ..., 35, \text{ or } 36)$
$= \dfrac{36-20}{36}$
$= \dfrac{16}{36}$
$= \frac{4}{9}$

d P(multiple of 13)
$= \text{P}(13 \text{ or } 26)$
$= \dfrac{2}{36}$
$= \frac{1}{18}$

e P(odd multiple of 3)
$= \text{P}(3, 9, 15, 21, 27, \text{ or } 33)$
$= \dfrac{6}{36}$
$= \frac{1}{6}$

f P(number containing a 1)
$= \text{P}(1, 10, 11, ..., 18, 19, 21, \text{ or } 31)$
$= \dfrac{13}{36}$

g P(multiple of both 4 and 6)
$= \text{P(multiple of 12)}$
$= \text{P}(12, 24, \text{ or } 36)$
$= \dfrac{3}{36}$
$= \frac{1}{12}$

h P(multiple of 4 or 6, or both)
$= \text{P}(4, 6, 8, 12, 16, 18, 20, 24, 28, 30, 32, \text{ or } 36)$
$= \dfrac{12}{36}$
$= \frac{1}{3}$

4 **a** $\text{P(on a Tuesday)} = \frac{1}{7}$

b $\text{P(on a weekend)} = \frac{2}{7}$

c $\text{P(in July)} = \dfrac{4 \times 31}{3 \times 365 + 1 \times 366}$ {over a 4 year period}
$= \frac{124}{1461}$

d $\text{P(in January or February)} = \dfrac{4 \times 31 + 3 \times 28 + 1 \times 29}{3 \times 365 + 1 \times 366}$ {over a 4 year period}
$= \frac{237}{1461} \quad (= \frac{79}{487})$

e P(in month containing the letter "a")
$= \text{P(in January, February, March, April, May, or August)}$
$= \dfrac{3 \times (31 + 28 + 31 + 30 + 31 + 31) + (31 + 29 + 31 + 30 + 31 + 31)}{3 \times 365 + 1 \times 366}$ {over a 4 year period}
$= \frac{729}{1461} \quad (= \frac{243}{487})$

5 Let G denote "a girl" and B denote "a boy".

a Possible orders are: {GGG, GGB, GBG, BGG, GBB, BGB, BBG, BBB}

b **i** P(all boys) $=$ P(BBB) $= \frac{1}{8}$

ii P(all girls) $=$ P(GGG) $= \frac{1}{8}$

iii P(boy, then girl, then girl)
$=$ P(BGG)
$= \frac{1}{8}$

iv P(2 girls and a boy)
$=$ P(GGB or GBG or BGG)
$= \frac{3}{8}$

v P(girl is eldest)
$=$ P(GGG or GBG or GBB or GGB)
$= \dfrac{4}{8}$
$= \frac{1}{2}$

vi P(at least one boy)
$= \frac{7}{8}$ {all except GGG}

6 **a** {ABCD, ABDC, ACBD, ACDB, ADBC, ADCB, BACD, BADC, BCAD, BCDA, BDAC, BDCA, CABD, CADB, CBAD, CBDA, CDAB, CDBA, DABC, DACB, DBAC, DBCA, DCAB, DCBA}

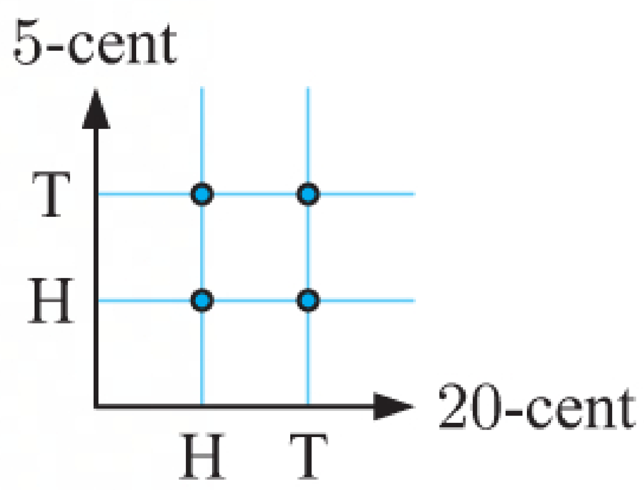
b **i** P(A sits on one end) $= \dfrac{12}{24} = \frac{1}{2}$

ii P(B sits on one of the two middle seats) $= \dfrac{12}{24} = \frac{1}{2}$

iii P(A and B are together) $= \dfrac{12}{24} = \frac{1}{2}$

iv P(A, B, and C are together) $= \dfrac{12}{24} = \frac{1}{2}$

7



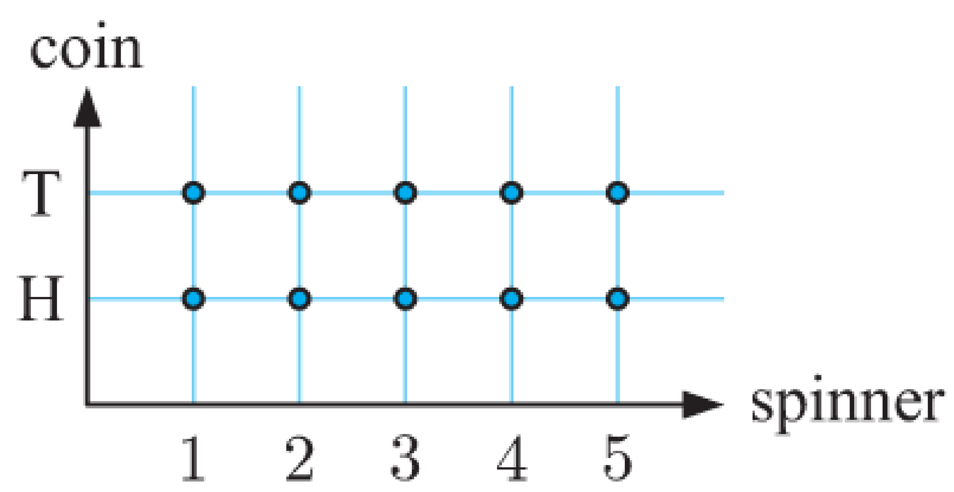
a P(2 heads) $= \frac{1}{4}$

b P(2 tails) $= \frac{1}{4}$

c P(exactly one tail)
$=$ P(HT or TH)
$= \dfrac{2}{4}$
$= \frac{1}{2}$

d P(at most one tail)
$=$ P(HT or TH or HH)
$= \frac{3}{4}$

8 **a**

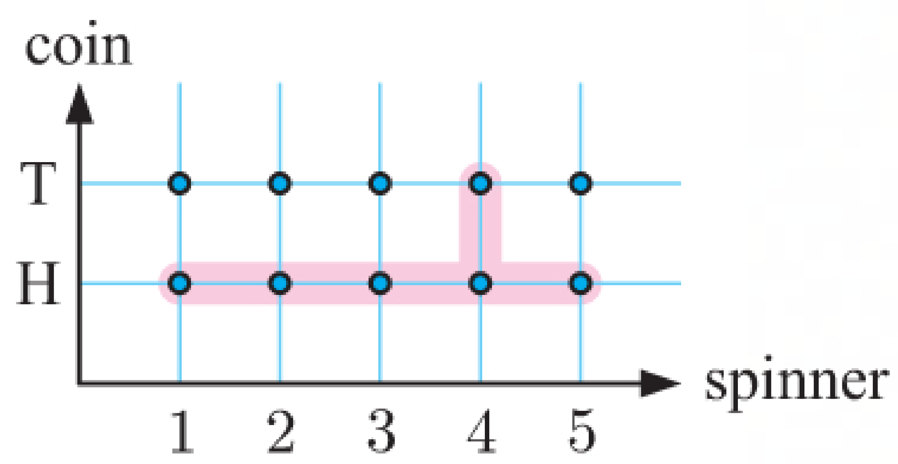


b **i** P(H and 5) $= \frac{1}{10}$

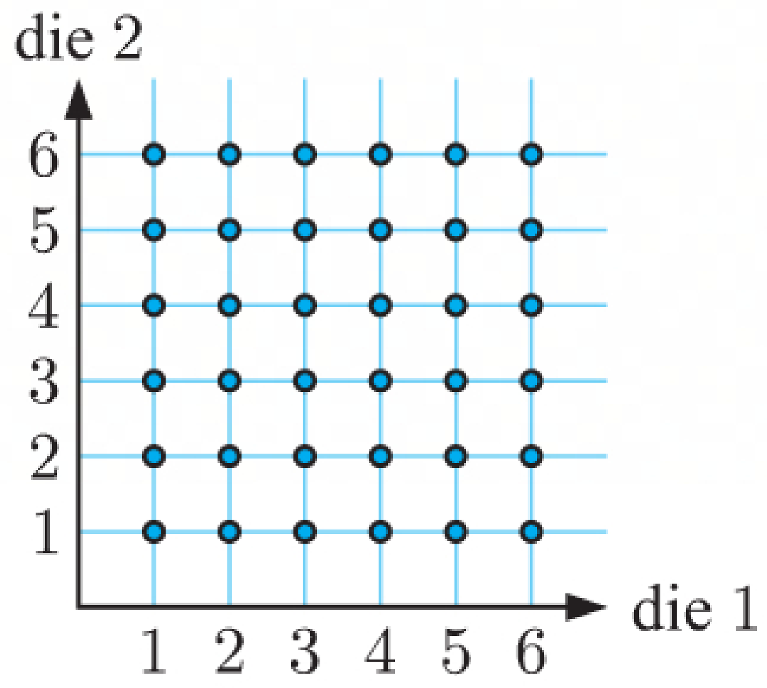
ii P(T and prime number)
$=$ P(T2, T3, or T5)
$= \frac{3}{10}$

iii P(an even number)
$=$ P(H2, T2, H4, or T4)
$= \dfrac{4}{10}$
$= \frac{2}{5}$

iv P(H or 4)
$= \dfrac{6}{10}$
$= \frac{3}{5}$ {shaded}



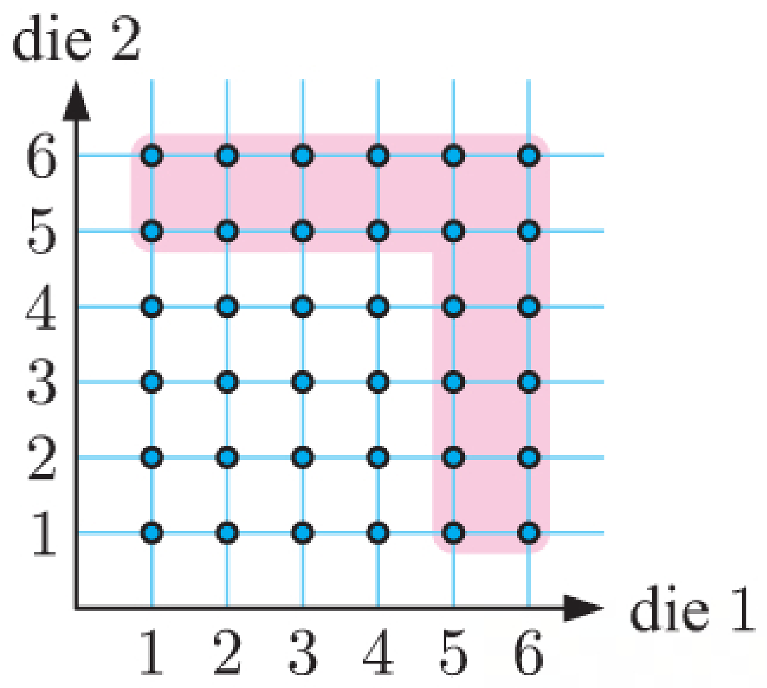
9



a P(two 3s)
$= \text{P}((3, 3))$
$= \frac{1}{36}$

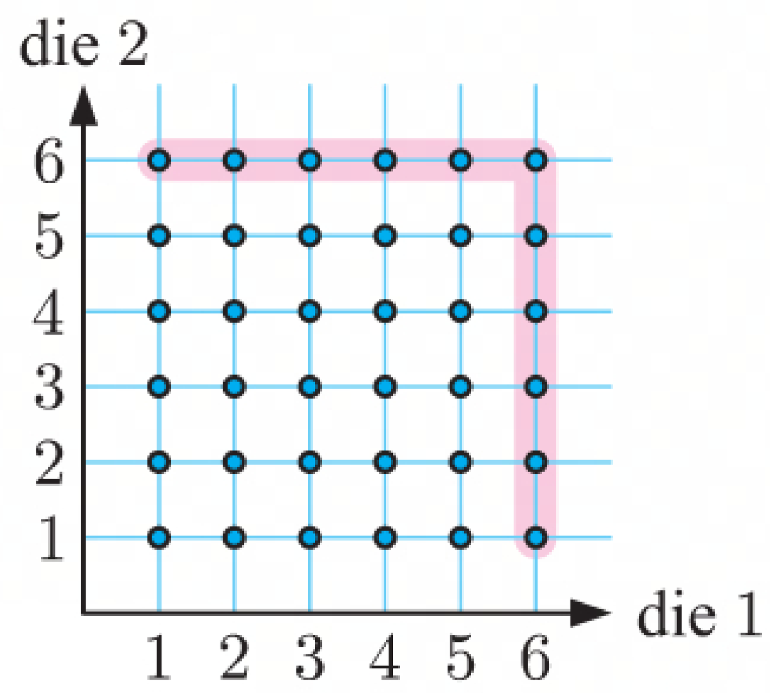
b P(a 5 and a 6)
$= \text{P}((5, 6), (6, 5))$
$= \frac{2}{36}$
$= \frac{1}{18}$

c



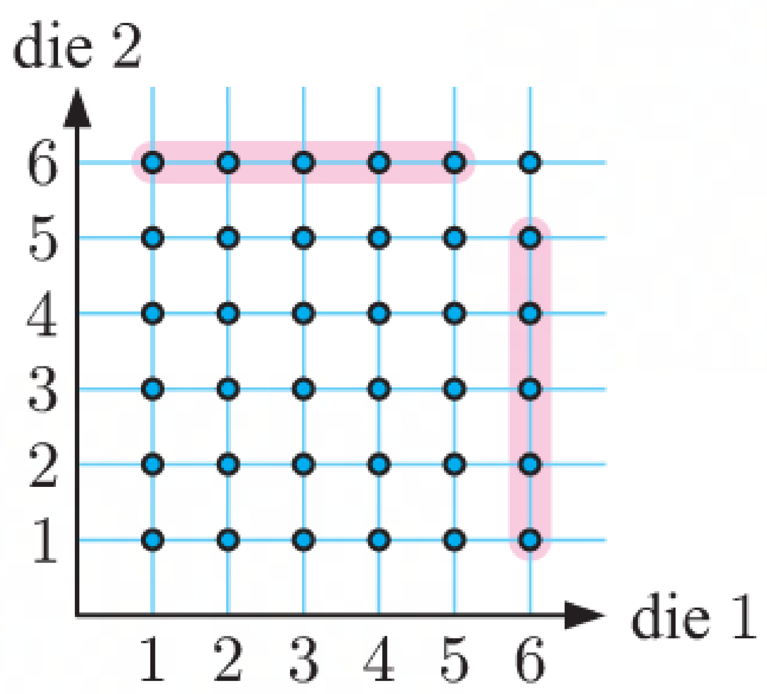
$\text{P(a 5 or a 6 or both)} = \frac{20}{36}$
$= \frac{5}{9}$

d



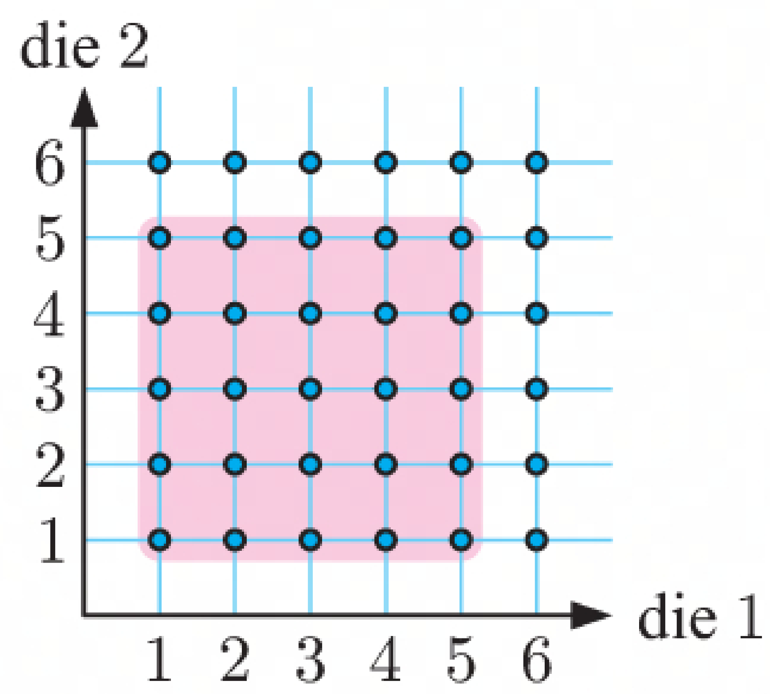
$\text{P(at least one 6)} = \frac{11}{36}$

e



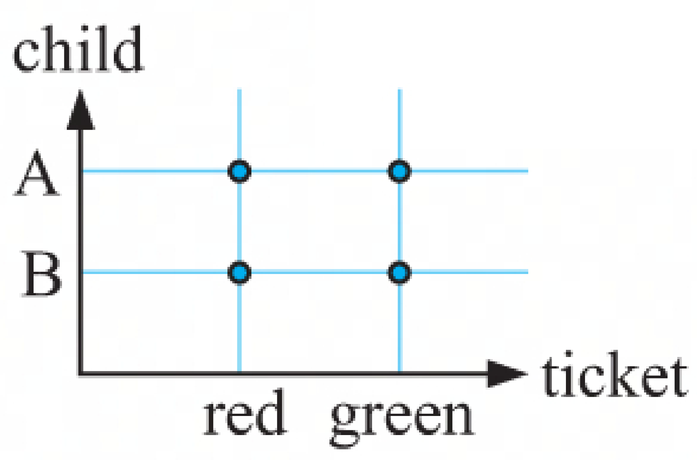
$\text{P(exactly one 6)} = \frac{10}{36}$
$= \frac{5}{18}$

f

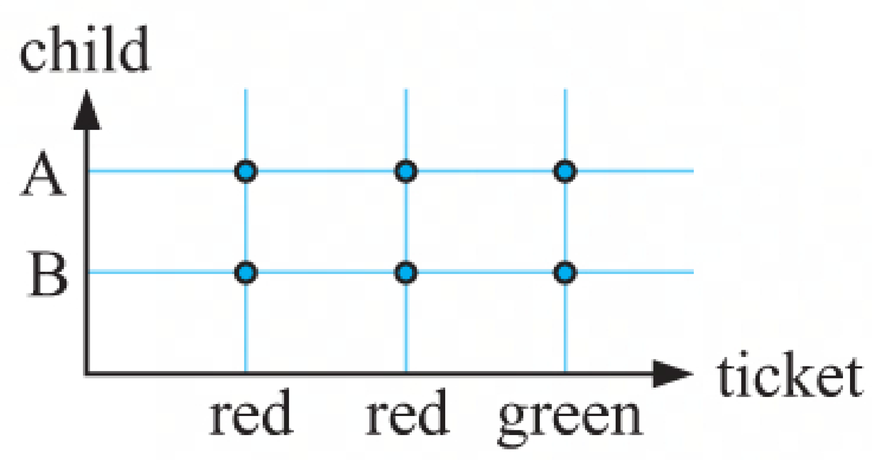


$\text{P(no sixes)} = \frac{25}{36}$

10 **a** **A**



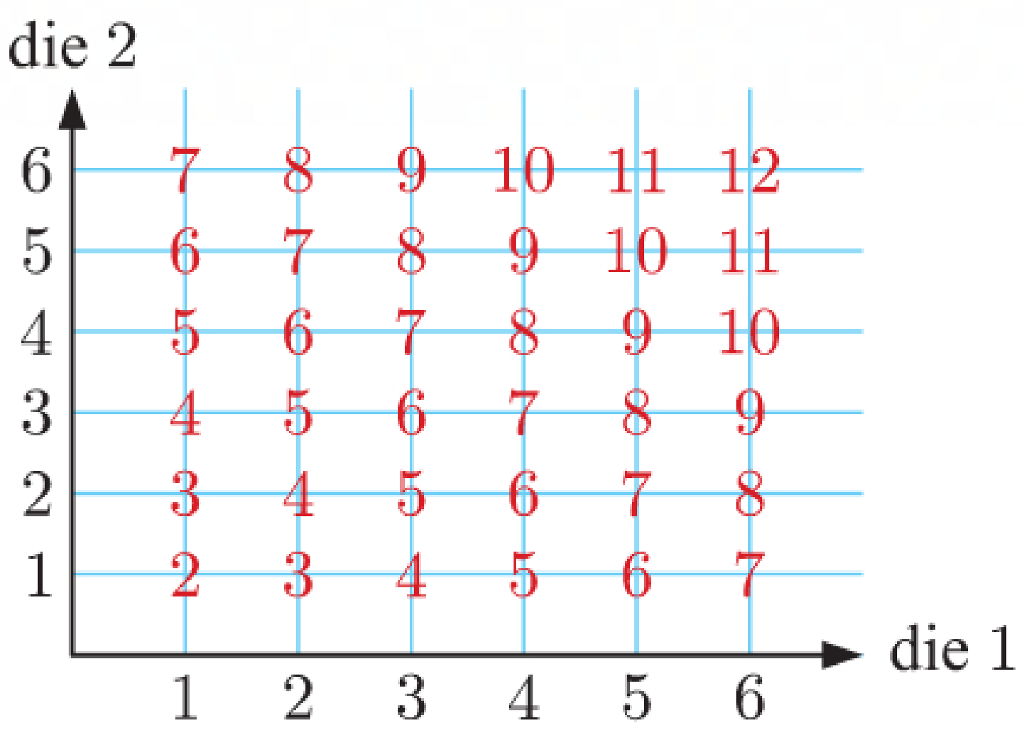
B



Both grids show the sample space correctly, although **B** is more useful for calculating probabilities.

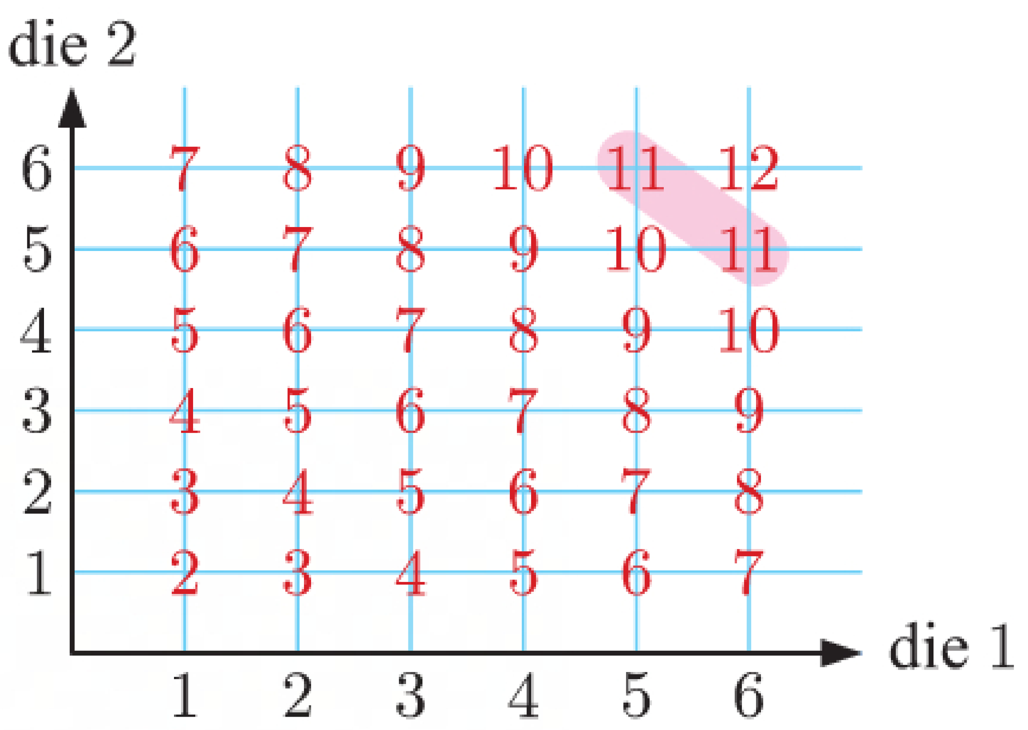
b $\text{P(child B selects green ticket)} = \frac{1}{6}$ {using grid **B**}

11 **a**



b There are 36 outcomes in the sample space.

i

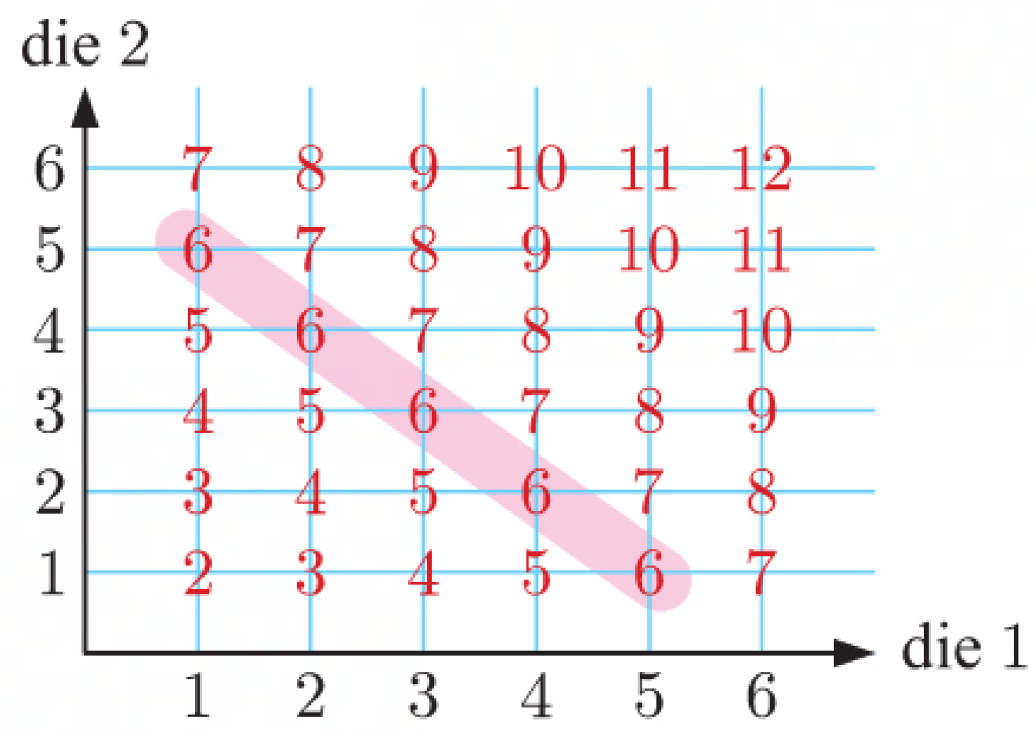


Two of the outcomes are 11.

$\therefore$ P(sum of dice is 11) $= \dfrac{2}{36}$

$= \frac{1}{18}$

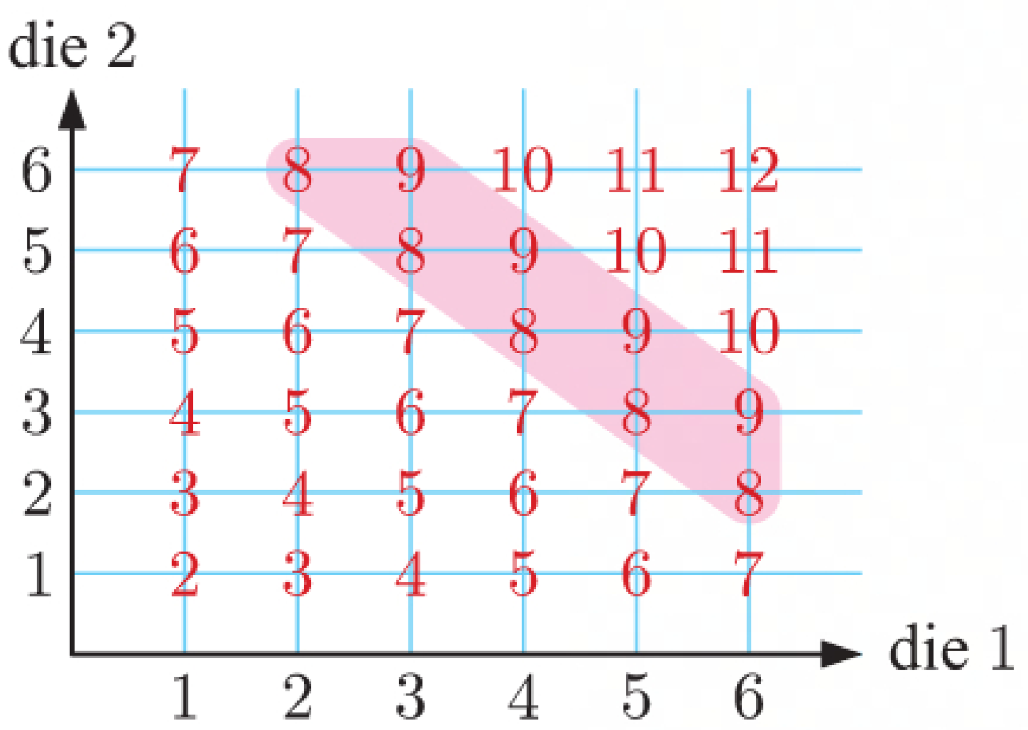
ii



Five of the outcomes are 6.

$\therefore$ P(sum of dice is 6) $= \frac{5}{36}$

iii

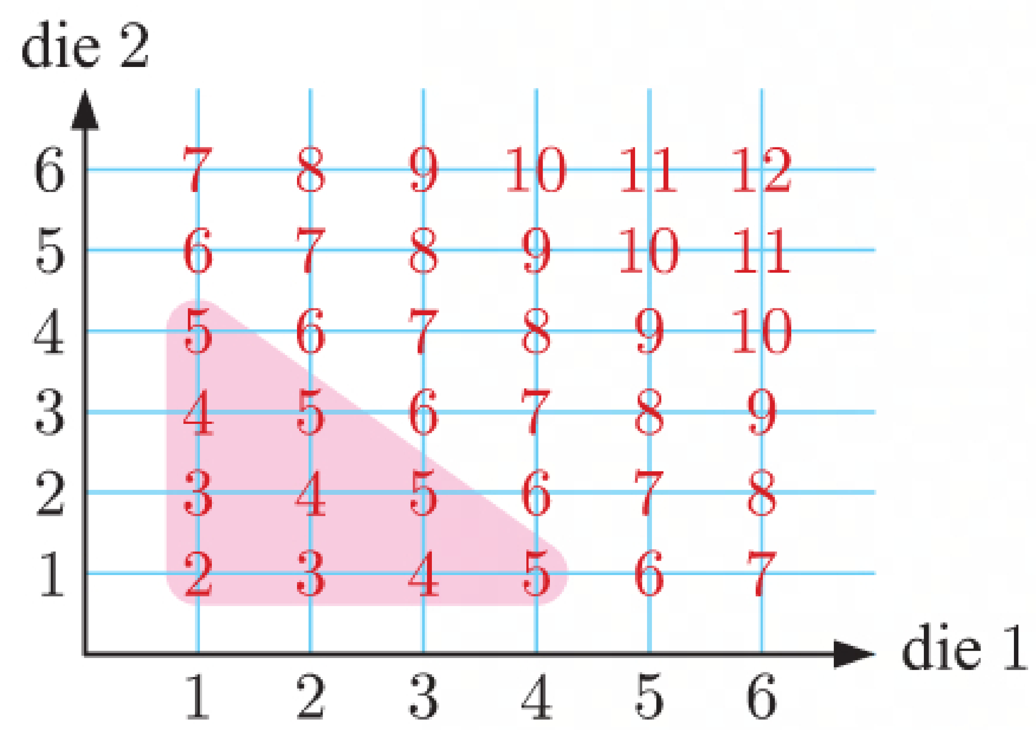


Nine of the outcomes are 8 or 9.

$\therefore$ P(sum of dice is 8 or 9) $= \dfrac{9}{36}$

$= \frac{1}{4}$

iv

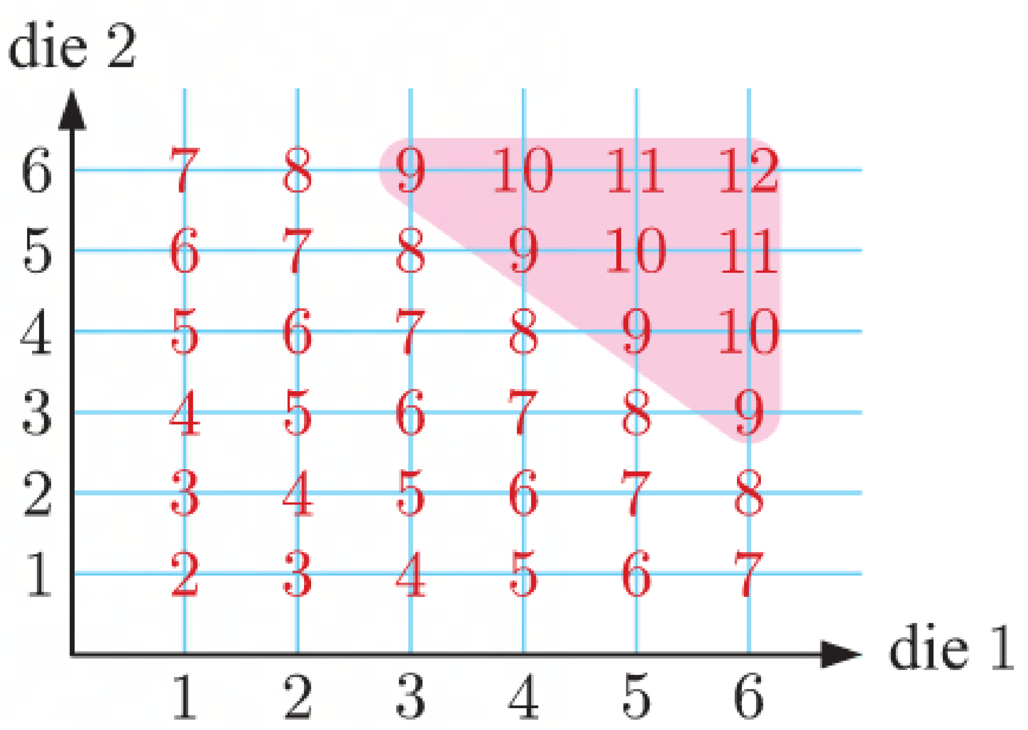


Ten of the outcomes are less than 6.

$\therefore$ P(sum of dice is less than 6) $= \dfrac{10}{36}$

$= \frac{5}{18}$

v



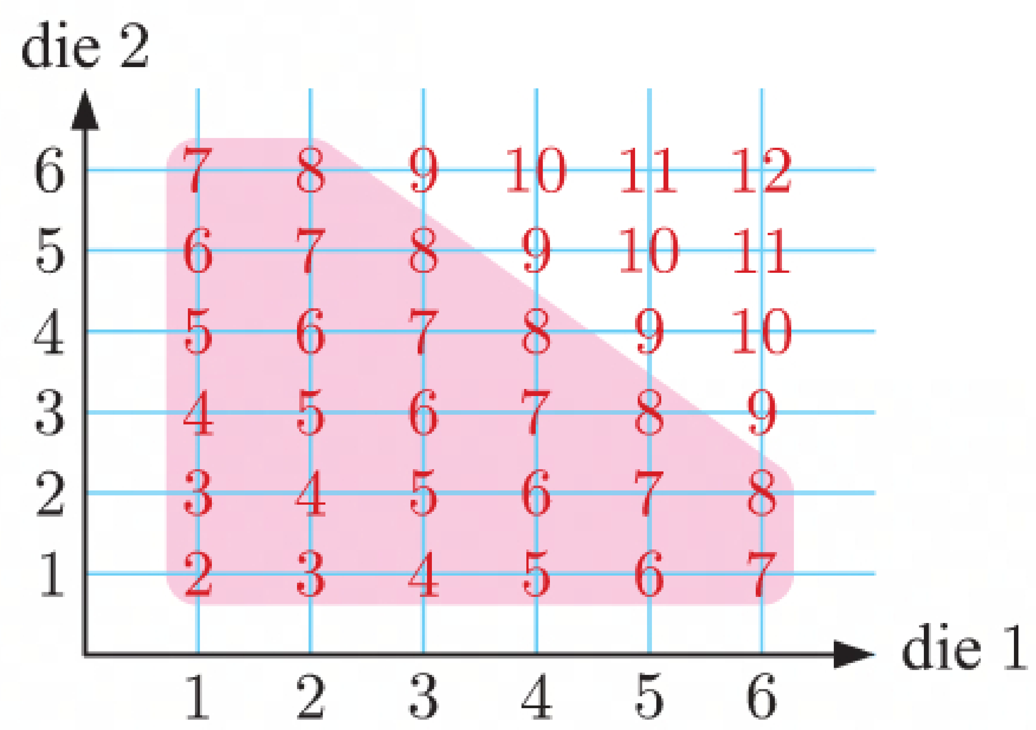
Ten of the outcomes are greater than 8.

$\therefore$ P(sum of dice is greater than 8)

$= \dfrac{10}{36}$

$= \frac{5}{18}$

vi



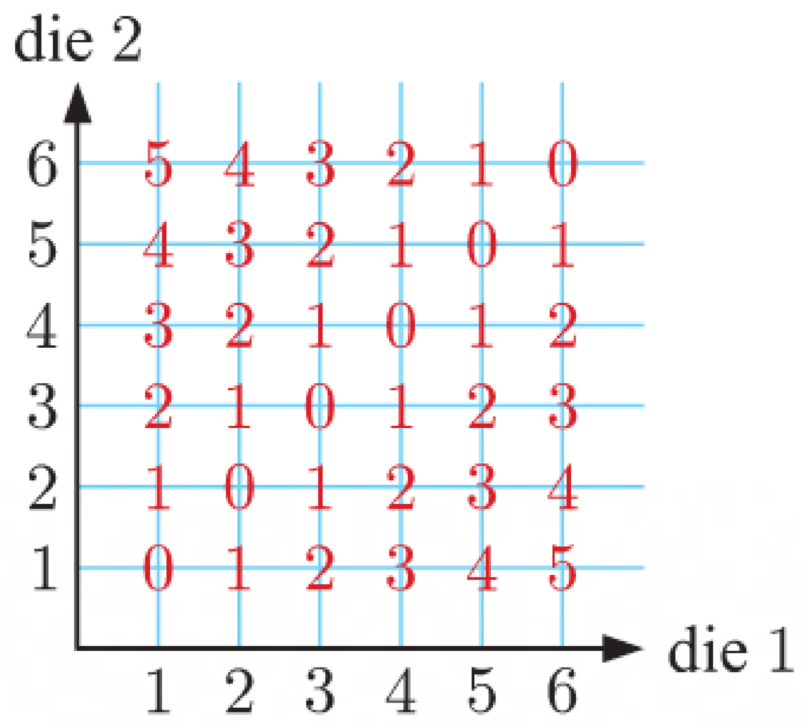
26 of the outcomes are no more than 8.

$\therefore$ P(sum of dice is no more than 8)

$= \dfrac{26}{36}$

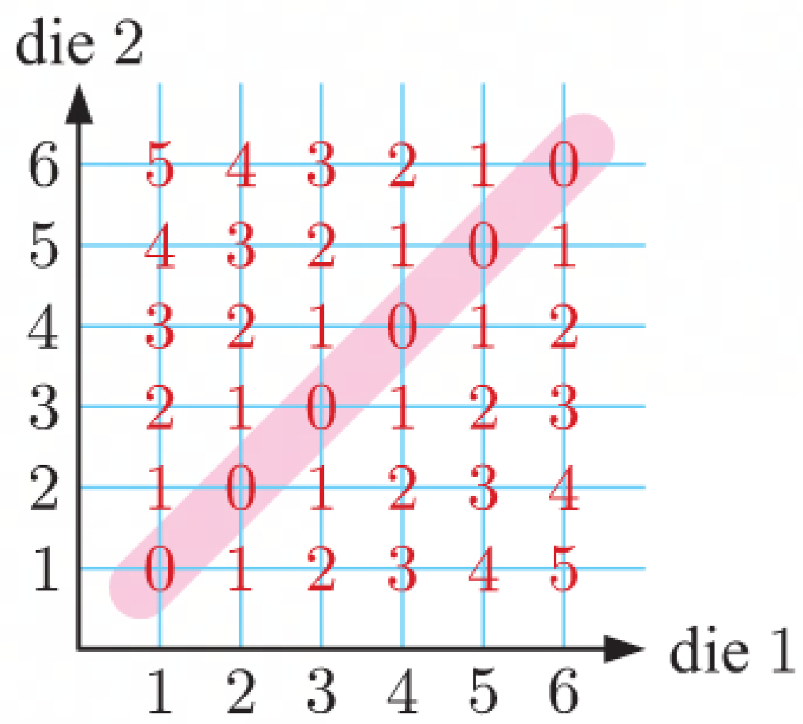
$= \frac{13}{18}$

12 **a**



b There are 36 outcomes in the sample space.

i

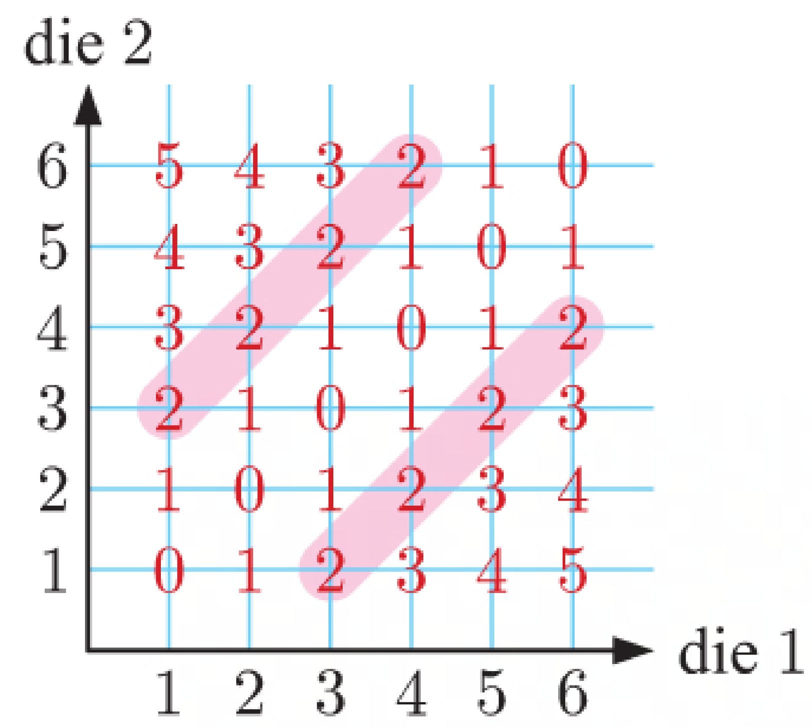


Six of the outcomes are 0.

$\therefore$ P(resulting value is 0) $= \dfrac{6}{36}$

$= \frac{1}{6}$

ii

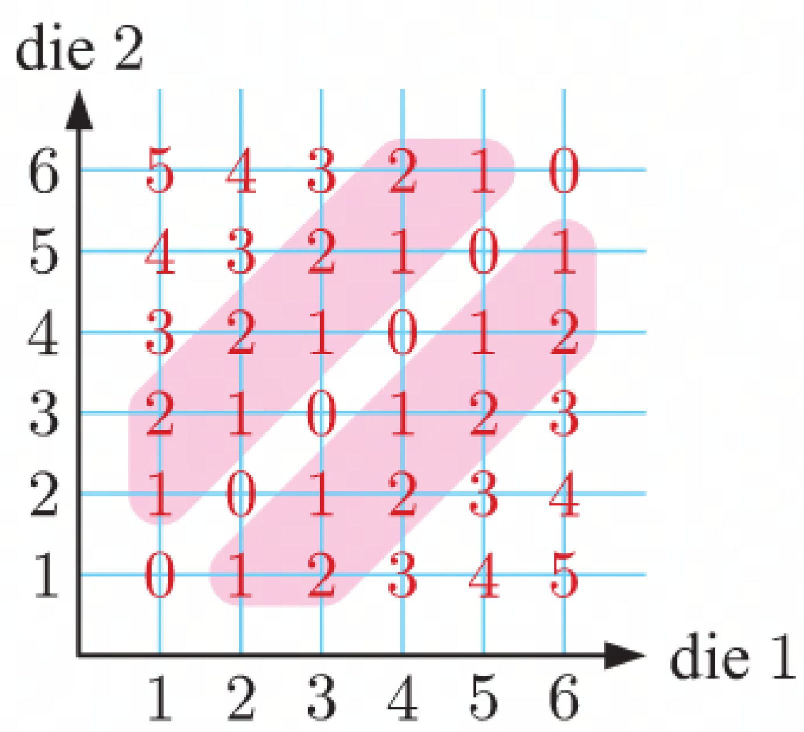


Eight of the outcomes are 2.

$\therefore$ P(resulting value is 2) $= \dfrac{8}{36}$

$= \frac{2}{9}$

iii

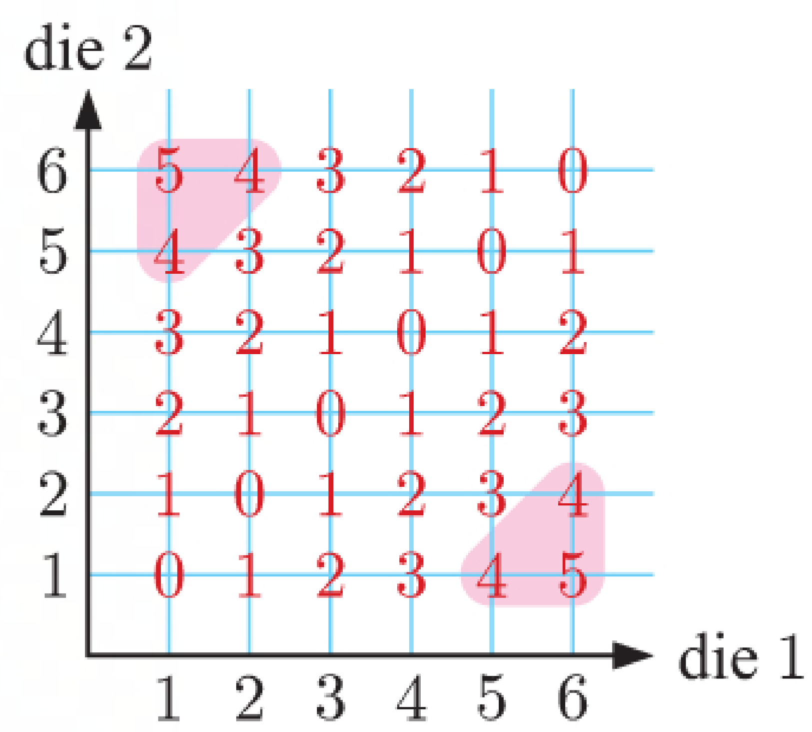


18 of the outcomes are 1 or 2.

$\therefore$ P(resulting value is 1 or 2) $= \dfrac{18}{36}$

$= \frac{1}{2}$

iv



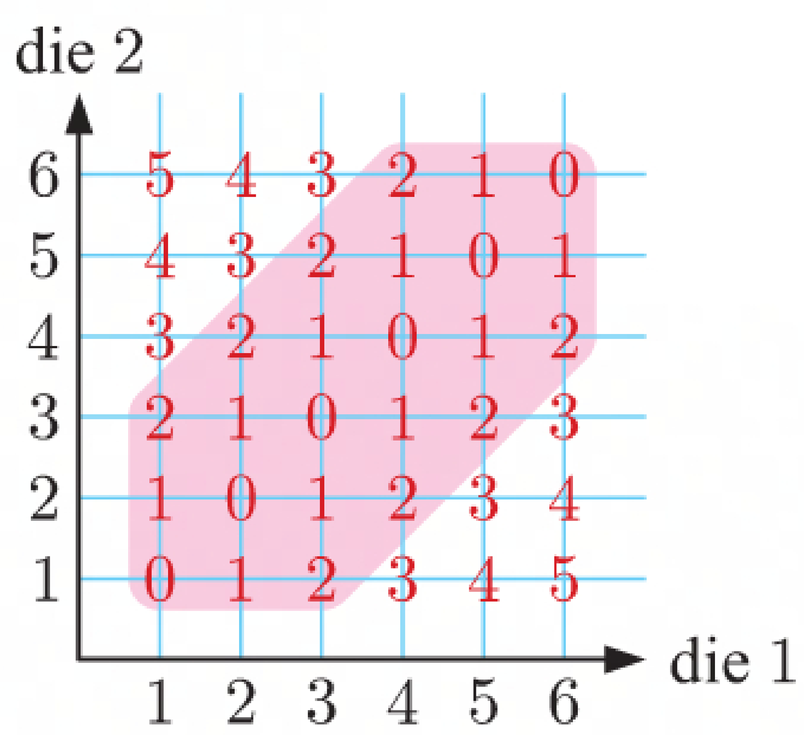
Six of the outcomes are more than 3.

$\therefore$ P(resulting value is more than 3)

$= \dfrac{6}{36}$

$= \frac{1}{6}$

v

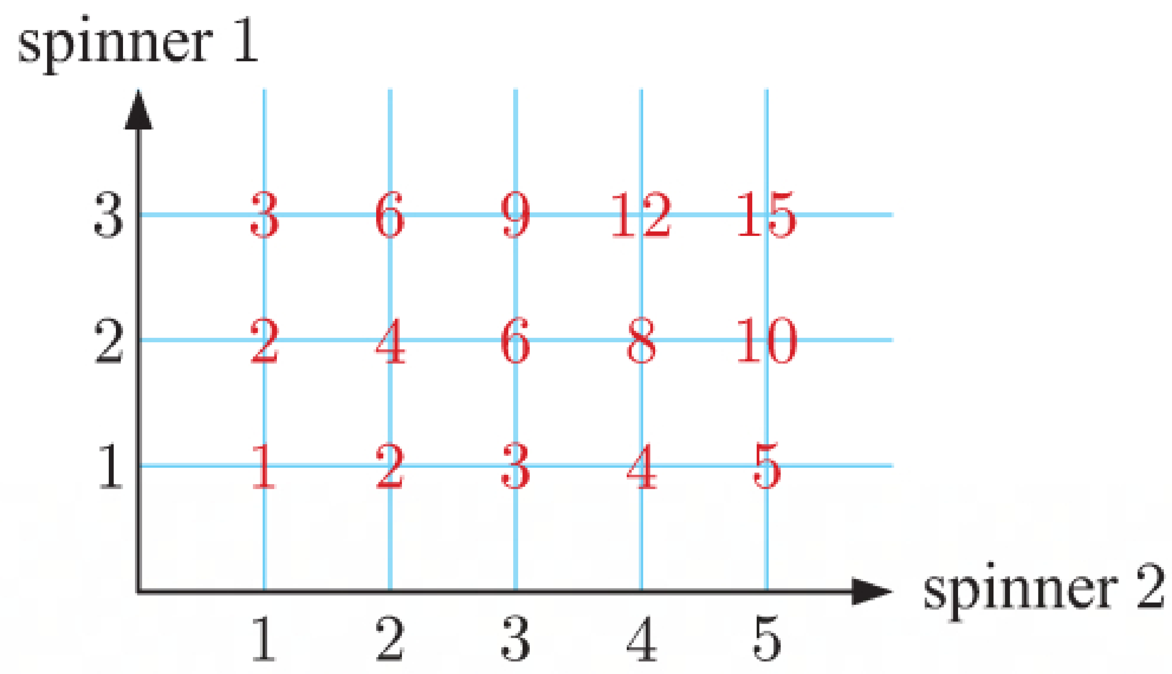


24 of the outcomes are less than 3.

$\therefore$ P(resulting value is less than 3) $= \dfrac{24}{36}$

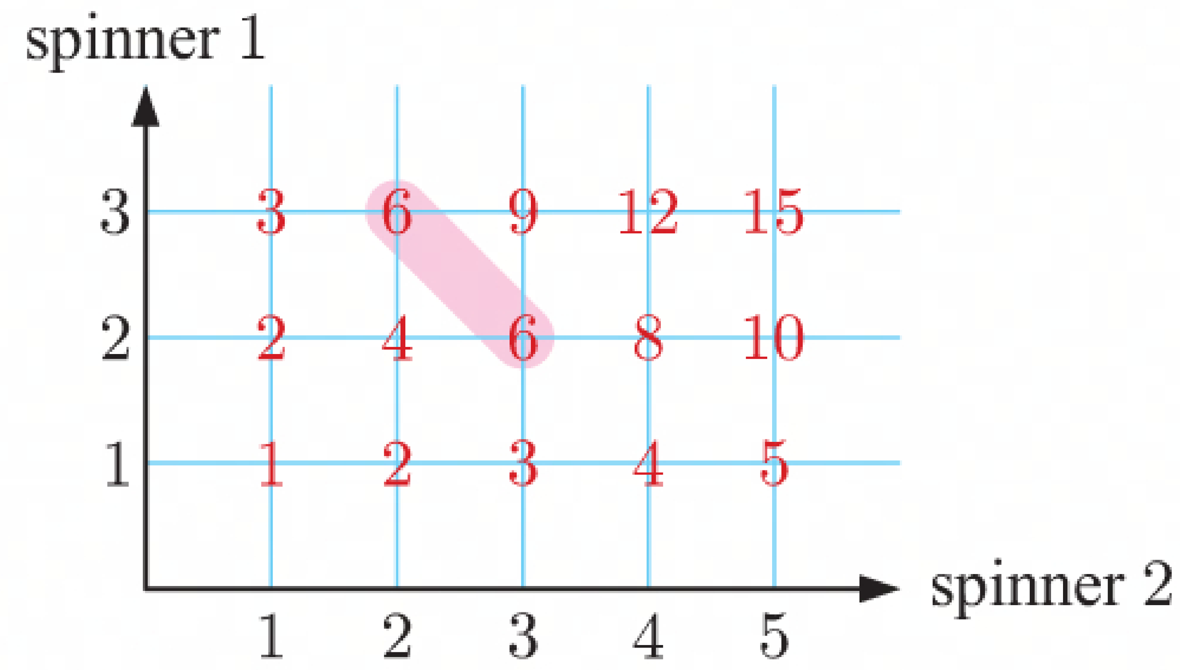
$= \frac{2}{3}$

13 **a**



b There are 15 outcomes in the sample space.

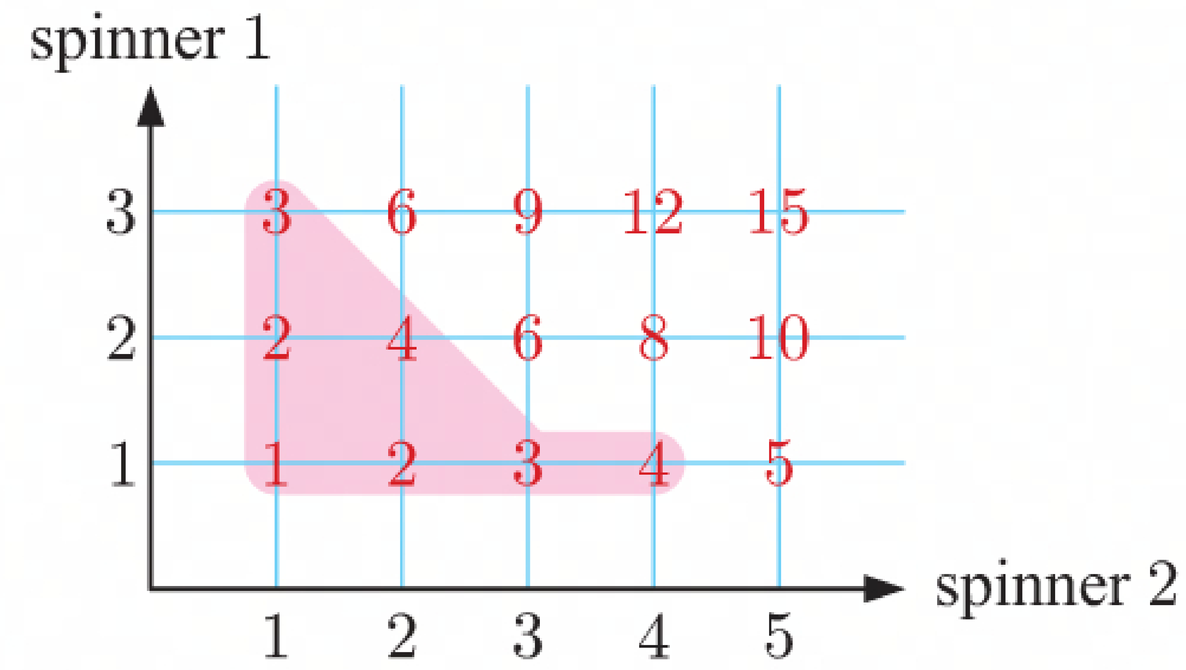
i



Two of the outcomes are 6.

$\therefore$ P(result is 6) $= \frac{2}{15}$

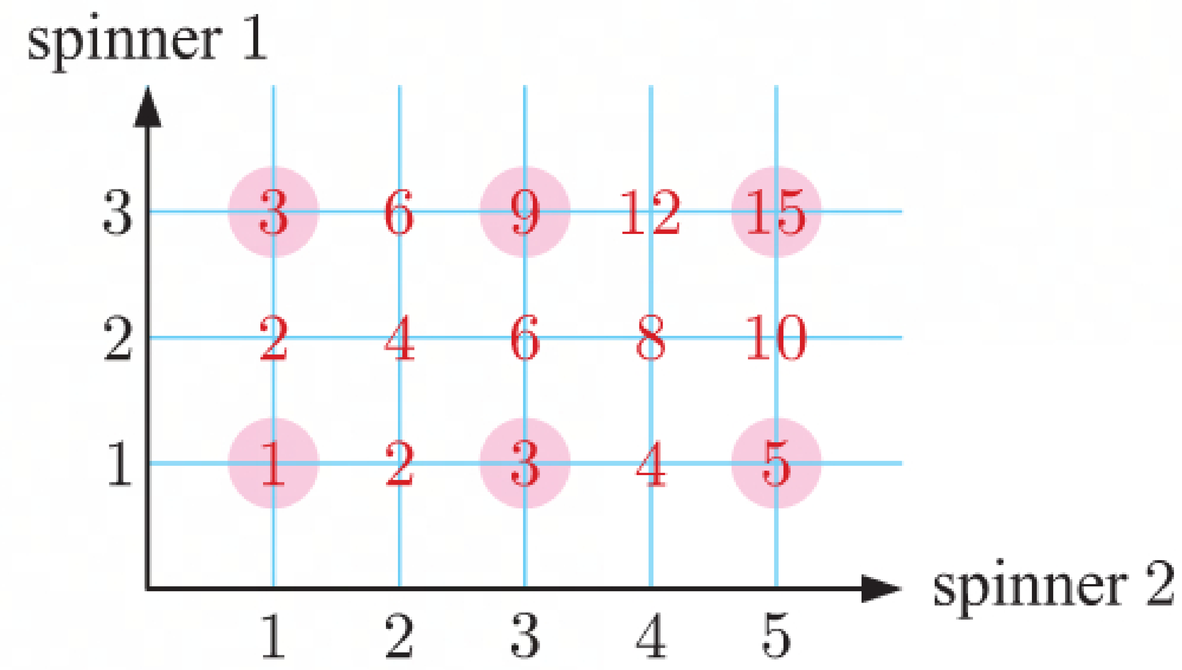
ii



Seven of the outcomes are less than 5.

$\therefore$ P(result is less than 5) $= \frac{7}{15}$

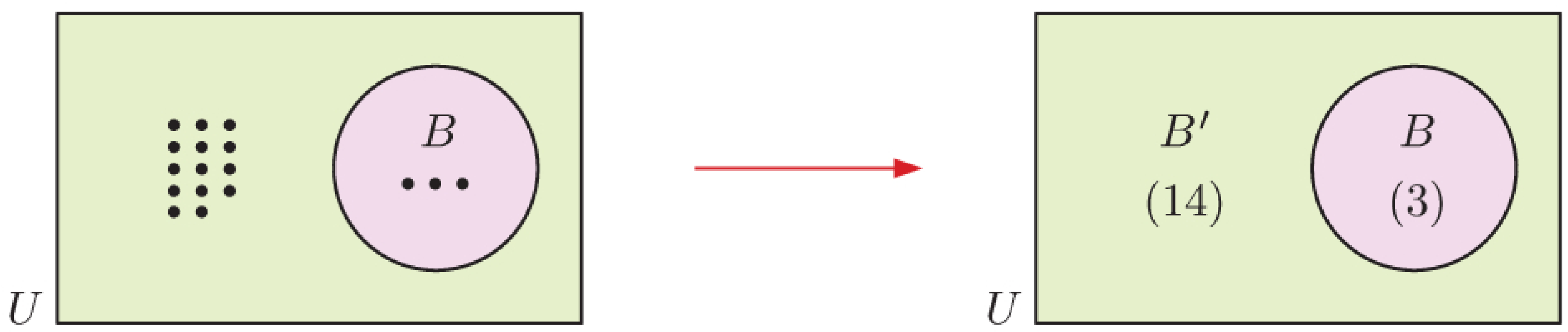
iii



Six of the outcomes are odd.

$$\therefore \quad \text{P(result is odd)} = \frac{6}{15} = \frac{2}{5}$$

14



$n(U) = 17, \quad n(B) = 3$

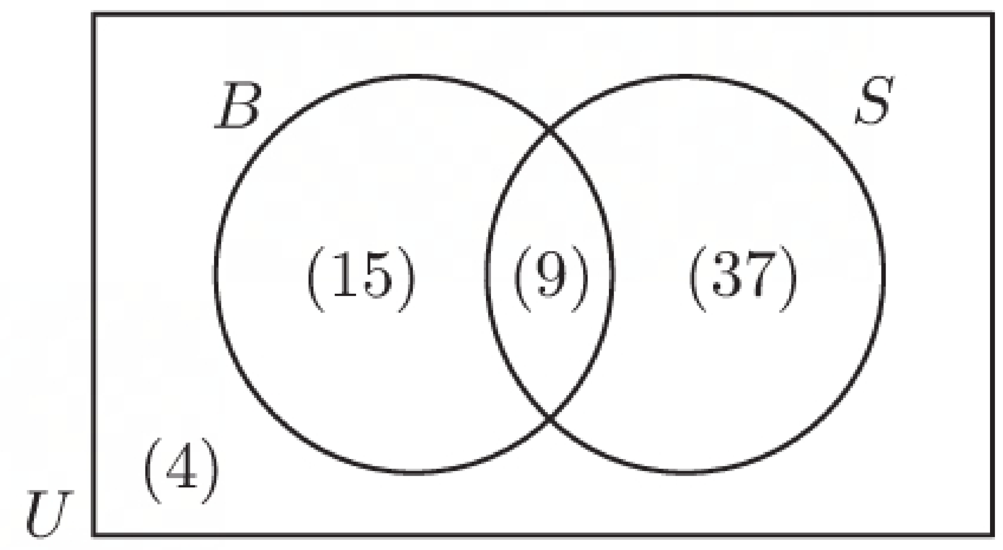
a $$\text{P(black wool)} = \frac{n(B)}{n(U)} = \frac{3}{17}$$

b $$\text{P(not black wool)} = \frac{n(B')}{n(U)} = \frac{14}{17}$$

15 $n(U) = 15 + 9 + 37 + 4 = 65,\quad n(B) = 15 + 9 = 24,\quad n(S) = 9 + 37 = 46$

a P(likes both activities) $= \dfrac{n(B \cap S)}{n(U)}$

$= \frac{9}{65}$



b P(likes neither activity) $= \dfrac{n(B \cup S)'}{n(U)}$

$= \frac{4}{65}$

c P(likes exactly one activity) $= \dfrac{15 + 37}{65}$

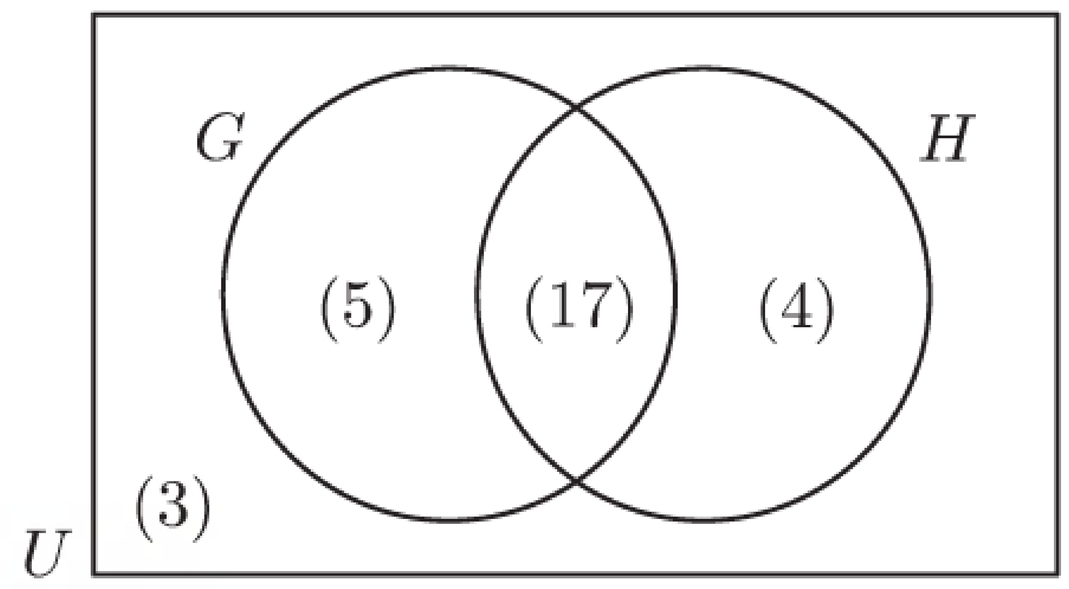
$= \dfrac{52}{65}$

$= \frac{4}{5}$

16 $n(U) = 5 + 17 + 4 + 3 = 29,\quad n(G) = 5 + 17 = 22,\quad n(H) = 17 + 4 = 21$

a P(studies both subjects) $= \dfrac{n(G \cap H)}{n(U)}$

$= \frac{17}{29}$



b P(studies at least one subject) $= \dfrac{n(G \cup H)}{n(U)}$

$= \dfrac{5 + 17 + 4}{29}$

$= \frac{26}{29}$

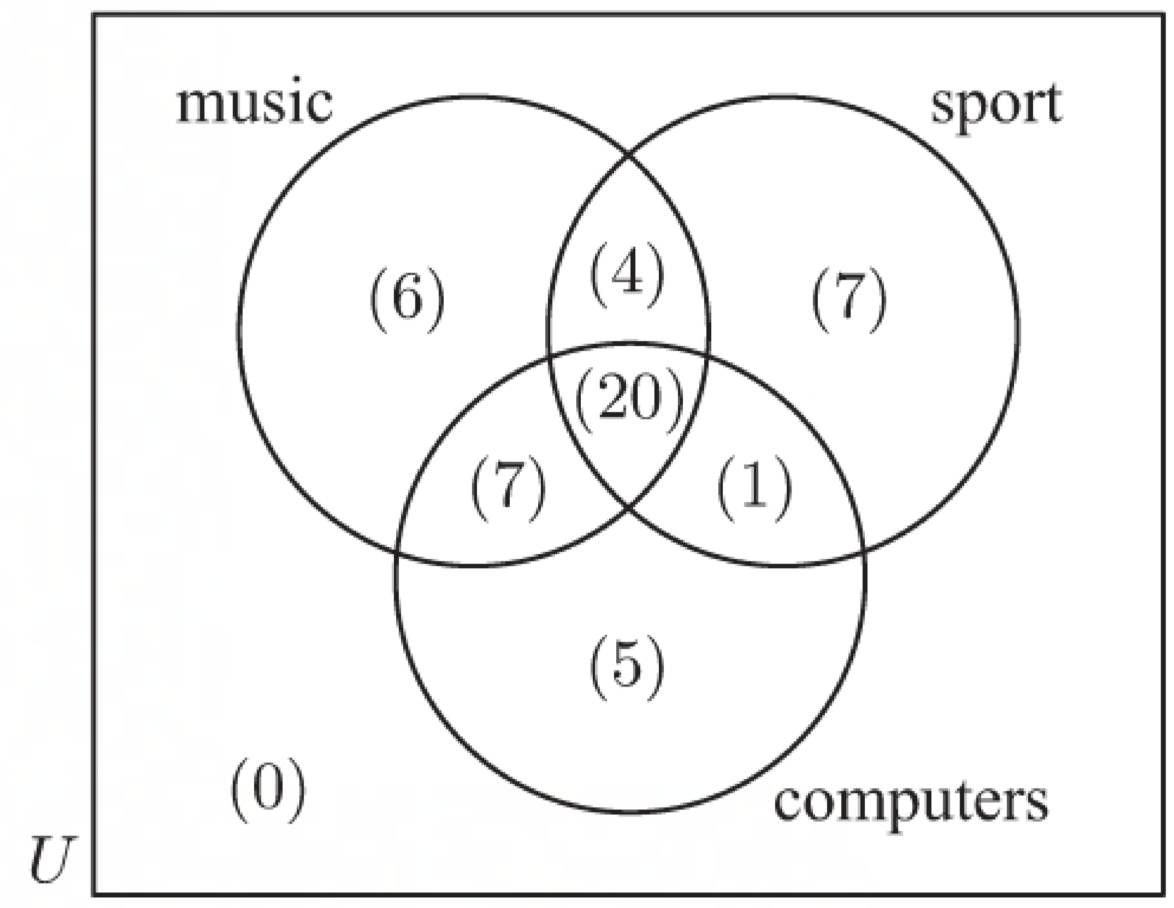
c P(studies only Geography) $= \dfrac{n(G \cap H')}{n(U)}$

$= \frac{5}{29}$

17 $n(U) = 50$

a P(interested in music) $= \dfrac{6 + 4 + 20 + 7}{n(U)}$

$= \frac{37}{50}$



b P(interested in music, sport, and computers)

$= \dfrac{20}{n(U)}$

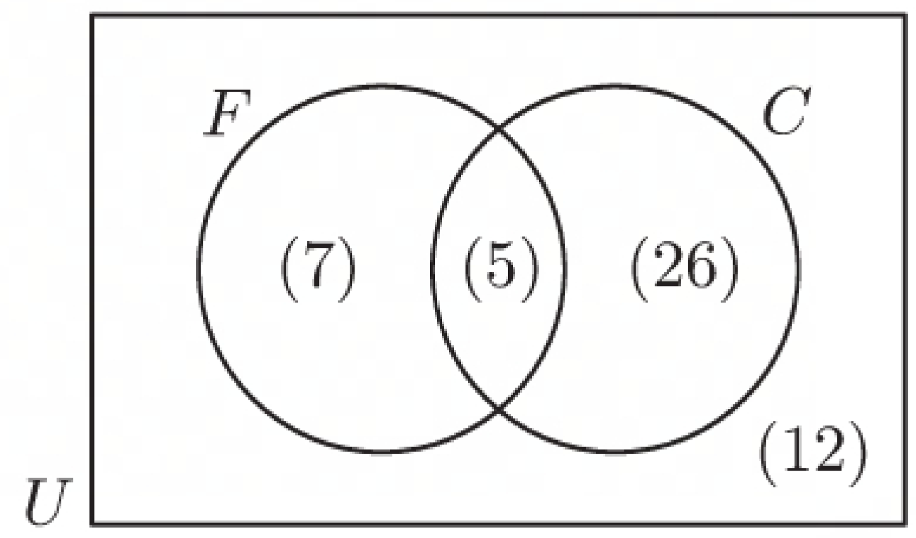
$= \dfrac{20}{50}$

$= \frac{2}{5}$

c P(not interested in computers) $= \dfrac{6 + 4 + 7 + 0}{n(U)}$

$= \frac{17}{50}$

18 **a** Let F represent the event "the man gave flowers" and C represent the event "the man gave chocolates".



$n(F \cap C) = 5$

$\therefore \; n(F \cap C') = 12 - 5 = 7$

and $\; n(F' \cap C) = 31 - 5 = 26$

$\therefore \; n(F' \cap C') = 50 - 5 - 7 - 26 = 12$

b **i** P(C or F)

$= \dfrac{7+5+26}{50}$

$= \dfrac{38}{50}$

$= \frac{19}{25}$

ii P(C but not F)

$= \dfrac{26}{50}$

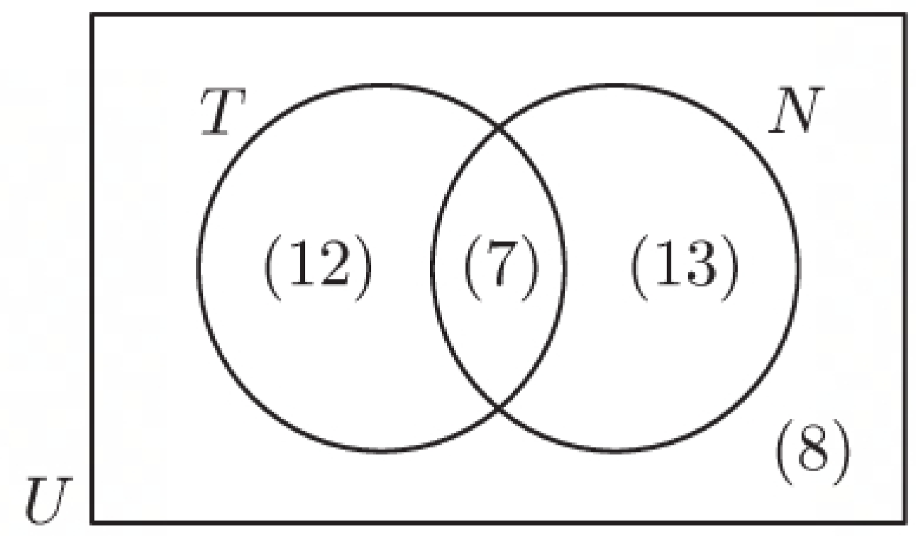
$= \frac{13}{25}$

iii P(neither C nor F)

$= \dfrac{12}{50}$

$= \frac{6}{25}$

19 Let T represent the event "a student plays tennis" and N represent the event "a student plays netball".



$n(T) = 19, \quad n(N) = 20, \quad n(T' \cap N') = 8, \quad n(U) = 40$

$n(T \cap N') = 19 - n(T \cap N)$

$n(T' \cap N) = 20 - n(T \cap N)$

$$n(T \cap N') + n(T \cap N) + n(T' \cap N) + n(T' \cap N') = n(U)$$

$$\therefore \; 19 - n(T \cap N) + n(T \cap N) + 20 - n(T \cap N) + 8 = 40$$

$$\therefore \; 47 - n(T \cap N) = 40$$

$$\therefore \; n(T \cap N) = 7$$

a P(plays tennis) $= \frac{19}{40}$

b P(does not play netball) $= \dfrac{12+8}{40}$

$= \dfrac{20}{40}$

$= \frac{1}{2}$

c P(plays at least one) $= \dfrac{12+7+13}{40}$

$= \dfrac{32}{40}$

$= \frac{4}{5}$

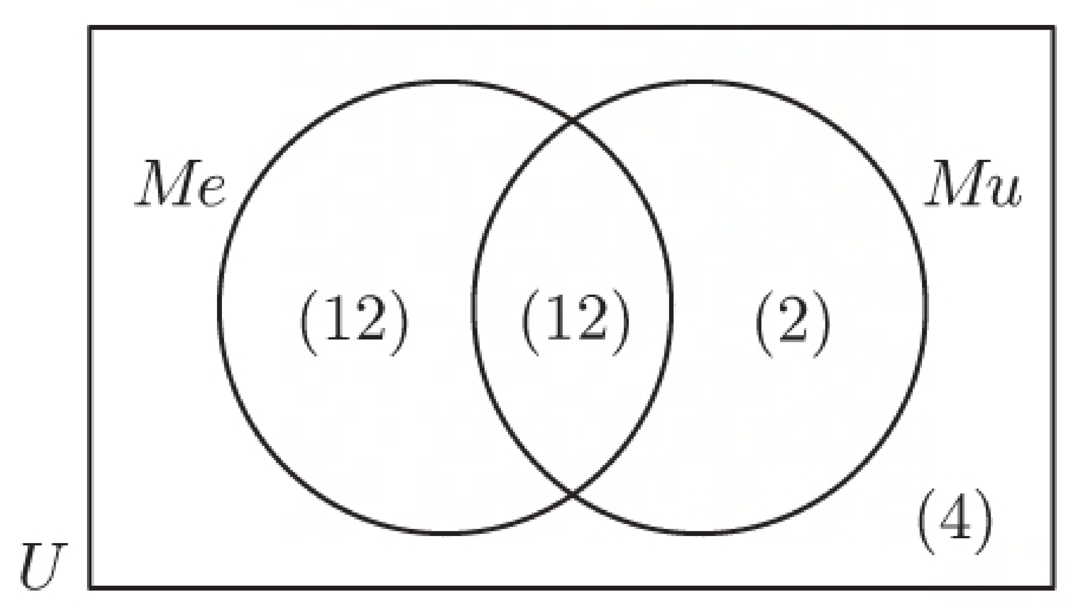
d P(plays exactly one) $= \dfrac{12+13}{40}$

$= \dfrac{25}{40}$

$= \frac{5}{8}$

e P(plays netball, but not tennis) $= \frac{13}{40}$

20 Let Me represent the event "a child had measles" and Mu represent the event "a child had mumps".



$n(Me) = 24, \quad n(Me \cap Mu) = 12, \quad n(Me \cup Mu) = 26,$
$n(U) = 30$

$n(Me \cap Mu') = 24 - 12 = 12$

$n(Me' \cap Mu) = 26 - 24 = 2$

$n(Me' \cap Mu') = 30 - 26 = 4$

a P(child has had mumps) $= \dfrac{12+2}{30}$

$= \dfrac{14}{30}$

$= \frac{7}{15}$

b P(child has had mumps but not measles)

$= \dfrac{2}{30}$

$= \frac{1}{15}$

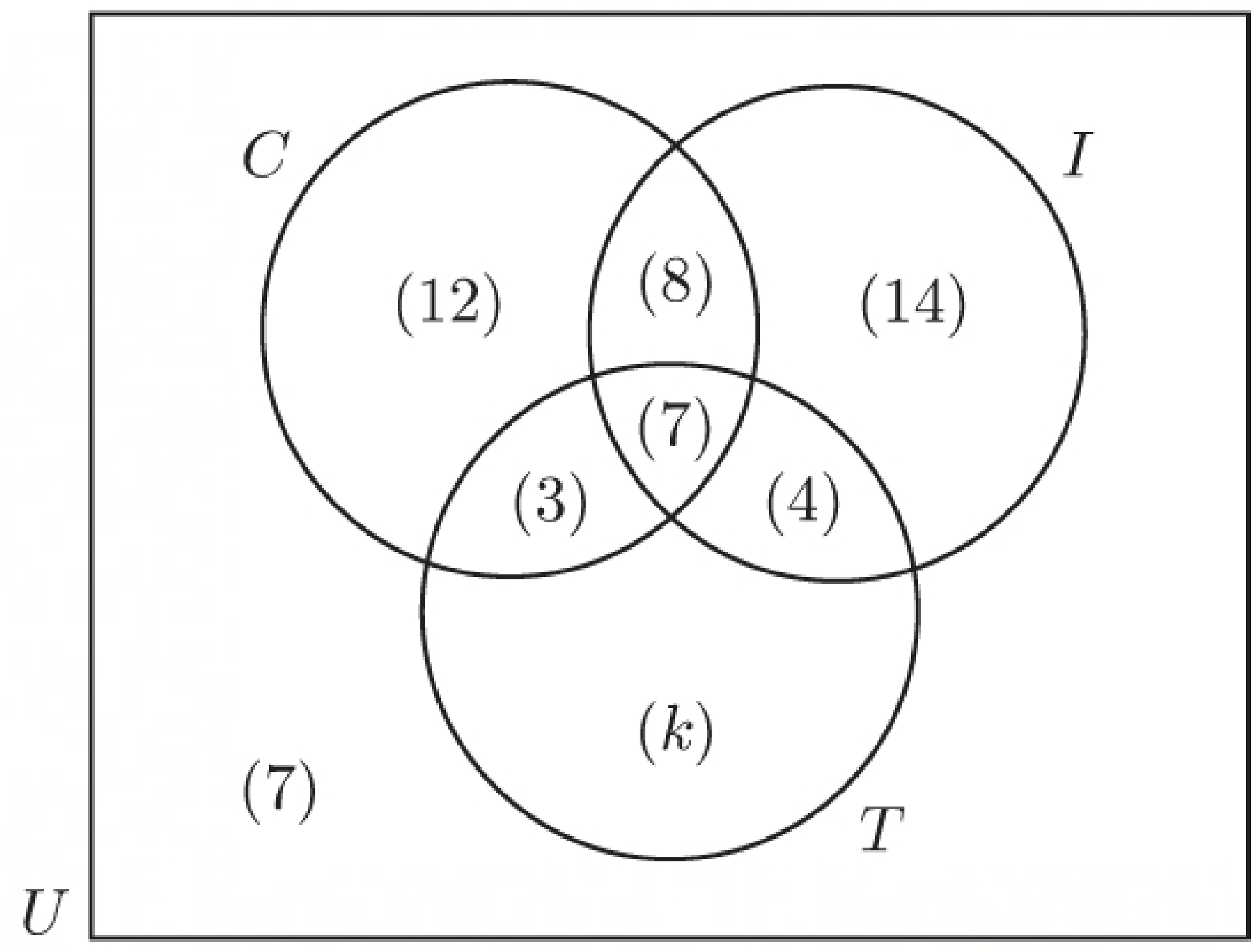
c P(child has had neither mumps nor measles) $= \dfrac{4}{30}$

$= \frac{2}{15}$

21 **a** $n(U) = 12 + 8 + 7 + 3 + 14 + 4 + k + 7 = 60$

$\therefore\ k + 55 = 60$

$\therefore\ k = 5$



b **i** P(member likes only Italian) $= \dfrac{14}{60}$

$= \frac{7}{30}$

ii P(member likes Italian and Thai) $= \dfrac{7+4}{60}$

$= \frac{11}{60}$

iii P(member likes none of these foods) $= \frac{7}{60}$

iv P(member likes at least one of these foods) = 1 − P(member likes none of these foods)

$= 1 - \dfrac{7}{60}$

$= \frac{53}{60}$

v P(member likes all of these foods) $= \frac{7}{60}$

vi P(member likes Chinese and Italian, but not Thai) $= \dfrac{8}{60}$

$= \frac{2}{15}$

vii P(member likes Thai or Italian) $= \dfrac{3+7+4+5+8+14}{60}$

$= \frac{41}{60}$

viii P(member likes exactly one of these foods) $= \dfrac{12+14+5}{60}$

$= \frac{31}{60}$

22

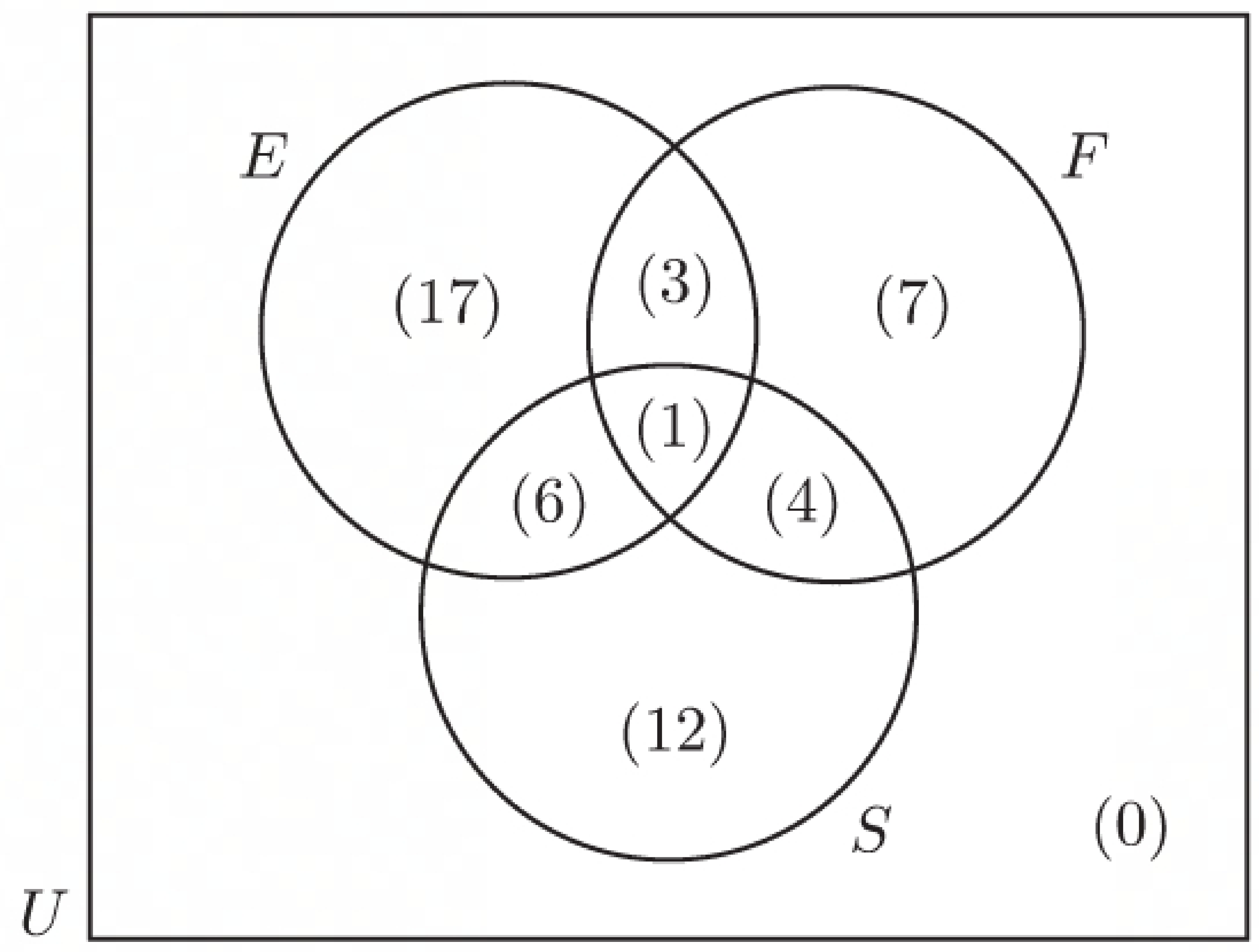
Languages	*Delegates*
English only	17
French only	7
Spanish only	12
English and French only	3
English and Spanish only	6
French and Spanish only	4
English, French, and Spanish	1

a Let E represent the event "a delegate had a conversation in English",
F represent the event "a delegate had a conversation in French",
and S represent the event "a delegate had a conversation in Spanish".

$$\begin{aligned} n(E \cup F \cup S) &= 17 + 3 + 1 + 6 + 7 + 4 + 12 \\ &= 50 \end{aligned}$$

So, every delegate had a conversation in at least one language.

$\therefore \quad n(E' \cap F' \cap S') = 0$



b **i** $$\text{P(delegate had a conversation in English)} = \frac{17+3+1+6}{50} = \frac{27}{50}$$

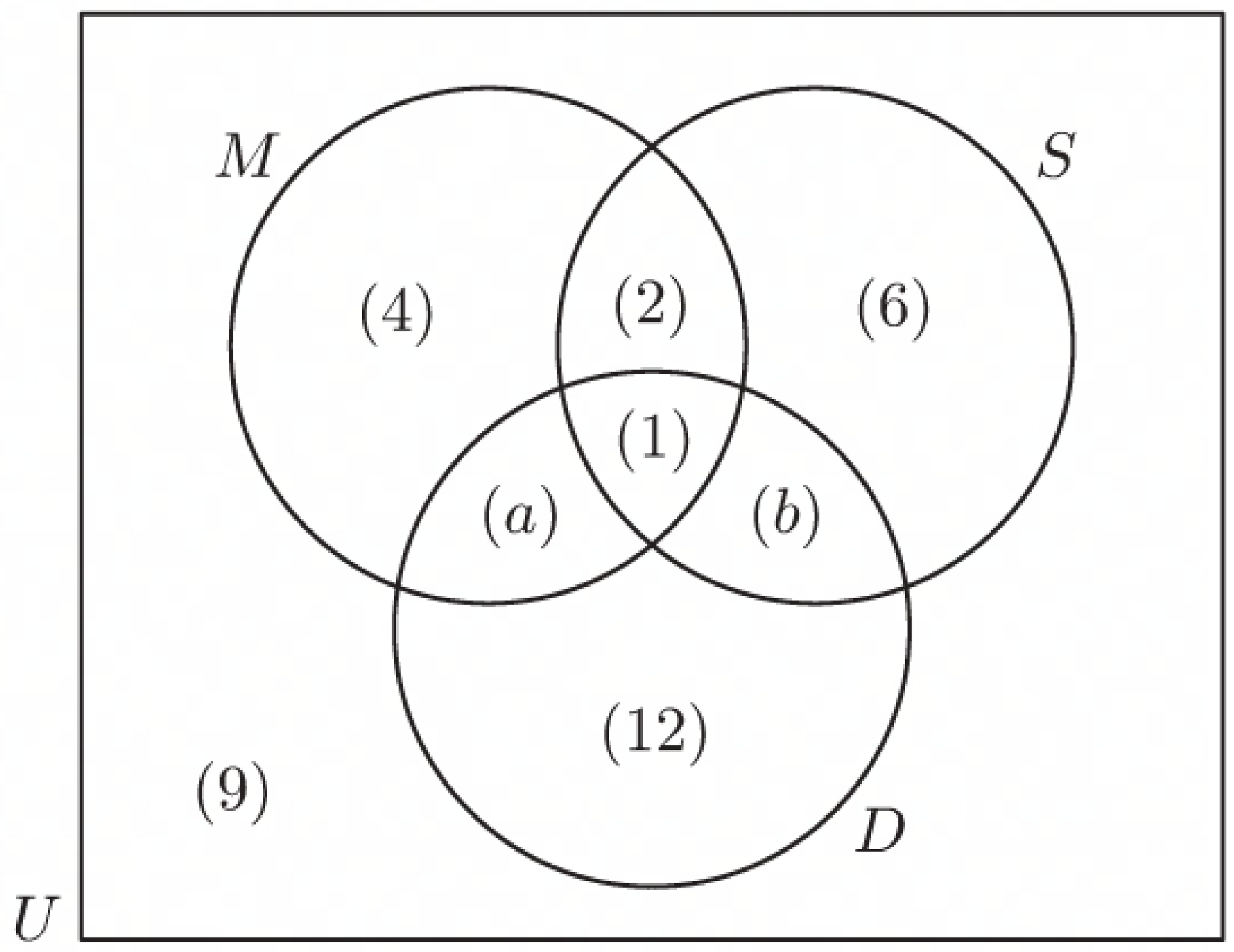
ii $$\text{P(delegate had a conversation in French)} = \frac{3+7+4+1}{50} = \frac{15}{50} = \frac{3}{10}$$

iii $$\text{P(delegate had a conversation in Spanish, but not in English)} = \frac{12+4}{50} = \frac{16}{50} = \frac{8}{25}$$

iv $$\text{P(delegate had a conversation in French, but not in Spanish)} = \frac{3+7}{50} = \frac{10}{50} = \frac{1}{5}$$

v $$\text{P(delegate had a conversation in French, and also one in English)} = \frac{3+1}{50} = \frac{4}{50} = \frac{2}{25}$$

23



a $4 + 2 + 1 + a = 10$ {10 watched a movie}

$\therefore \; a = 3$

$4 + 2 + 1 + 3 + 6 + 12 + 9 + b = 40$ {40 individuals in total}

$\therefore \; 37 + b = 40$

$\therefore \; b = 3$

b **i** $\text{P(watched sport)} = \dfrac{6+2+1+3}{40} = \dfrac{12}{40} = \frac{3}{10}$

ii $\text{P(watched drama and sport)} = \dfrac{3+1}{40} = \dfrac{4}{40} = \frac{1}{10}$

iii P(watched a movie but not sport)

$= \dfrac{4+3}{40}$

$= \frac{7}{40}$

iv P(watched drama but not a movie)

$= \dfrac{12+3}{40}$

$= \dfrac{15}{40}$

$= \frac{3}{8}$

v $\text{P(watched drama or a movie)} = \dfrac{12+3+3+1+4+2}{40} = \dfrac{25}{40} = \frac{5}{8}$

EXERCISE 11E

1 $n = 90$ attempts

$p = \text{P(saving a penalty attempt)} = \frac{3}{10}$

The goalkeeper would expect to save $np = 90 \times \frac{3}{10} = 27$ penalties.

2 $n = 68$ attempts

$p = \text{P(scores a goal)} = 0.23$

Brayden would expect to score $np = 68 \times 0.23 \approx 16$ times.

3 **a** $\quad \text{P(one coin falls heads)} = \frac{1}{2}$

$\therefore \text{P(both coins fall heads)} = \frac{1}{2} \times \frac{1}{2}$

$= \frac{1}{4}$

b $n = 200$ times

$p = \text{P(both coins fall heads)}$

We would expect the 2 coins to both fall heads on $np = 200 \times \frac{1}{4} = 50$ occasions.

4 $n = 5 \times 7 = 35$ days

$p = \text{P(snow falling on any particular day)} = \frac{3}{7}$

Udo could expect to see snow falling on $np = 35 \times \frac{3}{7} = 15$ days.

5 $n = 180$ times

$p = \text{P(rolling a double with two dice)}$

$= \text{P(rollling two 1s or two 2s or two 3s or two 4s or two 5s or two 6s)}$

$= \frac{6}{36}$ {6 of the possible 36 outcomes}

$= \frac{1}{6}$

You would expect to get a double on $np = 180 \times \frac{1}{6} = 30$ occasions.

6 Total number of voters in poll $= 165 + 87 + 48$

$= 300$

A	B	C
165	87	48

a **i** $\text{P(voter will vote for A)} \approx \frac{165}{300}$

≈ 0.55

ii $\text{P(voter will vote for B)} \approx \frac{87}{300}$

≈ 0.29

iii $\text{P(voter will vote for C)} \approx \frac{48}{300}$

≈ 0.16

b $n = 7500$ people

i We would expect $np \approx 7500 \times 0.55 \approx 4125$ people to vote for A.

ii We would expect $np \approx 7500 \times 0.29 \approx 2175$ people to vote for B.

iii We would expect $np \approx 7500 \times 0.16 \approx 1200$ people to vote for C.

INVESTIGATION 3 THE ADDITION LAW OF PROBABILITY

1 $U = \{x \mid x \text{ is a positive integer less than } 100\}$,

$A = \{\text{multiples of 7 in } U\}$, $B = \{\text{multiples of 5 in } U\}$

a **i** $A = \{7, 14, 21, 28, 35, 42, 49, 56, 63, 70, 77, 84, 91, 98\}$

There are 14 elements in A.

ii $B = \{5, 10, 15, 20, 25, 30, 35, 40, 45, 50, 55, 60, 65, 70, 75, 80, 85, 90, 95\}$

There are 19 elements in B.

iii $A \cap B = \{35, 70\}$

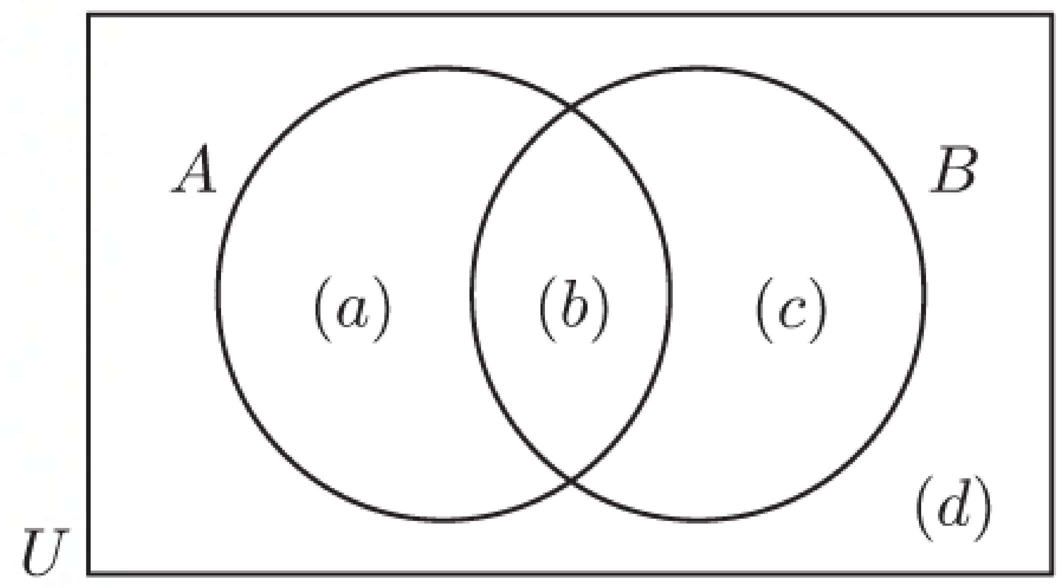
There are 2 elements in $A \cap B$.

iv $A \cup B = \{5, 7, 10, 14, 15, 20, 21, 25, 28, 30, 35, 40, 42, 45, 49, 50, 55, 56, 60, 63, 65, 70, 75, 77, 80, 84, 85, 90, 91, 95, 98\}$

There are 31 elements in $A \cup B$.

b $n(A) + n(B) - n(A \cap B) = 14 + 19 - 2$
$= 31$
$= n(A \cup B)$

2 $n(A) + n(B) - n(A \cap B) = (a + b) + (b + c) - b$
$= a + b + c$
$= n(A \cup B)$



3 $n(A \cup B) = n(A) + n(B) - n(A \cap B)$

$\therefore \dfrac{n(A \cup B)}{n(U)} = \dfrac{n(A) + n(B) - n(A \cap B)}{n(U)}$

$\therefore \dfrac{n(A \cup B)}{n(U)} = \dfrac{n(A)}{n(U)} + \dfrac{n(B)}{n(U)} - \dfrac{n(A \cap B)}{n(U)}$

$\therefore \mathrm{P}(A \cup B) = \mathrm{P}(A) + \mathrm{P}(B) - \mathrm{P}(A \cap B)$

EXERCISE 11F

1 $\mathrm{P}(A \cup B) = \mathrm{P}(A) + \mathrm{P}(B) - \mathrm{P}(A \cap B)$
$= 0.2 + 0.4 - 0.05$
$= 0.55$

2 $\mathrm{P}(A \cup B) = \mathrm{P}(A) + \mathrm{P}(B) - \mathrm{P}(A \cap B)$
$\therefore 0.9 = 0.4 + \mathrm{P}(B) - 0.1$
$\therefore \mathrm{P}(B) = 0.6$

3 $\mathrm{P}(X \cup Y) = \mathrm{P}(X) + \mathrm{P}(Y) - \mathrm{P}(X \cap Y)$
$\therefore 0.9 = 0.6 + 0.5 - \mathrm{P}(X \cap Y)$
$\therefore \mathrm{P}(X \cap Y) = 0.2$

4 **a** $\mathrm{P}(A \cup B) = \mathrm{P}(A) + \mathrm{P}(B) - \mathrm{P}(A \cap B)$
$\therefore 0.7 = 0.25 + 0.45 - \mathrm{P}(A \cap B)$
$\therefore \mathrm{P}(A \cap B) = 0$

b Since $\mathrm{P}(A \cap B) = 0$, A and B are mutually exclusive events.

5 $\mathrm{P}(A \cup B) = \mathrm{P}(A) + \mathrm{P}(B)$ {since A and B are mutually exclusive}
$\therefore 0.8 = \mathrm{P}(A) + 0.45$
$\therefore \mathrm{P}(A) = 0.35$

6 **a** It is impossible for a number to be both greater than 11 *and* less than 8.
$\therefore$ A and B are mutually exclusive.

b **i** $P(A) = P(\text{number drawn is 12, 13, 14, or 15})$
$= \frac{4}{15}$

ii $P(B) = P(\text{number drawn is 1, 2, 3, 4, 5, 6, or 7})$
$= \frac{7}{15}$

iii $P(A \cup B) = P(A) + P(B)$ {since A and B are mutually exclusive}
$= \frac{4}{15} + \frac{7}{15}$
$= \frac{11}{15}$

7 **a** $P(F) = \frac{11}{25}$ **b** $P(S) = \frac{12}{25}$ **c** $P(D) = \frac{8}{25}$

d $P(C) = \frac{7}{25}$ **e** $P(N) = \frac{4}{25}$

f $P(F \cup S) = P(F) + P(S)$ {since F and S are mutually exclusive}
$= \frac{11}{25} + \frac{12}{25}$
$= \frac{23}{25}$

g $P(F \cup D)$ cannot be found as we do not know how many students are both 15 *and* own a dog.

h $P(C \cup N) = P(C) + P(N)$ {since C and N are mutually exclusive}
$= \frac{7}{25} + \frac{4}{25}$
$= \frac{11}{25}$

i $P(C \cup D)$ cannot be found as we do not know how many students own both a cat *and* a dog.

j $P(D \cup N) = P(D) + P(N)$ {since D and N are mutually exclusive}
$= \frac{8}{25} + \frac{4}{25}$
$= \frac{12}{25}$

8 $P(A \cup B) = P(A) + P(B)$ {A and B are mutually exclusive}
$P(A' \cup B') = P(A') + P(B')$ {A' and B' are mutually exclusive}

Now, $P(A') = 1 - P(A)$ and $P(B') = 1 - P(B)$

$$\begin{aligned}\therefore\ P(A' \cup B') &= 1 - P(A) + 1 - P(B)\\ &= 2 - (P(A) + P(B))\\ &= 2 - P(A \cup B)\end{aligned}$$

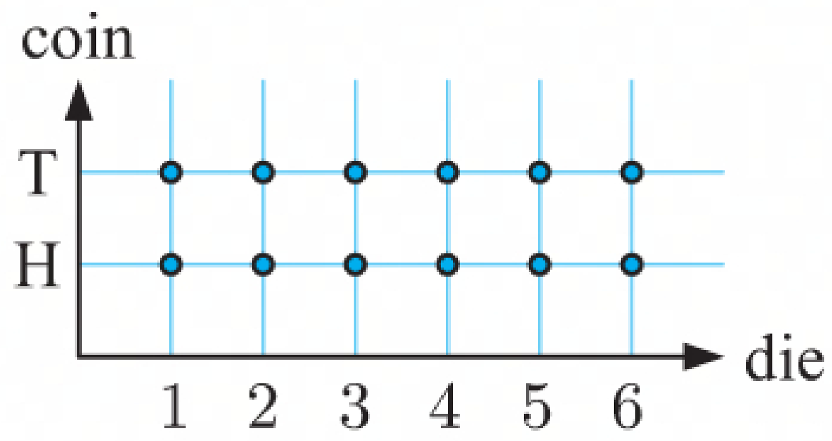
$\therefore\ P(A \cup B) + P(A' \cup B') = 2$

$\therefore\ P(A \cup B) = 1$ {since $P(X) \leqslant 1$ and $1 + 1 = 2$}

INVESTIGATION 4 INDEPENDENT EVENTS

1 **a** The outcomes of the coin toss and dice roll have no effect on each other. The events are independent.

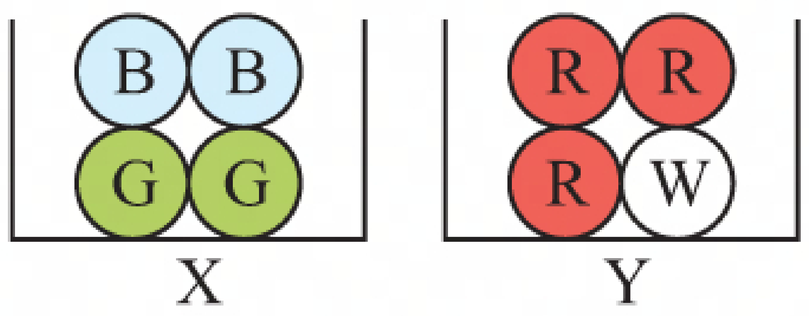
b



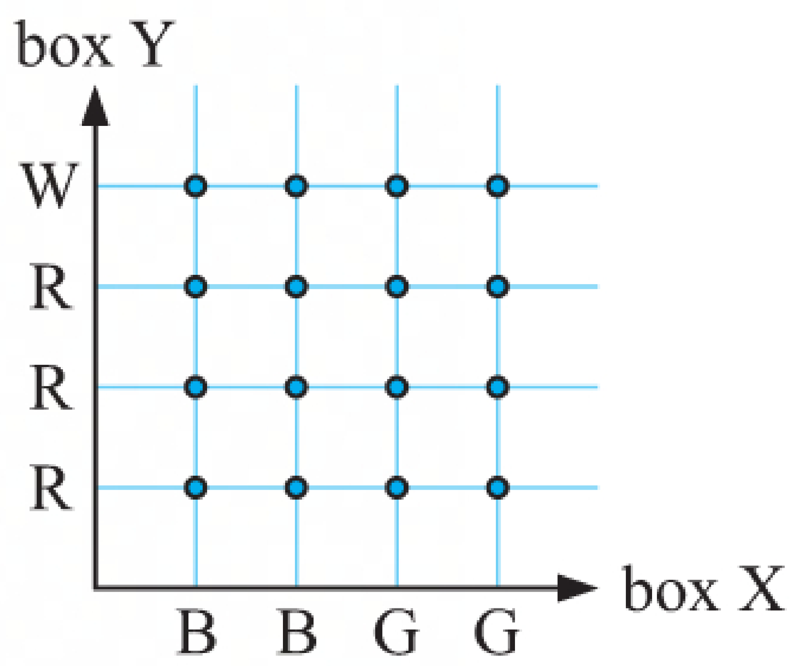
c

	A	B	$P(A)$	$P(B)$	$P(A \cap B)$
i	head	4	$\frac{1}{2}$	$\frac{1}{6}$	$\frac{1}{12}$
ii	head	odd number	$\frac{1}{2}$	$\frac{3}{6} = \frac{1}{2}$	$\frac{3}{12} = \frac{1}{4}$
iii	tail	number greater than 1	$\frac{1}{2}$	$\frac{5}{6}$	$\frac{5}{12}$
iv	tail	number less than 3	$\frac{1}{2}$	$\frac{2}{6} = \frac{1}{3}$	$\frac{2}{12} = \frac{1}{6}$

2 **a** The outcome of the draw from either box does not affect the outcome of the other. The events are independent.



b



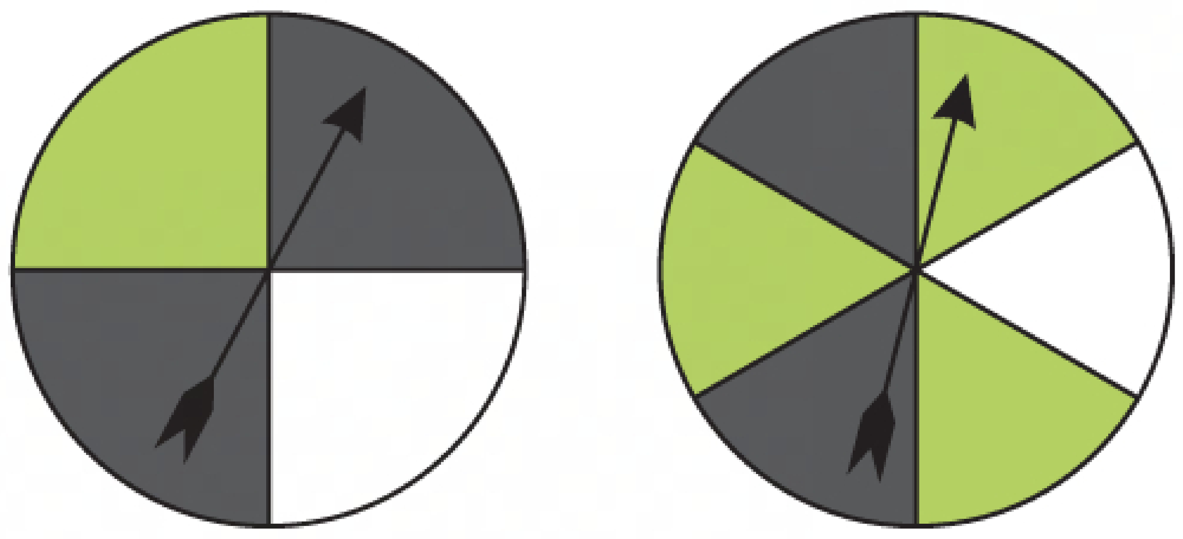
c

	A	B	$P(A)$	$P(B)$	$P(A \cap B)$
i	green from box X	red from box Y	$\frac{2}{4} = \frac{1}{2}$	$\frac{3}{4}$	$\frac{6}{16} = \frac{3}{8}$
ii	green from box X	white from box Y	$\frac{2}{4} = \frac{1}{2}$	$\frac{1}{4}$	$\frac{2}{16} = \frac{1}{8}$
iii	blue from box X	red from box Y	$\frac{2}{4} = \frac{1}{2}$	$\frac{3}{4}$	$\frac{6}{16} = \frac{3}{8}$
iv	blue from box X	white from box Y	$\frac{2}{4} = \frac{1}{2}$	$\frac{1}{4}$	$\frac{2}{16} = \frac{1}{8}$

3 From our answers to **1** and **2**, we can see that if A and B are independent events, then $P(A \cap B) = P(A) \times P(B)$.

EXERCISE 11G

1



a P(green with 1 and white with 2)
$= \text{P(green with 1)} \times \text{P(white with 2)}$ {events are independent}
$= \frac{1}{4} \times \frac{1}{6}$
$= \frac{1}{24}$

b P(black with both)
$= \text{P(black with 1)} \times \text{P(black with 2)}$ {events are independent}
$= \frac{2}{4} \times \frac{2}{6}$
$= \frac{4}{24}$
$= \frac{1}{6}$

2 **a** P(H, then H, then H)
$= \text{P}(\text{H} \cap \text{H} \cap \text{H})$
$= \text{P(H)} \times \text{P(H)} \times \text{P(H)}$ {events are independent}
$= \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2}$
$= \frac{1}{8}$

b P(T, then H, then T)
$= \text{P}(\text{T} \cap \text{H} \cap \text{T})$
$= \text{P(T)} \times \text{P(H)} \times \text{P(T)}$ {events are independent}
$= \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2}$
$= \frac{1}{8}$

3 Let A be the event of photocopier A malfunctioning, and B be the event of photocopier B malfunctioning.

a P(both malfunction)
$= \text{P}(A \cap B)$
$= \text{P}(A) \times \text{P}(B)$ {events are independent}
$= 0.08 \times 0.12$
$= 0.0096$

b P(both work effectively)
$= \text{P}(A' \cap B')$
$= \text{P}(A') \times \text{P}(B')$ {events are independent}
$= 0.92 \times 0.88$
$= 0.8096$

4 Let J be the event of Jiri hitting the target, and B be the event of Benita hitting the target.

a P(both hit target)
$= \text{P}(J \cap B)$
$= \text{P}(J) \times \text{P}(B)$ {events are independent}
$= 0.7 \times 0.8$
$= 0.56$

b P(both miss target)
$= \text{P}(J' \cap B')$
$= \text{P}(J') \times \text{P}(B')$ {events are independent}
$= 0.3 \times 0.2$
$= 0.06$

c P(Jiri hits but Benita misses)
$= \mathrm{P}(J \cap B')$
$= \mathrm{P}(J) \times \mathrm{P}(B')$
{events are independent}
$= 0.7 \times 0.2$
$= 0.14$

d P(Benita hits but Jiri misses)
$= \mathrm{P}(B \cap J')$
$= \mathrm{P}(B) \times \mathrm{P}(J')$
{events are independent}
$= 0.8 \times 0.3$
$= 0.24$

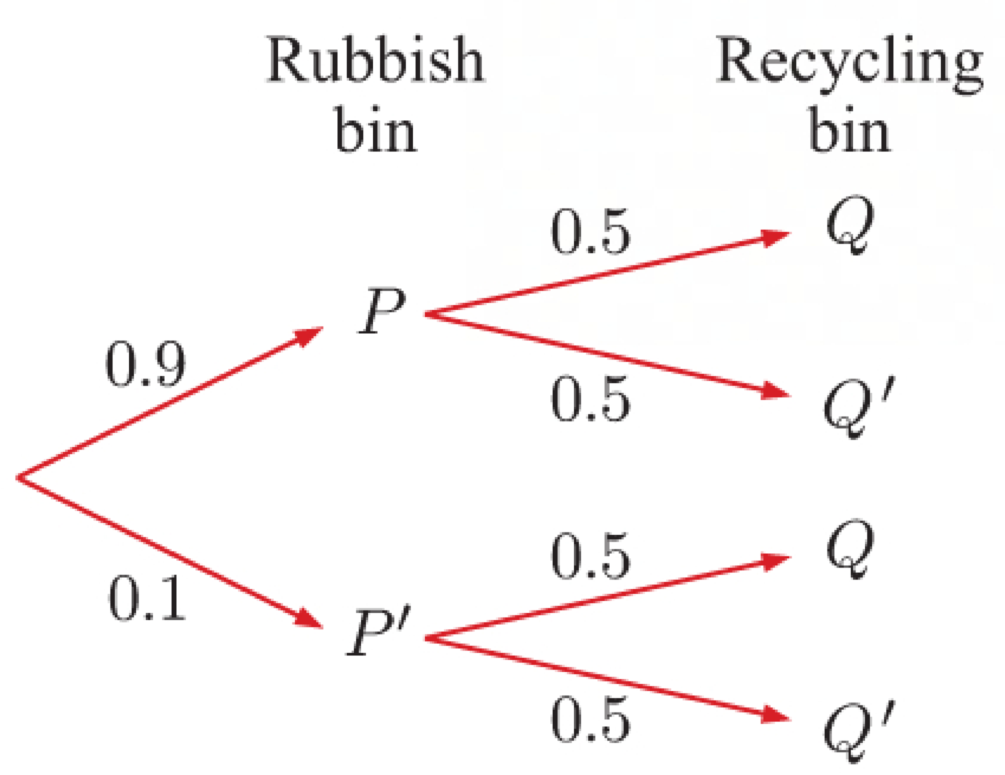
5 Let H be the event the archer hits the bullseye.
$\therefore \quad \mathrm{P}(H) = \frac{2}{5}, \quad \mathrm{P}(H') = \frac{3}{5}$

a P(3 hits)
$= \mathrm{P}(H \cap H \cap H)$
$= \mathrm{P}(H) \times \mathrm{P}(H) \times \mathrm{P}(H)$
{events are independent}
$= \frac{2}{5} \times \frac{2}{5} \times \frac{2}{5}$
$= \frac{8}{125}$

b P(2 hits then a miss)
$= \mathrm{P}(H \cap H \cap H')$
$= \mathrm{P}(H) \times \mathrm{P}(H) \times \mathrm{P}(H')$
{events are independent}
$= \frac{2}{5} \times \frac{2}{5} \times \frac{3}{5}$
$= \frac{12}{125}$

c P(all misses)
$= \mathrm{P}(H' \cap H' \cap H')$
$= \mathrm{P}(H') \times \mathrm{P}(H') \times \mathrm{P}(H')$
{events are independent}
$= \frac{3}{5} \times \frac{3}{5} \times \frac{3}{5}$
$= \frac{27}{125}$

6 **a** Let P be the event that the rubbish bin is full, and Q be the event that the recycling bin is full.

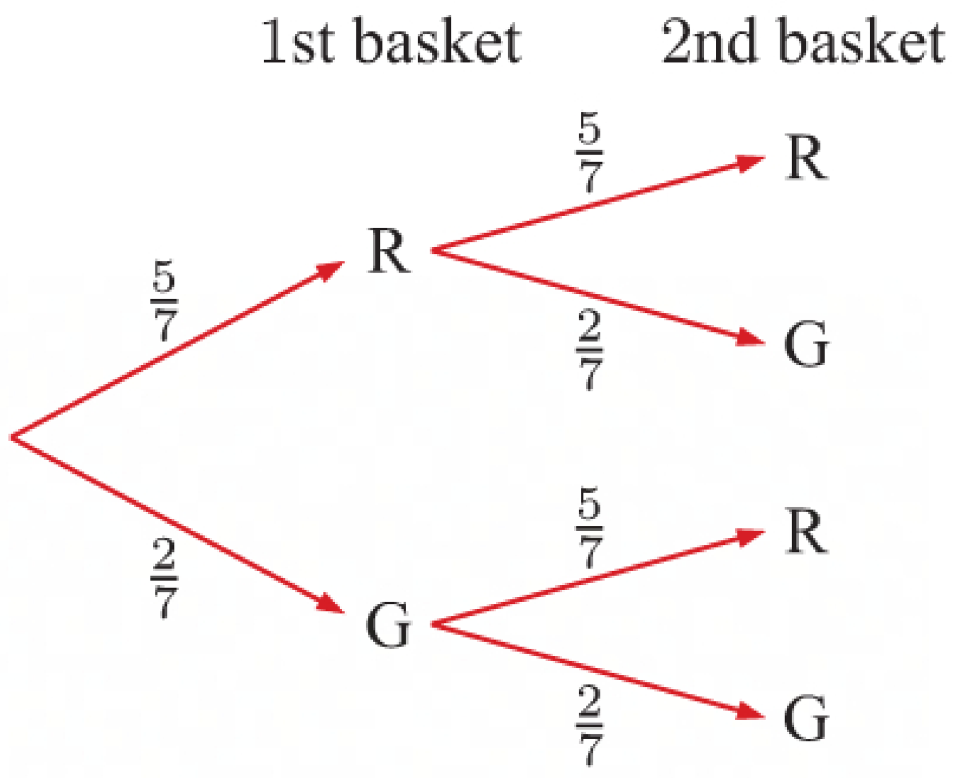


outcome	probability
P and Q	$0.9 \times 0.5 = 0.45$
P and Q'	$0.9 \times 0.5 = 0.45$
P' and Q	$0.1 \times 0.5 = 0.05$
P' and Q'	$0.1 \times 0.5 = 0.05$
total	1.00

b **i** P(both bins are full) $= \mathrm{P}(P \cap Q)$
$= 0.9 \times 0.5$
$= 0.45$

ii P(recycling bin is full but rubbish bin is not) $= \mathrm{P}(P' \cap Q)$
$= 0.1 \times 0.5$
$= 0.05$

7 **a** Let R be the event that Celia chooses a red apple, and G be the event that Celia chooses a green apple.



outcome	probability
R and R	$\frac{5}{7} \times \frac{5}{7} = \frac{25}{49}$
R and G	$\frac{5}{7} \times \frac{2}{7} = \frac{10}{49}$
G and R	$\frac{2}{7} \times \frac{5}{7} = \frac{10}{49}$
G and G	$\frac{2}{7} \times \frac{2}{7} = \frac{4}{49}$
total	$\frac{49}{49} = 1$

b **i** P(Celia chooses two red apples)

$= P(R \cap R)$

$= \frac{5}{7} \times \frac{5}{7}$

$= \frac{25}{49}$

ii P(Celia chooses one red and one green apple)

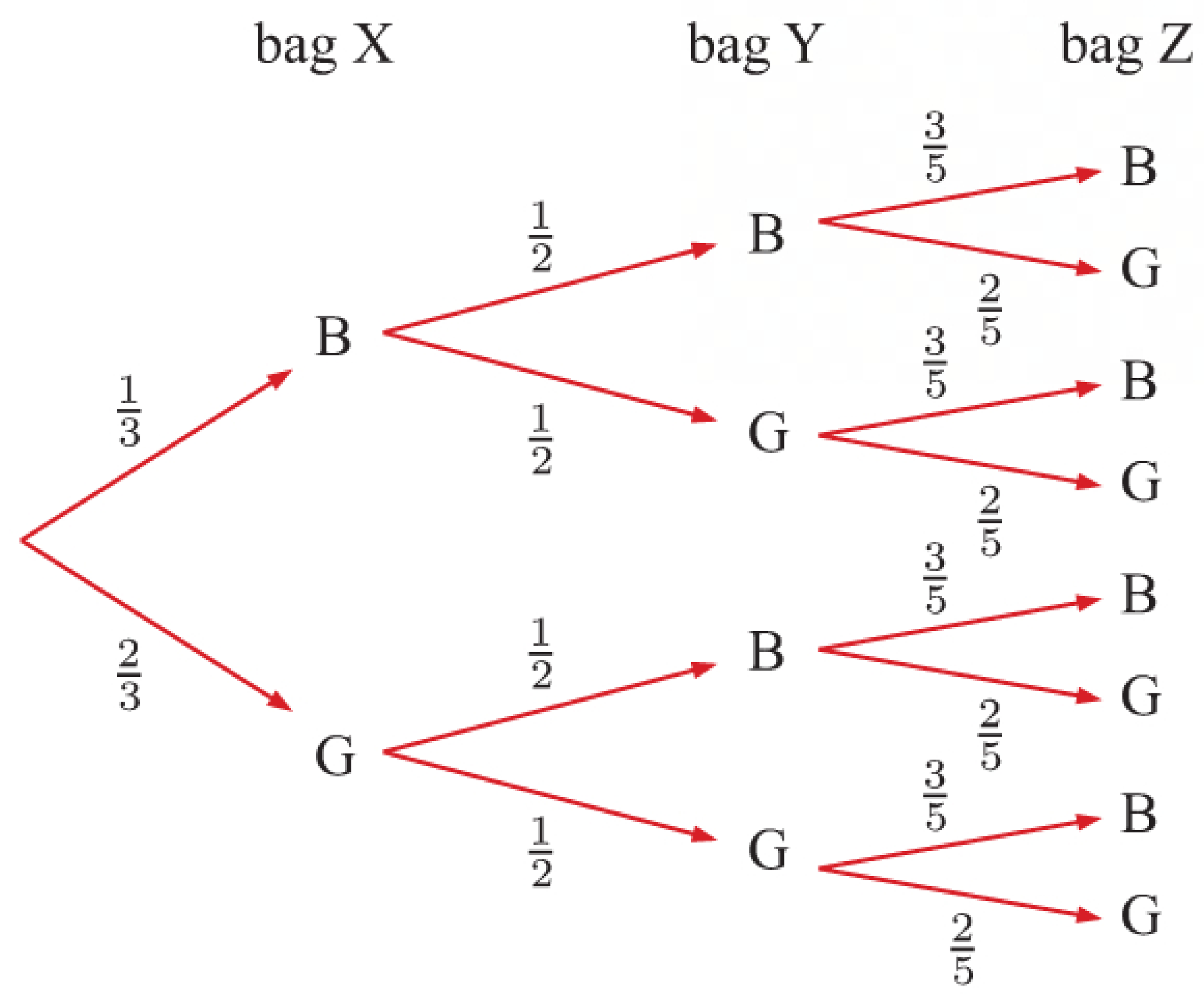
$= P(R \cap G) + P(G \cap R)$

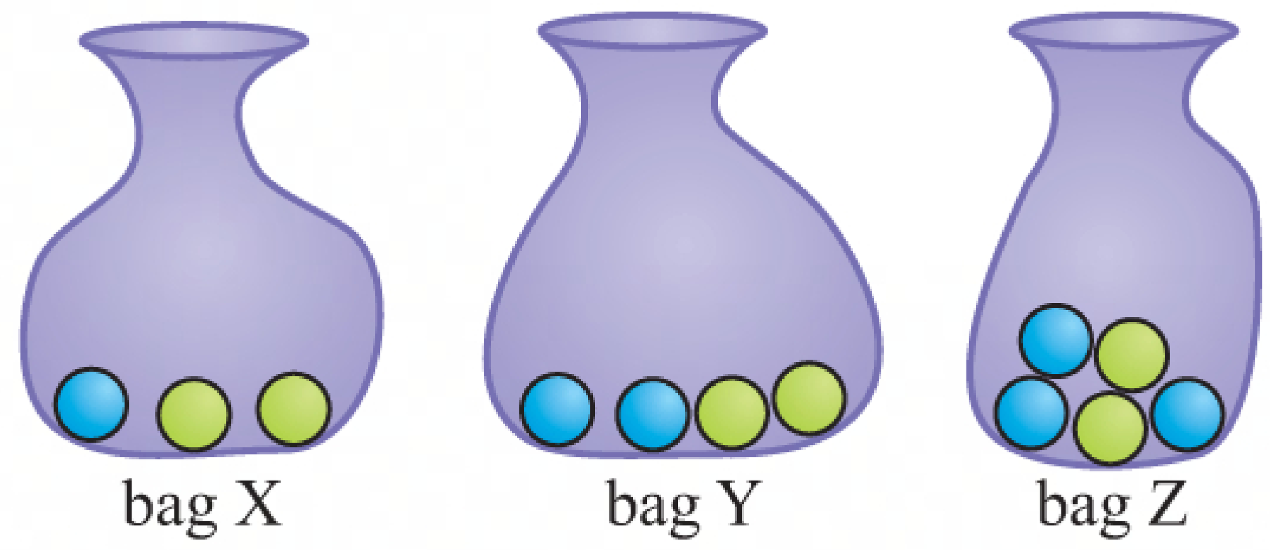
$= \frac{5}{7} \times \frac{2}{7} + \frac{2}{7} \times \frac{5}{7}$

$= \frac{10}{49} + \frac{10}{49}$

$= \frac{20}{49}$

8 **a** Let B represent a blue ball being drawn, and G represent a green ball being drawn.





b **i** P(3 blue balls are drawn) $= P(B \cap B \cap B)$

$= \frac{1}{3} \times \frac{1}{2} \times \frac{3}{5}$

$= \frac{3}{30}$

$= \frac{1}{10}$

ii P(green balls are drawn from bags Y and Z) $= P(B \cap G \cap G) + P(G \cap G \cap G)$

$= \frac{1}{3} \times \frac{1}{2} \times \frac{2}{5} + \frac{2}{3} \times \frac{1}{2} \times \frac{2}{5}$

$= \frac{2}{30} + \frac{4}{30}$

$= \frac{6}{30}$

$= \frac{1}{5}$

iii P(at least one blue ball is drawn) $= 1 - $ P(no blue balls are drawn)

$= 1 - P(G \cap G \cap G)$

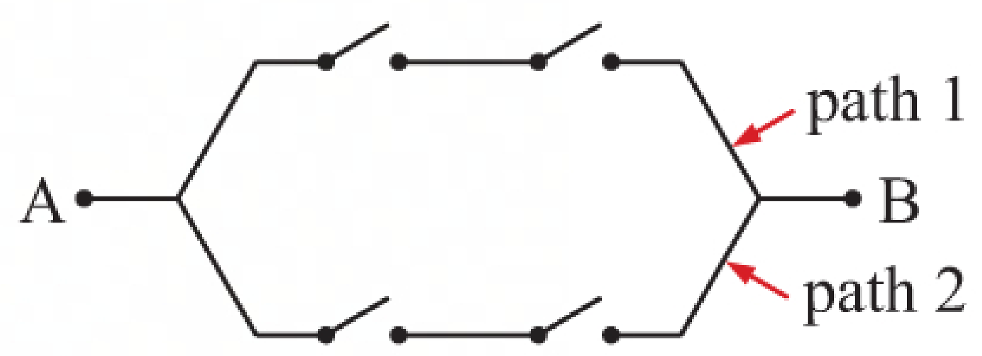
$= 1 - \frac{2}{3} \times \frac{1}{2} \times \frac{2}{5}$

$= 1 - \frac{4}{30}$

$= \frac{26}{30}$

$= \frac{13}{15}$

9



a In order for the current to flow from A to B, both switches on the top of the network (path 1), or both switches on the bottom of the network (path 2), need to be closed.

P(current flows from A to B)
= P(path 1 is closed ∪ path 2 is closed)
= P(path 1 is closed) + P(path 2 is closed) − P(both paths 1 *and* 2 are closed)
$= p \times p + p \times p - p \times p \times p \times p$
$= p^2 + p^2 - p^4$
$= 2p^2 - p^4$

b We need to solve $2p^2 - p^4 \geqslant \frac{1}{2}$

$$\begin{aligned}
\text{Let}\quad X &= p^2\\
\therefore\quad 2X - X^2 &\geqslant \tfrac{1}{2}\\
\therefore\quad X^2 - 2X &\leqslant -\tfrac{1}{2}\\
\therefore\quad X^2 - 2X + (-1)^2 &\leqslant -\tfrac{1}{2} + (-1)^2 \qquad \{\text{completing the square}\}\\
\therefore\quad (X-1)^2 &\leqslant \tfrac{1}{2}\\
\therefore\quad -\tfrac{1}{\sqrt{2}} \leqslant X - 1 &\leqslant \tfrac{1}{\sqrt{2}}\\
\therefore\quad 1 - \tfrac{1}{\sqrt{2}} \leqslant X &\leqslant 1 + \tfrac{1}{\sqrt{2}}\\
\therefore\quad 1 - \tfrac{1}{\sqrt{2}} \leqslant p^2 &\leqslant 1 + \tfrac{1}{\sqrt{2}}\\
\therefore\quad \sqrt{1 - \tfrac{1}{\sqrt{2}}} \leqslant p &\leqslant \sqrt{1 + \tfrac{1}{\sqrt{2}}} \qquad \{\text{as } p \geqslant 0\}
\end{aligned}$$

$\therefore\ p = \sqrt{1 - \frac{1}{\sqrt{2}}} \approx 0.541$ is the least value for which $2p^2 - p^4 \geqslant \frac{1}{2}$

10 Kane should choose to play Penny - Quentin - Penny
To win 2 matches in a row, Kane must win the middle match, so he should play against the weaker player in this match.

EXERCISE 11H

1 **a** P(both are red) = P(first is red ∩ second is red)

= P(first is red) × P(second is red given that the first is red)

$= \frac{7}{10} \times \frac{6}{9}$

$= \frac{42}{90}$

$= \frac{7}{15}$

b P(first is green and second is red)

$= \text{P(first is green)} \times \text{P(second is red given that the first is green)}$

$= \frac{3}{10} \times \frac{7}{9}$

$= \frac{21}{90}$

$= \frac{7}{30}$

2 **a** $\text{P(both are blue)} = \text{P(first is blue and second is blue)}$

$= \text{P(first is blue)} \times \text{P(second is blue given that the first is blue)}$

$= \frac{4}{10} \times \frac{3}{9}$

$= \frac{12}{90}$

$= \frac{2}{15}$

b P(first is blue and second is white)

$= \text{P(first is blue)} \times \text{P(second is white given that the first is blue)}$

$= \frac{4}{10} \times \frac{6}{9}$

$= \frac{24}{90}$

$= \frac{4}{15}$

3 **a** P(all strawberry creams)

$= \text{P(1st is a strawberry cream} \cap \text{2nd is a strawberry cream} \cap \text{3rd is a strawberry cream)}$

$= \frac{8}{12} \times \frac{7}{11} \times \frac{6}{10}$

$= \frac{336}{1320}$

$= \frac{14}{55}$

b P(none are strawberry creams)

$= \text{P(1st is not a strawberry cream} \cap \text{2nd is not a strawberry cream}$

$\cap \text{ 3rd is not a strawberry cream)}$

$= \frac{4}{12} \times \frac{3}{11} \times \frac{2}{10}$

$= \frac{24}{1320}$

$= \frac{1}{55}$

4 **a** $\text{P(wins first prize)} = \frac{3}{100}$

b P(wins 1st and 2nd prize)

$= \text{P(wins 1st prize)} \times \text{P(wins 2nd prize given that he won 1st prize)}$

$= \frac{3}{100} \times \frac{2}{99}$

$\approx 0.000\,606$

c $\text{P(wins all 3 prizes)} = \text{P(wins 1st prize} \cap \text{wins 2nd prize} \cap \text{wins 3rd prize)}$

$= \frac{3}{100} \times \frac{2}{99} \times \frac{1}{98}$

$\approx 0.000\,006\,18$

d P(wins none of the prizes)

$= \text{P(does not win 1st prize} \cap \text{does not win 2nd prize} \cap \text{does not win 3rd prize)}$

$= \frac{97}{100} \times \frac{96}{99} \times \frac{95}{98}$

≈ 0.912

5 **a** P(does not contain captain)

$= \text{P(1st player selected is not the captain} \cap \text{2nd player selected is not the captain}$
$\cap \text{3rd player selected is not the captain)}$

$= \frac{6}{7} \times \frac{5}{6} \times \frac{4}{5}$

$= \frac{120}{210}$

$= \frac{4}{7}$

b P(does not contain captain or vice captain)

$= \text{P(1st player selected is neither the captain nor vice captain}$
$\cap \text{2nd player selected is neither the captain nor vice captain}$
$\cap \text{3rd player selected is neither the captain nor vice captain)}$

$= \frac{5}{7} \times \frac{4}{6} \times \frac{3}{5}$

$= \frac{60}{210}$

$= \frac{2}{7}$

6 **a** $\text{P(two boys)} = \text{P(first selected is a boy} \cap \text{second selected is a boy)}$

$= \frac{5}{7} \times \frac{4}{6}$

$= \frac{20}{42}$

$= \frac{10}{21}$

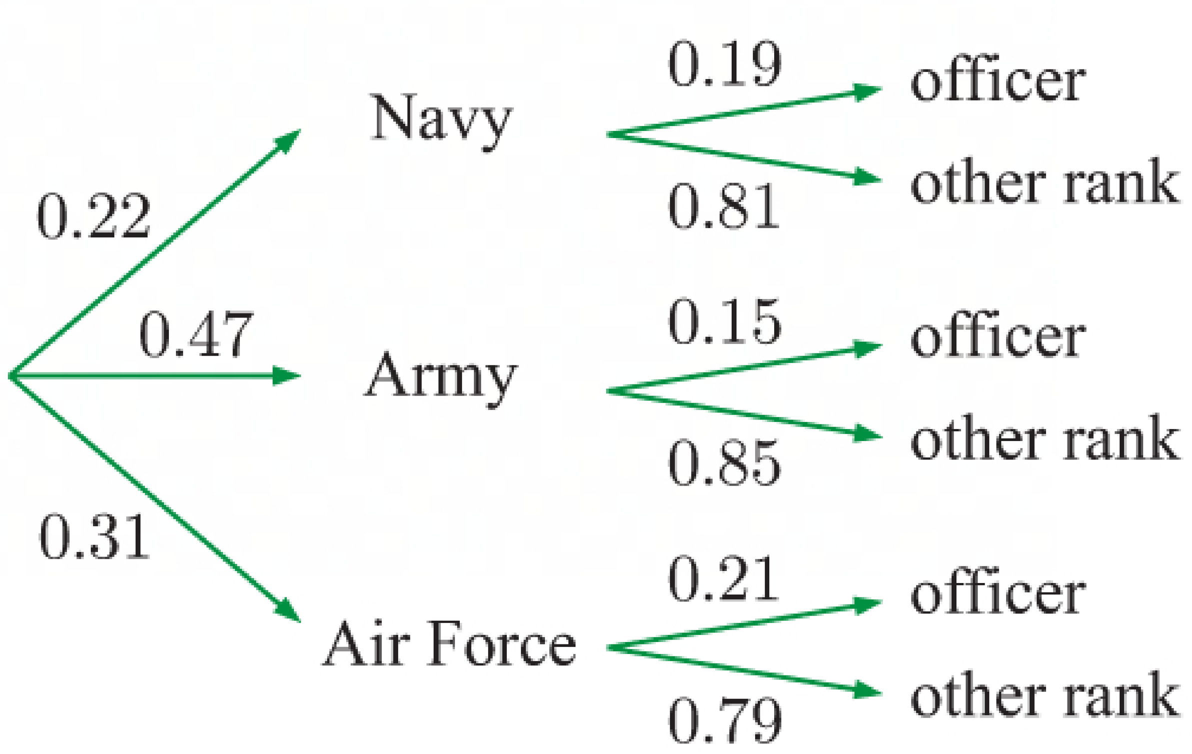
b $\text{P(eldest two students)} = \text{P(either of the two eldest students} \cap \text{the remaining eldest student)}$

$= \frac{2}{7} \times \frac{1}{6}$

$= \frac{2}{42}$

$= \frac{1}{21}$

7 **a**



b **i** P(officer)

$= P(N \cap O) + P(A \cap O) + P(AF \cap O)$ {where N represents Navy,
A represents Army,
AF represents Air Force, and
O represents officer}

$= 0.22 \times 0.19 + 0.47 \times 0.15 + 0.31 \times 0.21$

$= 0.1774$

ii P(not an officer in the navy)

$= 1 - P(\text{officer in the navy})$

$= 1 - P(N \cap O)$

$= 1 - 0.22 \times 0.19$

$= 0.9582$

iii P(not an army or air force officer)

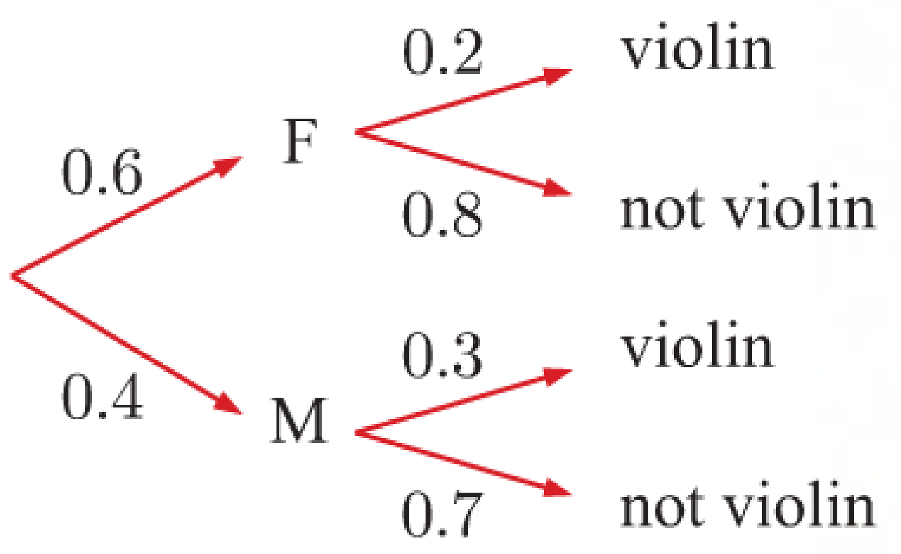
$= 1 - P(\text{army or air force officer})$

$= 1 - (P(A \cap O) + P(AF \cap O))$

$= 1 - (0.47 \times 0.15 + 0.31 \times 0.21)$

$= 0.8644$

8



a P(male and does not play violin)

$= P(M \cap \text{not violin})$

$= 0.4 \times 0.7$

$= 0.28$

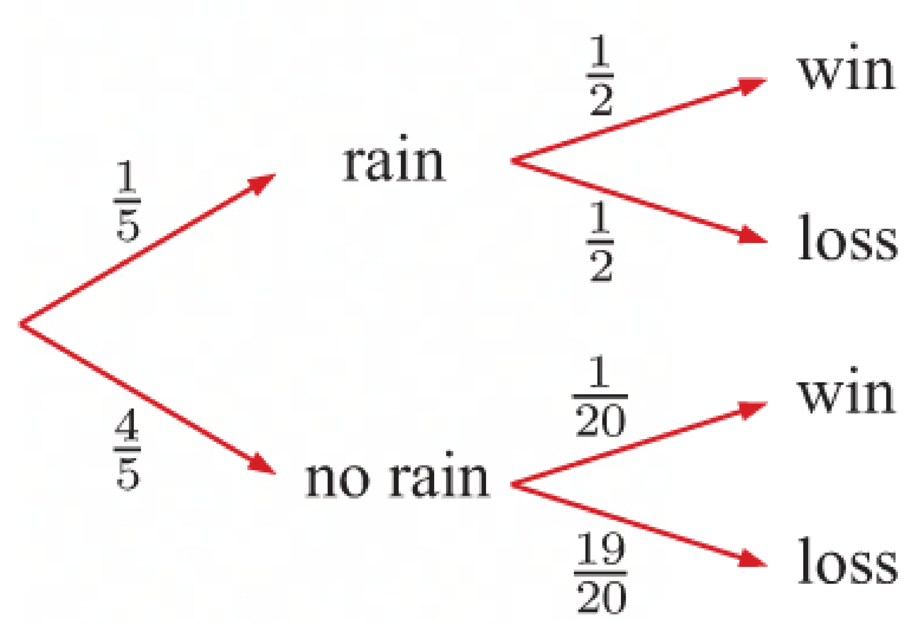
b P(plays the violin)

$= P(F \cap \text{violin}) + P(M \cap \text{violin})$

$= 0.6 \times 0.2 + 0.4 \times 0.3$

$= 0.24$

9 **a**



b P(Mudlark wins)

$= P(\text{rain} \cap \text{win}) + P(\text{no rain} \cap \text{win})$

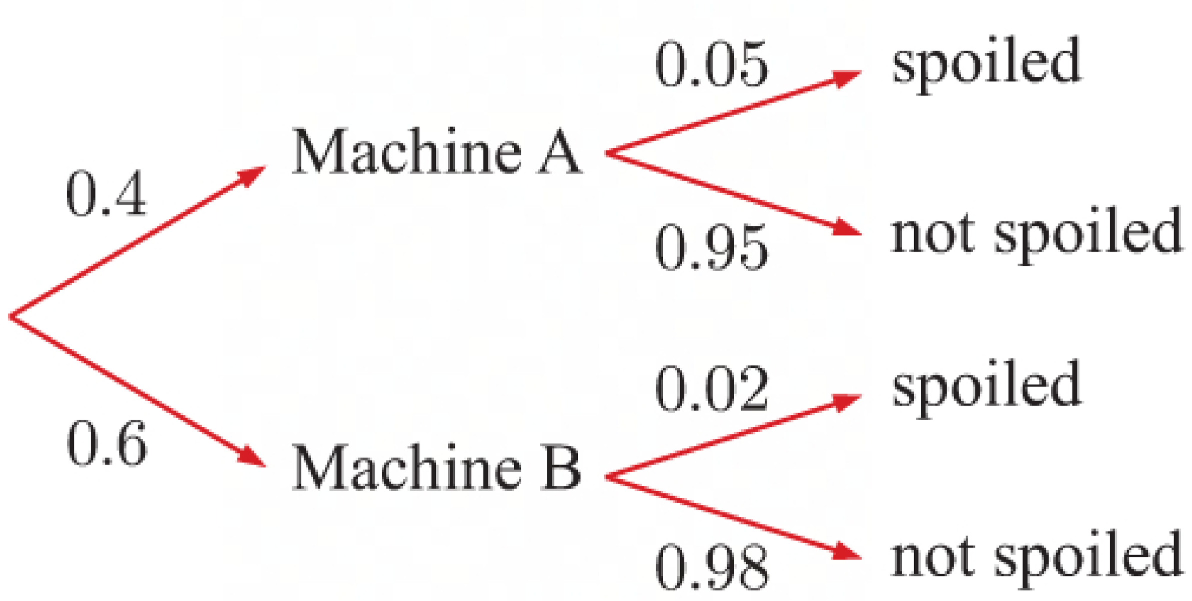
$= \frac{1}{5} \times \frac{1}{2} + \frac{4}{5} \times \frac{1}{20}$

$= \frac{1}{10} + \frac{4}{100}$

$= \frac{14}{100}$

$= \frac{7}{50}$

10



P(next bottle is spoiled)

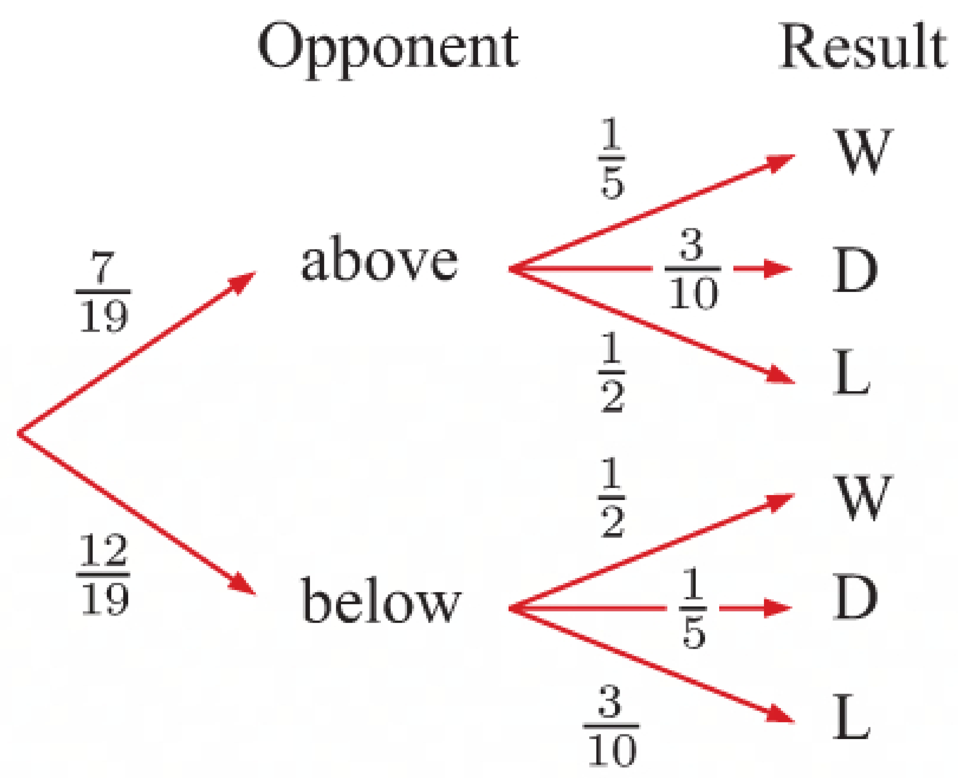
$= P(\text{from A} \cap \text{spoiled}) + P(\text{from B} \cap \text{spoiled})$

$= 0.4 \times 0.05 + 0.6 \times 0.02$

$= 0.020 + 0.012$

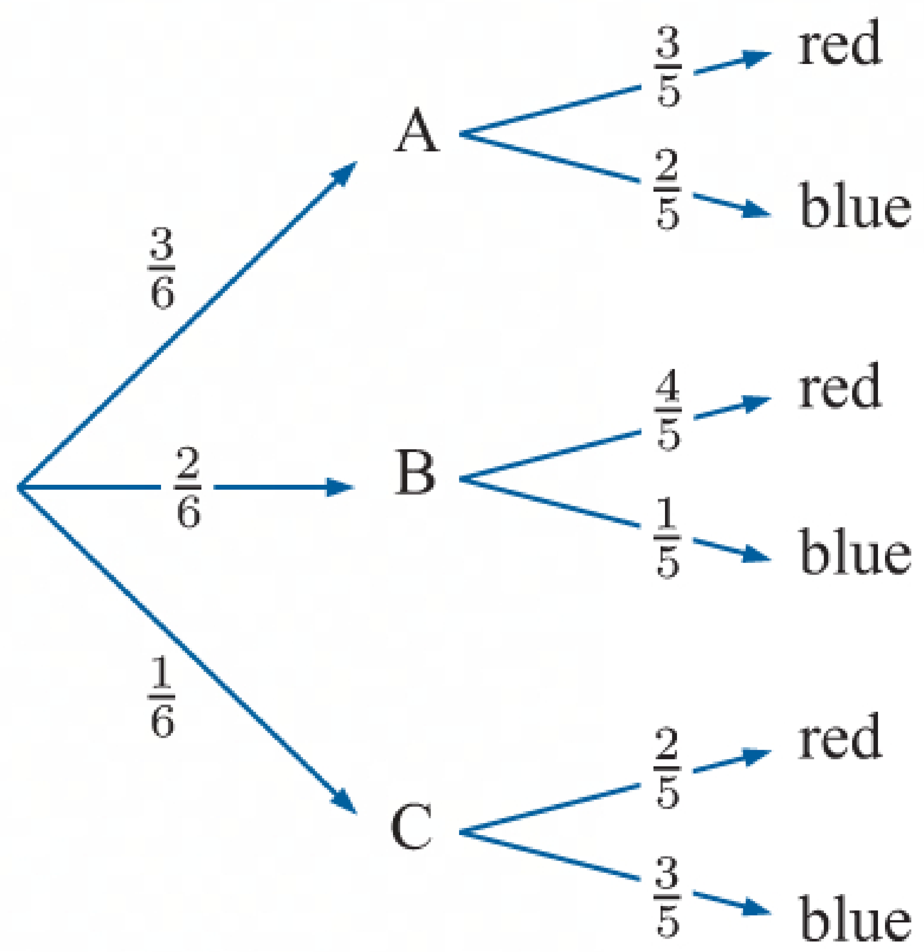
$= 0.032$

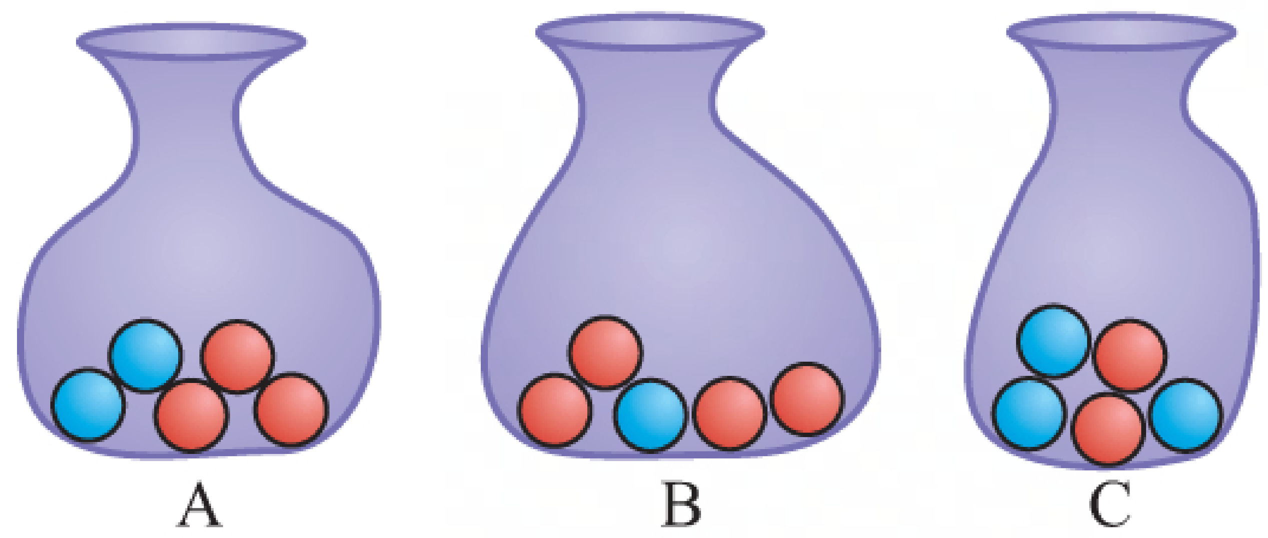
11 Tottenham is in 8th place, so there are 7 teams above Tottenham and 12 teams below Tottenham.



$$\begin{aligned}\therefore\ \text{P(draw)} &= \text{P(above} \cap \text{draw)} + \text{P(below} \cap \text{draw)}\\ &= \tfrac{7}{19} \times \tfrac{3}{10} + \tfrac{12}{19} \times \tfrac{1}{5}\\ &= \tfrac{21}{190} + \tfrac{24}{190}\\ &= \tfrac{45}{190}\\ &= \tfrac{9}{38}\end{aligned}$$

12





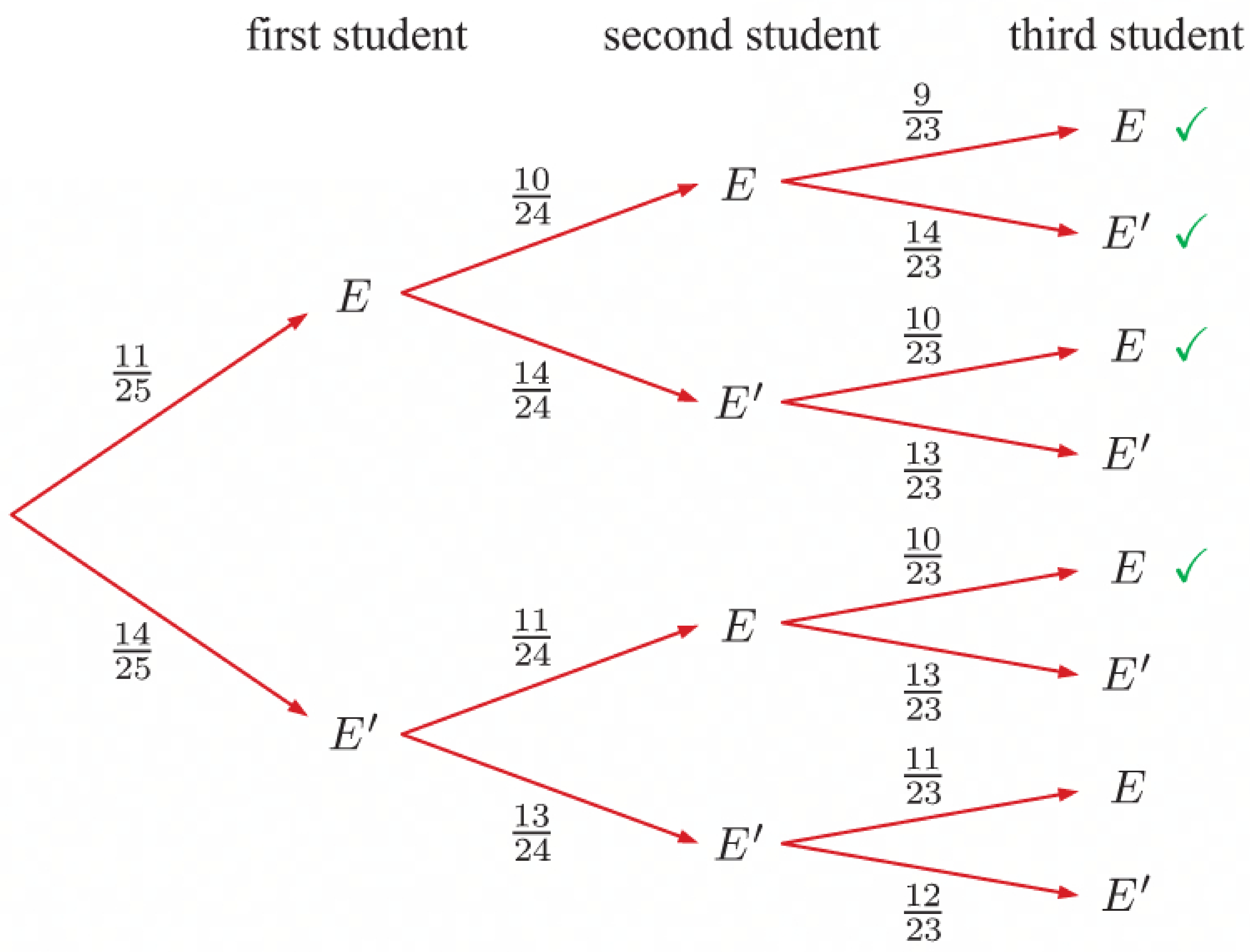
a $\text{P(blue)} = \text{P(A} \cap \text{blue)} + \text{P(B} \cap \text{blue)} + \text{P(C} \cap \text{blue)}$

$$\begin{aligned}&= \tfrac{3}{6} \times \tfrac{2}{5} + \tfrac{2}{6} \times \tfrac{1}{5} + \tfrac{1}{6} \times \tfrac{3}{5}\\ &= \frac{6+2+3}{30}\\ &= \tfrac{11}{30}\end{aligned}$$

b $\text{P(red)} = 1 - \text{P(blue)}$

$$\begin{aligned}&= 1 - \tfrac{11}{30}\\ &= \tfrac{19}{30}\end{aligned}$$

13 Let E represent a student who participates in extra-curricular activities, and E' represent a student who does not participate in extra-curricular activities.



P(at least two students selected also participate in extra-curricular activities)

$= \text{P}(EEE) + \text{P}(EEE') + \text{P}(EE'E) + \text{P}(E'EE)$

$= \left(\frac{11}{25} \times \frac{10}{24} \times \frac{9}{23}\right) + \left(\frac{11}{25} \times \frac{10}{24} \times \frac{14}{23}\right) + \left(\frac{11}{25} \times \frac{14}{24} \times \frac{10}{23}\right) + \left(\frac{14}{25} \times \frac{11}{24} \times \frac{10}{23}\right)$

{branches marked ✓}

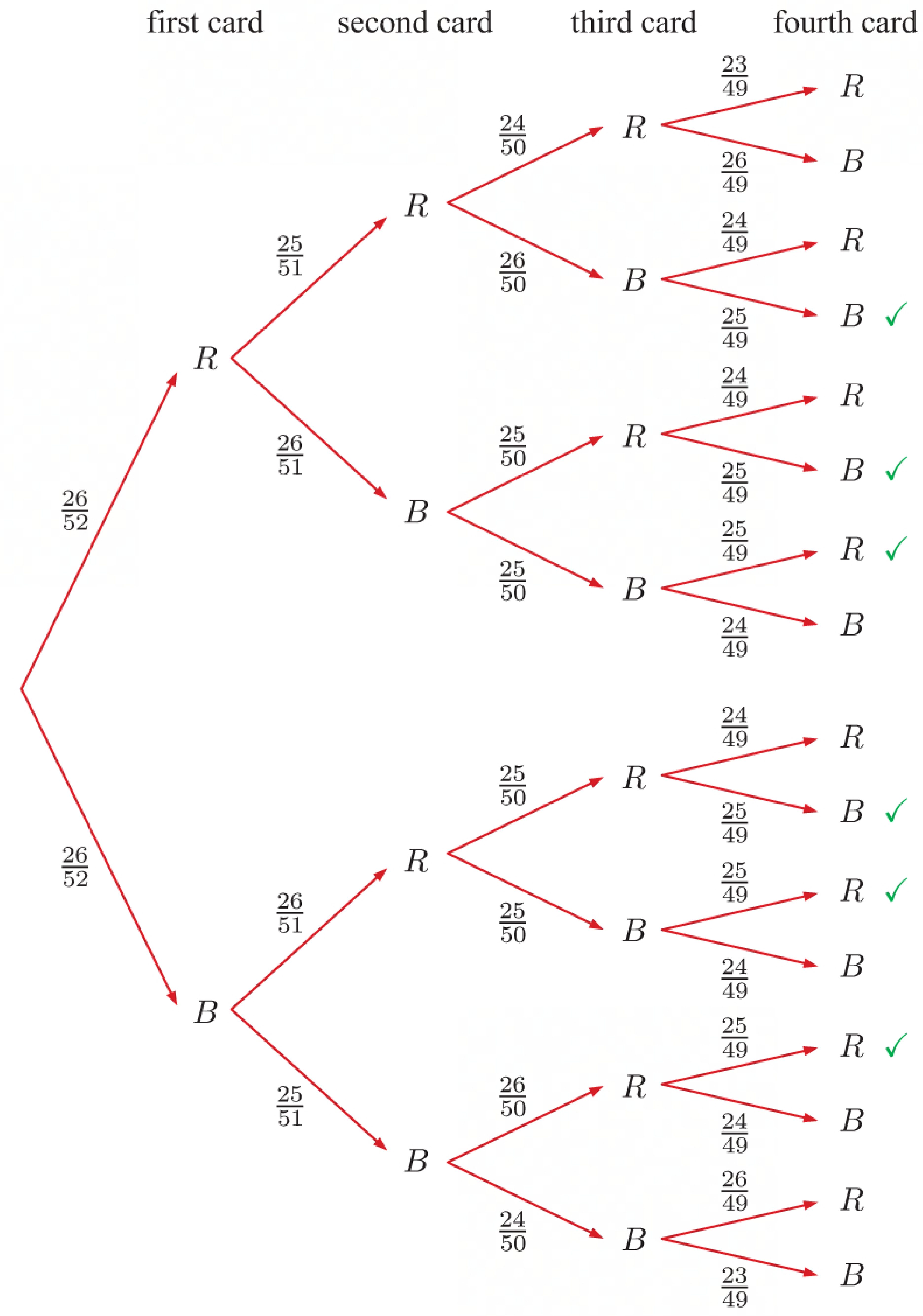
$= \dfrac{990 + 1540 + 1540 + 1540}{13\,800}$

$= \dfrac{5610}{13\,800}$

$= \dfrac{187}{460}$

≈ 0.407

14 Let R represent drawing a red card and B represent drawing a black card.



a P(two red cards are drawn)

$$= \mathrm{P}(RRBB) + \mathrm{P}(RBRB) + \mathrm{P}(RBBR) + \mathrm{P}(BRRB) + \mathrm{P}(BRBR) + \mathrm{P}(BBRR)$$

$$= \left(\tfrac{26}{52} \times \tfrac{25}{51} \times \tfrac{26}{50} \times \tfrac{25}{49}\right) + \left(\tfrac{26}{52} \times \tfrac{26}{51} \times \tfrac{25}{50} \times \tfrac{25}{49}\right) + \left(\tfrac{26}{52} \times \tfrac{26}{51} \times \tfrac{25}{50} \times \tfrac{25}{49}\right)$$

$$+ \left(\tfrac{26}{52} \times \tfrac{26}{51} \times \tfrac{25}{50} \times \tfrac{25}{49}\right) + \left(\tfrac{26}{52} \times \tfrac{26}{51} \times \tfrac{25}{50} \times \tfrac{25}{49}\right) + \left(\tfrac{26}{52} \times \tfrac{25}{51} \times \tfrac{26}{50} \times \tfrac{25}{49}\right)$$

{branches marked ✓}

$$= \frac{16\,250 + 16\,250 + 16\,250 + 16\,250 + 16\,250 + 16\,250}{249\,900}$$

$$= \frac{97\,500}{249\,900}$$

$$= \frac{325}{833}$$

$$\approx 0.390$$

b $\text{P}(\text{at least one black card is drawn}) = 1 - \text{P}(\text{no black cards are drawn})$

$$= 1 - \text{P}(RRRR)$$

$$= 1 - \tfrac{26}{52} \times \tfrac{25}{51} \times \tfrac{24}{50} \times \tfrac{23}{49}$$

$$= 1 - \tfrac{13\,800}{249\,900}$$

$$= \tfrac{236\,100}{249\,900}$$

$$= \tfrac{787}{833}$$

$$\approx 0.945$$

ACTIVITY 1 PÓLYA'S URN

1 **a** There are initially 3 black balls and 4 white balls in the urn.
If a black ball is drawn first, then there are 4 black balls and 4 white balls in the urn for the second draw.

$$\therefore \quad \text{P}(\text{2 black balls in succession}) = \tfrac{3}{7} \times \tfrac{4}{8} = \tfrac{3}{14}$$

b If the first two draws are both black, then there are 5 black balls and 4 white balls in the urn for the third draw.

$\therefore$ P(3 black balls in succession)

$$= \text{P}(\text{2 black balls in first two draws}) \times \text{P}(\text{black ball in third draw})$$

$$= \tfrac{3}{14} \times \tfrac{5}{9} \quad \{\text{using } \textbf{a}\}$$

$$= \tfrac{5}{42}$$

2 **a** The proportion of black balls in the urn approaches a steady value but this value differs with each repetition. The proportion of black balls varies randomly, symmetric about the value 0.5.

b When there are initially far *more* black balls than white balls in the urn, the proportion of black balls consistently approaches a value close to 1. It is initially more likely to draw and add black balls, making it even more likely to draw and add black balls as more balls are drawn.
When there are initially far *fewer* black balls than white balls in the urn, the proportion of black balls consistently approaches a value close to 0. It is initially more likely to draw and add white balls, making it even more likely to draw and add white balls as more balls are drawn.

EXERCISE 11I

1 **a** $$\text{P}(A \mid B) = \frac{\text{P}(A \cap B)}{\text{P}(B)} = \frac{0.1}{0.4} = \tfrac{1}{4}$$

b $$\text{P}(A \cup B) = \text{P}(A) + \text{P}(B) - \text{P}(A \cap B)$$

$$\therefore \quad 0.5 = 0.3 + 0.4 - \text{P}(A \cap B)$$

$$\therefore \quad \text{P}(A \cap B) = 0.2$$

Now $$\text{P}(A \mid B) = \frac{\text{P}(A \cap B)}{\text{P}(B)} = \frac{0.2}{0.4} = \tfrac{1}{2}$$

c $\text{P}(A \mid B) = \dfrac{\text{P}(A \cap B)}{\text{P}(B)}$

$= \dfrac{0}{\text{P}(B)}$ {since A and B are mutually exclusive}

$= 0$

2 Let C represent a cloudy day and R represent a rainy day.

$\text{P}(C) = 0.4, \quad \text{P}(C \cap R) = 0.2$

$$\begin{aligned}\text{P}(R \mid C) &= \frac{\text{P}(R \cap C)}{\text{P}(C)}\\ &= \frac{0.2}{0.4}\\ &= \tfrac{1}{2}\end{aligned}$$

The probability that it will be rainy on a day when it is cloudy is $\frac{1}{2}$.

3 **a** Let M represent a student who studies Mathematics, and P represent a student who studies Physics.

$n(M) = 40, \quad n(P) = 32, \quad n(M' \cap P') = 0, \quad n(U) = 50$

$n(U) = n(M \cup P) + n(M' \cap P')$

$\therefore \; 50 = n(M \cup P) + 0$

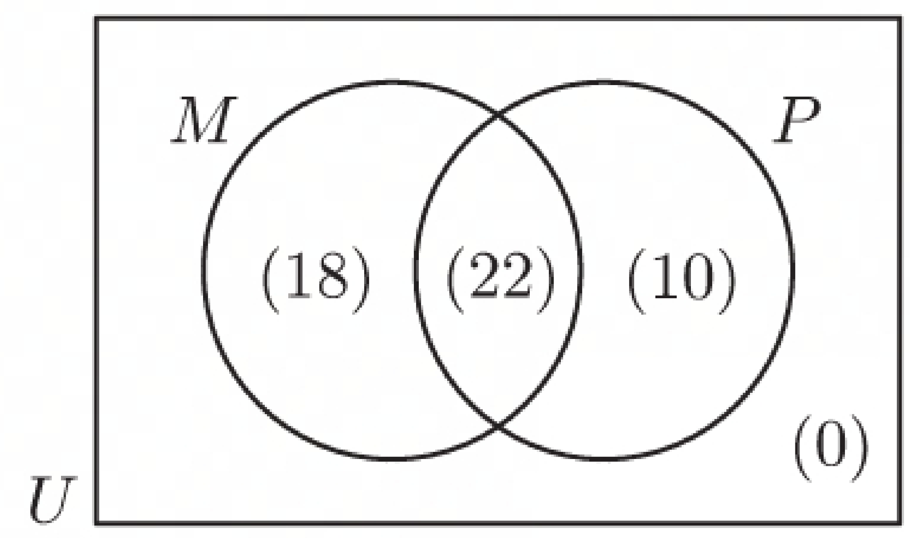
$\therefore \; n(M \cup P) = 50$

$n(M \cup P) = n(M) + n(P) - n(M \cap P)$

$\therefore \; 50 = 40 + 32 - n(M \cap P)$

$\therefore \; n(M \cap P) = 22$

$\therefore \; n(M \cap P') = 40 - 22 = 18$ and $n(M' \cap P) = 32 - 22 = 10$



So, 22 students study both subjects.

b **i** $\text{P}(M \cap P') = \dfrac{18}{50} = \dfrac{9}{25}$

ii $$\begin{aligned}\text{P}(P \mid M) &= \frac{\text{P}(P \cap M)}{\text{P}(M)}\\ &= \frac{\frac{22}{50}}{\frac{40}{50}}\\ &= \frac{22}{40}\\ &= \tfrac{11}{20}\end{aligned}$$

4 **a** Let D represent a boy who has dark hair, and B represent a boy who has brown eyes.

$n(D) = 23, \quad n(B) = 18, \quad n(B \cup D) = 26, \quad n(U) = 40$

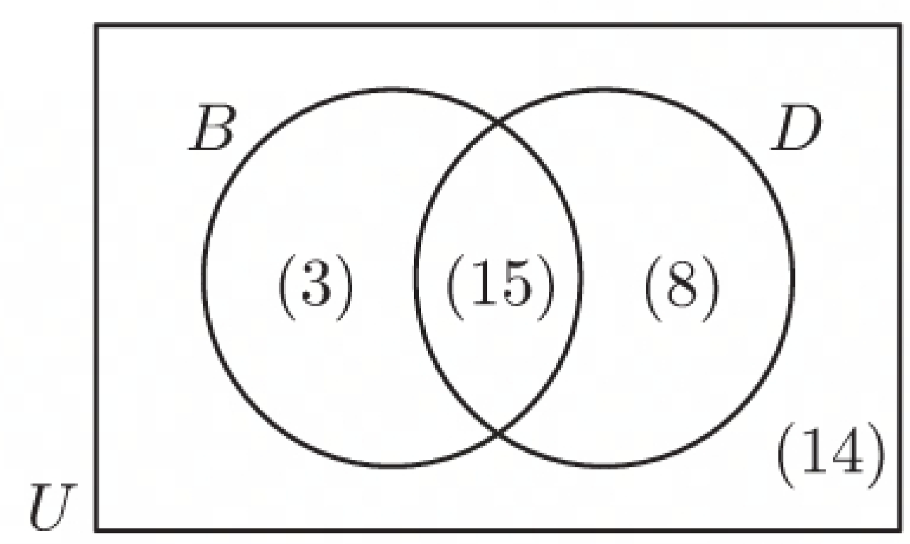
$$\begin{aligned}n(B' \cap D') &= n(U) - n(B \cup D)\\ &= 40 - 26\\ &= 14\end{aligned}$$

$n(B \cup D) = n(D) + n(B) - n(B \cap D)$

$\therefore \; 26 = 23 + 18 - n(B \cap D)$

$\therefore \; n(B \cap D) = 15$

$\therefore \; n(B \cap D') = 18 - 15 = 3$ and $n(B' \cap D) = 23 - 15 = 8$



b **i** P(dark hair and brown eyes)

$= \text{P}(B \cap D)$

$= \dfrac{15}{40}$

$= \frac{3}{8}$

ii P(brown eyes given dark hair)

$= \text{P}(B \mid D)$

$= \dfrac{\text{P}(B \cap D)}{\text{P}(D)}$

$= \dfrac{\frac{15}{40}}{\frac{23}{40}}$

$= \frac{15}{23}$

5 **a** Let S represent a hiker who was sunburnt and B represent a hiker who was bitten by ants.

$n(S) = 23, \quad n(B) = 22, \quad n(S \cap B) = 5, \quad n(U) = 50$

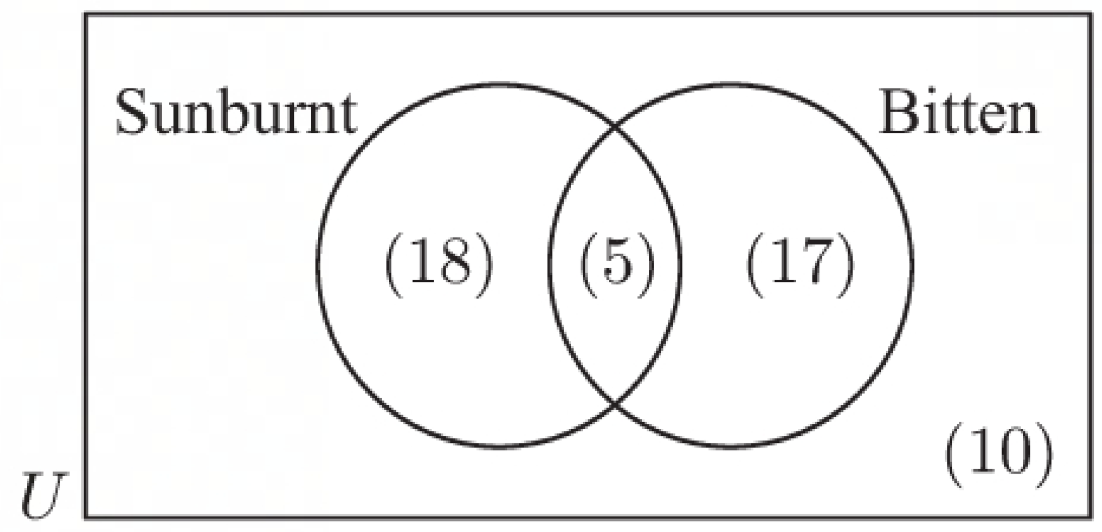
$\therefore \quad n(S \cap B') = 23 - 5 = 18$

and $\quad n(S' \cap B) = 22 - 5 = 17$

$n(S' \cap B') = n(U) - n(S \cup B)$

$= 50 - 5 - 18 - 17$

$= 10$



b **i** P(hiker avoided being bitten) $= 1 - \text{P}(B)$

$= 1 - \dfrac{22}{50}$

$= \dfrac{28}{50}$

$= \frac{14}{25}$

ii P(hiker was bitten or sunburnt or both) $= \text{P}(S \cup B)$

$= \dfrac{18 + 5 + 17}{50}$

$= \dfrac{40}{50}$

$= \frac{4}{5}$

iii P(hiker was bitten given he or she was sunburnt) $= \text{P}(B \mid S)$

$= \dfrac{\text{P}(B \cap S)}{\text{P}(S)}$

$= \dfrac{\frac{5}{50}}{\frac{23}{50}}$

$= \frac{5}{23}$

iv P(hiker was sunburnt given he or she was not bitten) $= \text{P}(S \mid B')$

$= \dfrac{\text{P}(S \cap B')}{\text{P}(B')}$

$= \dfrac{\frac{18}{50}}{\frac{18+10}{50}}$

$= \dfrac{18}{28}$

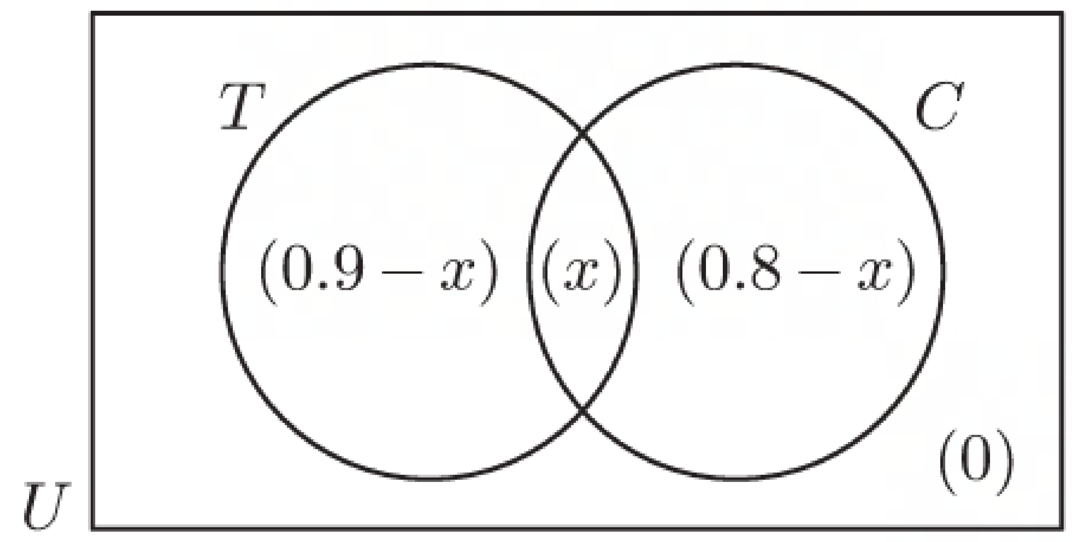
$= \frac{9}{14}$

6 Let T represent a family who had a TV set, and C represent a family who had a computer.

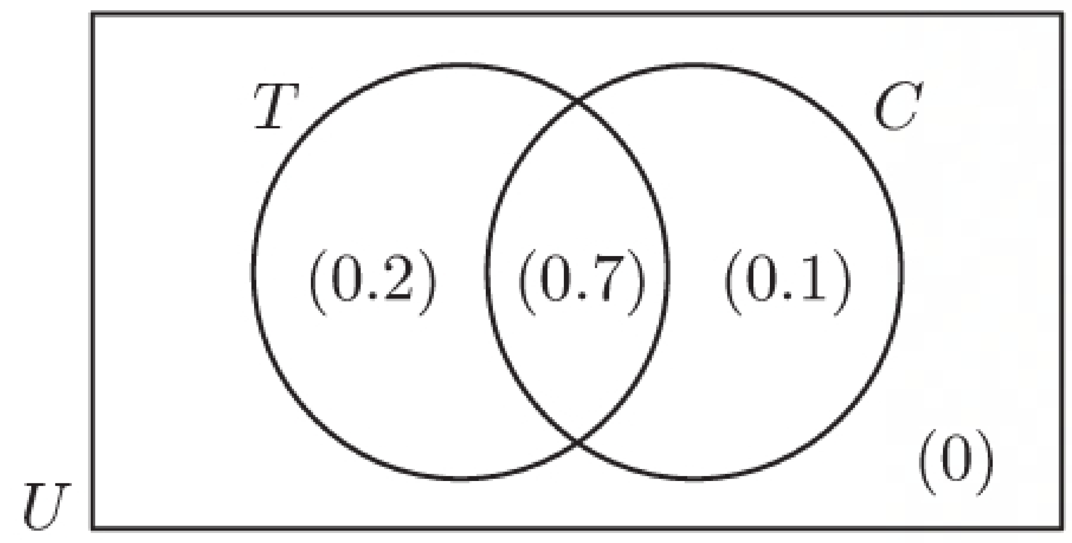
Let the proportion of families in $T \cap C$ be x.

$\therefore$ the proportion in $T \cap C'$ is $0.9 - x$ and
the proportion in $T' \cap C$ is $0.8 - x$.

The proportion in $T' \cap C'$ is 0.

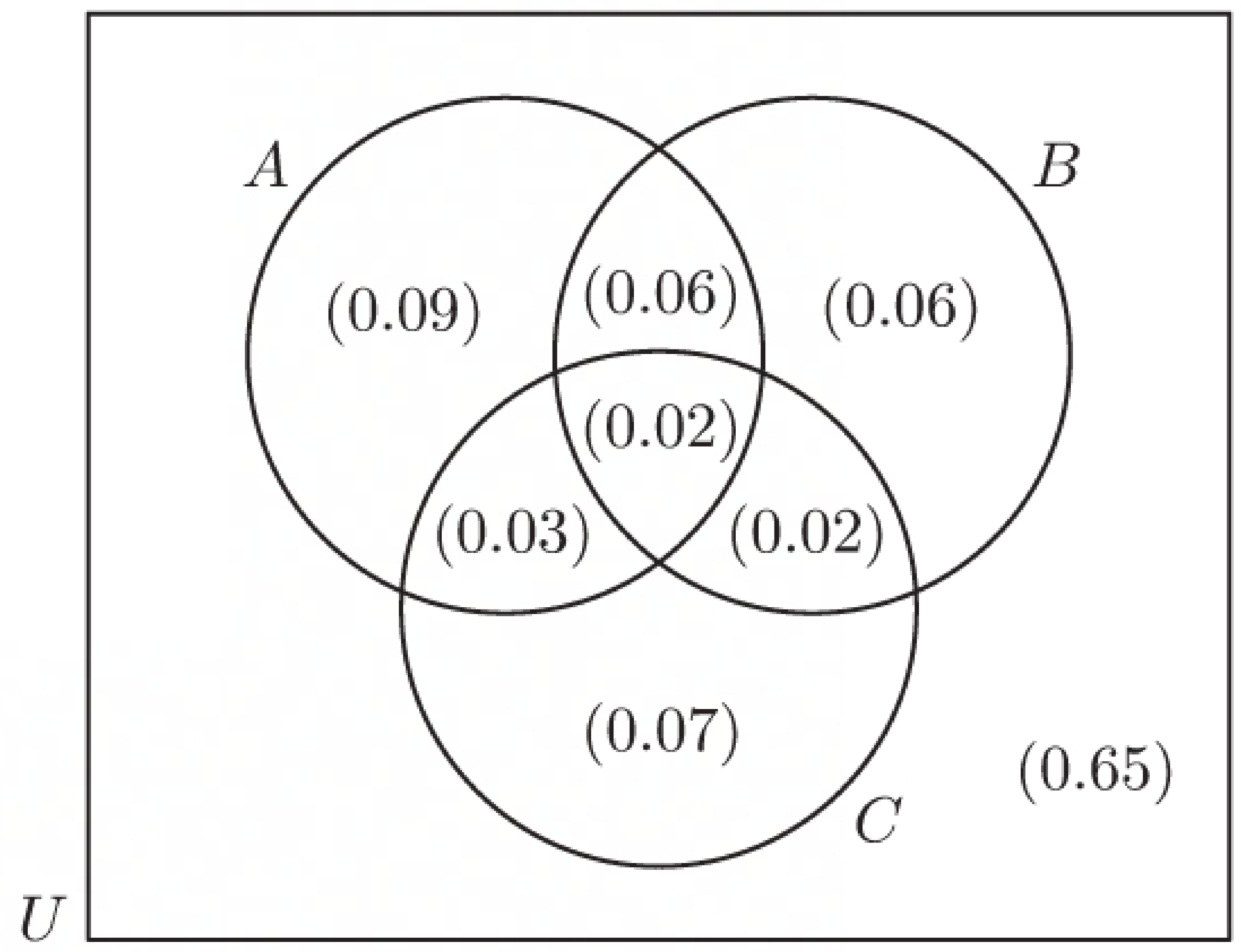


$$\therefore \quad (0.9 - x) + x + (0.8 - x) = 1$$
$$\therefore \quad 1.7 - x = 1$$
$$\therefore \quad x = 0.7$$



$$\begin{aligned} P(T \mid C) &= \frac{P(T \cap C)}{P(C)} \\ &= \frac{0.7}{0.8} \\ &= \tfrac{7}{8} \end{aligned}$$

7 Let A represent a person who reads newspaper A, B represent a person who reads newspaper B, and C represent a person who reads newspaper C.



The proportion of people in:

$$\begin{aligned} A \cap B \cap C \quad &\text{is} \quad 0.02 \\ A \cap B \cap C' \quad &\text{is} \quad 0.08 - 0.02 = 0.06 \\ A' \cap B \cap C \quad &\text{is} \quad 0.04 - 0.02 = 0.02 \\ A \cap B' \cap C \quad &\text{is} \quad 0.05 - 0.02 = 0.03 \\ A \cap B' \cap C' \quad &\text{is} \quad 0.2 - 0.06 - 0.02 - 0.03 = 0.09 \\ A' \cap B \cap C' \quad &\text{is} \quad 0.16 - 0.06 - 0.02 - 0.02 = 0.06 \\ A' \cap B' \cap C \quad &\text{is} \quad 0.14 - 0.02 - 0.02 - 0.03 = 0.07 \end{aligned}$$

$A' \cap B' \cap C'$ is $1 - 0.09 - 0.06 - 0.02 - 0.03 - 0.06 - 0.02 - 0.07 = 0.65$

a $$\begin{aligned} \text{P(person reads none of the papers)} &= 0.65 \\ &= \tfrac{13}{20} \end{aligned}$$

b $$\begin{aligned} \text{P(person reads at least one of the papers)} &= 1 - \text{P(person reads none of the papers)} \\ &= 1 - \tfrac{13}{20} \\ &= \tfrac{7}{20} \end{aligned}$$

c $$\begin{aligned} \text{P(person reads exactly one of the papers)} &= 0.09 + 0.06 + 0.07 \\ &= 0.22 \\ &= \tfrac{11}{50} \end{aligned}$$

d $$\begin{aligned} \text{P(person reads } A \text{ or } B \text{ or both)} &= 0.09 + 0.06 + 0.02 + 0.03 + 0.06 + 0.02 \\ &= 0.28 \\ &= \tfrac{7}{25} \end{aligned}$$

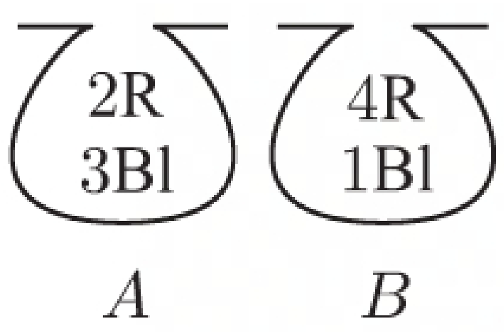
e P(person reads A, given that person reads at least one paper)

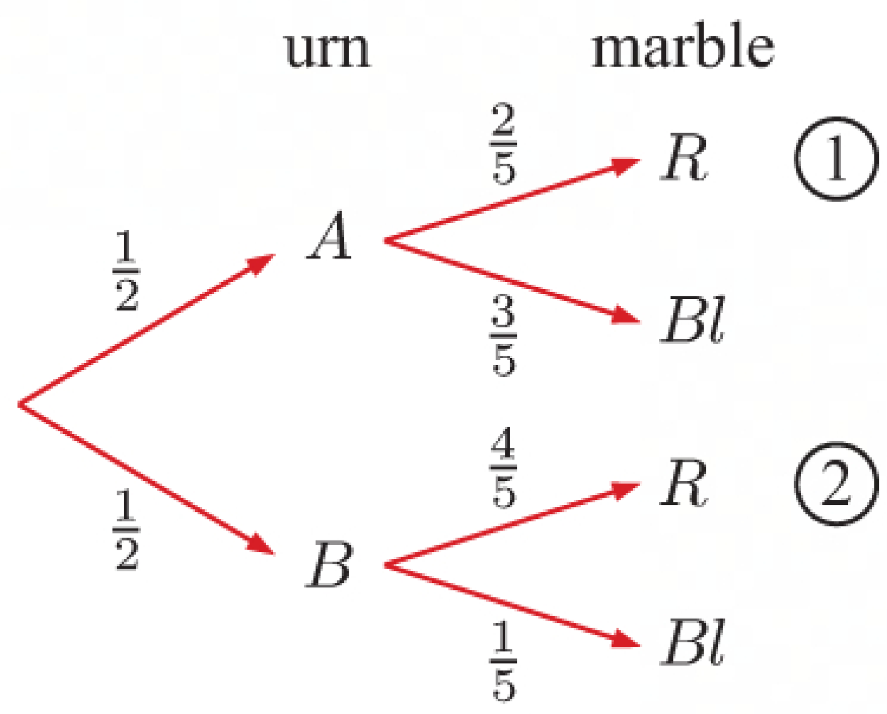
$$
\begin{aligned}
&= \mathrm{P}(A \mid (A \cup B \cup C)) \\
&= \frac{\mathrm{P}(A \cap (A \cup B \cup C))}{\mathrm{P}(A \cup B \cup C)} \\
&= \frac{0.2}{0.35} \\
&= \tfrac{4}{7}
\end{aligned}
$$

f P(person reads C, given that person reads either A or B or both)

$$
\begin{aligned}
&= \mathrm{P}(C \mid (A \cup B)) \\
&= \frac{\mathrm{P}(C \cap (A \cup B))}{\mathrm{P}(A \cup B)} \\
&= \frac{0.03 + 0.02 + 0.02}{0.09 + 0.06 + 0.02 + 0.03 + 0.06 + 0.02} \\
&= \frac{0.07}{0.28} \\
&= \tfrac{1}{4}
\end{aligned}
$$

8 Let A represent urn A, B represent urn B, R represent a red marble, and Bl represent a blue marble.





a $\mathrm{P}(R) = \underbrace{\mathrm{P}(A \cap R)}_{\text{branch 1}} + \underbrace{\mathrm{P}(B \cap R)}_{\text{branch 2}}$

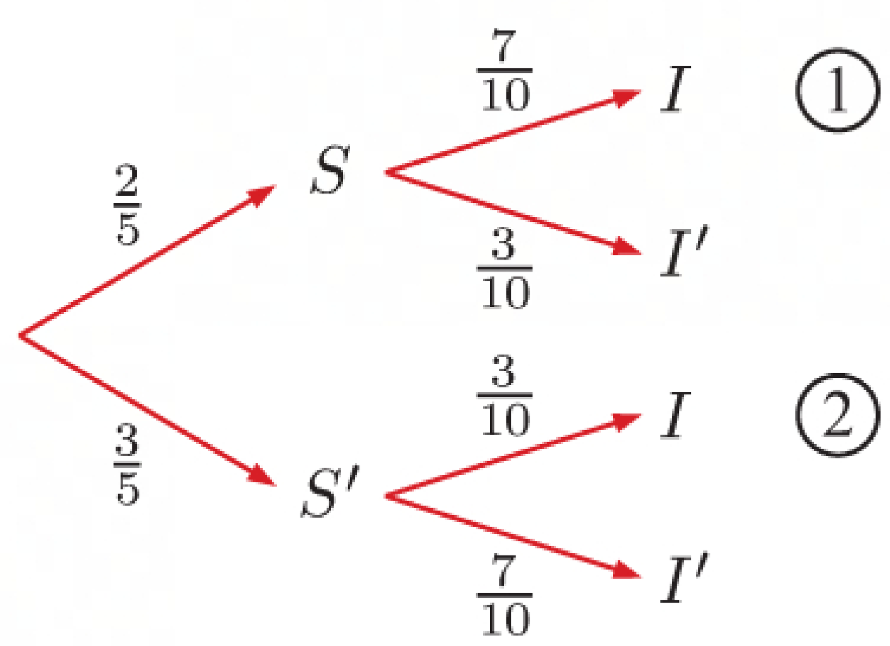
$$
\begin{aligned}
&= \tfrac{1}{2} \times \tfrac{2}{5} + \tfrac{1}{2} \times \tfrac{4}{5} \\
&= \tfrac{3}{5}
\end{aligned}
$$

b $\mathrm{P}(B \mid R) = \dfrac{\mathrm{P}(B \cap R)}{\mathrm{P}(R)}$

$$
= \frac{\frac{1}{2} \times \frac{4}{5}}{\frac{3}{5}} \qquad
\begin{array}{l} \leftarrow \text{branch 2} \\ \leftarrow \text{from } \mathbf{a} \end{array}
$$

$$
= \tfrac{2}{3}
$$

9 Let S represent Greta going shopping and I represent Greta having an ice cream.



a $\mathrm{P}(I) = \underbrace{\mathrm{P}(S \cap I)}_{\text{branch 1}} + \underbrace{\mathrm{P}(S' \cap I)}_{\text{branch 2}}$

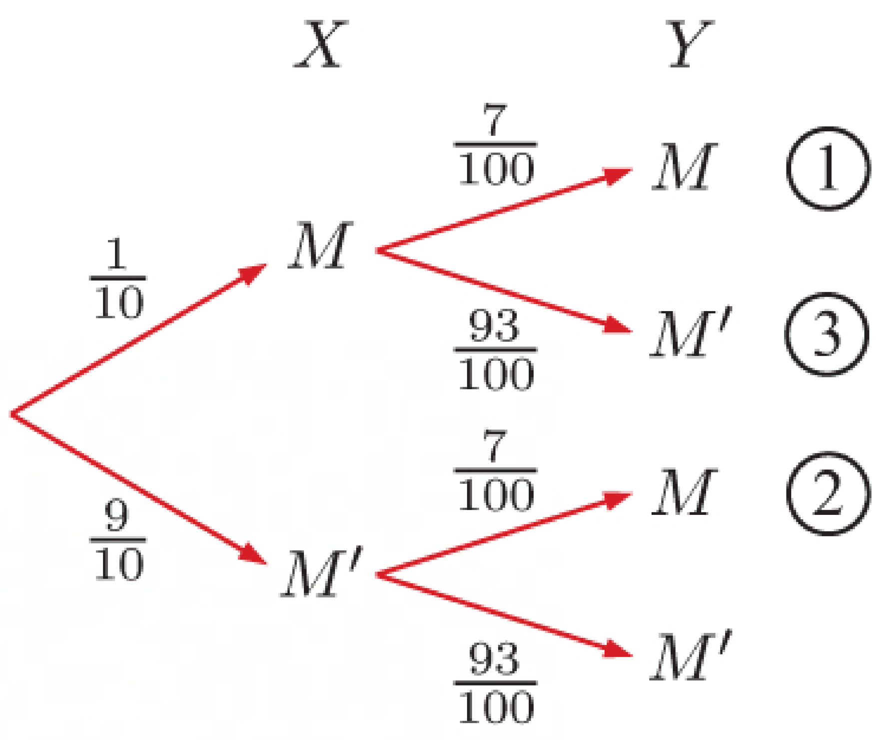
$$
\begin{aligned}
&= \tfrac{2}{5} \times \tfrac{7}{10} + \tfrac{3}{5} \times \tfrac{3}{10} \\
&= \tfrac{23}{50}
\end{aligned}
$$

b $\mathrm{P}(S \mid I) = \dfrac{\mathrm{P}(S \cap I)}{\mathrm{P}(I)}$

$$
= \frac{\frac{2}{5} \times \frac{7}{10}}{\frac{23}{50}} \qquad
\begin{array}{l} \leftarrow \text{branch 1} \\ \leftarrow \text{from } \mathbf{a} \end{array}
$$

$$
= \tfrac{14}{23}
$$

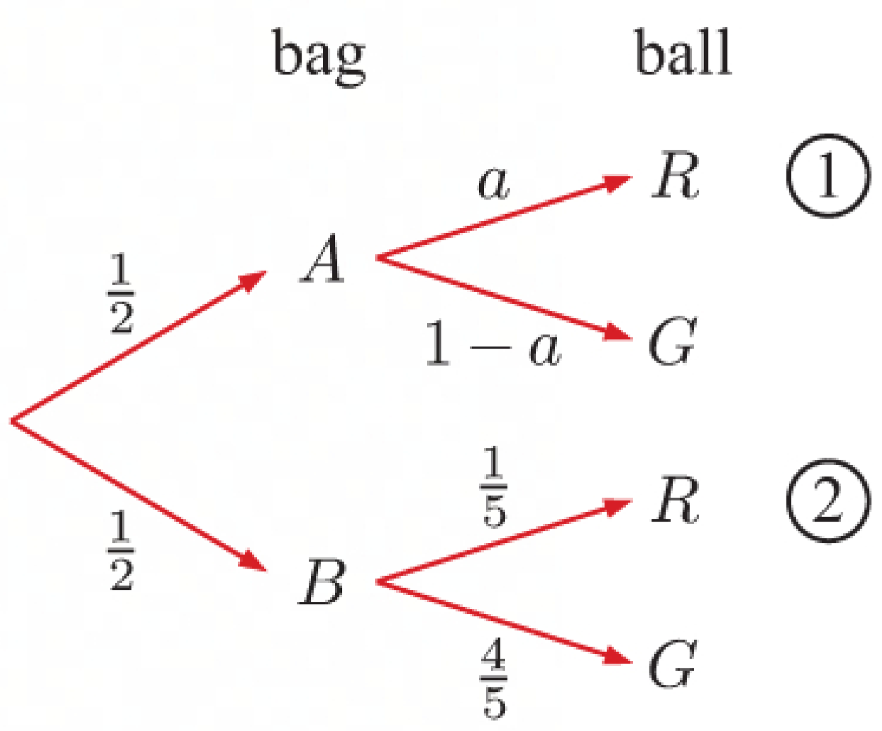
10 Let X represent machine X, Y represent machine Y, and M represent a machine malfunctioning.



a $$\begin{aligned} \mathrm{P}(X \mid \text{exactly one malfunctioned}) &= \frac{\mathrm{P}(X \cap \text{exactly one malfunctioned})}{\mathrm{P}(\text{exactly one malfunctioned})} \\ &= \frac{\frac{1}{10} \times \frac{93}{100}}{\frac{1}{10} \times \frac{93}{100} + \frac{9}{10} \times \frac{7}{100}} \begin{array}{l} \longleftarrow \text{branch } ③ \\ \longleftarrow \text{branches } ③ \text{ and } ② \end{array} \\ &= \frac{93}{93+63} \\ &= \frac{93}{156} \\ &= \frac{31}{52} \end{aligned}$$

b $$\begin{aligned} &\mathrm{P}(Y \mid \text{at least one malfunctioned}) \\ &= \frac{\mathrm{P}(Y \cap \text{at least one malfunctioned})}{\mathrm{P}(\text{at least one malfunctioned})} \\ &= \frac{\frac{1}{10} \times \frac{7}{100} + \frac{9}{10} \times \frac{7}{100}}{\frac{1}{10} \times \frac{7}{100} + \frac{1}{10} \times \frac{93}{100} + \frac{9}{10} \times \frac{7}{100}} \begin{array}{l} \longleftarrow \text{branches } ① \text{ and } ② \\ \longleftarrow \text{branches } ①, ③, \text{ and } ② \end{array} \\ &= \frac{7+63}{7+93+63} \\ &= \frac{70}{163} \end{aligned}$$

11 Let A represent bag A, B represent bag B, R represent a red ball, and G represent a green ball. Let the proportion of red balls in bag A be a.



$$P(R) = \underbrace{P(A \cap R)}_{\text{branch } 1} + \underbrace{P(B \cap R)}_{\text{branch } 2}$$

$$\therefore \quad \tfrac{1}{3} = \tfrac{1}{2} \times a + \tfrac{1}{2} \times \tfrac{1}{5}$$

$$= \frac{a}{2} + \tfrac{1}{10}$$

$$\therefore \quad \frac{a}{2} = \tfrac{1}{3} - \tfrac{1}{10}$$

$$= \frac{10 - 3}{30}$$

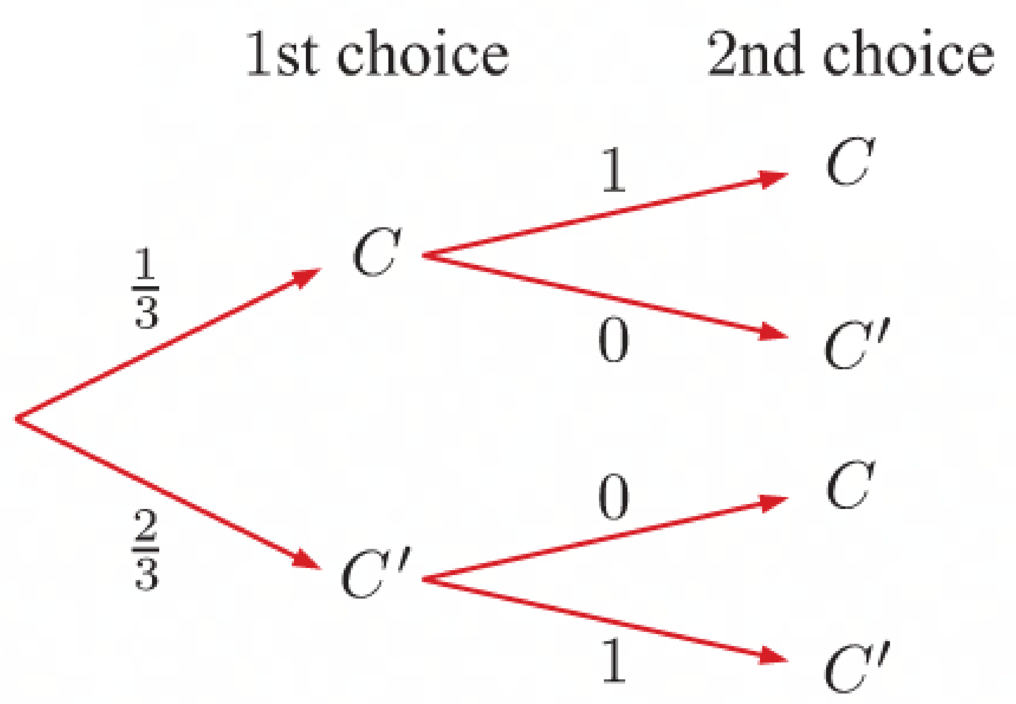
$$\therefore \quad \frac{a}{2} = \tfrac{7}{30}$$

$$\therefore \quad a = \tfrac{7}{15}$$

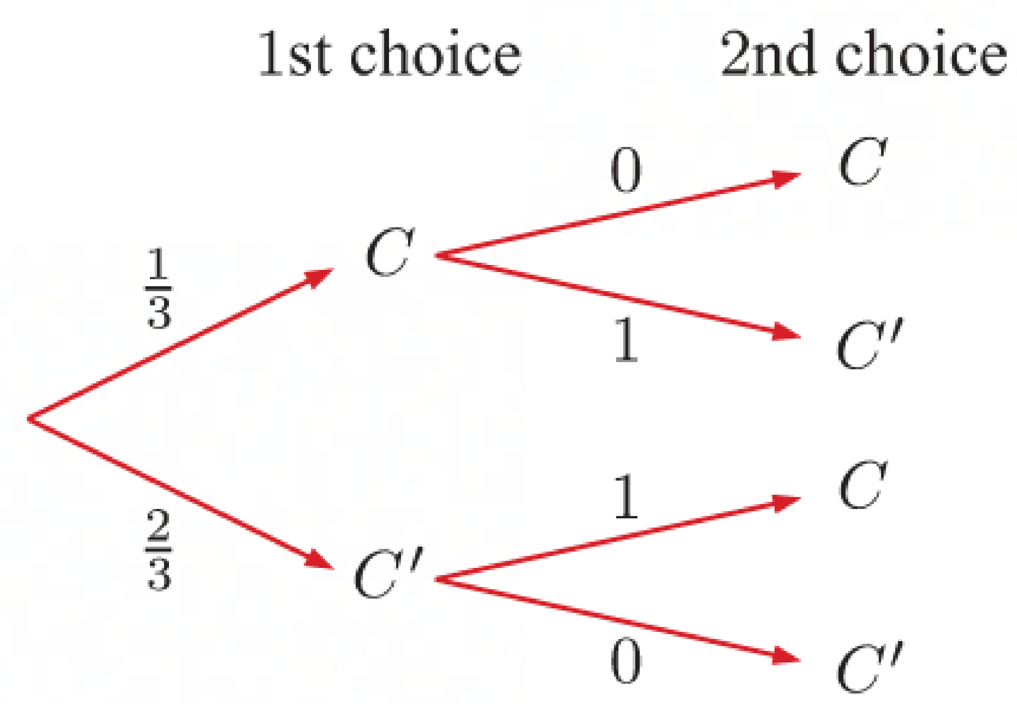
So, the proportion of red balls in bag A is $\frac{7}{15}$.

ACTIVITY 2 THE MONTY HALL PROBLEM

1 If the contestant *does not* switch his or her choice, then the tree diagram is:



If the contestant *does* switch his or her choice, then the tree diagram is:



2 **a** The contestant can choose 1 out of 3 doors and the car is behind one of these.

$\therefore$ P(contestant's first choice has the car) $= \frac{1}{3}$

b Given that the contestant changes their guess, we consider the second tree diagram in **1**.

P(contestant's second choice has the car | change their guess)

$= P(C' \cap C)$ {in 2nd tree diagram}

$= \frac{2}{3} \times 1$

$= \frac{2}{3}$

3 **a** The audience member sees one of the incorrect doors open, so must choose between the two remaining closed doors, one of which has the car behind it.

$\therefore$ P(audience member chooses the car) $= \frac{1}{2}$

b The contestant has the ability to switch or not switch from their original choice, unlike the audience member.

The door chosen by the contestant is never one of the doors opened by the host, so the contestant has a $\frac{2}{3}$ chance of winning the car if they switch their choice, as shown in **2 b**.

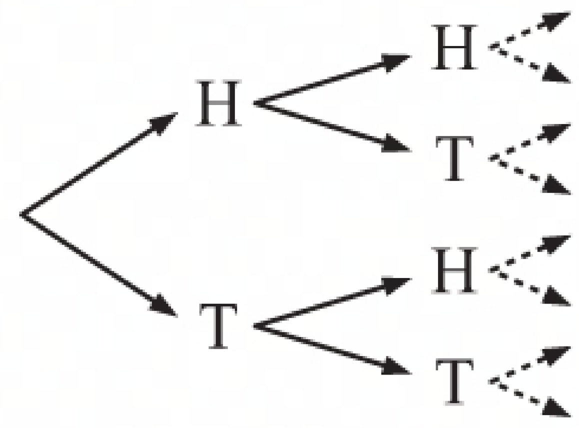
The audience member only sees two doors and has no other information, so the audience member's chance of guessing correctly is $\frac{1}{2}$.

ACTIVITY 3 PENNEY'S GAME

2 **a** The players can choose: HH, HT, TH, or TT.

b We cannot draw a complete tree diagram showing all the possible outcomes because the game could theoretically continue indefinitely.

For example, if Player 1 chooses HH and Player 2 chooses TT, a game with coin tosses HTHTHTHT.... would never end.



c **i** Player 1 - TH versus Player 2 - HT

If the first toss is T, then Player 1 must win.

If the first toss is H, then Player 2 must win.

The game is therefore fair.

ii Player 1 - HH versus Player 2 - HT

After many trials, the game appears to be fair.

d Player 1 - HH versus Player 2 - HT

i The sequence chosen by each player begins with H.

$\therefore$ an initial toss T helps neither player.

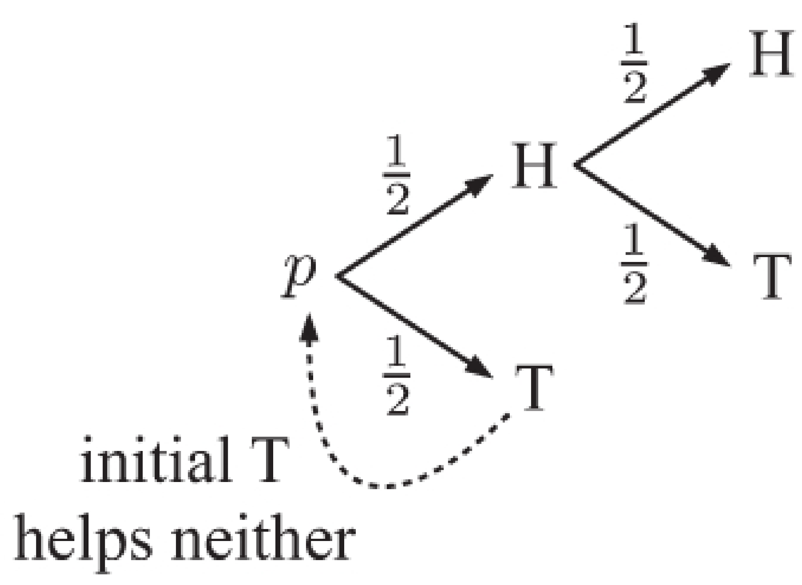
Once the first H appears, the result is determined by the next toss. If it is H then Player 1 wins. If it is T then Player 2 wins.

$\therefore$ the game is fair.

ii Let $p = \text{P(HH beats HT)}$

Using the tree diagram,

$$p = \text{P(H)} \times \text{P(HH beats HT} \mid \text{H)}$$
$$+ \text{P(T)} \times \text{P(HH beats HT} \mid \text{T)}$$
$$= \tfrac{1}{2}\left[\tfrac{1}{2}(1) + \tfrac{1}{2}(0)\right] + \tfrac{1}{2}p$$



iii
$$p = \tfrac{1}{2}\left[\tfrac{1}{2} + 0\right] + \tfrac{1}{2}p$$
$$\therefore \tfrac{1}{2}p = \tfrac{1}{4}$$
$$\therefore p = \tfrac{1}{2}$$

$\therefore$ the game is fair.

e Player 1 - HH versus Player 2 - TH

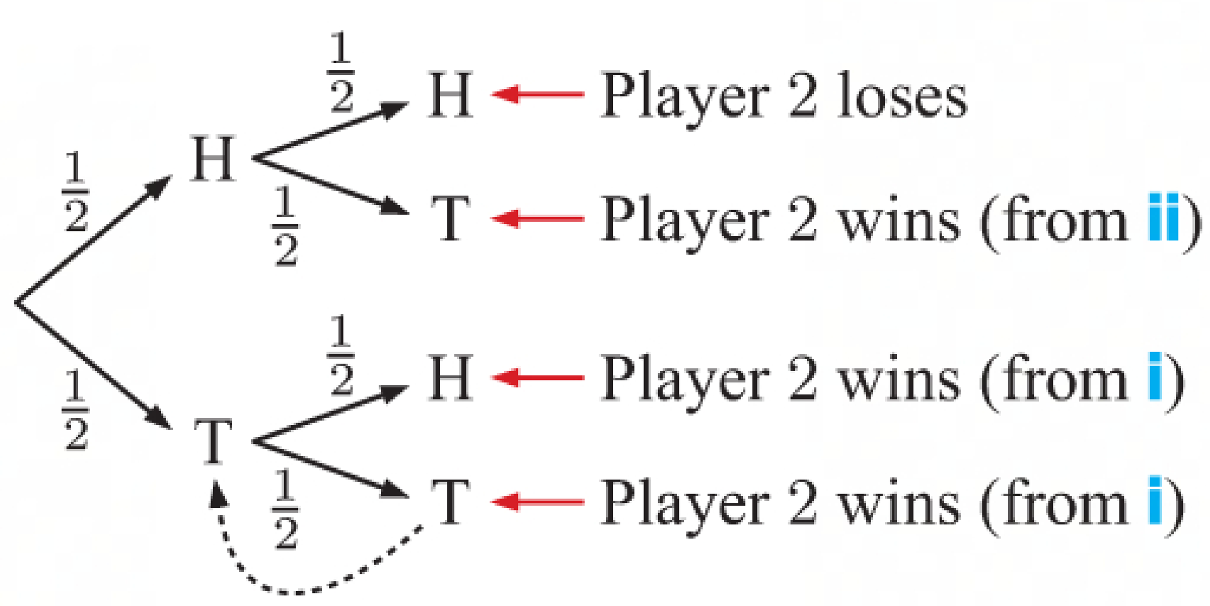
i An initial toss T means that as soon as H is tossed, Player 2 will immediately win, and therefore Player 1 cannot win.

ii If the first two tosses are HT, neither player has won, but we now have a T as in **i** above. As soon as H is tossed, Player 2 will immediately win, and therefore Player 1 cannot win.

iii P(TH beats HH)

$$= \text{P(player 2 wins)}$$
$$= \tfrac{1}{2}\left[\tfrac{1}{2} \times 0 + \tfrac{1}{2} \times 1\right] + \tfrac{1}{2}\left[\tfrac{1}{2} \times 1 + \tfrac{1}{2} \times 1\right]$$
$$= \tfrac{3}{4}$$

$\therefore$ the game is not fair.

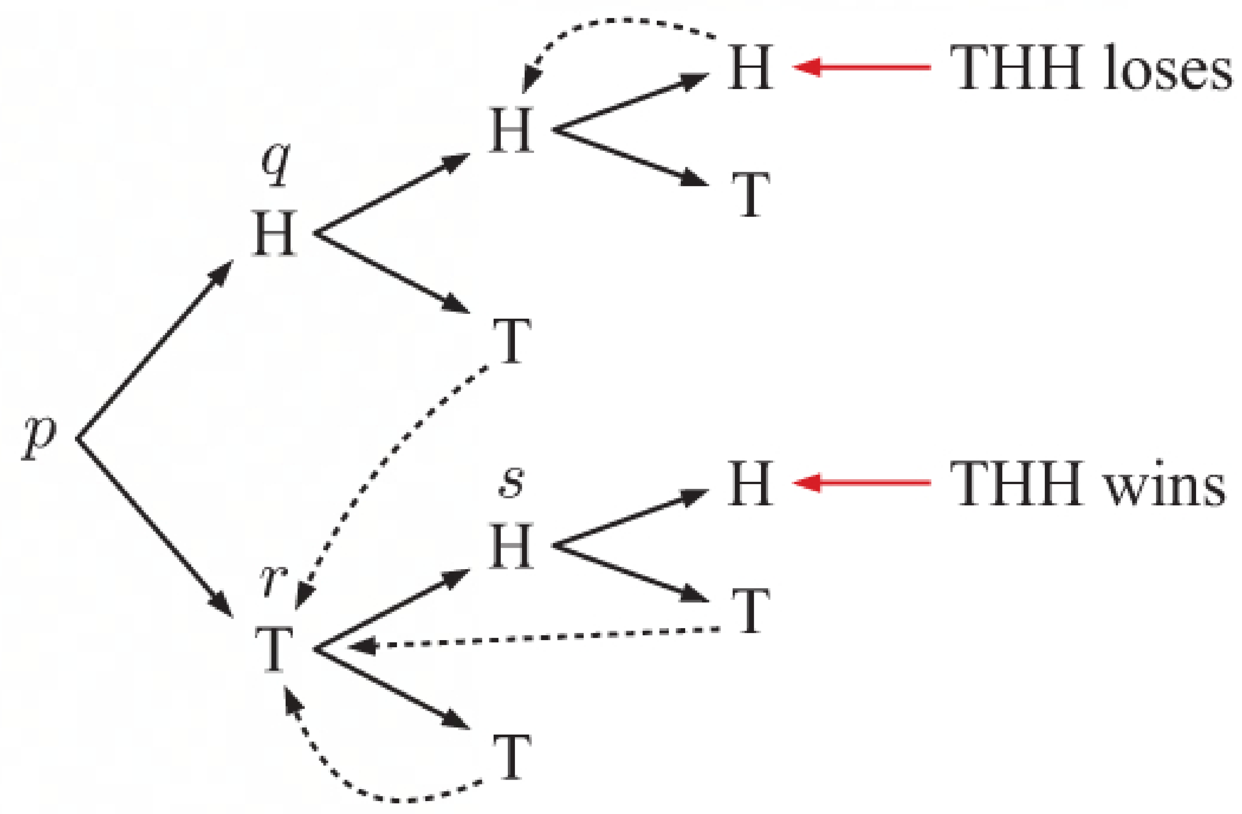


3 **a** The players can choose:
HHH, HHT, HTH, HTT, TTT, TTH, THT, or THH.

b Player 1 - HHT versus Player 2 - THH

i The game does not seem fair. Player 2 wins more often than Player 1.

ii We let: $p = \text{P(THH beats HHT)}$
$q = \text{P(THH beats HHT} \mid \text{H)}$
$r = \text{P(THH beats HHT} \mid \text{T)}$
$s = \text{P(THH beats HHT} \mid \text{TH)}$



$$\begin{aligned}
p &= \text{P(THH beats HHT)} \\
&= \text{P(H)} \times \text{P(THH beats HHT} \mid \text{H)} &&+ \text{P(T)} \times \text{P(THH beats HHT} \mid \text{T)} \\
&= \tfrac{1}{2}q &&+ \tfrac{1}{2}r \qquad \text{.... (1)}
\end{aligned}$$

$$\begin{aligned}
q &= \text{P(THH beats HHT} \mid \text{H)} \\
&= \text{P(H)} \times \text{P(THH beats HHT} \mid \text{HH)} &&+ \text{P(T)} \times \text{P(THH beats HHT} \mid \text{HT)} \\
&= \tfrac{1}{2} \times 0 &&+ \tfrac{1}{2} \times \text{P(THH beats HHT} \mid \text{T)} \\
&= \tfrac{1}{2}(0) &&+ \tfrac{1}{2}r \qquad \text{.... (2)}
\end{aligned}$$

$$\begin{aligned}
r &= \text{P(THH beats HHT} \mid \text{T)} \\
&= \text{P(H)} \times \text{P(THH beats HHT} \mid \text{TH)} &&+ \text{P(T)} \times \text{P(THH beats HHT} \mid \text{TT)} \\
&= \tfrac{1}{2}s &&+ \tfrac{1}{2} \times \text{P(THH beats HHT} \mid \text{T)} \\
&= \tfrac{1}{2}s &&+ \tfrac{1}{2}r \qquad \text{.... (3)}
\end{aligned}$$

$$\begin{aligned}
s &= \text{P(THH beats HHT} \mid \text{TH)} \\
&= \text{P(H)} \times 1 &&+ \text{P(T)} \times \text{P(THH beats HHT} \mid \text{THT)} \\
&= \tfrac{1}{2} \times 1 &&+ \tfrac{1}{2} \times \text{P(THH beats HHT} \mid \text{T)} \\
&= \tfrac{1}{2}(1) &&+ \tfrac{1}{2}r \qquad \text{.... (4)}
\end{aligned}$$

iii Using (3), $\tfrac{1}{2}r = \tfrac{1}{2}s$

$\therefore\ r = s$

Substituting into (4), $s = \tfrac{1}{2} + \tfrac{1}{2}s$

$\therefore\ \tfrac{1}{2}s = \tfrac{1}{2}$

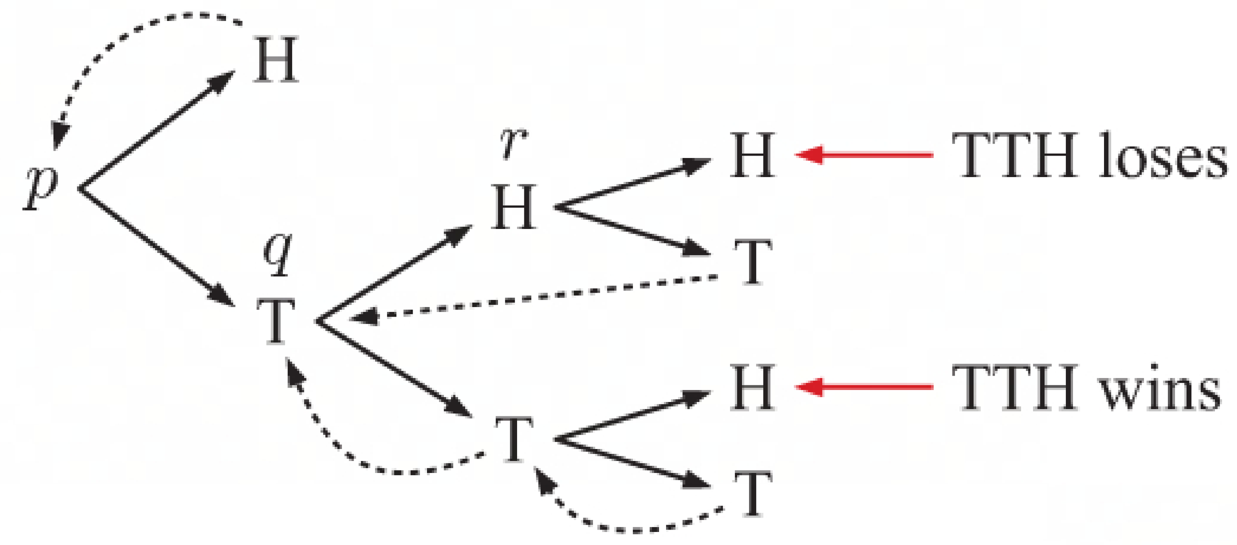
$\therefore\ s = 1$ and $r = 1$

Substituting into (2), $q = \tfrac{1}{2}$

Substituting into (1), $p = \tfrac{1}{2}(\tfrac{1}{2}) + \tfrac{1}{2}(1) = \tfrac{3}{4}$

$\therefore\ \text{P(THH beats HHT)} = \tfrac{3}{4}$

c **i** We let $p = \text{P(TTH beats THH)}$
$q = \text{P(TTH beats THH | T)}$
$r = \text{P(TTH beats THH | TH)}$



$$\begin{aligned}
p &= \text{P(TTH beats THH)} \\
&= \text{P(H)} \times \text{P(TTH beats THH | H)} + \text{P(T)} \times \text{P(TTH beats THH | T)} \\
&= \tfrac{1}{2} \times \text{P(TTH beats THH)} + \tfrac{1}{2}q \\
&= \tfrac{1}{2}p + \tfrac{1}{2}q
\end{aligned}$$

$\therefore \ \frac{1}{2}p = \frac{1}{2}q$

$\therefore \ p = q$ .... (1)

But $q = \text{P(TTH beats THH | T)}$
$= \text{P(H)} \times \text{P(TTH beats THH | TH)} + \text{P(T)} \times \text{P(TTH beats THH | TT)}$
$= \frac{1}{2}r + \frac{1}{2} \times 1$ .... (2)

Also, $r = \text{P(TTH beats THH | TH)}$
$= \text{P(H)} \times 0 + \text{P(T)} \times \text{P(TTH beats THH | THT)}$
$= \frac{1}{2} \times 0 + \frac{1}{2} \times \text{P(TTH beats THH | T)}$
$= 0 + \frac{1}{2}q$ .... (3)

Substituting (3) into (2), $q = \frac{1}{2}(\frac{1}{2}q) + \frac{1}{2}(1)$

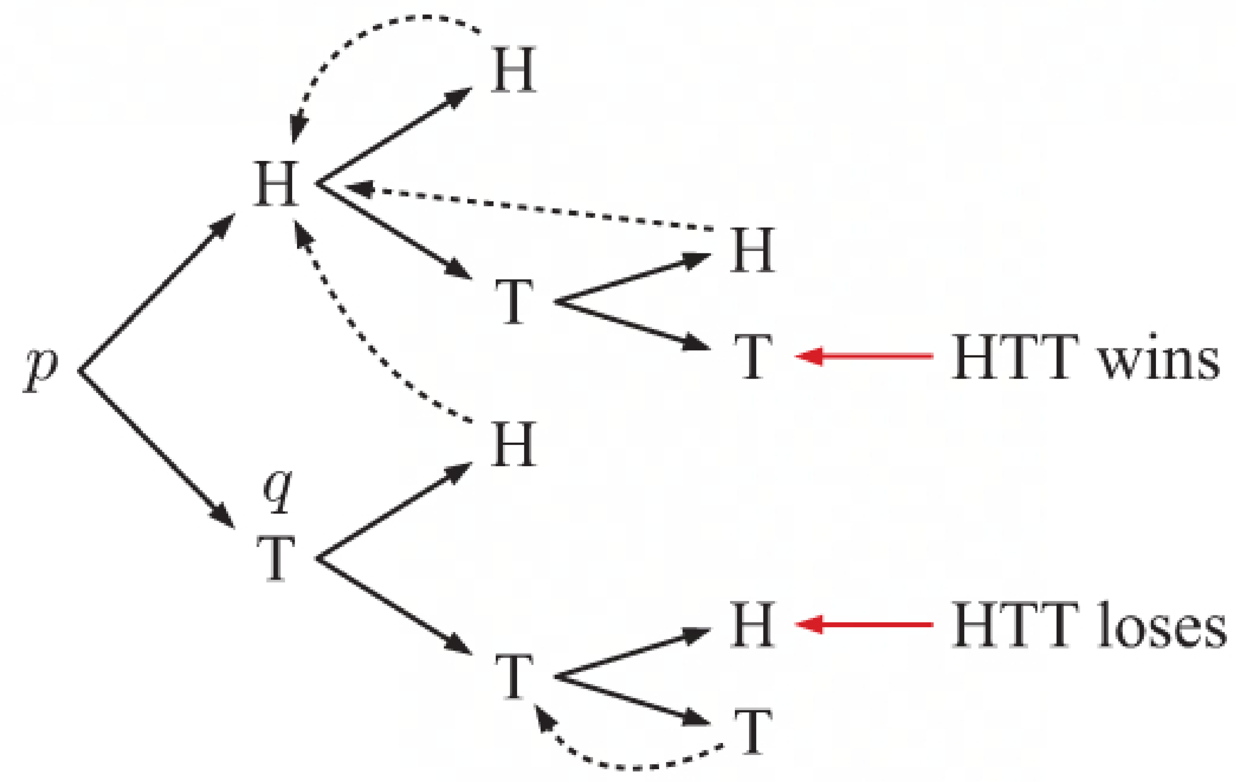
$\therefore \ \frac{3}{4}q = \frac{1}{2}$

$\therefore \ q = \frac{2}{3}$

$\therefore \ p = \frac{2}{3}$ {using (1)}

$\therefore$ P(TTH beats THH) $= \frac{2}{3}$

ii We let $p = \text{P(HTT beats TTH)}$
$q = \text{P(HTT beats TTH | T)}$



$p = \text{P(HTT beats TTH)}$
$= \text{P(H)} \times \text{P(HTT beats TTH | H)} + \text{P(T)} \times \text{P(HTT beats TTH | T)}$
$= \frac{1}{2} \times 1 + \frac{1}{2}q$ .... (1)

$$
\begin{aligned}
q &= \text{P(HTT beats TTH} \mid \text{T)} \\
&= \text{P(H)} \times \text{P(HTT beats TTH} \mid \text{TH)} + \text{P(T)} \times \text{P(HTT beats TTH} \mid \text{TT)} \\
&= \tfrac{1}{2} \times \text{P(HTT beats TTH} \mid \text{H)} \qquad + \tfrac{1}{2} \times 0 \\
&= \tfrac{1}{2} \times 1 \qquad + \tfrac{1}{2} \times 0 \\
&= \tfrac{1}{2}
\end{aligned}
$$

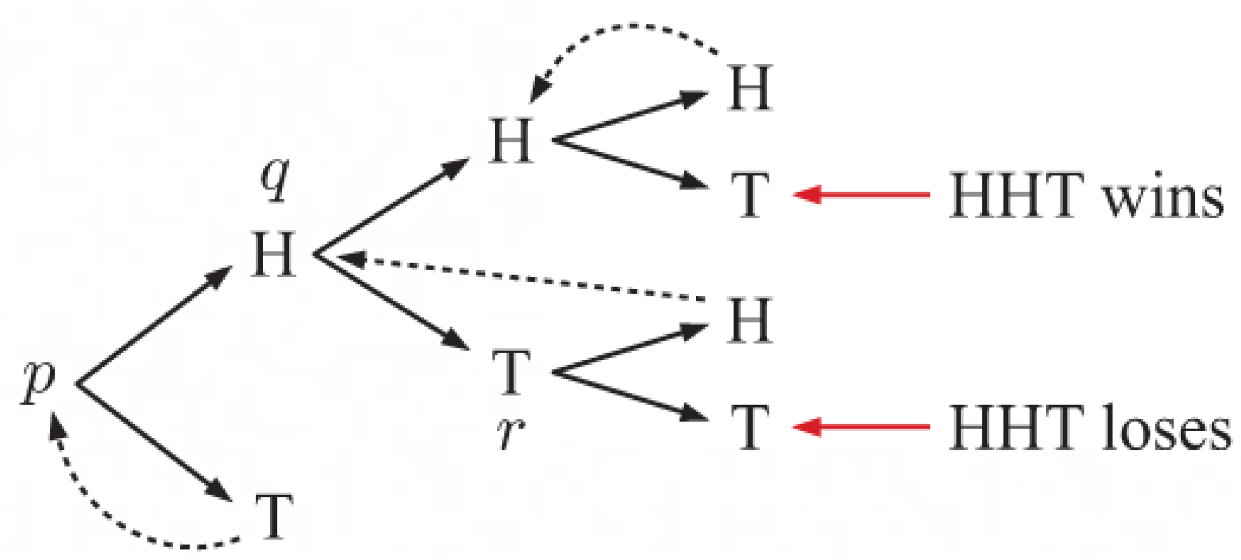
Substituting into (1), $p = \frac{1}{2} + \frac{1}{2}(\frac{1}{2}) = \frac{3}{4}$

$\therefore$ P(HTT beats TTH) $= \frac{3}{4}$

iii We let $p =$ P(HHT beats HTT)

$q =$ P(HHT beats HTT | H)

$r =$ P(HHT beats HTT | HT)



$$
\begin{aligned}
p &= \text{P(HHT beats HTT)} \\
&= \text{P(H)} \times \text{P(HHT beats HTT} \mid \text{H)} \quad + \text{P(T)} \times \text{P(HHT beats HTT} \mid \text{T)} \\
&= \tfrac{1}{2}q \qquad + \tfrac{1}{2} \times \text{P(HHT beats HTT)} \\
&= \tfrac{1}{2}q \qquad + \tfrac{1}{2}p
\end{aligned}
$$

$\therefore \frac{1}{2}p = \frac{1}{2}q$

$\therefore p = q$ .... (1)

But $q =$ P(HHT beats HTT | H)

$$
\begin{aligned}
&= \text{P(H)} \times \text{P(HHT beats HTT} \mid \text{HH)} \quad + \text{P(T)} \times \text{P(HHT beats HTT} \mid \text{HT)} \\
&= \tfrac{1}{2} \times 1 \qquad + \tfrac{1}{2}r \qquad \text{.... (2)}
\end{aligned}
$$

Also, $r =$ P(HHT beats HTT | HT)

$$
\begin{aligned}
&= \text{P(H)} \times \text{P(HHT beats HTT} \mid \text{HTH)} + \text{P(T)} \times 0 \\
&= \tfrac{1}{2} \times \text{P(HHT beats HTT} \mid \text{H)} \qquad + \tfrac{1}{2} \times 0 \\
&= \tfrac{1}{2}q \qquad + 0 \qquad \text{.... (3)}
\end{aligned}
$$

Substituting (3) into (2), $q = \frac{1}{2} + \frac{1}{2}(\frac{1}{2}q)$

$\therefore \frac{3}{4}q = \frac{1}{2}$

$\therefore q = \frac{2}{3}$

$\therefore p = \frac{2}{3}$ {using (1)}

$\therefore$ P(HHT beats HTT) $= \frac{2}{3}$

d Each game in **b** and **c** is not fair.
If we try to rank the four sequences in order, we obtain:

$$\begin{tikzcd} \text{THH} \arrow[r, "\text{is stronger than}"] & \text{HHT} \arrow[d, "\text{is stronger than}"] \\ \text{TTH} \arrow[u, "\text{is stronger than}"] & \text{HTT} \arrow[l, "\text{is stronger than}"] \end{tikzcd}$$

The four sequences cannot be ranked in order of best to worst, but rather form a circle of preferences.

4 If Player 1 picks their sequence first, Player 2 can *always* pick a "better" sequence which gives them a greater than 50% chance of winning.

5 A non-transitive game is a game whose strategies include at least one circle of preferences.
The most widely played non-transitive game is "Rock, Paper, Scissors".

EXERCISE 11J

1 $\begin{aligned} P(R \cap S) &= P(R) + P(S) - P(R \cup S) \\ &= 0.4 + 0.5 - 0.7 \\ &= 0.2 \end{aligned}$ Also, $\begin{aligned} P(R) \times P(S) &= 0.4 \times 0.5 \\ &= 0.2 \end{aligned}$

So, $P(R \cap S) = P(R) \times P(S)$ and hence R and S are independent events.

2 **a** **i** $\begin{aligned} P(A \cap B) &= P(A) + P(B) - P(A \cup B) \\ &= \tfrac{2}{5} + \tfrac{1}{3} - \tfrac{1}{2} \\ &= \tfrac{7}{30} \end{aligned}$

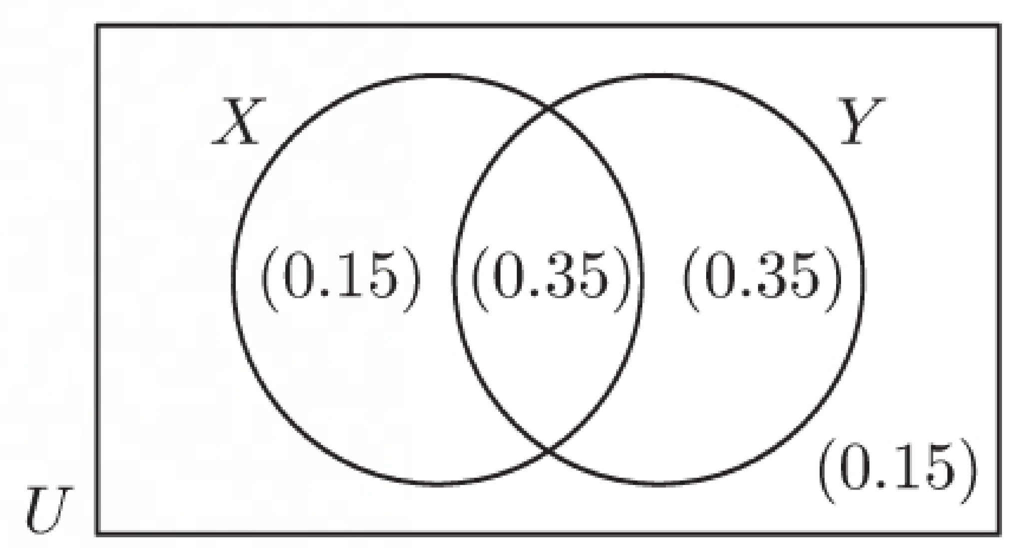
ii $\begin{aligned} P(B \mid A) &= \frac{P(B \cap A)}{P(A)} \\ &= \frac{\frac{7}{30}}{\frac{2}{5}} \\ &= \tfrac{7}{12} \end{aligned}$

iii $\begin{aligned} P(A \mid B) &= \frac{P(A \cap B)}{P(B)} \\ &= \frac{\frac{7}{30}}{\frac{1}{3}} \\ &= \tfrac{7}{10} \end{aligned}$

b A and B are not independent as $P(A \mid B) \neq P(A)$.

3 **a** As X and Y are independent,
then $\begin{aligned} P(X \cap Y) &= P(X) \times P(Y) \\ &= 0.5 \times 0.7 \\ &= 0.35 \end{aligned}$

$\therefore$ P(both X and Y) $= 0.35$



b $\text{P}(X \text{ or } Y \text{ or both}) = \text{P}(X \cup Y)$
$= \text{P}(X) + \text{P}(Y) - \text{P}(X \cap Y)$
$= 0.5 + 0.7 - 0.35$
$= 0.85$

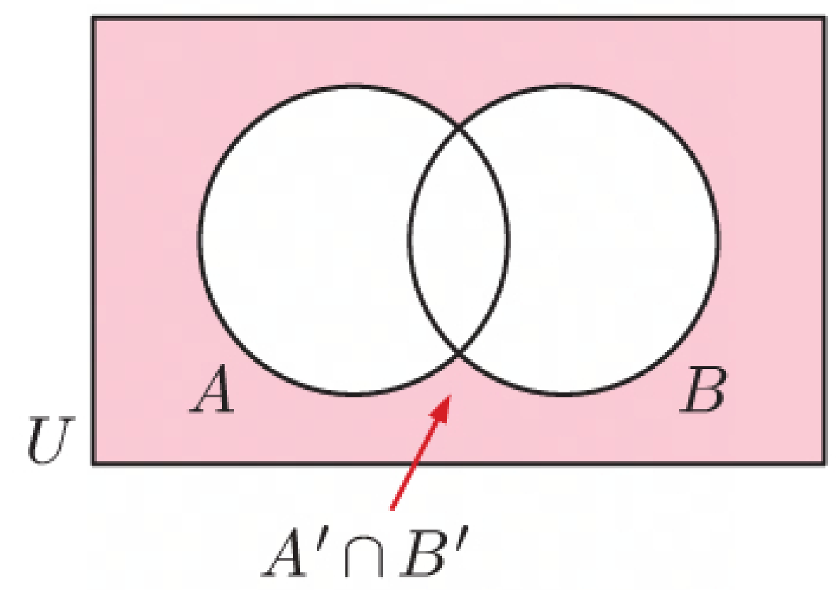
c $\text{P}(\text{neither } X \text{ nor } Y) = 1 - \text{P}(X \cup Y)$
$= 1 - 0.85$
$= 0.15$

d $\text{P}(X \text{ but not } Y) = 0.15$

e $\text{P}(X \mid Y) = \dfrac{\text{P}(X \cap Y)}{\text{P}(Y)}$
$= \dfrac{0.35}{0.7}$
$= 0.5$

4 A and B are independent, so $\text{P}(A \cap B) = \text{P}(A)\ \text{P}(B)$ .... $(*)$

Now $\text{P}(A' \cap B')$
$= 1 - \text{P}(A \cup B)$
$= 1 - [\text{P}(A) + \text{P}(B) - \text{P}(A \cap B)]$
$= 1 - \text{P}(A) - \text{P}(B) + \text{P}(A \cap B)$
$= 1 - \text{P}(A) - \text{P}(B) + \text{P}(A)\ \text{P}(B)$ {using $(*)$}
$= 1 - \text{P}(A) - \text{P}(B)[1 - \text{P}(A)]$
$= [1 - \text{P}(A)]\,[1 - \text{P}(B)]$
$= \text{P}(A')\ \text{P}(B')$

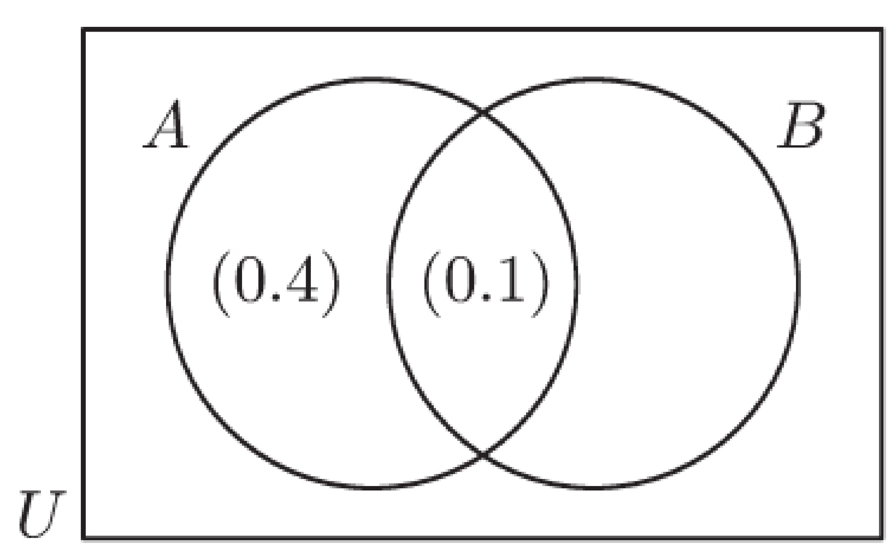


$\therefore$ A' and B' are also independent.

5 For events A and B to be independent, $\text{P}(A \cap B) = \text{P}(A) \times \text{P}(B)$.

For events A and B to be mutually exclusive, $\text{P}(A \cap B) = 0$
$\therefore\ 0 = \frac{5}{7} \times \text{P}(B)$
$\therefore\ \text{P}(B) = 0$

6



$\text{P}(A) = \text{P}(A \cap B) + \text{P}(A \cap B')$
$= 0.1 + 0.4$
$\therefore\ \text{P}(A) = 0.5$

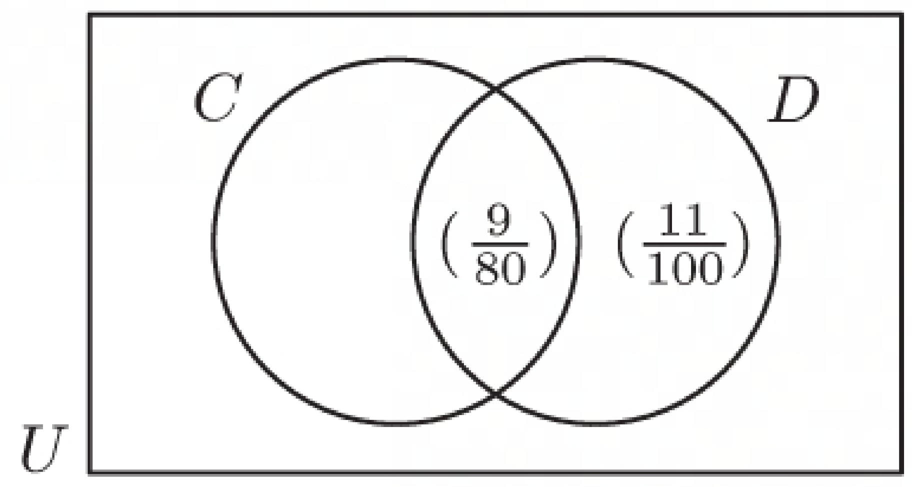
and $\text{P}(A \cap B) = \text{P}(A) \times \text{P}(B)$ {A and B are independent}
$\therefore\ 0.1 = 0.5 \times \text{P}(B)$
$\therefore\ \text{P}(B) = 0.2$

Now $\text{P}(A \cup B') = \text{P}(A) + \text{P}(B') - \text{P}(A \cap B')$
$= 0.5 + 0.8 - 0.4$
$= 0.9$

7 $$\text{P}(D \cap C) = \text{P}(D \mid C)\,\text{P}(C) = \tfrac{1}{4} \times \tfrac{9}{20} = \tfrac{9}{80}$$

Similarly, $\text{P}(D \cap C') = \text{P}(D \mid C')\,\text{P}(C') = \tfrac{1}{5} \times \tfrac{11}{20} = \tfrac{11}{100}$

$\therefore$ the Venn diagram is:



a $\text{P}(D) = \frac{9}{80} + \frac{11}{100} = \frac{89}{400}$

b $\text{P}(D) \neq \text{P}(D \mid C)$, so C and D are not independent events.

8 If X and Y are independent events, then $\text{P}(X \cap Y) = \text{P}(X) \times \text{P}(Y)$

$\therefore$ if $A \cap B$ and $A \cup B$ are independent,

$$\text{P}((A \cap B) \cap (A \cup B)) = \text{P}(A \cap B) \times \text{P}(A \cup B)$$

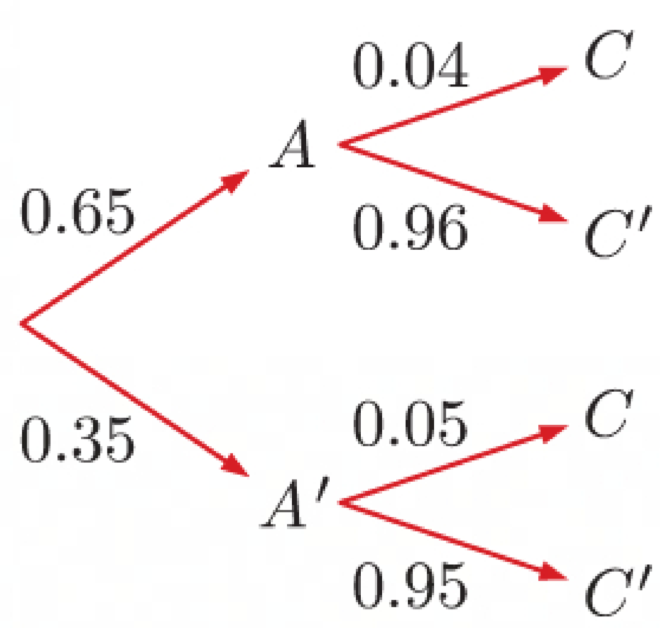
$$\therefore\ \text{P}(A \cap B) = \text{P}(A \cap B) \times \text{P}(A \cup B) \qquad \{\text{since } (A \cap B) \subseteq (A \cup B)\}$$

$$\therefore\ \text{P}(A \cap B)[1 - \text{P}(A \cup B)] = 0$$

$$\therefore\ \text{P}(A \cup B) = 1 \quad \text{or} \quad \text{P}(A \cap B) = 0$$

EXERCISE 11K

1 Let A be the event that a cup of coffee came from machine A,
and C be the event that the cup was underfilled.



a $\text{P(cup of coffee is underfilled)} = \text{P}(C)$

$$= \text{P}(C \mid A)\,\text{P}(A) + \text{P}(C \mid A')\,\text{P}(A')$$
$$= 0.04 \times 0.65 + 0.05 \times 0.35$$
$$= 0.0435$$

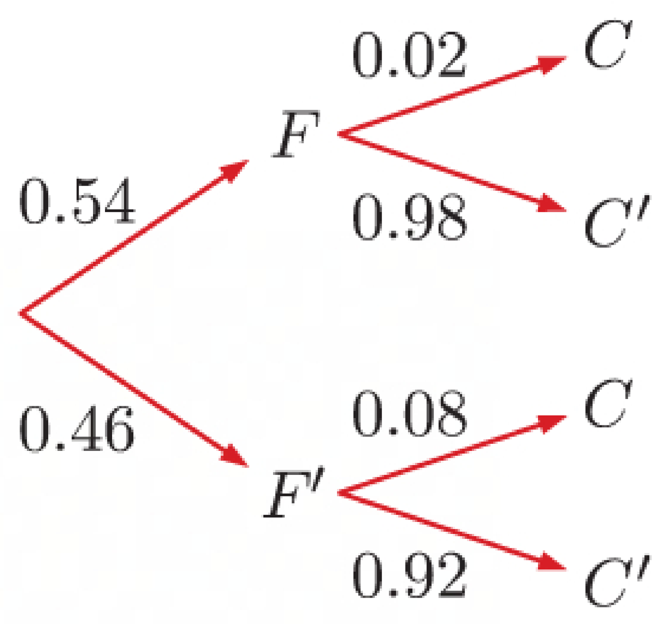
b $\text{P(came from machine A} \mid \text{it is underfilled)} = \text{P}(A \mid C)$

$$= \frac{\text{P}(C \mid A)\,\text{P}(A)}{\text{P}(C)} \qquad \{\text{Bayes' theorem}\}$$

$$= \frac{0.04 \times 0.65}{0.0435} \qquad \{\text{using } \textbf{a}\}$$

$$\approx 0.598$$

2 Let F be the event that a student is female,
and C be the event that a student is colour-blind.



a $\quad$ P(student is male | student is colour-blind)

$= \mathrm{P}(F' \mid C)$

$= \dfrac{\mathrm{P}(C \mid F')\,\mathrm{P}(F')}{\mathrm{P}(C \mid F)\,\mathrm{P}(F) \;+\; \mathrm{P}(C \mid F')\,\mathrm{P}(F')}$ $\quad$ {Bayes' theorem}

$= \dfrac{0.08 \times 0.46}{0.02 \times 0.54 \;+\; 0.08 \times 0.46}$

≈ 0.773

b $\quad$ P(student is female | student is not colour-blind)

$= \mathrm{P}(F \mid C')$

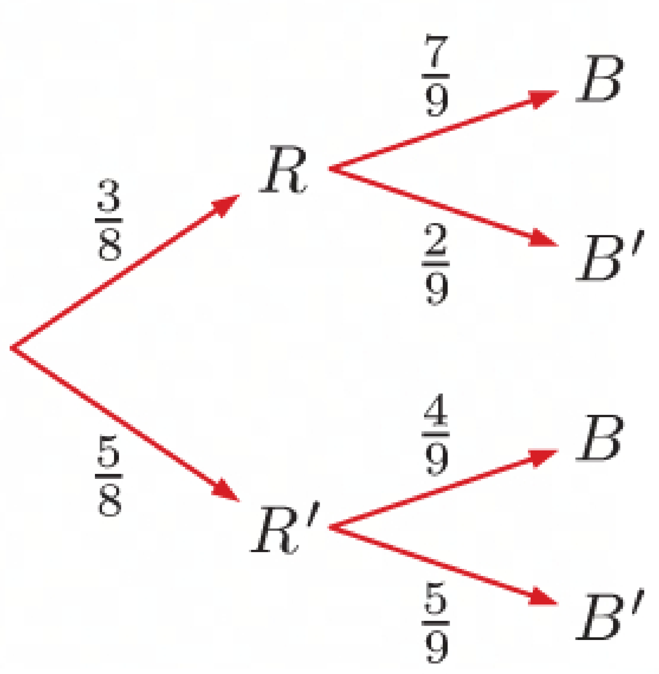
$= \dfrac{\mathrm{P}(C' \mid F)\,\mathrm{P}(F)}{\mathrm{P}(C' \mid F)\,\mathrm{P}(F) \;+\; \mathrm{P}(C' \mid F')\,\mathrm{P}(F')}$ $\quad$ {Bayes' theorem}

$= \dfrac{0.98 \times 0.54}{0.98 \times 0.54 \;+\; 0.92 \times 0.46}$

≈ 0.556

3 Let R be the event that the first marble is red,
B be the event that the second marble is blue,
and S be the event that both marbles are the same colour.

$\therefore \quad S = \underbrace{(R \cap B')}_{\text{both red}} \cup \underbrace{(R' \cap B)}_{\text{both blue}}$



As "both red" and "both blue" are mutually exclusive events, $\quad \mathrm{P}(S) = \mathrm{P}(R \cap B') + \mathrm{P}(R' \cap B)$

$= \mathrm{P}(B' \mid R)\,\mathrm{P}(R) \;+\; \mathrm{P}(B \mid R')\,\mathrm{P}(R')$

$= \frac{2}{9} \times \frac{3}{8} \;+\; \frac{4}{9} \times \frac{5}{8}$

$= \frac{13}{36}$

So, $\quad$ P(both marbles are blue | both are same colour)

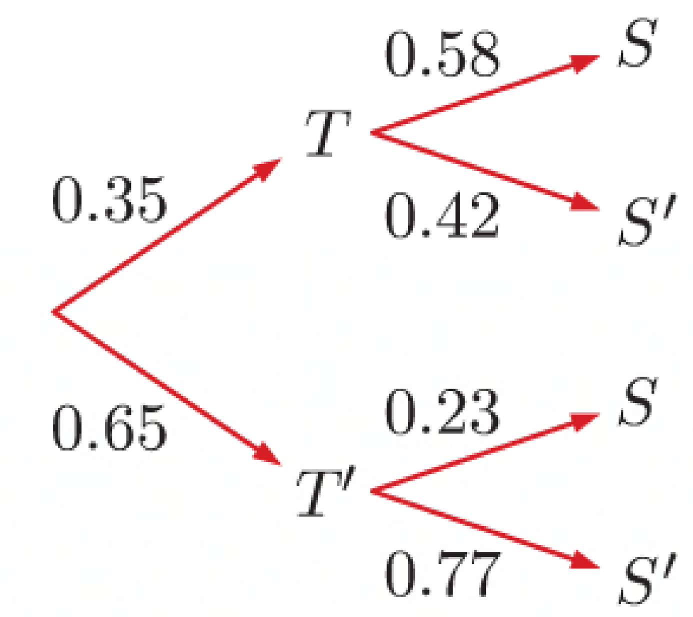
$= \mathrm{P}((R' \cap B) \mid S)$

$= \dfrac{\mathrm{P}(S \mid (R' \cap B))\,\mathrm{P}(R' \cap B)}{\mathrm{P}(S)}$ $\quad$ {Bayes' theorem}

$= \dfrac{1 \times \frac{4}{9} \times \frac{5}{8}}{\frac{13}{36}}$ $\quad$ {as $(R' \cap B) \subseteq S$}

$= \frac{10}{13}$

4 Let T be the event that a deer carries the TPC gene,
and S be the event that a deer carries the SD gene.



P(does not carry TPC gene | does carry SD gene)

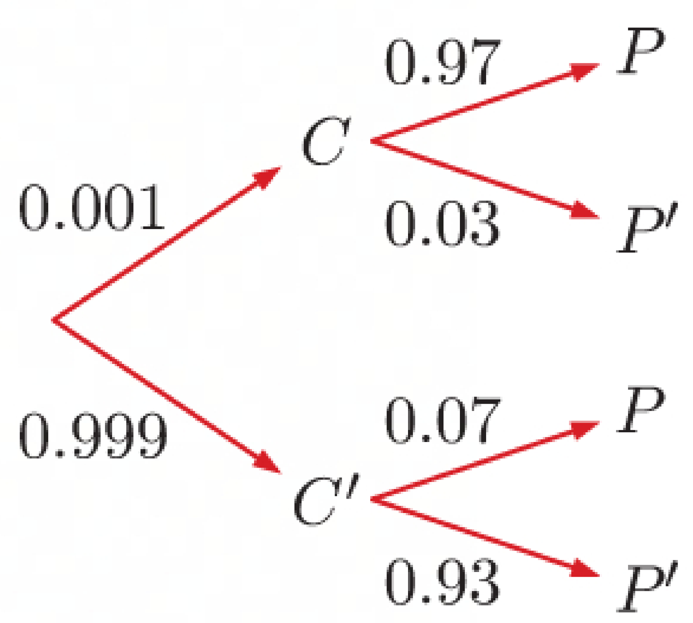
$$= \mathrm{P}(T' \mid S)$$

$$= \frac{\mathrm{P}(S \mid T')\,\mathrm{P}(T')}{\mathrm{P}(S \mid T)\,\mathrm{P}(T) \;+\; \mathrm{P}(S \mid T')\,\mathrm{P}(T')} \qquad \{\text{Bayes' theorem}\}$$

$$= \frac{0.23 \times 0.65}{0.58 \times 0.35 + 0.23 \times 0.65}$$

$$\approx 0.424$$

5 Let C be the event that a person has the cancer,
and P be the event that the test results are positive for the cancer.



P(patient has cancer | tests positive for cancer)

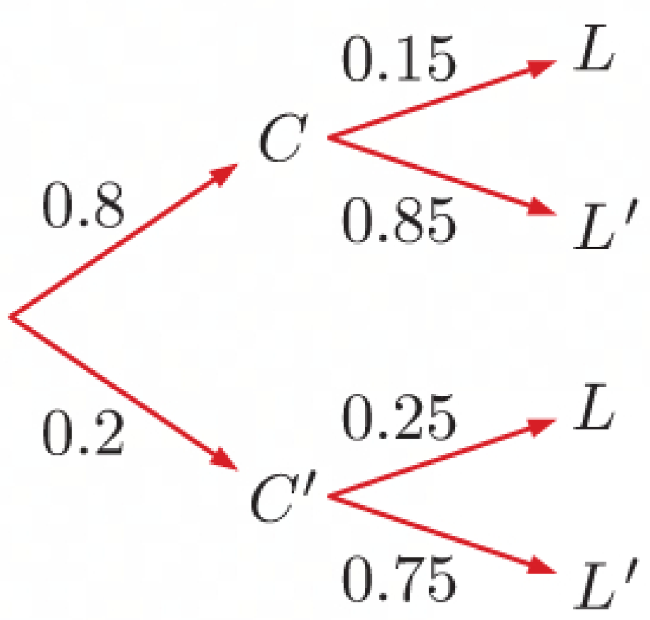
$$= \mathrm{P}(C \mid P)$$

$$= \frac{\mathrm{P}(P \mid C)\,\mathrm{P}(C)}{\mathrm{P}(P \mid C)\,\mathrm{P}(C) \;+\; \mathrm{P}(P \mid C')\,\mathrm{P}(C')} \qquad \{\text{Bayes' theorem}\}$$

$$= \frac{0.97 \times 0.001}{0.97 \times 0.001 + 0.07 \times 0.999}$$

$$\approx 0.0137$$

6 Let C be the event that the man drives his car to work,
and L be the event that the man is late.



P(the man rode his bicycle | the man is early)

$$= \mathrm{P}(C' \mid L')$$

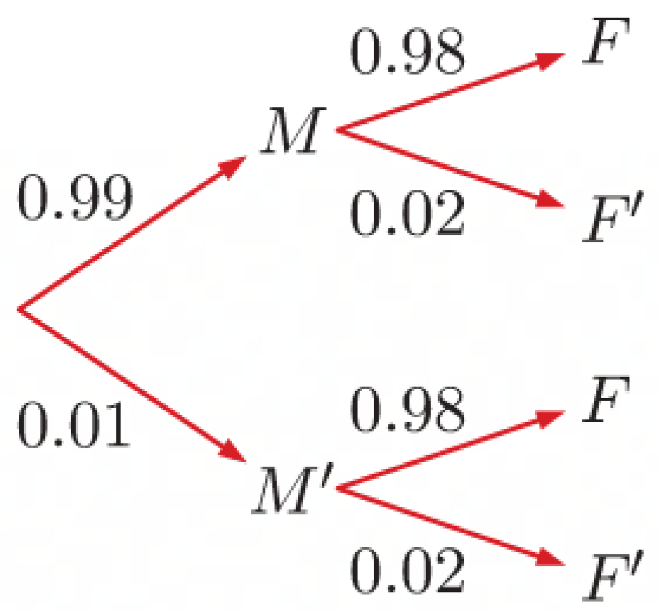
$$= \frac{\mathrm{P}(L' \mid C')\,\mathrm{P}(C')}{\mathrm{P}(L' \mid C)\,\mathrm{P}(C) \;+\; \mathrm{P}(L' \mid C')\,\mathrm{P}(C')} \qquad \{\text{Bayes' theorem}\}$$

$$= \frac{0.75 \times 0.2}{0.85 \times 0.8 + 0.75 \times 0.2}$$

$$= \frac{0.15}{0.83}$$

$$= \tfrac{15}{83}$$

7 Let M be the event that Hiran's mother is alive after ten years,
F be the event that Hiran's father is alive after ten years,
and O be the event that only one of them is alive after ten years.



$$\therefore \quad O = \underbrace{(M \cap F')}_{\text{only mother alive}} \cup \underbrace{(M' \cap F)}_{\text{only father alive}}$$

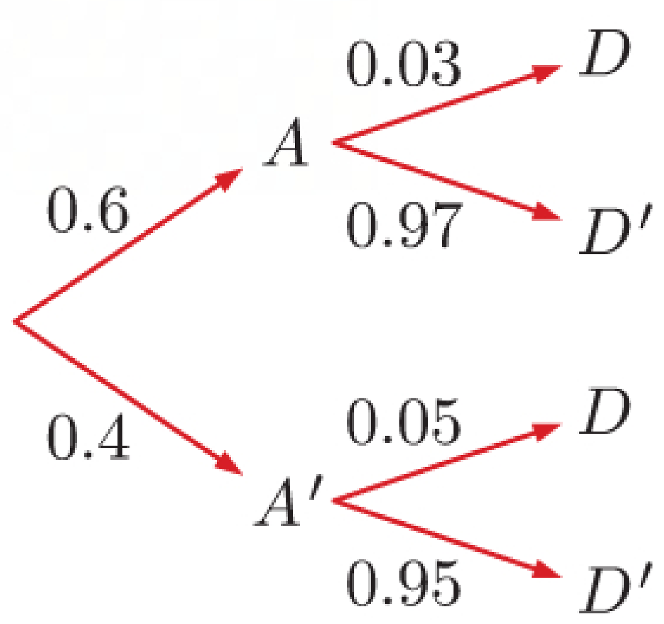
As "only mother alive" and "only father alive" are mutually exclusive events,

$$\begin{aligned} \text{P}(O) &= \text{P}(M \cap F') \cup \text{P}(M' \cap F) \\ &= \text{P}(F' \mid M)\,\text{P}(M) + \text{P}(F \mid M')\,\text{P}(M') \\ &= 0.02 \times 0.99 + 0.98 \times 0.01 \\ &= 0.0296 \end{aligned}$$

So, P(Hiran's mother is alive after ten years | only one is alive after ten years)

$$\begin{aligned} &= \text{P}(M \mid O) \\ &= \frac{\text{P}(O \mid M)\,\text{P}(M)}{\text{P}(O)} \qquad \{\text{Bayes' theorem}\} \\ &= \frac{\text{P}(F' \mid M)\,\text{P}(M)}{\text{P}(O)} \qquad \{\text{as } O \mid M = F' \mid M\} \\ &= \frac{0.02 \times 0.99}{0.0296} \\ &= \frac{0.0198}{0.0296} \\ &= \tfrac{99}{148} \end{aligned}$$

8 Let A be the event that a bottle is produced by machine A,
and D be the event that a bottle is defective.



a P(bottle comes from machine A | bottle is defective)

$$\begin{aligned} &= \text{P}(A \mid D) \\ &= \frac{\text{P}(D \mid A)\,\text{P}(A)}{\text{P}(D \mid A)\,\text{P}(A) + \text{P}(D \mid A')\,\text{P}(A')} \qquad \{\text{Bayes' theorem}\} \\ &= \frac{0.03 \times 0.6}{0.03 \times 0.6 + 0.05 \times 0.4} \\ &= \frac{0.018}{0.038} \\ &= \tfrac{9}{19} \end{aligned}$$

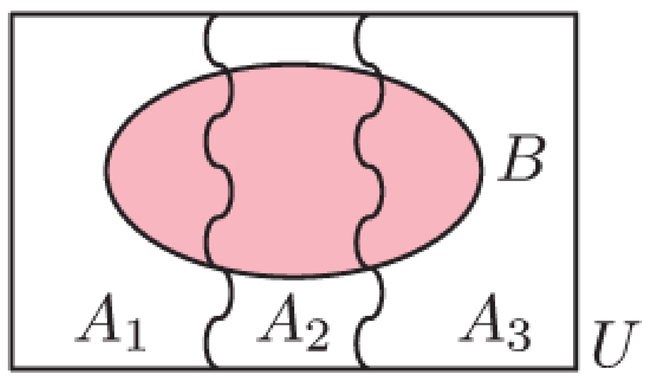
b P(bottle comes from machine B | bottle is defective) $= \mathrm{P}(A' \mid D)$

$$= 1 - \mathrm{P}(A \mid D)$$

$$= 1 - \tfrac{9}{19} \qquad \{\text{using } \mathbf{a}\}$$

$$= \tfrac{10}{19}$$

9 **a** $B = (B \cap A_1) \cup (B \cap A_2) \cup (B \cap A_3)$



and $B \cap A_1$, $B \cap A_2$, $B \cap A_3$ are pairwise disjoint.

$$\therefore\ \mathrm{P}(B) = \mathrm{P}(B \cap A_1) + \mathrm{P}(B \cap A_2) + \mathrm{P}(B \cap A_3)$$

$$= \mathrm{P}(B \mid A_1)\,\mathrm{P}(A_1) + \mathrm{P}(B \mid A_2)\,\mathrm{P}(A_2) + \mathrm{P}(B \mid A_3)\,\mathrm{P}(A_3)$$

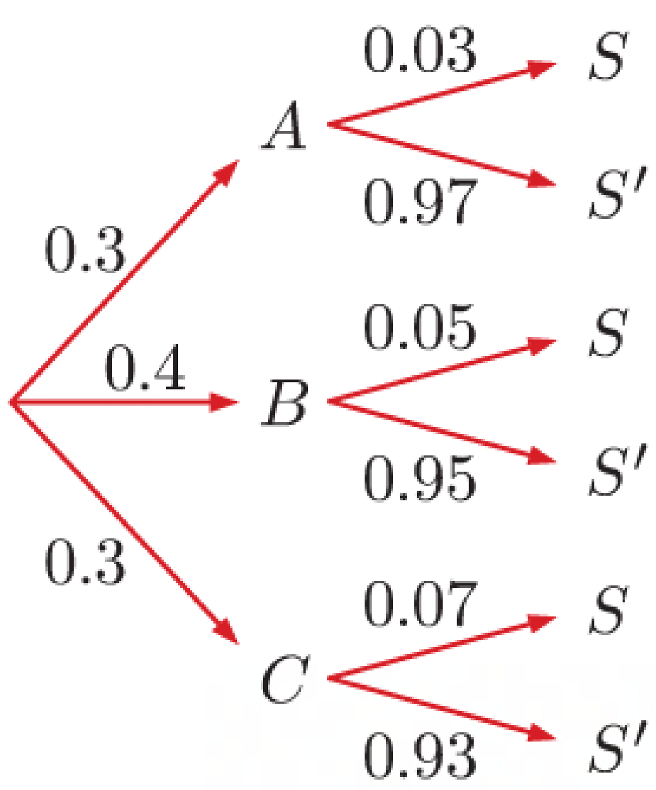
$$\left\{\text{since } \mathrm{P}(X \mid Y) = \frac{\mathrm{P}(X \cap Y)}{\mathrm{P}(Y)} \ \Rightarrow\ \mathrm{P}(X \cap Y) = \mathrm{P}(X \mid Y)\,\mathrm{P}(Y)\right\}$$

b $$\mathrm{P}(A_i \mid B) = \frac{\mathrm{P}(A_i \cap B)}{\mathrm{P}(B)} = \frac{\mathrm{P}(B \mid A_i)\,\mathrm{P}(A_i)}{\mathrm{P}(B)}, \qquad i \in \{1, 2, 3\}$$

where $\mathrm{P}(B) = \mathrm{P}(B \mid A_1)\,\mathrm{P}(A_1) + \mathrm{P}(B \mid A_2)\,\mathrm{P}(A_2) + \mathrm{P}(B \mid A_3)\,\mathrm{P}(A_3)$ {using **a**}

$$= \sum_{j=1}^{3} \mathrm{P}(B \mid A_j)\,\mathrm{P}(A_j)$$

10 Let A be the event that a newspaper is printed by press A,
B be the event that a newspaper is printed by press B,
C be the event that a newspaper is printed by press C,
and S be the event that a newspaper has streaks.



a P(newspaper does not have streaks)

$= \mathrm{P}(S')$

$= \mathrm{P}(S' \mid A)\,\mathrm{P}(A) + \mathrm{P}(S' \mid B)\,\mathrm{P}(B) + \mathrm{P}(S' \mid C)\,\mathrm{P}(C)$ {using **9 b**}

$= 0.97 \times 0.3 + 0.95 \times 0.4 + 0.93 \times 0.3$

$= 0.95$

b P(printed by press A | newspaper does not have streaks)

$= \mathrm{P}(A \mid S')$

$= \dfrac{\mathrm{P}(S' \mid A)\,\mathrm{P}(A)}{\mathrm{P}(S')}$ {Bayes' theorem}

$= \dfrac{0.97 \times 0.3}{0.95}$ {using **a**}

≈ 0.306

c P(printed by press A or press C | newspaper does have streaks)

$= \mathrm{P}((A \cup C) \mid S)$

$= 1 - \mathrm{P}(B \mid S)$

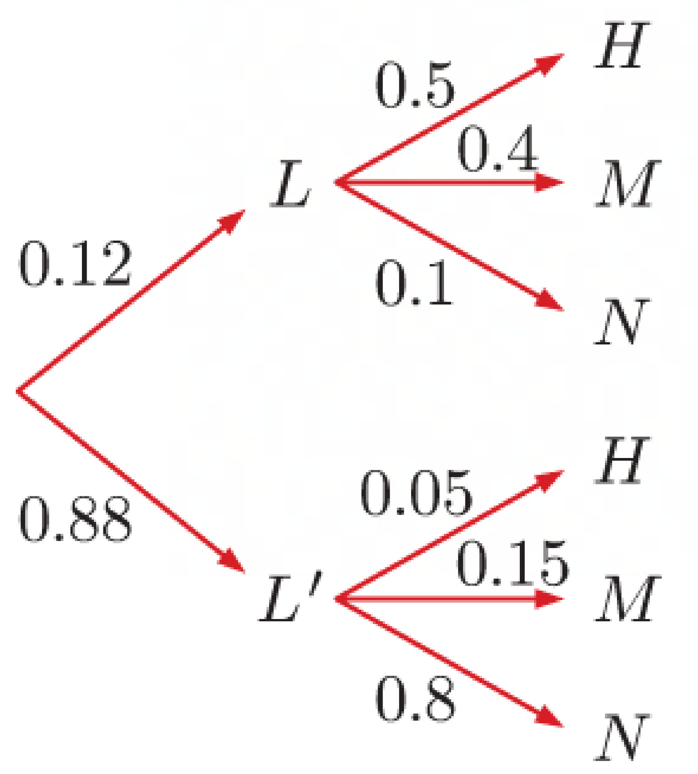
$= 1 - \dfrac{\mathrm{P}(S \mid B)\,\mathrm{P}(B)}{\mathrm{P}(S)}$ {Bayes' theorem}

$= 1 - \dfrac{\mathrm{P}(S \mid B)\,\mathrm{P}(B)}{1 - \mathrm{P}(S')}$

$= 1 - \dfrac{0.05 \times 0.4}{1 - 0.95}$ {using **a**}

$= 0.6$

11 Let L be the event that a person over 60 has lung cancer,
H be the event that a person over 60 was a heavy smoker,
M be the event that a person over 60 was a moderate smoker,
and N be the event that a person over 60 was a non-smoker.



a P(person over 60 was a heavy smoker) $= \mathrm{P}(H)$

$= \mathrm{P}(H \mid L)\,\mathrm{P}(L) + \mathrm{P}(H \mid L')\,\mathrm{P}(L')$

$= 0.5 \times 0.12 + 0.05 \times 0.88$

$= 0.104$

b P(person has lung cancer | person was a moderate smoker)

$= \mathrm{P}(L \mid M)$

$= \dfrac{\mathrm{P}(M \mid L)\,\mathrm{P}(L)}{\mathrm{P}(M \mid L)\,\mathrm{P}(L) + \mathrm{P}(M \mid L')\,\mathrm{P}(L')}$ {Bayes' theorem}

$= \dfrac{0.4 \times 0.12}{0.4 \times 0.12 + 0.15 \times 0.88}$

≈ 0.267

c P(person has lung cancer | person was a non-smoker)

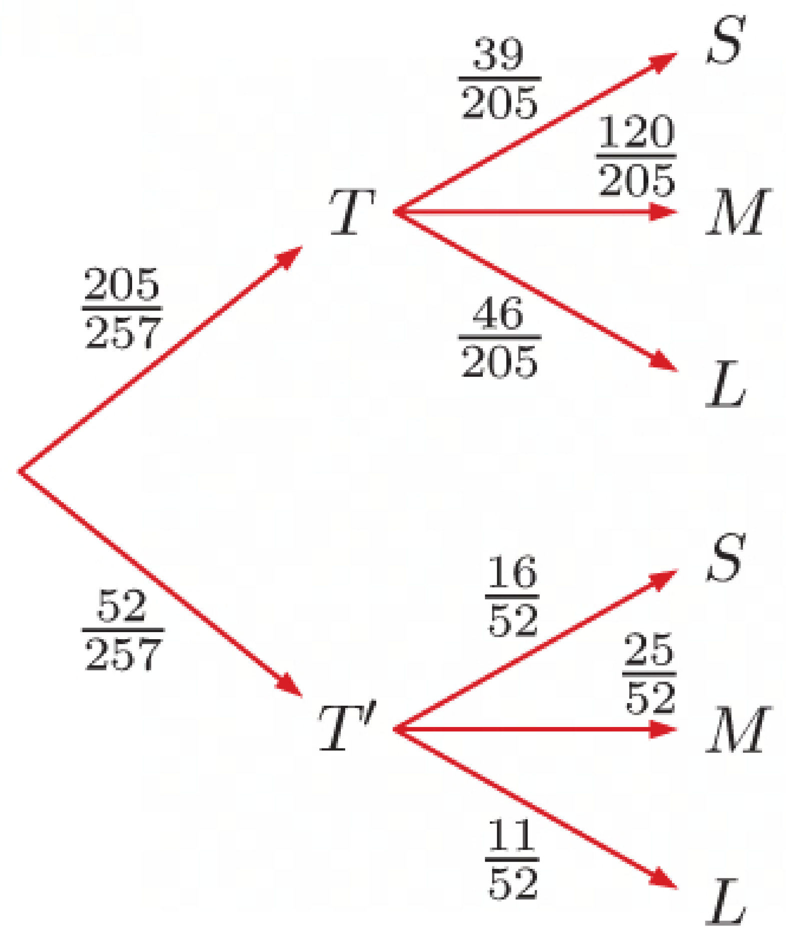
$= \mathrm{P}(L \mid N)$

$= \dfrac{\mathrm{P}(N \mid L)\,\mathrm{P}(L)}{\mathrm{P}(N \mid L)\,\mathrm{P}(L) + \mathrm{P}(N \mid L')\,\mathrm{P}(L')}$ {Bayes' theorem}

$= \dfrac{0.1 \times 0.12}{0.1 \times 0.12 + 0.8 \times 0.88}$

≈ 0.0168

12 Let T be the event that a person at the conference is a teacher,
S be the event that a person was given a small t-shirt,
M be the event that a person was given a medium t-shirt,
and L be the event that a person was given a large t-shirt.



a P(person was given a large t-shirt | person is a teacher) $= P(L \mid T)$
$= \frac{46}{205}$

b P(person is a teacher | person was given a large t-shirt)

$= P(T \mid L)$

$= \dfrac{P(L \mid T)\, P(T)}{P(L \mid T)\, P(T) + P(L \mid T')\, P(T')}$ {Bayes' theorem}

$= \dfrac{\frac{46}{205} \times \frac{205}{257}}{\frac{46}{205} \times \frac{205}{257} + \frac{11}{52} \times \frac{52}{257}}$

$= \frac{46}{57}$

c Bayes' theorem tells us that $P(L \mid T) = P(T \mid L)\, \dfrac{P(L)}{P(T)}$.

Our answers to **a** and **b** differ since $P(L) = P(L \mid T)\, P(T) + P(L \mid T')\, P(T')$
$= \frac{46}{205} \times \frac{205}{257} + \frac{11}{52} \times \frac{52}{257}$
$= \frac{57}{257} \neq P(T)$

REVIEW SET 11A

1 Total frequency $= 2 + 5 + 9 + 5 + 4 + 4 + 1 = 30$

Number of emails	*Frequency*
0	2
1	5
2	9
3	5
4	4
5	4
6	1

a P(Katie will send 5 emails) $= \dfrac{4}{30}$
≈ 0.13

b P(Katie will send less than 3 emails)

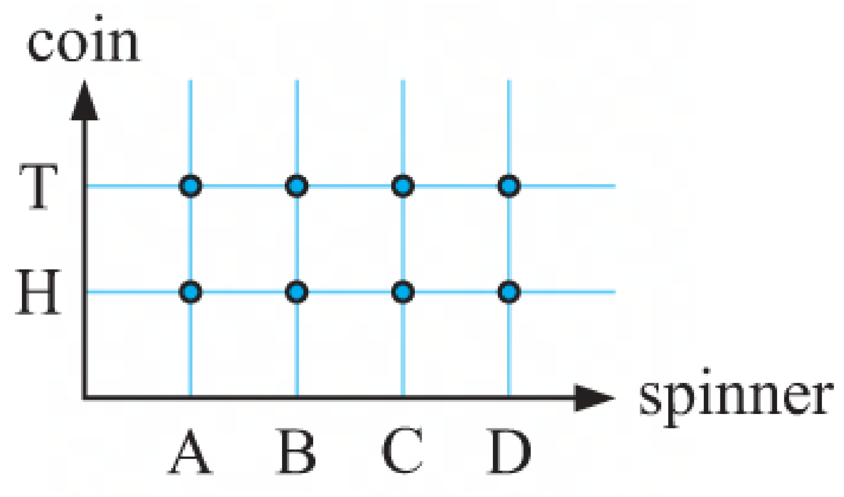
$=$ P(Katie will send 0, 1, or 2 emails)

$= \dfrac{2+5+9}{30}$

$= \dfrac{16}{30}$

≈ 0.53

2 **a**



b **i** P(a head and consonant)
$= \text{P}(\text{H} \cap \text{B}) + \text{P}(\text{H} \cap \text{C}) + \text{P}(\text{H} \cap \text{D})$
$= \frac{1}{8} + \frac{1}{8} + \frac{1}{8}$
$= \frac{3}{8}$

ii $\text{P(a tail and C)} = \frac{1}{8}$

iii $\text{P(a tail or a vowel or both)} = \text{P(T)} + \text{P(A)} - \text{P}(\text{T} \cap \text{A})$
$= \frac{4}{8} + \frac{2}{8} - \frac{1}{8}$
$= \frac{5}{8}$

3 **a** Two events are independent if the occurrence of either event does not affect the probability that the other occurs. For A and B independent, $\text{P}(A \cap B) = \text{P}(A) \times \text{P}(B)$.

b Two events A and B are mutually exclusive if they have no common outcomes.
$\text{P}(A \cup B) = \text{P}(A) + \text{P}(B)$

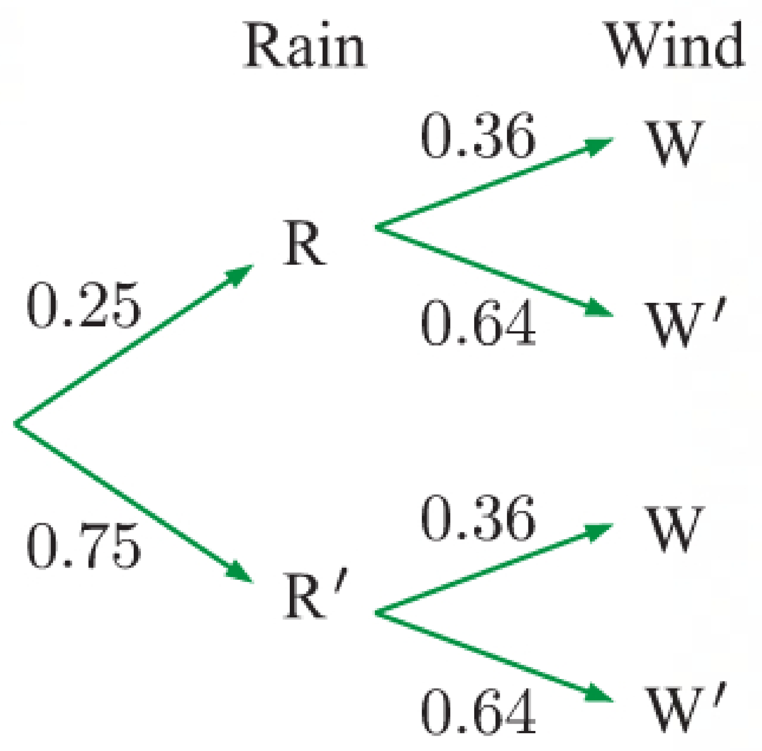
4 Let A represent student A solving the problem, B represent student B solving the problem, and C represent student C solving the problem.
$\text{P}(A) = 0.1, \quad \text{P}(B) = 0.2, \quad \text{P}(C) = 0.3$
$\text{P(at least one student solves it)} = 1 - \text{P(no-one solves it)}$
$= 1 - \text{P}(A' \cap B' \cap C')$
$= 1 - (0.9 \times 0.8 \times 0.7)$
$= 0.496$

5 $\text{P}(A) = x$ and $\text{P}(B') = 0.43$

a $\text{P}(A \cup B) = \text{P}(A) + \text{P}(B)$ {mutually exclusive events}
$= \text{P}(A) + [1 - \text{P}(B')]$
$= x + 1 - 0.43$
$= x + 0.57$

b $\text{P}(A \cup B) = x + 0.57 = 0.73$
$\therefore \quad x = 0.16$

6 **a**



b **i** P(rain and wind)
$= \mathrm{P}(\mathrm{R} \cap \mathrm{W})$
$= 0.25 \times 0.36$
$= 0.09$

ii P(rain or wind or both)
$= \mathrm{P}(\mathrm{R} \cap \mathrm{W}) + \mathrm{P}(\mathrm{R} \cap \mathrm{W}') + \mathrm{P}(\mathrm{R}' \cap \mathrm{W})$
$= 0.25 \times 0.36 \ + \ 0.25 \times 0.64 \ + \ 0.75 \times 0.36$
$= 0.52$

c It is assumed that the events rain and wind are independent.

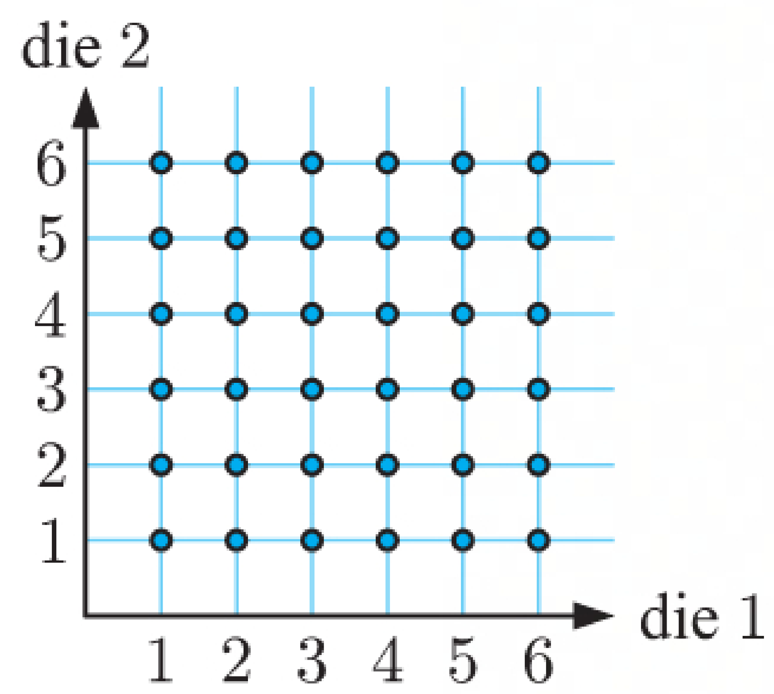
7 $\mathrm{P}(Y) = 0.35$ and $\mathrm{P}(X \cup Y) = 0.8$

a $\mathrm{P}(X \cap Y) = 0$ {X and Y are mutually exclusive events}

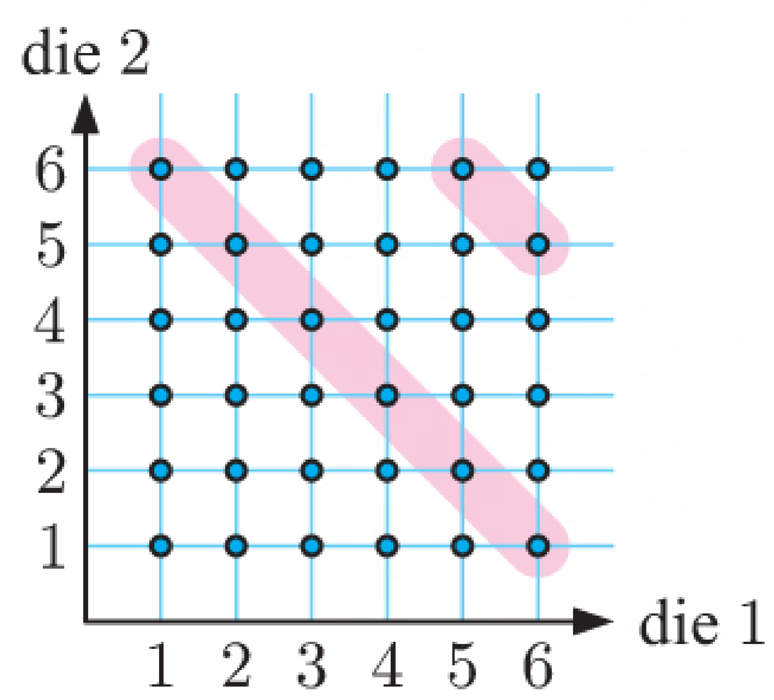
b $\mathrm{P}(X \cup Y) = \mathrm{P}(X) + \mathrm{P}(Y)$
$\therefore \ 0.8 = \mathrm{P}(X) + 0.35$
$\therefore \ \mathrm{P}(X) = 0.45$

c $\mathrm{P}(X \text{ or } Y \text{ but not both}) = \mathrm{P}(X \text{ or } Y)$ {X and Y mutually exclusive}
$= \mathrm{P}(X \cup Y)$
$= 0.8$

8 **a**

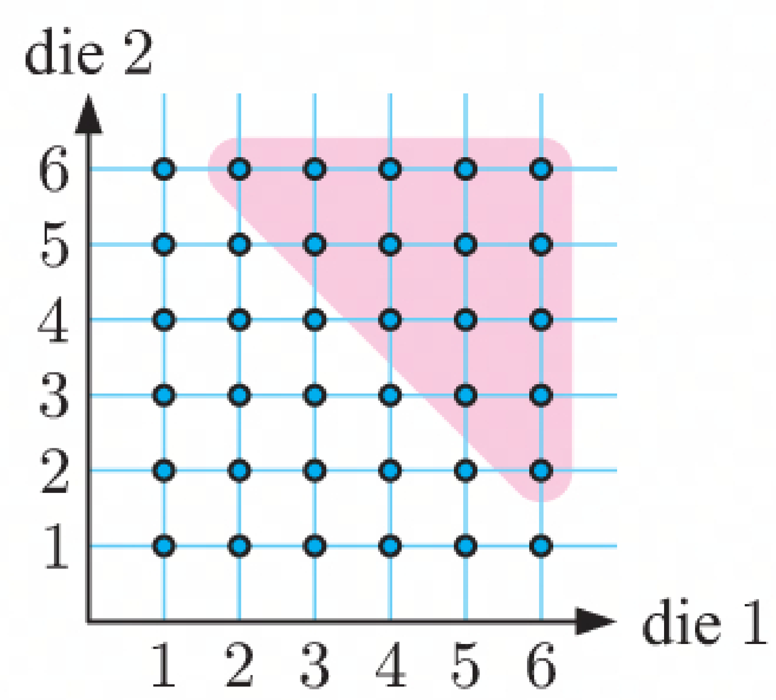


b **i**



$\mathrm{P}(\text{sum of 7 or 11}) = \dfrac{8}{36}$ {shaded}
$= \frac{2}{9}$

ii



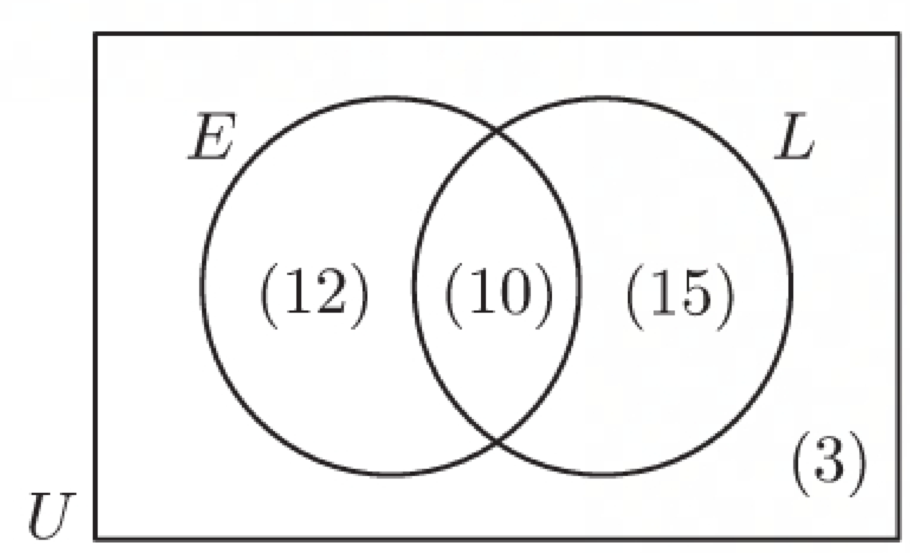
$\mathrm{P}(\text{sum of at least 8}) = \dfrac{15}{36}$ {shaded}
$= \frac{5}{12}$

9 **a** Let E represent the event that a student studies Economics, and L represent the event that a student studies Law.

$n(E) = 22, \quad n(L) = 25, \quad n(E' \cap L') = 3, \quad n(U) = 40$

$n(E \cup L) = n(U) - n(E' \cap L')$
$= 40 - 3$
$= 37$

$n(E \cup L) = n(E) + n(L) - n(E \cap L)$
$\therefore \ 37 = 22 + 25 - n(E \cap L)$
$\therefore \ n(E \cap L) = 10$
$\therefore \ n(E \cap L') = 22 - 10 = 12$ and $n(E' \cap L) = 25 - 10 = 15$



b **i** $P(E \cap L) = \frac{10}{40}$
$= \frac{1}{4}$

ii $P(\text{at least one}) = \frac{12+10+15}{40}$
$= \frac{37}{40}$

iii $P(E \mid L) = \frac{P(E \cap L)}{P(L)}$
$= \frac{10}{25}$
$= \frac{2}{5}$

10 $n = 5000$ seeds
$p = P(\text{tomato seed will germinate}) = 0.87$
$np = 5000 \times 0.87 = 4350$ tomato seeds are expected to germinate.

11 Total number of marbles $= 3 + 4 + 5 = 12$

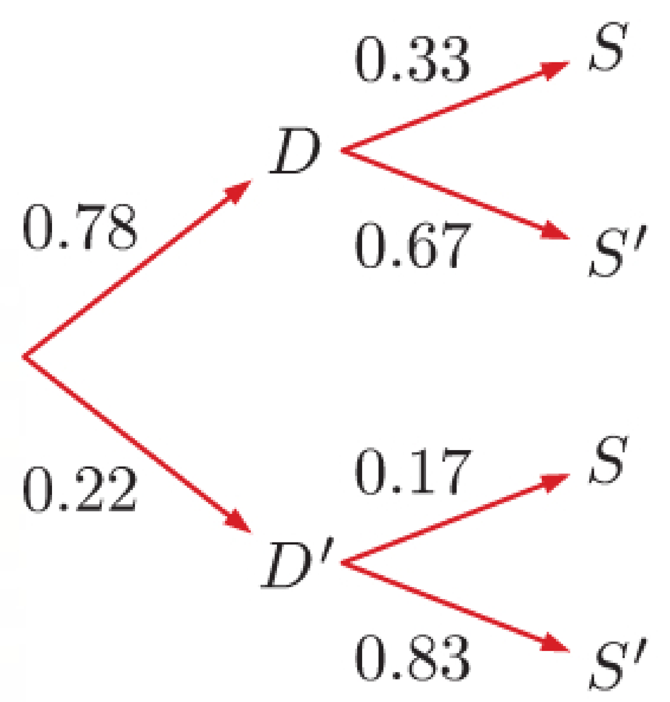
a $P(\text{both are blue}) = \frac{5}{12} \times \frac{5}{12}$
$= \frac{25}{144}$

b $P(\text{they are the same colour}) = P(2 \text{ reds}) + P(2 \text{ yellows}) + P(2 \text{ blues})$
$= \left(\frac{3}{12} \times \frac{3}{12}\right) + \left(\frac{4}{12} \times \frac{4}{12}\right) + \left(\frac{5}{12} \times \frac{5}{12}\right)$
$= \frac{9+16+25}{144}$
$= \frac{50}{144}$
$= \frac{25}{72}$

c $P(\text{at least one is red})$
$= P(2 \text{ reds}) + P(1 \text{ red, } 1 \text{ yellow}) + P(1 \text{ yellow, } 1 \text{ red}) + P(1 \text{ red, } 1 \text{ blue}) + P(1 \text{ blue, } 1 \text{ red})$
$= \left(\frac{3}{12} \times \frac{3}{12}\right) + \left(\frac{3}{12} \times \frac{4}{12}\right) + \left(\frac{4}{12} \times \frac{3}{12}\right) + \left(\frac{3}{12} \times \frac{5}{12}\right) + \left(\frac{5}{12} \times \frac{3}{12}\right)$
$= \frac{9+12+12+15+15}{144}$
$= \frac{63}{144}$
$= \frac{7}{16}$

d $P(\text{exactly one is yellow})$
$= P(1 \text{ red, } 1 \text{ yellow}) + P(1 \text{ yellow, } 1 \text{ red}) + P(1 \text{ yellow, } 1 \text{ blue}) + P(1 \text{ blue, } 1 \text{ yellow})$
$= \left(\frac{3}{12} \times \frac{4}{12}\right) + \left(\frac{4}{12} \times \frac{3}{12}\right) + \left(\frac{4}{12} \times \frac{5}{12}\right) + \left(\frac{5}{12} \times \frac{4}{12}\right)$
$= \frac{12+12+20+20}{144}$
$= \frac{64}{144}$
$= \frac{4}{9}$

12 Let S be the event that a person applying for an executive position is successful, and D be the event that the applicant has a university degree.



P(applicant does not have university degree | applicant is successful)

$= \mathrm{P}(D' \mid S)$

$= \dfrac{\mathrm{P}(S \mid D')\,\mathrm{P}(D')}{\mathrm{P}(S \mid D)\,\mathrm{P}(D) + \mathrm{P}(S \mid D')\,\mathrm{P}(D')}$ {Bayes' theorem}

$= \dfrac{0.17 \times 0.22}{0.33 \times 0.78 + 0.17 \times 0.22}$

≈ 0.127

13 **a**

	Female	*Male*	*Total*
Smoker	20	40	60
Non-smoker	70	70	140
Total	90	110	200

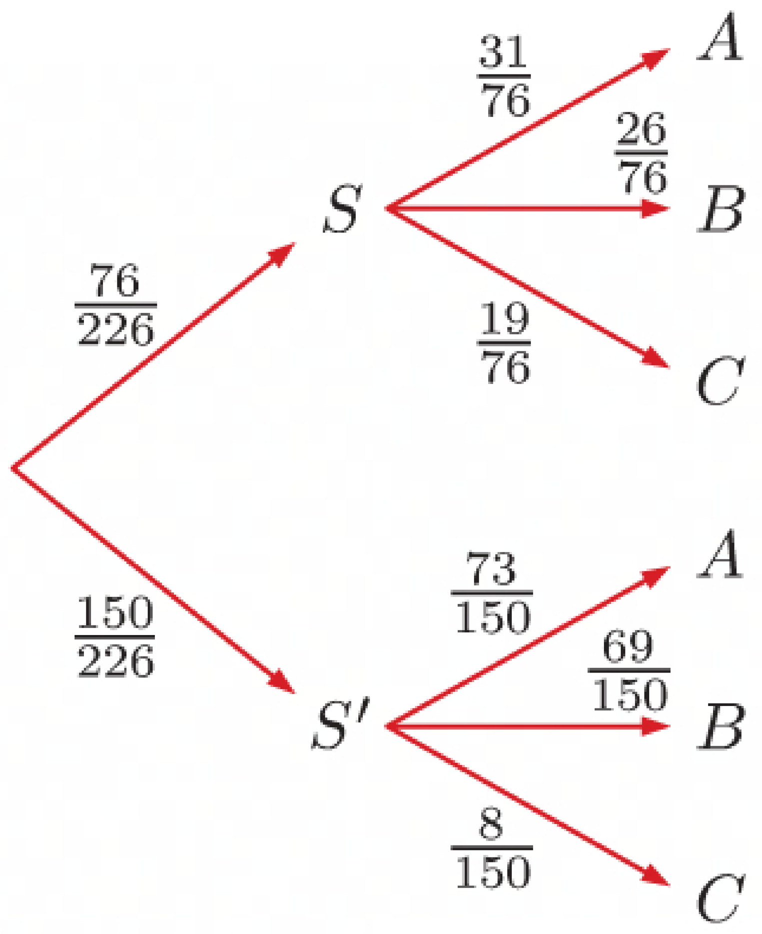
b **i** P(a female non-smoker) $= \dfrac{70}{200}$
$= \frac{7}{20}$

ii P(male | non-smoker) $= \dfrac{70}{140}$
$= \frac{1}{2}$

c **i** P(both are non-smoker females) $= \frac{70}{200} \times \frac{69}{199}$
$= \frac{4830}{39\,800}$
≈ 0.121

ii P(one is smoker and other is non-smoker) $= \frac{60}{200} \times \frac{140}{199} + \frac{140}{200} \times \frac{60}{199}$
$= \frac{16\,800}{39\,800}$
≈ 0.422

14 Let S be the event that a politician is a Senator,
A be the event that a politician is a member of the Coalition,
B be the event that a politician is a member of the Opposition,
and C be the event that a politician is a crossbencher.



P(politician is an MP | politician is a member of the Opposition)

$= \text{P}(S' \mid B)$

$$= \frac{\text{P}(B \mid S')\,\text{P}(S')}{\text{P}(B \mid S)\,\text{P}(S) + \text{P}(B \mid S')\,\text{P}(S')} \qquad \{\text{Bayes' theorem}\}$$

$$= \frac{\frac{69}{150} \times \frac{150}{226}}{\frac{26}{76} \times \frac{76}{226} + \frac{69}{150} \times \frac{150}{226}}$$

$= \frac{69}{95}$

15 $\text{P}(A) = \frac{2}{5}$, $\text{P}(B) = \frac{3}{10}$, $\text{P}(B \mid A) = \frac{1}{2}$

a $$\text{P}(B \mid A) = \frac{\text{P}(B \cap A)}{\text{P}(A)}$$

$$\therefore \quad \frac{1}{2} = \frac{\text{P}(B \cap A)}{\frac{2}{5}}$$

$\therefore$ $\text{P}(B \cap A) = \frac{1}{5}$

$\therefore$ $\text{P}(A \cap B) = \frac{1}{5}$

b $\text{P}(B \mid A) = \frac{1}{2} \neq \text{P}(B)$

$\therefore$ A and B are not independent.

c $$\text{P}(A \mid B) = \frac{\text{P}(A \cap B)}{\text{P}(B)}$$

$$= \frac{\frac{1}{5}}{\frac{3}{10}}$$

$= \frac{2}{3}$

16 We need to find the probability that one of each of the six faces appears upmost when the six cubes are rolled.
Each cube is identical, so the probability of any of the six faces of the picture appearing on the upmost face is $\frac{1}{6}$.
For the first cube, any of the six faces can appear upmost. For the second cube, one of the six faces is already upmost so there are five other faces possible. For the third cube, two of the six faces are already upmost so there are four other faces possible, and so on.

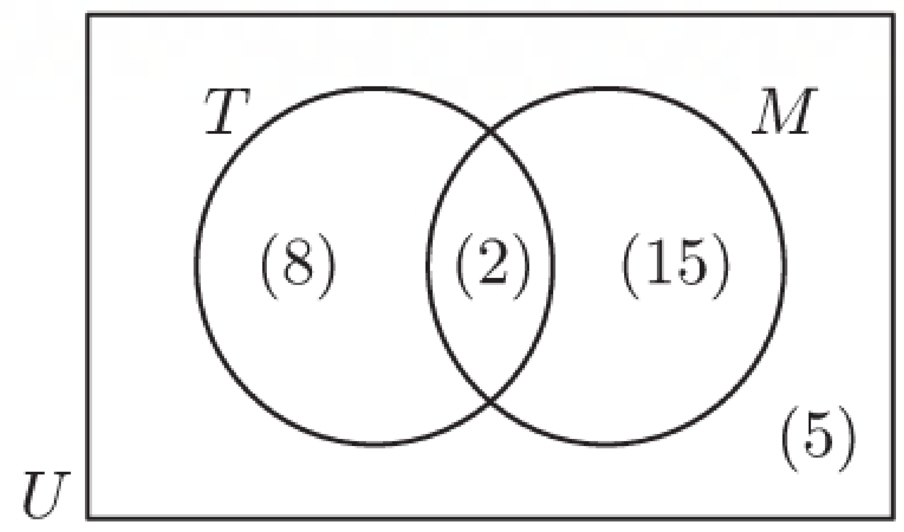
P(upmost faces can be assembled into the original image)
$= \text{P(one of each of the six faces appears upmost when all six cubes are rolled)}$
$= \frac{6}{6} \times \frac{5}{6} \times \frac{4}{6} \times \frac{3}{6} \times \frac{2}{6} \times \frac{1}{6}$
$= \dfrac{720}{46\,656}$
$= \frac{5}{324}$

REVIEW SET 11B

1 **a** $n(T) = 10, \quad n(M) = 17, \quad n((T \cup M)') = 5, \quad n(U) = 30$

$n(T \cup M) = n(U) - n((T \cup M)')$
$= 30 - 5$
$= 25$

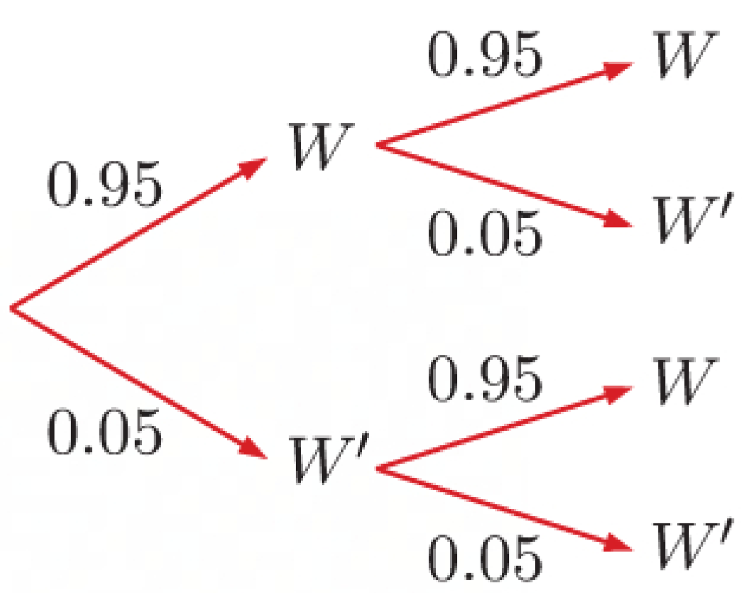
$n(T \cup M) = n(T) + n(M) - n(T \cap M)$
$\therefore \ 25 = 10 + 17 - n(T \cap M)$
$\therefore \ n(T \cap M) = 2$
$\therefore \ n(T \cap M') = 10 - 2 = 8$ and $n(T' \cap M) = 17 - 2 = 15$



b **i** $\text{P}(T \cap M) = \dfrac{2}{30}$
$= \frac{1}{15}$

ii $\text{P}((T \cap M) \mid M) = \dfrac{\text{P}((T \cap M) \cap M)}{\text{P}(M)}$
$= \dfrac{\frac{2}{30}}{\frac{17}{30}}$
$= \frac{2}{17}$

2 Let W represent the photocopier working.



P(works on at least one day)
$= \text{P}(W \cap W) + \text{P}(W \cap W') + \text{P}(W' \cap W)$
$= 0.95 \times 0.95 + 0.95 \times 0.05 + 0.05 \times 0.95$
$= 0.9975$

or P(works on at least one day)
$= 1 - \text{P(does not work on either day)}$
$= 1 - 0.05 \times 0.05$
$= 1 - 0.0025$
$= 0.9975$

3 $P(A) = 0.4$ and $P(B) = 0.7$

a $P(A \cap B) = P(A) \times P(B)$ $\{A$ and B are independent events$\}$

$= 0.4 \times 0.7$

$= 0.28$

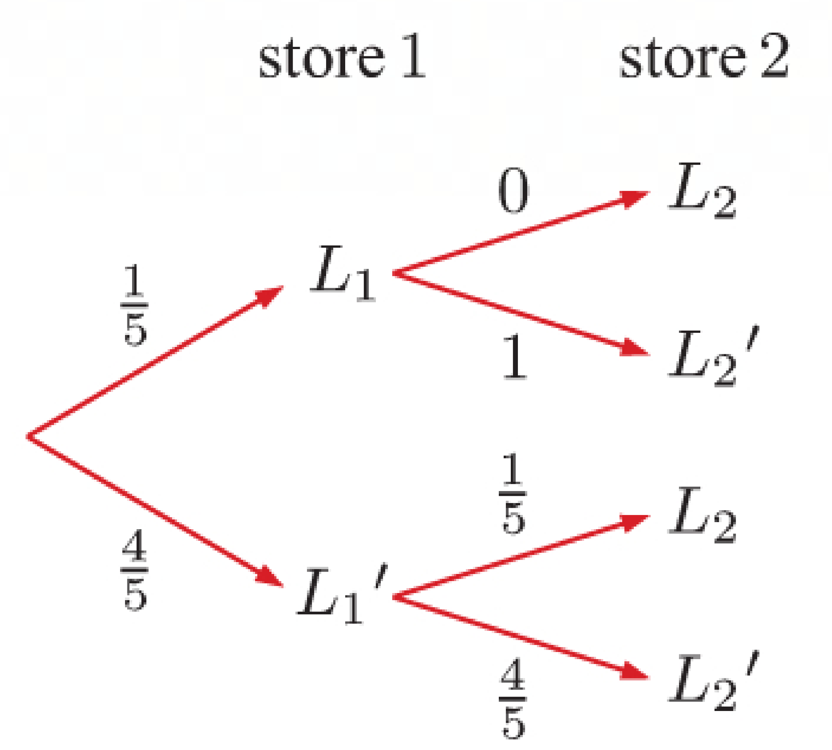
For A and B to be mutually exclusive, $P(A \cap B)$ must be equal to 0. This is not the case, so A and B cannot be mutually exclusive.

b $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$= 0.4 + 0.7 - 0.28$

$= 0.82$

4 Let L_i be the event that the salesman leaves his sunglasses in store i.



P(salesman left sunglasses in first store given that he left them in one of the stores)

$= P(L_1 \mid L_1 \text{ or } L_2)$

$= \dfrac{P(L_1 \cap (L_1 \cup L_2))}{P(L_1 \cup L_2)}$

$= \dfrac{P(L_1)}{P((L_1 \cap L_2') \cup (L_1' \cap L_2))}$

$= \dfrac{\frac{1}{5}}{\frac{1}{5} \times 1 + \frac{4}{5} \times \frac{1}{5}}$

$= \frac{5}{9}$

5 **a** 15 men prefer tea, so $50 - 15 = 35$ men prefer coffee.

$\therefore$ $n(C \cap M) = 35$ and $n(C' \cap M) = 15$

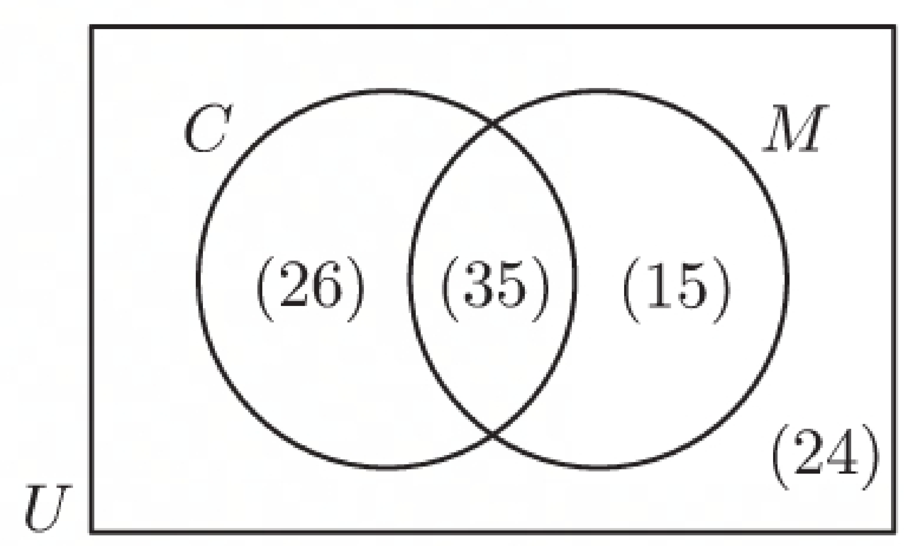
24 women prefer tea, so $50 - 24 = 26$ women prefer coffee.

$\therefore$ $n(C \cap M') = 26$

$n(C' \cap M') = n(U) - n(C \cup M)$

$= 100 - 26 - 35 - 15$

$= 24$

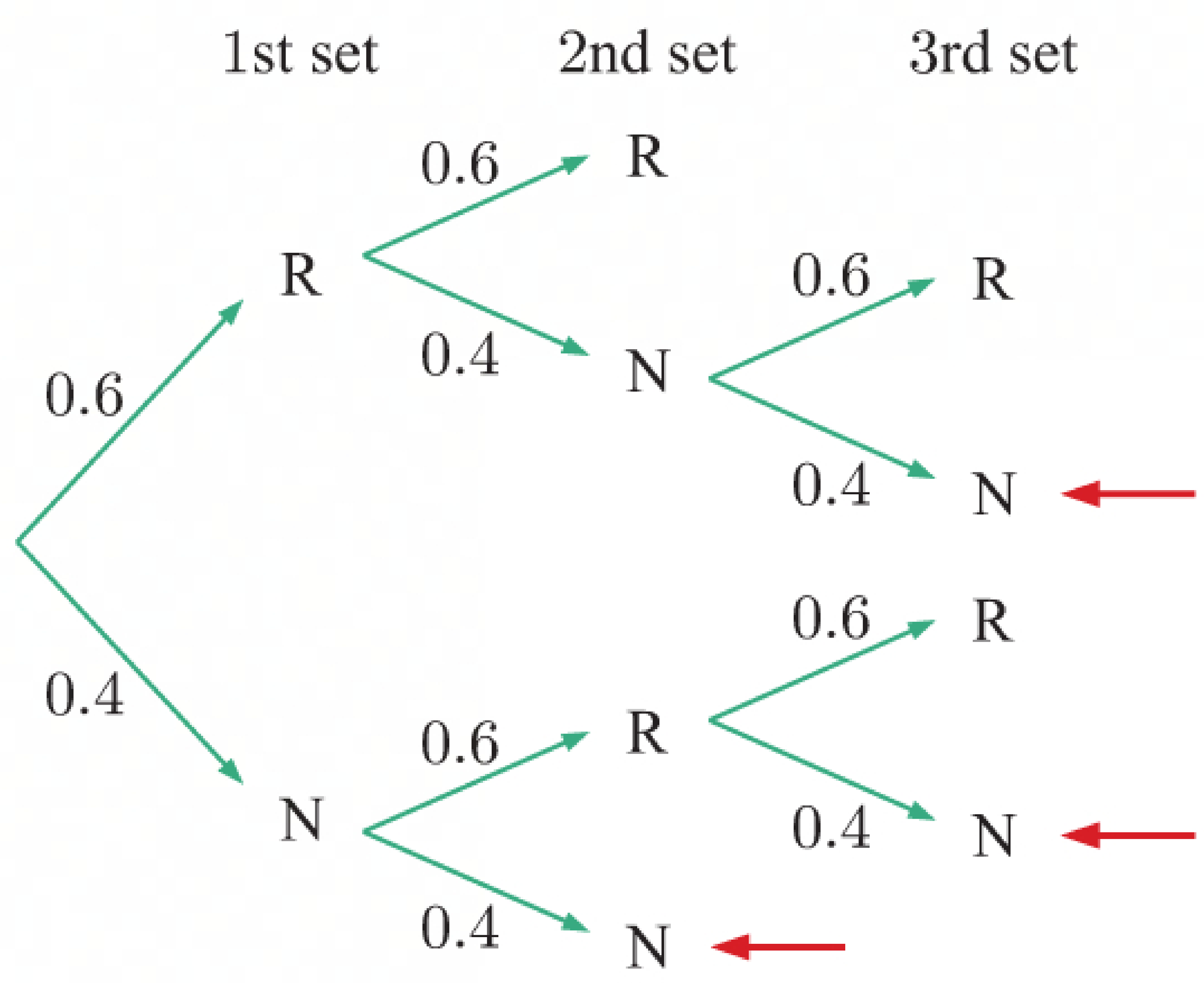


b $P(M \mid C) = \dfrac{P(M \cap C)}{P(C)}$

$= \dfrac{\frac{35}{100}}{\frac{26+35}{100}}$

$= \frac{35}{61}$

6 **a** Let R represent Rolf winning a set and N represent Niklas winning a set.



b $\text{P(Niklas will win the match)} = \text{P}(\text{R} \cap \text{N} \cap \text{N}) + \text{P}(\text{N} \cap \text{R} \cap \text{N}) + \text{P}(\text{N} \cap \text{N})$ {branches marked ←}

$= 0.6 \times 0.4 \times 0.4 + 0.4 \times 0.6 \times 0.4 + 0.4 \times 0.4$

$= 0.352$

7 $\text{P}(A) = 0.8$ and $\text{P}(B) = 0.65$

a $\text{P}(A \cup B) = \text{P}(A) + \text{P}(B) - \text{P}(A \cap B)$

$= 0.8 + 0.65 - [\text{P}(A) \times \text{P}(B)]$ {A and B are independent events}

$= 1.45 - (0.8 \times 0.65)$

$= 0.93$

b $\text{P}(A \mid B) = \text{P}(A)$ {A and B are independent events}

$= 0.8$

c $\text{P}(A' \mid B') = \dfrac{\text{P}(A' \cap B')}{\text{P}(B')}$

$= \dfrac{\text{P}((A \cup B)')}{\text{P}(B')}$

$= \dfrac{1 - 0.93}{1 - 0.65}$

$= \dfrac{0.07}{0.35}$

$= 0.2$

d $\text{P}(B \mid A) = \text{P}(B)$ {A and B are independent events}

$= 0.65$

8 **a** P(win first 3 prizes) = P(win 1st prize) × P(win 2nd prize given that you won 1st prize)

× P(win 3rd prize given that you won 1st and 2nd prizes)

$= \frac{4}{500} \times \frac{3}{499} \times \frac{2}{498}$

$\approx 0.000\,000\,193$

b P(win at least one of first 3 prizes)

= 1 − P(do not win any of first 3 prizes)

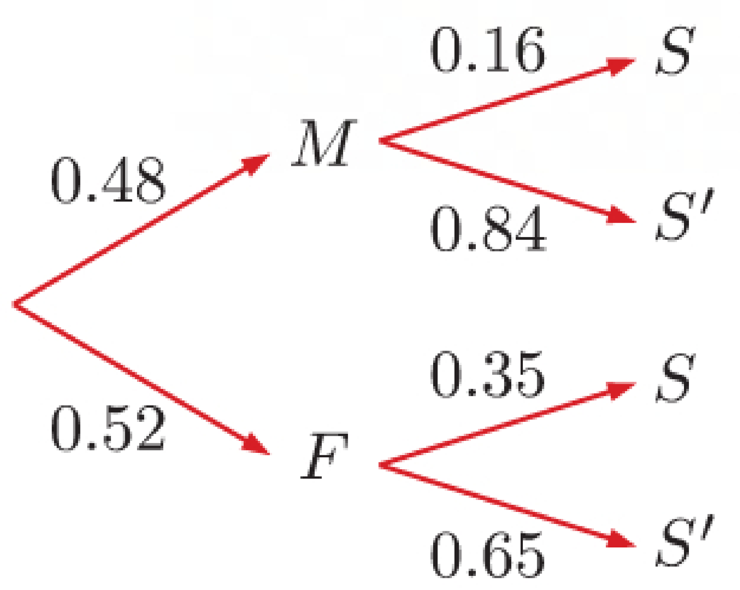
= 1 − [P(do not win 1st prize) × P(do not win 2nd prize given that you did not win 1st prize)

× P(do not win 3rd prize given that you did not win 1st or 2nd prizes)]

$= 1 - \left(\frac{496}{500} \times \frac{495}{499} \times \frac{494}{498}\right)$

≈ 0.0239

9 Let M represent a male student, F represent a female student, and S represent a student who participates in the survey.



a P(student will participate in the survey)
$= \mathrm{P}(M \cap S) + \mathrm{P}(F \cap S)$
$= 0.48 \times 0.16 + 0.52 \times 0.35$
$= 0.2588$

b P(student is female | will participate in survey)
$= \mathrm{P}(F \mid S)$
$= \dfrac{\mathrm{P}(F \cap S)}{\mathrm{P}(S)}$
$= \dfrac{0.52 \times 0.35}{0.2588}$
≈ 0.703

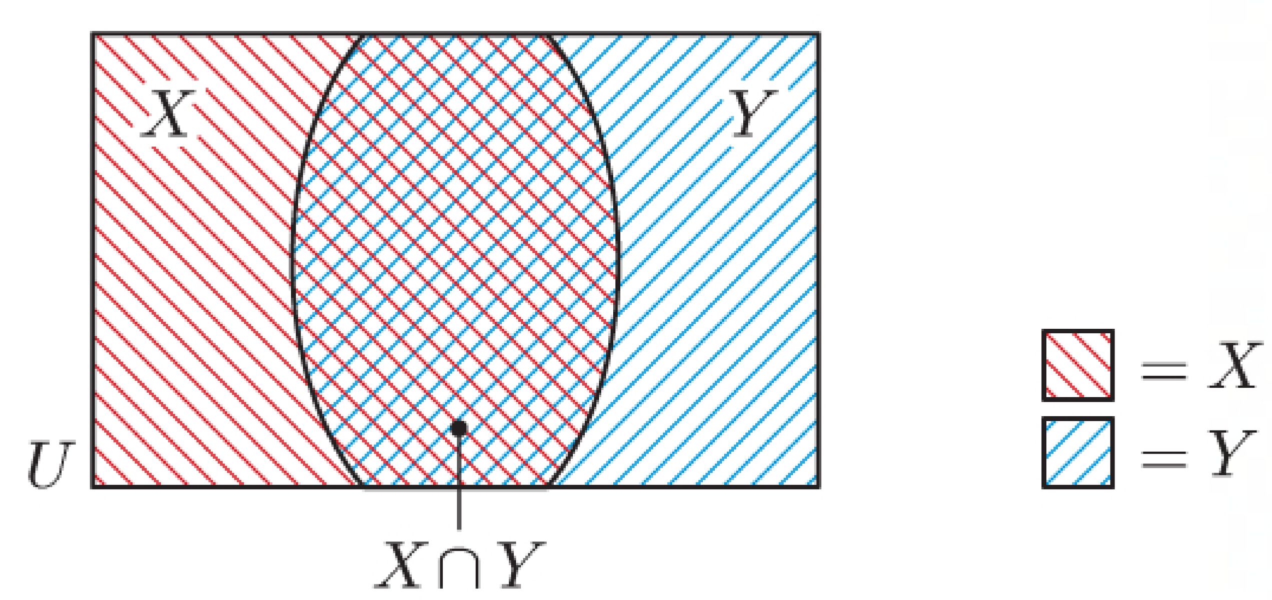
10 $\mathrm{P}(A) = \frac{3}{7}$ and $\mathrm{P}(B') = \frac{2}{3}$

a $\mathrm{P}(B) = 1 - \mathrm{P}(B')$
$= 1 - \frac{2}{3}$
$= \frac{1}{3}$

b **i** $\mathrm{P}(A \cup B) = \mathrm{P}(A) + \mathrm{P}(B)$ {A and B are mutually exclusive}
$= \frac{3}{7} + \frac{1}{3}$
$= \frac{16}{21}$

ii $\mathrm{P}(A \cup B) = \mathrm{P}(A) + \mathrm{P}(B) - \mathrm{P}(A \cap B)$
$= \frac{3}{7} + \frac{1}{3} - [\mathrm{P}(A) \times \mathrm{P}(B)]$ {A and B are independent}
$= \frac{16}{21} - \left(\frac{3}{7} \times \frac{1}{3}\right)$
$= \dfrac{16 - 3}{21}$
$= \frac{13}{21}$

11 Since $X' \cap Y' = \varnothing$, every element of U is in either X or Y or both.
$\therefore$ we can construct the Venn diagram alongside:



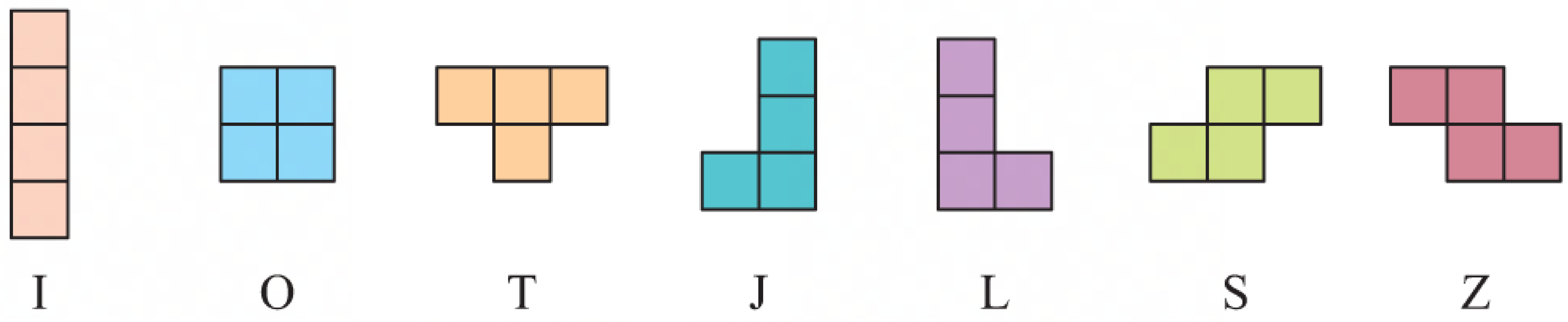
Now $\mathrm{P}(X' \mid Y) = \dfrac{\mathrm{P}(X' \cap Y)}{\mathrm{P}(Y)}$

$\therefore \frac{2}{3} = \dfrac{\mathrm{P}(X' \cap Y)}{\frac{5}{6}}$

$\therefore \mathrm{P}(X' \cap Y) = \frac{2}{3} \times \frac{5}{6}$
$= \frac{5}{9}$

$\mathrm{P}(X) = 1 - \mathrm{P}(X' \cap Y)$
$= 1 - \frac{5}{9}$
$= \frac{4}{9}$

12



a P(player is given at least one "I") $= 1 - $ P(player is not given any "I"s)

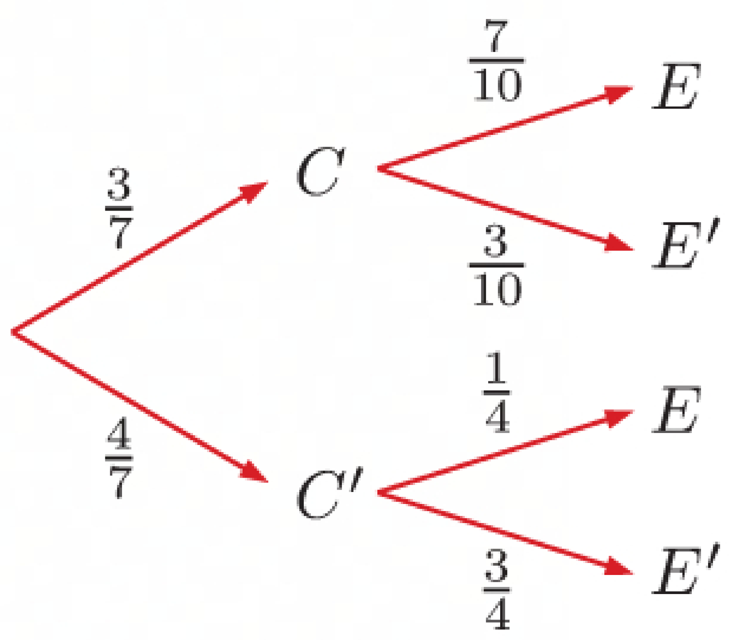
$$= 1 - \left(\tfrac{6}{7} \times \tfrac{6}{7} \times \tfrac{6}{7} \times \tfrac{6}{7} \times \tfrac{6}{7} \times \tfrac{6}{7} \times \tfrac{6}{7}\right)$$

$$= 1 - \frac{279\,936}{823\,543}$$

$$\approx 0.660$$

b The expected number of "O"s in a sequence of 100 pieces $= \frac{100}{7} \approx 14.3$ or 14 pieces.

13 Let C represent Jon going cycling, and E represent Jon having eggs for breakfast.



a P(Jon has eggs for breakfast) $= \mathrm{P}(C \cap E) + \mathrm{P}(C' \cap E)$

$$= \tfrac{3}{7} \times \tfrac{7}{10} + \tfrac{4}{7} \times \tfrac{1}{4}$$

$$= \tfrac{3}{10} + \tfrac{1}{7}$$

$$= \frac{21 + 10}{70}$$

$$= \tfrac{31}{70}$$

b P(Jon goes cycling | he has eggs for breakfast) $= \mathrm{P}(C \mid E)$

$$= \frac{\mathrm{P}(C \cap E)}{\mathrm{P}(E)}$$

$$= \frac{\frac{3}{7} \times \frac{7}{10}}{\frac{31}{70}}$$

$$= \tfrac{21}{31}$$

14 After 2 pregnancies the woman has given birth to 3 children.

$\therefore$ she must have had twins and then a single baby, *or* a single baby and then twins.

The probabilities of having twins or a single baby remain constant. The probability of having twins first is the same as having twins second.

$\therefore$ P(woman had twins first) $= \frac{1}{2}$

15 We extend the table to include totals for each row and column.

	Red	*Yellow*	*Blue*	*Total*
Large	12	5	9	26
Medium	15	8	10	33
Small	24	11	6	41
Total	51	24	25	100

a **i** There are 100 balloons in total in the pack.

ii There are 33 medium balloons in the pack.

b **i** $P(\text{balloon is not yellow}) = P(\text{balloon is red or blue})$

$$= \frac{51+25}{100}$$

$$= \frac{76}{100}$$

$$= \frac{19}{25}$$

ii $P(\text{balloon is either medium or small}) = \dfrac{33+41}{100}$

$$= \frac{74}{100}$$

$$= \frac{37}{50}$$

c **i** $P(\text{both balloons are red})$

$= P(\text{first balloon is red}) \times P(\text{second balloon is red given that first is red})$

$= \frac{51}{100} \times \frac{50}{99}$

$= \frac{51}{198}$

$= \frac{17}{66}$

ii $P(\text{neither of the balloons are large})$

$= P(\text{first balloon is medium or small})$

$\quad \times P(\text{second balloon is medium or small given that first is medium or small})$

$= \dfrac{33+41}{100} \times \dfrac{33+41-1}{99}$

$= \dfrac{74}{100} \times \dfrac{73}{99}$

$= \dfrac{5402}{9900}$

$= \frac{2701}{4950}$

iii $P(\text{exactly one balloon is blue})$

$= P(\text{first balloon is blue and second balloon is red})$

$\quad + P(\text{first balloon is blue and second balloon is yellow})$

$\quad + P(\text{first balloon is red and second balloon is blue})$

$\quad + P(\text{first balloon is yellow and second balloon is blue})$

$= \frac{25}{100} \times \frac{51}{99} + \frac{25}{100} \times \frac{24}{99} + \frac{51}{100} \times \frac{25}{99} + \frac{24}{100} \times \frac{25}{99}$

$= \dfrac{1275+600+1275+600}{9900}$

$= \dfrac{3750}{9900}$

$= \frac{25}{66}$

iv P(at least one of the balloons is blue)

$= 1 - \text{P(neither of the balloons is blue)}$

$= 1 - [\text{P(first balloon is red and second balloon is yellow)}$

$\quad + \text{P(first balloon is yellow and second balloon is red)}$

$\quad + \text{P(both balloons are red)} + \text{P(both balloons are yellow)}]$

$= 1 - \left(\frac{51}{100} \times \frac{24}{99} + \frac{24}{100} \times \frac{51}{99} + \frac{51}{100} \times \frac{50}{99} + \frac{24}{100} \times \frac{23}{99}\right)$

$= 1 - \frac{5550}{9900}$

$= \frac{4350}{9900}$

$= \frac{29}{66}$

d **i** P(all three balloons are small and yellow)

$= \text{P(first balloon is small and yellow)}$

$\quad \times \text{P(second balloon is small and yellow given that first is small and yellow)}$

$\quad \times \text{P(third balloon is small and yellow given that first two are small and yellow)}$

$= \frac{11}{100} \times \frac{10}{99} \times \frac{9}{98}$

$= \frac{1}{980}$

ii P(exactly two balloons are medium and red)

$= \text{P(first two are medium and red, third is not)}$

$\quad + \text{P(first and third are medium and red, second is not)}$

$\quad + \text{P(second and third are medium and red, first is not)}$

$= \frac{15}{100} \times \frac{14}{99} \times \frac{85}{98} + \frac{15}{100} \times \frac{85}{99} \times \frac{14}{98} + \frac{85}{100} \times \frac{15}{99} \times \frac{14}{98}$

$= \frac{53\,550}{970\,200}$

$= \frac{17}{308}$

16 **b** Of 100 000 females born, 98 956 are still alive at age 15.
Of 100 000 males born, 98 555 are still alive at age 15.

$\therefore$ P(reaching the age of 15) $= \dfrac{98\,956 + 98\,555}{200\,000}$

$= \dfrac{197\,511}{200\,000}$

$= 0.987\,555$

≈ 0.988

c **i** There are 98 555 boys alive at age 15, and 53 942 still alive at 75.

P(15 year old boy will reach age 75) $= \dfrac{53\,942}{98\,555}$

≈ 0.547

ii There are 98 956 females alive at age 15, and 72 656 still alive at age 75.

P(15 year old girl will *not* reach age 75) $= 1 - \dfrac{72\,656}{98\,956}$

$= \dfrac{26\,300}{98\,956}$

≈ 0.266

d In general, females live longer.

e A 20 year old is expected to live much longer than 30 more years, so it is unlikely the insurance company will have to pay out the policy. A 50 year old however is expected to live for only another 26.45 years (males) or 31.59 years (females), so the insurance company may have to pay out the policy.

g For "third world" countries with poverty, lack of sanitation, and so on, the tables would show a significantly lower life expectancy.

Chapter 12

SAMPLING AND DATA

EXERCISE 12A

1 The sample size is only 7 patients which is far too small to draw reliable conclusions about the drug's effectiveness for all patients.

2
- The sample size is very small and may not be representative of the whole population.
- The sample was taken in a Toronto shopping mall. People living outside of the city are probably not represented.

3
a The sample is likely to under-represent full-time weekday working voters.
b The members of the golf club may not be representative of the whole electorate.
c Only people who catch the train between 7 am and 9 am such as full-time workers or students will be sampled.
d The voters in the street may not be representative of those in the whole electorate.

4
a The sample size of only 10 sheep from a population of 2000 is far too small, so this may produce a coverage error.
b With only 10 sheep being weighed, any errors in the measuring of weights will have more impact on the results, so this may produce a measurement error.

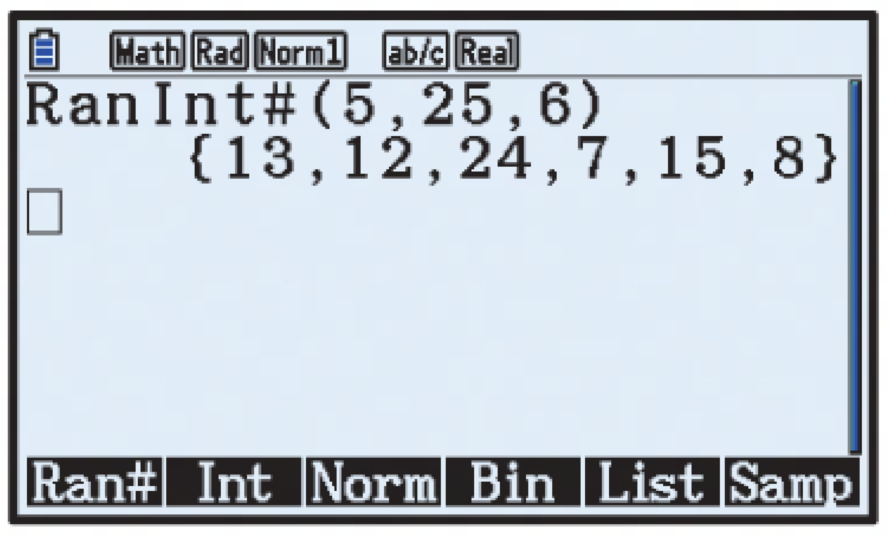
5
a The whole population is being considered, not just a sample. There will be no sampling error as this is a census.
b Two of the sons have used a different method of counting the number of apples from the other two sons. This is likely to produce a measurement error.

6
a Many of the workers may not return or even complete the survey, which may produce a significant non-response error.
b There may be more responses to an online survey as many workers would feel that it is easier to complete a survey online rather than on paper and mailing it back. Responses would also be received more quickly however some workers may not have internet access and will therefore be unable to complete the survey.

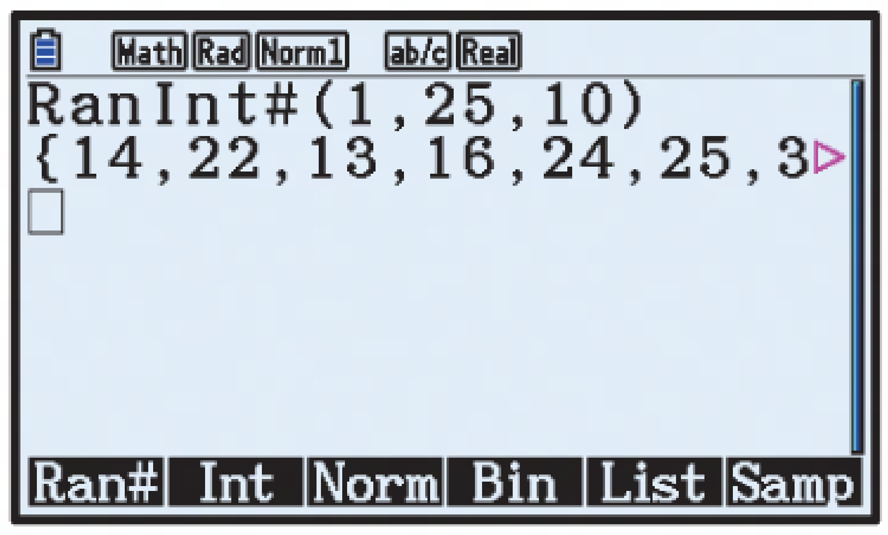
7
a Yes, the non-response error in this situation is likely to produce a biased sample. Members with strong negative opinions regarding the management structure of the organisation are more likely to respond.
b No, the feedback from the survey is still valid. Although it might be biased, the feedback might bring certain issues to attention.

EXERCISE 12B

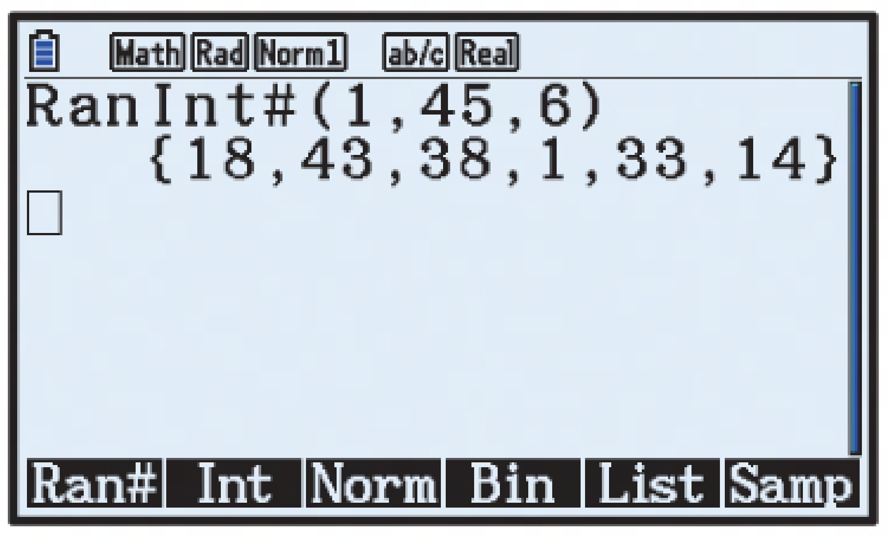
Note: The solutions given for questions **1** and **3** are sample solutions only - many solutions are possible. The random numbers generated in the solutions to these questions are different from those generated in the answers given in the back of the book.

1 **a** 

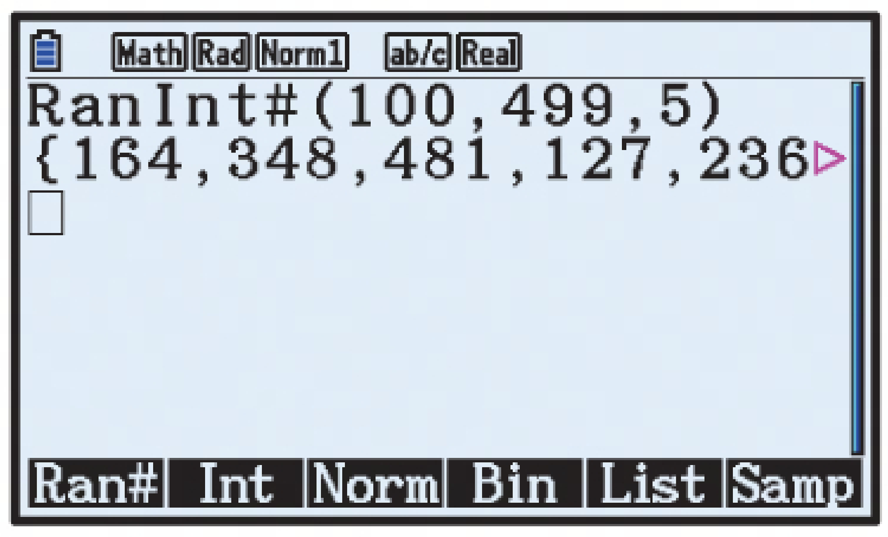
The numbers randomly generated are 13, 12, 24, 7, 15, and 8.

b 

The numbers randomly generated are 14, 22, 13, 16, 24, 25, 3, 4, 23, and 5.

c 

The numbers randomly generated are 18, 43, 38, 1, 33, and 14.

d 

The numbers randomly generated are 164, 348, 481, 127, and 236.

2 **a** $2\% = \frac{2}{100} = \frac{1}{50}$

So, every 50th block of chocolate will be sampled.
The first block to be sampled is the 17th block.
So, the first 5 blocks to be sampled are the 17th, 67th, 117th, 167th, and 217th blocks.

b Total number of blocks sampled $= 2\%$ of $80\,000$
$= 0.02 \times 80\,000$
$= 1600$ blocks of chocolate

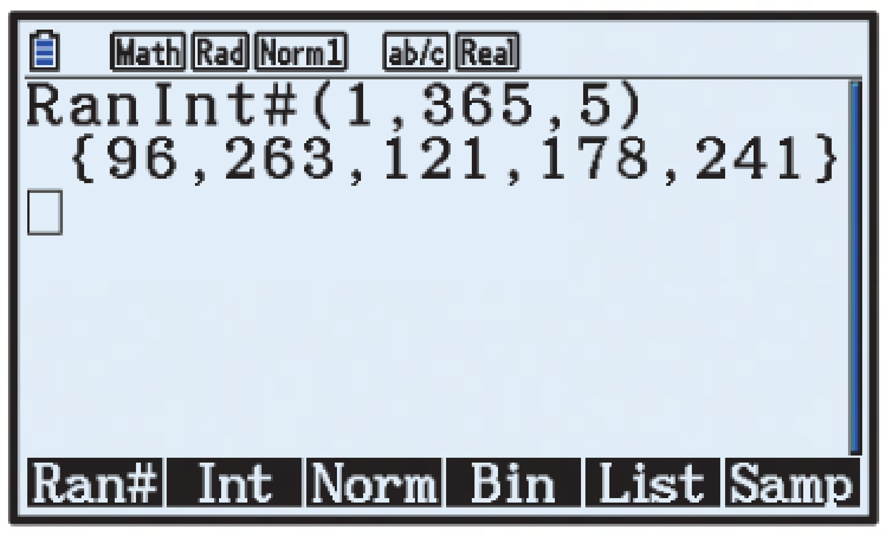
We use the following calendar for 2019 to answer question **3**.

CALENDAR 2019

January	February	March	April	May	June
1 Tu (1) Wk 1	1 Fr (32)	1 Fr (60)	1 Mo (91)	1 We (121)	1 Sa (152)
2 We (2)	2 Sa (33)	2 Sa (61)	2 Tu (92) Wk 14	2 Th (122)	2 Su (153)
3 Th (3)	3 Su (34)	3 Su (62)	3 We (93)	3 Fr (123)	3 Mo (154)
4 Fr (4)	4 Mo (35)	4 Mo (63)	4 Th (94)	4 Sa (124)	4 Tu (155) Wk 23
5 Sa (5)	5 Tu (36) Wk 6	5 Tu (64) Wk 10	5 Fr (95)	5 Su (125)	5 We (156)
6 Su (6)	6 We (37)	6 We (65)	6 Sa (96)	6 Mo (126)	6 Th (157)
7 Mo (7)	7 Th (38)	7 Th (66)	7 Su (97)	7 Tu (127) Wk 19	7 Fr (158)
8 Tu (8) Wk 2	8 Fr (39)	8 Fr (67)	8 Mo (98)	8 We (128)	8 Sa (159)
9 We (9)	9 Sa (40)	9 Sa (68)	9 Tu (99) Wk 15	9 Th (129)	9 Su (160)
10 Th (10)	10 Su (41)	10 Su (69)	10 We (100)	10 Fr (130)	10 Mo (161)
11 Fr (11)	11 Mo (42)	11 Mo (70)	11 Th (101)	11 Sa (131)	11 Tu (162) Wk 24
12 Sa (12)	12 Tu (43) Wk 7	12 Tu (71) Wk 11	12 Fr (102)	12 Su (132)	12 We (163)
13 Su (13)	13 We (44)	13 We (72)	13 Sa (103)	13 Mo (133)	13 Th (164)
14 Mo (14)	14 Th (45)	14 Th (73)	14 Su (104)	14 Tu (134) Wk 20	14 Fr (165)
15 Tu (15) Wk 3	15 Fr (46)	15 Fr (74)	15 Mo (105)	15 We (135)	15 Sa (166)
16 We (16)	16 Sa (47)	16 Sa (75)	16 Tu (106) Wk 16	16 Th (136)	16 Su (167)
17 Th (17)	17 Su (48)	17 Su (76)	17 We (107)	17 Fr (137)	17 Mo (168)
18 Fr (18)	18 Mo (49)	18 Mo (77)	18 Th (108)	18 Sa (138)	18 Tu (169) Wk 25
19 Sa (19)	19 Tu (50) Wk 8	19 Tu (78) Wk 12	19 Fr (109)	19 Su (139)	19 We (170)
20 Su (20)	20 We (51)	20 We (79)	20 Sa (110)	20 Mo (140)	20 Th (171)
21 Mo (21)	21 Th (52)	21 Th (80)	21 Su (111)	21 Tu (141) Wk 21	21 Fr (172)
22 Tu (22) Wk 4	22 Fr (53)	22 Fr (81)	22 Mo (112)	22 We (142)	22 Sa (173)
23 We (23)	23 Sa (54)	23 Sa (82)	23 Tu (113) Wk 17	23 Th (143)	23 Su (174)
24 Th (24)	24 Su (55)	24 Su (83)	24 We (114)	24 Fr (144)	24 Mo (175)
25 Fr (25)	25 Mo (56)	25 Mo (84)	25 Th (115)	25 Sa (145)	25 Tu (176) Wk 26
26 Sa (26)	26 Tu (57) Wk 9	26 Tu (85) Wk 13	26 Fr (116)	26 Su (146)	26 We (177)
27 Su (27)	27 We (58)	27 We (86)	27 Sa (117)	27 Mo (147)	27 Th (178)
28 Mo (28)	28 Th (59)	28 Th (87)	28 Su (118)	28 Tu (148) Wk 22	28 Fr (179)
29 Tu (29) Wk 5		29 Fr (88)	29 Mo (119)	29 We (149)	29 Sa (180)
30 We (30)		30 Sa (89)	30 Tu (120) Wk 18	30 Th (150)	30 Su (181)
31 Th (31)		31 Su (90)		31 Fr (151)	

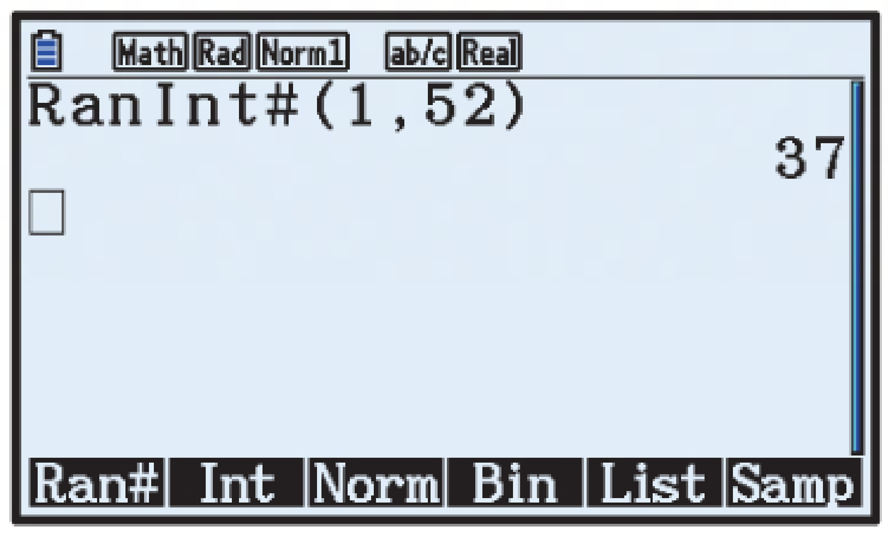
July	August	September	October	November	December
1 Mo (182)	1 Th (213)	1 Su (244)	1 Tu (274) Wk 40	1 Fr (305)	1 Su (335)
2 Tu (183) Wk 27	2 Fr (214)	2 Mo (245)	2 We (275)	2 Sa (306)	2 Mo (336)
3 We (184)	3 Sa (215)	3 Tu (246) Wk 36	3 Th (276)	3 Su (307)	3 Tu (337) Wk 49
4 Th (185)	4 Su (216)	4 We (247)	4 Fr (277)	4 Mo (308)	4 We (338)
5 Fr (186)	5 Mo (217)	5 Th (248)	5 Sa (278)	5 Tu (309) Wk 45	5 Th (339)
6 Sa (187)	6 Tu (218) Wk 32	6 Fr (249)	6 Su (279)	6 We (310)	6 Fr (340)
7 Su (188)	7 We (219)	7 Sa (250)	7 Mo (280)	7 Th (311)	7 Sa (341)
8 Mo (189)	8 Th (220)	8 Su (251)	8 Tu (281) Wk 41	8 Fr (312)	8 Su (342)
9 Tu (190) Wk 28	9 Fr (221)	9 Mo (252)	9 We (282)	9 Sa (313)	9 Mo (343)
10 We (191)	10 Sa (222)	10 Tu (253) Wk 37	10 Th (283)	10 Su (314)	10 Tu (344) Wk 50
11 Th (192)	11 Su (223)	11 We (254)	11 Fr (284)	11 Mo (315)	11 We (345)
12 Fr (193)	12 Mo (224)	12 Th (255)	12 Sa (285)	12 Tu (316) Wk 46	12 Th (346)
13 Sa (194)	13 Tu (225) Wk 33	13 Fr (256)	13 Su (286)	13 We (317)	13 Fr (347)
14 Su (195)	14 We (226)	14 Sa (257)	14 Mo (287)	14 Th (318)	14 Sa (348)
15 Mo (196)	15 Th (227)	15 Su (258)	15 Tu (288) Wk 42	15 Fr (319)	15 Su (349)
16 Tu (197) Wk 29	16 Fr (228)	16 Mo (259)	16 We (289)	16 Sa (320)	16 Mo (350)
17 We (198)	17 Sa (229)	17 Tu (260) Wk 38	17 Th (290)	17 Su (321)	17 Tu (351) Wk 51
18 Th (199)	18 Su (230)	18 We (261)	18 Fr (291)	18 Mo (322)	18 We (352)
19 Fr (200)	19 Mo (231)	19 Th (262)	19 Sa (292)	19 Tu (323) Wk 47	19 Th (353)
20 Sa (201)	20 Tu (232) Wk 34	20 Fr (263)	20 Su (293)	20 We (324)	20 Fr (354)
21 Su (202)	21 We (233)	21 Sa (264)	21 Mo (294)	21 Th (325)	21 Sa (355)
22 Mo (203)	22 Th (234)	22 Su (265)	22 Tu (295) Wk 43	22 Fr (326)	22 Su (356)
23 Tu (204) Wk 30	23 Fr (235)	23 Mo (266)	23 We (296)	23 Sa (327)	23 Mo (357)
24 We (205)	24 Sa (236)	24 Tu (267) Wk 39	24 Th (297)	24 Su (328)	24 Tu (358) Wk 52
25 Th (206)	25 Su (237)	25 We (268)	25 Fr (298)	25 Mo (329)	25 We (359)
26 Fr (207)	26 Mo (238)	26 Th (269)	26 Sa (299)	26 Tu (330) Wk 48	26 Th (360)
27 Sa (208)	27 Tu (239) Wk 35	27 Fr (270)	27 Su (300)	27 We (331)	27 Fr (361)
28 Su (209)	28 We (240)	28 Sa (271)	28 Mo (301)	28 Th (332)	28 Sa (362)
29 Mo (210)	29 Th (241)	29 Su (272)	29 Tu (302) Wk 44	29 Fr (333)	29 Su (363)
30 Tu (211) Wk 31	30 Fr (242)	30 Mo (273)	30 We (303)	30 Sa (334)	30 Mo (364)
31 We (212)	31 Sa (243)		31 Th (304)		31 Tu (365) Wk 53

3 **a** We select 5 random numbers between 1 and 365 inclusive.



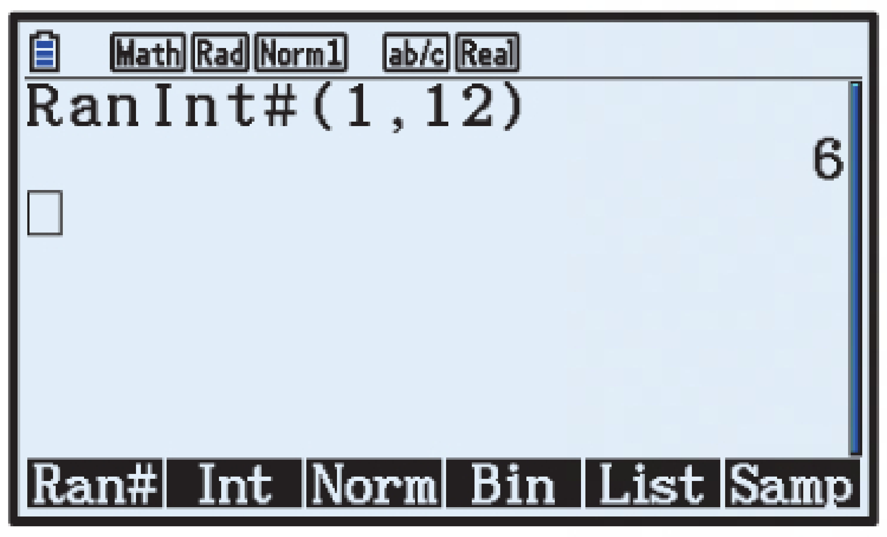
The randomly generated numbers are 96, 263, 121, 178, and 241.
Looking at the calendar, these numbers correspond to the dates 6th April, 20th September, 1st May, 27th June, and 29th August.

b We select a random number between 1 and 52 inclusive.



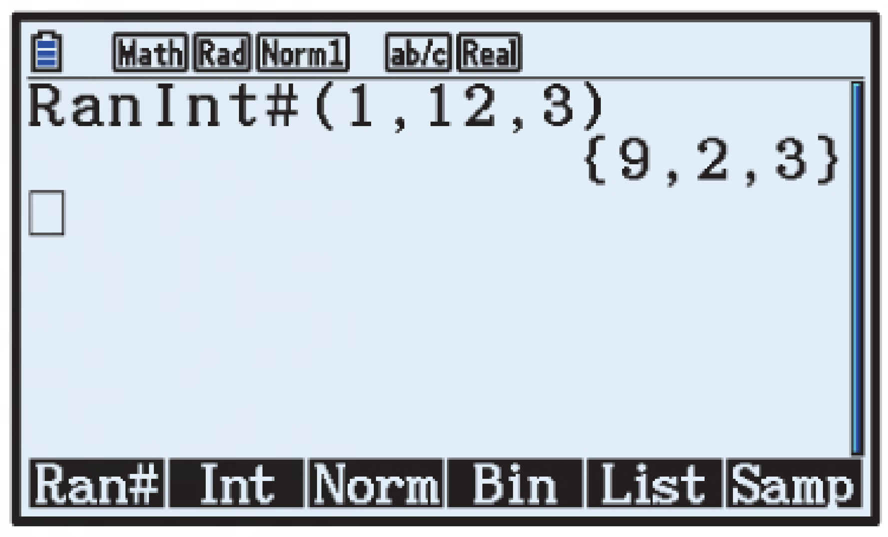
The randomly generated number is 37.
Looking at the calendar, we take the week which starts on the Monday that lies in the 37th week. This corresponds to the week Monday 16th September to Sunday 22nd September.

c We select a random number between 1 and 12 inclusive.



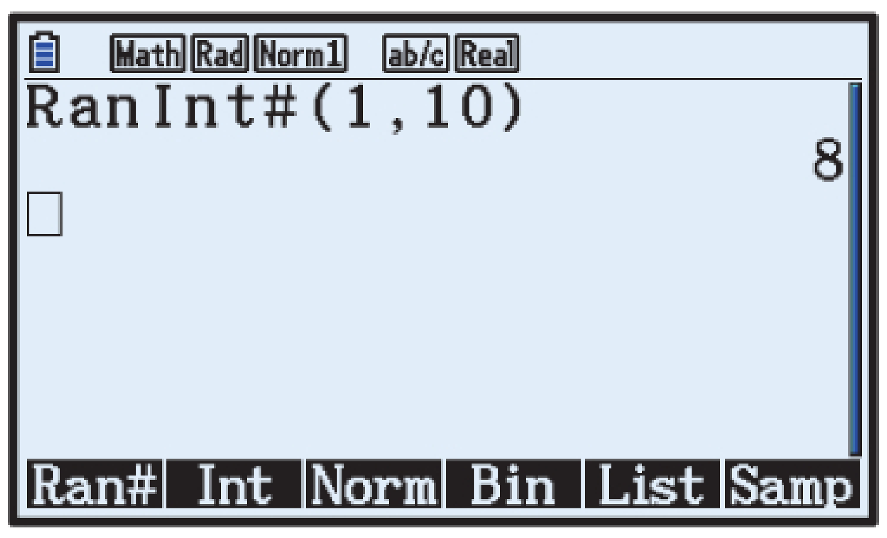
The randomly generated number is 6. The 6th month of the year is June, so the sample is the month of June.

d We select 3 random numbers between 1 and 12 inclusive.



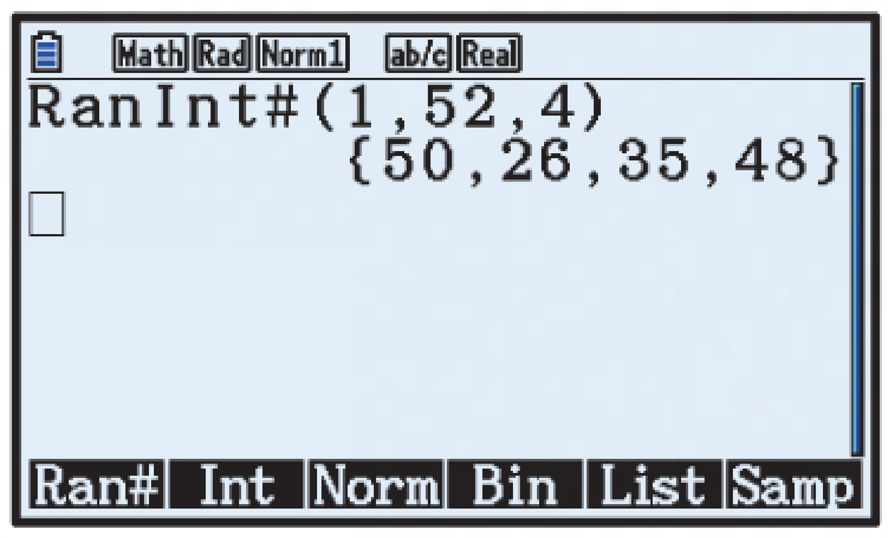
The randomly generated numbers are 9, 2, and 3.
The 9th, 2nd, and 3rd months of the year are September, February, and March.

e We select a random number between 1 and 10 inclusive for the starting month.



The randomly generated number is 8. The 8th month of the year is August, so the sample is the months of August, September, and October.

f We select 4 random numbers between 1 and 52 inclusive.



The randomly generated numbers are 50, 26, 35, and 48.
Looking at the calendar, we take the Wednesday that lies in these weeks.
This corresponds to the dates 11th December, 26th June, 28th August, and 27th November.

4 **a** The sampling method used is convenience sampling, as the first 40 people through the gate are more convenient to sample than every 80th person for example.

b The first 40 people through the gate will probably spend more time at the show, and so are more likely to spend more than €20. Also, the sample size is relatively small, being about 1.1% of the total number of visitors.

c We could use a systematic sampling technique in which every 10th person through the gate is surveyed. The sample size would therefore be 10% of the total population.

5 **a** The sampling method used is systematic sampling as people have been selected at regular intervals.

b
$$\begin{aligned}1 \text{ year} &= 365 \text{ days}\\ &= \frac{365}{28} \text{ lots of 28 days}\\ &= 13 \text{ lots of 28 days} + 1 \text{ day remaining}\end{aligned}$$

So, including the first Monday sampled, there will be $13 + 1 = 14$ days in her sample.

c $28 \text{ days} = 4 \times 7$ days, and the first day sampled is a Monday. So the sample consists of every fourth Monday in the year. Only visitors who use the library on Mondays will be counted. Mondays may not be representative of all the days in a week, so the sample may be biased.

6 **a**
$$\begin{aligned}\text{Total number of members} &= 80 + 60 + 20\\ &= 160 \text{ members}\end{aligned}$$

b For the sample, we want:

$$\begin{aligned}\text{number of tennis members} &= \tfrac{80}{160} \times 40 = 20\\ \text{number of lawn bowls members} &= \tfrac{60}{160} \times 40 = 15\\ \text{number of croquet members} &= \tfrac{20}{160} \times 40 = 5\end{aligned}$$

So, the club should sample 20 tennis members, 15 lawn bowls members, and 5 croquet members.

7 Total number of staff $= 10 + 24 + 65 + 98 + 28 = 225$

For the sample, we want:

$$\begin{aligned}\text{number of departmental managers} &= \tfrac{10}{225} \times 30 \approx 1.33 \approx 1\\ \text{number of supervisors} &= \tfrac{24}{225} \times 30 = 3.2 \approx 3\\ \text{number of senior sales staff} &= \tfrac{65}{225} \times 30 \approx 8.67 \approx 9\\ \text{number of junior sales staff} &= \tfrac{98}{225} \times 30 \approx 13.07 \approx 13\\ \text{number of shelf packers} &= \tfrac{28}{225} \times 30 \approx 3.73 \approx 4\end{aligned}$$

Now $1 + 3 + 9 + 13 + 4 = 30$ which is the required sample size.

So, 1 departmental manager, 3 supervisors, 9 senior sales staff, 13 junior sales staff, and 4 shelf packers should be selected for the sample.

8 **a** It is easier for Mona to survey her own home room class, so this is a convenience sample.

b Mona's sample will not be representative of all of the classes in the school. Mona's survey may be influenced by her friends in her class. Mona's sample will therefore be biased.

c A stratified sample of students from every class may be a more appropriate sampling method.

9 **a** Not all students selected for the sample will be comfortable discussing the topic, so it may not be practical for Lucian to use a simple random sample or systematic sample.

b Lucian should use a quota sample so that individuals may be specifically selected rather than randomly selected as in a stratified sample.

10 **a** This is considered to be a census because all of the Year 11 and Year 12 students were asked, not just a sample of them.

b 96 students said they had smoked.

$$\text{Proportion of all students who said they had smoked} = \frac{\text{number who said they had smoked}}{\text{total number of students}}$$
$$= \frac{96}{200}$$
$$= 0.48$$

c **i** The sample size would be too small to be representative of the whole population.

ii The sample size would be too small to be representative of the whole population.

iii This sampling technique would be valid but at 50% of the population, it is an unnecessarily large sample size.

iv This is a valid sampling technique with an appropriate sample size.

v This is a valid sampling technique with an appropriate sample size.

vi This is a valid sampling technique with an appropriate sample size.

d **v** is simple random sampling, **iii** and **iv** are systematic sampling, and **vi** is stratified or quota sampling.

EXERCISE 12C

Note: The solutions given for questions **1** to **6** are sample solutions only - many solutions are possible.

1 **a** Posing "Is your shirt red, blue, yellow, or white?" as a structured question does not allow for colours which are different from those given.

b The question could be rewritten as "What colour is your shirt?".

c A person's response to this question may be subjective which may be a problem. For example, one person may interpret a colour as blue whereas another person may interpret it as purple. A shirt may also be more than one colour which could lead to difficulties in interpreting the colour.

2 **a** The question "Do you have any allergies?" could be interpreted as:

- "Do you have any medically diagnosed allergies?"
- "Do you have any life threatening allergies?"
- "Do you have any food allergies?"
- "Do you think you have any allergies?"

The question also does not specify if it is a structured yes/no type of question or if the respondent should list specific allergies.

b The question could be rewritten as "Do you have any food or other type of allergies (medically diagnosed or otherwise), and if so, what are they?".

3 **a** The question "Do you have any pets?" could be interpreted as:

- "Do you have any animals in your household?"
- "Do you have any animals in your care at home or elsewhere?"

The question does not specify if it is a structured yes/no type of question or if it includes livestock or only domestic animals.

b The question could be rewritten as "Do you have any domesticated animals in your household (not including livestock), and if so, what are they?".

4 **a** The journalist's question is misleading as it only mentions the proposed cuts to education, not the proposal to move those funds to health. This may produce a measurement error as the respondents are unlikely to give their views about the whole proposal.

b The question could be worded differently, such as "What are your views about the Government's proposal to move funding from education to health?".

5 **a** The question "Where do you live?" is likely to have a high non-response error as many respondents would not be comfortable giving their address to someone they do not know.

b The question could be improved by being more specific, such as asking for the general area or suburb only. For example:

- "Which suburb do you live in?"
- "Which state do you live in?"

Giving a reason why this information is needed should also improve the response rate.

6 **a** **i** The question contains a double negative which could confuse respondents. The word "infectious" suggests that *not* immunising is undesirable behaviour, thus the question is biased.

ii The question could be rewritten as "Have you been immunised against meningococcal disease?".

b **i** The question asks for two things:

- whether climate change is a major issue
- the respondent's political opinion on climate change.

It is not clear whether the respondent's response will reflect their general or political opinion on the issue.

The phrase "thrown around by politicians" is also rather emotive, thus the question is biased.

ii The question could be rewritten as "Do you believe that climate change is an important issue?"

c **i** The question uses a positive fact about fair trade cocoa to try to persuade the respondent into answering "yes". So the question is biased.

It is also very long and takes a long time to get to the point.

ii The question could be rewritten as "Do you believe that fair trade certified chocolate should be more expensive than uncertified chocolate?".

EXERCISE 12D

1 **a** The number of brothers a person has takes exact number values.

$\therefore$ this is a discrete variable which could take the values 0, 1, 2, 3, ...

b The colours of lollies in a packet is a categorical variable which could have the categories red, yellow, orange, green, and so on.

c The time children spend brushing their teeth each day is a numerical variable which can be measured. The data can take any value between certain limits.

$\therefore$ this is a continuous variable which could take any value from 0 to 15 minutes.

d The height of the trees in a garden is a numerical variable which can be measured. The data can take any value between certain limits.

$\therefore$ this is a continuous variable which could take any value from 0 to 25 metres.

e The brand of car a person drives is a categorical variable which could have the categories Ford, BMW, Renault, and so on.

f The number of petrol pumps at a service station takes exact number values.
$\therefore$ this is a discrete variable which could take the values 1, 2, 3, ...

g The most popular holiday destinations is a categorical variable which could have the categories Australia, Hawaii, Dubai, and so on.

h The scores out of 10 in a diving competition take exact number values.
$\therefore$ this is a discrete variable which could take the values 0.0, 0.5, 1.0, ..., 9.5, 10.0.

i The amount of water a person drinks each day is a numerical variable which can be measured. The data can take any value between certain limits.
$\therefore$ this is a continuous variable which could take any value from 0 to 4 litres.

j The number of hours spent per week at work is a numerical variable which can be measured. The data can take any value between certain limits.
$\therefore$ this is a continuous variable which could take any value from 0 to 80 hours.

k The average temperatures of various cities is a numerical variable which can be measured. The data can take any value between certain limits.
$\therefore$ this is a continuous numerical variable which could take any value from $-20°C$ to $35°C$.

l The items students ate for breakfast before coming to school is a categorical variable which could have the categories cereal, toast, fruit, rice, eggs, and so on.

m The number of televisions in each house takes exact number values.
$\therefore$ this is a discrete variable which could take the values 0, 1, 2, ...

2 The player's *name* is a categorical variable.

The player's *age* can be measured and can take any value between certain limits, so it is a continuous variable.

The player's *height* can be measured and can take any value between certain limits, so it is a continuous variable.

The player's *country* is a categorical variable as it describes which country the player is playing for.

The player's *tournament wins* can be counted, so it is a discrete variable.

The player's *average serving speed* can be measured and can take any value between certain limits, so it is a continuous variable.

The player's *ranking* can be counted, so it is a discrete variable.

The player's *career prize money* can be counted, so it is a discrete variable.

EXERCISE 12E

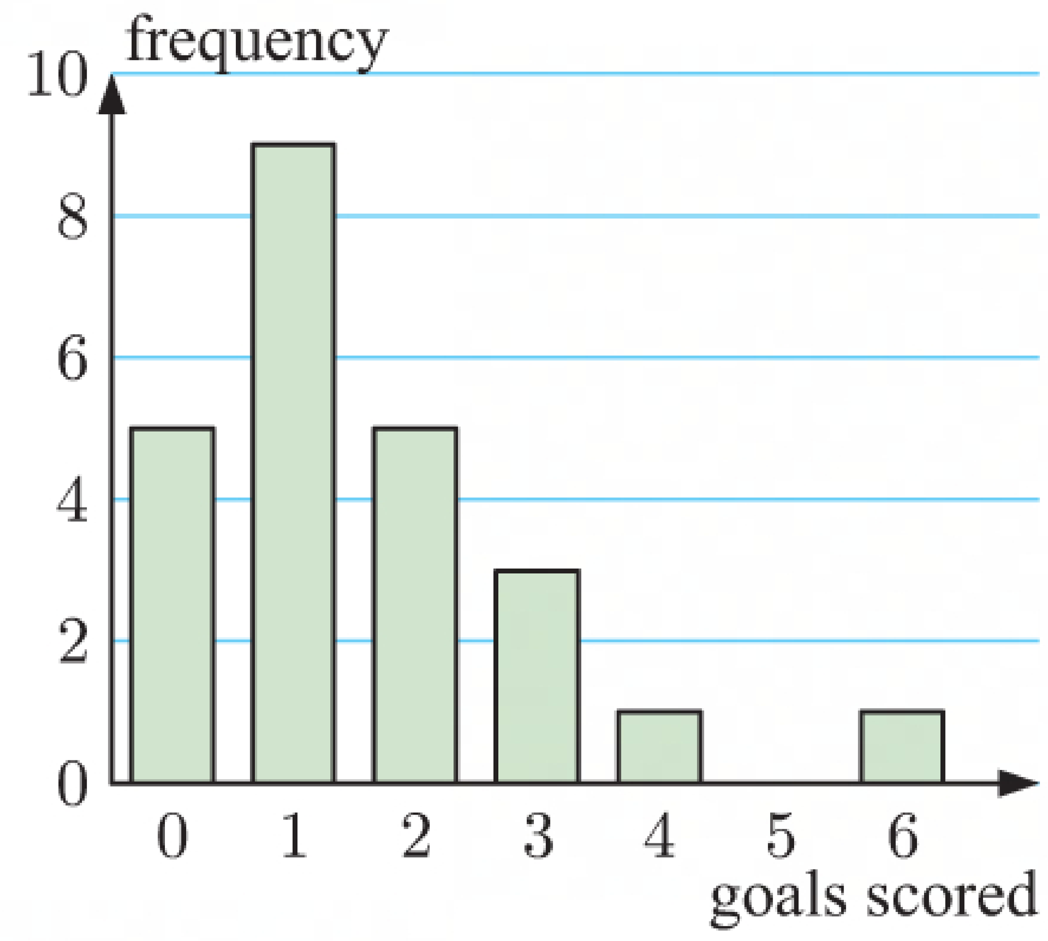
1 **a** The variable being considered is the number of goals scored in a game.

b The data is discrete as only a whole number of goals can be scored. The variable is counted, not measured.

c

Goals scored	Tally	Frequency	Relative Frequency
0	||||	5	$\frac{5}{24} \approx 0.208$
1	|||| ||||	9	$\frac{9}{24} = 0.375$
2	||||	5	$\frac{5}{24} \approx 0.208$
3	|||	3	$\frac{3}{24} = 0.125$
4	|	1	$\frac{1}{24} \approx 0.042$
5		0	$\frac{0}{24} = 0.000$
6	|	1	$\frac{1}{24} \approx 0.042$
	Total	24	

d

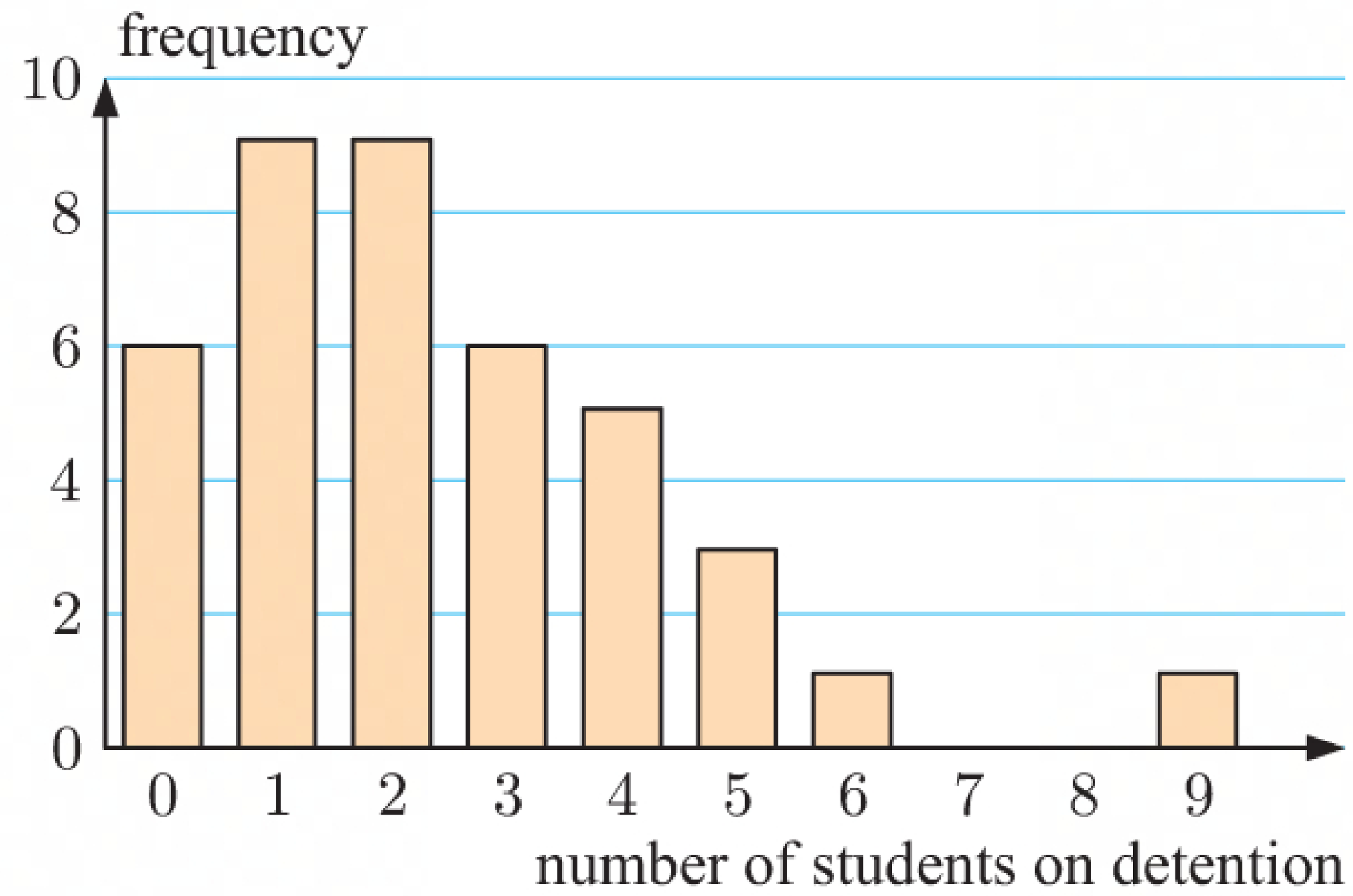


e The modal score for the team is 1 goal.

f The data is positively skewed with one outlier (6 goals).

g The Flames failed to score in $\frac{5}{24} \times 100\% \approx 20.8\%$ of games.

2 **a**



b The modal number of students on detention in a week is 1 and 2 students.

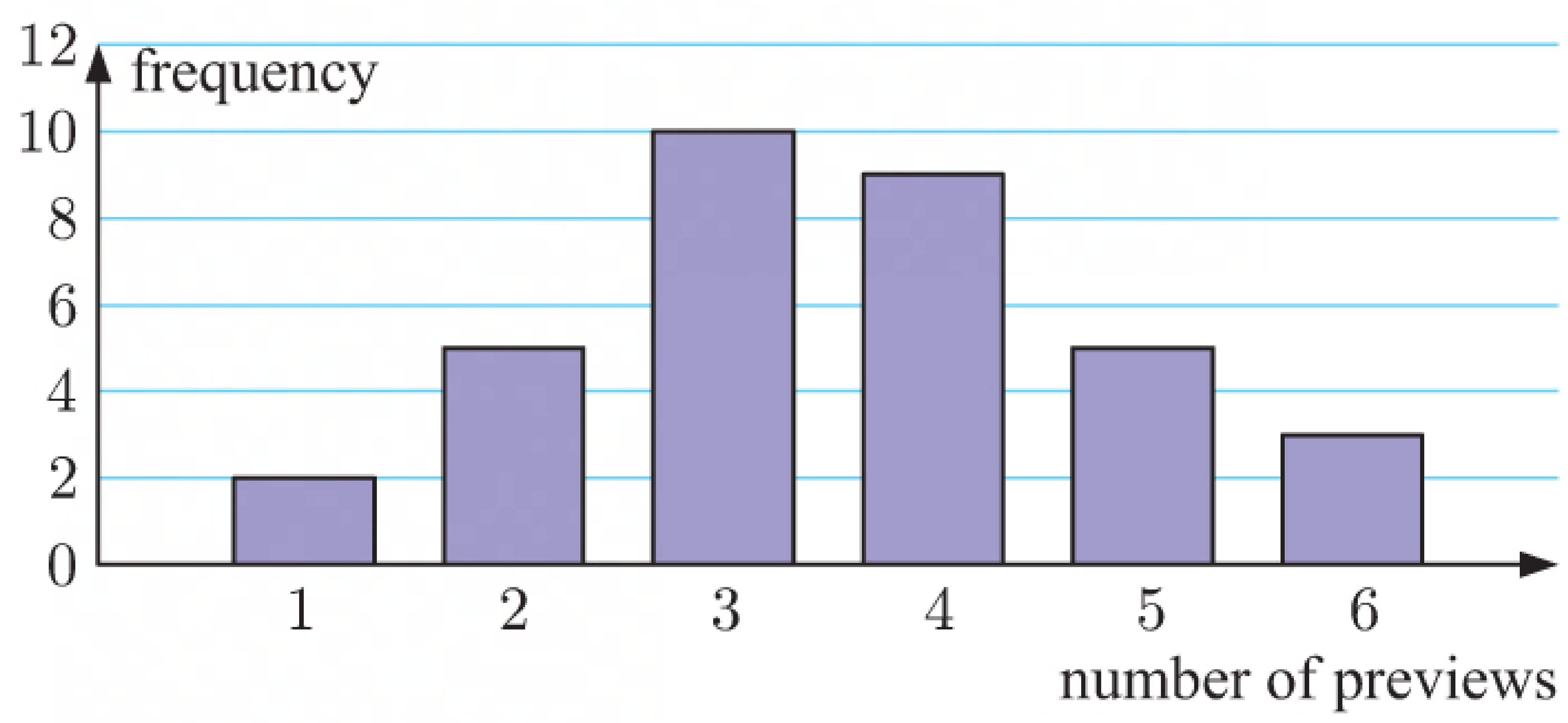
c The data is positively skewed with one outlier (9 students).

d There were more than 4 students on detention in $\frac{3+1+1}{40} \times 100\% = 12\frac{1}{2}\%$ of weeks.

3 **a**

Number of previews	*Tally*	*Frequency*
1	$\vert\vert$	2
2	$\cancel{\vert\vert\vert\vert}$	5
3	$\cancel{\vert\vert\vert\vert}$ $\cancel{\vert\vert\vert\vert}$	10
4	$\cancel{\vert\vert\vert\vert}$ $\vert\vert\vert\vert$	9
5	$\cancel{\vert\vert\vert\vert}$	5
6	$\vert\vert\vert$	3
	Total	34

b

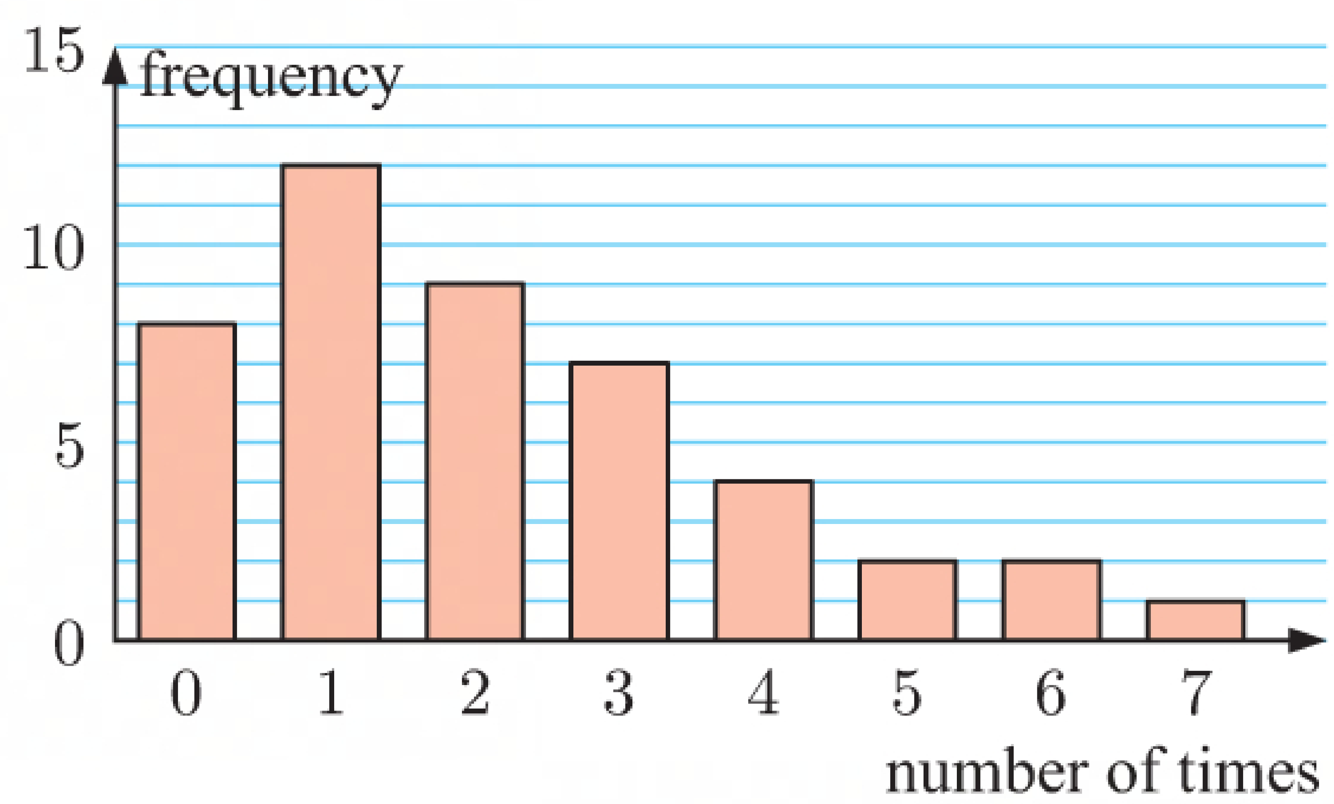


c The mode of the data is 3 previews.

d The data is symmetrical with no outliers.

e At least 3 previews were shown on $\dfrac{10+9+5+3}{34} \times 100\% \approx 79.4\%$ of occasions.

4 **a** $8+12+9+7+4+2+2+1 = 45$ people were surveyed.



b The mode of the data is 1 time.

c 8 people did not eat out at all last week.

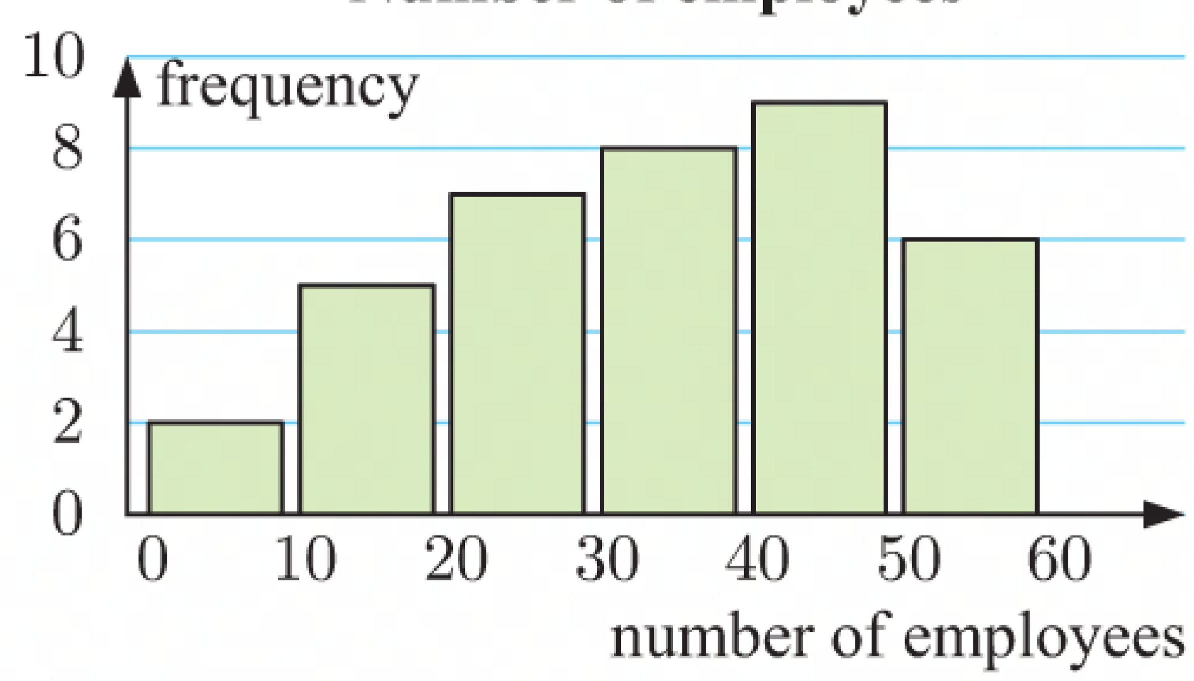
d $\dfrac{4+2+2+1}{45} \times 100\% = 20\%$ of people surveyed ate out more than three times last week.

e The data is positively skewed with no outliers.

EXERCISE 12F

1 **a** $2+5+7+8+9+6 = 37$ businesses were surveyed.

Number of employees



b The modal class is 40 - 49 employees.

c The data is negatively skewed.

d $\dfrac{2+5+7}{37} \times 100\% \approx 37.8\%$ of businesses surveyed had less than 30 employees.

e No, it is not possible to determine the highest number of employees a business had. We can only say that it was in the interval 50 - 59 employees.

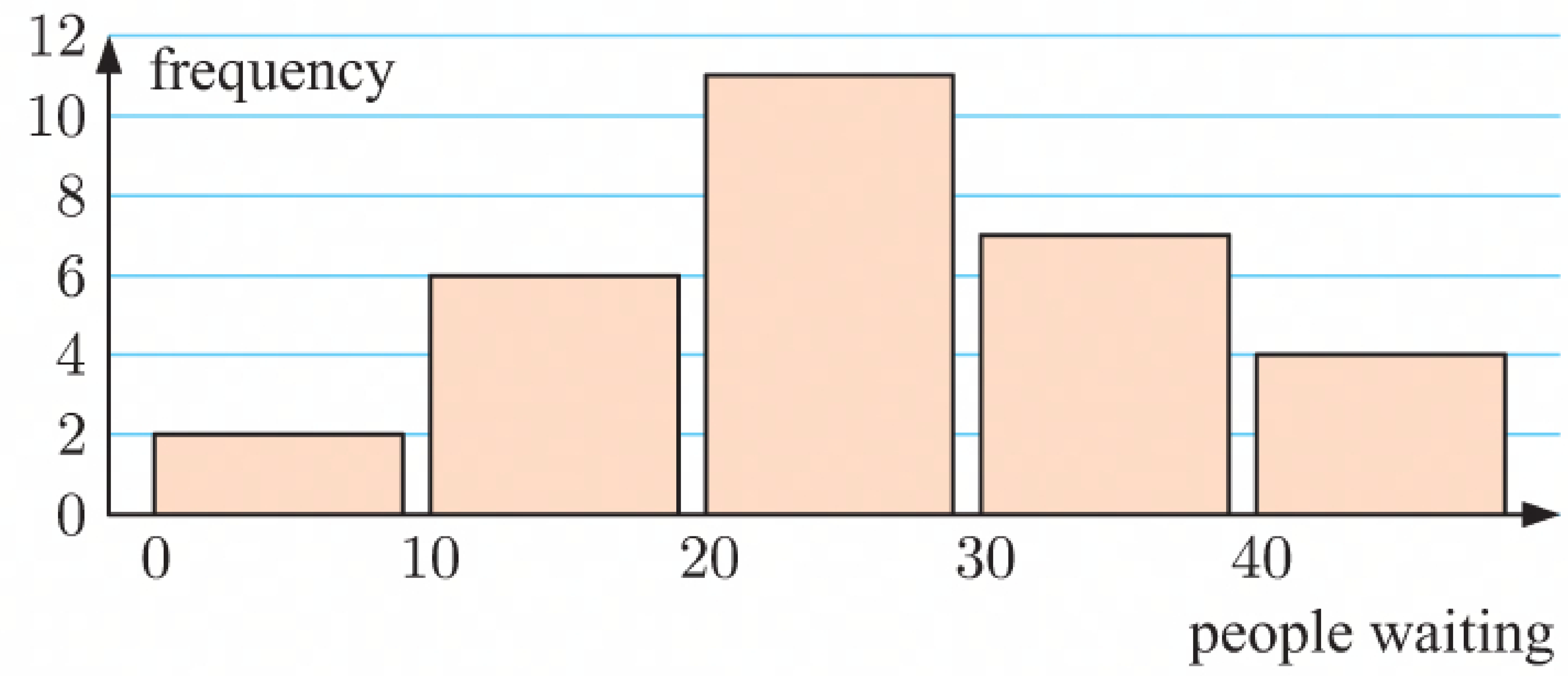
2 **a**

People waiting	*Tally*	*Frequency*	*Relative Frequency*
0 - 9	||	2	$\frac{2}{30} \approx 0.067$
10 - 19	~~||||~~ |	6	$\frac{6}{30} = 0.200$
20 - 29	~~||||~~ ~~||||~~ |	11	$\frac{11}{30} \approx 0.367$
30 - 39	~~||||~~ ||	7	$\frac{7}{30} \approx 0.233$
40 - 49	||||	4	$\frac{4}{30} \approx 0.133$
	Total	30	

b There were less than 10 people at the station on 2 days.

c There were at least 30 people at the station on $\frac{7+4}{30} \times 100\% \approx 36.7\%$ of days.

d

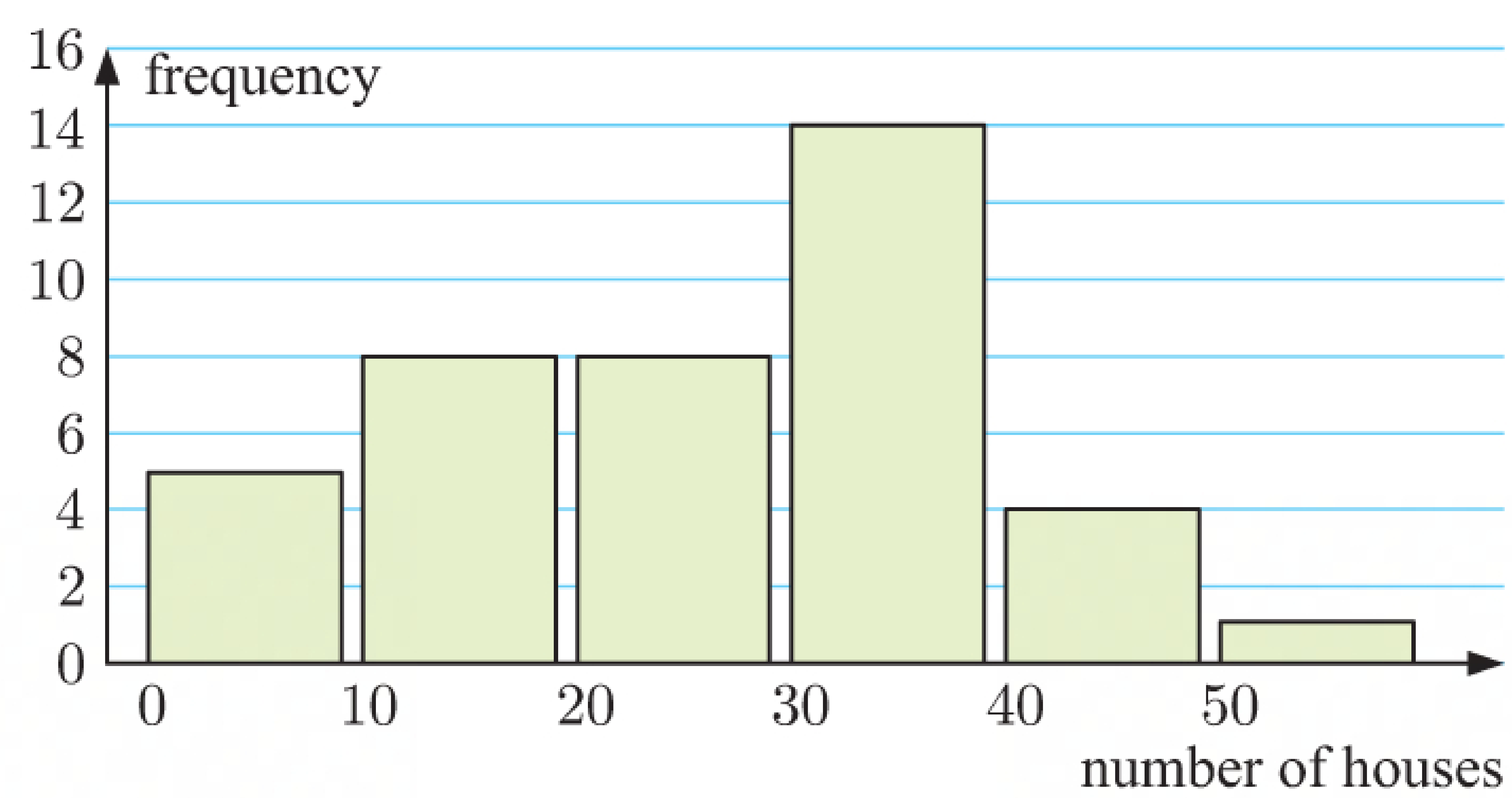


e The modal class of the data is 20 - 29 people.

3 **a**

Number of houses	*Tally*	*Frequency*
0 - 9	~~||||~~	5
10 - 19	~~||||~~ |||	8
20 - 29	~~||||~~ |||	8
30 - 39	~~||||~~ ~~||||~~ ||||	14
40 - 49	||||	4
50 - 59	|	1
	Total	40

b



c The modal class is 30 - 39 houses.

d $\frac{8+14+4+1}{40} \times 100\% = 67.5\%$ of the streets contain at least 20 houses.

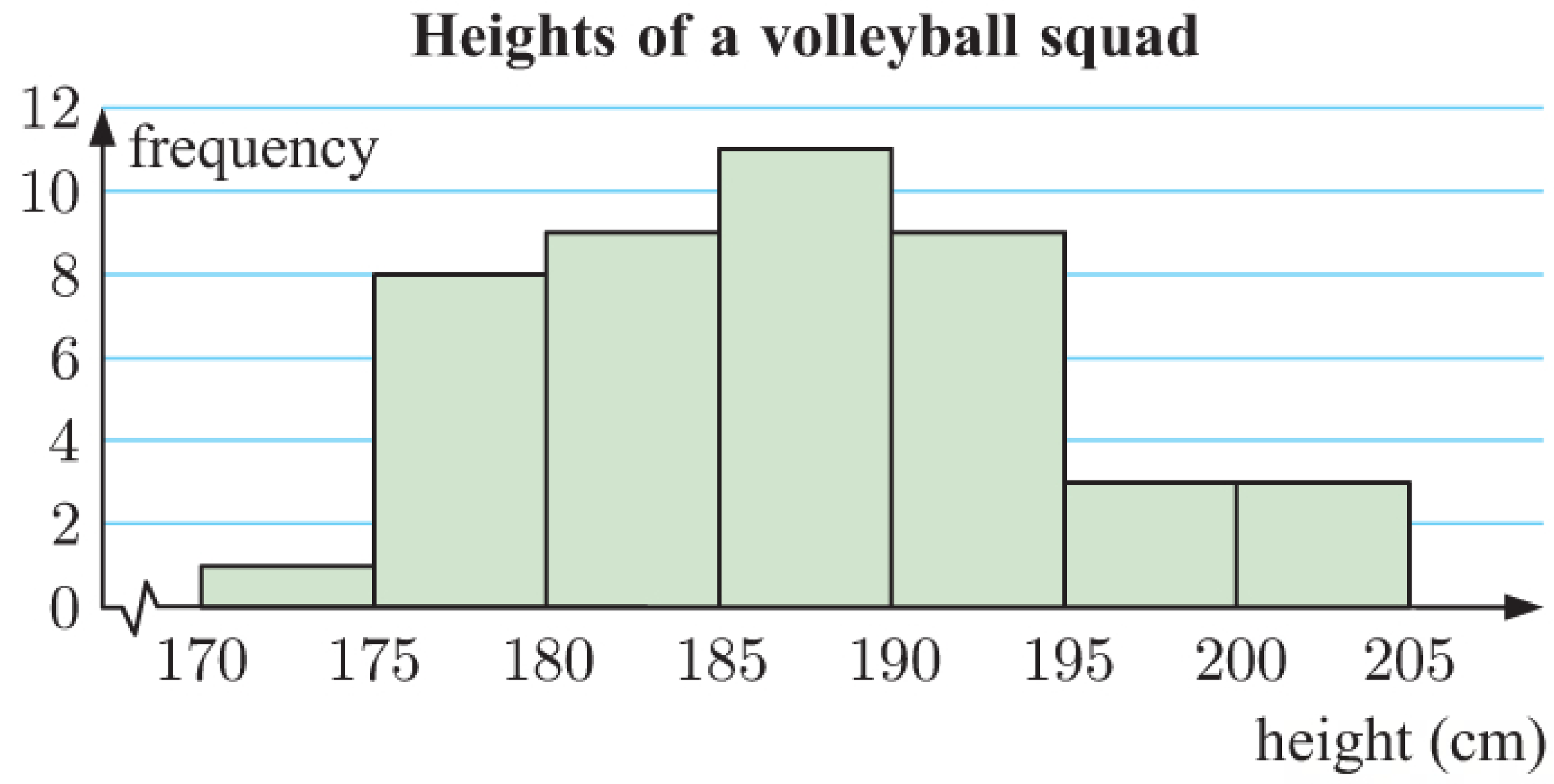
EXERCISE 12G

1

Height (H cm)	*Frequency*
$170 \leqslant H < 175$	1
$175 \leqslant H < 180$	8
$180 \leqslant H < 185$	9
$185 \leqslant H < 190$	11
$190 \leqslant H < 195$	9
$195 \leqslant H < 200$	3
$200 \leqslant H < 205$	3

a Height is a continuous variable as it is measured on a continuous scale.

b



c The modal class $185 \leqslant H < 190$ cm occurs most frequently. More volleyball players have heights in this interval than in any other interval.

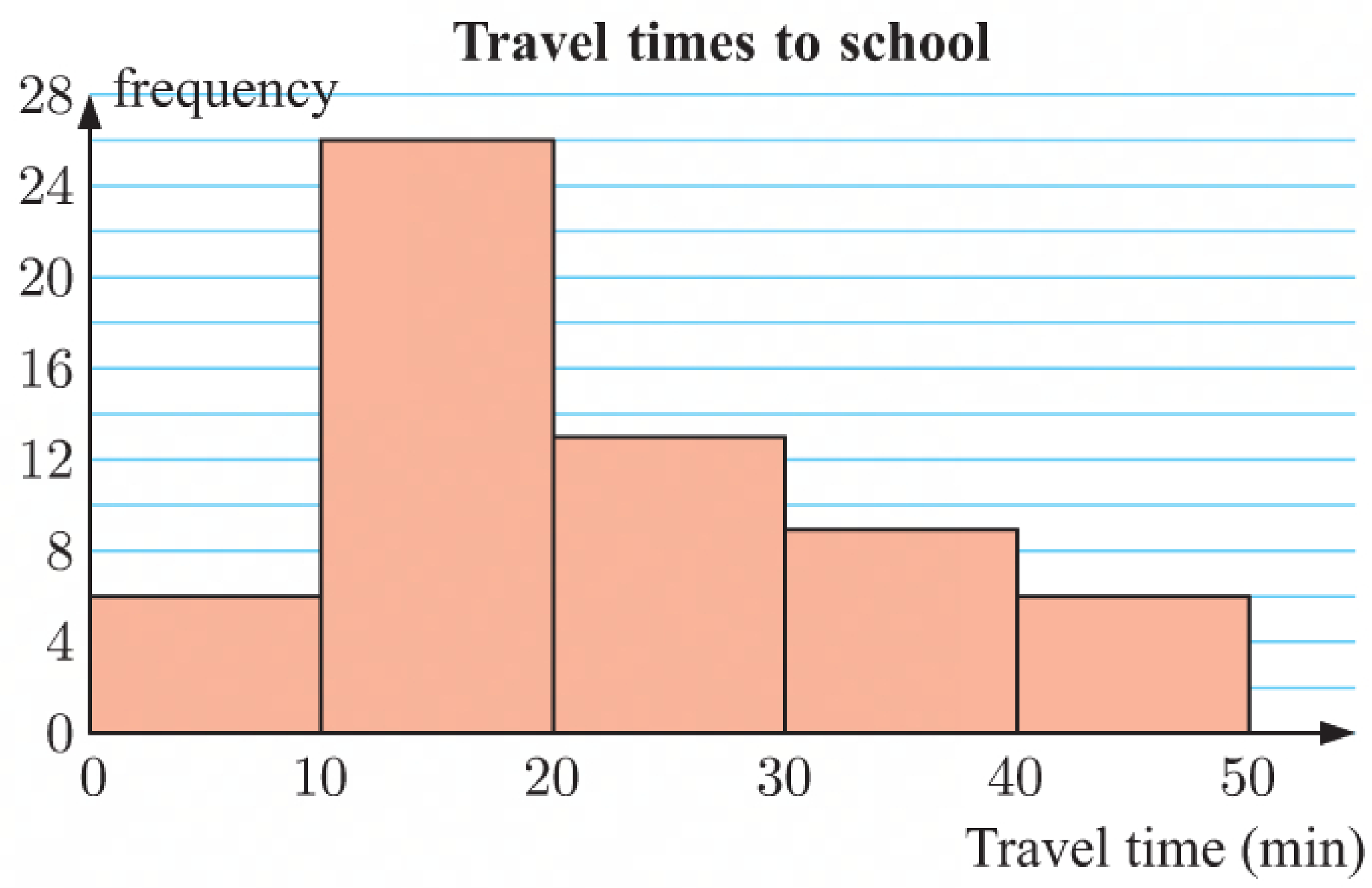
d The data is slightly positively skewed.

2 **a** Travel time is a continuous variable, even though times have been rounded to the nearest minute.

b

Travel time (min)	*Tally*	*Frequency*
$0 \leqslant t < 10$	~~||||~~ |	6
$10 \leqslant t < 20$	~~||||~~ ~~||||~~ ~~||||~~ ~~||||~~ ~~||||~~ |	26
$20 \leqslant t < 30$	~~||||~~ ~~||||~~ |||	13
$30 \leqslant t < 40$	~~||||~~ ||||	9
$40 \leqslant t < 50$	~~||||~~ |	6
	Total	60

c



d The data is positively skewed.

e The modal travelling time is $10 \leqslant t < 20$ minutes. More students have travel times in this interval than in any other interval.

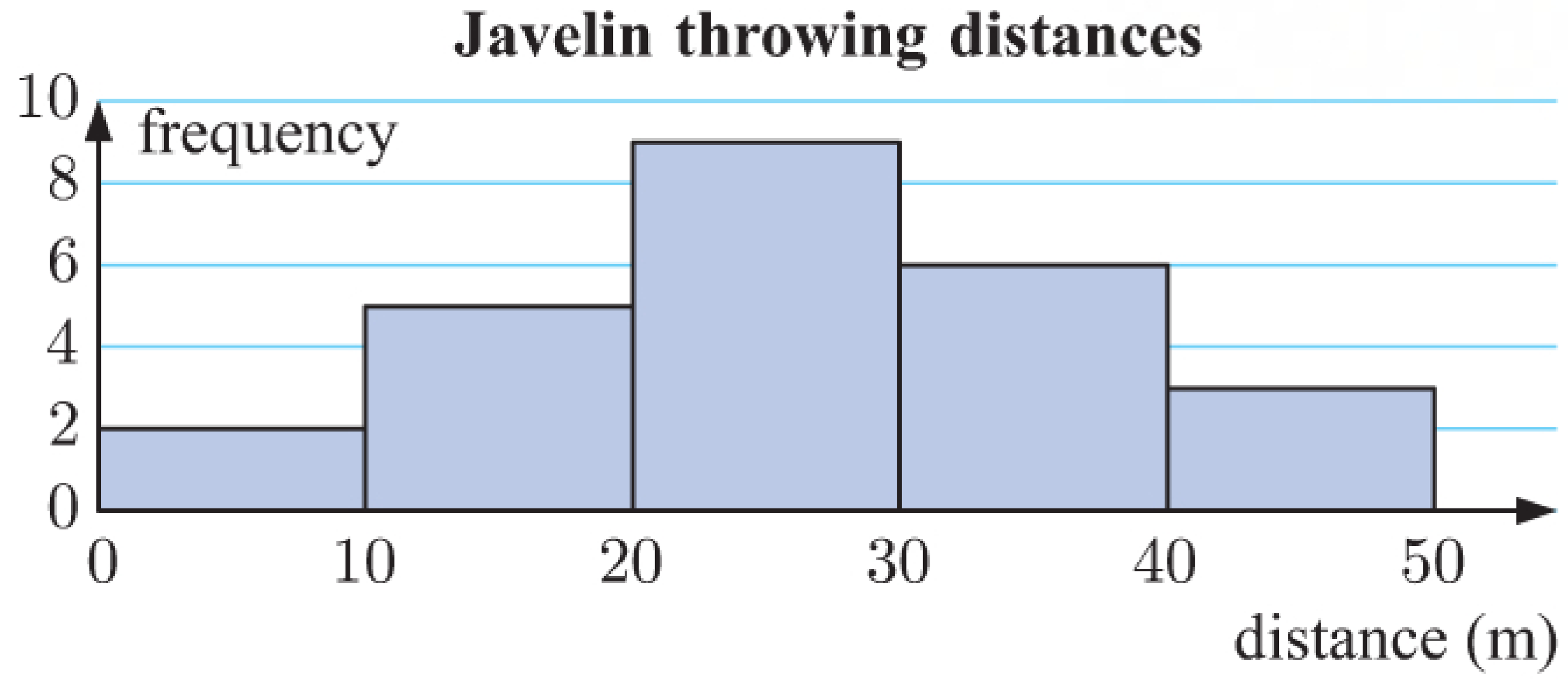
3 **a** The variable *distance* is continuous, even though distances have been rounded to the nearest 10 cm.

The shortest distance is 7.4 m and the longest is 42.9 m, so we will use class intervals of width 10 m: $0 \leqslant d < 10$, $10 \leqslant d < 20$, ..., $40 \leqslant d < 50$.

b

Distance (m)	*Tally*	*Frequency*
$0 \leqslant d < 10$	||	2
$10 \leqslant d < 20$	~~||||~~	5
$20 \leqslant d < 30$	~~||||~~ ||||	9
$30 \leqslant d < 40$	~~||||~~ |	6
$40 \leqslant d < 50$	|||	3
	Total	25

c



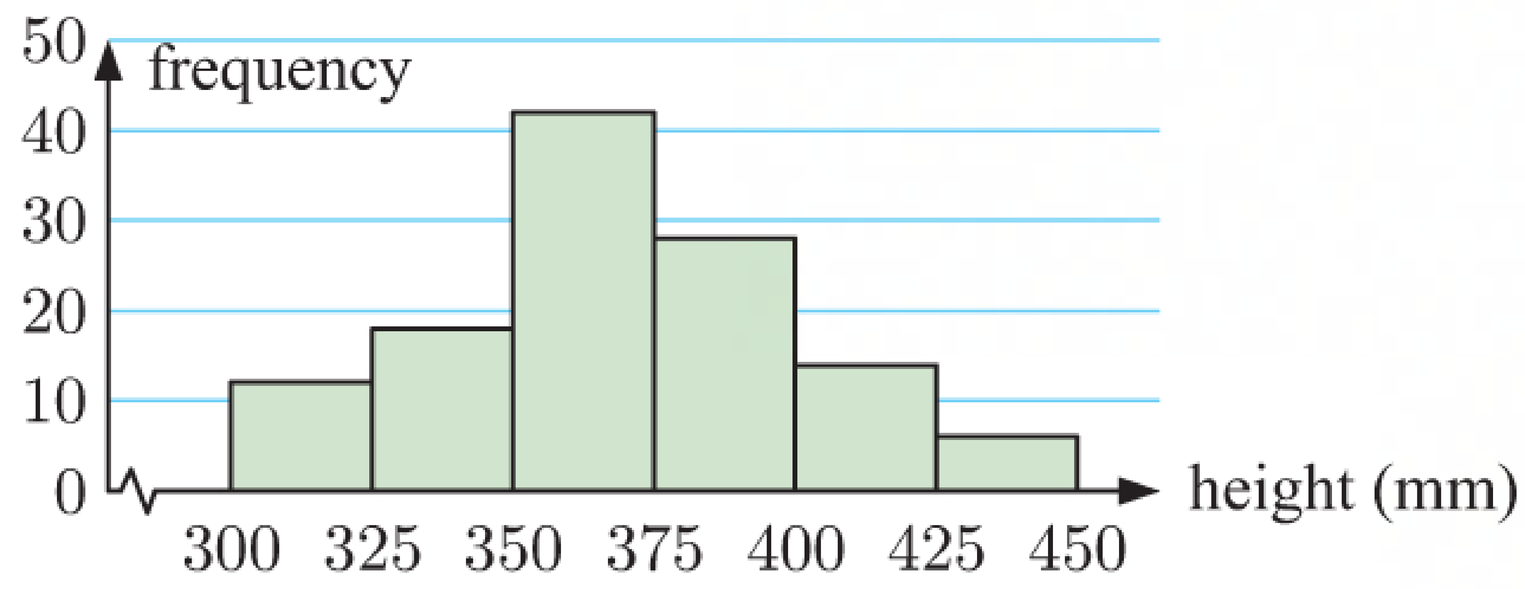
d The modal class is $20 \leqslant d < 30$ m. More athletes achieved distances in this interval than in any other interval.

e $\dfrac{6+3}{25} \times 100\% = 36\%$ of athletes threw the javelin 30 m or further.

4

Height (h mm)	*Frequency*
$300 \leqslant h < 325$	12
$325 \leqslant h < 350$	18
$350 \leqslant h < 375$	42
$375 \leqslant h < 400$	28
$400 \leqslant h < 425$	14
$425 \leqslant h < 450$	6

a **Heights of 6-month old seedlings at a nursery**



b $14 + 6 = 20$ seedlings are 400 mm or higher.

c $12 + 18 + 42 + 28 + 14 + 6 = 120$ seedlings have been measured.

$\dfrac{42 + 28}{120} \times 100\% \approx 58.3\%$ of the seedlings are between 350 mm and 400 mm high.

d **i** $\dfrac{12 + 18 + 42 + 28}{120} \times 100\% \approx 83.3\%$ of the seedlings are less than 400 mm high.

So in a population of 1462 seedlings, we would expect

83.3% of $1462 \approx 0.833 \times 1462$

≈ 1218 seedlings to be less than 400 mm high.

ii $\dfrac{28 + 14}{120} \times 100\% = 35\%$ of the seedlings are between 375 mm and 425 mm high.

So in a population of 1462 seedlings, we would expect

35% of $1462 = 0.35 \times 1462$

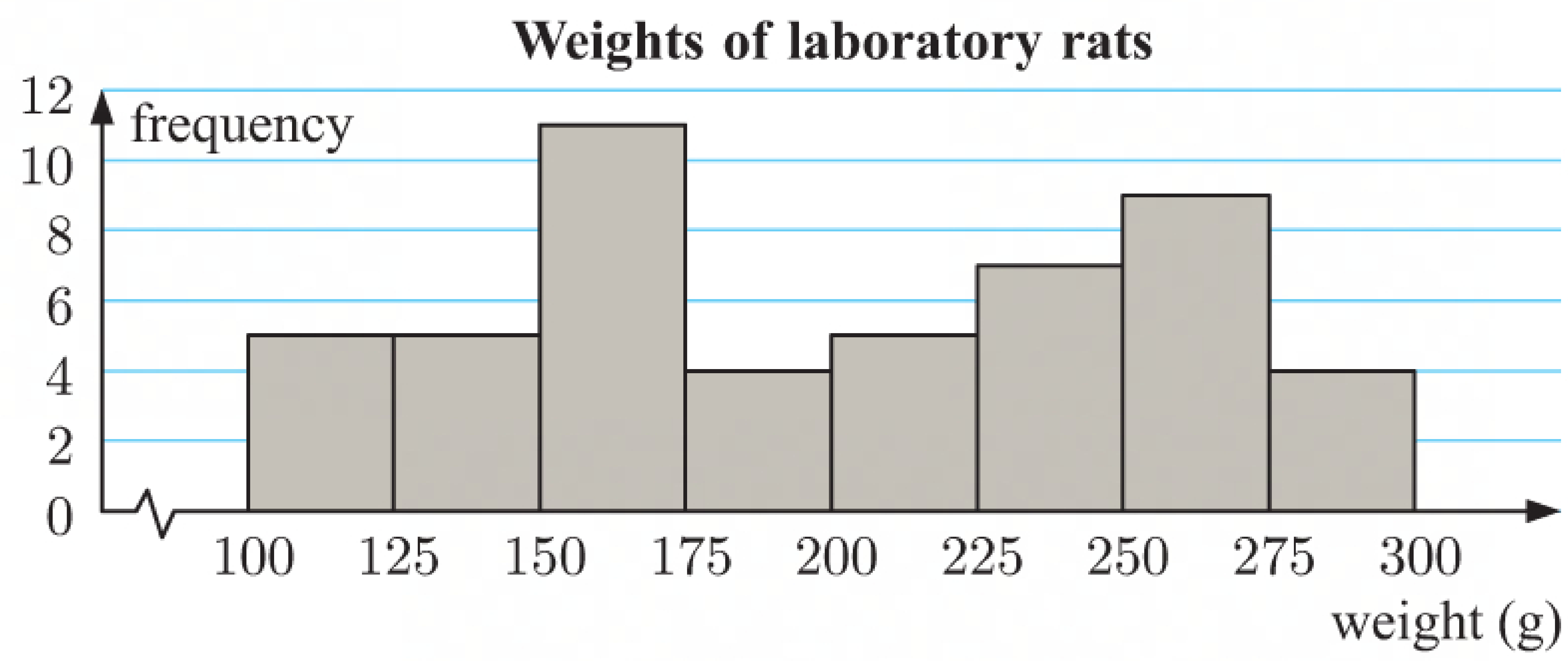
≈ 512 seedlings to be between 375 mm and 425 mm high.

5 **a** The variable *weight* is a continuous variable, even though weights have been recorded to the nearest gram.
The lowest weight is 100 g and the highest is 295 g, so we will use class intervals of width 25 g: $100 \leqslant w < 125$, $125 \leqslant w < 150$, ..., $275 \leqslant w < 300$.

b

Weight (g)	*Tally*	*Frequency*
$100 \leqslant w < 125$	~~||||~~	5
$125 \leqslant w < 150$	~~||||~~	5
$150 \leqslant w < 175$	~~||||~~ ~~||||~~ |	11
$175 \leqslant w < 200$	||||	4
$200 \leqslant w < 225$	~~||||~~	5
$225 \leqslant w < 250$	~~||||~~ ||	7
$250 \leqslant w < 275$	~~||||~~ ||||	9
$275 \leqslant w < 300$	||||	4
	Total	50

c



d $\dfrac{5+5+11+4}{50} \times 100\% = 50\%$ of the rats weigh less than 200 grams.

REVIEW SET 12A

1 **a** Students studying Italian may have an Italian background so surveying these students may produce a biased result.

b Andrew could survey a randomly selected group of students as they entered the school grounds one morning. This should ensure that the results will be more representative of the whole population of interest.

2 **a** As there are 1800 members in the club, it would be too expensive and time consuming to question all members.

Age range	*Members*
under 18	257
18 - 39	421
40 - 54	632
55 - 70	356
over 70	134

b For the sample, we want:

$$\text{number of under 18s} = \tfrac{257}{1800} \times 350 \approx 50.0 \approx 50$$
$$\text{number of } 18 - 39\text{s} = \tfrac{421}{1800} \times 350 \approx 81.9 \approx 82$$
$$\text{number of } 40 - 54\text{s} = \tfrac{632}{1800} \times 350 \approx 122.9 \approx 123$$
$$\text{number of } 55 - 70\text{s} = \tfrac{356}{1800} \times 350 \approx 69.2 \approx 69$$
$$\text{number of over 70s} = \tfrac{134}{1800} \times 350 \approx 26.1 \approx 26$$

Now $50 + 82 + 123 + 69 + 26 = 350$ which is the required sample size.
So, the club should survey 50 members aged under 18, 82 members aged 18 - 39, 123 members aged 40 - 54, 69 members aged 55 - 70, and 26 members aged over 70.

3 **a** The number of pages in a daily newspaper takes exact number values.
$\therefore$ this is a discrete variable.

b The maximum daily temperature in a city is a numerical variable which can be measured. The data can take any value between certain limits.
$\therefore$ this is a continuous variable.

c The manufacturer of a television is a categorical variable.

d A person's favourite flavour of ice cream is a categorical variable.

e The position taken by a player on a lacrosse field is a categorical variable.

f The time it takes to run one kilometre is a numerical variable which can be measured. The data can take any value between certain limits.
$\therefore$ this is a continuous variable.

g The length of a person's feet is a numerical variable which can be measured. The data can take any value between certain limits.
$\therefore$ this is a continuous variable.

h A person's shoe size takes exact number values.
$\therefore$ this is a discrete variable.

i The cost of a bicycle takes exact number values.
$\therefore$ this is a discrete variable.

4 **a** It is easier for the police officers to test drivers on a major road due to higher volumes of traffic, so this is convenience sampling.

b Yes, the sample will be biased as people are more likely to be drinking on a Saturday night. It is sensible for this sample to be biased since drink-driving is illegal.

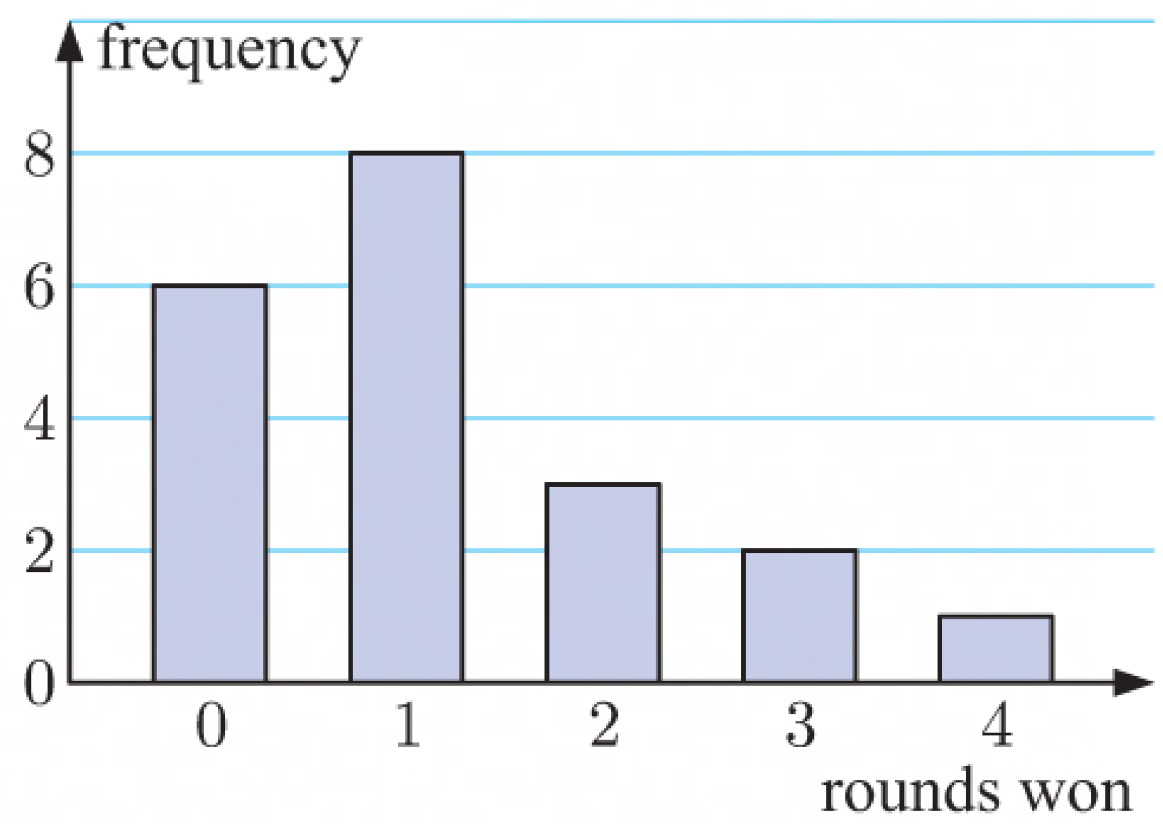
5 **a** The question "Are you healthy?" could be interpreted as:
- "Do you consider yourself to be healthy?"
- "Are you not currently suffering from any health conditions?"
- "Do you eat a balanced diet and exercise regularly?"
- "Do you take any medication for any health conditions?"

b The question could be rewritten as "Do you eat a balanced diet and exercise regularly?"

6 **a** The *number of rounds won* takes exact number values.
$\therefore$ this is a discrete variable.

b The modal number of rounds won is 1 round.

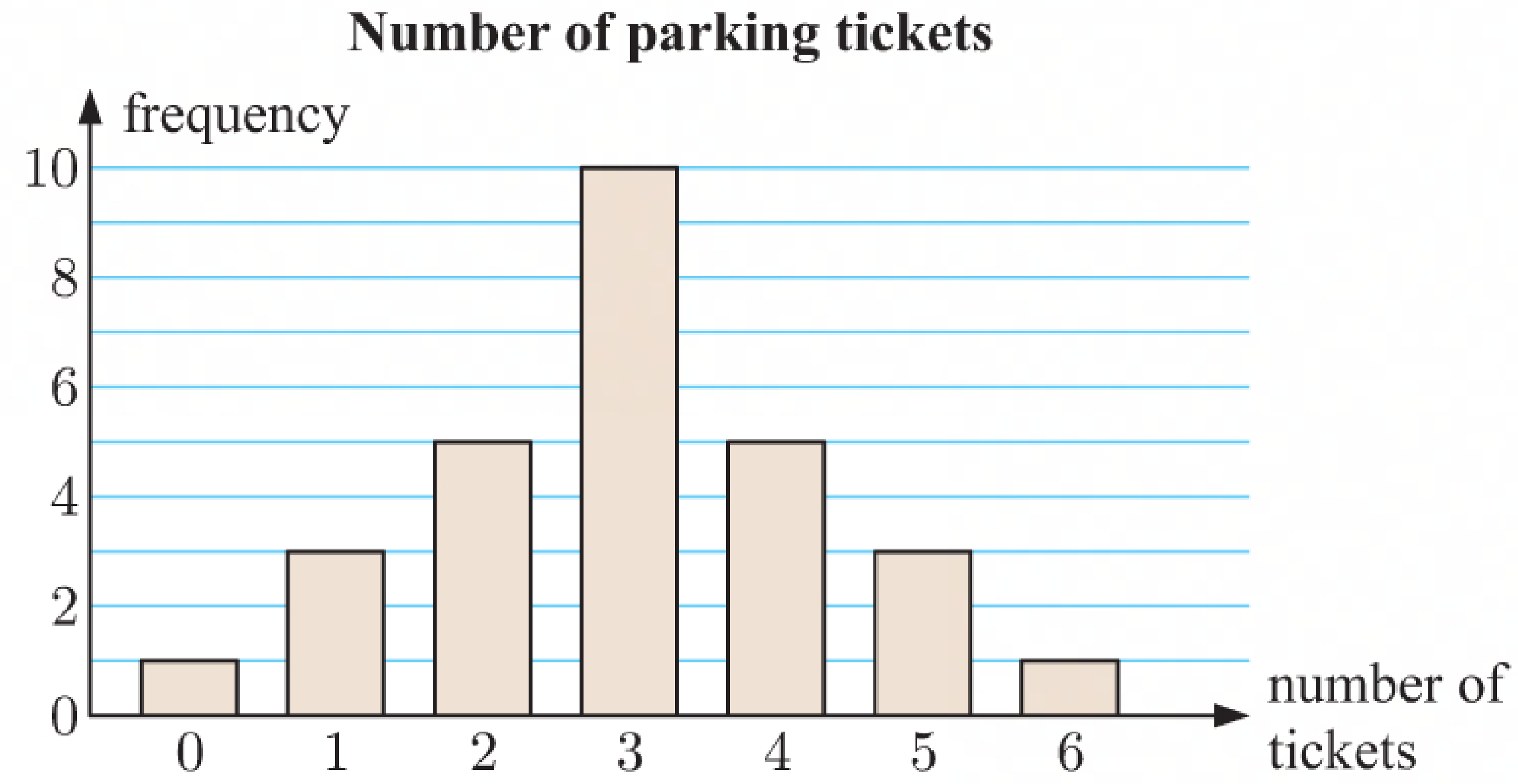
c The data is positively skewed with no outliers.



7 **a**

Number of tickets	*Tally*	*Frequency*
0	|	1
1	|||	3
2	~~||||~~	5
3	~~||||~~ ~~||||~~	10
4	~~||||~~	5
5	|||	3
6	|	1

b



c The data is symmetric with no outliers.

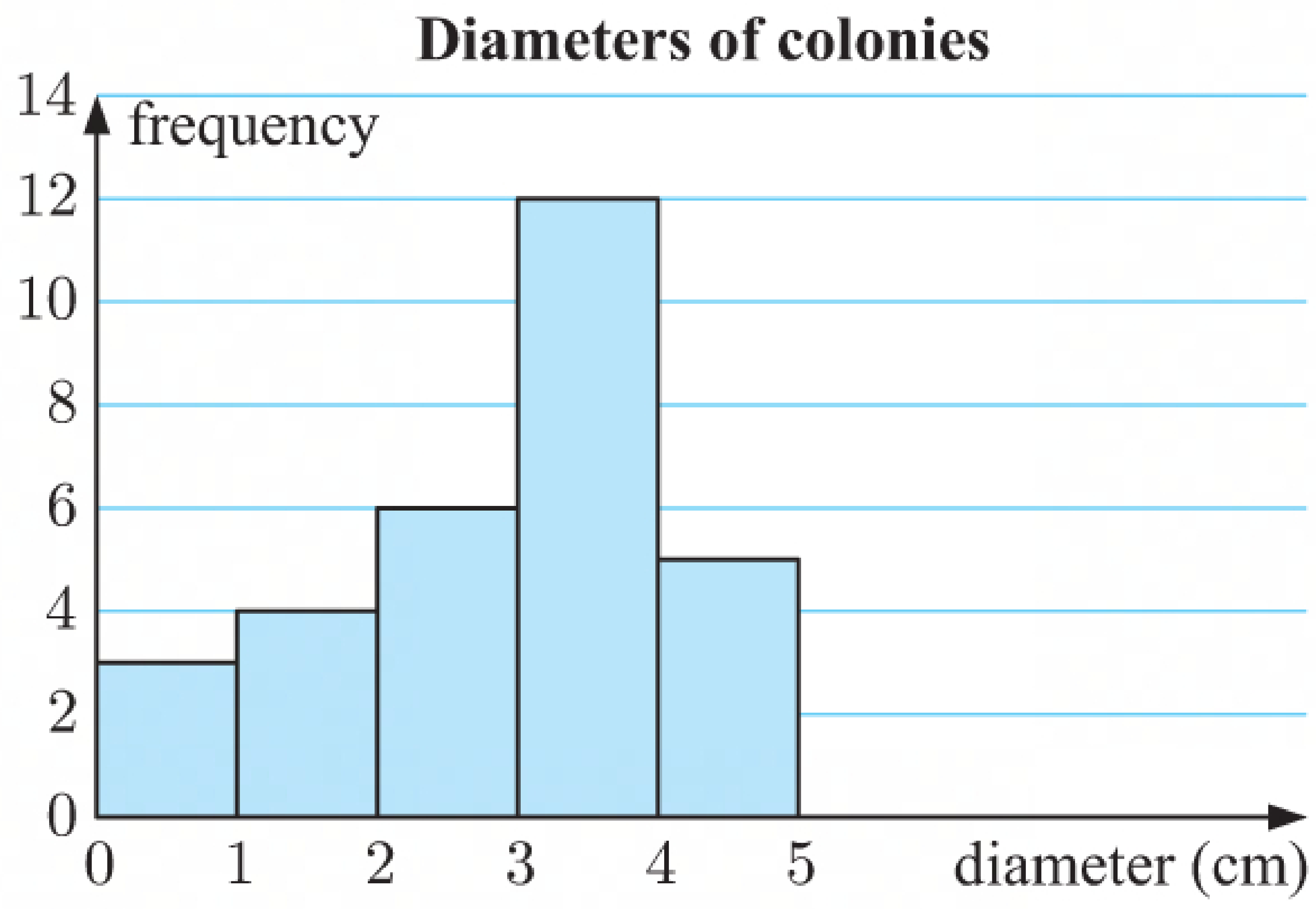
8 **a** The *diameter of bacteria colonies* is a numerical variable which can be measured. The data can take any value between certain limits.

$\therefore$ this is a continuous variable.

b The shortest diameter is 0.4 cm and the longest is 4.9 cm, so we will use class intervals of width 1 cm.

Diameter (d cm)	*Tally*	*Frequency*
$0 \leqslant d < 1$	|||	3
$1 \leqslant d < 2$	||||	4
$2 \leqslant d < 3$	~~||||~~ |	6
$3 \leqslant d < 4$	~~||||~~ ~~||||~~ ||	12
$4 \leqslant d < 5$	~~||||~~	5
	Total	30

c



d The modal class is $3 \leqslant d < 4$ cm. More bacteria have diameters in this interval than in any other interval.

e The data is slightly negatively skewed.

REVIEW SET 12B

1 **a** The number of pages in a book takes exact number values.

$\therefore$ this is a discrete variable.

b The distance travelled by hikers in one day is a numerical variable which can be measured. The data can take any value between certain limits.

$\therefore$ this is a continuous variable.

c The attendance figures for a music festival take exact number values.

$\therefore$ this is a discrete variable.

2 **a** The houses have been selected at regular intervals, so systematic sampling has been used.

b A house will be visited if the last digit in its number is equal to the random number chosen by the promoter, with the random number 10 corresponding to the digit 0. Each house therefore has a 1 in 10 chance of being visited.

c Once the first house number has been chosen, the remaining houses chosen must all have the same second digit in their house number, that is, they are not randomly chosen. For example, it is impossible for two consecutively numbered houses to be selected for the sample. So this is not a simple random sample.

3 **a** Petra's teacher colleagues are quite likely to ignore the emailed questionnaire as emails are easy to ignore. So, Petra's questionnaire may produce a high non-response error.

b It is likely that the teachers who have responded will have strong opinions either for or against the general student behaviour. These responses may therefore not be representative of all teachers' views. Petra may therefore be likely to encounter a coverage error.

4 The question could be rewritten as:

"Did you learn about our services via:

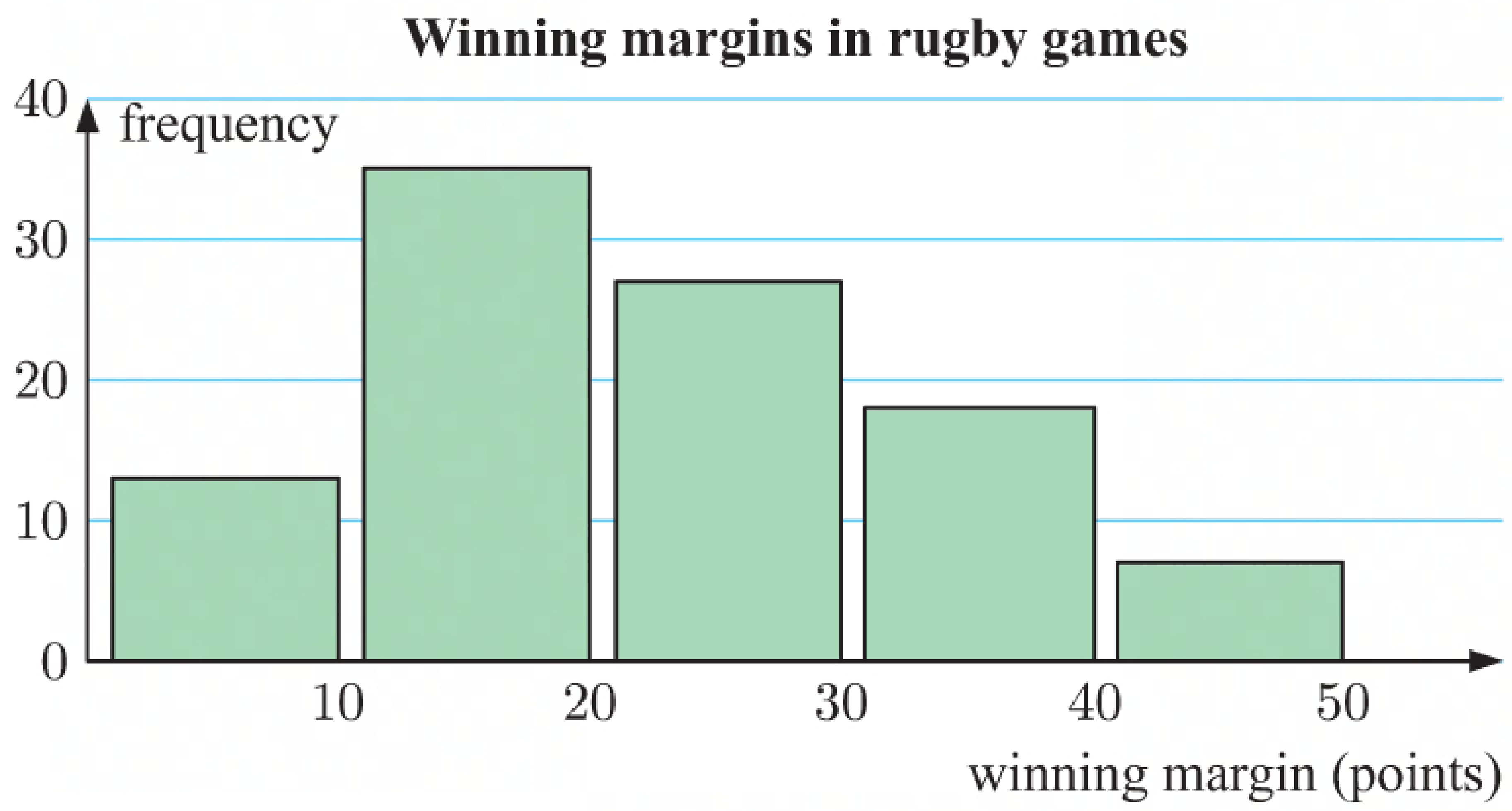
- friends/family
- the internet
- newspaper
- television
- elsewhere?"

5 **a** The tone is not neutral and it is a structured question. The only responses possible are yes or no.

b The question could be rewritten as "How would you describe your general behaviour when you were a child?".

6

Margin (points)	1 - 10	11 - 20	21 - 30	31 - 40	41 - 50
Frequency	13	35	27	18	7



7 **a** The *mass of a horse*, m kg, is a quantitative variable which can be measured. The data can take any value between certain limits.

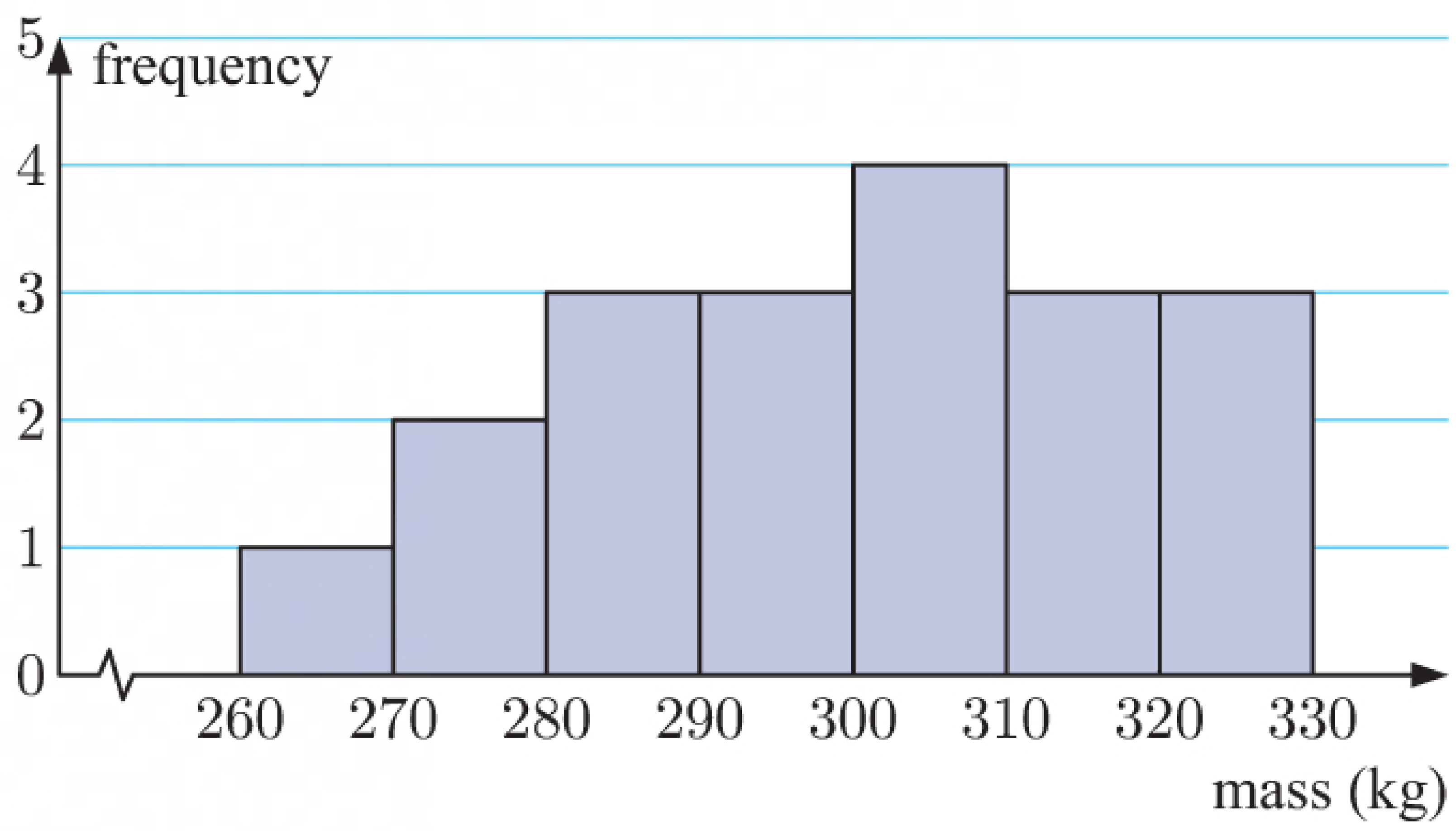
$\therefore$ this is a continuous quantitative variable.

b

Mass (m kg)	*Frequency*
$260 \leqslant m < 270$	1
$270 \leqslant m < 280$	2
$280 \leqslant m < 290$	3
$290 \leqslant m < 300$	3
$300 \leqslant m < 310$	4
$310 \leqslant m < 320$	3
$320 \leqslant m < 330$	3

c The modal class is $300 \leqslant m < 310$ kg. More horses have a mass in this interval than in any other interval.

d



e The data is slightly negatively skewed.

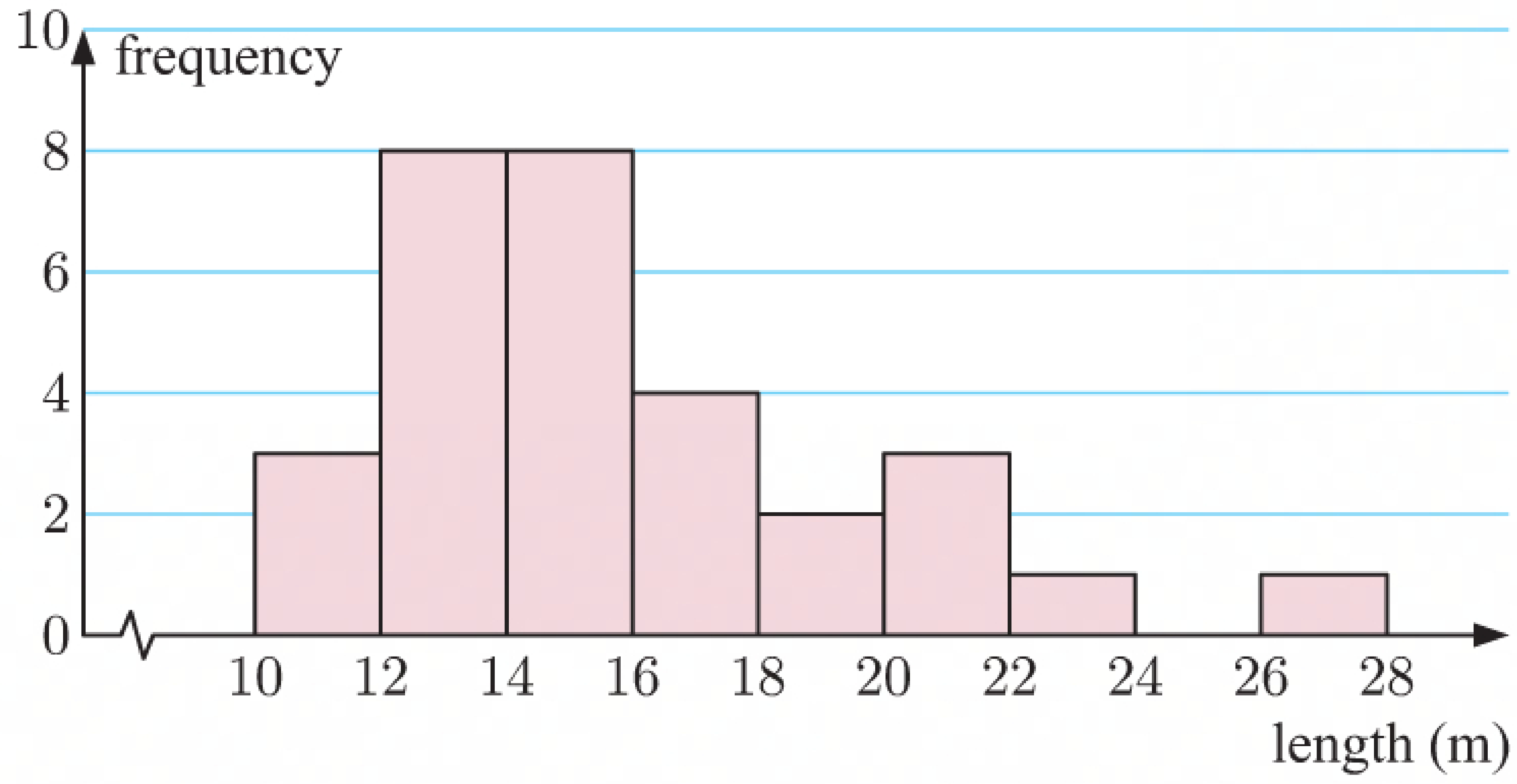
8 **a** The lengths of yachts is a numerical variable which can be measured. The data can take any value between certain limits.

$\therefore$ this is a continuous variable.

b The shortest length is 10.1 m and the longest is 27.4 m, so we will use class intervals of width 2 m.

Length (l m)	*Frequency*
$10 \leqslant l < 12$	3
$12 \leqslant l < 14$	8
$14 \leqslant l < 16$	8
$16 \leqslant l < 18$	4
$18 \leqslant l < 20$	2
$20 \leqslant l < 22$	3
$22 \leqslant l < 24$	1
$24 \leqslant l < 26$	0
$26 \leqslant l < 28$	1

c



d The data is positively skewed with one outlier (27.4 m).

Chapter 13

STATISTICS

EXERCISE 13A

1 **a** 1 is the data value which occurs most often, so the mode is 1 cup.

b As $n = 15$, $\dfrac{n+1}{2} = 8$

The ordered data set is: $\cancel{0}\ \cancel{0}\ \cancel{0}\ \cancel{1}\ \cancel{1}\ \cancel{1}\ \cancel{1}\ \boxed{2}\ \cancel{2}\ \cancel{2}\ \cancel{3}\ \cancel{3}\ \cancel{3}\ \cancel{4}\ \cancel{4}$

8th value

$\therefore$ median $= 2$ cups

c $\text{mean} = \dfrac{2+3+1+1+ \dots + 1+4}{15}$ ← sum of all the data values; ← 15 data values

$= \dfrac{27}{15}$

$= 1.8$ cups

2 **a** **i** $\text{mean} = \dfrac{2+3+3+ \dots +8+9+9}{23}$ ← sum of all the data values; ← 23 data values

$= \dfrac{129}{23}$

≈ 5.61

ii As $n = 23$, $\dfrac{n+1}{2} = 12$

The ordered data set is:

$\cancel{2}\ \cancel{3}\ \cancel{3}\ \cancel{3}\ \cancel{4}\ \cancel{4}\ \cancel{4}\ \cancel{5}\ \cancel{5}\ \cancel{5}\ \cancel{5}\ \boxed{6}\ \cancel{6}\ \cancel{6}\ \cancel{6}\ \cancel{6}\ \cancel{7}\ \cancel{7}\ \cancel{8}\ \cancel{8}\ \cancel{8}\ \cancel{9}\ \cancel{9}$

12th value

$\therefore$ median $= 6$

iii 6 is the data value which occurs most often, so the mode is 6.

b **i** $\text{mean} = \dfrac{10+12+12+ \dots +19+20+21}{15}$ ← sum of all the data values; ← 15 data values

$= \dfrac{245}{15}$

≈ 16.3

ii As $n = 15$, $\dfrac{n+1}{2} = 8$

The ordered data set is:

$\cancel{10}\ \cancel{12}\ \cancel{12}\ \cancel{15}\ \cancel{15}\ \cancel{16}\ \cancel{16}\ \boxed{17}\ \cancel{18}\ \cancel{18}\ \cancel{18}\ \cancel{18}\ \cancel{19}\ \cancel{20}\ \cancel{21}$

8th value

$\therefore$ median $= 17$

iii 18 is the data value which occurs most often, so the mode is 18.

c **i** $$\text{mean} = \frac{22.4 + 24.6 + 21.8 + ... + 25.3 + 29.5 + 23.5}{11}$$

← sum of all the data values
← 11 data values

$$= \frac{273}{11}$$
$$\approx 24.8$$

ii As $n = 11$, $\dfrac{n+1}{2} = 6$

The ordered data set is:

~~21.8 22.4 23.5 23.5 24.6~~ 24.9 ~~25.0 25.3 26.1 26.4 29.5~~

6th value

median $= 24.9$

iii 23.5 is the data value which occurs most often, so the mode is 23.5.

3 $$\text{mean} = \frac{\text{sum of all data values}}{\text{the number of data values}} = \frac{63}{7} = 9$$

4 $$\text{Gordon's mean} = \frac{160 + 175 + ... + 175 + 155}{10} = \frac{1590}{10} = 159$$

$$\text{Ruth's mean} = \frac{157 + 181 + ... + 168 + 148}{10} = \frac{1640}{10} = 164$$

So, Ruth had the higher mean score.

5 **a** $$\text{mean of set A} = \frac{3+4+4+5+ .... + 10}{13} = \frac{84}{13} \approx 6.46$$

$$\text{mean of set B} = \frac{3+4+4+5+ .... + 15}{13} = \frac{89}{13} \approx 6.85$$

b As $n = 13$ for both data sets, the median is the $\left(\dfrac{13+1}{2}\right)$th data value.

$\therefore$ the median is the 7th data value for each data set.

median of set A $= 7$, median of set B $= 7$

c The data sets are the same except for the last value, and the last value of set A is less than that of set B. So, the mean of set A is less than that of set B.
The middle value of both data sets is the same, so the median is the same.

6 **a** **i** mean number of motichoor ladoo

$$= \frac{62 + 76 + 55 + 65 + ... + 54}{31} = \frac{2079}{31} \approx 67.1 \text{ motichoor ladoo}$$

mean number of malai jamun

$$= \frac{37 + 52 + 71 + 59 + ... + 76}{31} = \frac{1663}{31} \approx 53.6 \text{ malai jamun}$$

ii As $n = 31$ for both data sets, the median is the $\left(\frac{31+1}{2}\right)$th data value.

$\therefore$ the median is the 16th data value for each data set.

For the motichoor ladoo, the ordered data set is:

16th value ↓

~~47 48 49 50 54 55 56 58 60 61 62 63 63 65 67~~ 69
~~70 71 72 74 74 75 76 76 77 78 79 81 82 82 85~~

$\therefore$ median = 69 motichoor ladoo

For the malai jamun, the ordered data set is:

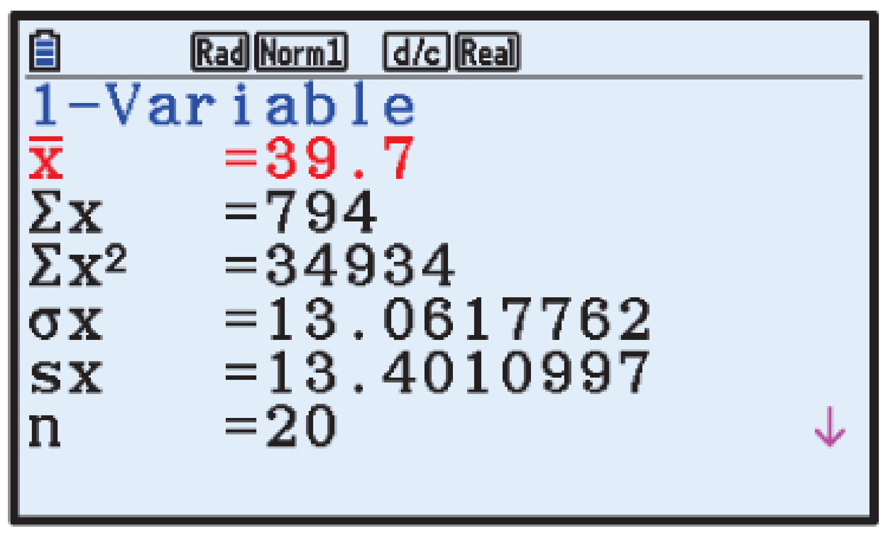
16th value ↓

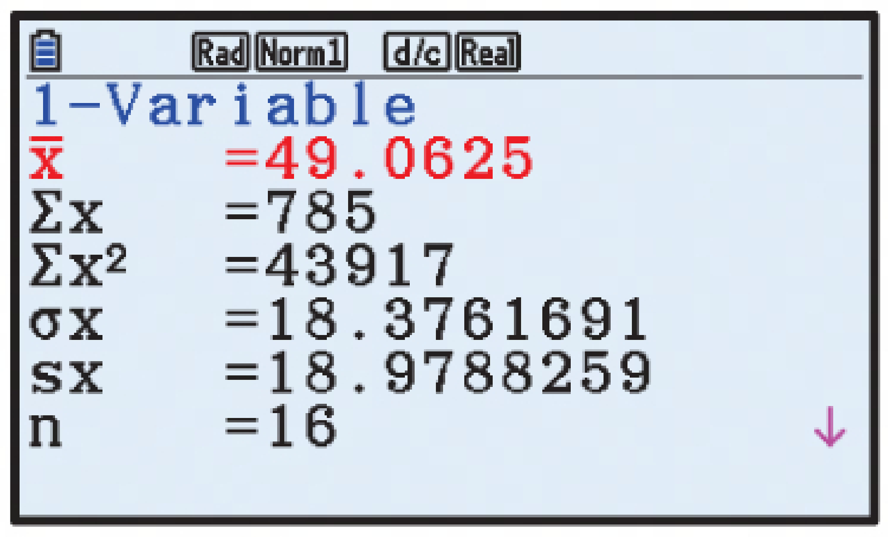
~~37 38 38 39 41 43 44 45 46 47 48 49 50 50 51~~ 52
~~53 54 55 56 57 59 60 61 63 67 68 71 72 73 76~~

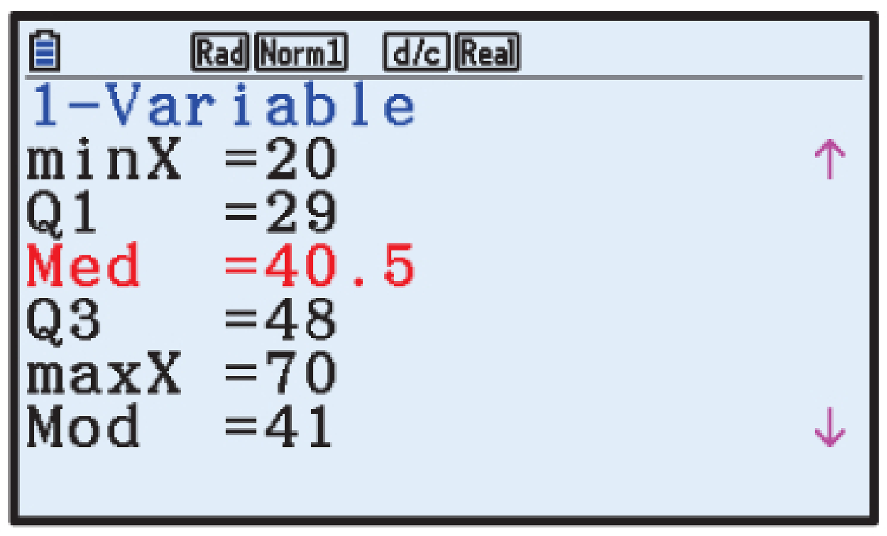
$\therefore$ median = 52 malai jamun

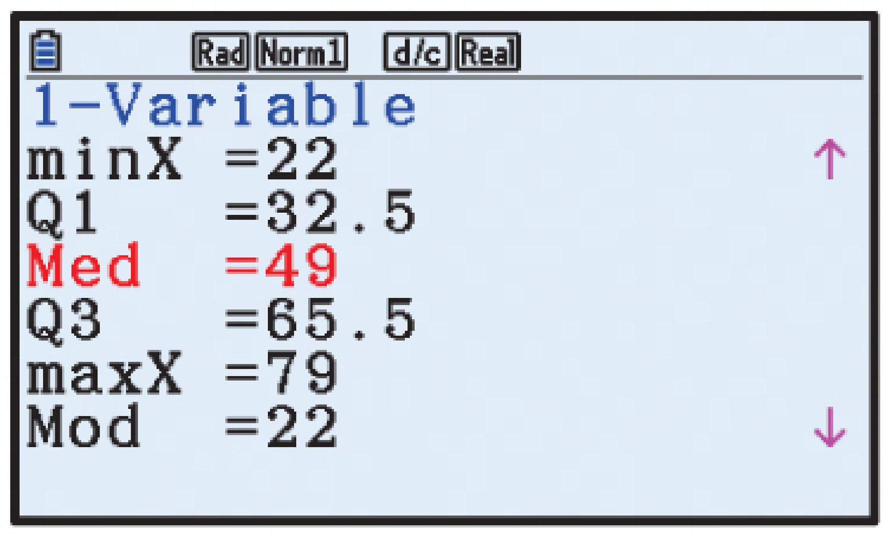
b The motichoor ladoo were more popular as the mean and median are both higher for motichoor ladoo than for malai jamun.

7 a **Bus** **Tram**









mean = 39.7 passengers
median = 40.5 passengers

mean $\approx$ 49.1 passengers
median = 49 passengers

b The tram data has a higher mean and median, but since there were more bus trips on the day and more people travelled by bus in total, we conclude the bus is more popular.

8 a
$$\text{mean number of points} = \frac{43+55+41+37}{4} = \frac{176}{4} = 44 \text{ points}$$

b Let the score for the next match be x points.

$$\therefore \quad \text{mean number of points for first 5 matches} = \frac{43+55+41+37+x}{5}$$

$$\therefore \quad 44 = \frac{176+x}{5}$$

$$\therefore \quad 220 = 176 + x$$

$$\therefore \quad x = 44$$

So the team needs to score 44 points in their next match.

c **i** If the team scores only 25 points in the fifth match, this will decrease their overall mean score since 25 is lower than the mean of 44 for the first four matches.

ii $$\begin{aligned}\text{mean number of points} &= \frac{43+55+41+37+25}{5}\\ &= 40.2 \text{ points}\end{aligned}$$

9 $$\begin{aligned}\frac{\text{total sales for the year}}{12} &= €15\,467\\ \therefore\ \text{total sales for the year} &= €15\,467 \times 12\\ &= €185\,604\end{aligned}$$

10 $$\begin{aligned}\overline{x} &= \frac{\sum_{i=1}^{n} x_i}{n}\\ \therefore\ 11.6 &= \frac{\sum_{i=1}^{10} x_i}{10}\\ \therefore\ \sum_{i=1}^{10} x_i &= 116\end{aligned}$$

11 $$\begin{aligned}\frac{\text{total number of goals for first 14 matches}}{14} &= 16.5\\ \therefore\ \text{total number of goals for first 14 matches} &= 16.5 \times 14\\ &= 231\end{aligned}$$

$$\begin{aligned}\text{netballer's average for whole season} &= \frac{231+21+24}{16}\\ &= \frac{276}{16}\\ &= 17.25 \text{ goals per game}\end{aligned}$$

12 $$\begin{aligned}\frac{5+9+11+12+13+14+17+x}{8} &= 12\\ \therefore\ \frac{81+x}{8} &= 12\\ \therefore\ 81+x &= 96\\ \therefore\ x &= 15\end{aligned}$$

13 $$\begin{aligned}\frac{3+0+a+a+4+a+6+a+3}{9} &= 4\\ \therefore\ \frac{4a+16}{9} &= 4\\ \therefore\ 4a+16 &= 36\\ \therefore\ 4a &= 20\\ \therefore\ a &= 5\end{aligned}$$

14 Let Aruna's eighth test mark be x.

$$\begin{aligned}\frac{29+36+32+38+35+34+39+x}{8} &= 35\\ \therefore\ \frac{243+x}{8} &= 35\\ \therefore\ 243+x &= 280\\ \therefore\ x &= 37\end{aligned}$$

So, Aruna scored 37 marks out of 40 for the eighth test.

15 $\dfrac{\text{sum of sample of 10 measurements}}{10} = 15.7$

$\therefore$ sum of sample of 10 measurements $= 15.7 \times 10$
$= 157$

$\dfrac{\text{sum of sample of 20 measurements}}{20} = 14.3$

$\therefore$ sum of sample of 20 measurements $= 14.3 \times 20$
$= 286$

$$\text{Mean of all 30 measurements} = \frac{157+286}{30} = \frac{443}{30} \approx 14.8$$

16 As $n = 9$, $\dfrac{n+1}{2} = 5$, so the median is the 5th ordered data value.

The median is 12, so 12 must be one of the unknown measurements. Let the other unknown measurement be a.

$\therefore$ the measurements are 7, 9, 11, 12, 13, 14, 17, 19, and a.

Now, $\dfrac{7+9+11+12+13+14+17+19+a}{9} = 12$ {since mean = 12}

$\therefore$ $102 + a = 108$

$\therefore$ $a = 6$

So, the other two measurements are 6 and 12.

INVESTIGATION 1 EFFECTS OF OUTLIERS

1 The data set is: 4, 5, 6, 6, 6, 7, 7, 8, 9, 10

a $\text{mean} = \dfrac{4+5+6+ .... +9+10}{10} = \dfrac{68}{10} = 6.8$

b 6 is the data value which occurs most often, so the mode is 6.

c As $n = 10$, $\dfrac{n+1}{2} = 5.5$

The ordered data set is: ~~4 5 6 6~~ $\underbrace{6 \quad 7}_{\text{two middle data values}}$ ~~7 8 9 10~~

$\therefore$ the median $= \dfrac{6+7}{2} = 6.5$

2 The data set is: 4, 5, 6, 6, 6, 7, 7, 8, 9, 10, 100

a $\text{mean} = \dfrac{4+5+6+ .... +10+100}{11} = \dfrac{168}{11} \approx 15.3$

b 6 is the data value which occurs most often, so the mode is 6.

c As $n = 11$, $\dfrac{n+1}{2} = 6$

The ordered data set is: $\cancel{4\ \ 5\ \ 6\ \ 6\ \ 6}\ \ \underset{\substack{\uparrow \\ \text{6th value}}}{\boxed{7}}\ \ \cancel{7\ \ 8\ \ 9\ \ 10\ \ 100}$

$\therefore$ the median $= 7$

3 **a** The presence of the extreme value increases the mean by more than double.

b The presence of the extreme value has no effect on the mode.

c The presence of the extreme value increases the median slightly. If the two middle values in **1 c** were the same, then the median would not have changed.

4 The mean is most affected by the inclusion of an outlier.

EXERCISE 13B

1 **a** mean selling price $= \dfrac{\$346\,400 + \$327\,600 + ... + \$331\,400 + \$362\,500}{10}$

$= \dfrac{\$3\,637\,700}{10}$

$= \$363\,770$

Since $n = 10$, $\dfrac{n+1}{2} = 5.5$

So the median is the average of the 5th and 6th ordered data values.

The ordered data set is:

$\cancel{327\,600\ \ 329\,500\ \ 331\,400\ \ 332\,400}\ \ \underbrace{\boxed{346\,400\ \ 348\,000}}_{\text{two middle data values}}\ \ \cancel{362\,500\ \ 392\,500\ \ 411\,000\ \ 456\,400}$

$\therefore$ median selling price $= \dfrac{\$346\,400 + \$348\,000}{2}$

$= \$347\,200$

The mean has been affected by the extreme values (the two values greater than \$400 000).

b **i** If you were a vendor wanting to sell your house, you would use the mean as it is higher, and you want to sell at the highest price possible.

ii If you were looking to buy a house in the district, you would use the median as it is lower, and is more representative of a typical selling price in the area.

2 **a** \$33 000 is the data value which occurs the most often, so the modal salary is \$33 000.

mean salary $= \dfrac{\$33\,000 + \$56\,000 + ... + \$33\,000 + \$42\,000}{10}$

$= \dfrac{\$393\,000}{10}$

$= \$39\,300$

Since $n = 10$, $\dfrac{n+1}{2} = 5.5$

So the median is the average of the 5th and 6th ordered data values.

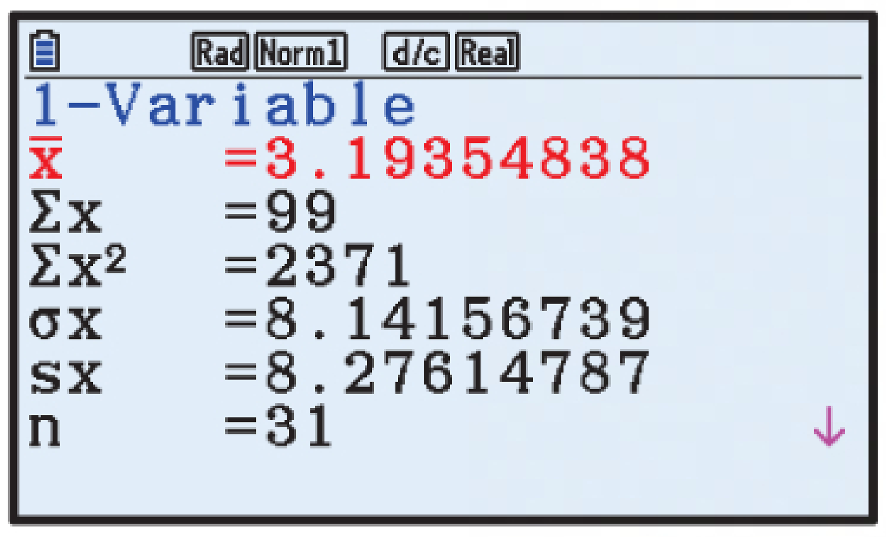
The ordered data set is:

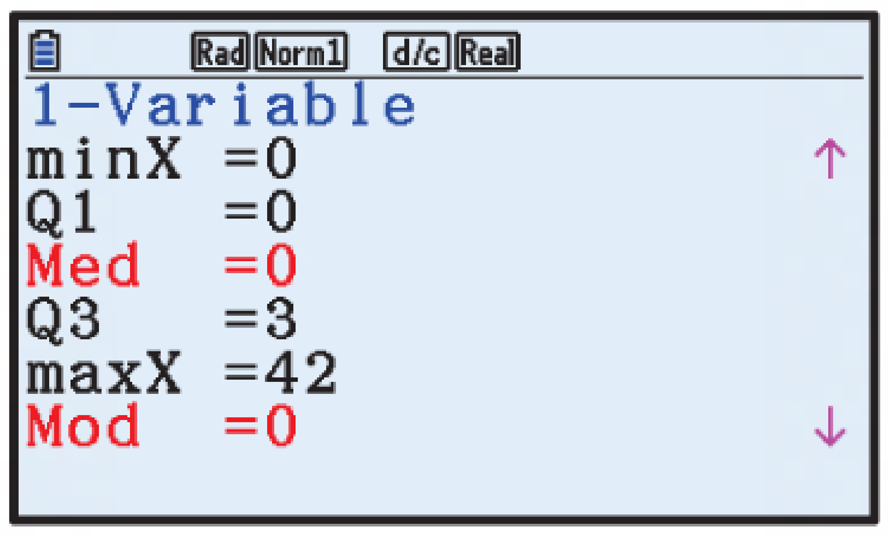
$$\cancel{33\,000}\ \ \cancel{33\,000}\ \ \cancel{33\,000}\ \ \cancel{33\,000}\ \ \underbrace{33\,000\ \ 34\,000}_{\text{two middle data values}}\ \ \cancel{42\,000}\ \ \cancel{48\,000}\ \ \cancel{48\,000}\ \ \cancel{56\,000}$$

$$\therefore\ \text{median salary} = \frac{\$33\,000 + \$34\,000}{2}$$
$$= \$33\,500$$

b The mode is the lowest value and it does not take the higher values into account. So the mode is an unsatisfactory measure of centre in this case.

c No, the median does not take the higher values into account. It is too close to the lower end of the distribution. So it is not a satisfactory measure of centre for this data set.

3 **a** 



So the mean is ≈ 3.19 mm, the median is 0 mm, and the mode is 0 mm.

b The data is very positively skewed which means the median is not in the centre. Therefore the median is not the most suitable measure of centre for this data set.

c The mode is the lowest value, and it does not take the higher values into account. So it is not the most suitable measure of centre for this data set.

d There are two outliers. They are 21 mm and 42 mm.

e No, the outliers should not be removed as they are genuine data values.

4 **a** $$\text{mean number of children} = \frac{2+2+2+\dots.+2+3+2}{30}$$
$$= \frac{61}{30}$$
$$\approx 2.03$$

Since $n = 30$, $\dfrac{n+1}{2} = 15.5$

So the median is the average of the 15th and 16th ordered data values.

The ordered data set is:

$$\cancel{1}\ \cancel{1}\ \cancel{1}\ \cancel{1}\ \cancel{1}\ \cancel{1}\ \cancel{1}\ \cancel{1}\ \cancel{1}\ \cancel{1}\ \cancel{1}\ \cancel{2}\ \cancel{2}\ \cancel{2}\ \underbrace{2\ \ 2}_{\text{two middle data values}}\ \cancel{2}\ \cancel{2}\ \cancel{2}\ \cancel{2}\ \cancel{2}\ \cancel{2}\ \cancel{3}\ \cancel{3}\ \cancel{3}\ \cancel{3}\ \cancel{4}\ \cancel{4}\ \cancel{4}\ \cancel{4}$$

$$\therefore\ \text{median number of children} = \frac{2+2}{2}$$
$$= 2$$

Both 1 and 2 are the data values which occur the most often, so the modal number of children per family is 1 and 2.

b Yes, the mode is a useful statistic in this case as Esmé can then offer a "family package" to match the most common number of children per family.

c Esmé should include 2 children per family in the package, since this is one of the modes; it is also the median and is very close to the mean.

EXERCISE 13C

1

Number of people (x)	*Frequency* (f)	*Product* (xf)	*Cumulative frequency*
1	13	13	13
2	8	16	21
3	4	12	25
4	5	20	30
Total	$\sum f = 30$	$\sum xf = 61$	

a Looking down the frequency column, the highest frequency is 13. This corresponds to 1 person, so the mode is 1 person.

b There are 30 data values, so $n = 30$. $\frac{n+1}{2} = 15.5$, so the median is the average of the 15th and 16th ordered data values.
From the cumulative frequency column, the 14th to 21st ordered data values are 2 people.
$\therefore$ the 15th and 16th ordered data values are 2 people.
$\therefore$ median $= \frac{2+2}{2} = 2$ people

c $\overline{x} = \frac{\sum xf}{\sum f}$
$= \frac{61}{30}$
≈ 2.03 people

2

Number of phone calls (x)	*Frequency* (f)	*Product* (xf)	*Cumulative frequency*
0	5	0	5
1	8	8	13
2	13	26	26
3	8	24	34
4	6	24	40
5	3	15	43
6	3	18	46
7	2	14	48
8	1	8	49
9	0	0	49
10	0	0	49
11	1	11	50
Total	$\sum f = 50$	$\sum xf = 148$	

a **i** $\overline{x} = \dfrac{\sum xf}{\sum f}$

$= \dfrac{148}{50}$

$= 2.96$ phone calls

ii There are 50 data values, so $n = 50$. $\dfrac{n+1}{2} = 25.5$, so the median is the average of the 25th and 26th ordered data values.

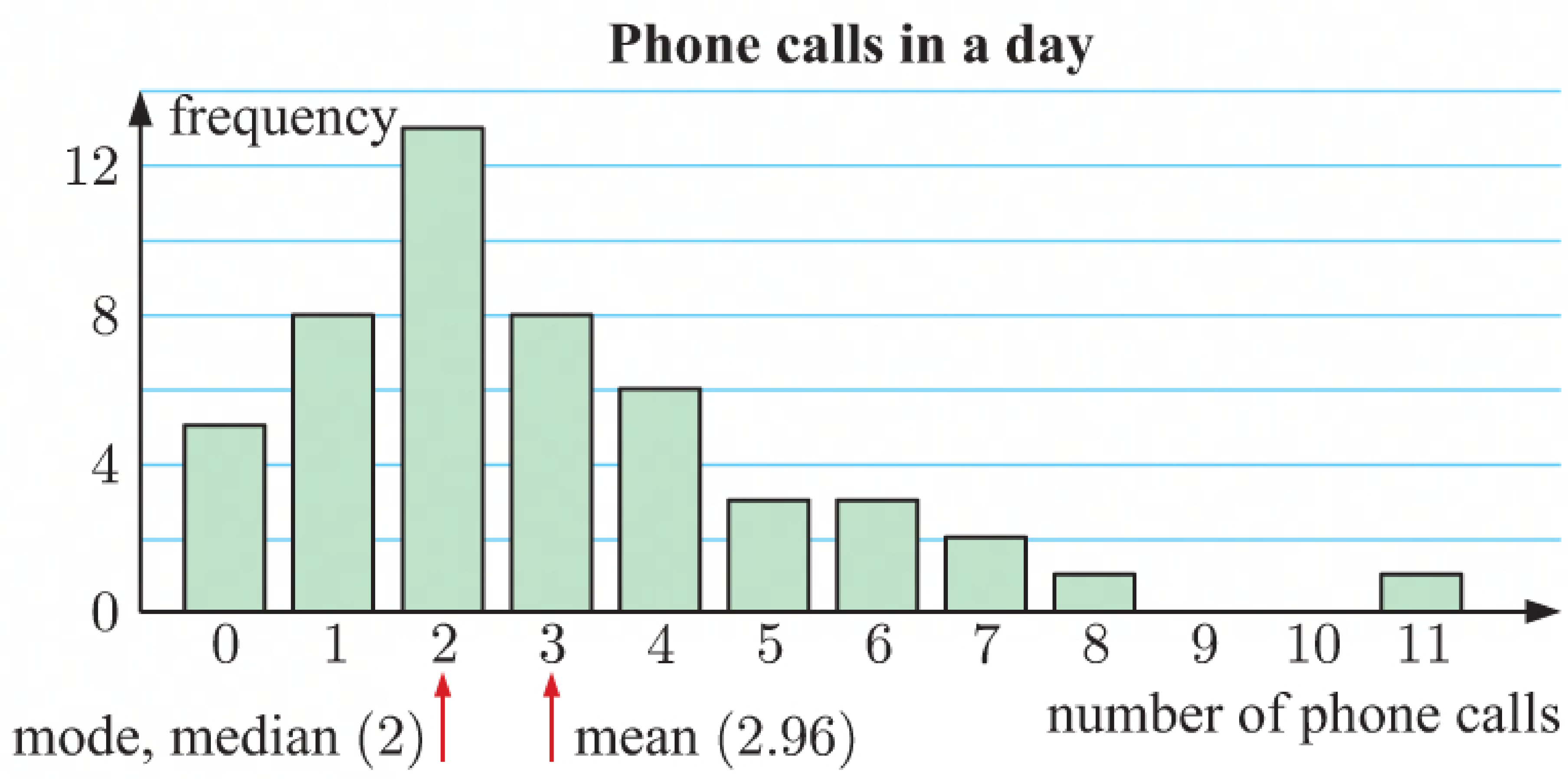
From the cumulative frequency column, the 14th to 26th ordered data values are 2 phone calls.

$\therefore$ the 25th and 26th ordered data values are 2 phone calls.

$\therefore$ median $= \dfrac{2+2}{2} = 2$ phone calls

iii Looking down the frequency column, the highest frequency is 13. This corresponds to 2 phone calls, so the mode is 2 phone calls.

b



c The distribution is positively skewed, with one outlier (11 phone calls).

d The mean is larger than the median as the mean is affected by outliers and larger data values, unlike the median.

e The mean would be the most suitable measure of centre for this data set as it best represents all of the data.

3

Number of children (x)	*Frequency* (f)	*Product* (xf)	*Cumulative frequency*
1	5	5	5
2	28	56	33
3	15	45	48
4	8	32	56
5	2	10	58
6	1	6	59
Total	$\sum f = 59$	$\sum xf = 154$	

a **i** $\overline{x} = \dfrac{\sum xf}{\sum f}$

$= \dfrac{154}{59}$

≈ 2.61 children

ii Looking down the frequency column, the highest frequency is 28. This corresponds to 2 children, so the mode is 2 children.

iii There are 59 data values, so $n = 59$. $\dfrac{n+1}{2} = 30$, so the median is the 30th ordered data value.

From the cumulative frequency column, the 6th to 33rd ordered data values are 2 children.

$\therefore$ the 30th data value is 2 children.

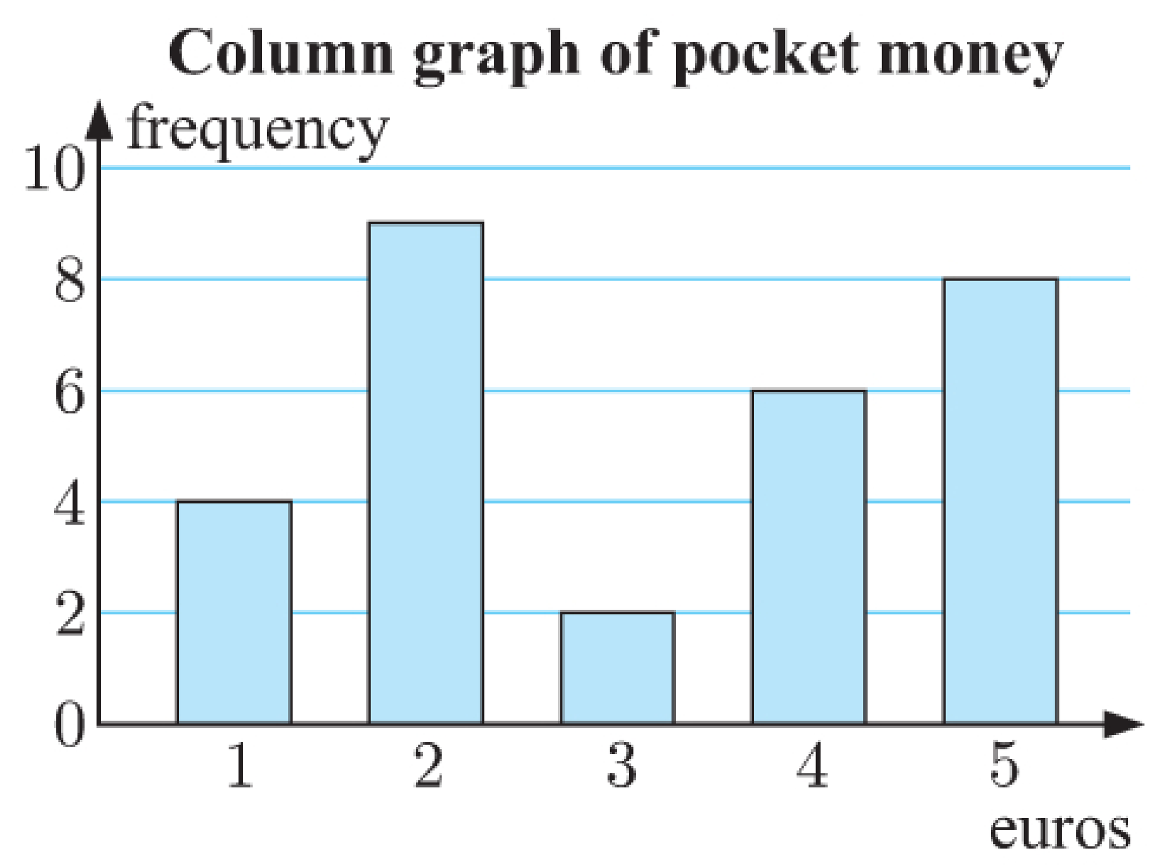
$\therefore$ the median is 2 children.

b As the mean of this data set is ≈ 2.61 children, this school has more children per family than the average British family.

c The data is positively skewed.

d The values at the higher end of the data have increased the mean more than the mode and median.

4



a

Pocket money ($€x$)	*Frequency* (f)	*Product* (xf)	*Cumulative frequency*
1	4	4	4
2	9	18	13
3	2	6	15
4	6	24	21
5	8	40	29
Total	$\sum f = 29$	$\sum xf = 92$	

b There are 29 children in total.

c **i** $\overline{x} = \dfrac{\sum xf}{\sum f}$

$= \dfrac{92}{29}$

$\approx €3.17$

ii There are 29 data values, so $n = 29$. $\dfrac{n+1}{2} = 15$, so the median is the 15th ordered data value.

From the cumulative frequency column, the 14th and 15th ordered data values are €3.

$\therefore$ the 15th data value is €3.

$\therefore$ the median is €3.

iii Looking down the frequency column, the highest frequency is 9. This corresponds to €2, so the mode is €2.

d The mode can be found easily using the graph only, as it is represented by the highest column on the graph.

5 The 31 measurements in order are:
$\{15$ values below $10\}$, 10.1, 10.4, 10.7, 10.9, $\{12$ values above $11\}$

There are 31 data values, so $n = 31$. $\frac{n+1}{2} = 16$, so the median is the 16th ordered data value.

$\therefore$ the median is 10.1 cm.

6 **a**

Salary ($\$x$)	*Frequency* (f)	*Product* (xf)	*Cumulative frequency*
56 000	10	560 000	10
70 000	6	420 000	16
84 000	3	252 000	19
100 000	1	100 000	20
Total	$\sum f = 20$	$\sum xf = 1\,332\,000$	

i There are 20 data values, so $n = 20$. $\frac{n+1}{2} = 10.5$, so the median is the average of the 10th and 11th ordered data values.
From the cumulative frequency column, the first 10 ordered data values are \$56 000, and the 11th to 16th ordered data values are \$70 000.

$\therefore$ the 10th data value is \$56 000 and the 11th data value is \$70 000.

$$\therefore \text{ the median} = \frac{\$56\,000 + \$70\,000}{2} = \$63\,000$$

ii Looking down the frequency column, the highest frequency is 10. This corresponds to a salary of \$56 000, so the mode is \$56 000.

iii
$$\overline{x} = \frac{\sum xf}{\sum f} = \frac{\$1\,332\,000}{20} = \$66\,600$$

b The boss would use the mean value to argue against a pay rise, as it is the largest of the measures, and takes all salaries into account.

7

Score	2	3	4	5	6	7	8
Frequency	0	2	3	5	x	4	1

a
$$\overline{x} = \frac{\sum xf}{\sum f}$$

$$\therefore 5.45 = \frac{2\times 0 + 3\times 2 + 4\times 3 + 5\times 5 + 6\times x + 7\times 4 + 8\times 1}{0+2+3+5+x+4+1}$$

$$\therefore 5.45 = \frac{6x+79}{x+15}$$

$$\therefore 5.45(x+15) = 6x+79$$

$$\therefore 5.45x + 81.75 = 6x + 79$$

$$\therefore 0.55x = 2.75$$

$$\therefore x = 5$$

b There were $5 + 15 = 20$ students in total.

$$\text{Percentage of students who passed} = \frac{\text{number of students who passed}}{\text{total number of students}} \times 100\%$$
$$= \frac{5+5+4+1}{20} \times 100\%$$
$$= \frac{15}{20} \times 100\%$$
$$= 75\%$$

INVESTIGATION 2 MID-INTERVAL VALUES

1

Marks	*Frequency* (f)	*Lowest possible result* (x)	*Product* (xf)
0 - 9	2	0	0
10 - 19	31	10	310
20 - 29	73	20	1460
30 - 39	85	30	2550
40 - 49	28	40	1120
Total	$\sum f = 219$		$\sum xf = 5440$

$$\overline{x} = \frac{\sum xf}{\sum f}$$
$$= \frac{5440}{219}$$
$$\approx 24.8$$

The mean Physics examination mark must be *at least* 24.8 .

2

Marks	*Frequency* (f)	*Highest possible result* (x)	*Product* (xf)
0 - 9	2	9	18
10 - 19	31	19	589
20 - 29	73	29	2117
30 - 39	85	39	3315
40 - 49	28	49	1372
Total	$\sum f = 219$		$\sum xf = 7411$

$$\overline{x} = \frac{\sum xf}{\sum f}$$
$$= \frac{7411}{219}$$
$$\approx 33.8$$

The mean Physics examination mark must be *at most* 33.8 .

3 **a**

Marks	*Frequency* (f)	*Mid-interval value* (x)	*Product* (xf)
0 - 9	2	4.5	9
10 - 19	31	14.5	449.5
20 - 29	73	24.5	1788.5
30 - 39	85	34.5	2932.5
40 - 49	28	44.5	1246
Total	$\sum f = 219$		$\sum xf = 6425.5$

$$\begin{aligned}\overline{x} &= \frac{\sum xf}{\sum f} \\ &= \frac{6425.5}{219} \\ &\approx 29.3\end{aligned}$$

b The result in **a** is halfway between the lower and upper limits in **1** and **2**.

c The mean Physics examination mark was approximately 29.3 .

4 The accuracy of the mid-interval estimate will depend on how the data values are distributed in each class interval. For example, the estimate will be more accurate if the data is uniformly or symmetrically distributed in each class interval.
Using a greater number of narrower class intervals will also improve the accuracy of the estimate. This is because the mid-interval value will be more likely to be close to the actual data values.

EXERCISE 13D

1

Time (t min)	*Frequency* (f)	*Midpoint* (x)	*Product* (xf)
$0 \leqslant t < 10$	17	5	85
$10 \leqslant t < 20$	10	15	150
$20 \leqslant t < 30$	9	25	225
$30 \leqslant t < 40$	4	35	140
Total	$\sum f = 40$		$\sum xf = 600$

a Simone made 40 phone calls during the week.

b $$\begin{aligned}\overline{x} &= \frac{\sum xf}{\sum f} \\ &= \frac{600}{40} \\ &= 15\end{aligned}$$

$\therefore$ the mean length of the calls was about 15 minutes.

2

Score	*Frequency* (f)	*Midpoint* (x)	*Product* (xf)
0 - 9	2	4.5	9
10 - 19	5	14.5	72.5
20 - 29	7	24.5	171.5
30 - 39	27	34.5	931.5
40 - 49	9	44.5	400.5
Total	$\sum f = 50$		$\sum xf = 1585$

$$\overline{x} = \frac{\sum xf}{\sum f} = \frac{1585}{50} = 31.7$$

$\therefore$ the mean score was about 31.7.

3

Amount of petrol (P L)	*Frequency* (f)	*Midpoint* (x)	*Product* (xf)
$2000 < P \leqslant 3000$	4	2500	10 000
$3000 < P \leqslant 4000$	4	3500	14 000
$4000 < P \leqslant 5000$	9	4500	40 500
$5000 < P \leqslant 6000$	14	5500	77 000
$6000 < P \leqslant 7000$	23	6500	149 500
$7000 < P \leqslant 8000$	16	7500	120 000
Total	$\sum f = 70$		$\sum xf = 411\,000$

a There were 70 service stations involved in the survey.

b The total amount of petrol sold was about 411 000 L, or 411 kL.

c
$$\overline{x} = \frac{\sum xf}{\sum f} = \frac{411\,000}{70} \approx 5870$$

$\therefore$ the mean amount of petrol sold for the day was about 5870 L.

d The modal class is $6000 < P \leqslant 7000$ L. This is the most frequently occurring amount of petrol sold at a service station in one day.

4 **a**

Runs scored	*Tally*	*Frequency* (f)	*Midpoint* (x)	*Product* (xf)
0 - 9	卌 卌 |	11	4.5	49.5
10 - 19	卌 |||	8	14.5	116
20 - 29	卌 |||	8	24.5	196
30 - 39	||	2	34.5	69
	Total	$\sum f = 29$		$\sum xf = 430.5$

b
$$\overline{x} = \frac{\sum xf}{\sum f} = \frac{430.5}{29} \approx 14.8$$

$\therefore$ the mean number of runs scored was about 14.8.

c exact mean number of runs scored

$= \dfrac{17+5+22+13+\dots.+25+9}{29}$

$= \dfrac{432}{29}$

≈ 14.9

The exact mean ≈ 14.9 is very close to the estimated mean in **b**, which means the estimate was very accurate.

5

Waiting time (t min)	*Frequency* (f)	*Midpoint* (x)	*Product* (xf)
$0 \leqslant t < 1$	$p = 24$	0.5	12
$1 \leqslant t < 2$	42	1.5	63
$2 \leqslant t < 3$	50	2.5	125
$3 \leqslant t < 4$	78	3.5	273
$4 \leqslant t < 5$	60	4.5	270
$5 \leqslant t < 6$	30	5.5	165
$6 \leqslant t < 7$	16	6.5	104
Total	$\sum f = 300$		$\sum xf = 1012$

a Total number of customers $= p + 42 + 50 + 78 + 60 + 30 + 16$

$\therefore\ 300 = p + 276$

$\therefore\ p = 24$

b $\overline{x} = \dfrac{\sum xf}{\sum f}$

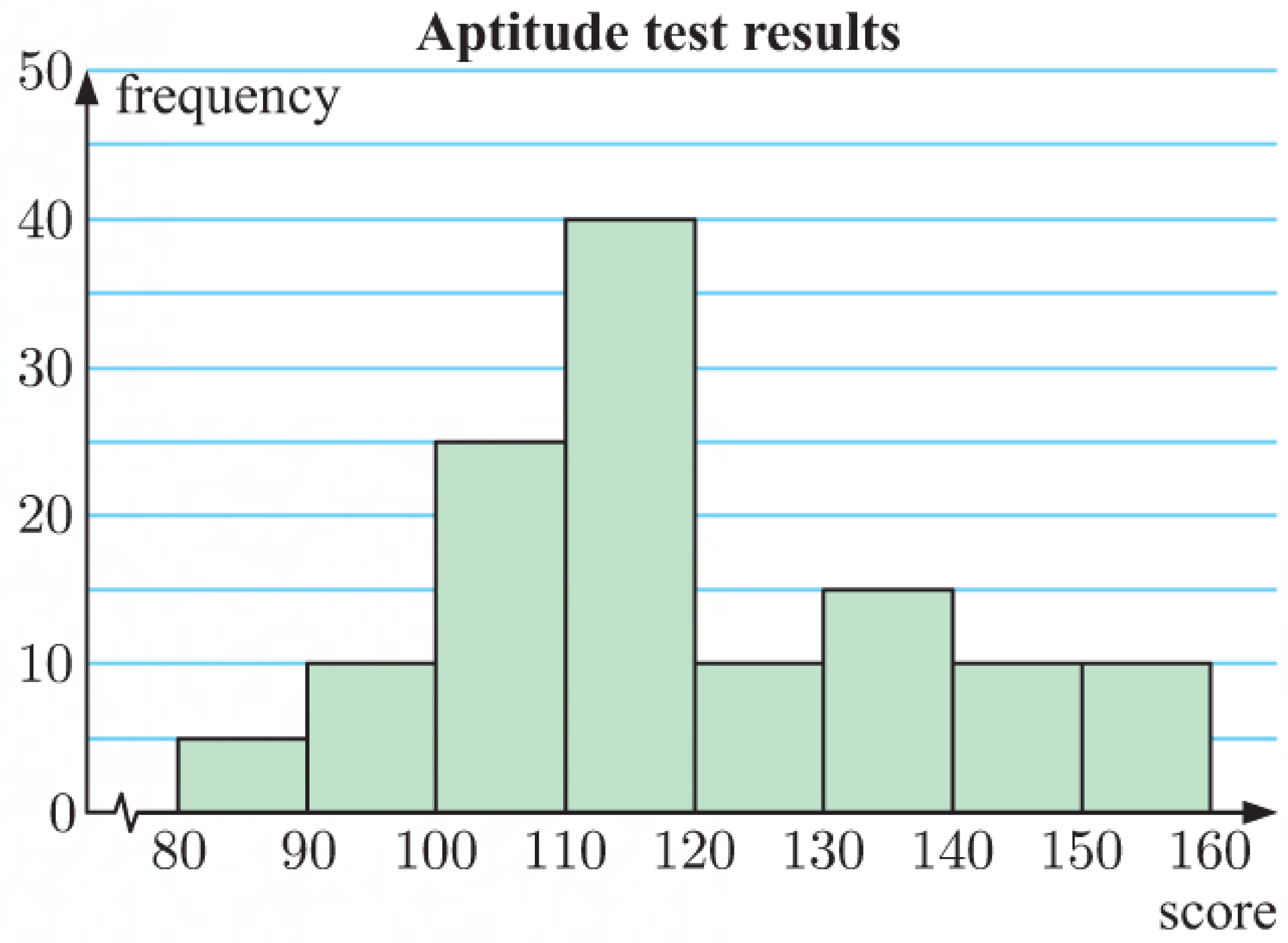
$= \dfrac{1012}{300}$

≈ 3.37

$\therefore$ the mean waiting time is about 3.37 minutes.

c $\dfrac{30+16}{300} \times 100\% \approx 15.3\%$ of customers waited for at least 5 minutes.

6



Score	*Frequency* (f)	*Midpoint* (x)	*Product* (xf)
$80 \leqslant s < 90$	5	85	425
$90 \leqslant s < 100$	10	95	950
$100 \leqslant s < 110$	25	105	2625
$110 \leqslant s < 120$	40	115	4600
$120 \leqslant s < 130$	10	125	1250
$130 \leqslant s < 140$	15	135	2025
$140 \leqslant s < 150$	10	145	1450
$150 \leqslant s < 160$	10	155	1550
Total	$\sum f = 125$		$\sum xf = 14\,875$

a 125 people took the test.

b
$$\overline{x} = \frac{\sum xf}{\sum f} = \frac{14\,875}{125} = 119$$

$\therefore$ the mean score was about 119.

c $\dfrac{5+10}{125} = \dfrac{15}{125} = \dfrac{3}{25}$ of the people scored less than 100 for the test.

d $\dfrac{15+10+10}{125} = \dfrac{35}{125} = 28\%$ of the people scored at least 130 for the test.

EXERCISE 13E

1 **a** The ordered data set is: 5 6 9 10 11 13 15 16 18 20 21 (11 data values)

i Since $n = 11$, $\dfrac{n+1}{2} = 6$ $\therefore$ the median is the 6th data value.

~~5 6 9 10 11~~ 13 ~~15 16 18 20 21~~

$\therefore$ median $= 13$

ii Since the median is a data value we now ignore it and split the remaining data into two:

$\underbrace{5\ \ 6\ \ 9\ \ 10\ \ 11}_{\text{lower half}}$ $\underbrace{15\ \ 16\ \ 18\ \ 20\ \ 21}_{\text{upper half}}$

Q_1 = median of lower half = 9
Q_3 = median of upper half = 18

iii range = maximum − minimum
= 21 − 5
= 16

iv IQR = $Q_3 - Q_1$
= 18 − 9
= 9

b The ordered data set is:
7 7 10 13 14 15 18 19 21 21 23 24 24 26 (14 data values)

i Since $n = 14$, $\dfrac{n+1}{2} = 7.5$ $\therefore$ the median is the average of the 7th and 8th data values.

$\cancel{7\ \ 7\ \ 10\ \ 13\ \ 14\ \ 15}$ 18 19 $\cancel{21\ \ 21\ \ 23\ \ 24\ \ 24\ \ 26}$

$$\therefore \text{ median} = \frac{\text{7th value} + \text{8th value}}{2} = \frac{18+19}{2} = 18.5$$

ii We have an even number of data values, so we include all data values when we split the data set into two:

$\underbrace{7\ \ 7\ \ 10\ \ 13\ \ 14\ \ 15\ \ 18}_{\text{lower half}}$ $\underbrace{19\ \ 21\ \ 21\ \ 23\ \ 24\ \ 24\ \ 26}_{\text{upper half}}$

Q_1 = median of lower half = 13
Q_3 = median of upper half = 23

iii range = maximum − minimum
= 26 − 7
= 19

iv IQR = $Q_3 - Q_1$
= 23 − 13
= 10

c The ordered data set is: 15 19 21 24 29 32 38 43 (8 data values)

i Since $n = 8$, $\dfrac{n+1}{2} = 4.5$ $\therefore$ the median is the average of the 4th and 5th data values.

$\cancel{15\ \ 19\ \ 21}$ 24 29 $\cancel{32\ \ 38\ \ 43}$

$$\therefore \text{ median} = \frac{\text{4th value} + \text{5th value}}{2} = \frac{24+29}{2} = 26.5$$

ii We have an even number of data values, so we include all data values when we split the data set into two:

$$\overbrace{15\ \ 19\ \ 21\ \ 24}^{\text{lower half}}\quad \overbrace{29\ \ 32\ \ 38\ \ 43}^{\text{upper half}}$$

$Q_1 = \text{median of lower half} = \dfrac{19+21}{2} = 20$

$Q_3 = \text{median of upper half} = \dfrac{32+38}{2} = 35$

iii range = maximum − minimum
$= 43 - 15$
$= 28$

iv $\text{IQR} = Q_3 - Q_1$
$= 35 - 20$
$= 15$

d The ordered data set is: 20 26 28 32 33 41 45 52 57 69 (10 data values)

i Since $n = 10$, $\dfrac{n+1}{2} = 5.5$ $\therefore$ the median is the average of the 5th and 6th data values.

$$\cancel{20\ \ 26\ \ 28\ \ 32}\ \ 33\ \ 41\ \ \cancel{45\ \ 52\ \ 57\ \ 69}$$

$$\therefore\ \text{median} = \frac{\text{5th value} + \text{6th value}}{2}$$
$$= \frac{33+41}{2}$$
$$= 37$$

ii We have an even number of data values, so we include all data values when we split the data set into two:

$$\overbrace{20\ \ 26\ \ 28\ \ 32\ \ 33}^{\text{lower half}}\quad \overbrace{41\ \ 45\ \ 52\ \ 57\ \ 69}^{\text{upper half}}$$

$Q_1 = \text{median of lower half} = 28$
$Q_3 = \text{median of upper half} = 52$

iii range = maximum − minimum
$= 69 - 20$
$= 49$

iv $\text{IQR} = Q_3 - Q_1$
$= 52 - 28$
$= 24$

2 The ordered data sets are:

Jane:	\$29	\$29	\$29	\$31	\$34	\$35	\$36	\$36	\$38	\$40	\$42	\$47	(12 data values)
Ashley:	\$19	\$19	\$23	\$24	\$24	\$26	\$26	\$32	\$35	\$40	\$42	\$59	(12 data values)

a *Jane*:

$$\text{mean} = \frac{\$29 + \$29 + \$29 + ... + \$40 + \$42 + \$47}{12}$$
$$= \frac{\$426}{12}$$
$$= \$35.50$$

Since $n = 12$, $\dfrac{n+1}{2} = 6.5$ $\therefore$ the median is the average of the 6th and 7th data values.

$\cancel{\$29\ \ \$29\ \ \$29\ \ \$31\ \ \$34}$ $\$35\ \ \$36$ $\cancel{\$36\ \ \$38\ \ \$40\ \ \$42\ \ \$47}$

$$\begin{aligned}\therefore\ \text{median} &= \frac{\text{6th value} + \text{7th value}}{2}\\ &= \frac{\$35 + \$36}{2}\\ &= \$35.50\end{aligned}$$

Ashley:

$$\begin{aligned}\text{mean} &= \frac{\$19 + \$19 + \$23 + \ldots + \$40 + \$42 + \$59}{12}\\ &= \frac{\$369}{12}\\ &= \$30.75\end{aligned}$$

Since $n = 12$, $\dfrac{n+1}{2} = 6.5$ $\therefore$ the median is the average of the 6th and 7th data values.

$\cancel{\$19\ \ \$19\ \ \$23\ \ \$24\ \ \$24}$ $\$26\ \ \$26$ $\cancel{\$32\ \ \$35\ \ \$40\ \ \$42\ \ \$59}$

$$\begin{aligned}\therefore\ \text{median} &= \frac{\text{6th value} + \text{7th value}}{2}\\ &= \frac{\$26 + \$26}{2}\\ &= \$26\end{aligned}$$

b *Jane*:

$$\begin{aligned}\text{range} &= \text{maximum} - \text{minimum}\\ &= \$47 - \$29\\ &= \$18\end{aligned}$$

We have an even number of data values, so we include all data values when we split the data set into two:

$$\overbrace{\$29\ \ \$29\ \ \$29\ \ \$31\ \ \$34\ \ \$35}^{\text{lower half}}\quad \overbrace{\$36\ \ \$36\ \ \$38\ \ \$40\ \ \$42\ \ \$47}^{\text{upper half}}$$

$$Q_1 = \text{median of lower half} = \frac{\$29 + \$31}{2} = \$30$$

$$Q_3 = \text{median of upper half} = \frac{\$38 + \$40}{2} = \$39$$

$$\begin{aligned}\text{IQR} &= Q_3 - Q_1\\ &= \$39 - \$30\\ &= \$9\end{aligned}$$

Ashley:

$$\begin{aligned}\text{range} &= \text{maximum} - \text{minimum}\\ &= \$59 - \$19\\ &= \$40\end{aligned}$$

We have an even number of data values, so we include all data values when we split the data set into two:

$$\underbrace{\$19\ \ \$19\ \ \$23\ \ \$24\ \ \$24\ \ \$26}_{\text{lower half}}\quad \underbrace{\$26\ \ \$32\ \ \$35\ \ \$40\ \ \$42\ \ \$59}_{\text{upper half}}$$

$$Q_1 = \text{median of lower half} = \frac{\$23 + \$24}{2} = \$23.50$$

$$Q_3 = \text{median of upper half} = \frac{\$35 + \$40}{2} = \$37.50$$

$$\begin{aligned}\text{IQR} &= Q_3 - Q_1\\ &= \$37.50 - \$23.50\\ &= \$14\end{aligned}$$

c The mean and median are much higher for Jane than for Ashley.
$\therefore$ Jane generally pays more for her telephone bills.

d The range and IQR are much higher for Ashley than for Jane.
$\therefore$ Ashley has the greater variability in telephone bills.

3 The ordered data set is:
7 7 9 10 11 11 12 13 14 14 15 15 18 18 19 20 20 22 25 67 (20 data values)

a
$$\begin{aligned}\text{range} &= \text{maximum} - \text{minimum}\\ &= 67 - 7\\ &= 60\end{aligned}$$

We have an even number of data values, so we include all data values when we split the data set into two:

$$\underbrace{7\ \ 7\ \ 9\ \ 10\ \ 11\ \ 11\ \ 12\ \ 13\ \ 14\ \ 14}_{\text{lower half}}\quad \underbrace{15\ \ 15\ \ 18\ \ 18\ \ 19\ \ 20\ \ 20\ \ 22\ \ 25\ \ 67}_{\text{upper half}}$$

$$Q_1 = \text{median of lower half} = \frac{11 + 11}{2} = 11$$

$$Q_3 = \text{median of upper half} = \frac{19 + 20}{2} = 19.5$$

$$\begin{aligned}\text{IQR} &= Q_3 - Q_1\\ &= 19.5 - 11\\ &= 8.5\end{aligned}$$

b The outlier is the value 67.

c If the data value 67 is removed, then $\text{range} = 25 - 7 = 18$

Since $n = 20 - 1 = 19$, $\dfrac{n+1}{2} = 10$ $\therefore$ the median is the 10th data value.

~~7 7 9 10 11 11 12 13 14~~ 14 ~~15 15 18 18 19 20 20 22 25~~

$\therefore$ median $= 14$

Since the median is a data value, we now ignore it and split the remaining data into two:

$$\overbrace{7\ \ 7\ \ 9\ \ 10\ \ \boxed{11}\ \ 11\ \ 12\ \ 13\ \ 14}^{\text{lower half}}\quad \overbrace{15\ \ 15\ \ 18\ \ 18\ \ \boxed{19}\ \ 20\ \ 20\ \ 22\ \ 25}^{\text{upper half}}$$

Q_1 = median of lower half = 11
Q_3 = median of upper half = 19

$$\begin{aligned}\text{IQR} &= Q_3 - Q_1\\ &= 19 - 11\\ &= 8\end{aligned}$$

d The range is much more affected by the outlier than the IQR.

4 The ordered data sets are:

Derrick: 210 380 400 415 420 420 425 425 430 435 440 445 445 450 450 (15 data values)

Gareth: 330 340 340 360 370 420 430 450 460 460 470 480 480 490 500 (15 data values)

a *Derrick*:

$$\begin{aligned}\text{range} &= \text{maximum} - \text{minimum}\\ &= 450 - 210\\ &= 240 \text{ minutes}\end{aligned}$$

Since $n = 15$, $\dfrac{n+1}{2} = 8$ $\therefore$ the median is the 8th data value.

~~210 380 400 415 420 420 425~~ 425 ~~430 435 440 445 445 450 450~~

$\therefore$ median = 425

Since the median is a data value, we now ignore it and split the remaining data into two:

$$\overbrace{210\ \ 380\ \ 400\ \ \boxed{415}\ \ 420\ \ 420\ \ 425}^{\text{lower half}}\quad \overbrace{430\ \ 435\ \ 440\ \ \boxed{445}\ \ 445\ \ 450\ \ 450}^{\text{upper half}}$$

Q_1 = median of lower half = 415
Q_3 = median of upper half = 445

$$\begin{aligned}\text{IQR} &= Q_3 - Q_1\\ &= 445 - 415\\ &= 30 \text{ minutes}\end{aligned}$$

Gareth:

$$\begin{aligned}\text{range} &= \text{maximum} - \text{minimum}\\ &= 500 - 330\\ &= 170 \text{ minutes}\end{aligned}$$

Since $n = 15$, $\dfrac{n+1}{2} = 8$ $\therefore$ the median is the 8th data value.

~~330 340 340 360 370 420 430~~ 450 ~~460 460 470 480 480 490 500~~

$\therefore$ median = 450

Since the median is a data value, we now ignore it and split the remaining data into two:

$$\underbrace{330\ \ 340\ \ 340\ \ 360\ \ 370\ \ 420\ \ 430}_{\text{lower half}}\quad \underbrace{460\ \ 460\ \ 470\ \ 480\ \ 480\ \ 490\ \ 500}_{\text{upper half}}$$

$Q_1 =$ median of lower half $= 360$

$Q_3 =$ median of upper half $= 480$

$$\begin{aligned} \text{IQR} &= Q_3 - Q_1 \\ &= 480 - 360 \\ &= 120 \text{ minutes} \end{aligned}$$

b **i** Gareth's data has the lower range.

ii Derrick's data has the lower interquartile range.

c The interquartile range is more appropriate than the range for determining who is generally the more consistent sleeper as it is less affected by outliers.

5 The ordered data set is: $a\ \ b\ \ c\ \ d\ \ e\ \ f\ \ g\ \ h\ \ i\ \ j\ \ k\ \ l\ \ m$ (13 data values)

a Since $n = 13$, $\dfrac{n+1}{2} = 7$ $\therefore$ the median is the 7th data value.

$$\cancel{a}\ \ \cancel{b}\ \ \cancel{c}\ \ \cancel{d}\ \ \cancel{e}\ \ \cancel{f}\ \ g\ \ \cancel{h}\ \ \cancel{i}\ \ \cancel{j}\ \ \cancel{k}\ \ \cancel{l}\ \ \cancel{m}$$

$\therefore$ median $= g$

b **i**
$$\begin{aligned} \text{range} &= \text{maximum} - \text{minimum} \\ &= m - a \end{aligned}$$

ii Since the median is a data value we now ignore it and split the remaining data into two:

$$\underbrace{a\ \ b\ \ c\ \ d\ \ e\ \ f}_{\text{lower half}}\quad \underbrace{h\ \ i\ \ j\ \ k\ \ l\ \ m}_{\text{upper half}}$$

$Q_1 =$ median of lower half $= \dfrac{c+d}{2}$

$Q_3 =$ median of upper half $= \dfrac{j+k}{2}$

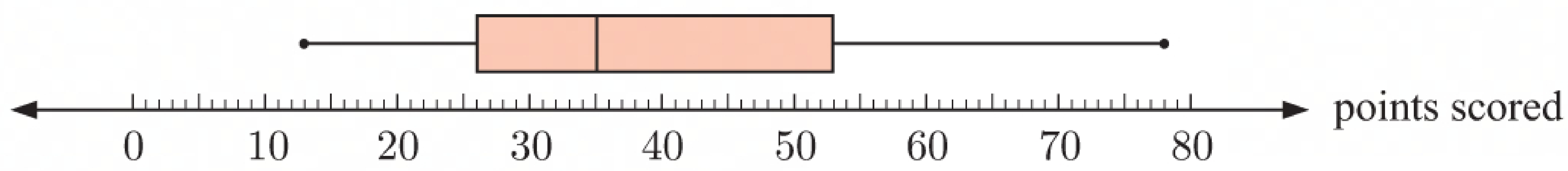
$$\begin{aligned} \text{IQR} &= Q_3 - Q_1 \\ &= \left(\frac{j+k}{2}\right) - \left(\frac{c+d}{2}\right) \end{aligned}$$

6

	Measure	median	mode	range	interquartile range
	Original value	9	7	13	6
a	*New value*	$9+2$ $= 11$	$7+2$ $= 9$	$(\max+2) - (\min+2)$ $= \max - \min$ $= 13$	$(Q_3+2) - (Q_1+2)$ $= Q_3 - Q_1$ $= 6$
b	*New value*	9×2 $= 18$	7×2 $= 14$	$(2\times\max) - (2\times\min)$ $= 2(\max - \min)$ $= 2 \times 13$ $= 26$	$(2\times Q_3) - (2\times Q_1)$ $= 2(Q_3 - Q_1)$ $= 2 \times 6$ $= 12$

EXERCISE 13F

1



a median $= 35$ points

b maximum value $= 78$ points

c minimum value $= 13$ points

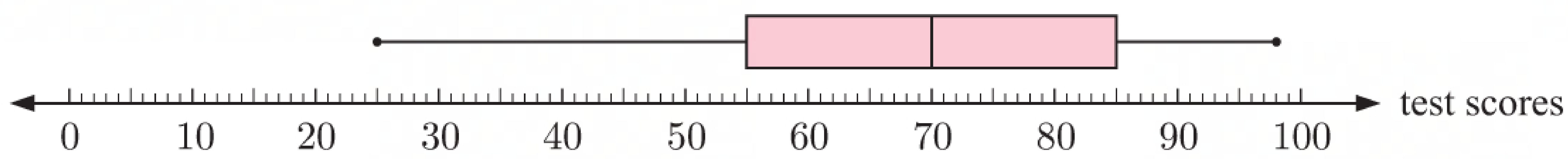
d upper quartile $= 53$ points

e lower quartile $= 26$ points

f range $=$ maximum $-$ minimum
$= 78 - 13$
$= 65$ points

g $\text{IQR} = Q_3 - Q_1$
$= 53 - 26$
$= 27$ points

2



a **i** The highest mark scored for the test was 98, and the lowest mark was 25.

ii Half of the class scored a mark greater than or equal to 70 marks.

iii The top 25% of the class scored at least 85 marks.

iv The middle half of the class had scores between 55 and 85 marks.

b range $=$ maximum $-$ minimum
$= 98 - 25$
$= 73$ marks

$\text{IQR} = Q_3 - Q_1$
$= 85 - 55$
$= 30$ marks

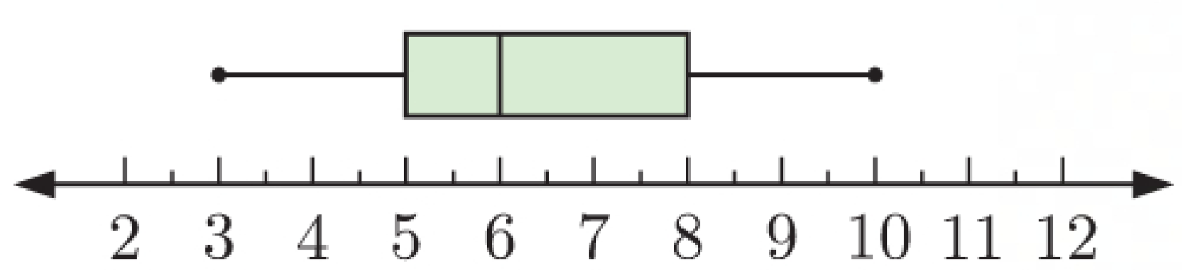
3 **a** **i** The ordered data set is:

3 4 5 5 5 6 6 6 7 7 8 8 9 10 (14 data values)

$Q_1 = 5$ median $= 6$ $Q_3 = 8$

So the five-number summary is:
$$\begin{cases} \text{minimum} = 3 & Q_1 = 5 \\ \text{median} = 6 & Q_3 = 8 \\ \text{maximum} = 10 & \end{cases}$$

ii



iii range $=$ maximum $-$ minimum
$= 10 - 3$
$= 7$

$\text{IQR} = Q_3 - Q_1$
$= 8 - 5$
$= 3$

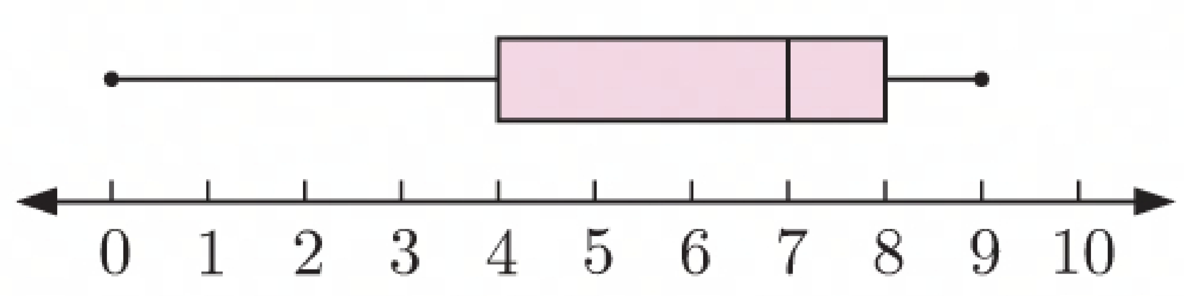
b **i** The ordered data set is:

0 1 2 3 4 5 6 6 7 7 7 8 8 8 8 8 8 9 9 (19 data values)

$Q_1 = 4$ median $= 7$ $Q_3 = 8$

So the five-number summary is:
$$\begin{cases} \text{minimum} = 0 & Q_1 = 4 \\ \text{median} = 7 & Q_3 = 8 \\ \text{maximum} = 9 & \end{cases}$$

ii



iii range = maximum − minimum
$= 9 - 0$
$= 9$

IQR $= Q_3 - Q_1$
$= 8 - 4$
$= 4$

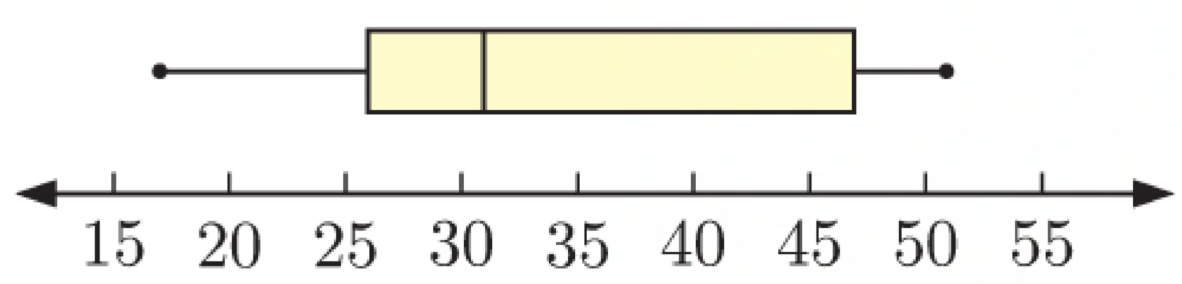
c **i** The ordered data set is:

17 20 23 26 **26** 28 30 31 **31 31** 33 35 44 **47** 47 49 49 51 (18 data values)

$Q_1 = 26$ median $= 31$ $Q_3 = 47$

So the five-number summary is: minimum $= 17$, $Q_1 = 26$, median $= 31$, $Q_3 = 47$, maximum $= 51$

ii



iii range = maximum − minimum
$= 51 - 17$
$= 34$

IQR $= Q_3 - Q_1$
$= 47 - 26$
$= 21$

4 **a** The ordered data set is:

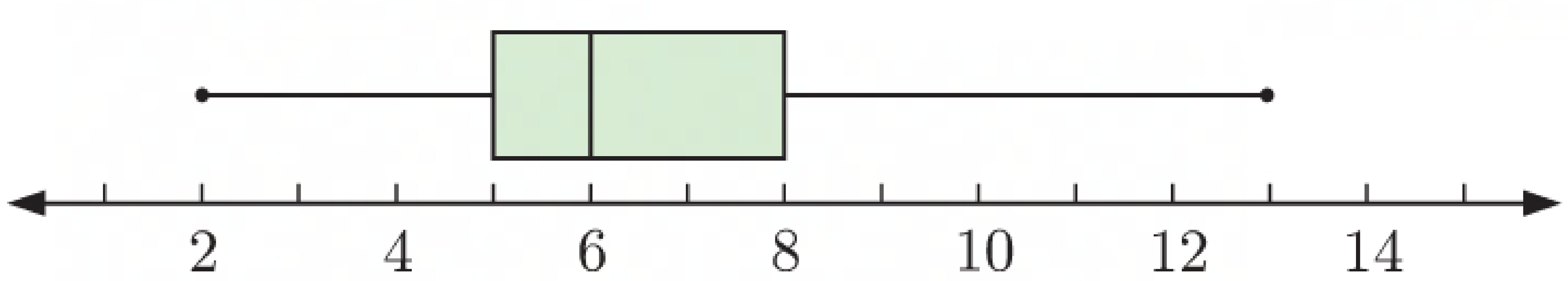
2 3 3 4 4 4 4 **5 5** 5 5 5 5 5 6 6 **6** 6 6 6 7 7 7 7 **8 8** 8 9 9 9 10 12 13
(33 data values)

$Q_1 = 5$ median $= 6$ $Q_3 = 8$

So, median = 6 beans, $Q_1 = 5$ beans, $Q_3 = 8$ beans.

b IQR $= Q_3 - Q_1$
$= 8 - 5$
$= 3$ beans

c



5 **a**

Number of bolts	33	34	35	36	37	38	39	40
Frequency	1	5	7	13	12	8	0	1

The ordered data set is:

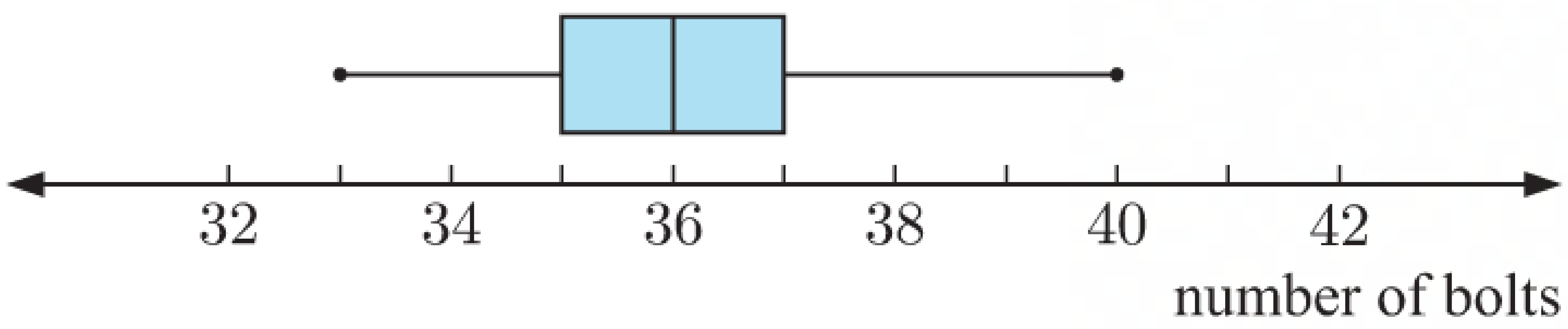
minimum $= 33$ $Q_1 = 35$ median $= 36$

33 34 34 34 34 34 35 35 35 35 35 **35** 35 36 36 36 36 36 36 36 36 36 36 **36**
36 36 37 37 37 37 37 37 37 37 37 **37** 37 37 38 38 38 38 38 38 38 38 **40**

$Q_3 = 37$ maximum $= 40$

(47 data values)

So the box plot is:



b range = maximum − minimum
$= 40 - 33$
$= 7$ bolts

$\text{IQR} = Q_3 - Q_1$
$= 37 - 35$
$= 2$ bolts

EXERCISE 13G

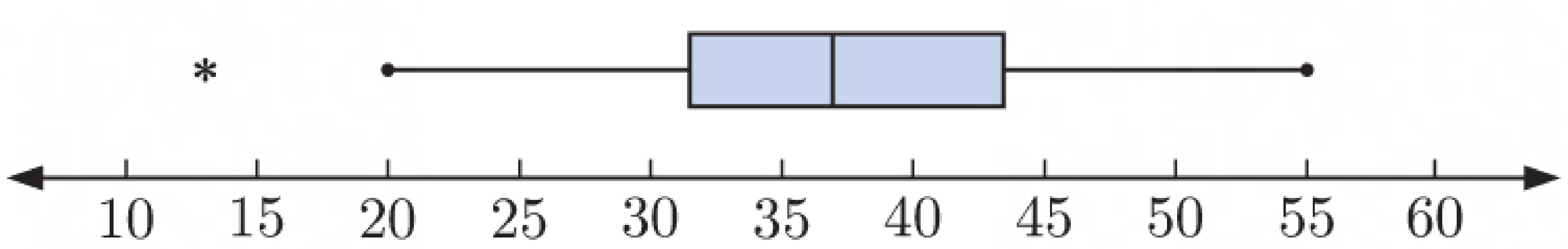
1 **a** $\text{IQR} = Q_3 - Q_1$
$= 43.5 - 31.5$
$= 12$

b lower boundary = lower quartile $- 1.5 \times \text{IQR}$
$= 31.5 - 1.5 \times 12$
$= 13.5$

upper boundary = upper quartile $+ 1.5 \times \text{IQR}$
$= 43.5 + 1.5 \times 12$
$= 61.5$

c 13 is below the lower boundary, so it is an outlier.
20, 52, and 55 are all within the two boundary values, so none of these data values are outliers.

d



2 **a** The ordered data set is:

3 5 6 7 7 8 8 9 9 9 10 10 10 11 11 12 12 13 13 13 14 14 16 18 22

$Q_1 = 8$ median $= 10$ $Q_3 = 13$ $\{n = 25\}$

So the median is 10, the lower quartile is 8, and the upper quartile is 13.

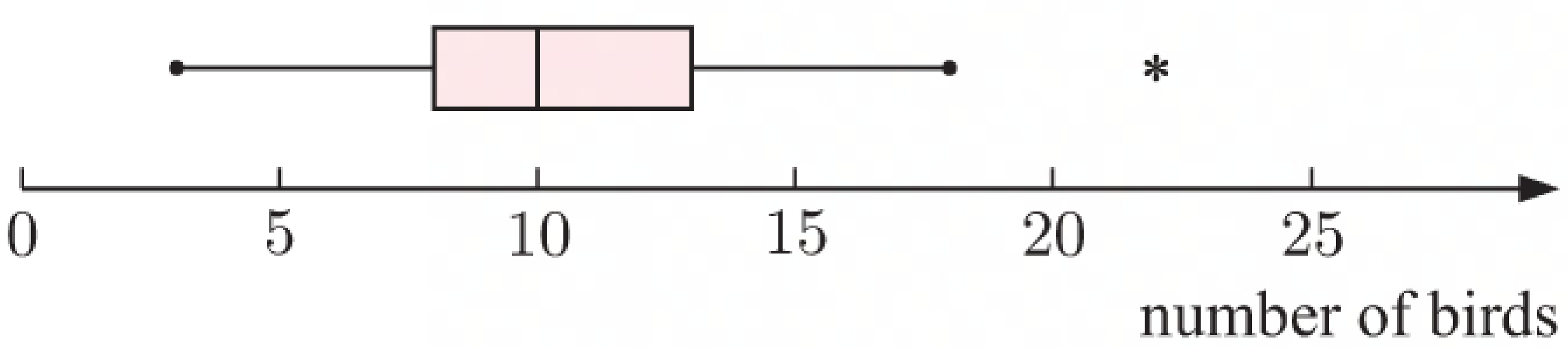
b $\text{IQR} = Q_3 - Q_1$
$= 13 - 8$
$= 5$

c lower boundary
= lower quartile $- 1.5 \times \text{IQR}$
$= 8 - 1.5 \times 5$
$= 0.5$

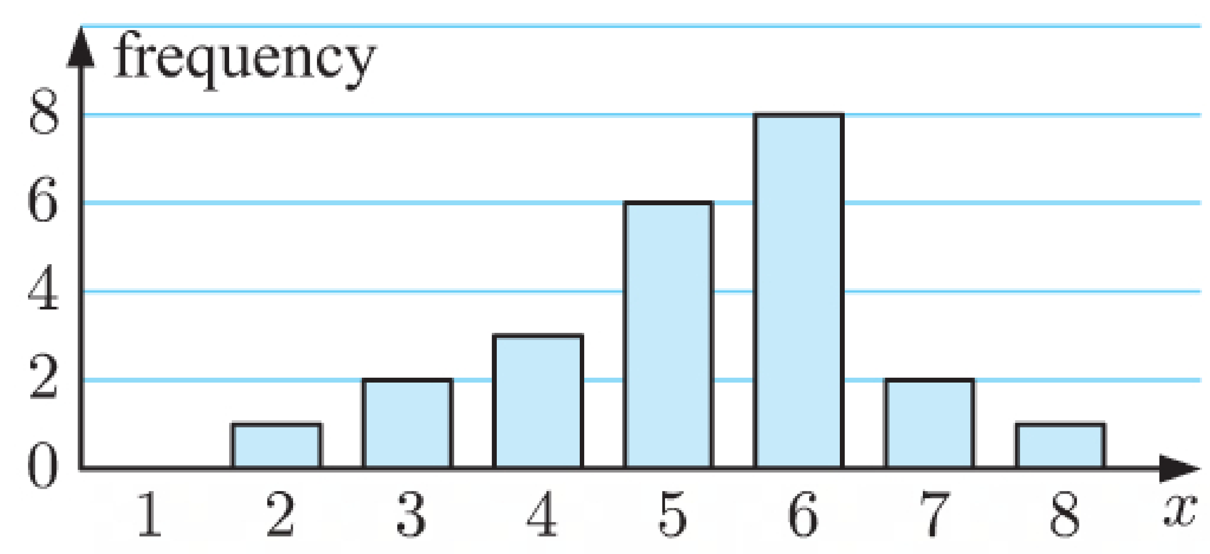
upper boundary
= upper quartile $+ 1.5 \times \text{IQR}$
$= 13 + 1.5 \times 5$
$= 20.5$

22 is above the upper boundary, so it is an outlier.

d



3 **a**

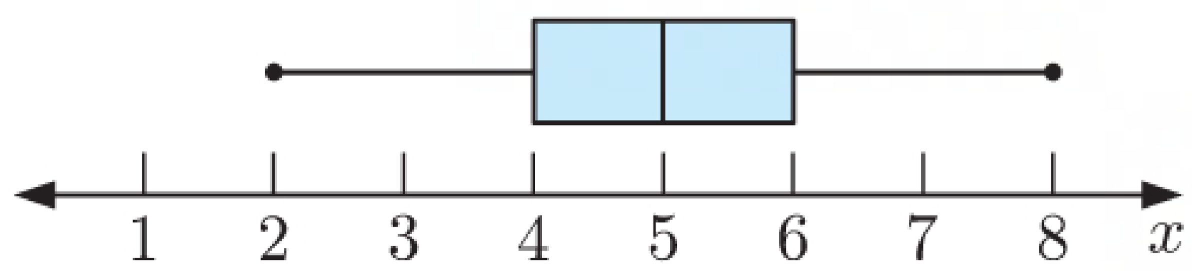


The data displayed in this graph has a minimum of 2 and a maximum of 8.

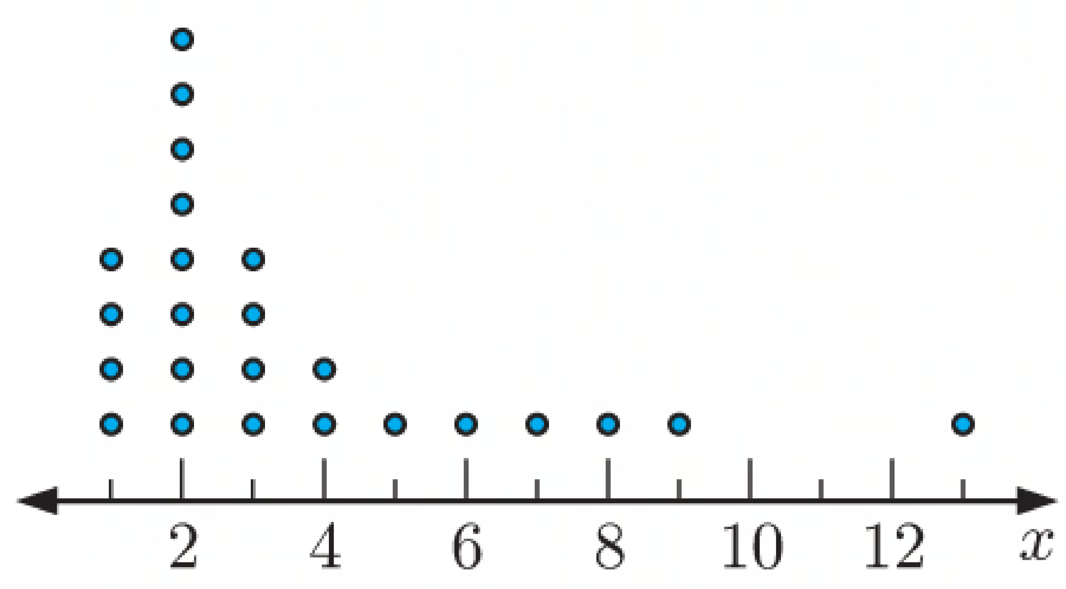
There are no outliers.

These characteristics match box plot **A**.

A



b

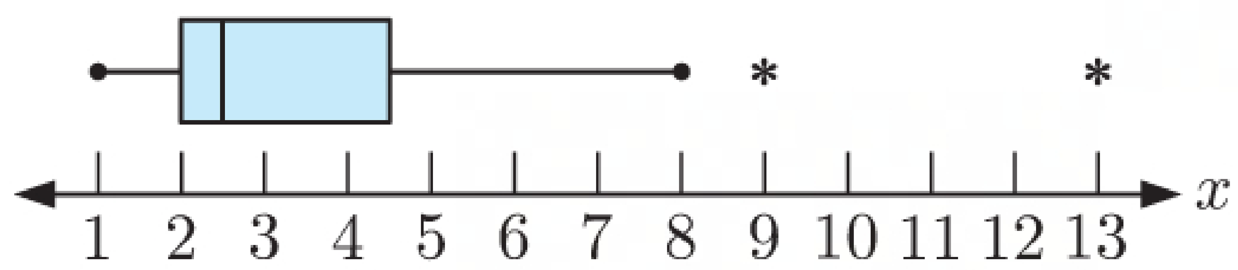


The data displayed in this graph has a minimum of 1 and a maximum of 13.

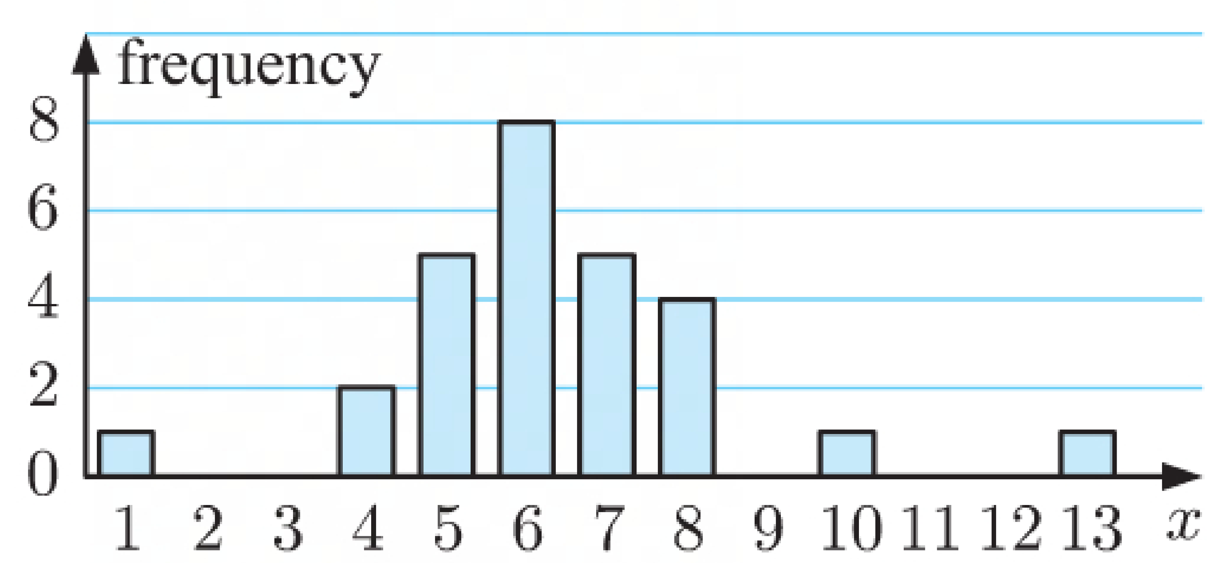
13 is clearly an outlier, and there are no outliers at the lower end of the data set.

These characteristics match box plot **D**.

D



c

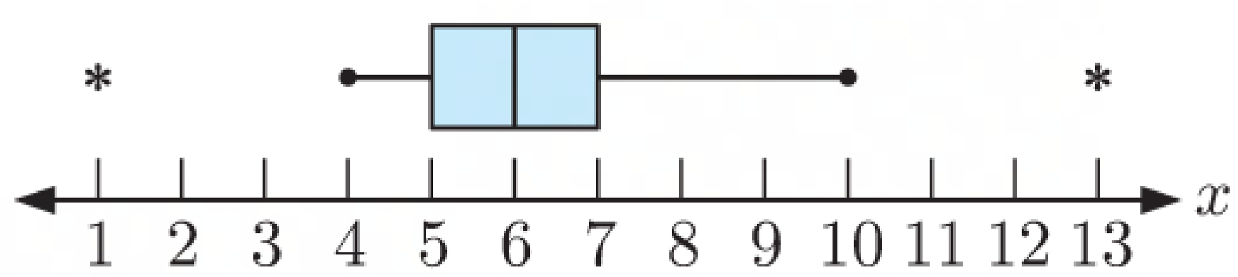


The data displayed in this graph has a minimum of 1 and a maximum of 13.

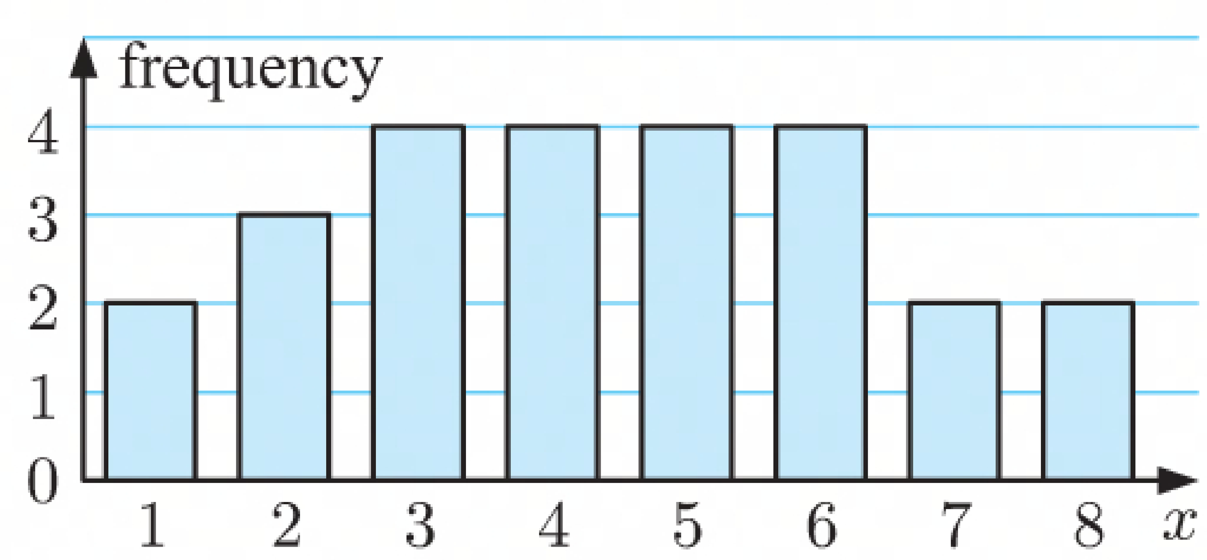
1 and 13 are clear outliers.

These characteristics match box plot **C**.

C



d

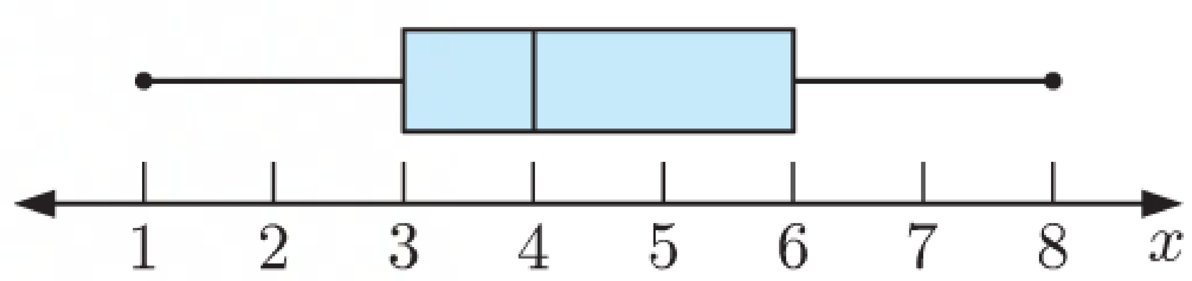


The data displayed in this graph has a minimum of 1 and a maximum of 8.

There are no outliers.

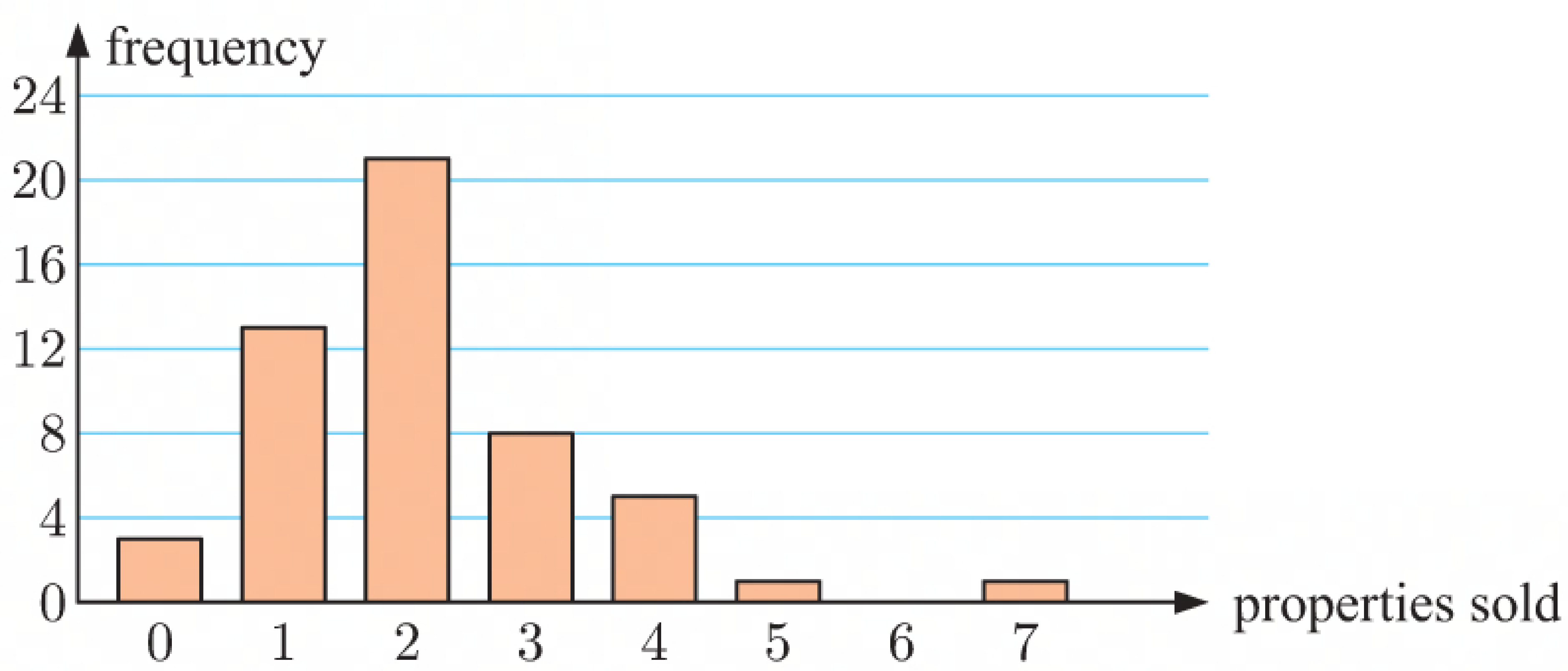
These characteristics match box plot **B**.

B



4 **a**

Properties sold by a real estate agent



b From the column graph, 7 properties sold appears to be an outlier.

c Since $n = 52$, we have an even number of data values, so we include all data values when we split the data set into two groups of 26 data values.

For the lower half of the data set, $n = 26$, so $\frac{n+1}{2} = 13.5$

$\therefore$ Q_1 is the average of the 13th and 14th data values.

$$\begin{aligned} Q_1 &= \text{median of lower half} \\ &= \frac{\text{13th value} + \text{14th value}}{2} \\ &= \frac{1+1}{2} \qquad \{\text{from } \mathbf{a}\text{, the 4th to 16th values are 1}\} \\ &= 1 \end{aligned}$$

For the upper half of the data set, we need to find the average of the $26 + 13 = 39\text{th}$ and $26 + 14 = 40\text{th}$ data values.

$$\begin{aligned} Q_3 &= \text{median of upper half} \\ &= \frac{\text{39th value} + \text{40th value}}{2} \\ &= \frac{3+3}{2} \qquad \{\text{from } \mathbf{a}\text{, the 38th to 45th values are 3}\} \\ &= 3 \end{aligned}$$

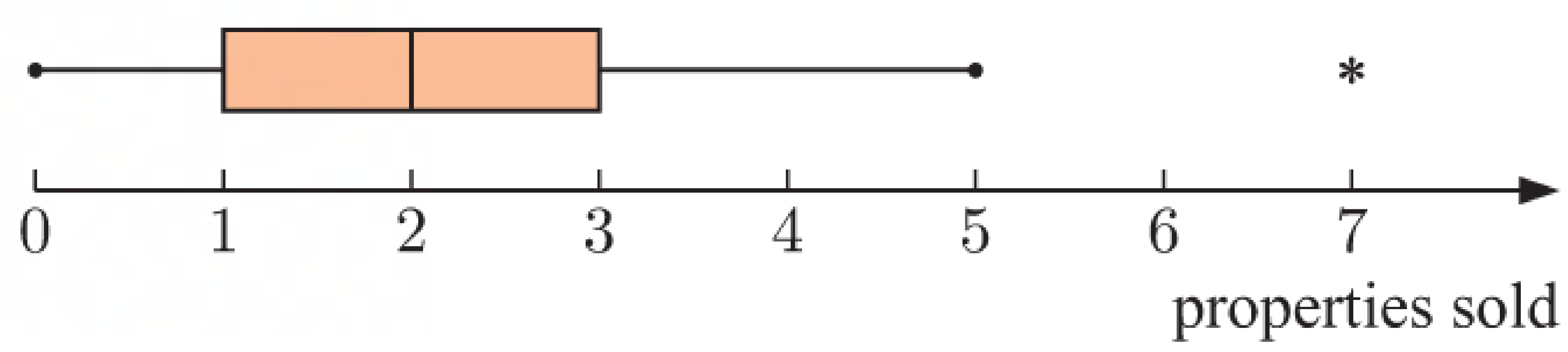
$$\begin{aligned} \text{IQR} &= Q_3 - Q_1 \\ &= 3 - 1 \\ &= 2 \end{aligned}$$

lower boundary
$= \text{lower quartile} - 1.5 \times \text{IQR}$
$= 1 - 1.5 \times 2$
$= -2$

upper boundary
$= \text{upper quartile} + 1.5 \times \text{IQR}$
$= 3 + 1.5 \times 2$
$= 6$

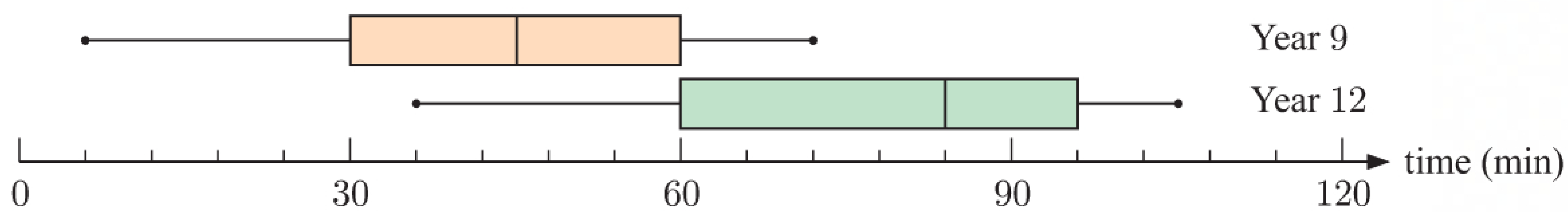
7 properties sold is above the upper boundary, so it is an outlier.

d



EXERCISE 13H

1



a

Statistic	*Year 9*	*Year 12*
minimum	6	36
Q_1	30	60
median	45	84
Q_3	60	96
maximum	72	105

b **i** Year 9: range $= 72 - 6$
$= 66$ min

Year 12: range $= 105 - 36$
$= 69$ min

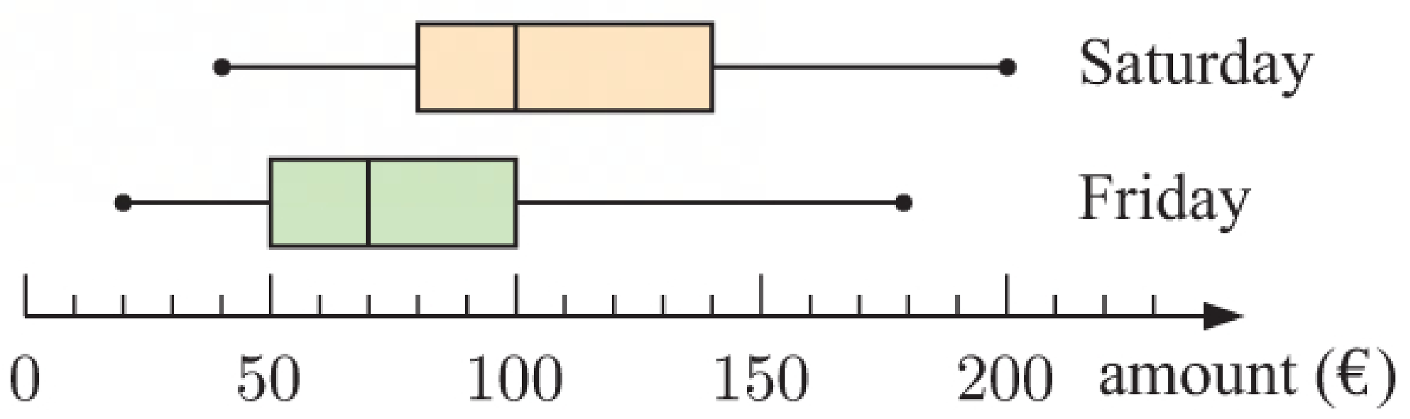
ii Year 9: IQR $= Q_3 - Q_1$
$= 60 - 30$
$= 30$ min

Year 12: IQR $= Q_3 - Q_1$
$= 96 - 60$
$= 36$ min

c **i** We cannot tell if Year 12 students spend about twice as much time on homework as Year 9 students since the mean was not calculated.

ii It is true that over 25% of Year 9 students spend less time on homework than all Year 12 students since the lower quartile for the Year 9 students is less than the minimum value for the Year 12 students.

2



a Friday: min = €20, Q_1 = €50, median = €70, Q_3 = €100, max = €180
Saturday: min = €40, Q_1 = €80, median = €100, Q_3 = €140, max = €200

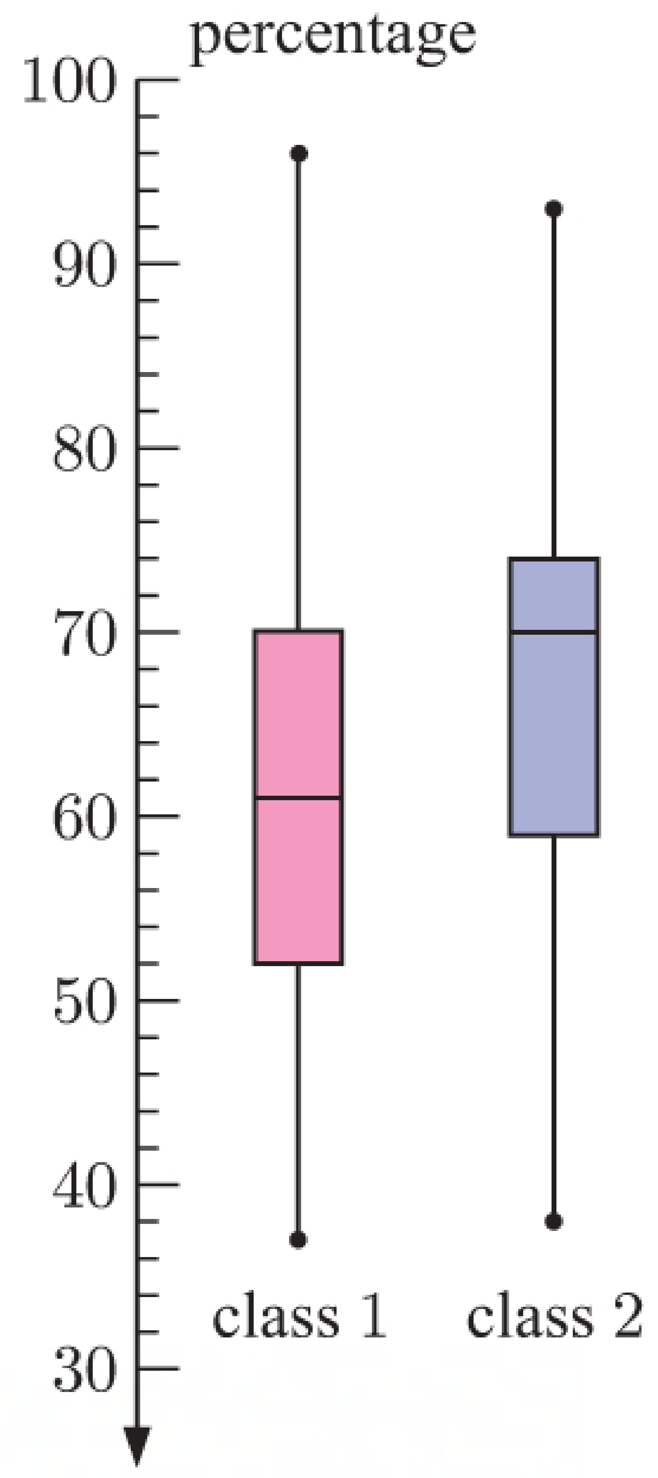
b **i** Friday: range = €180 − €20
= €160

Saturday: range = €200 − €40
= €160

ii Friday: IQR $= Q_3 - Q_1$
= €100 − €50
= €50

Saturday: IQR $= Q_3 - Q_1$
= €140 − €80
= €60

3 **a** **i** The highest mark was in class 1 (96%).
ii The lowest mark was in class 1 (37%).
iii Class 1 had the larger spread of marks.



b IQR of class 1 $= Q_3 - Q_1$
$= 70\% - 52\%$
$= 18\%$

c range of class 2 = maximum − minimum
$= 93\% - 38\%$
$= 55\%$

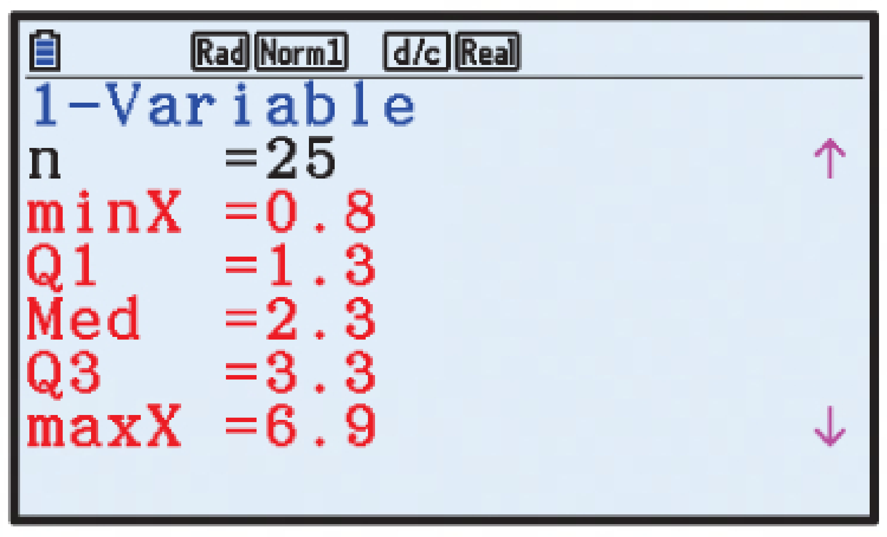
d **i** 70% is the upper quartile of class 1.
$\therefore$ 25% of the students in class 1 received an achievement award.

ii 70% is the median for class 2.
$\therefore$ 50% of the students in class 2 received an achievement award.

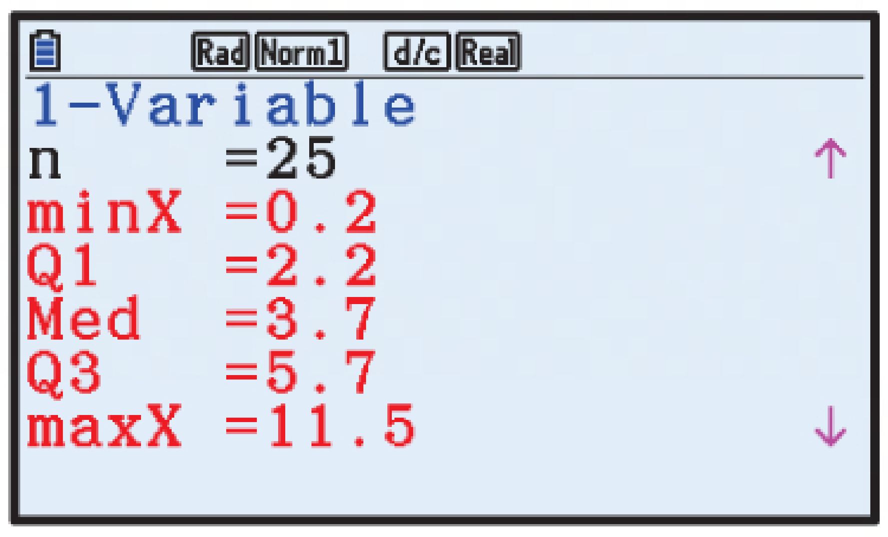
e **i** The marks in class 1 were slightly positively skewed.
ii The marks in class 2 were negatively skewed.

f The students in class 2 generally scored higher marks.
The marks in class 1 were more varied.

4 **a** *Kirsten*:



Erika:

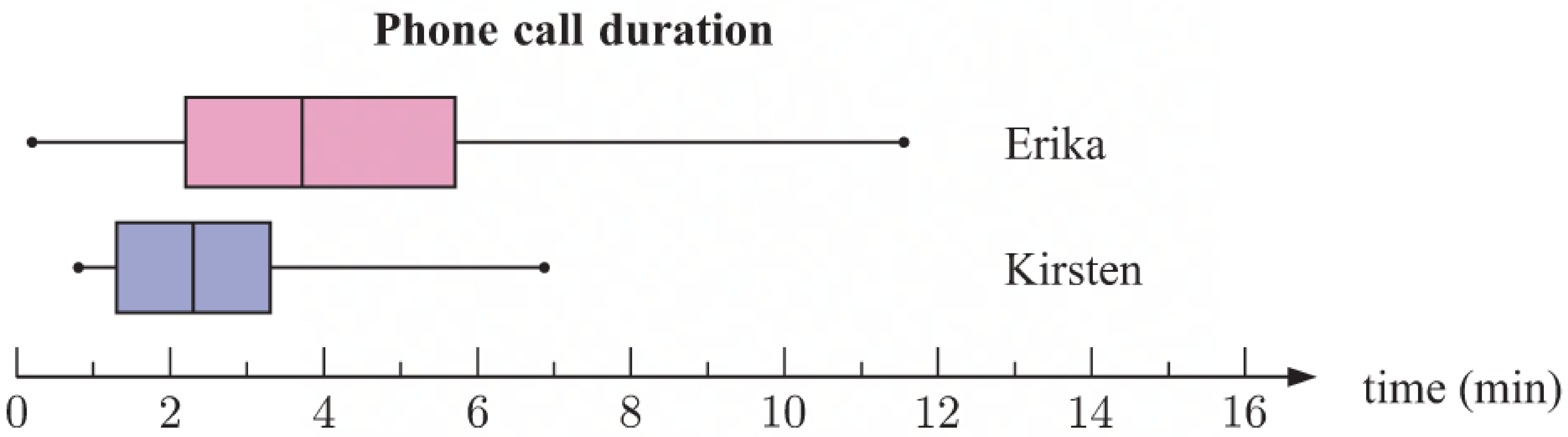


The five-number summaries are:

Kirsten: minimum $= 0.8$ min
$Q_1 = 1.3$ min
median $= 2.3$ min
$Q_3 = 3.3$ min
maximum $= 6.9$ min

Erika: minimum $= 0.2$ min
$Q_1 = 2.2$ min
median $= 3.7$ min
$Q_3 = 5.7$ min
maximum $= 11.5$ min

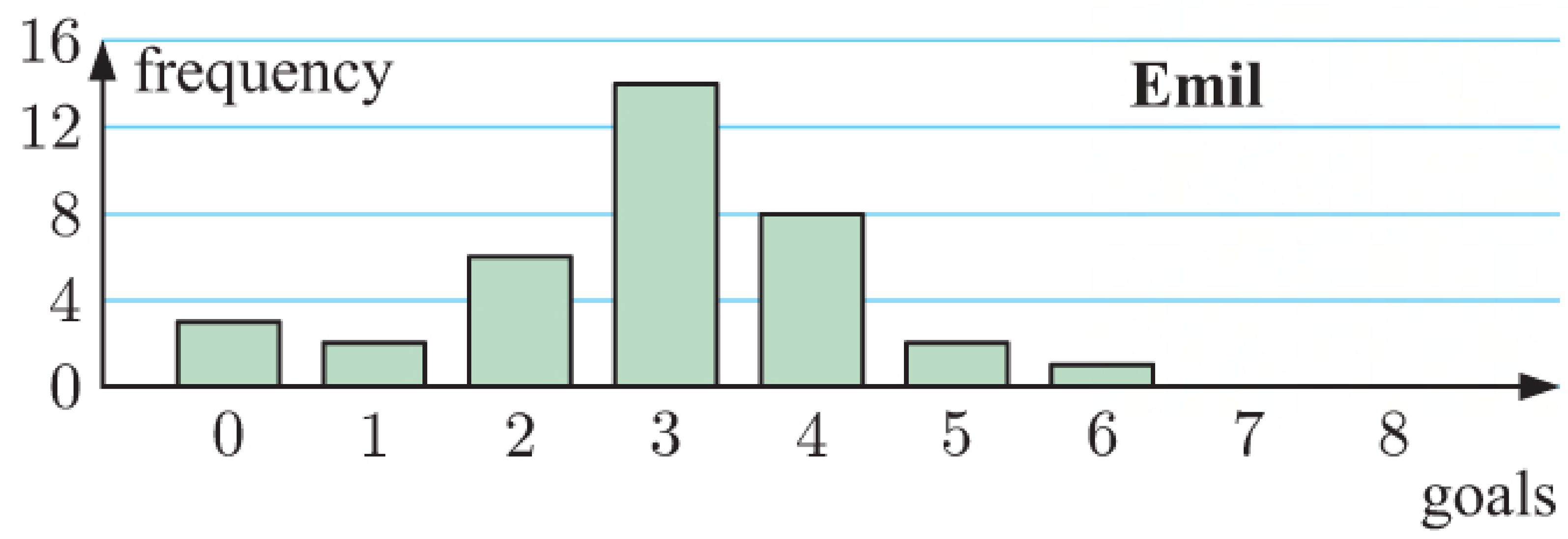
b

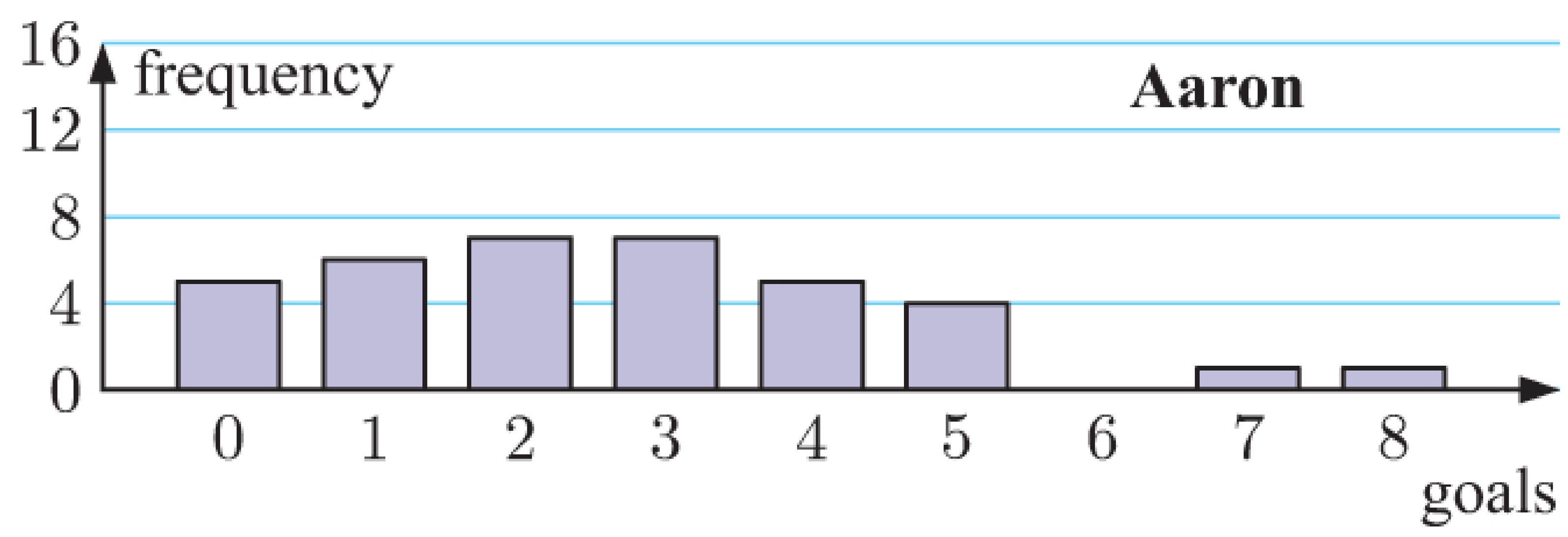


c Both sets of data are positively skewed. Erika's phone calls were more varied in their duration, but tended to be longer than Kirsten's.

5 **a** The number of goals scored is an exact number value.
$\therefore$ this is a discrete numerial variable.

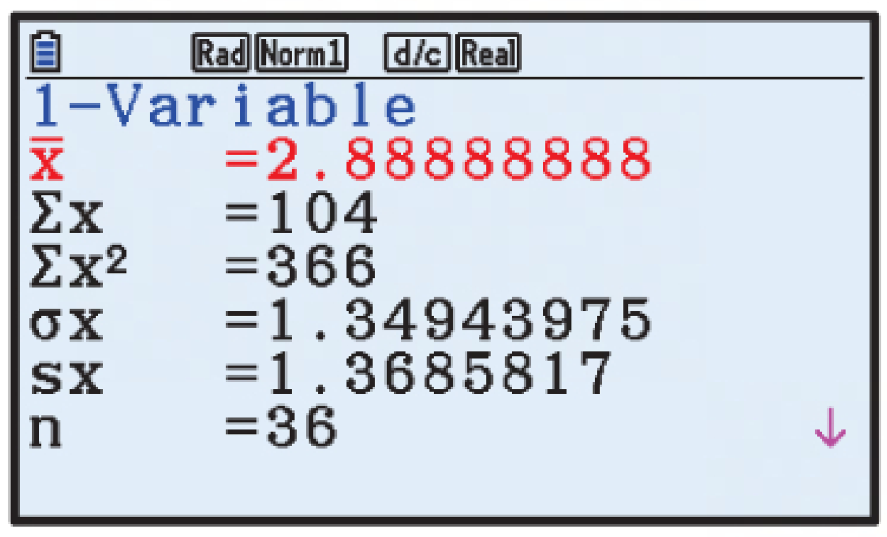
c

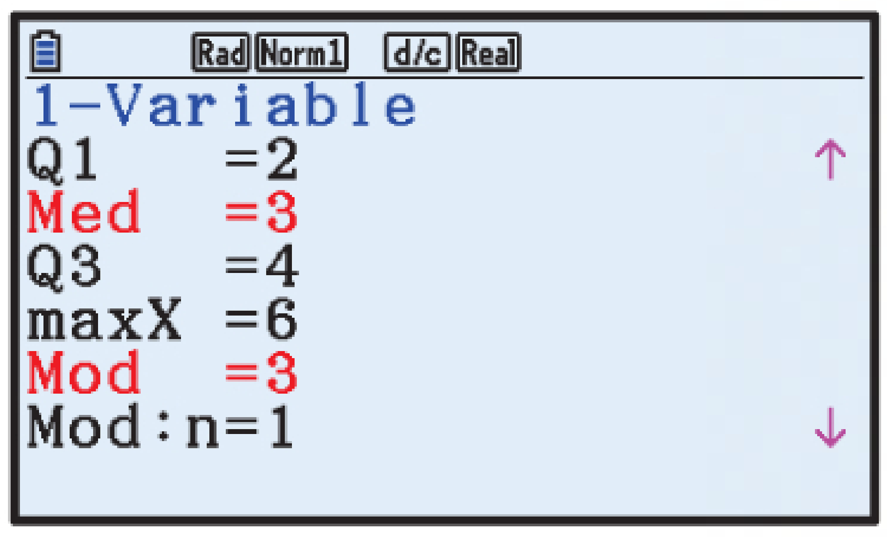




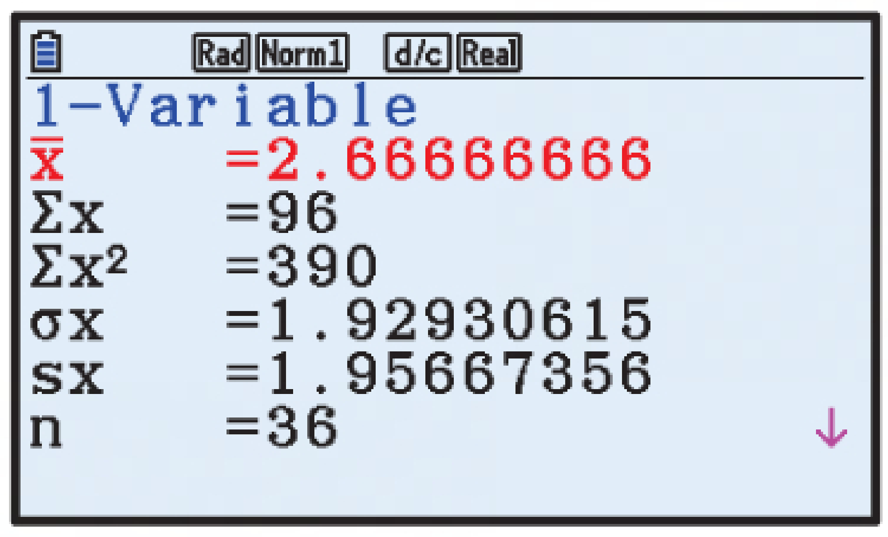
d Emil's distribution is approximately symmetrical. Aaron's distribution is positively skewed.

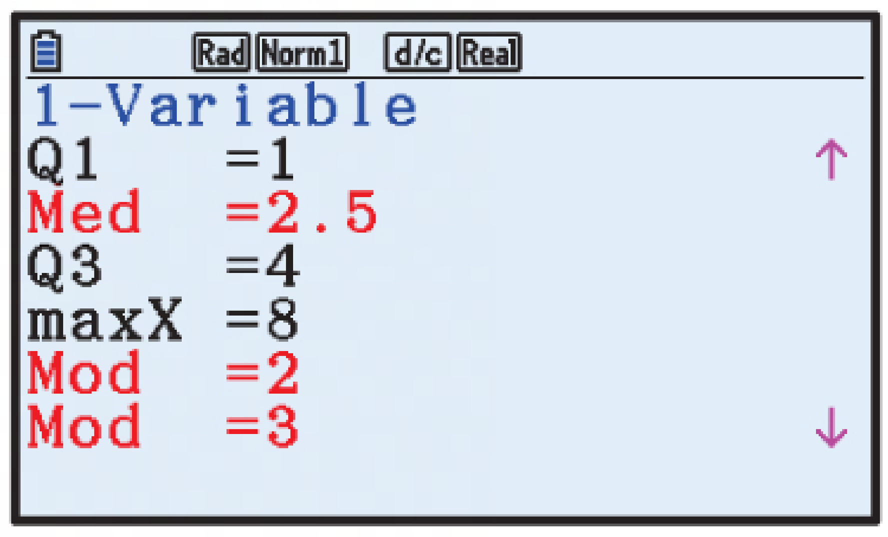
e *Emil*:





Aaron:



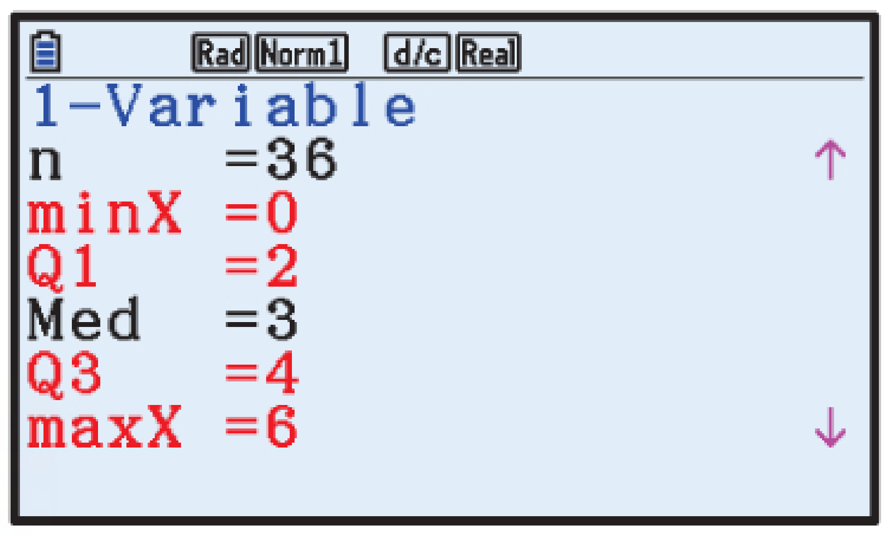


Emil: mean ≈ 2.89 goals, median $= 3$ goals, mode $= 3$ goals

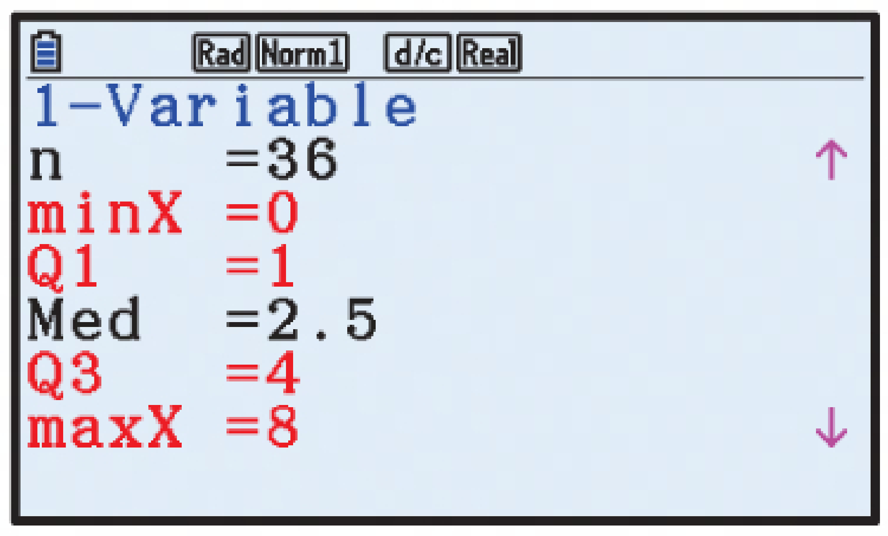
Aaron: mean ≈ 2.67 goals, median $= 2.5$ goals, mode $= 2$ and 3 goals

Emil's mean and median are slightly higher than Aaron's, and Emil has a clear mode of 3 goals, whereas Aaron has two modes of 2 and 3 goals.

f *Emil*:



Aaron:



Emil:

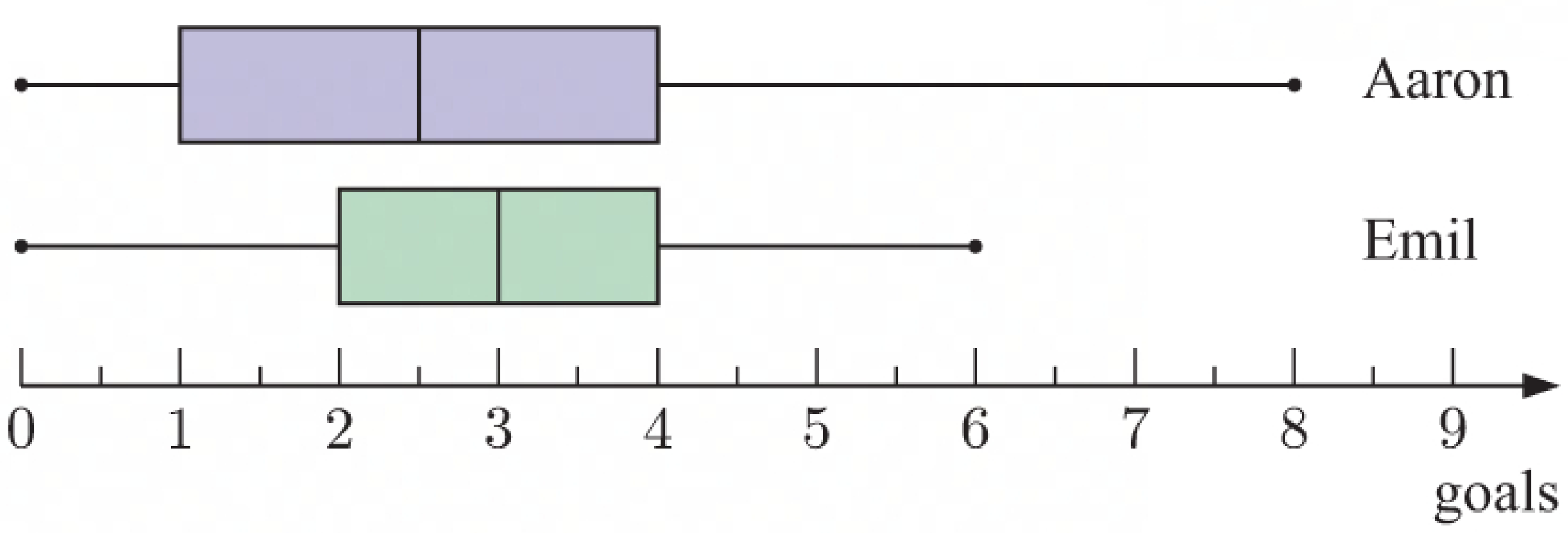
$$\text{range} = 6 - 0 = 6 \text{ goals} \qquad \text{IQR} = Q_3 - Q_1 = 4 - 2 = 2 \text{ goals}$$

Aaron:

$$\text{range} = 8 - 0 = 8 \text{ goals} \qquad \text{IQR} = Q_3 - Q_1 = 4 - 1 = 3 \text{ goals}$$

The range and IQR are lower for Emil than for Aaron. So Emil's data set demonstrates less variability than Aaron's.

g

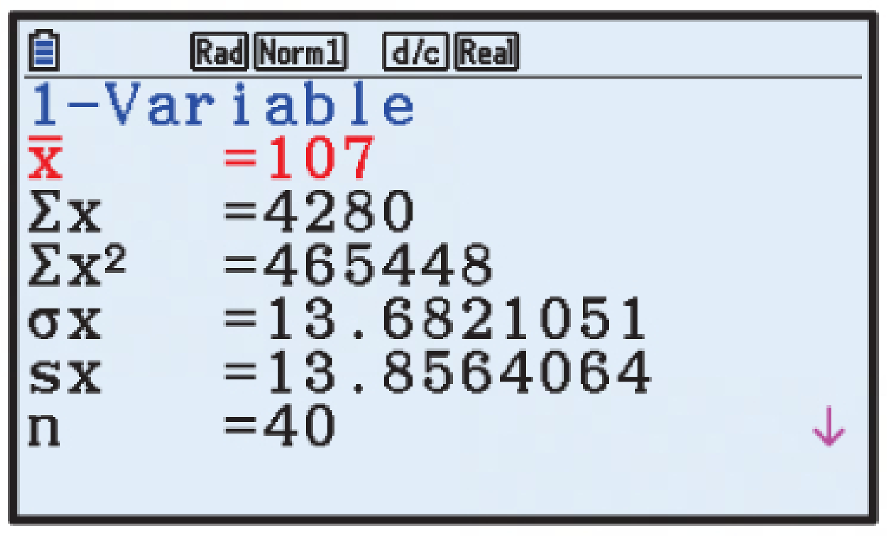


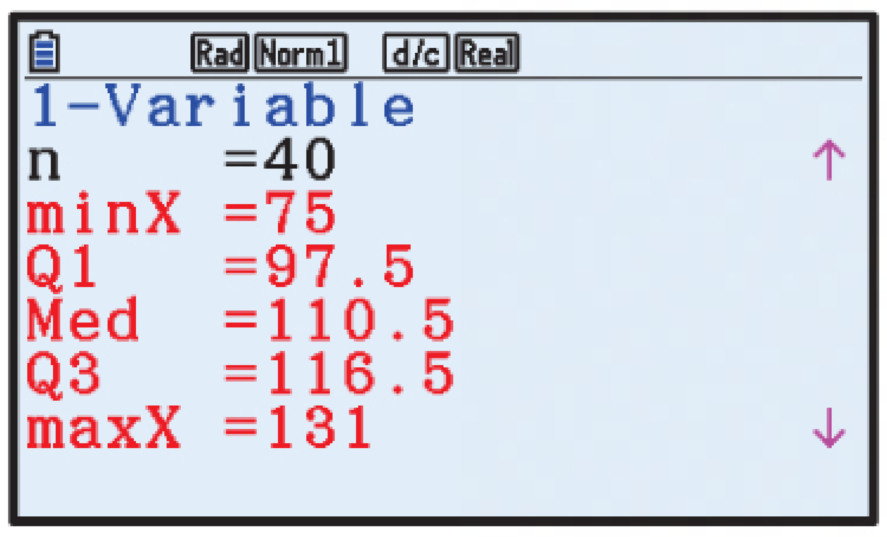
h Emil generally scores more goals than Aaron and is a more consistent goal scorer than Aaron.

6 **a** The lifespan of the globes is a numerical variable which is measured.

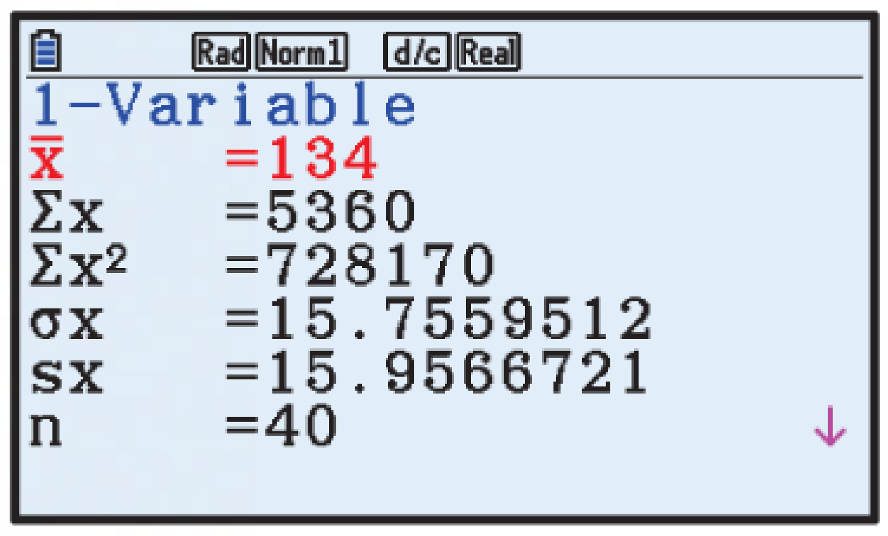
$\therefore$ this is a continuous variable.

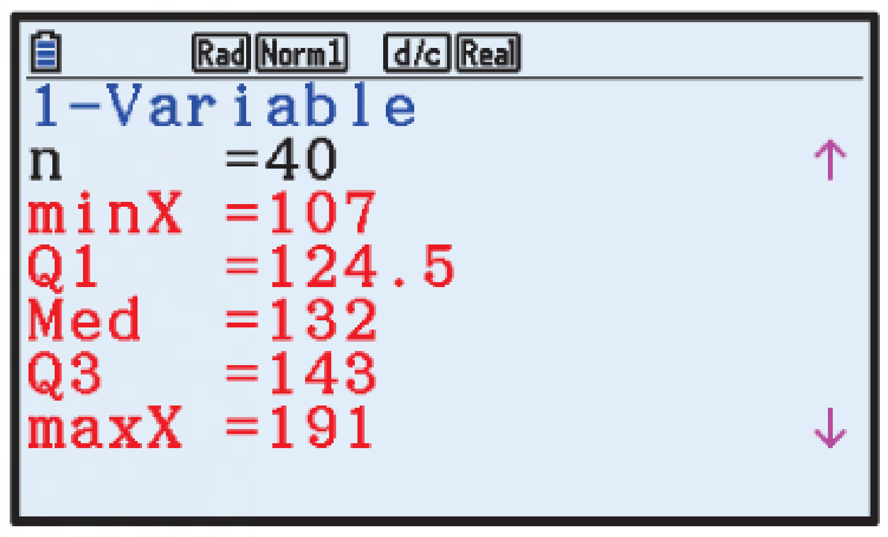
b *Old type*:





New type:



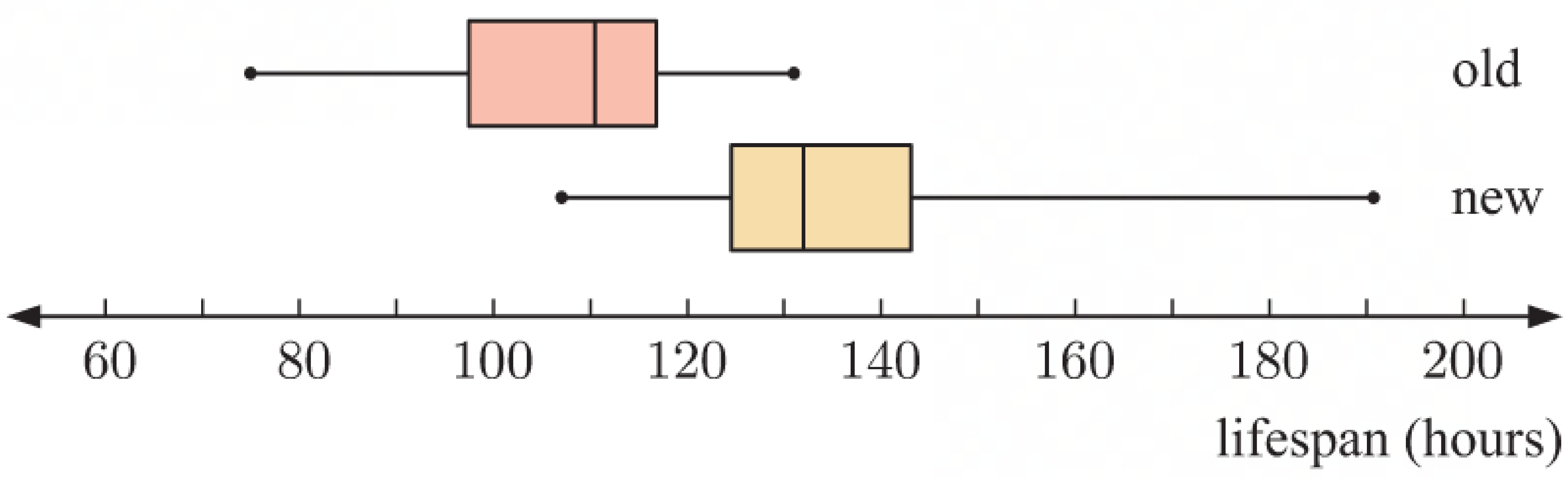


Old type: mean = 107 hours
median = 110.5 hours
range = 56 hours
IQR = 19 hours

New type: mean = 134 hours
median = 132 hours
range = 84 hours
IQR = 18.5 hours

The "new" type of light globe has a higher mean and median than the "old" type.
The IQR is relatively unchanged going from "old" to "new", however, the range of the "new" type is greater, suggesting greater variability.

c



d The "old" type data is negatively skewed. The "new" type data is positively skewed.

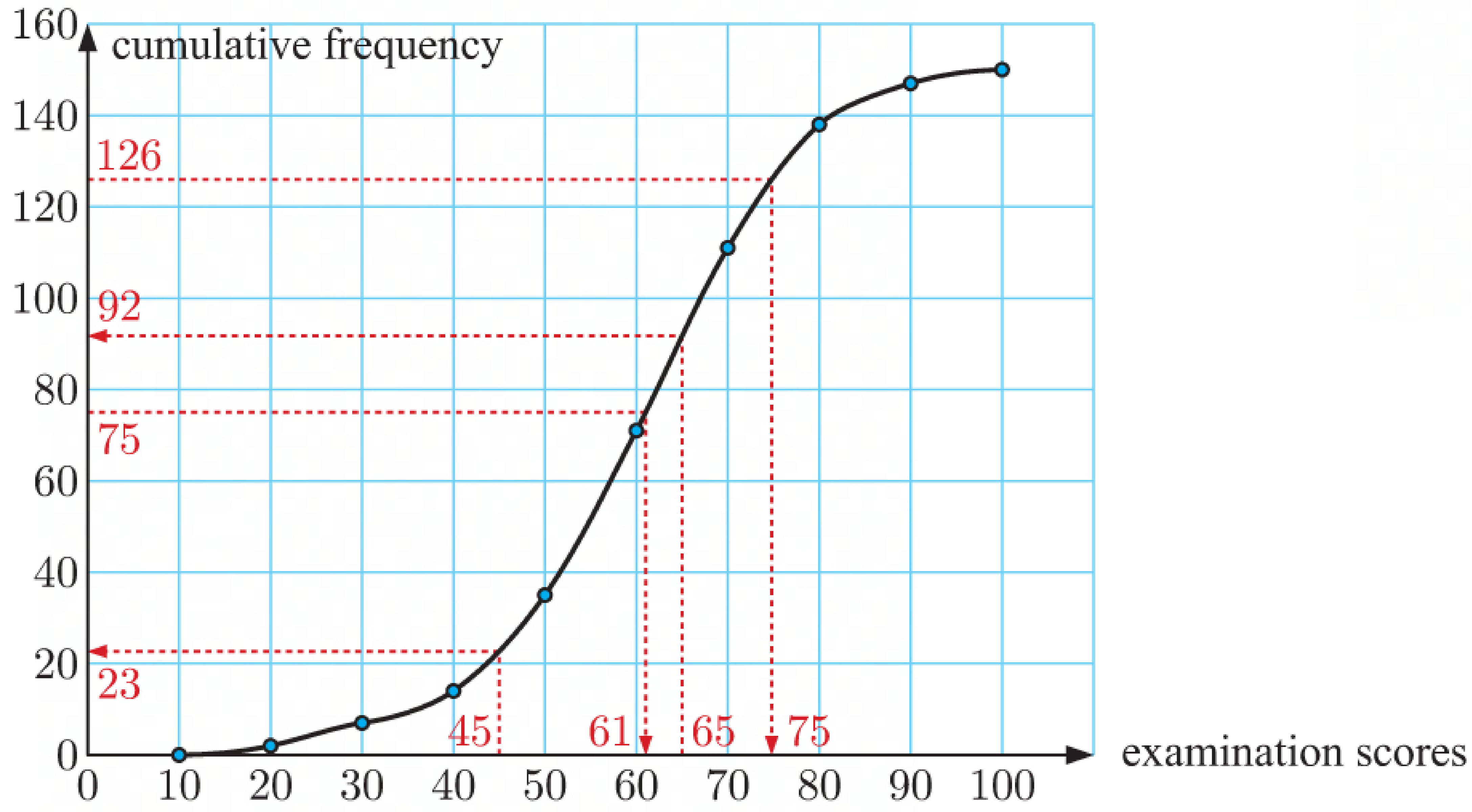
e The "new" type of light globes do last longer than the "old" type. From **c**, both the mean and median for the "new" type are close to 20% greater than that of the "old" type. The manufacturer's claim appears to be valid.

EXERCISE 13I

1

Score (x)	*Frequency*	*Cumulative frequency*
$10 \leqslant x < 20$	2	2
$20 \leqslant x < 30$	5	7
$30 \leqslant x < 40$	7	14
$40 \leqslant x < 50$	21	35
$50 \leqslant x < 60$	36	71
$60 \leqslant x < 70$	40	111
$70 \leqslant x < 80$	27	138
$80 \leqslant x < 90$	9	147
$90 \leqslant x < 100$	3	150

a



b The median is the 50th percentile. As 50% of 150 is 75, we start with the cumulative frequency 75 and find the corresponding examination score.
The median ≈ 61 marks.

c Approximately 92 students scored 65 marks or less.

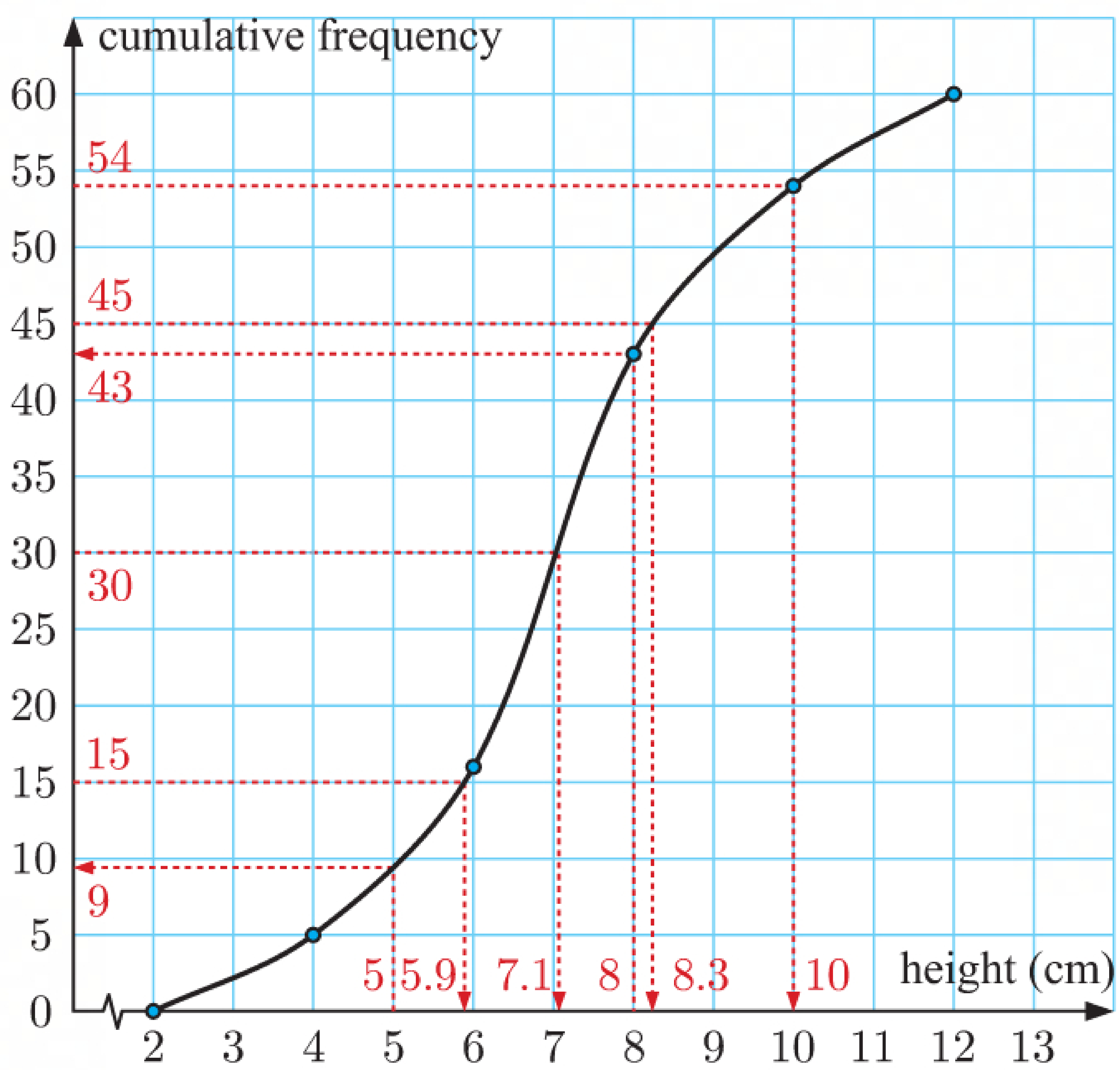
d From the table, $36 + 40 = 76$ students scored at least 50 but less than 70 marks.

e Approximately 23 students scored 45 marks or less.
$\therefore$ approximately 23 students failed the examination.

f As 16% of 150 is 24, we start with the cumulative frequency $150 - 24 = 126$ and find the corresponding examination score.
The top 16% of students scored approximately 75 marks or more.
$\therefore$ the credit mark was approximately 75 marks.

2

Heights of seedlings



a Approximately 9 seedlings have heights of 5 cm or less.

b Approximately $60 - 43 = 17$ seedlings have heights of more than 8 cm.
$\therefore$ $\frac{17}{60} \times 100\% \approx 28.3\%$ of seedlings are taller than 8 cm.

c The median is the 50th percentile. As 50% of 60 is 30, we start with the cumulative frequency 30 and find the corresponding height.
The median $\approx$ 7.1 cm.

d Q_1 is the 25th percentile. As 25% of 60 is 15, we start with the cumulative frequency 15 and find the corresponding height.
$Q_1 \approx 5.9$ cm

Q_3 is the 75th percentile. As 75% of 60 is 45, we start with the cumulative frequency 45 and find the corresponding height.
$Q_3 \approx 8.3$ cm

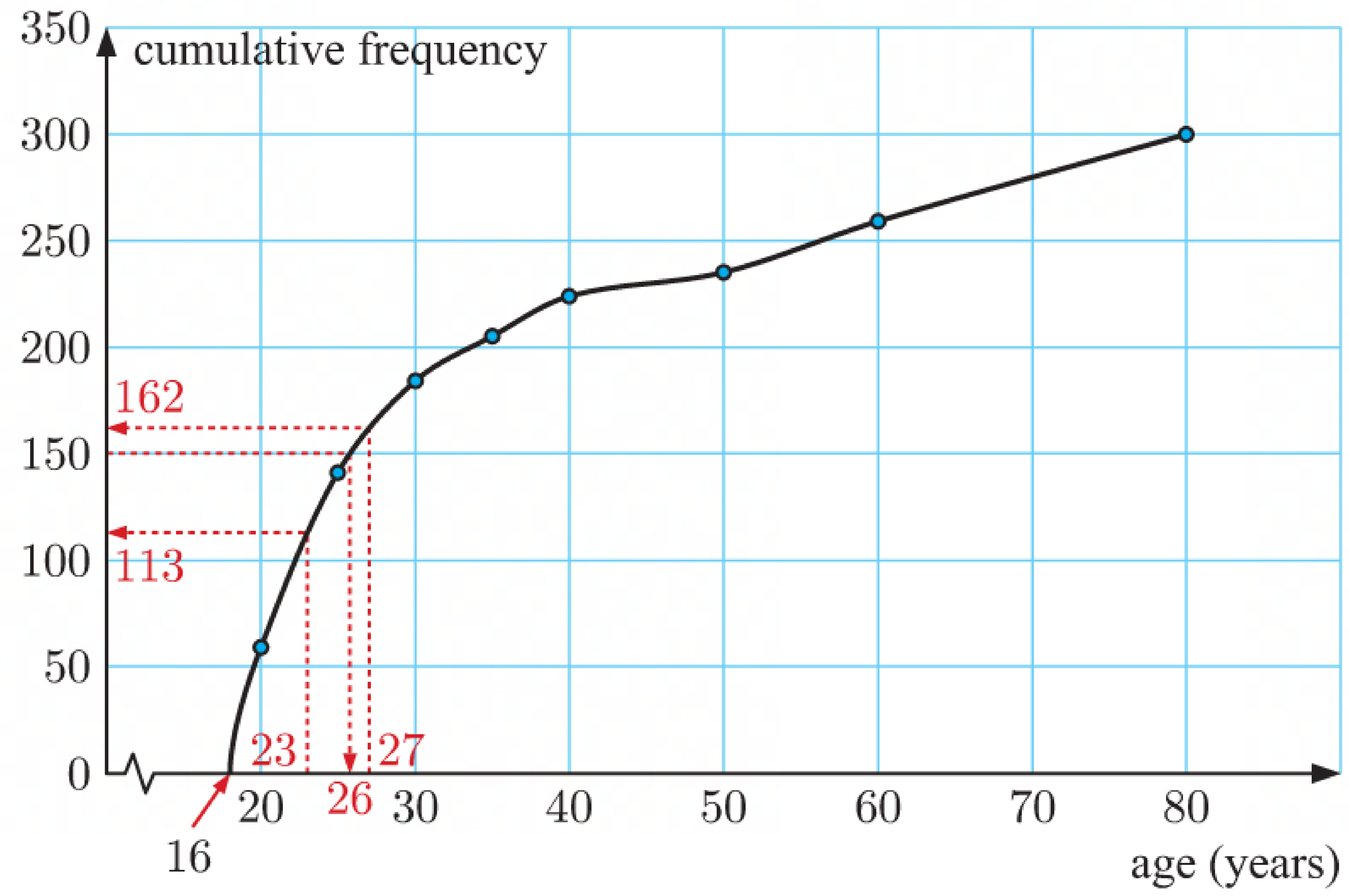
$$\begin{aligned} \text{IQR} &= Q_3 - Q_1 \\ &\approx 8.3 - 5.9 \\ &\approx 2.4 \text{ cm} \end{aligned}$$

e As 90% of 60 is 54, we start with the cumulative frequency 54 and find the corresponding height.
The 90th percentile $\approx$ 10 cm which means that 90% of the seedlings are shorter than approximately 10 cm.

3

Age (x *years*)	*Number of accidents*	*Cumulative frequency*
$16 \leqslant x < 20$	59	59
$20 \leqslant x < 25$	82	141
$25 \leqslant x < 30$	43	184
$30 \leqslant x < 35$	21	205
$35 \leqslant x < 40$	19	224
$40 \leqslant x < 50$	11	235
$50 \leqslant x < 60$	24	259
$60 \leqslant x < 80$	41	300

a



b The median is the 50th percentile. As 50% of 300 is 150, we start with the cumulative frequency 150 and find the corresponding age.
The median $\approx$ 26 years.

c Approximately 113 drivers involved in accidents had an age of 23 or less.
$\therefore \quad \frac{113}{300} \times 100\% \approx 37.7\%$ of drivers involved in accidents had an age of 23 or less.

d **i** Approximately 162 drivers involved in accidents were aged 27 years or less.

$\therefore$ P(driver involved in an accident is aged 27 years or less) $\approx \frac{162}{300}$

≈ 0.54

ii Approximately 150 drivers involved in accidents were aged 26 years or less, from **b**.

$\therefore$ $162 - 150 = 12$ drivers involved in accidents were aged 27 years.

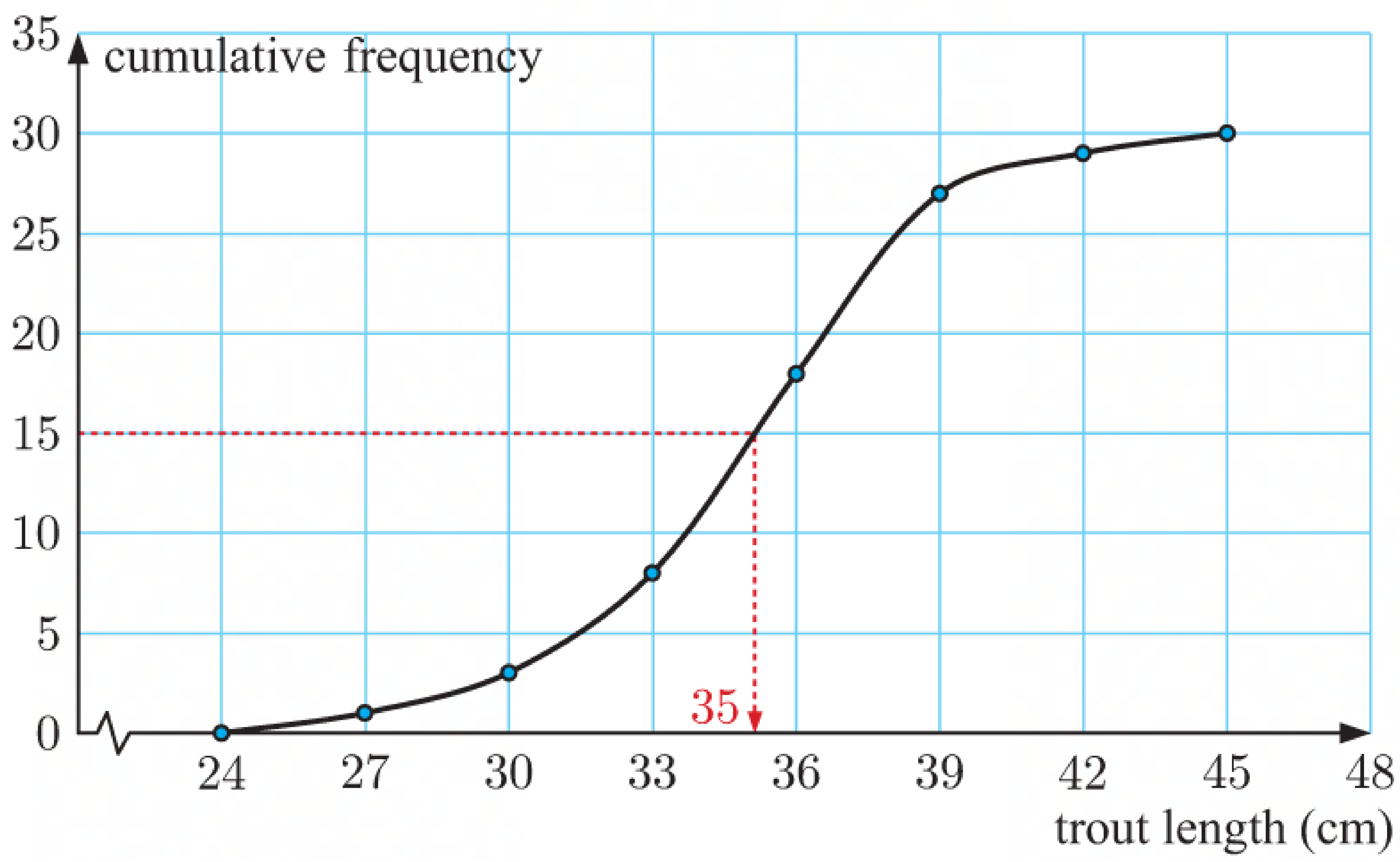
$\therefore$ P(driver involved in an accident is aged 27 years) $\approx \frac{12}{300}$

≈ 0.04

4 **a**

Length (cm)	*Frequency*	*Cumulative frequency*
$24 \leqslant x < 27$	1	1
$27 \leqslant x < 30$	2	3
$30 \leqslant x < 33$	5	8
$33 \leqslant x < 36$	10	18
$36 \leqslant x < 39$	9	27
$39 \leqslant x < 42$	2	29
$42 \leqslant x < 45$	1	30

b



c The median is the 50th percentile. As 50% of 30 is 15, we start with the cumulative frequency 15 and find the corresponding length.

The median $\approx$ 35 cm.

d There are 30 data values, so $n = 30$. $\frac{n+1}{2} = 15.5$, so the median is the average of the 15th and 16th ordered data values.

The ordered data set is:

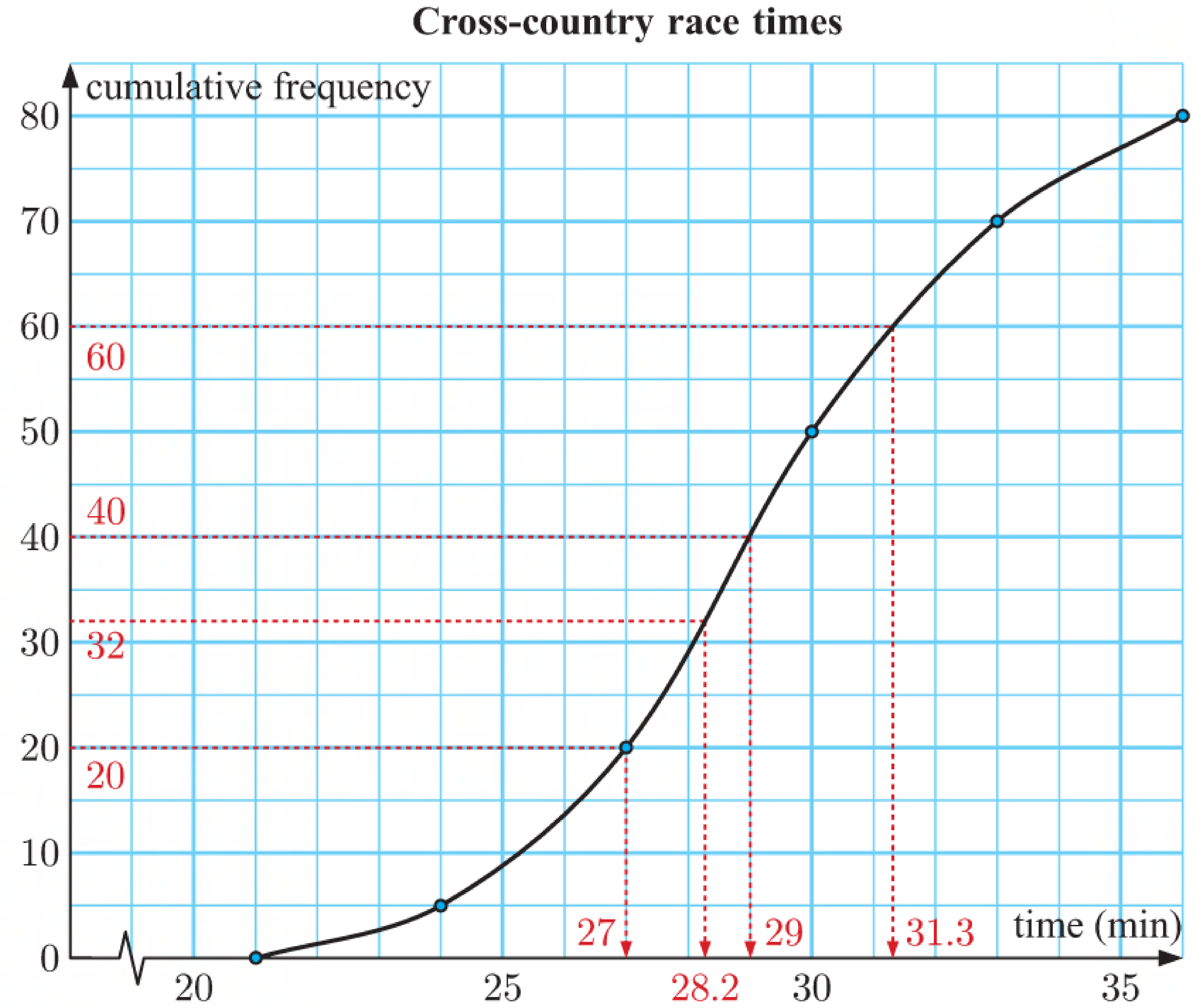
~~24 27 28 30 31 31 32 32 33 33 33 33 34 34~~ 34

35 ~~35 35 36 36 36 36 37 38 38 38 38 40 40 44~~

$$\begin{aligned}\text{median} &= \frac{\text{15th value} + \text{16th value}}{2} \\ &= \frac{34 + 35}{2} \\ &= 34.5 \text{ cm}\end{aligned}$$

The median found from the graph is a good approximation for the actual median.

5



a The lower quartile is the 25th percentile. As 25% of 80 is 20, we start with the cumulative frequency 20 and find the corresponding time.
$Q_1 \approx 27$ min

b The median is the 50th percentile. As 50% of 80 is 40, we start with the cumulative frequency 40 and find the corresponding time.
The median ≈ 29 min.

c The upper quartile is the 75th percentile. As 75% of 80 is 60, we start with the cumulative frequency 60 and find the corresponding time.
$Q_3 \approx 31.3$ min

d $\text{IQR} = Q_3 - Q_1$
$\approx 31.3 - 27$
≈ 4.3 min

e As 40% of 80 is 32, we start with the cumulative frequency 32 and find the corresponding time.
The 40th percentile ≈ 28.2 min.

f From the cumulative frequency curve we can obtain the following cumulative frequency table:

Time (t min)	$21 \leqslant t < 24$	$24 \leqslant t < 27$	$27 \leqslant t < 30$	$30 \leqslant t < 33$	$33 \leqslant t < 36$
Cumulative frequency	5	20	50	70	80

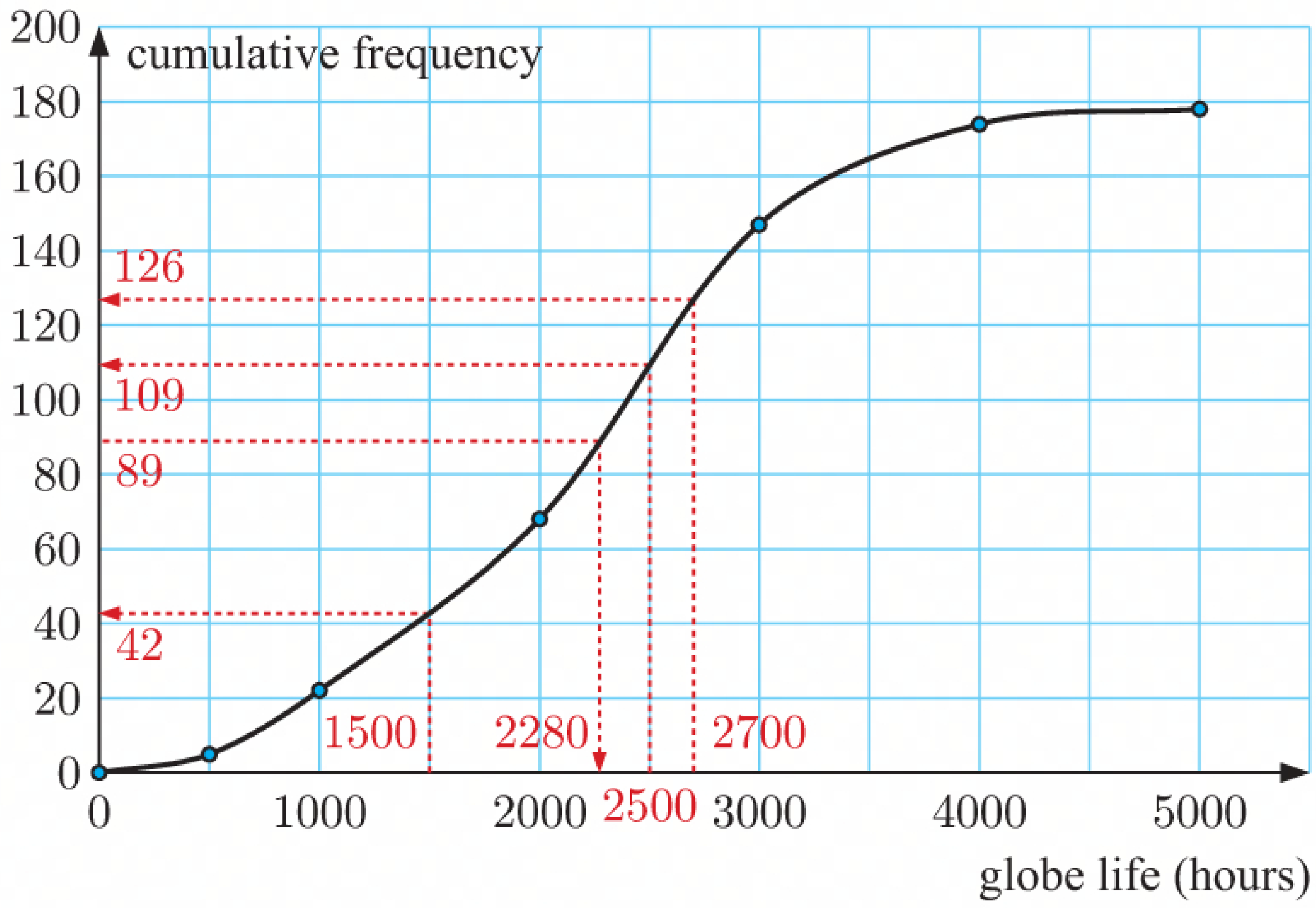
So, the table is:

Time (t min)	$21 \leqslant t < 24$	$24 \leqslant t < 27$	$27 \leqslant t < 30$	$30 \leqslant t < 33$	$33 \leqslant t < 36$
Number of competitors	$5 - 0 = 5$	$20 - 5 = 15$	$50 - 20 = 30$	$70 - 50 = 20$	$80 - 70 = 10$

6

Life (l hours)	*Number of globes*	*Cumulative frequency*
$0 \leqslant l < 500$	5	5
$500 \leqslant l < 1000$	17	22
$1000 \leqslant l < 2000$	46	68
$2000 \leqslant l < 3000$	79	147
$3000 \leqslant l < 4000$	27	174
$4000 \leqslant l < 5000$	4	178

a



b The median is the 50th percentile. As 50% of 178 is 89, we start with the cumulative frequency 89 and find the corresponding globe life.
The median $\approx$ 2280 hours.

c Approximately 126 globes had a life of 2700 hours or less.
$\therefore$ $\frac{126}{178} \times 100\% \approx 70.8\%$ of globes had a life of 2700 hours or less.

d Approximately 42 globes had a life of 1500 hours or less.
Approximately 109 globes had a life of 2500 hours or less.
$\therefore$ approximately $109 - 42 = 67$ globes had a life between 1500 and 2500 hours.

7

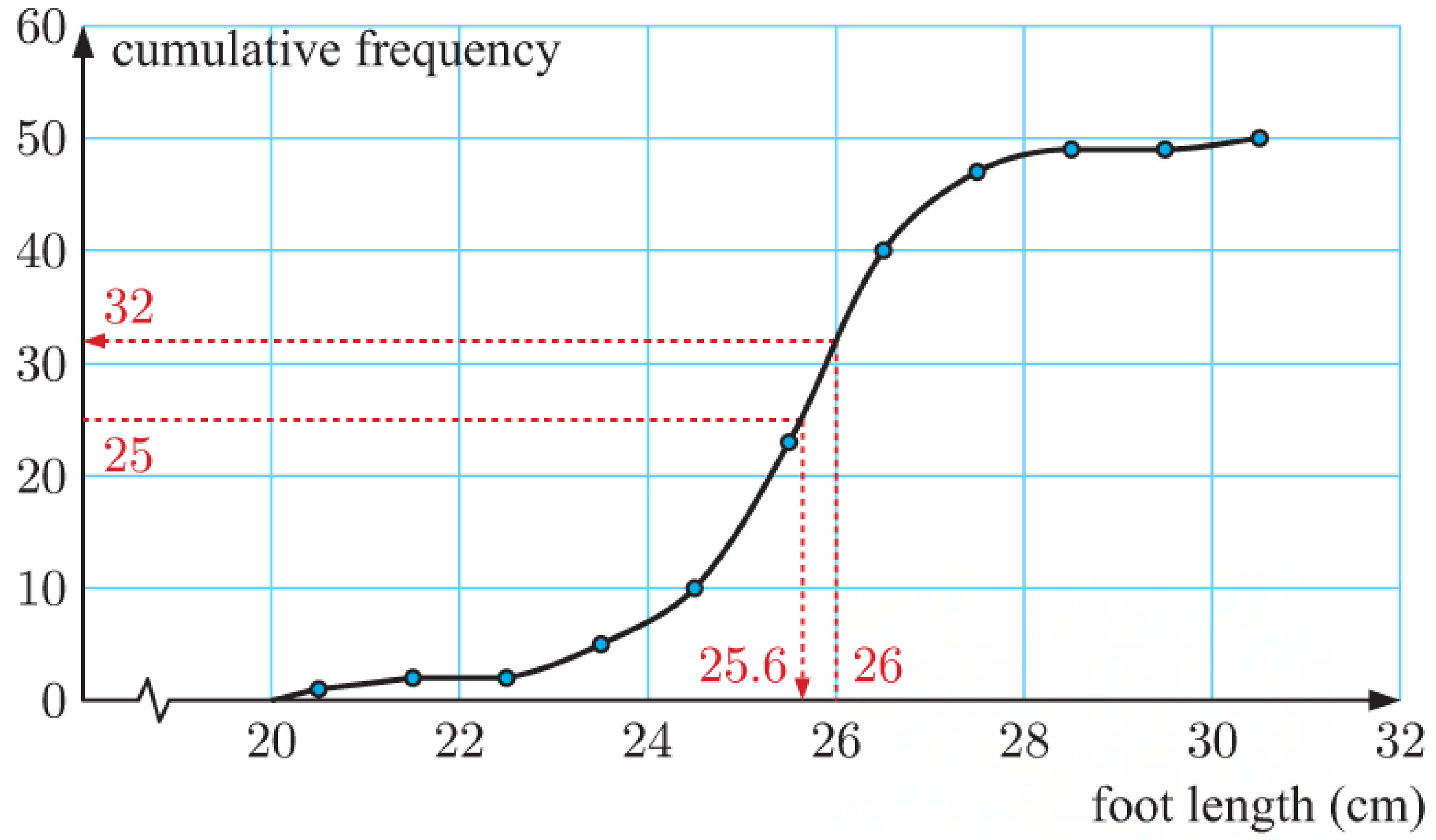
Foot length (cm)	20	21	22	23	24	25	26	27	28	29	30
Frequency	1	1	0	3	5	13	17	7	2	0	1

a Lengths are rounded to 20 cm if they are in the range $19.5 \leqslant l < 20.5$ cm.

b

Foot length (cm)	*Frequency*	*Cumulative frequency*
$19.5 \leqslant l < 20.5$	1	1
$20.5 \leqslant l < 21.5$	1	2
$21.5 \leqslant l < 22.5$	0	2
$22.5 \leqslant l < 23.5$	3	5
$23.5 \leqslant l < 24.5$	5	10
$24.5 \leqslant l < 25.5$	13	23
$25.5 \leqslant l < 26.5$	17	40
$26.5 \leqslant l < 27.5$	7	47
$27.5 \leqslant l < 28.5$	2	49
$28.5 \leqslant l < 29.5$	0	49
$29.5 \leqslant l < 30.5$	1	50

c



d **i** The median is the 50th percentile. As 50% of 50 is 25, we start with the cumulative frequency 25 and find the corresponding foot length.

The median foot length ≈ 25.6 cm.

ii Approximately 32 people had a foot length of 26 cm or less.

$\therefore$ approximately $50 - 32 = 18$ people had a foot length of 26 cm or more.

EXERCISE 13J

1 **a** The mean of data set A $= \dfrac{10+7+5+8+10}{5} = 8$

The mean of data set B $= \dfrac{4+12+11+14+1+6}{6} = 8$

So, each data set has mean 8, as required.

b Data set B appears to have a greater spread than data set A, as data set B has more values which are a long way from the mean, such as 1 and 14.

c *Data set A*:

The population variance $\sigma^2 = \dfrac{\sum(x-\mu)^2}{n}$

$= \dfrac{18}{5}$

$= 3.6$

The population standard deviation $\sigma = \sqrt{3.6}$

≈ 1.90

x	$x-\mu$	$(x-\mu)^2$
10	2	4
7	-1	1
5	-3	9
8	0	0
10	2	4
	Total	18

Data set B:

The population variance $\sigma^2 = \dfrac{\sum(x-\mu)^2}{n}$

$= \dfrac{130}{6}$

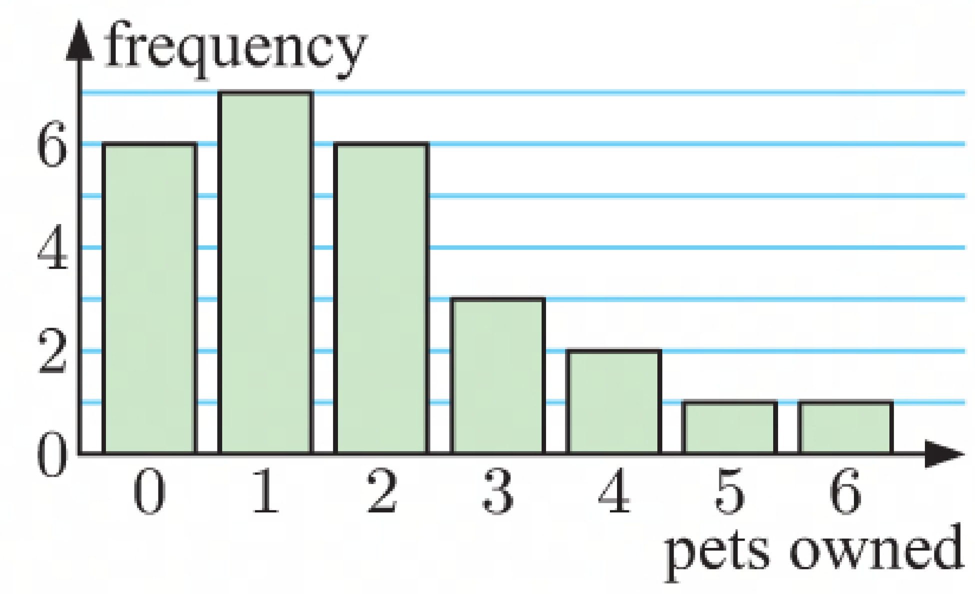
≈ 21.7

The population standard deviation $\sigma \approx \sqrt{21.7}$

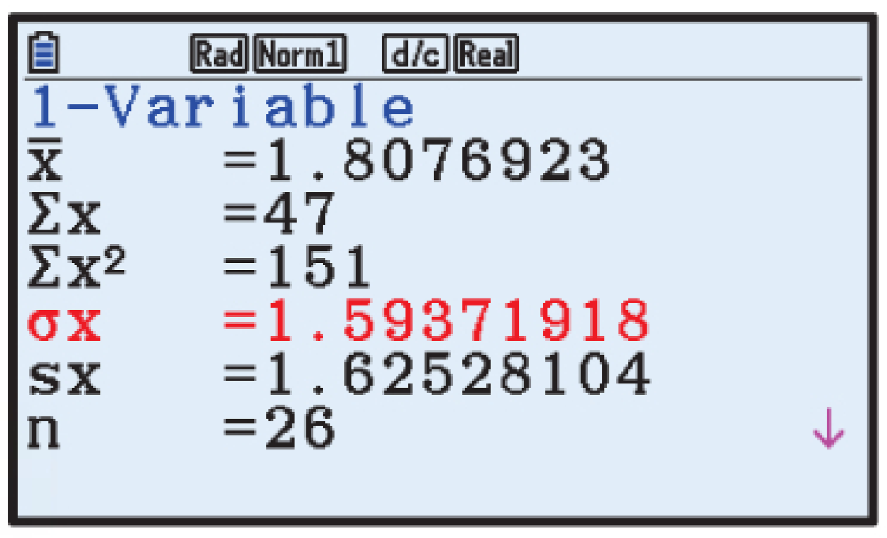
≈ 4.65

x	$x-\mu$	$(x-\mu)^2$
4	-4	16
12	4	16
11	3	9
14	6	36
1	-7	49
6	-2	4
	Total	130

2 **a** There is a high concentration of 0s and 1s in the data set. Looking at a column graph of the data, it appears the data is positively skewed.

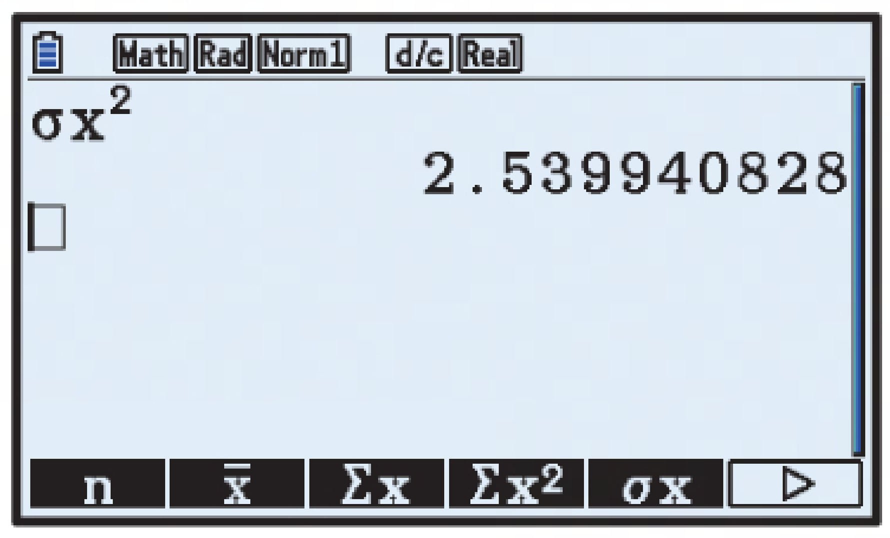


b Using technology:



The population standard deviation $\sigma \approx 1.59$ pets.

c Using technology:



The population variance $\sigma^2 \approx 2.54$.

3 **a** The mean $\mu = \dfrac{22+25+23+28+29+21+20+26}{8}$

$= 24.25$ years

The population standard deviation $\sigma = \sqrt{\dfrac{\sum(x-\mu)^2}{n}}$

$= \sqrt{\dfrac{75.5}{8}}$

≈ 3.07 years

x	$x-\mu$	$(x-\mu)^2$
22	-2.25	5.0625
25	0.75	0.5625
23	-1.25	1.5625
28	3.75	14.0625
29	4.75	22.5625
21	-3.25	10.5625
20	-4.25	18.0625
26	1.75	3.0625
	Total	75.5

b 4 years later, each team member will be 4 years older, so the ages of the members will be: 26, 29, 27, 32, 33, 25, 24, 30.

The new mean $\mu = \dfrac{26+29+27+32+33+25+24+30}{8}$

$= 28.25$ years

The new population standard deviation $\sigma = \sqrt{\dfrac{\sum(x-\mu)^2}{n}}$

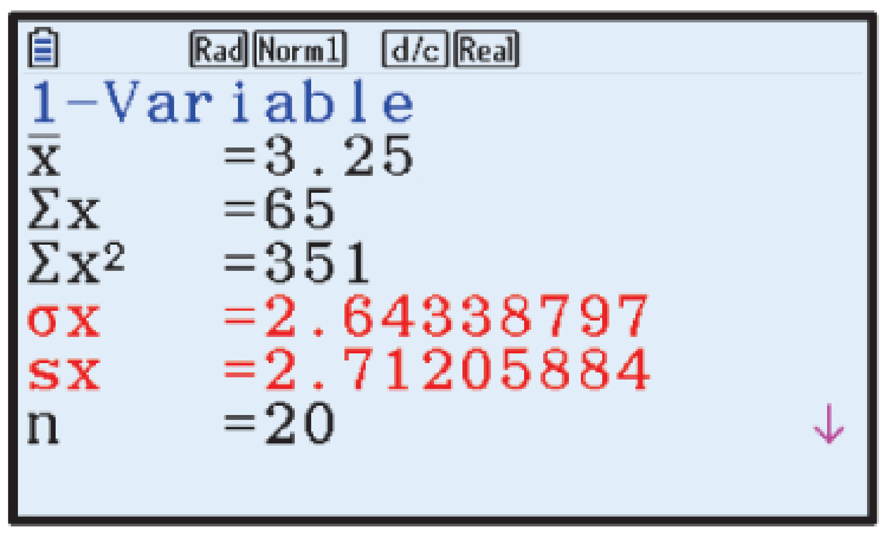
$= \sqrt{\dfrac{75.5}{8}}$

≈ 3.07 years

x	$x-\mu$	$(x-\mu)^2$
26	-2.25	5.0625
29	0.75	0.5625
27	-1.25	1.5625
32	3.75	14.0625
33	4.75	22.5625
25	-3.25	10.5625
24	-4.25	18.0625
30	1.75	3.0625
	Total	75.5

c If each data value is increased or decreased by the same amount, then the mean will also be increased or decreased by that amount, however the population standard deviation will be unchanged.

4 Using technology:



The population standard deviation $\sigma \approx 2.64$ glasses, and the sample standard deviation $s \approx 2.71$ glasses.

5 **a** *Danny*:

mean number of hours spent on homework

$= \dfrac{3.5+3.5+4+2.5+3+3.5+3+1.5+3+4+2.5+4+4+3}{14}$

≈ 3.21 hours

Jennifer:

mean number of hours spent on homework

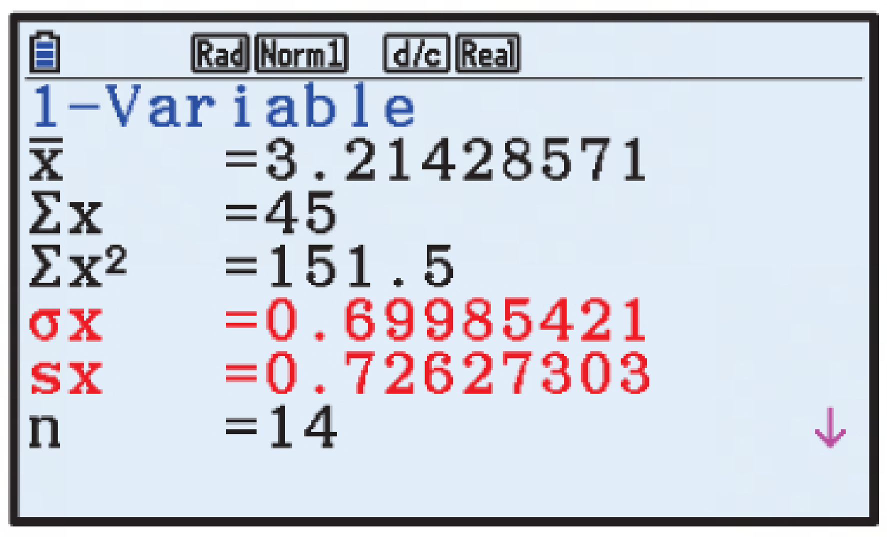
$= \dfrac{2.5+1+2.5+2+2+2.5+1.5+2+2+2.5+2+2+2+1.5}{14}$

$= 2$ hours

b Danny's mean is higher than Jennifer's, so Danny generally studies for longer.

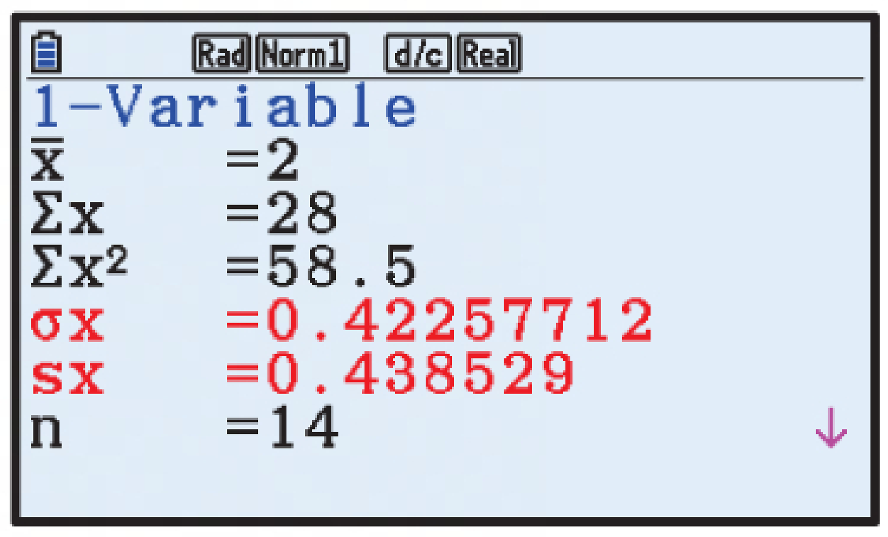
c Using technology:

Danny:



The population standard deviation $\sigma \approx 0.700$ hours, and the sample standard deviation $s \approx 0.726$ hours.

Jennifer:



The population standard deviation $\sigma \approx 0.423$ hours, and the sample standard deviation $s \approx 0.439$ hours.

d Jennifer's standard deviation is lower than Danny's, so there is less deviation from the mean for her data set. Jennifer therefore studies more consistently than Danny.

6 **a** Boys' mean time $= \dfrac{32.2 + 26.4 + 35.6 + ... + 38.9 + 29.0 + 31.3}{10}$

$= 32.02$ s

Girls' mean time $= \dfrac{36.2 + 33.5 + 28.1 + ... + 36.0 + 39.7 + 29.8}{10}$

$= 34.77$ s

The ordered data set for the boys is:

~~26.4 27.3 28.5 29.0~~ 30.8 31.3 ~~32.2 35.6 38.9 40.2~~ $\{n = 10\}$

Since $n = 10$, $\dfrac{n+1}{2} = 5.5$ $\therefore$ the median is the average of the 5th and 6th data values.

$\therefore$ median of boys' data $= \dfrac{\text{5th value} + \text{6th value}}{2}$

$= \dfrac{30.8 + 31.3}{2}$

$= 31.05$ s

The range of the boys' data = maximum value − minimum value

$= 40.2 - 26.4$

$= 13.8$ s

The ordered data set for the girls is:

~~28.1 29.8 31.6 33.5~~ 35.7 36.0 ~~36.2 37.3 39.7 39.8~~ $\{n = 10\}$

Since $n = 10$, $\dfrac{n+1}{2} = 5.5$ $\therefore$ the median is the average of the 5th and 6th data values.

$\therefore$ median of girls' data $= \dfrac{\text{5th value} + \text{6th value}}{2}$

$= \dfrac{35.7 + 36.0}{2}$

$= 35.85$ s

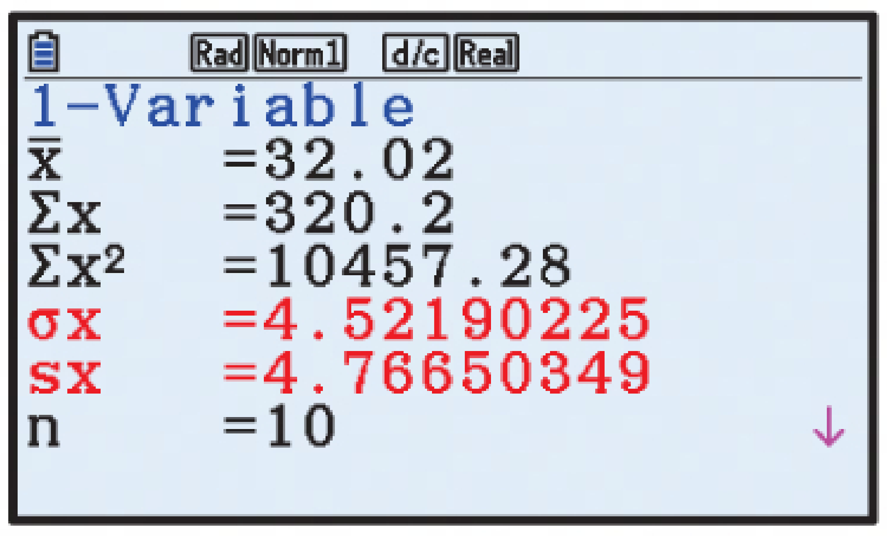
The range of the girls' data = maximum value − minimum value

$= 39.8 - 28.1$

$= 11.7$ s

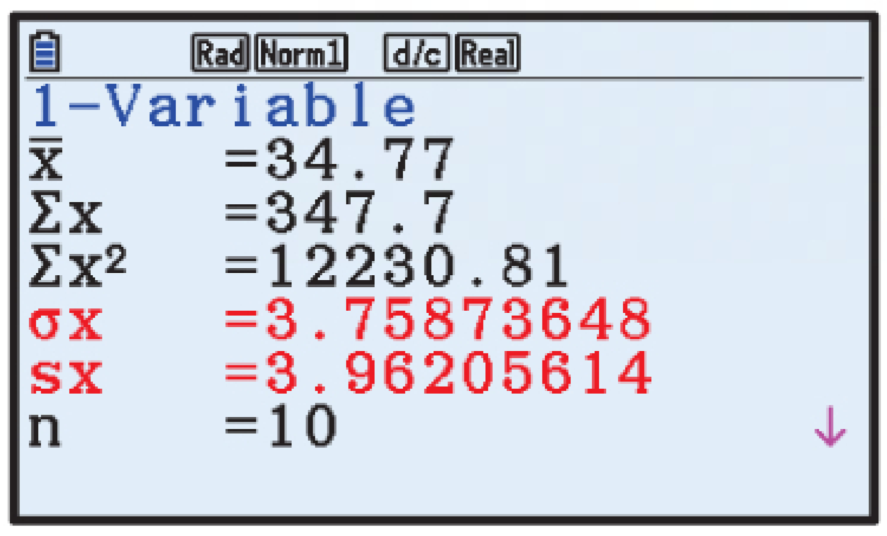
Using technology:

Boys:



The population standard deviation $\sigma \approx 4.52$ s, and the sample standard deviation $s \approx 4.77$ s.

Girls:



The population standard deviation $\sigma \approx 3.76$ s, and the sample standard deviation $s \approx 3.96$ s.

So, the table is:

	Boys	*Girls*
Mean $\overline{x}$	32.02 s	34.77 s
Median	31.05 s	35.85 s
Standard deviation	$\sigma \approx 4.52$ s $s \approx 4.77$ s	$\sigma \approx 3.76$ s $s \approx 3.96$ s
Range	13.8 s	11.7 s

b **i** The mean and median are lower for the boys, so the boys generally swim faster.

ii The standard deviation and range are higher for the boys, so the boys have the greater spread of swimming speeds.

c Tyson could improve the reliability of his findings by increasing his sample size.

7

Rockets	0	10	1	9	11	0	8	5	6	7
Bullets	4	3	4	1	4	11	7	6	12	5

a Rockets' mean number of runs $= \dfrac{0+10+1+ .... +5+6+7}{10}$

$= 5.7$ runs

Bullets' mean number of runs $= \dfrac{4+3+4+ .... +6+12+5}{10}$

$= 5.7$ runs

Range of Rockets' data $=$ maximum $-$ minimum

$= 11 - 0$

$= 11$ runs

Range of Bullets' data $=$ maximum $-$ minimum

$= 12 - 1$

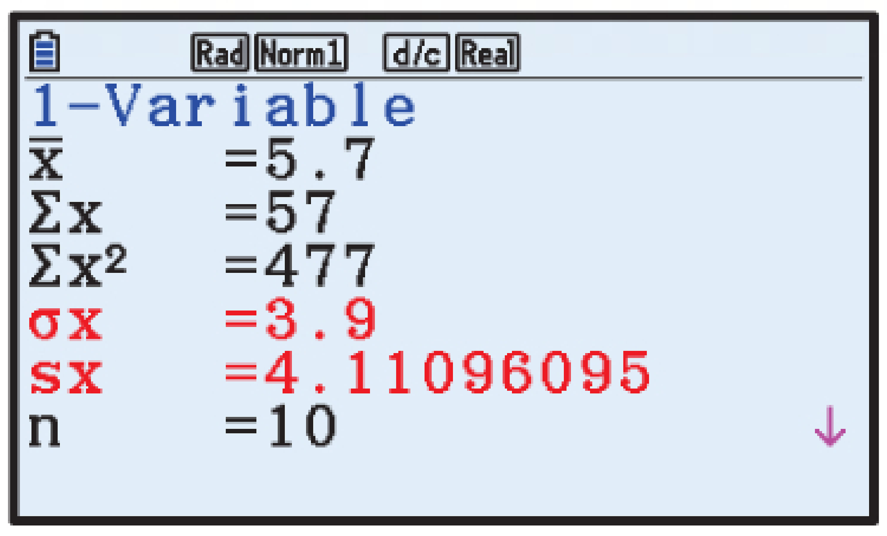
$= 11$ runs

So, the two teams have the same mean (5.7 runs) and range (11 runs) of runs scored.

b We suspect the Rockets' performance is more variable over the period since they twice scored zero runs.

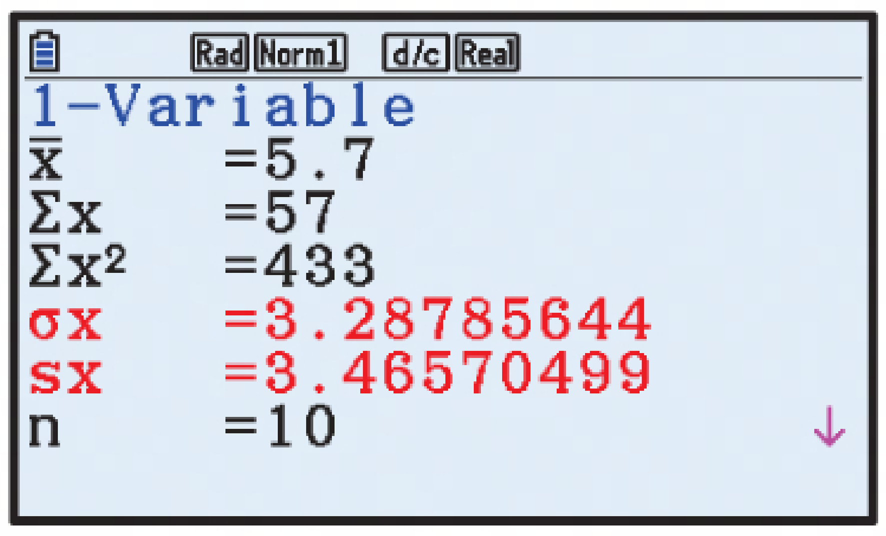
Using technology:

Rockets:



The population standard deviation $\sigma = 3.9$ runs, and the sample standard deviation $s \approx 4.11$ runs.

Bullets:



The population standard deviation $\sigma \approx 3.29$ runs, and the sample standard deviation $s \approx 3.47$ runs.

The standard deviation is higher for the Rockets which confirms our suspicion that the Rockets' performance is more variable.

c The standard deviation gives a better indication of variability as it takes all data values into account, not just the lowest and highest values.

8 **a** **i** *Museum*:

$$\text{Mean number of visitors} = \frac{1108 + 1019 + 850 + 1243 + ... + 1084 + 981}{31}$$

$$= \frac{28\,963}{31}$$

$$\approx 934 \text{ visitors}$$

Art gallery:

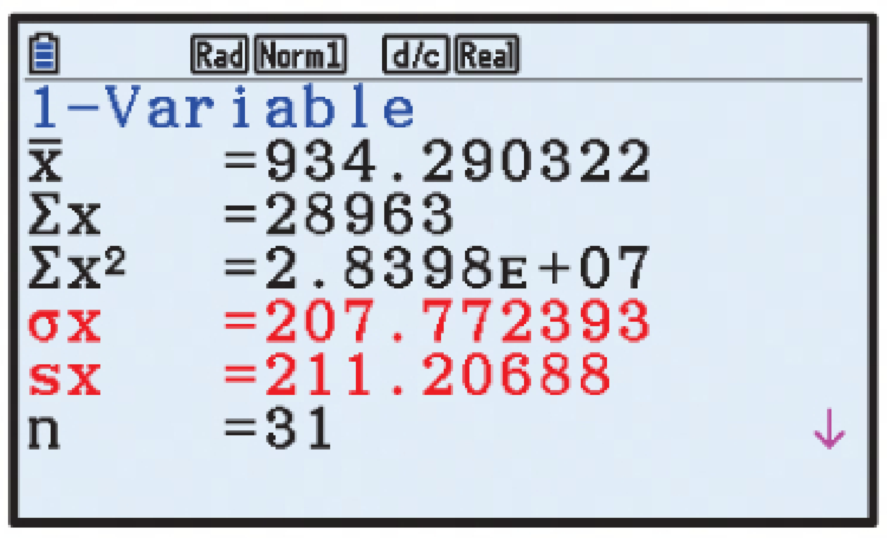
$$\text{Mean number of visitors} = \frac{1258 + 1107 + 1179 + 1302 + ... + 1259 + 1366}{31}$$

$$= \frac{38\,197}{31}$$

$$\approx 1230 \text{ visitors}$$

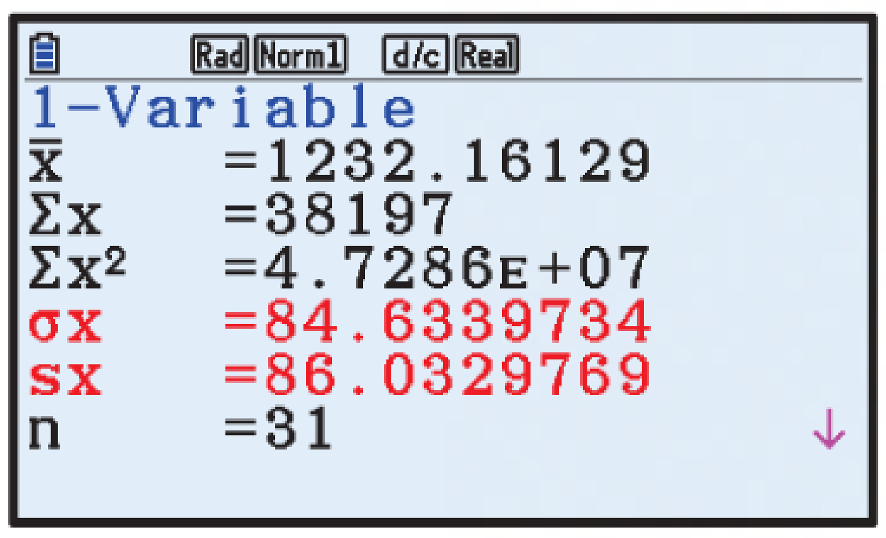
ii Using technology:

Museum:



The population standard deviation $\sigma \approx 208$ visitors, and the sample standard deviation $s \approx 211$ visitors.

Art gallery:



The population standard deviation $\sigma \approx 84.6$ visitors, and the sample standard deviation $s \approx 86.0$ visitors.

b The standard deviation was higher for the museum data, so the museum had the greater spread of visitor numbers.

c **i** "0" is an outlier in the *Museum* data.

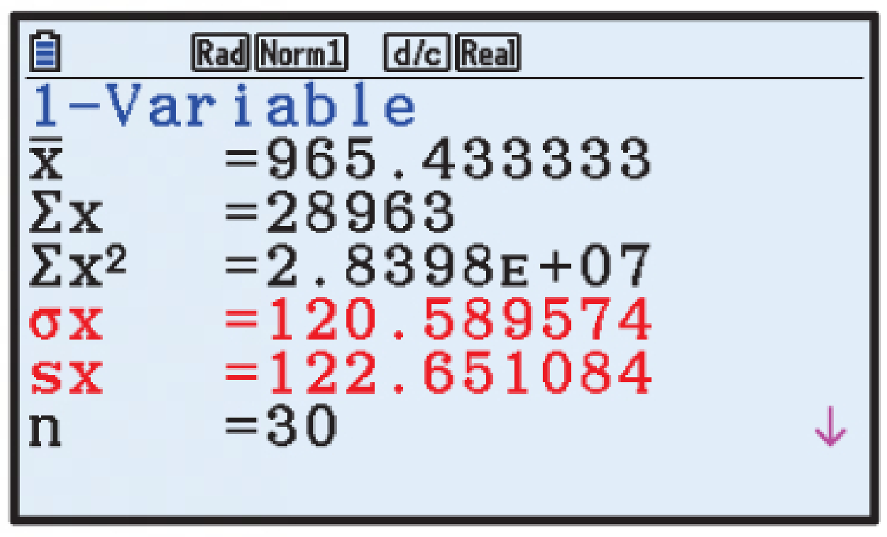
ii This outlier corresponds to Christmas Day, so the museum was probably closed which meant there were no visitors on that day.

iii Yes, it is reasonable to remove the outlier when comparing the numbers of visitors to these places. Even though the outlier is not an error, it is not a true reflection of the visitor count for a particular day.

iv New mean number of visitors to the museum $= \dfrac{28\,963}{30}$

≈ 965 visitors

Using technology:



The new population standard deviation $\sigma \approx 121$ visitors, and the new sample standard deviation $s \approx 123$ visitors.

v The outlier had greatly increased the standard deviation.

9 The population standard deviation σ and the sample standard deviation s are related by the formula

$\sigma = \sqrt{\dfrac{n-1}{n}}\, s$ where n is the sample size.

Suppose we take a sample of size 5 from population A, and $s_A = 0.5$

$$\therefore \quad \sigma_A = \sqrt{\frac{5-1}{5}}(0.5) \approx 0.447$$

Suppose we take a sample of size 500 from population B, and $s_B = 0.48$

$$\therefore \quad \sigma_B = \sqrt{\frac{500-1}{500}}(0.48) \approx 0.480$$

So, we have $s_A > s_B$, and $\sigma_B > \sigma_A$.

$\therefore$ Anna will not necessarily find that $\sigma_A > \sigma_B$ for the samples given that $s_A > s_B$.

10 Mean of 8 integers $= \dfrac{1+3+5+7+4+5+p+q}{8}$

$$\therefore \quad 5 = \frac{25+p+q}{8}$$

$$\therefore \quad 40 = 25+p+q$$

$$\therefore \quad p+q = 15$$

$$\therefore \quad p = 15-q \quad .... \; (*)$$

x	$x-\mu$	$(x-\mu)^2$
1	-4	16
3	-2	4
5	0	0
7	2	4
4	-1	1
5	0	0
p	$p-5$	$(p-5)^2$
q	$q-5$	$(q-5)^2$
	Total	$25+(p-5)^2+(q-5)^2$

Population variance $\sigma^2 = \dfrac{\sum(x-\mu)^2}{n}$

$$\therefore\ 5.25 = \frac{25+(p-5)^2+(q-5)^2}{8}$$

$$\therefore\ 42 = 25+(15-q-5)^2+(q-5)^2 \qquad \{\text{using } (*)\}$$

$$\therefore\ 17 = (10-q)^2+(q-5)^2$$

$$\therefore\ 17 = 100-20q+q^2+q^2-10q+25$$

$$\therefore\ 2q^2-30q+108=0$$

$$\therefore\ 2(q^2-15q+54)=0$$

$$\therefore\ 2(q-6)(q-9)=0$$

$$\therefore\ q=6 \quad \text{or} \quad q=9$$

$$\therefore\ p=15-6 \quad \text{or} \quad p=15-9$$

$$\therefore\ p=9 \quad \text{or} \quad p=6$$

but $p<q$ $\quad\therefore\ p=6$ and $q=9$

11

x	$x-\mu$	$(x-\mu)^2$
3	-3	9
9	3	9
5	-1	1
5	-1	1
6	0	0
4	-2	4
a	$a-6$	$(a-6)^2$
6	0	0
b	$b-6$	$(b-6)^2$
8	2	4
	Total	$28+(a-6)^2+(b-6)^2$

Mean of 10 integers $= \dfrac{3+9+5+5+6+4+a+6+b+8}{10}$

$$\therefore\ 6 = \frac{46+a+b}{10}$$

$$\therefore\ 60 = 46+a+b$$

$$\therefore\ a+b=14$$

$$\therefore\ a = 14-b \quad \text{....} \ (*)$$

$$\text{Population variance } \sigma^2 = \frac{\sum(x-\mu)^2}{n}$$

$$\therefore\ 3.2 = \frac{28 + (a-6)^2 + (b-6)^2}{10}$$

$$\therefore\ 32 = 28 + (14-b-6)^2 + (b-6)^2 \qquad \{\text{using } (*)\}$$

$$\therefore\ 4 = (8-b)^2 + (b-6)^2$$

$$\therefore\ 4 = 64 - 16b + b^2 + b^2 - 12b + 36$$

$$\therefore\ 2b^2 - 28b + 96 = 0$$

$$\therefore\ 2(b^2 - 14b + 48) = 0$$

$$\therefore\ 2(b-6)(b-8) = 0$$

$$\therefore\ b = 6 \quad \text{or} \quad b = 8$$

$$\therefore\ a = 14-6 \quad \text{or} \quad a = 14-8$$

$$\therefore\ a = 8 \quad \text{or} \quad a = 6$$

$$\text{but } a > b \quad \therefore\ a = 8 \text{ and } b = 6$$

12 **a**

$$\sum_{i=1}^{n}(x_i-\mu)^2 = \sum_{i=1}^{n}({x_i}^2 - 2x_i\mu + \mu^2)$$

$$= \sum_{i=1}^{n}({x_i}^2) - \sum_{i=1}^{n}(2x_i\mu) + \sum_{i=1}^{n}\mu^2$$

$$= \sum_{i=1}^{n}({x_i}^2) - 2\mu\sum_{i=1}^{n}x_i + \sum_{i=1}^{n}\mu^2$$

$$= \sum_{i=1}^{n}({x_i}^2) - 2\mu(x_1 + x_2 + \dots + x_n) + n\mu^2$$

$$= \sum_{i=1}^{n}({x_i}^2) - 2\mu(n\mu) + n\mu^2 \qquad \left\{\text{since } \mu = \frac{x_1 + x_2 + \dots + x_n}{n}\right\}$$

$$= \sum_{i=1}^{n}({x_i}^2) - 2n\mu^2 + n\mu^2$$

$$\therefore\ \sum_{i=1}^{n}(x_i-\mu)^2 = \sum_{i=1}^{n}({x_i}^2) - n\mu^2$$

b

$$\sum_{i=1}^{n}(x_i-\mu)^2 = \sum_{i=1}^{n}({x_i}^2) - n\mu^2 \qquad \{\text{from } \mathbf{a}\}$$

$$\therefore\ \sum_{i=1}^{25}(x_i-\mu)^2 = \sum_{i=1}^{25}({x_i}^2) - 25\mu^2$$

$$\therefore\ \sqrt{\frac{\sum_{i=1}^{25}(x_i-\mu)^2}{25}} = \sqrt{\frac{\sum_{i=1}^{25}({x_i}^2) - 25\mu^2}{25}}$$

$$\therefore\ 5.2 = \frac{\sqrt{2568.25 - 25\mu^2}}{5}$$

$$\therefore\ 26 = \sqrt{2568.25 - 25\mu^2}$$

$$\therefore\ 676 = 2568.25 - 25\mu^2$$

$$\therefore\ 25\mu^2 = 1892.25$$

$$\therefore\ \mu^2 = 75.69$$

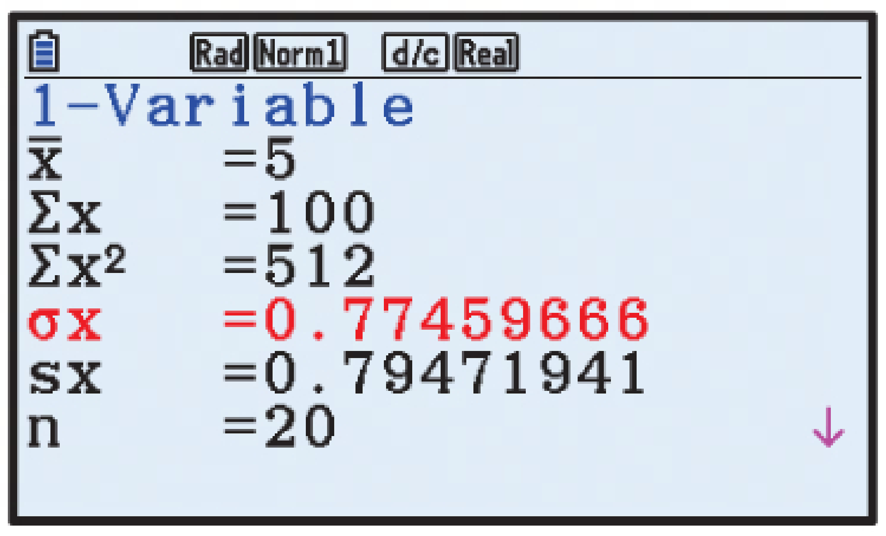
$$\therefore\ \mu = \pm\sqrt{75.69}$$

$$\therefore\ \mu = \pm 8.7$$

13

Value	*Frequency*
3	1
4	3
5	11
6	5
Total	20

Using technology:

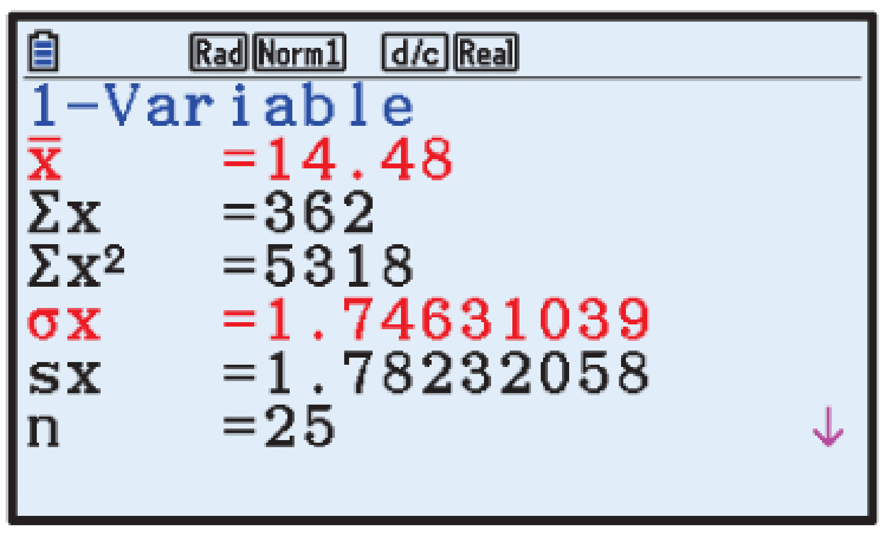


The population standard deviation $\sigma \approx 0.775$.

14

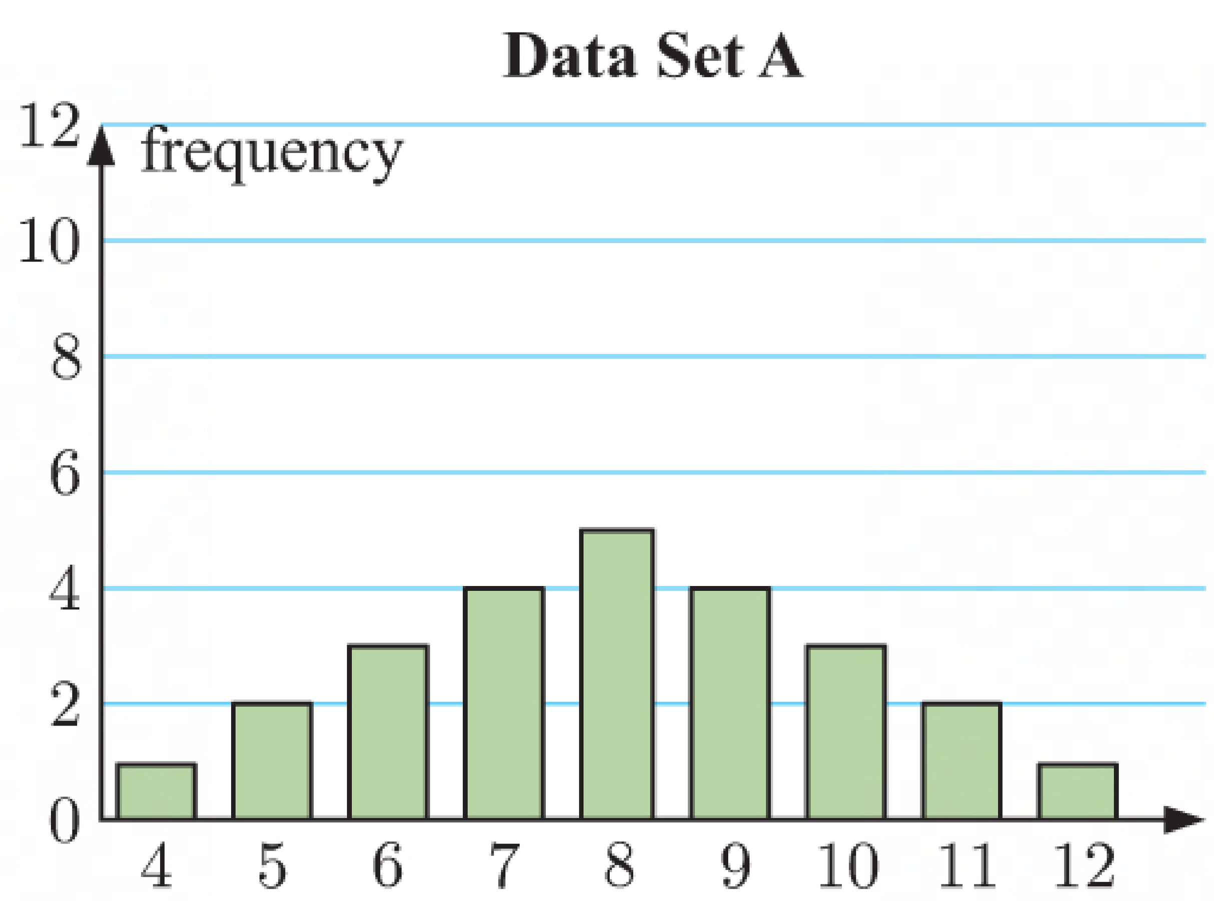
Age	11	12	13	14	15	16	17	18
Frequency	2	1	4	5	6	4	2	1

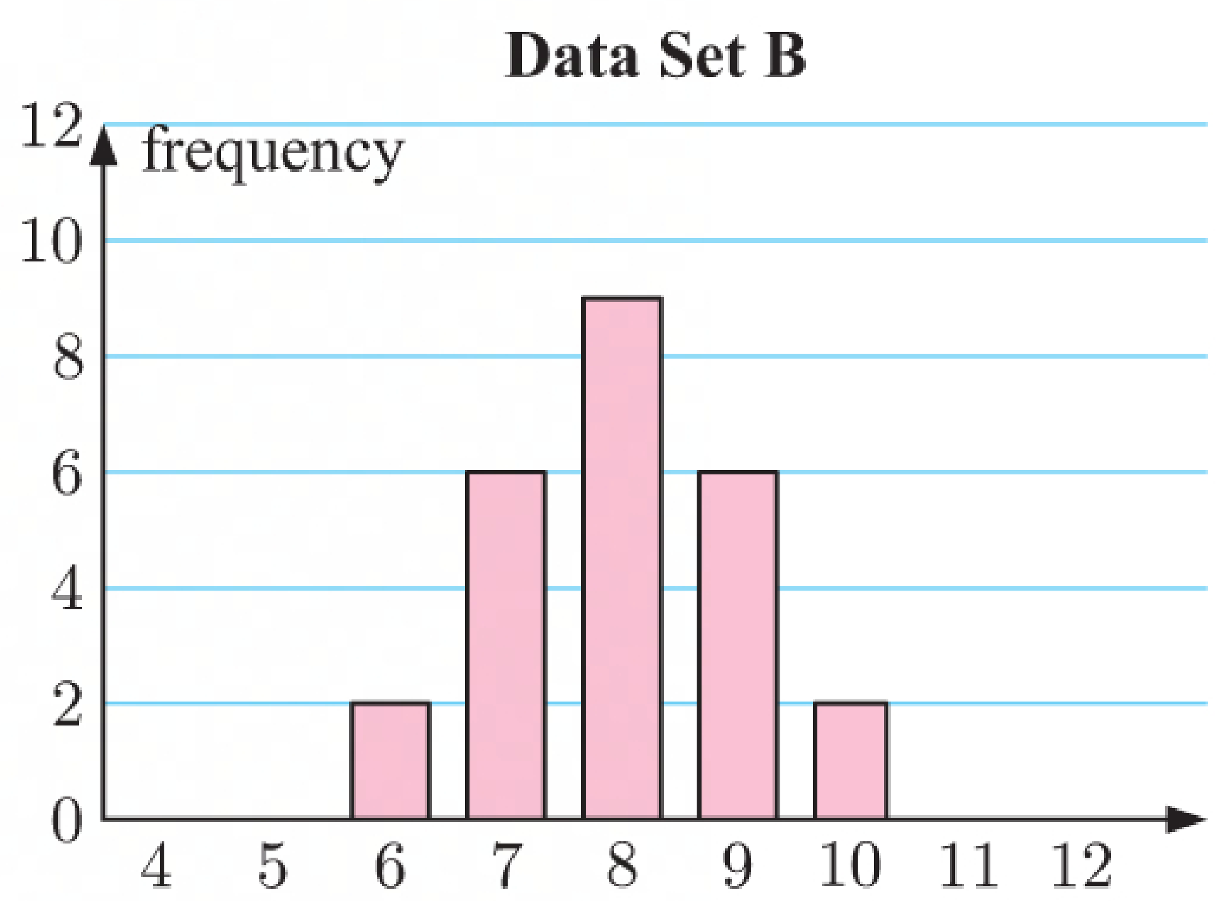
Using technology:



The mean age of squash players $\mu = 14.48$ years, and the population standard deviation $\sigma \approx 1.75$ years.

15

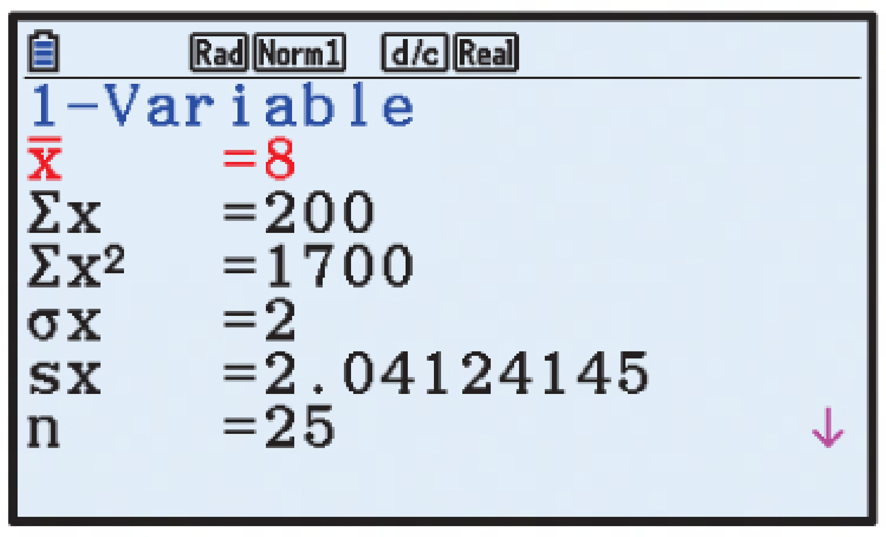




a By looking at the graphs, data set A appears to have a wider spread.

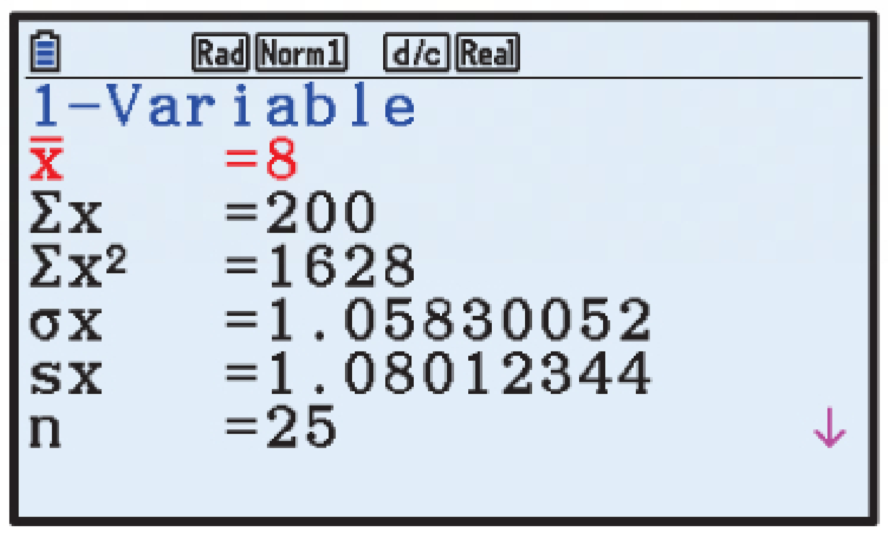
b Using technology:

Data set A:



The mean of data set A is 8.

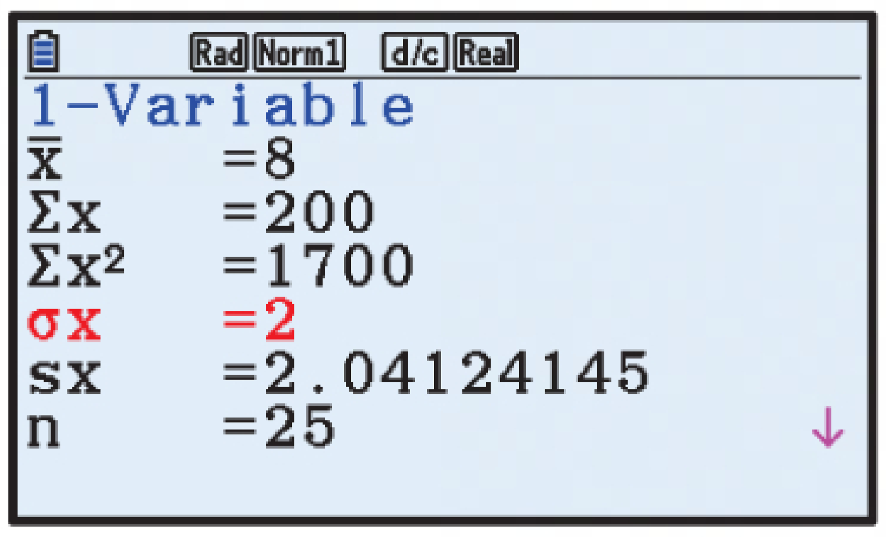
Data set B:



The mean of data set B is 8.

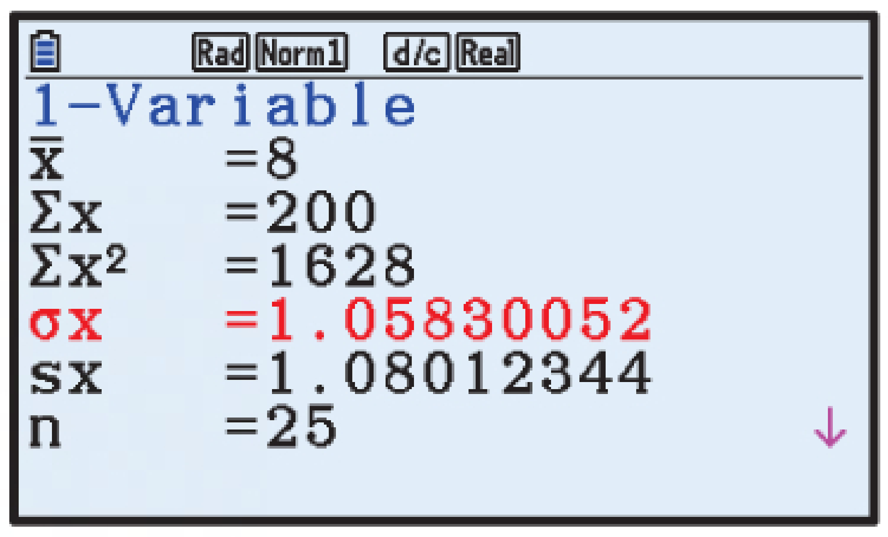
c Using technology:

Data set A:



The population standard deviation is $\sigma = 2$.

Data set B:



The population standard deviation is $\sigma \approx 1.06$.

The population standard deviation is higher for data set A than for data set B which confirms that data set A has a wider spread.

d

Data set	*Range*	*IQR*
A	8	3
B	4	2

The range only takes into account the maximum and minimum values.

The IQR only takes into account the upper and lower quartiles.

The standard deviation however is calculated using all of the data values, so it gives a better description of how the data is distributed than the range or IQR.

16

Score	*Females*	*Males*
12	0	1
13	0	0
14	0	2
15	0	3
16	2	4
17	6	2
18	5	0
19	1	1
20	1	0

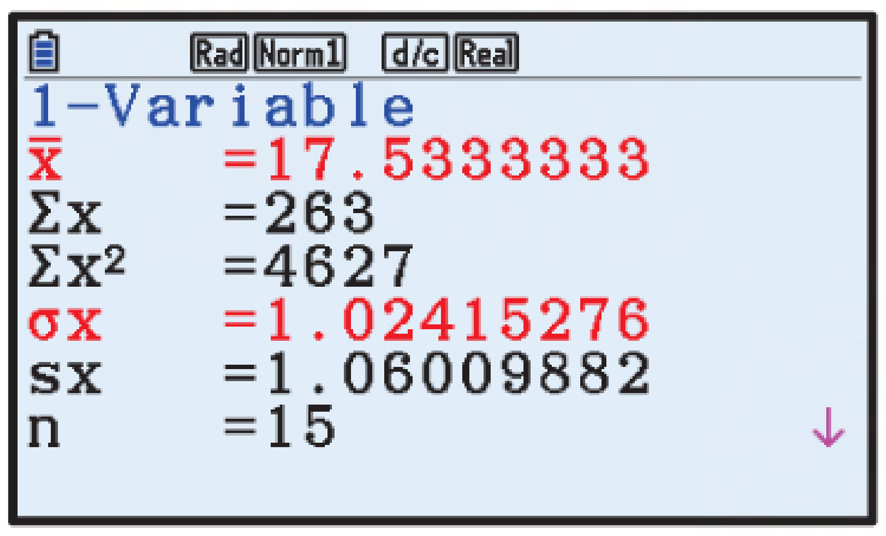
a The female students' marks are in the range 16 to 20 whereas the male students' marks are in the range 12 to 19.

i The females appear to have scored better in the test.

ii The males appear to have a greater spread of scores.

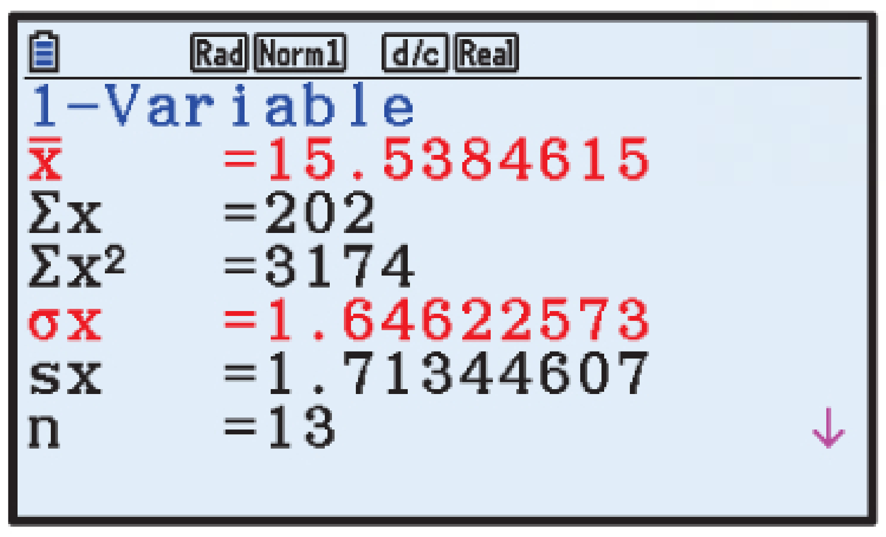
b Using technology:

Females:



The females' mean score $\mu \approx 17.5$ marks, and the population standard deviation $\sigma \approx 1.02$ marks.

Males:



The males' mean score $\mu \approx 15.5$ marks, and the population standard deviation $\sigma \approx 1.65$ marks.

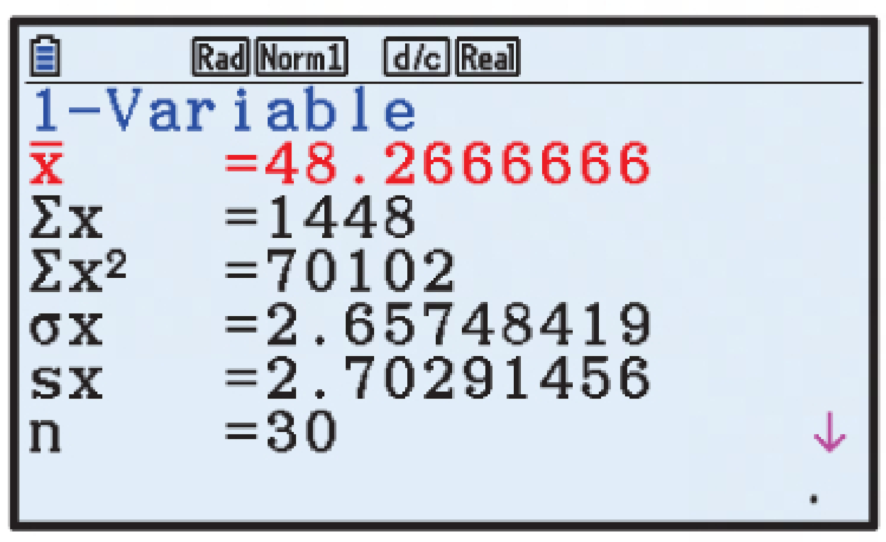
17 Jess' question is worded so that the respondent will not include themselves.

$\therefore$ the results for the mean will differ by 1, but the results for the standard deviation will be the same.

18

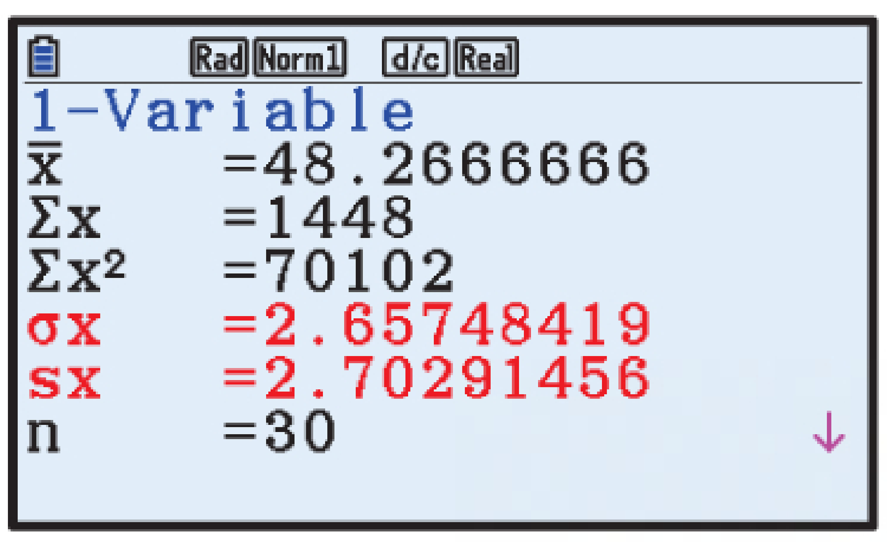
Class interval	*Mid-interval value*	*Frequency*
$40 \leqslant L < 42$	41	1
$42 \leqslant L < 44$	43	1
$44 \leqslant L < 46$	45	3
$46 \leqslant L < 48$	47	7
$48 \leqslant L < 50$	49	11
$50 \leqslant L < 52$	51	5
$52 \leqslant L < 54$	53	2

a Using technology:



The mean ≈ 48.3 cm.

b Using technology:

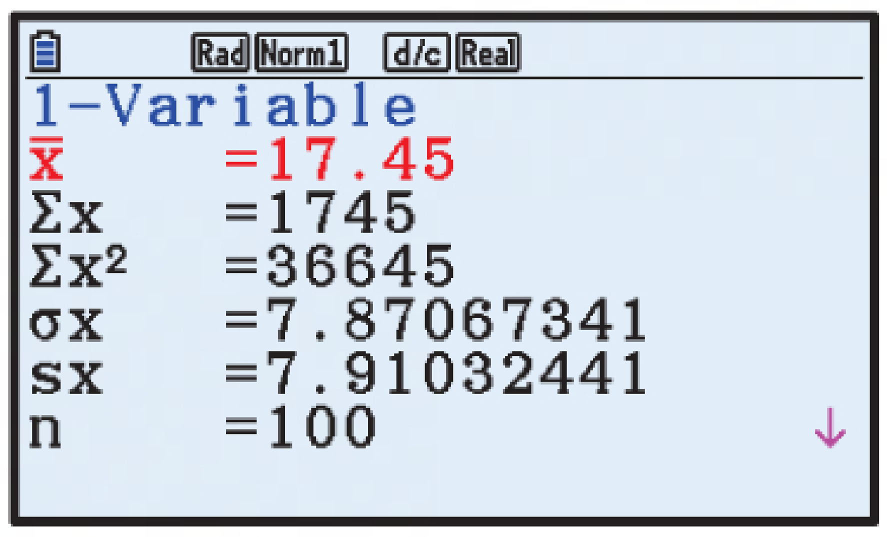


The population standard deviation $\sigma \approx 2.66$ cm, and the sample standard deviation $s \approx 2.70$ cm.

19

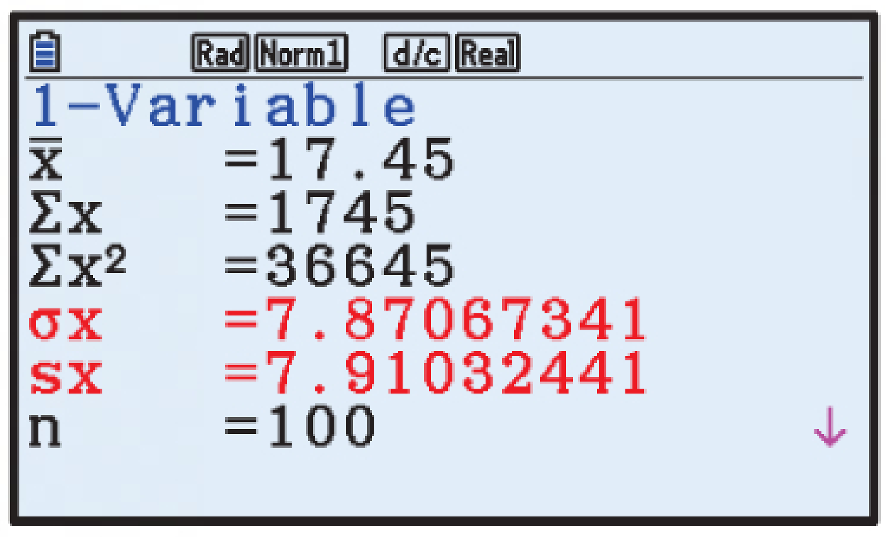
Class interval	*Mid-interval value*	*Frequency*
1 - 5	3	4
6 - 10	8	16
11 - 15	13	22
16 - 20	18	28
21 - 25	23	14
26 - 30	28	9
31 - 35	33	5
36 - 40	38	2

a Using technology:



The mean ≈ 17.45 vehicles.

b Using technology:

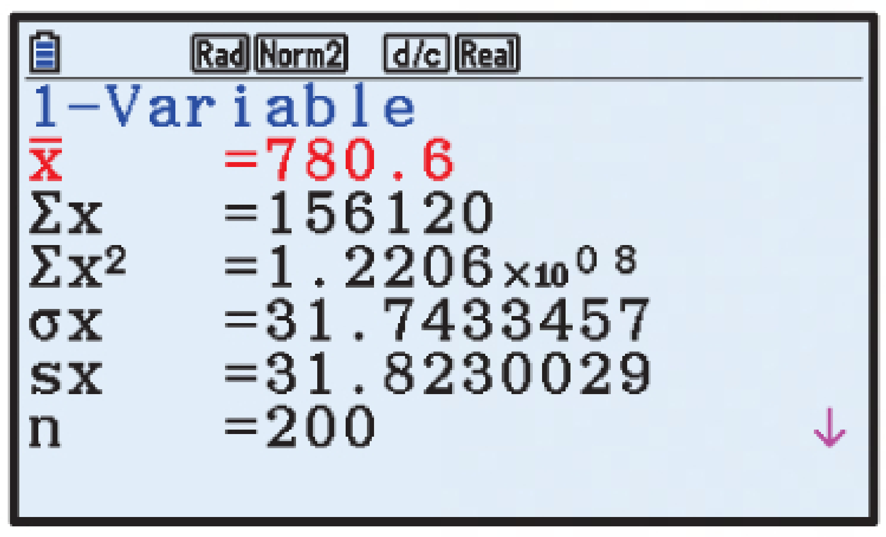


The population standard deviation $\sigma \approx 7.87$ vehicles, and the sample standard deviation $s \approx 7.91$ vehicles.

20

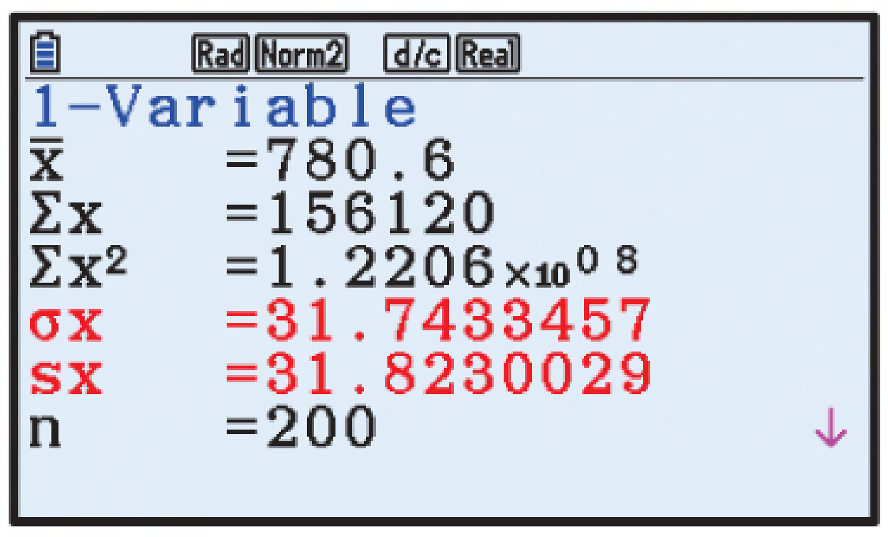
Class interval	*Mid-interval value*	*Frequency*
$720 \leqslant W < 740$	730	17
$740 \leqslant W < 760$	750	38
$760 \leqslant W < 780$	770	47
$780 \leqslant W < 800$	790	57
$800 \leqslant W < 820$	810	18
$820 \leqslant W < 840$	830	10
$840 \leqslant W < 860$	850	10
$860 \leqslant W < 880$	870	3

a Using technology:



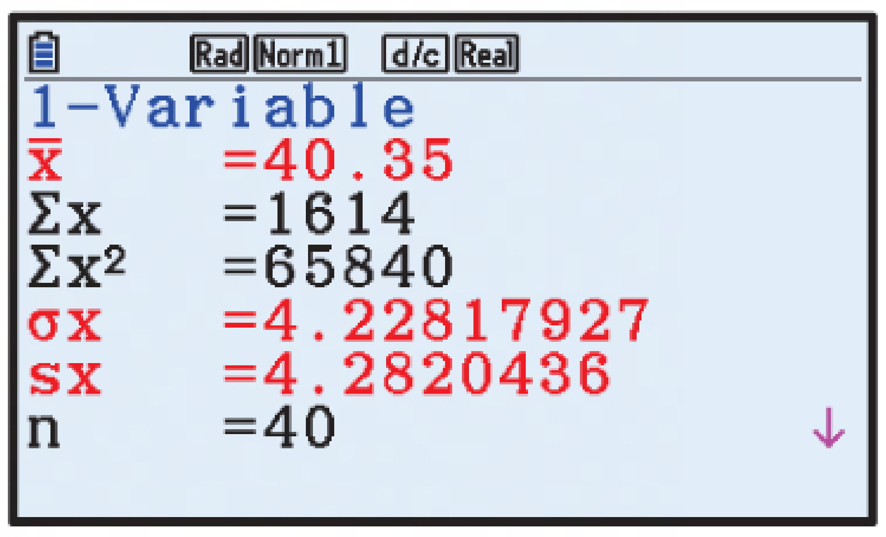
The mean $\approx \$780.60$.

b Using technology:



The population standard deviation $\sigma \approx \$31.74$, and the sample standard deviation $s \approx \$31.82$.

21 **a** Using technology:

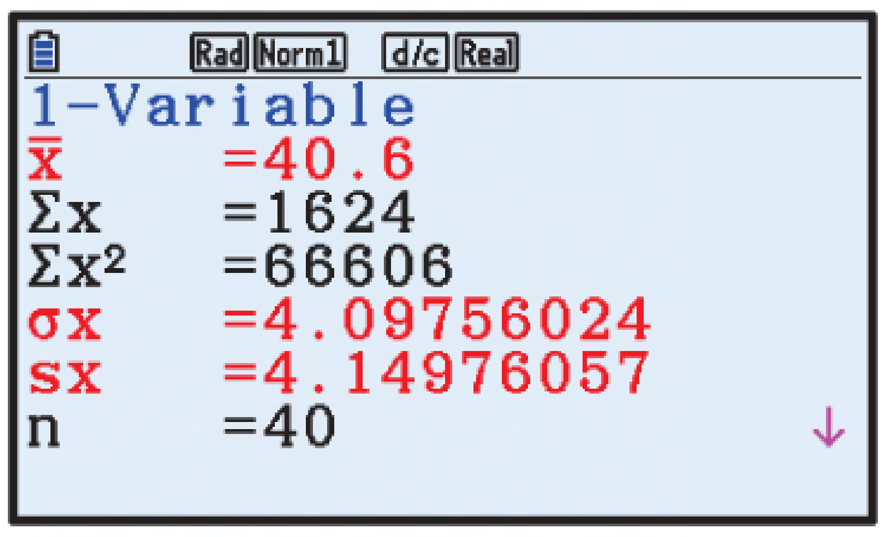


The mean $\overline{x} = 40.35$ hours, the population standard deviation $\sigma \approx 4.23$ hours, and the sample standard deviation $s \approx 4.28$ hours.

b

Class interval	*Mid-interval value*	*Frequency*
30 - 33	31.5	2
34 - 37	35.5	5
38 - 41	39.5	19
42 - 45	43.5	8
46 - 49	47.5	6

Using technology:



The mean $\overline{x} = 40.6$ hours, the population standard deviation $\sigma \approx 4.10$ hours, and the sample standard deviation $s \approx 4.15$ hours.
The mean is slightly higher for the class interval data set than for the raw data. The standard deviation is slightly lower for the class interval data set than for the raw data. The values for the mean and standard deviation for the class interval data set are therefore good approximations for the mean and standard deviation of the raw data.

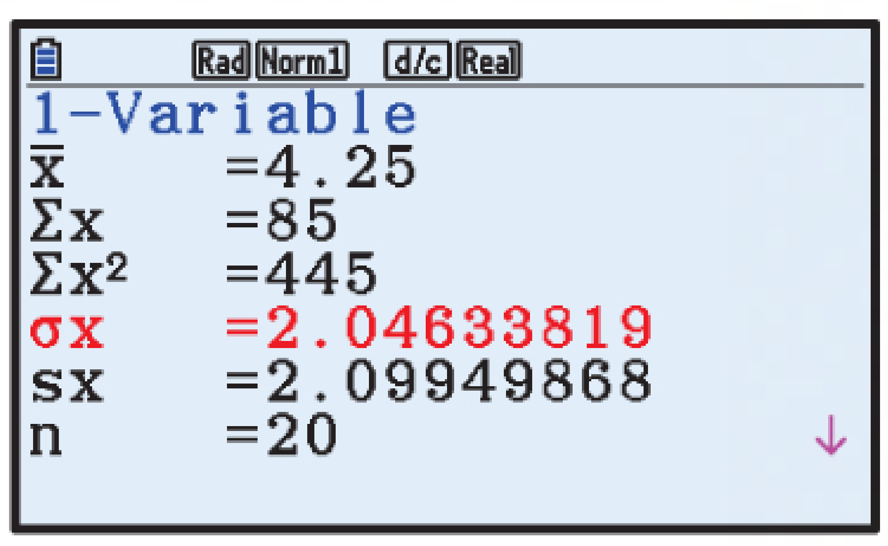
INVESTIGATION 3 TRANSFORMING DATA

1 For the data set:

4	2	3	3	5	2	9	7	3	5
2	1	5	3	6	6	3	3	6	7

$$\begin{aligned}\text{The mean} &= \frac{4+2+3+\ldots.+6+7}{20}\\ &= \frac{85}{20}\\ &= 4.25\end{aligned}$$

Using technology, the population standard deviation ≈ 2.05.

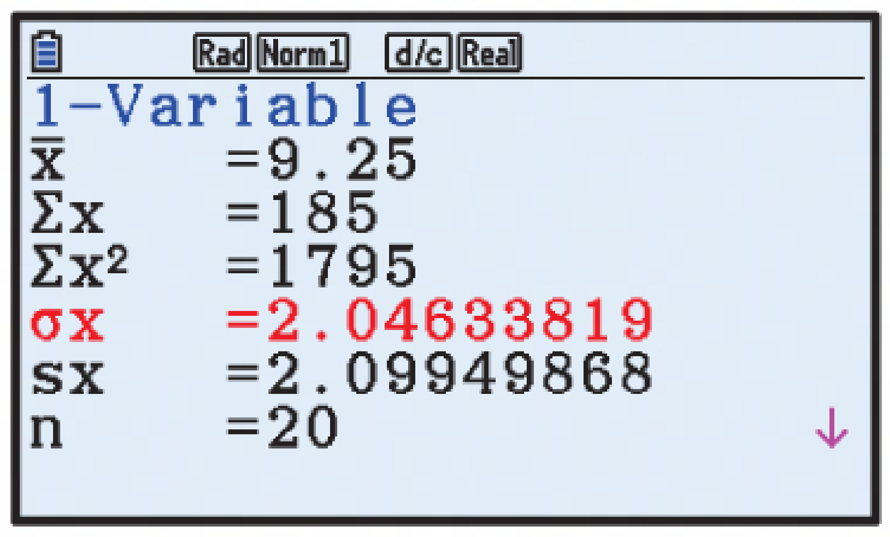


2 **a** The new data set is:

9	7	8	8	10	7	14	12	8	10
7	6	10	8	11	11	8	8	11	12

$$\begin{aligned}\text{The mean} &= \frac{9+7+8+\dots+11+12}{20}\\ &= \frac{185}{20}\\ &= 9.25\\ &= 4.25+5\end{aligned}$$

Using technology, the population standard deviation ≈ 2.05.



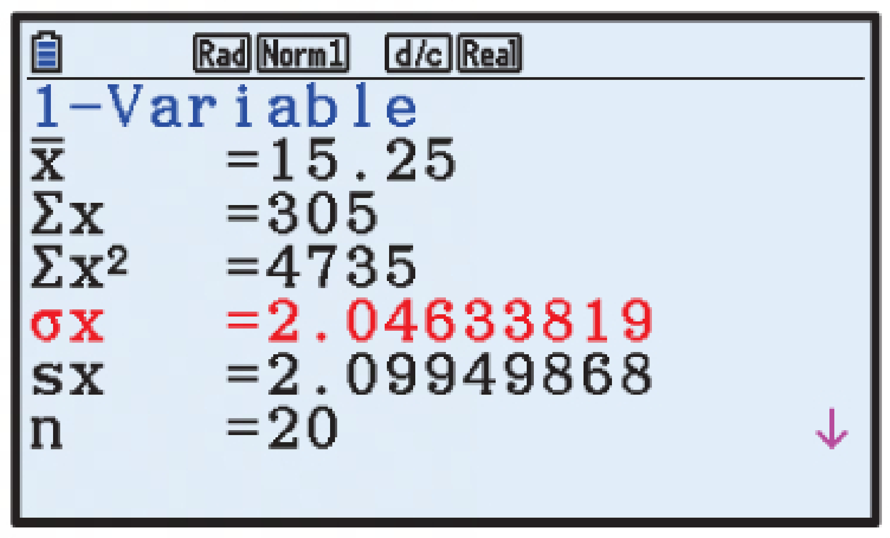
b If k is added to each data value, then k will be added to the original mean but the standard deviation will not change.

c **i** The new data set is:

15	13	14	14	16	13	20	18	14	16
13	12	16	14	17	17	14	14	17	18

$$\begin{aligned}\text{The mean} &= \frac{15+13+14+\dots+17+18}{20}\\ &= \frac{305}{20}\\ &= 15.25\\ &= 4.25+11\end{aligned}$$

Using technology,
the population standard deviation ≈ 2.05.

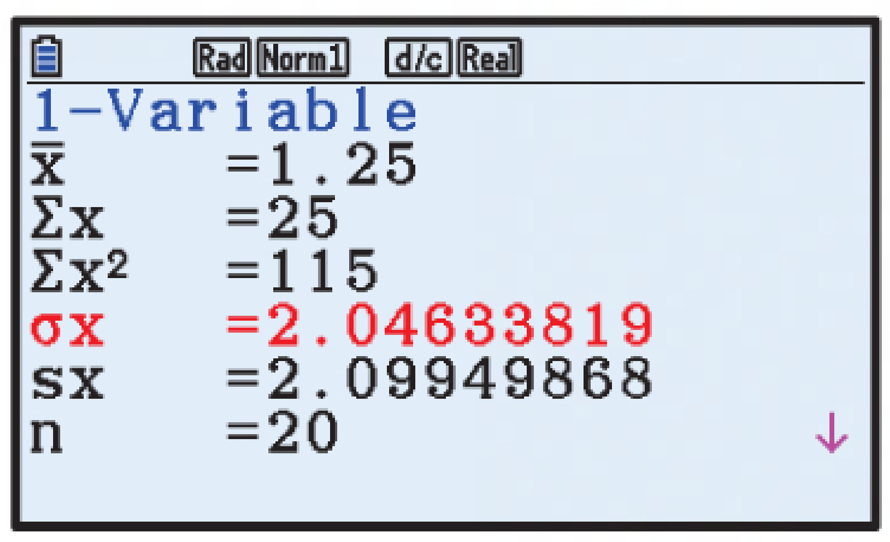


ii The new data set is:

1	−1	0	0	2	−1	6	4	0	2
−1	−2	2	0	3	3	0	0	3	4

$$\begin{aligned}\text{The mean} &= \frac{1+(-1)+0+\dots+3+4}{20}\\ &= \frac{25}{20}\\ &= 1.25\\ &= 4.25-3\end{aligned}$$

Using technology,
the population standard deviation ≈ 2.05.



3 **a** The new data set is:

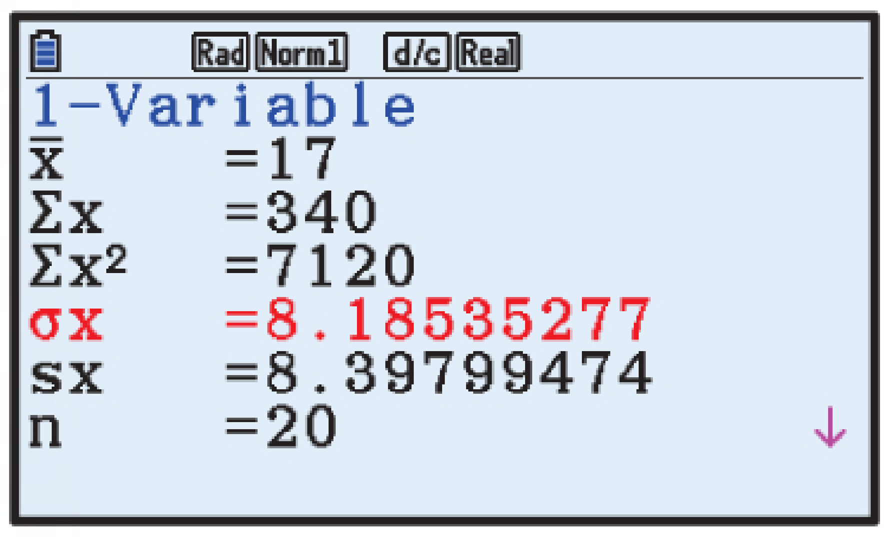
16	8	12	12	20	8	36	28	12	20
8	4	20	12	24	24	12	12	24	28

$$\begin{aligned}\text{The mean} &= \frac{16+8+12+....+24+28}{20}\\ &= \frac{340}{20}\\ &= 17\\ &= 4.25\times 4\end{aligned}$$

Using technology,

the population standard deviation ≈ 8.19

$\approx 2.05 \times 4$



b If each data value is multiplied by a, we expect the mean and standard deviation will be multiplied by a.

c **i** The new data set is:

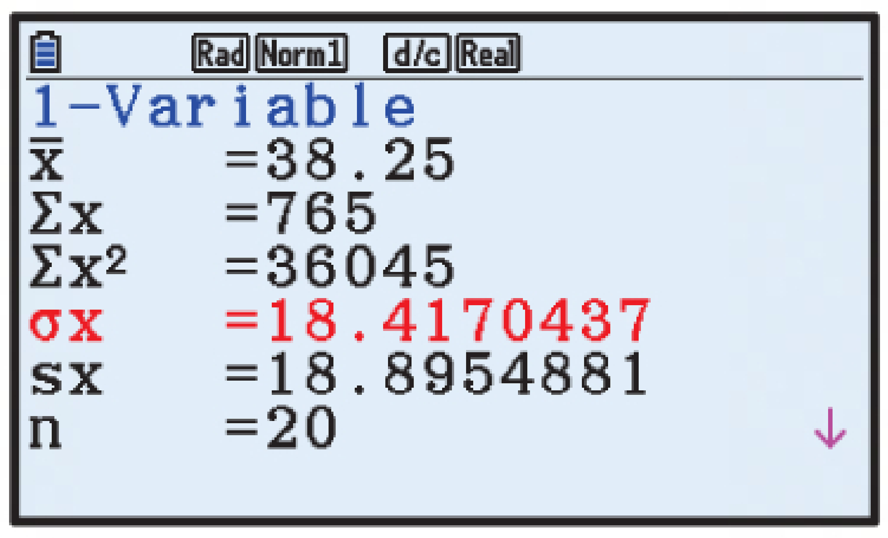
36	18	27	27	45	18	81	63	27	45
18	9	45	27	54	54	27	27	54	63

$$\begin{aligned}\text{The mean} &= \frac{36+18+27+....+54+63}{20}\\ &= \frac{765}{20}\\ &= 38.25\\ &= 4.25\times 9\end{aligned}$$

Using technology,

the population standard deviation ≈ 18.4

$\approx 2.05 \times 9$



ii The new data set is:

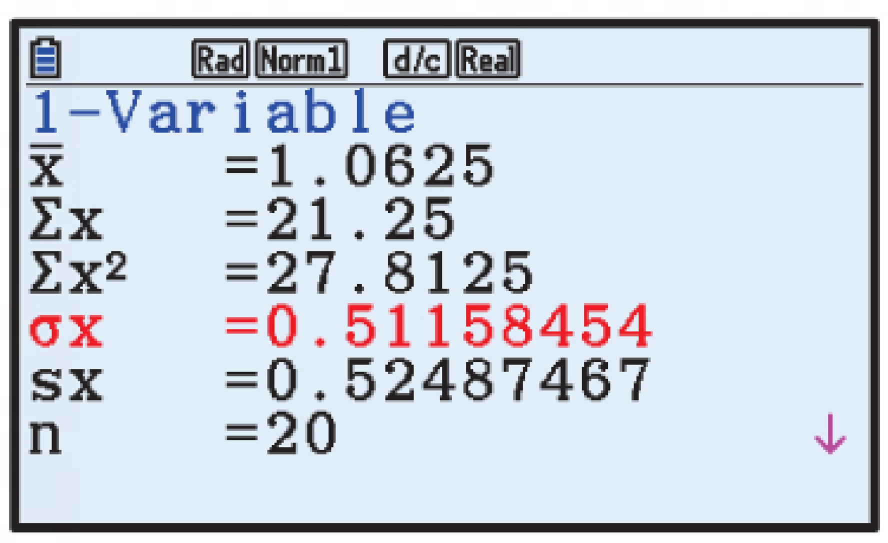
1	0.5	0.75	0.75	1.25	0.5	2.25	1.75	0.75	1.25
0.5	0.25	1.25	0.75	1.5	1.5	0.75	0.75	1.5	1.75

$$\begin{aligned}\text{The mean} &= \frac{1+0.5+0.75+....+1.5+1.75}{20}\\ &= \frac{21.25}{20}\\ &= 1.0625\\ &= 4.25\times \tfrac{1}{4}\end{aligned}$$

Using technology,

the population standard deviation ≈ 0.512

$\approx 2.05 \times \frac{1}{4}$

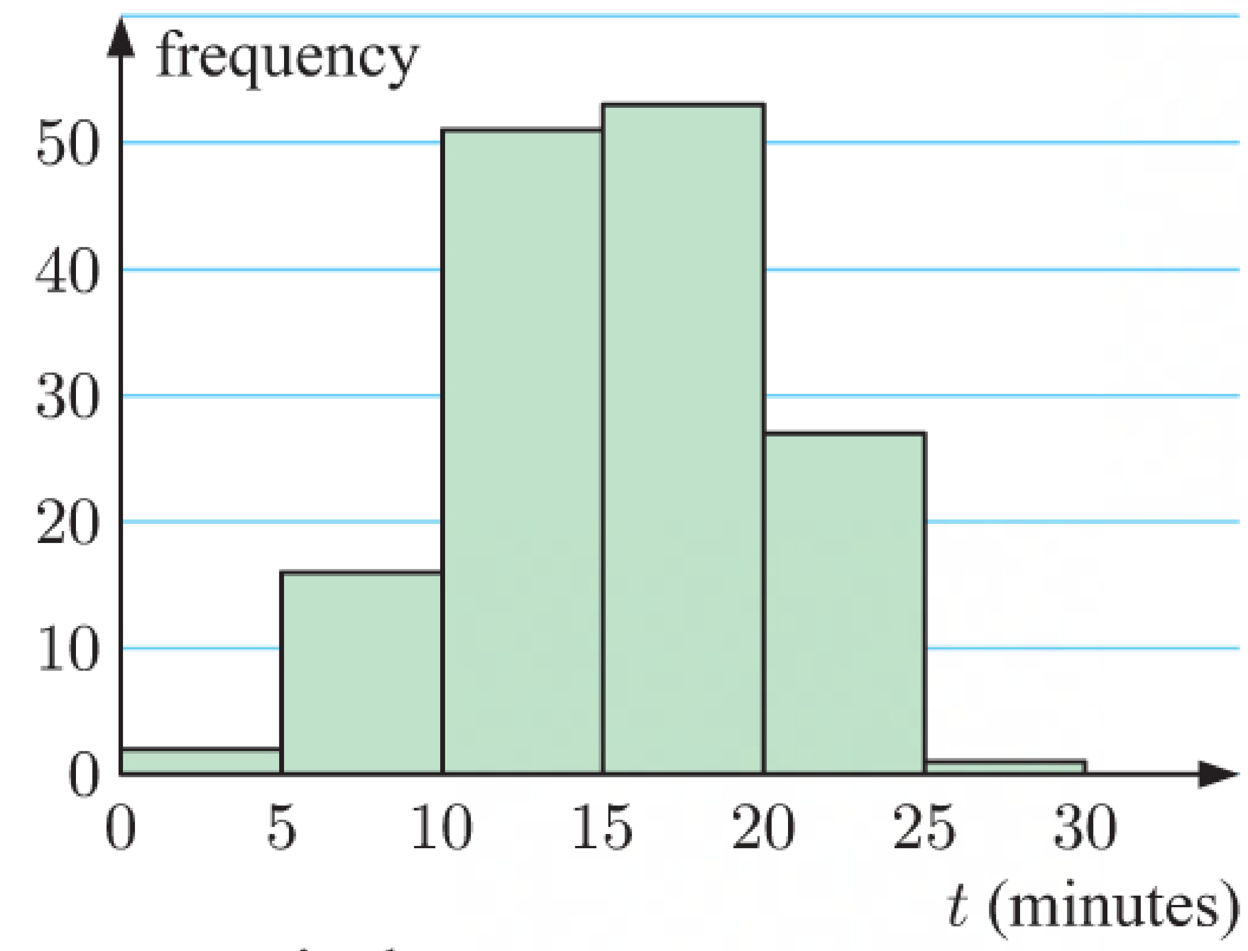


4 **a** mean $= a\mu$, standard deviation $= a\sigma$ **b** mean $= \mu + k$, standard deviation $= \sigma$
c mean $= a\mu + k$, standard deviation $= a\sigma$

INVESTIGATION 4 ESTIMATING THE VARIANCE AND STANDARD DEVIATION OF A POPULATION

1 **a**

Time (t minutes)	*Frequency* (f)
$0 \leqslant t < 5$	2
$5 \leqslant t < 10$	16
$10 \leqslant t < 15$	51
$15 \leqslant t < 20$	53
$20 \leqslant t < 25$	27
$25 \leqslant t < 30$	1
$t \geqslant 30$	0



The distribution of the data is approximately symmetrical.

b From the spreadsheet, the true population standard deviation ≈ 4.521.

$\therefore$ true population variance $=$ (true population standard deviation)^2

$\approx (4.521)^2$

≈ 20.439

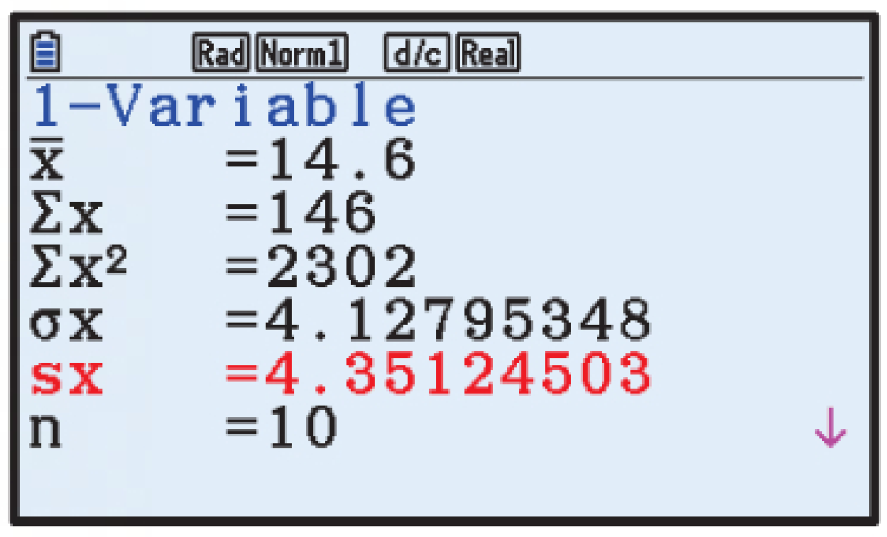
2 **a** *Sample 1*:

Using technology,

the sample standard deviation $s \approx 4.351$

$\therefore\ s^2 \approx (4.351)^2$

≈ 18.933



Repeating this process with the remaining samples, we get:

Sample	1	2	3	4	5	6
s	≈ 4.351	≈ 3.689	≈ 4.517	≈ 3.659	≈ 5.582	≈ 5.340
s^2	≈ 18.933	≈ 13.611	≈ 20.4	≈ 13.389	≈ 31.156	≈ 29.156

b

Sample	1	2	3	4	5	6
σ	≈ 4.128	3.5	≈ 4.285	≈ 3.471	≈ 5.295	≈ 5.123
σ^2	17.04	12.25	18.36	12.05	28.04	26.24

c To help us judge which estimates are closer to the true values, we calculate the absolute difference of each estimate from the true population values:

	Sample						*Average*
	1	2	3	4	5	6	
s	≈ 0.170	≈ 0.832	≈ 0.004	≈ 0.862	≈ 1.061	≈ 0.879	≈ 0.635
s^2	≈ 1.507	≈ 6.829	≈ 0.0404	≈ 7.051	≈ 10.716	≈ 8.716	≈ 5.810
σ	≈ 0.393	≈ 1.021	≈ 0.236	≈ 1.050	≈ 0.774	≈ 0.601	≈ 0.679
σ^2	≈ 3.399	≈ 8.189	≈ 2.079	≈ 8.389	≈ 7.601	≈ 5.801	≈ 5.910

So, on average s is closer to the true standard deviation and s^2 is closer to the true variance.

d Yes, the formulae for the sample statistics s and s^2 generally produce estimates which are closer to the true standard deviation and variance respectively.

3 From the spreadsheet:

	Average estimate	
Parameter	*Sample statistic*	*Population statistic*
Standard deviation	$s \approx 4.981$	$\sigma \approx 4.833$
Variance	$s^2 \approx 24.806$	$\sigma^2 \approx 23.358$

Based on these results, the sample estimates are generally closer to the true values $\sigma = 5$ and $\sigma^2 = 25$. This agrees with our answer to **2 c**.

4 **Note:** The following answers are examples only.

Changing the true mean $\mu = 40$, we obtain:

	Average estimate	
Parameter	*Sample statistic*	*Population statistic*
Standard deviation	$s \approx 4.997$	$\sigma \approx 4.851$
Variance	$s^2 \approx 24.970$	$\sigma^2 \approx 23.532$

The sample estimates are still generally closer to the true values.

Changing the true standard deviation $\sigma = 10$, we obtain:

	Average estimate	
Parameter	*Sample statistic*	*Population statistic*
Standard deviation	$s \approx 10.006$	$\sigma \approx 9.753$
Variance	$s^2 \approx 100.127$	$\sigma^2 \approx 95.121$

Again, the sample estimates are generally closer to the true values.

Considering the above results, changing μ or σ does not affect the conclusion.

5 Having accurate estimates of the variance and standard deviation of a population is important when we use these statistics in our inference.

For example, suppose we wanted to simulate the population using the standard deviation or variance as one of the parameters. We would want our estimates to be as close to the actual values as possible, so that our simulation matches what we observe in reality.

REVIEW SET 13A

1 **a** **i** mean $= \dfrac{0+2+3+3+4+5+5+6+6+7+7+8}{12}$

$= \dfrac{56}{12}$

≈ 4.67

ii As $n = 12$, $\dfrac{n+1}{2} = 6.5$, so the median is the average of the 6th and 7th ordered data values.

The ordered data set is: ~~0 2 3 3 4~~ 5 5 ~~6 6 7 7 8~~

$\therefore$ median $= \dfrac{\text{6th value} + \text{7th value}}{2} = \dfrac{5+5}{2} = 5$

b **i** mean $= \dfrac{2.9+3.1+3.7+3.8+3.9+3.9+4.0+4.5+4.7+5.4}{10}$

$= \dfrac{39.9}{10}$

$= 3.99$

ii As $n = 10$, $\dfrac{n+1}{2} = 5.5$, so the median is the average of the 5th and 6th ordered data values.

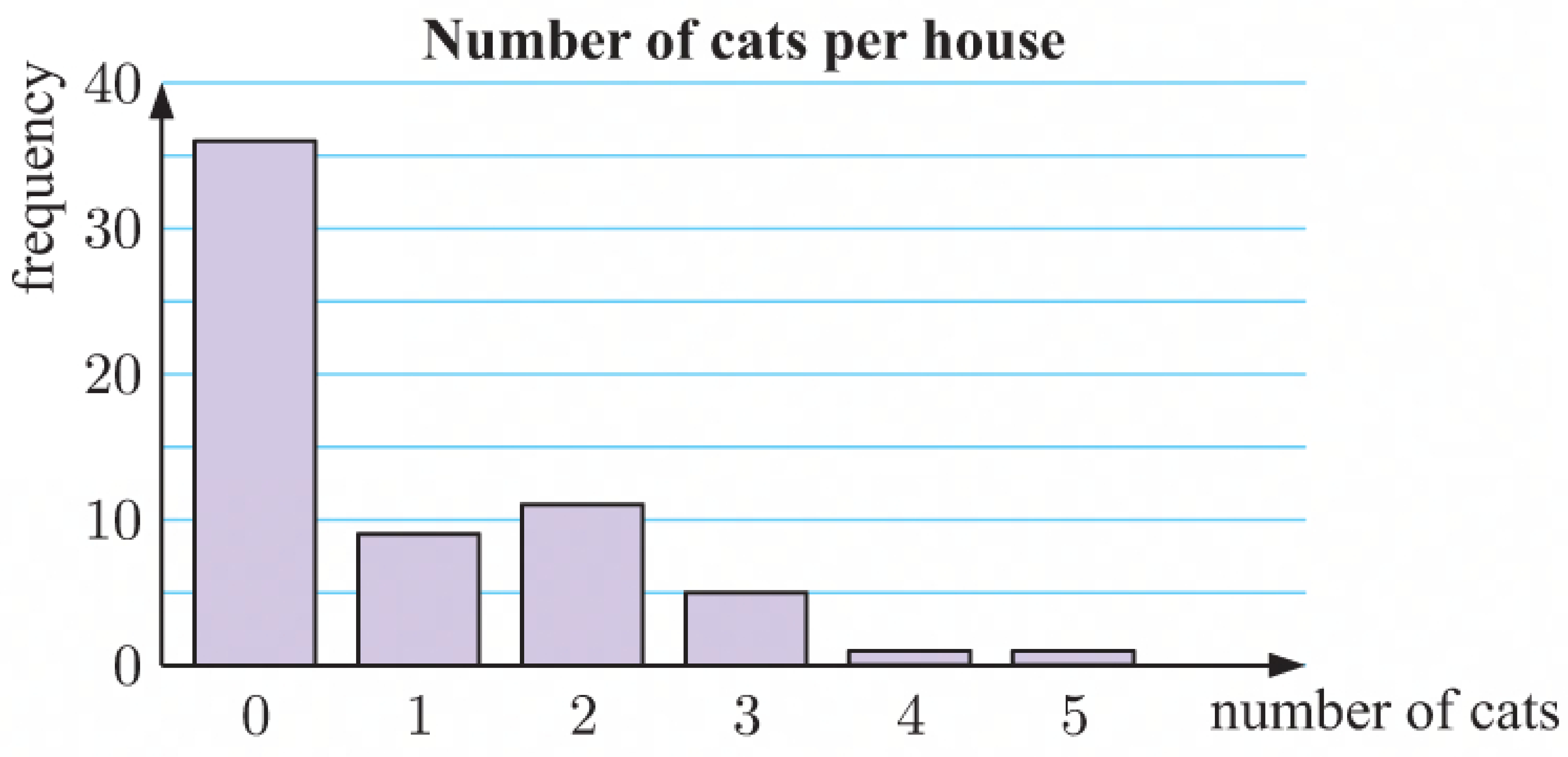
The ordered data set is: ~~2.9 3.1 3.7 3.8~~ 3.9 3.9 ~~4.0 4.5 4.7 5.4~~

$\therefore$ median $= \dfrac{\text{5th value} + \text{6th value}}{2} = \dfrac{3.9+3.9}{2} = 3.9$

2

Number of cats (x)	*Frequency* (f)	*Product* (xf)	*Cumulative frequency*
0	36	0	36
1	9	9	45
2	11	22	56
3	5	15	61
4	1	4	62
5	1	5	63
Total	$\sum f = 63$	$\sum xf = 55$	

a



b The data is positively skewed.

c **i** Looking down the frequency column, the highest frequency is 36. This corresponds to 0 cats, so the mode is 0 cats.

ii $\overline{x} = \dfrac{\sum xf}{\sum f}$

$= \dfrac{55}{63}$

≈ 0.873 cats

iii There are 63 data values, so $n = 63$. $\dfrac{n+1}{2} = 32$, so the median is the 32nd ordered data value.

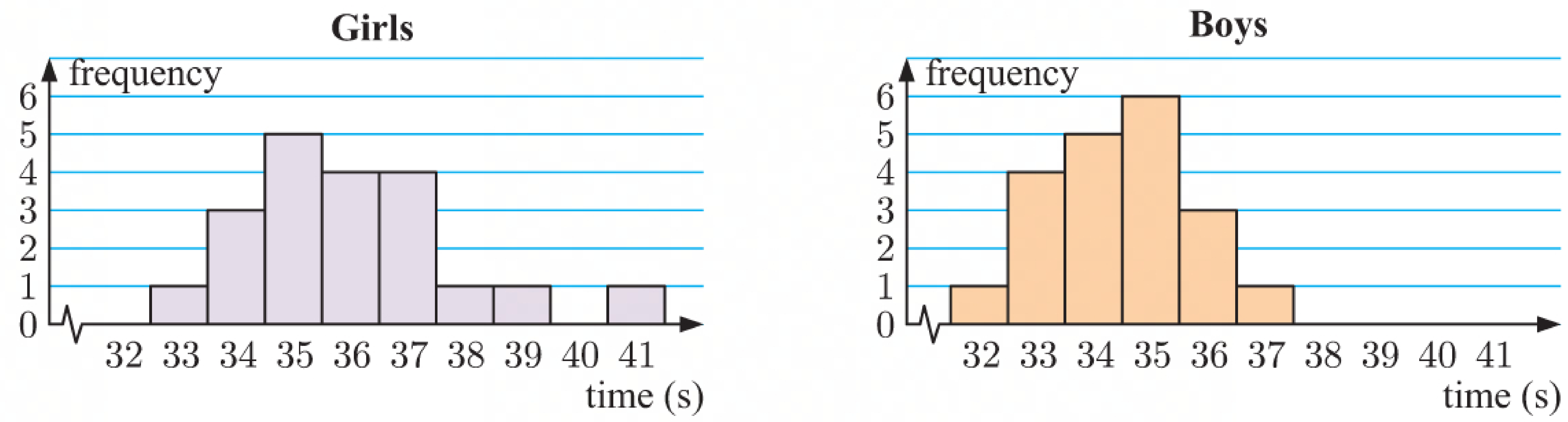
From the cumulative frequency column, the 1st to 36th ordered data values are 0 cats.

$\therefore$ the 32nd ordered data value is 0 cats.

$\therefore$ median $= 0$ cats

d The mean is the most appropriate measure of centre for this data as it does at least suggest that some people have cats, whereas the mode and median are both 0 which suggests that no one has any cats.

3



We first organise the data into tables:

Girls:

Time (s)	*Frequency* (f)	*Midpoint* (x)	*Product* (xf)	*Cumulative frequency*
32.5 - 33.5	1	33	33	1
33.5 - 34.5	3	34	102	4
34.5 - 35.5	5	35	175	9
35.5 - 36.5	4	36	144	13
36.5 - 37.5	4	37	148	17
37.5 - 38.5	1	38	38	18
38.5 - 39.5	1	39	39	19
39.5 - 40.5	0	40	0	19
40.5 - 41.5	1	41	41	20
Total	$\sum f = 20$		$\sum xf = 720$	

Boys:

Time (s)	*Frequency* (f)	*Midpoint* (x)	*Product* (xf)	*Cumulative frequency*
31.5 - 32.5	1	32	32	1
32.5 - 33.5	4	33	132	5
33.5 - 34.5	5	34	170	10
34.5 - 35.5	6	35	210	16
35.5 - 36.5	3	36	108	19
36.5 - 37.5	1	37	37	20
Total	$\sum f = 20$		$\sum xf = 689$	

a There are 20 data values for each data set, so $n = 20$. $\frac{n+1}{2} = 10.5$, so the median is the average of the 10th and 11th ordered data values.

From the cumulative frequency column for the girls' data set, the 10th to 13th ordered data values are 36 s.

$\therefore$ the 10th and 11th ordered data values are both 36 s.

$$\therefore \text{ median} = \frac{36+36}{2} = 36 \text{ s}$$

From the cumulative frequency column for the boys' data set, the 6th to 10th data values are 34 s and the 11th to 16th data values are 35 s.

$\therefore$ the 10th ordered data value is 34 s and the 11th ordered data value is 35 s.

$$\therefore \text{ median} = \frac{34+35}{2} = 34.5 \text{ s}$$

The mean of the girls' data set is $\overline{x} = \frac{\sum xf}{\sum f} = \frac{720}{20} = 36$ s

The mean of the boys' data set is $\overline{x} = \frac{\sum xf}{\sum f} = \frac{689}{20} = 34.45$ s

Looking at the histograms, the highest column for the girls' data is a frequency of 5 which corresponds to the interval 34.5 - 35.5 s. So the modal class is 34.5 - 35.5 s.

Similarly, the highest column for the boys' data is a frequency of 6 which corresponds to the interval 34.5 - 35.5 s. So the modal class is 34.5 - 35.5 s.

So, the table is:

Distribution	*Girls*	*Boys*
median	36 s	34.5 s
mean	36 s	34.45 s
modal class	34.5 - 35.5 s	34.5 - 35.5 s

b The girls' distribution is positively skewed and the boys' distribution is approximately symmetrical. The median and mean swim times for boys are both about 1.5 seconds lower than for girls. Despite this, the distributions have the same modal class because of the skewness in the girls' distribution.

The analysis supports the conjecture that boys generally swim faster than girls with less spread of times.

4 If the mode is 6, then one of the unknown numbers must be 6.
Suppose the other unknown number is x.

$$\therefore \quad \frac{4+6+9+6+3+x}{6} = 6 \quad \{\text{since mean} = 6\}$$
$$\therefore \quad 28 + x = 36$$
$$\therefore \quad x = 8$$

Since $a > b$, then $a = 8$ and $b = 6$.

5 **a**
$$\begin{aligned} \text{mean} &= \frac{(k-2)+k+(k+3)+(k+3)}{4} \\ &= \frac{4k+4}{4} \\ &= \frac{4(k+1)}{4} \\ &= k+1 \end{aligned}$$

b If each number in the data set is increased by 2, then the data set becomes $k,\ k+2,\ k+5,\ k+5$.

$$\begin{aligned} \text{new mean} &= \frac{k+(k+2)+(k+5)+(k+5)}{4} \\ &= \frac{4k+12}{4} \\ &= \frac{4(k+3)}{4} \\ &= k+3 \end{aligned}$$

6 **a** We do not know each individual data value, only the intervals they fall in, so we cannot calculate the mean winning margin exactly.

b

Margin (points)	*Frequency* (f)	*Midpoint* (x)	*Product* (xf)
1 - 10	13	5.5	71.5
11 - 20	35	15.5	542.5
21 - 30	27	25.5	688.5
31 - 40	18	35.5	639
41 - 50	7	45.5	318.5
Total	$\sum f = 100$		$\sum xf = 2260$

$$\begin{aligned} \overline{x} &= \frac{\sum xf}{\sum f} \\ &= \frac{2260}{100} \\ &= 22.6 \end{aligned}$$

$\therefore$ the mean winning margin is about 22.6 points.

7 **a** The ordered data set is:

3 7 8 10 11 13 14 14 14 15 15 16 18 18 19 19 19 22 28 31 $(n = 20)$

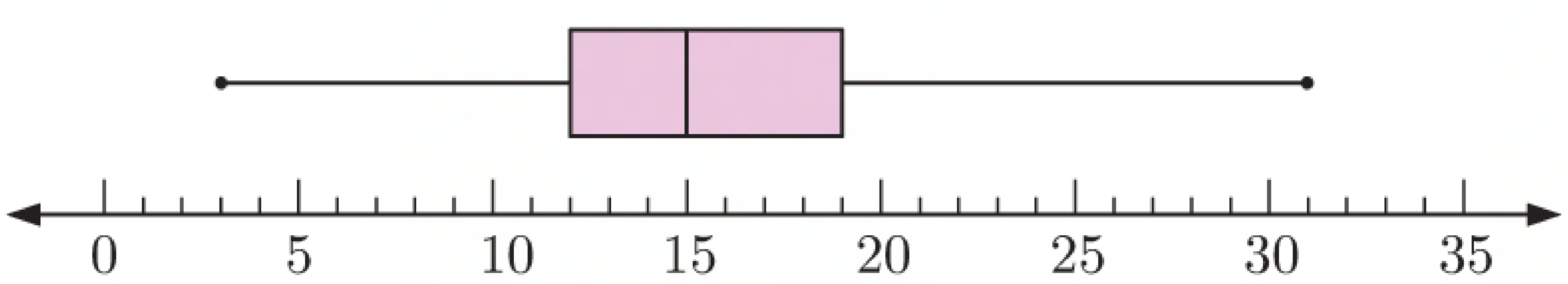
min $= 3$ $\quad$ $Q_1 = 12$ $\quad$ median $= 15$ $\quad$ $Q_3 = 19$ $\quad$ max $= 31$

So the five-number summary is: $\begin{cases} \text{minimum} = 3 & Q_1 = 12 \\ \text{median} = 15 & Q_3 = 19 \\ \text{maximum} = 31 & \end{cases}$

b range $=$ maximum $-$ minimum
$= 31 - 3$
$= 28$

IQR $= Q_3 - Q_1$
$= 19 - 12$
$= 7$

c



8 **a** Since $n = 20$, $\dfrac{n+1}{2} = 10.5$, so the median is the average of the 10th and 11th ordered data values.

The ordered data set is:

~~81 84 90 95 98 98 99 100 101~~ 101 102 ~~103 104 104 105 106 106 107 108 112~~

(20 data values)

$$\therefore \text{ median} = \frac{\text{10th value} + \text{11th value}}{2}$$
$$= \frac{101 + 102}{2}$$
$$= 101.5$$

b We have an even number of data values, so we include all data values when we split the data set into two:

$$\overbrace{81\ 84\ 90\ 95\ 98\ 98\ 99\ 100\ 101\ 101}^{\text{lower half}}\ \overbrace{102\ 103\ 104\ 104\ 105\ 106\ 106\ 107\ 108\ 112}^{\text{upper half}}$$

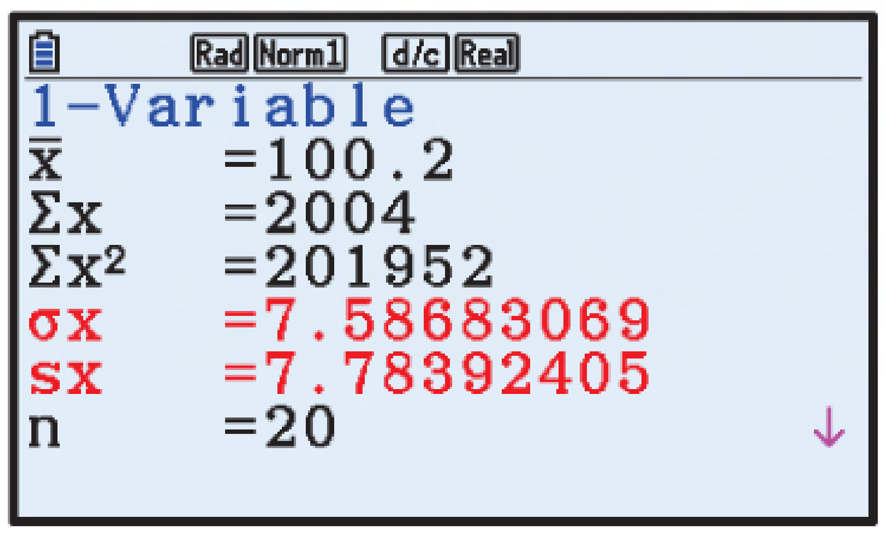
$Q_1 =$ median of lower half $= \dfrac{98 + 98}{2} = 98$

$Q_3 =$ median of upper half $= \dfrac{105 + 106}{2} = 105.5$

IQR $= Q_3 - Q_1$
$= 105.5 - 98$
$= 7.5$

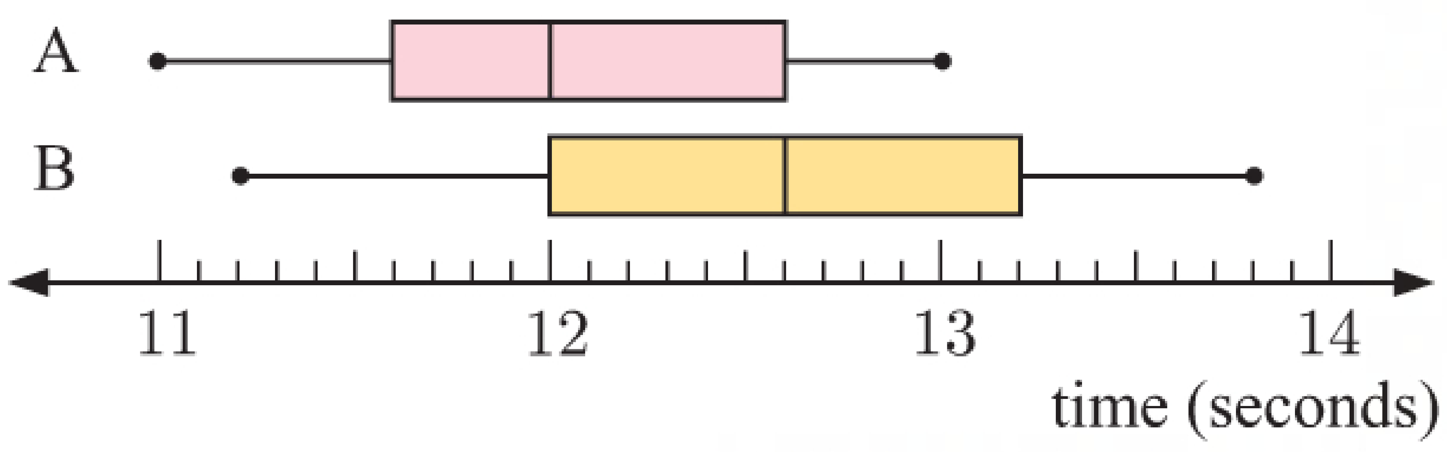
c mean $= \dfrac{90 + 106 + 84 + \ldots. + 102 + 98 + 101}{20}$
$= \dfrac{2004}{20}$
$= 100.2$

d Using technology:



The population standard deviation $\sigma \approx 7.59$, and the sample standard deviation $s \approx 7.78$.

9 **a**



Reading from the box plot, the five-number summaries are:

A: min = 11 s, $Q_1 = 11.6$ s, median = 12 s, $Q_3 = 12.6$ s, max = 13 s

B: min = 11.2 s, $Q_1 = 12$ s, median = 12.6 s, $Q_3 = 13.2$ s, max = 13.8 s

b A: range $= 13 - 11$
$= 2$ s

IQR $= Q_3 - Q_1$
$= 12.6 - 11.6$
$= 1$ s

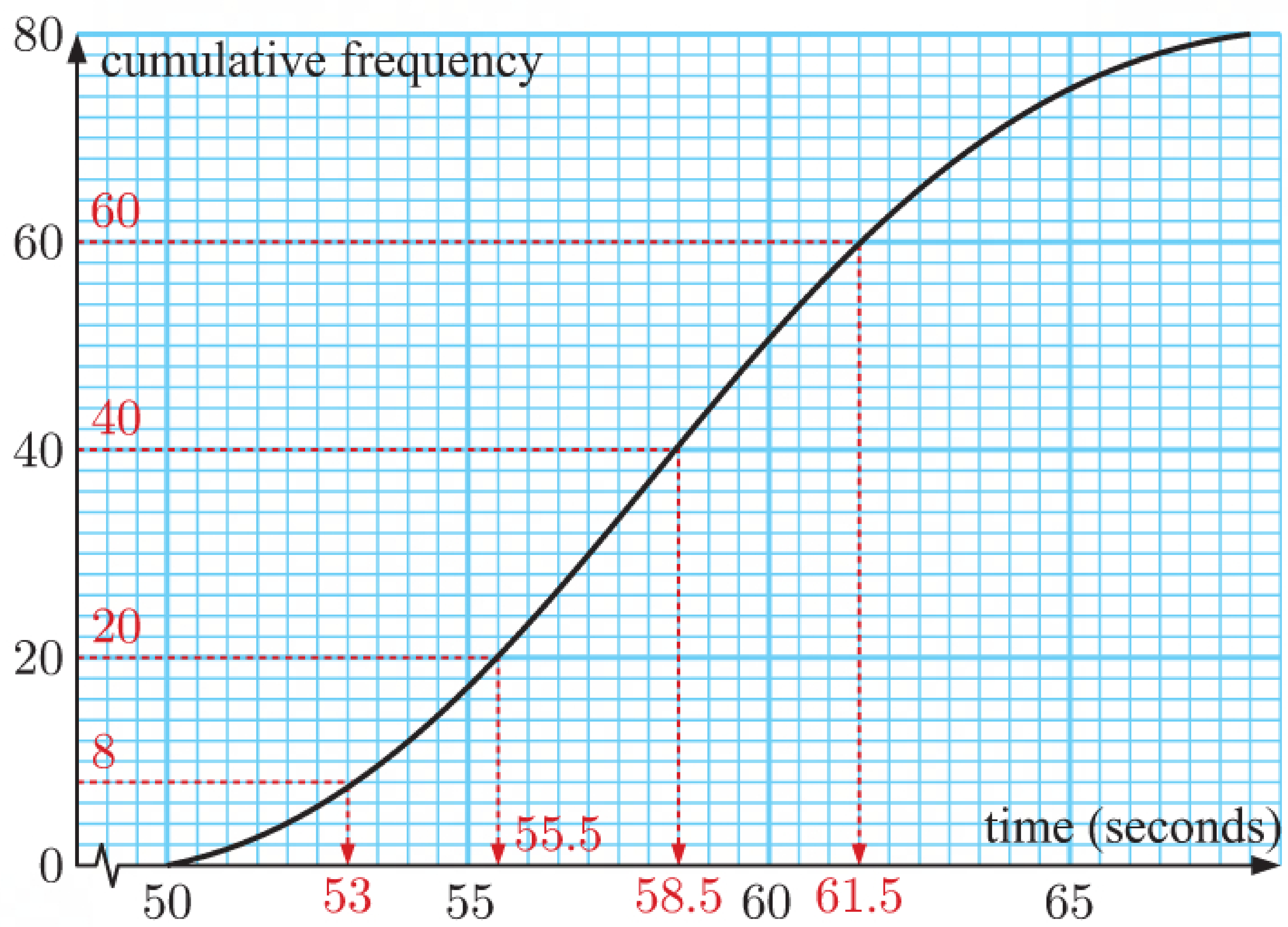
B: range $= 13.8 - 11.2$
$= 2.6$ s

IQR $= Q_3 - Q_1$
$= 13.2 - 12$
$= 1.2$ s

c **i** The members of squad A generally ran faster because their median time is lower.

ii The times in squad B are more varied because their range and IQR are higher.

10



a The median is the 50th percentile. As 50% of 80 is 40, we start with the cumulative frequency 40 and find the corresponding time.

The median ≈ 58.5 s.

b Q_1 is the 25th percentile. As 25% of 80 is 20, we start with the cumulative frequency 20 and find the corresponding time.

$Q_1 \approx 55.5$ s

Q_3 is the 75th percentile. As 75% of 80 is 60, we start with the cumulative frequency 60 and find the corresponding time.

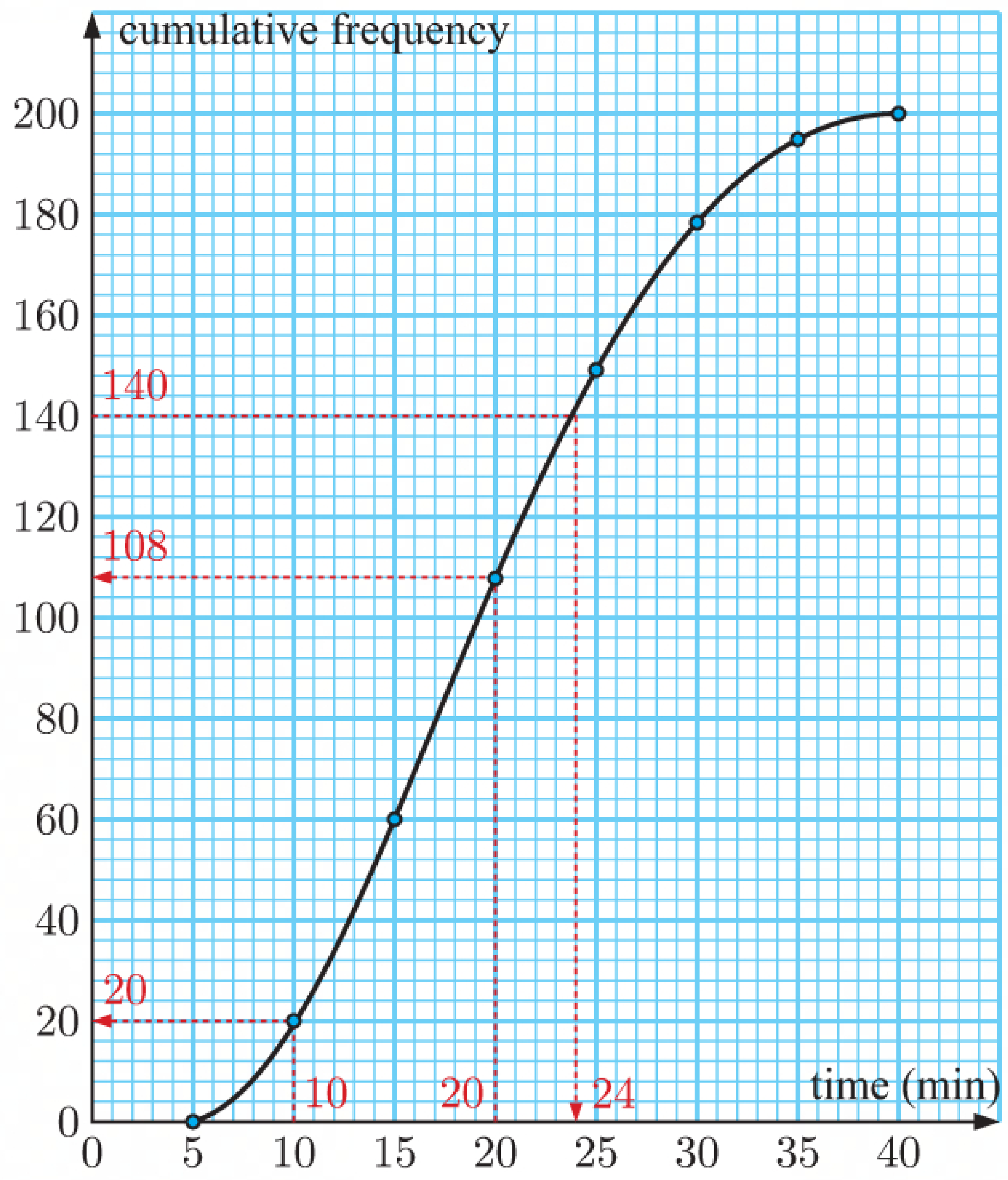
$Q_3 \approx 61.5$ s

$$\begin{aligned} \text{IQR} &= Q_3 - Q_1 \\ &\approx 61.5 \text{ s} - 55.5 \text{ s} \\ &\approx 6 \text{ s} \end{aligned}$$

c As 10% of 80 is 8, we start with the cumulative frequency 8 and find the corresponding time.

The top 10% of runners took less than approximately 53 s.

11



a Approximately 20 students took 10 minutes or less to travel to school by bus.

Approximately 108 students took 20 minutes or less to travel to school by bus.

$\therefore$ approximately $108 - 20 = 88$ students took between 10 and 20 minutes to travel to school by bus.

b As 30% of 200 is 60, we start with the cumulative frequency $200 - 60 = 140$ and find the corresponding time.

Approximately 30% of the students spent more than 24 minutes travelling to school.

$\therefore \quad m \approx 24$

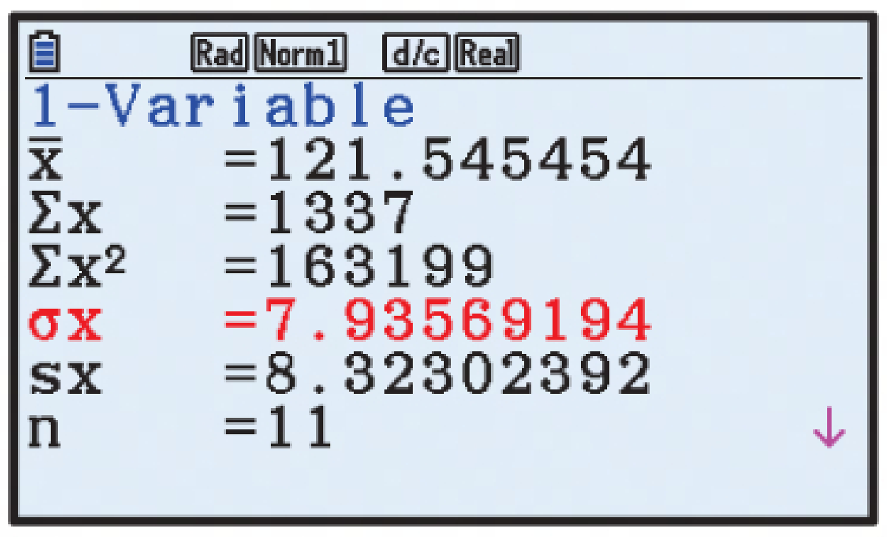
c From the cumulative frequency graph we can obtain the cumulative frequency table:

Time (t min)	*Cumulative frequency*
$5 \leqslant t < 10$	≈ 20
$10 \leqslant t < 15$	≈ 60
$15 \leqslant t < 20$	≈ 108
$20 \leqslant t < 25$	≈ 150
$25 \leqslant t < 30$	≈ 178
$30 \leqslant t < 35$	≈ 195
$35 \leqslant t < 40$	≈ 200

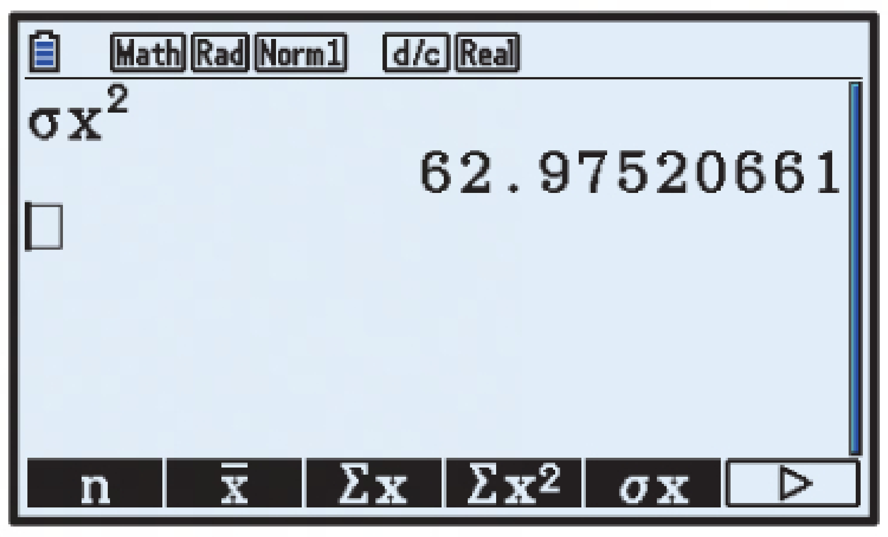
So, the table is:

Time (t min)	*Frequency*
$5 \leqslant t < 10$	$\approx 20 - 0 \approx 20$
$10 \leqslant t < 15$	$\approx 60 - 20 \approx 40$
$15 \leqslant t < 20$	$\approx 108 - 60 \approx 48$
$20 \leqslant t < 25$	$\approx 150 - 108 \approx 42$
$25 \leqslant t < 30$	$\approx 178 - 150 \approx 28$
$30 \leqslant t < 35$	$\approx 195 - 178 \approx 17$
$35 \leqslant t < 40$	$\approx 200 - 195 \approx 5$

12 **a** Using technology:

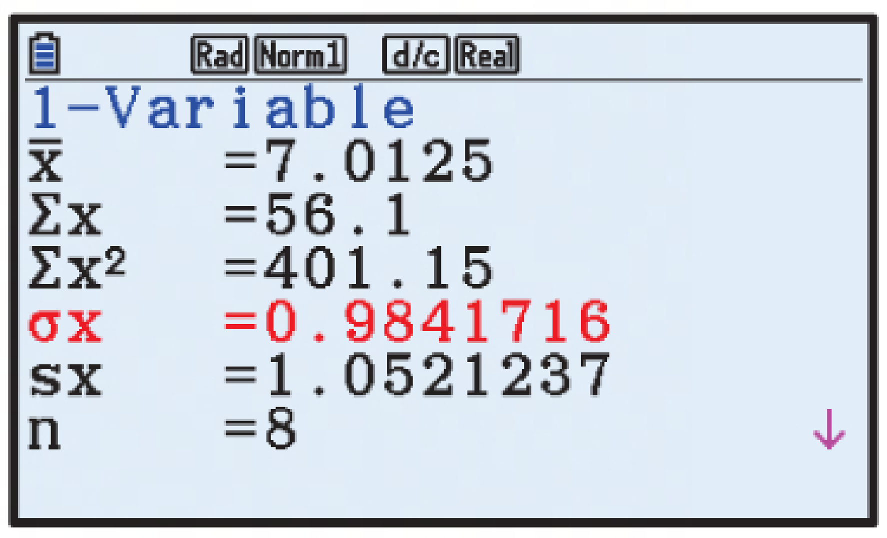


The population standard deviation $\sigma \approx 7.94$.

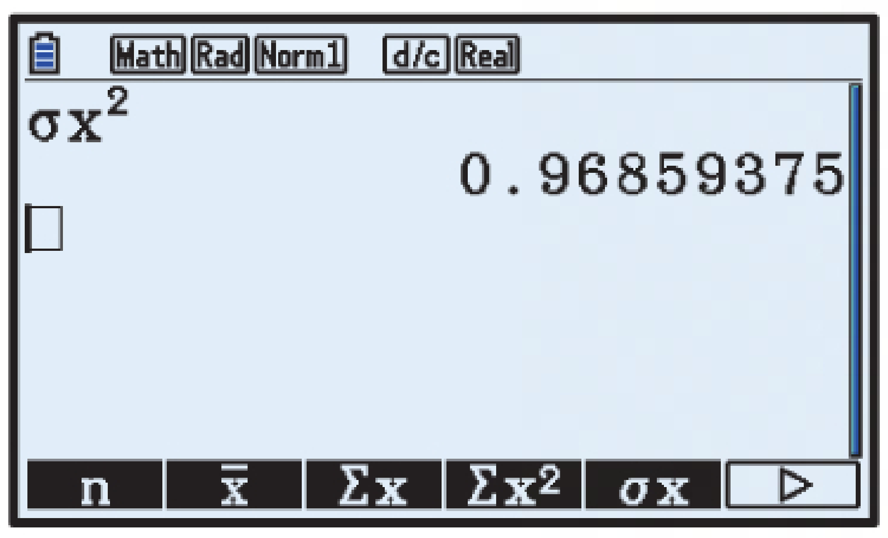


The population variance $\sigma^2 \approx 63.0$.

b Using technology:



The population standard deviation $\sigma \approx 0.984$.

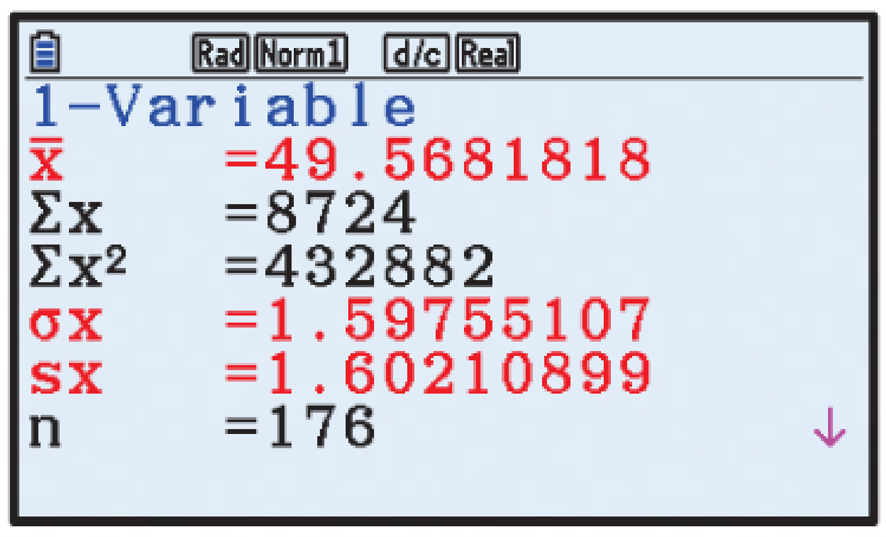


The population variance $\sigma^2 \approx 0.969$.

13

Number	47	48	49	50	51	52
Frequency	21	29	35	42	18	31

a Using technology:



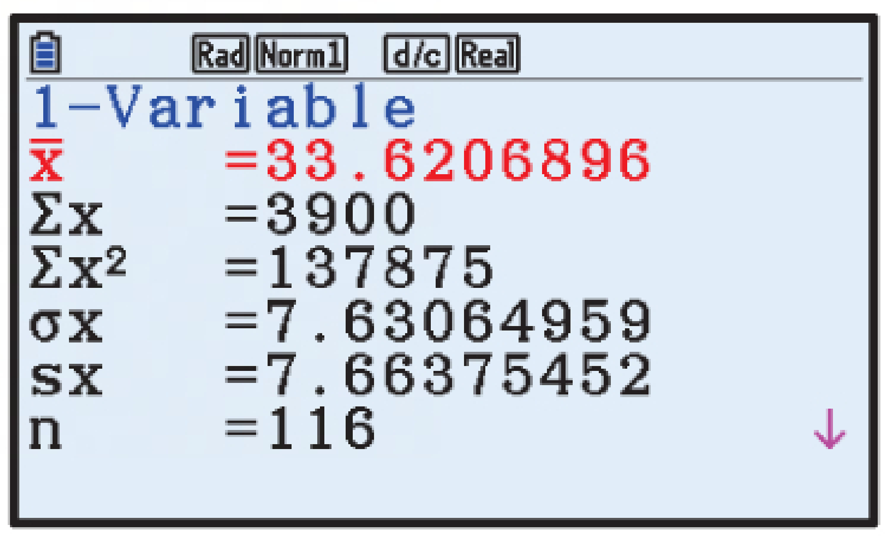
The mean number of matches in a box $\mu \approx 49.6$ matches, the population standard deviation $\sigma \approx 1.60$ matches, and the sample standard deviation $s \approx 1.60$ matches.

b Yes, this result does justify the claim that the average number of matches per box is 50, because the mean $\mu \approx 50$ matches.

14

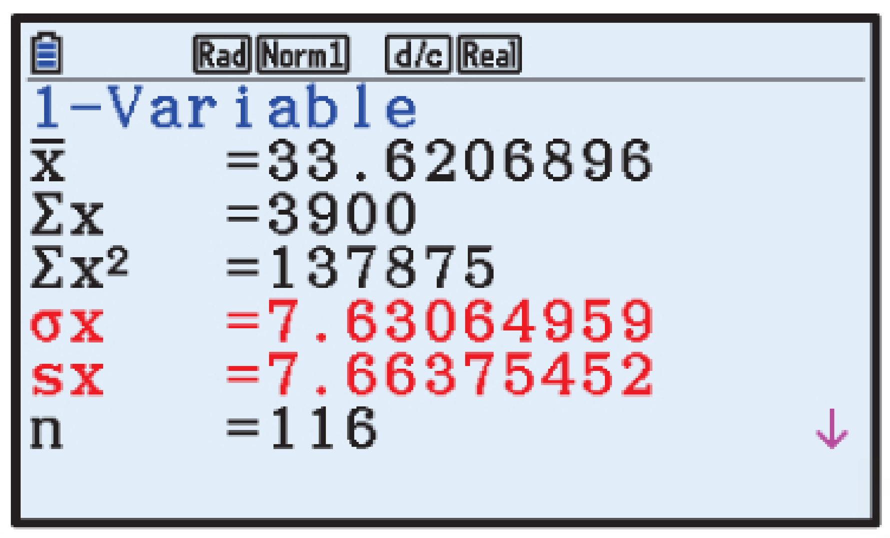
Class interval	*Mid-interval value*	*Frequency*
$15 \leqslant L < 20$	17.5	5
$20 \leqslant L < 25$	22.5	13
$25 \leqslant L < 30$	27.5	17
$30 \leqslant L < 35$	32.5	29
$35 \leqslant L < 40$	37.5	27
$40 \leqslant L < 45$	42.5	18
$45 \leqslant L < 50$	47.5	7

a Using technology:



The mean ≈ 33.6 L.

b Using technology:



The population standard deviation $\sigma \approx 7.63$ L, and the sample standard deviation $s \approx 7.66$ L.

15 **a** Extreme values will have less effect on the standard deviation of a larger population than on a smaller population.

$\therefore$ no, you would not expect the standard deviation for the whole population to be the same for one day as it is for one week.

b **i** The mean would be used to check that an average of 250 g of biscuits goes into each packet.

ii The standard deviation would be used to check the variability of the mass going into each packet.

c A low standard deviation means that the weight of biscuits in each packet is, on average, close to 250 g.

REVIEW SET 13B

1 **a** *Week 1*:

$$\text{mean} = \frac{16.4 + 15.2 + 16.3 + 16.3 + 17.1 + 15.5 + 14.9}{7}$$
$$= \frac{111.7}{7}$$
$$\approx 16.0 \text{ s}$$

As $n = 7$, $\frac{n+1}{2} = 4$

The ordered data set is: $\cancel{14.9}$ $\cancel{15.2}$ $\cancel{15.5}$ 16.3 $\cancel{16.3}$ $\cancel{16.4}$ $\cancel{17.1}$

4th value

$\therefore$ median $= 16.3$ s

Week 2:

$$\text{mean} = \frac{14.9 + 15.7 + 15.1 + 15.1 + 14.7 + 14.7 + 15.3}{7}$$
$$= \frac{105.5}{7}$$
$$\approx 15.1 \text{ s}$$

As $n = 7$, $\frac{n+1}{2} = 4$

The ordered data set is: $\cancel{14.7}$ $\cancel{14.7}$ $\cancel{14.9}$ 15.1 $\cancel{15.1}$ $\cancel{15.3}$ $\cancel{15.7}$

4th value

$\therefore$ median $= 15.1$ s

Week 3:

$$\text{mean} = \frac{14.3 + 14.2 + 14.6 + 14.6 + 14.3 + 14.3 + 14.4}{7}$$
$$= \frac{100.7}{7}$$
$$\approx 14.4 \text{ s}$$

As $n = 7$, $\frac{n+1}{2} = 4$

The ordered data set is: $\cancel{14.2}$ $\cancel{14.3}$ $\cancel{14.3}$ 14.3 $\cancel{14.4}$ $\cancel{14.6}$ $\cancel{14.6}$

4th value

$\therefore$ median $= 14.3$ s

Week 4:

$$\text{mean} = \frac{14.0 + 14.0 + 13.9 + 14.0 + 14.1 + 13.8 + 14.2}{7}$$
$$= \frac{98}{7}$$
$$= 14.0 \text{ s}$$

As $n = 7$, $\frac{n+1}{2} = 4$

The ordered data set is: ~~13.8 13.9 14.0~~ 14.0 ~~14.0 14.1 14.2~~

4th value

$\therefore$ median = 14.0 s

b Yes, Heike's mean and median times have gradually decreased each week which indicates that her speed has improved over the 4 week period.

2 **a** The mode is 5 as this is the data value which occurred most frequently.

b

Value (x)	*Frequency* (f)	*Product* (xf)	*Cumulative frequency*
1	10	10	10
2	7	14	17
3	8	24	25
4	5	20	30
5	12	60	42
6	8	48	50
Total	$\sum f = 50$	$\sum xf = 176$	

$$\overline{x} = \frac{\sum xf}{\sum f} = \frac{176}{50} = 3.52$$

c There are 50 data values, so $n = 50$. $\dfrac{n+1}{2} = 25.5$, so the median is the average of the 25th and 26th ordered data values.

From the cumulative frequency column, the 18th to 25th ordered data values are 3 and the 26th to 30th ordered data values are 4.

$\therefore$ the 25th ordered data value is 3 and the 26th ordered data value is 4.

$$\therefore \text{ median} = \frac{3+4}{2} = 3.5$$

3 **a**

Score (s)	*Frequency* (f)	*Product* (sf)	*Cumulative frequency*
2	3	6	3
5	2	10	5
x	4	$4x$	9
$x+6$	1	$x+6$	10
Total	$\sum f = 10$	$\sum sf = 5x+22$	

$$\overline{x} = \frac{\sum sf}{\sum f}$$

$$\therefore\ 5.7 = \frac{5x+22}{10}$$

$$\therefore\ 57 = 5x+22$$

$$\therefore\ 5x = 35$$

$$\therefore\ x = 7$$

b There are 10 data values, so $n = 10$. $\dfrac{n+1}{2} = 5.5$, so the median is the average of the 5th and 6th ordered data values.

From the cumulative frequency column, the 4th and 5th ordered data values are 5 and the 6th to 9th ordered data values are 7.

$\therefore$ the 5th ordered data value is 5 and the 6th ordered data value is 7.

$$\therefore \text{ median} = \frac{5+7}{2} = 6$$

4 If the mode is 7, then one of the unknown numbers must be 7 as there are currently an equal number of 6s and 7s in the list.

Suppose the other unknown number is x.

$$\therefore \quad \frac{6+8+7+7+5+7+6+8+6+9+6+7+7+x}{14} = 7 \qquad \{\text{since mean} = 7\}$$

$$\therefore \quad \frac{89+x}{14} = 7$$

$$\therefore \quad 89 + x = 98$$

$$\therefore \quad x = 9$$

$\therefore \quad p = 7$ and $q = 9$, or $p = 9$ and $q = 7$.

5

Number of patrons	*Frequency* (f)	*Midpoint* (x)	*Product* (xf)
250 - 299	14	274.5	3843
300 - 349	34	324.5	11 033
350 - 399	68	374.5	25 466
400 - 449	72	424.5	30 564
450 - 499	54	474.5	25 623
500 - 549	23	524.5	12 063.5
550 - 599	7	574.5	4021.5
Total	$\sum f = 272$		$\sum xf = 112\,614$

$$\overline{x} = \frac{\sum xf}{\sum f} = \frac{112\,614}{272} \approx 414$$

$\therefore$ the mean number of patrons per day is about 414.

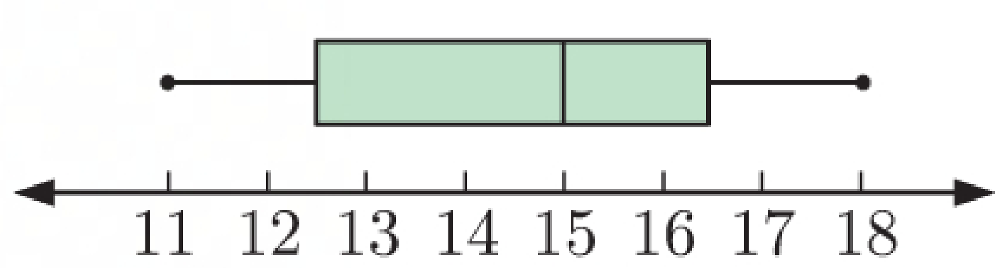
6 The ordered data set is:

11 12 12 13 14 14 15 15 15 16 17 17 18 (13 data values)

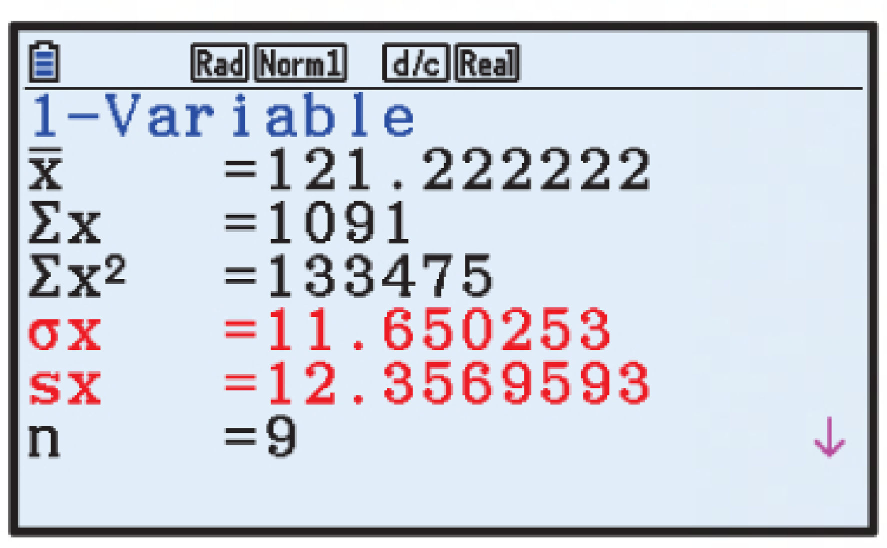
min = 11 $\quad Q_1 = 12.5 \quad$ median = 15 $\quad Q_3 = 16.5 \quad$ max = 18

The five-number summary is: $\begin{cases} \text{minimum} = 11 & Q_1 = 12.5 \\ \text{median} = 15 & Q_3 = 16.5 \\ \text{maximum} = 18 & \end{cases}$

So, the box and whisker diagram is:



7 **a** Using technology:



The population standard deviation $\sigma \approx 11.7$, and the sample standard deviation $s \approx 12.4$.

b The ordered data set is:

93 116 118 120 122 127 128 132 135 (9 data values)

$Q_1 = 117$ median $= 122$ $Q_3 = 130$

So, $Q_1 = 117$ and $Q_3 = 130$.

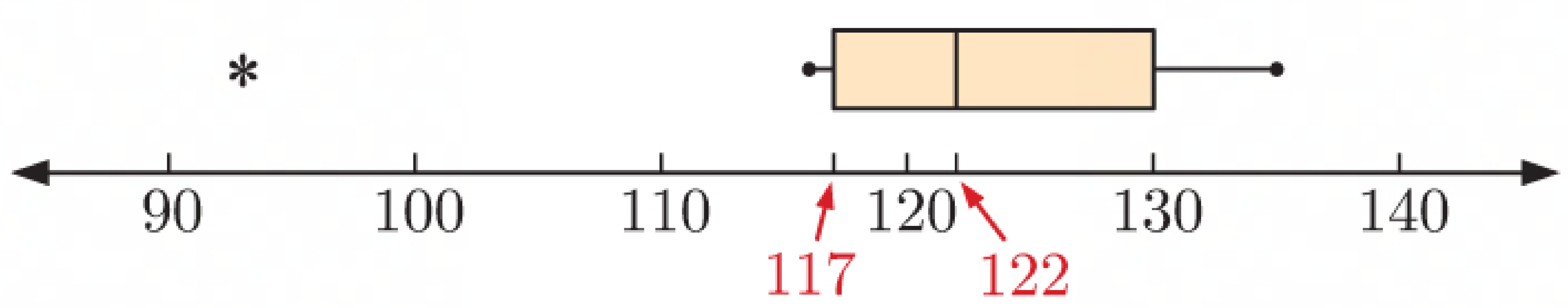
c $\text{IQR} = Q_3 - Q_1$
$= 130 - 117$
$= 13$

lower boundary
$=$ lower quartile $- 1.5 \times \text{IQR}$
$= 117 - 1.5 \times 13$
$= 97.5$

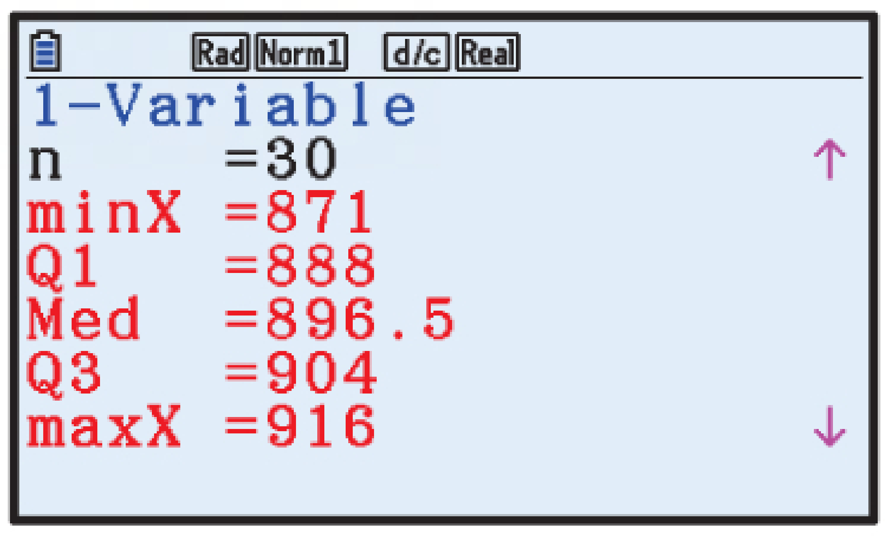
upper boundary
$=$ upper quartile $+ 1.5 \times \text{IQR}$
$= 130 + 1.5 \times 13$
$= 149.5$

93 is below the lower boundary, so it is an outlier.

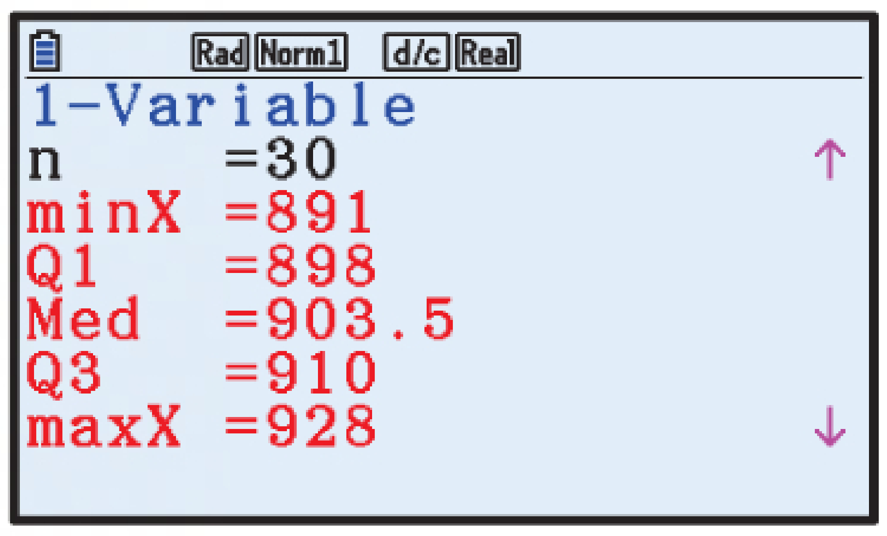
d



8 a *Brand X*:



Brand Y:



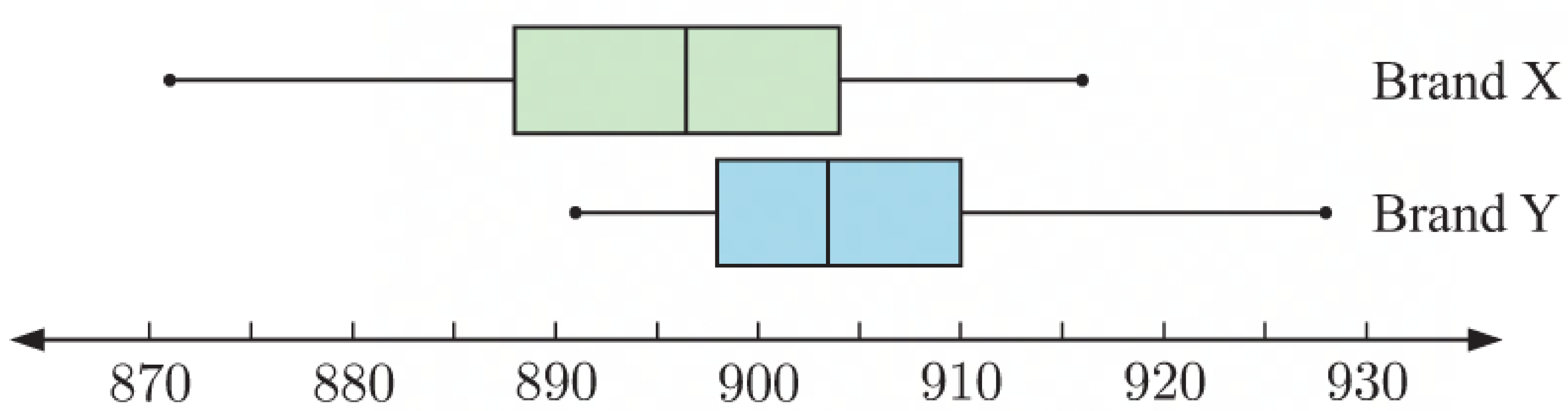
$\text{IQR} = Q_3 - Q_1$
$= 904 - 888$
$= 16$

$\text{IQR} = Q_3 - Q_1$
$= 910 - 898$
$= 12$

So, the table is:

	Brand X	*Brand Y*
min	871	891
Q_1	888	898
median	896.5	903.5
Q_3	904	910
max	916	928
IQR	16	12

b



c **i** The median is higher for brand Y than for brand X, so we would expect brand Y to have more peanuts per jar.

ii The IQR is lower for brand Y than for brand X, so we would expect brand Y to have a more consistent number of peanuts per jar.

9

Score	*Frequency*	*Cumulative frequency*
6	2	2
7	4	m
8	7	13
9	p	25
10	5	30

a $2+4=m$ and $13+p=25$
$\therefore\ m=6$ $\therefore\ p=12$

b The highest frequency is $p=12$. This corresponds to a score of 9, so the mode is 9.

There are 30 data values, so $n=30$. $\dfrac{n+1}{2}=15.5$, so the median is the average of the 15th and 16th ordered data values.

From the cumulative frequency column, the 14th to 25th ordered data values are 9.

$\therefore$ the 15th and 16th ordered data values are 9.

$$\therefore\ \text{median} = \frac{9+9}{2} = 9$$

$$\text{range} = \text{maximum} - \text{minimum} = 10-6 = 4$$

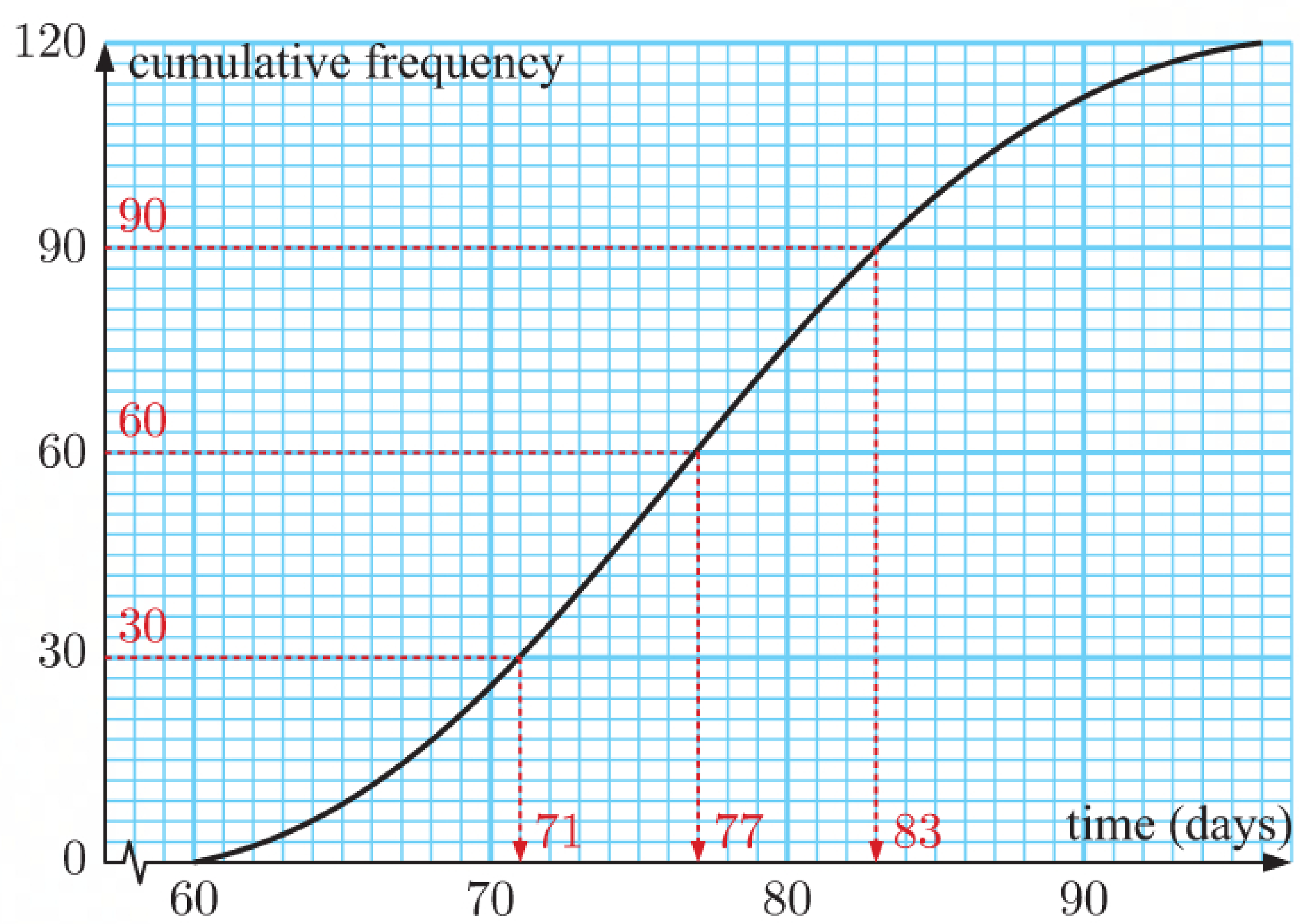
c The mean $\overline{x} = \dfrac{\text{sum of all data values}}{\text{number of data values}}$

$$= \frac{\sum xf}{\sum f}$$

$$= \frac{254}{30} \quad \{\text{there are 30 data values}\}$$

$$= \tfrac{127}{15}$$

10



a The median is the 50th percentile. As 50% of 120 is 60, we start with the cumulative frequency 60 and find the corresponding time.

The median $\approx$ 77 days.

b The lower quartile is the 25th percentile. As 25% of 120 is 30, we start with the cumulative frequency 30 and find the corresponding time.

$Q_1 \approx 71$ days

The upper quartile is the 75th percentile. As 75% of 120 is 90, we start with the cumulative frequency 90 and find the corresponding time.

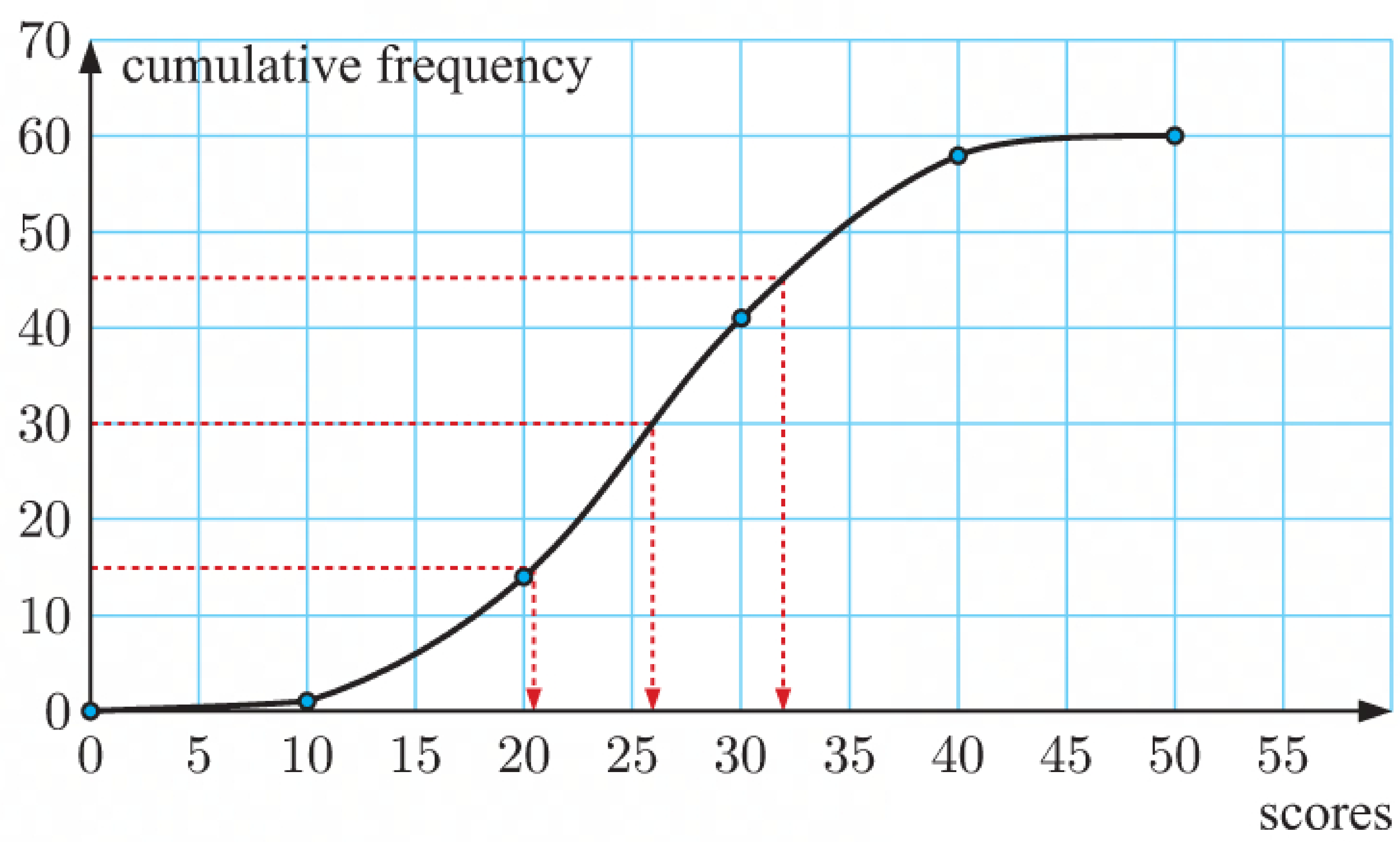
$Q_3 \approx 83$ days

$$\begin{aligned}\text{IQR} &= Q_3 - Q_1 \\ &\approx 83 - 71 \text{ days} \\ &\approx 12 \text{ days}\end{aligned}$$

11

Scores (x)	*Frequency*	*Cumulative frequency*
$0 \leqslant x < 10$	1	1
$10 \leqslant x < 20$	13	14
$20 \leqslant x < 30$	27	41
$30 \leqslant x < 40$	17	58
$40 \leqslant x < 50$	2	60

a



b **i** The median is the 50th percentile. As 50% of 60 is 30, we start with the cumulative frequency 30 and find the corresponding score.

The median $\approx$ 26.

ii The lower quartile is the 25th percentile. As 25% of 60 is 15, we start with the cumulative frequency 15 and find the corresponding score.

$Q_1 \approx 20$

The upper quartile is the 75th percentile. As 75% of 60 is 45, we start with the cumulative frequency 45 and find the corresponding score.

$Q_3 \approx 32$

$$\begin{aligned}\text{IQR} &= Q_3 - Q_1 \\ &\approx 32 - 20 \\ &\approx 12\end{aligned}$$

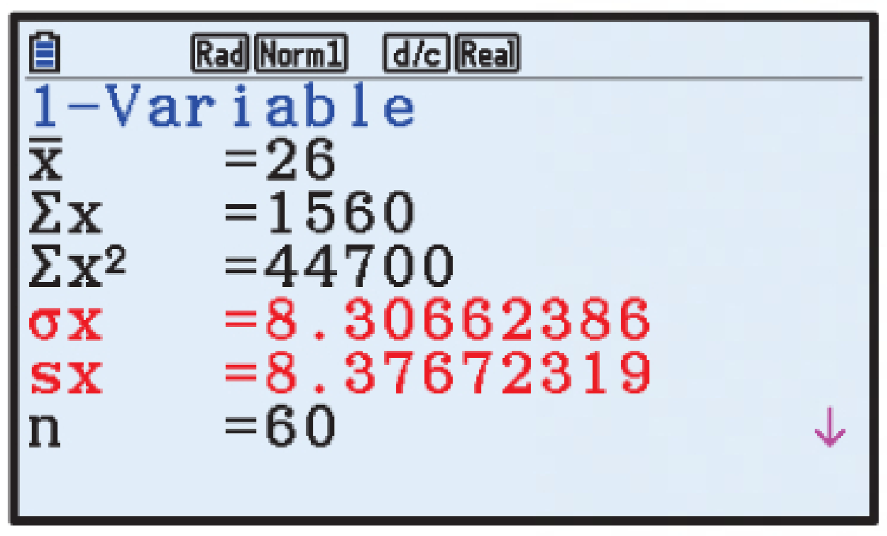
iii

Scores	*Frequency* (f)	*Midpoint* (x)	*Product* (xf)
$0 \leqslant x < 10$	1	5	5
$10 \leqslant x < 20$	13	15	195
$20 \leqslant x < 30$	27	25	675
$30 \leqslant x < 40$	17	35	595
$40 \leqslant x < 50$	2	45	90
Total	$\sum f = 60$		$\sum xf = 1560$

$$\overline{x} = \frac{\sum xf}{\sum f}$$
$$= \frac{1560}{60}$$
$$= 26$$

$\therefore$ the mean of the data set is about 26.

iv Using technology:



The population standard deviation $\sigma \approx 8.31$, and the sample standard deviation $s \approx 8.38$.

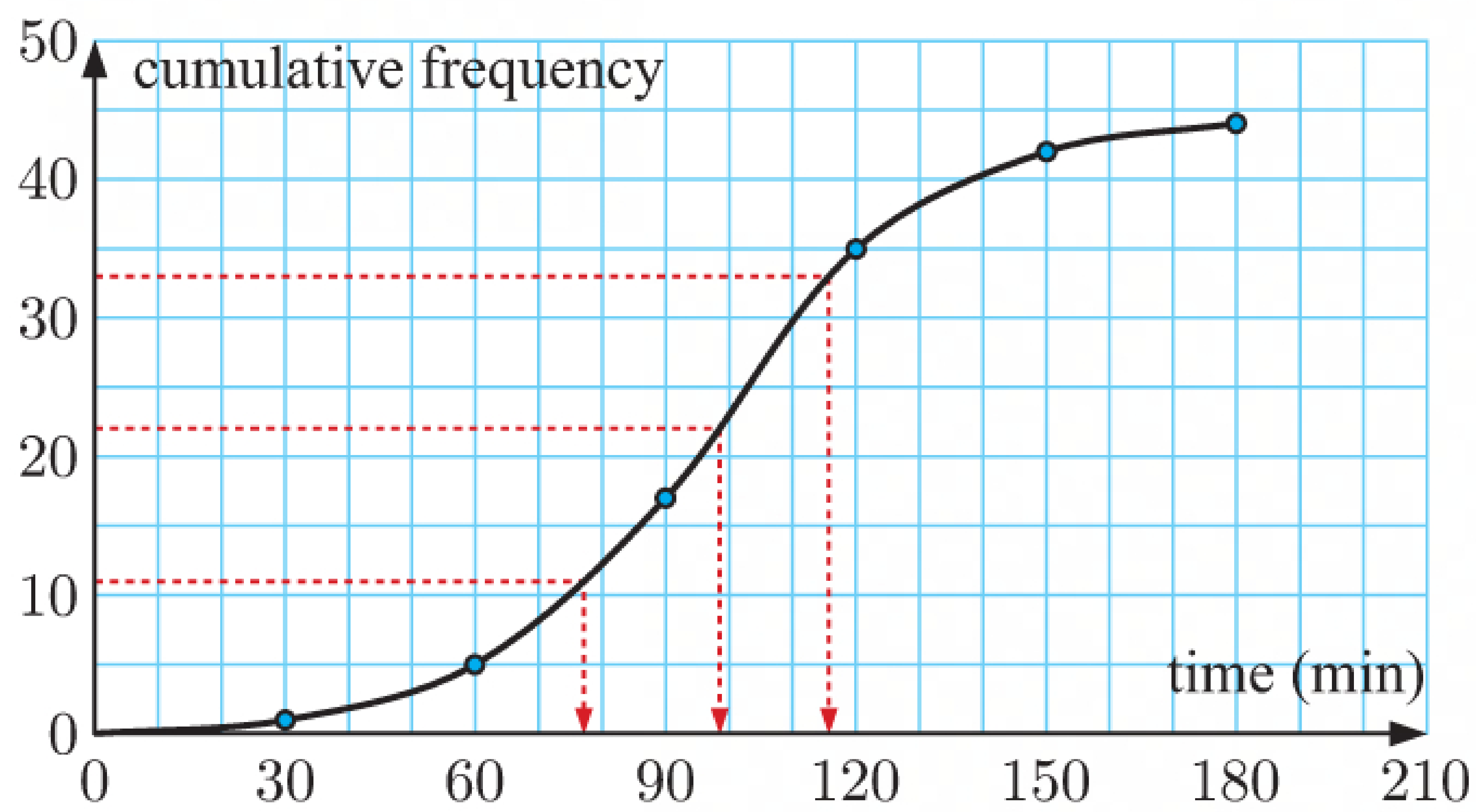
12

Completion time (t min)	*Number of players*	*Cumulative frequency*
$0 \leqslant t < 30$	1	1
$30 \leqslant t < 60$	4	5
$60 \leqslant t < 90$	12	17
$90 \leqslant t < 120$	18	35
$120 \leqslant t < 150$	7	42
$150 \leqslant t < 180$	2	44

a There were 44 players surveyed.

b The modal class is $90 \leqslant t < 120$ min as this is the completion time which occurred most often.

c



d **i** The median is the 50th percentile. As 50% of 44 is 22, we start with the cumulative frequency 22 and find the corresponding time.
The median $\approx$ 98 min.

ii

Completion time	*Frequency* (f)	*Midpoint* (x)	*Product* (xf)
$0 \leqslant t < 30$	1	15	15
$30 \leqslant t < 60$	4	45	180
$60 \leqslant t < 90$	12	75	900
$90 \leqslant t < 120$	18	105	1890
$120 \leqslant t < 150$	7	135	945
$150 \leqslant t < 180$	2	165	330
Total	$\sum f = 44$		$\sum xf = 4260$

$$\overline{x} = \frac{\sum xf}{\sum f}$$
$$= \frac{4260}{44}$$
$$\approx 96.8 \text{ min}$$

iii The game is considered too easy if either the mean or median completion time is below 90 minutes. Since both the median and mean are both above 90 minutes, then the game is not considered to be too easy.

e The lower quartile is the 25th percentile. As 25% of 44 is 11, we start with the cumulative frequency 11 and find the corresponding time.
$Q_1 \approx 78$ min

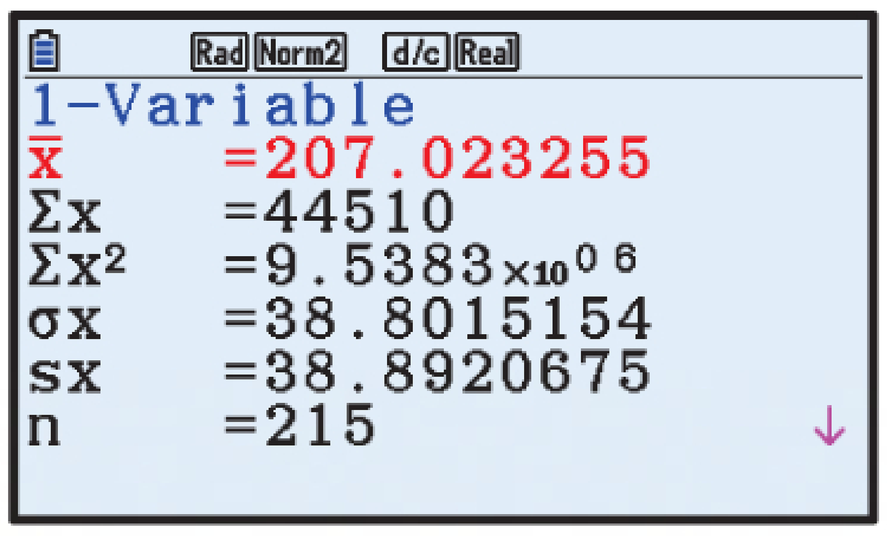
The upper quartile is the 75th percentile. As 75% of 44 is 33, we start with the cumulative frequency 33 and find the corresponding time.
$Q_3 \approx 116$ min

The middle 50% of players completed the game in times between 78 and 116 minutes.

13

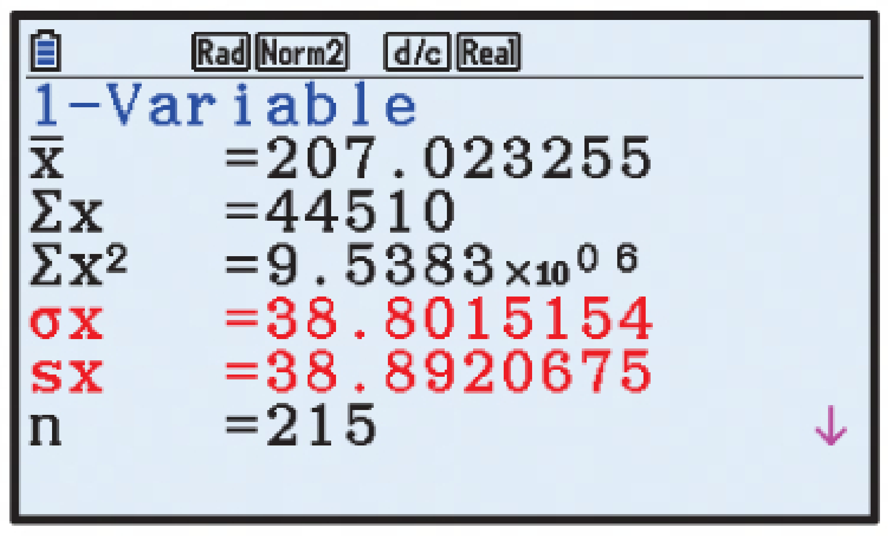
Class interval	*Mid-interval value*	*Frequency*
$140 \leqslant b < 160$	150	27
$160 \leqslant b < 180$	170	32
$180 \leqslant b < 200$	190	48
$200 \leqslant b < 220$	210	25
$220 \leqslant b < 240$	230	37
$240 \leqslant b < 260$	250	21
$260 \leqslant b < 280$	270	18
$280 \leqslant b < 300$	290	7

a Using technology:



The mean $\approx$ €207.02.

b Using technology:



The population standard deviation $\sigma \approx$ €38.80, and the sample standard deviation $s \approx$ €38.89.

14 **a** *Kevin*:

The mean time $\overline{x}$ taken by Kevin to complete a crossword puzzle

$$= \frac{37 + 53 + 47 + 33 + 39 + ... + 39 + 41}{20}$$

$$= \frac{824}{20}$$

$$= 41.2 \text{ minutes}$$

Felicity:

The mean time $\overline{x}$ taken by Felicity to complete a crossword puzzle

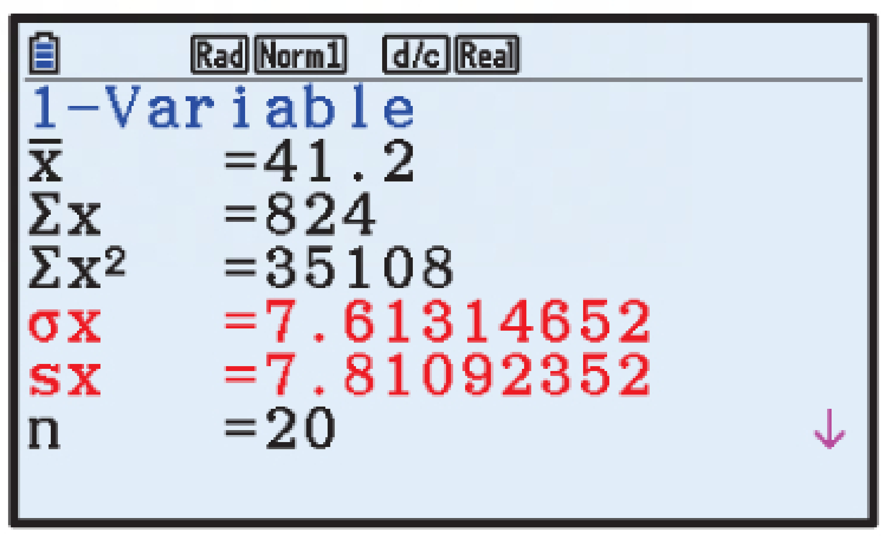
$$= \frac{33 + 36 + 41 + 26 + 52 + ... + 50 + 31}{20}$$

$$= \frac{790}{20}$$

$$= 39.5 \text{ minutes}$$

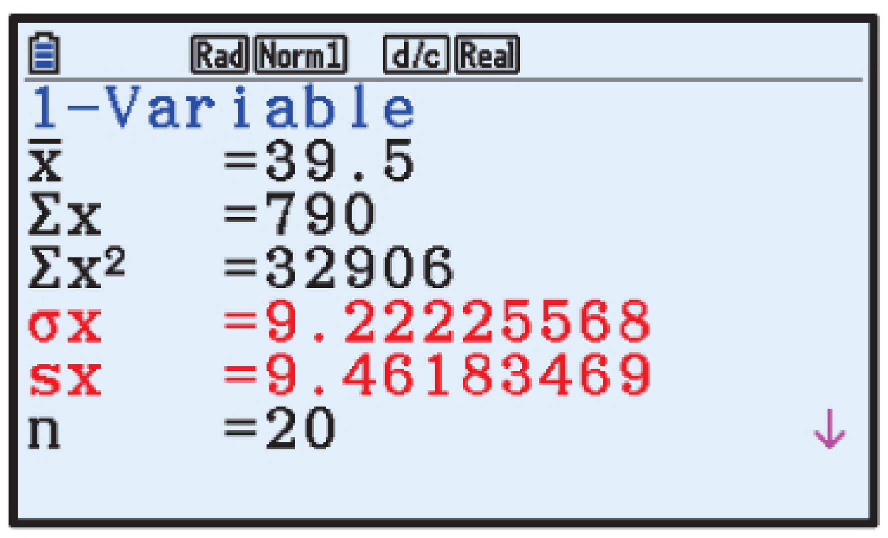
b Using technology:

Kevin:



The population standard deviation $\sigma \approx 7.61$ minutes, and the sample standard deviation $s \approx 7.81$ minutes.

Felicity:



The population standard deviation $\sigma \approx 9.22$ minutes, and the sample standard deviation $s \approx 9.46$ minutes.

c Felicity's mean time is lower than Kevin's, so Felicity generally solves crossword puzzles faster.

d Kevin's population standard deviation is lower than Felicity's, so Kevin is more consistent in his time taken to solve the puzzles.

15 $s^2 = \dfrac{\sum(x-\overline{x})^2}{n-1}$ and $\sigma^2 = \dfrac{\sum(x-\mu)^2}{n}$

$\therefore\ 4.1 = \dfrac{\sum(x-\overline{x})^2}{n-1}$ $\qquad \therefore\ 3.69 = \dfrac{\sum(x-\mu)^2}{n}$

$\therefore\ 4.1(n-1) = \sum(x-\overline{x})^2$ $\qquad \therefore\ \sum(x-\mu)^2 = 3.69n$ .... (2)

$\therefore\ \sum(x-\overline{x})^2 = 4.1n - 4.1$ .... (1)

Equating (1) and (2) where $\overline{x} = \mu$ gives $4.1n - 4.1 = 3.69n$

$$\therefore\ 0.41n = 4.1$$

$$\therefore\ n = 10$$

So, the data set has 10 data values.

Chapter 14

QUADRATIC FUNCTIONS

EXERCISE 14A

1 **a** $y = x^2 - 3x + 1$

x	-2	-1	0	1	2
y	11	5	1	-1	-1

b $y = x^2 + 2x - 5$

x	-2	-1	0	1	2
y	-5	-6	-5	-2	3

c $y = 2x^2 - x + 3$

x	-4	-2	0	2	4
y	39	13	3	9	31

d $y = -3x^2 + 2x + 4$

x	-4	-2	0	2	4
y	-52	-12	4	-4	-36

2 **a** When $x = 0$,

$$\begin{aligned} y &= 2(0)^2 + 5 \\ &= 5 \end{aligned}$$

$\therefore$ $(0, 4)$ does not satisfy the function $y = 2x^2 + 5$.

b When $x = 2$,

$$\begin{aligned} y &= (2)^2 - 3(2) + 2 \\ &= 4 - 6 + 2 \\ &= 0 \end{aligned}$$

$\therefore$ $(2, 0)$ satisfies the function $y = x^2 - 3x + 2$.

c When $x = -1$,

$$\begin{aligned} y &= -(-1)^2 + 2(-1) - 5 \\ &= -1 - 2 - 5 \\ &= -8 \end{aligned}$$

$\therefore$ $(-1, -8)$ satisfies the function $y = -x^2 + 2x - 5$.

d When $x = 3$,

$$\begin{aligned} y &= -2(3)^2 - 3 + 6 \\ &= -18 - 3 + 6 \\ &= -15 \end{aligned}$$

$\therefore$ $(3, -15)$ satisfies the function $y = -2x^2 - x + 6$.

e When $x = 2$,

$$\begin{aligned} y &= 3(2)^2 - 4(2) + 10 \\ &= 12 - 8 + 10 \\ &= 14 \end{aligned}$$

$\therefore$ $(2, 10)$ does not satisfy the function $y = 3x^2 - 4x + 10$.

f When $x = 2$,

$$\begin{aligned} y &= -\tfrac{1}{2}(2)^2 + 4(2) - 1 \\ &= -2 + 8 - 1 \\ &= 5 \end{aligned}$$

$\therefore$ $(2, 5)$ satisfies the function $y = -\frac{1}{2}x^2 + 4x - 1$.

3 **a** If $y = 4$ then

$$\begin{aligned} x^2 + 3x + 6 &= 4 \\ \therefore\ x^2 + 3x + 2 &= 0 \\ \therefore\ (x+1)(x+2) &= 0 \\ \therefore\ x &= -1 \text{ or } -2 \end{aligned}$$

b If $y = 3$ then

$$\begin{aligned} x^2 - 4x + 7 &= 3 \\ \therefore\ x^2 - 4x + 4 &= 0 \\ \therefore\ (x-2)^2 &= 0 \\ \therefore\ x &= 2 \end{aligned}$$

c If $y = -4$ then

$$\begin{aligned} x^2 - 6x + 1 &= -4 \\ \therefore\ x^2 - 6x + 5 &= 0 \\ \therefore\ (x-1)(x-5) &= 0 \\ \therefore\ x &= 1 \text{ or } 5 \end{aligned}$$

d If $y = 4$ then

$$\begin{aligned} 2x^2 + 5x + 1 &= 4 \\ \therefore\ 2x^2 + 5x - 3 &= 0 \\ \therefore\ (2x-1)(x+3) &= 0 \\ \therefore\ x &= \tfrac{1}{2} \text{ or } -3 \end{aligned}$$

e If $y = 1$ then

$$\begin{aligned} \tfrac{1}{2}x^2 + \tfrac{5}{2}x - 2 &= 1 \\ \therefore\ \tfrac{1}{2}x^2 + \tfrac{5}{2}x - 3 &= 0 \\ \therefore\ x^2 + 5x - 6 &= 0 \\ \therefore\ (x+6)(x-1) &= 0 \\ \therefore\ x &= -6 \text{ or } 1 \end{aligned}$$

f If $y = 2$ then

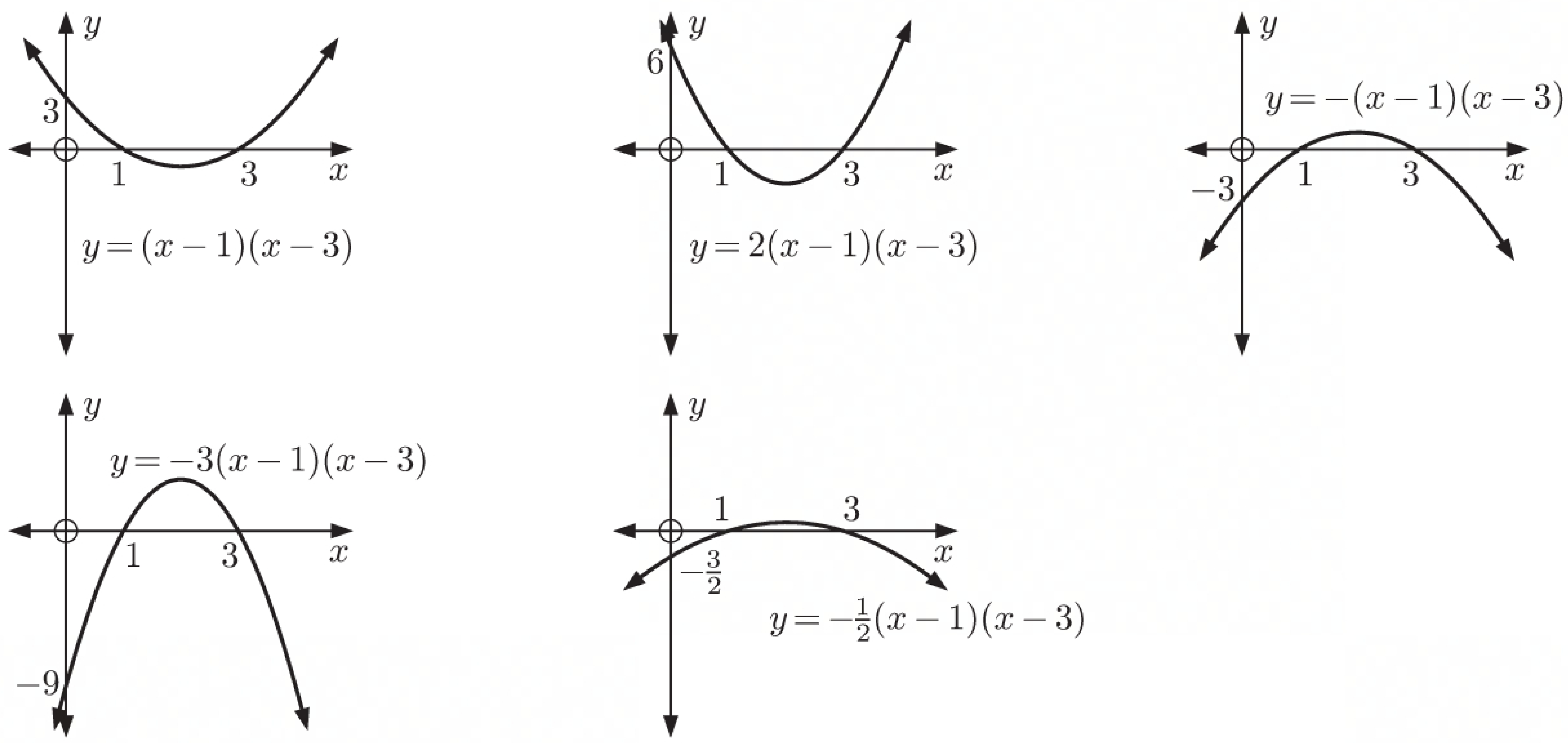
$$\begin{aligned} -\tfrac{1}{2}x^2 + 2x - 1 &= 2 \\ \therefore\ -\tfrac{1}{2}x^2 + 2x - 3 &= 0 \\ \therefore\ x^2 - 4x + 6 &= 0 \end{aligned}$$

which has $\Delta = (-4)^2 - 4(1)(6)$
$= -8 < 0$

$\therefore$ the equation has no real solutions.

INVESTIGATION 1 GRAPHING $y = a(x-p)(x-q)$

1 **a**



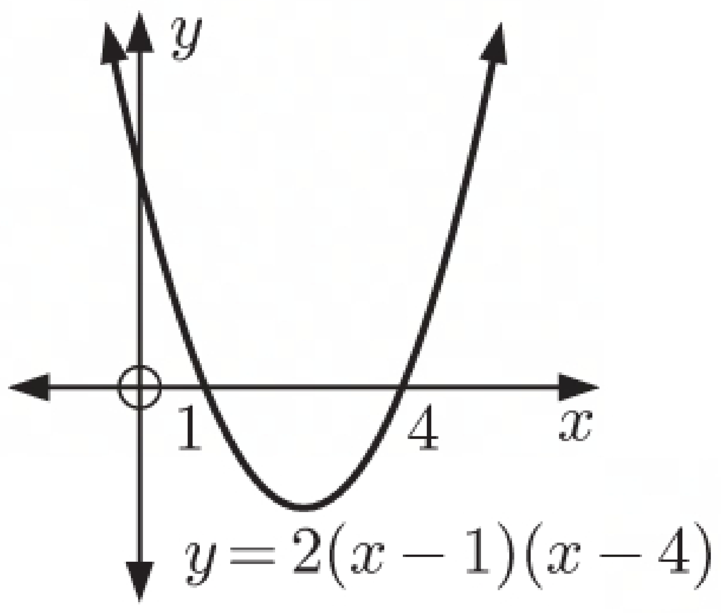
b The x-intercepts for each function in **a** are 1 and 3.

c For $y = a(x-1)(x-3)$: When $a > 0$ the shape is  .

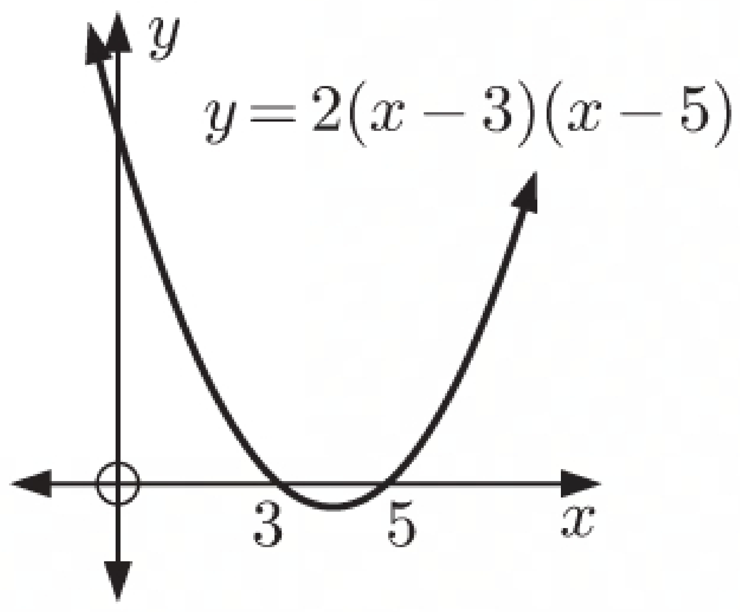
When $a < 0$ the shape is  .

As $|a|$ increases, the graph becomes narrower.

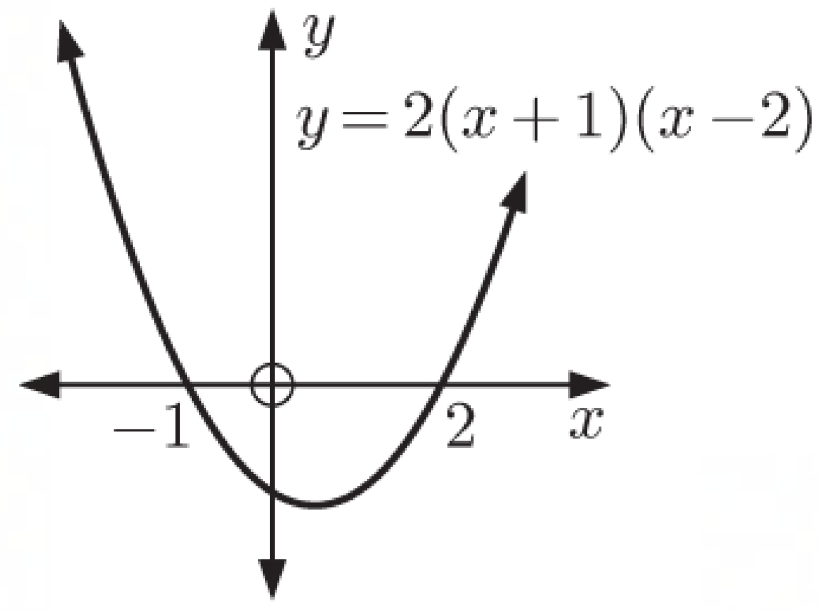
2 **a, b**



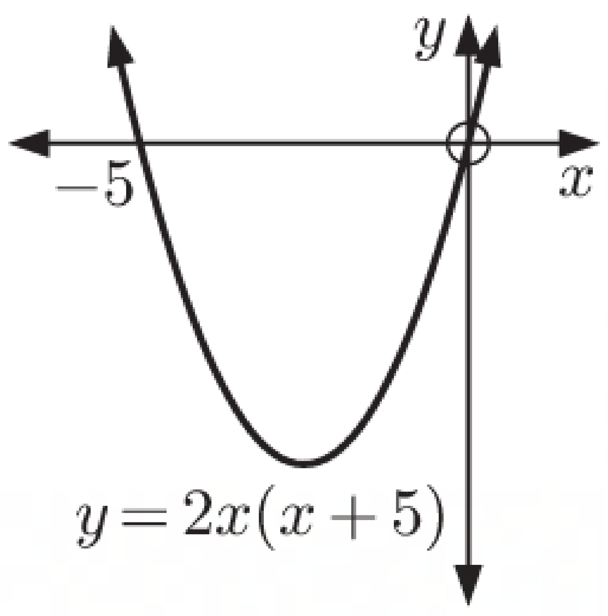
x-intercepts 1 and 4



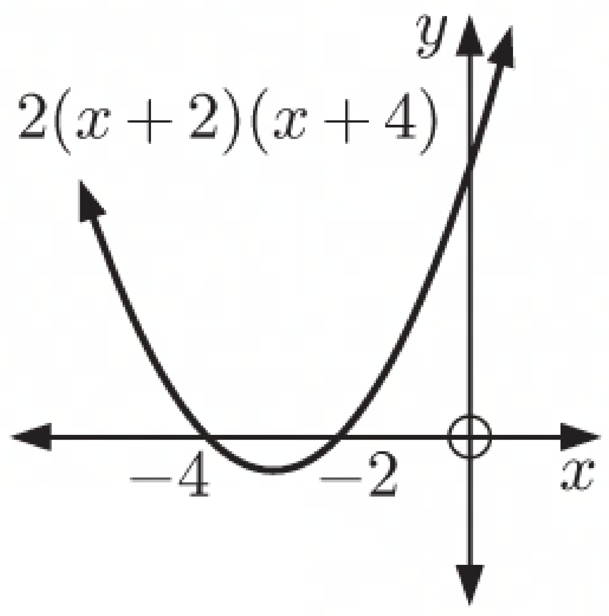
x-intercepts 3 and 5



x-intercepts -1 and 2



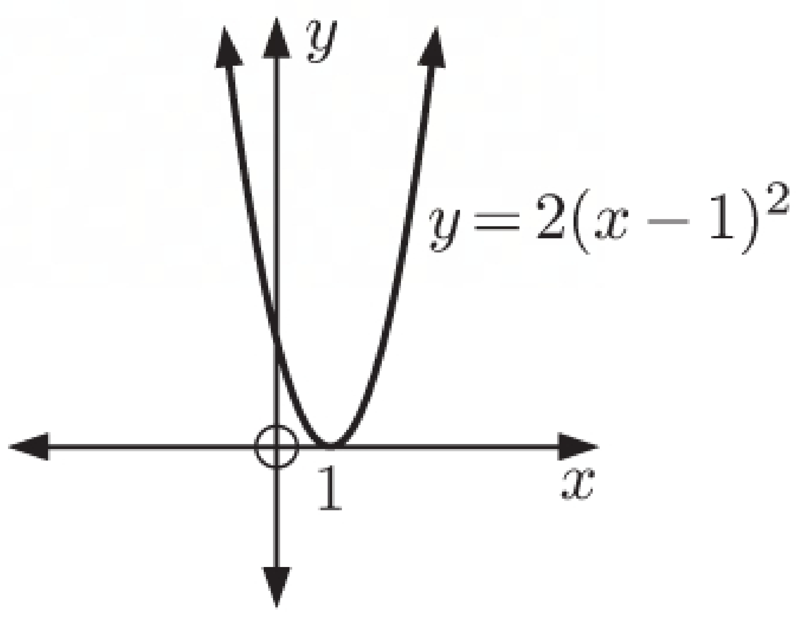
x-intercepts 0 and -5



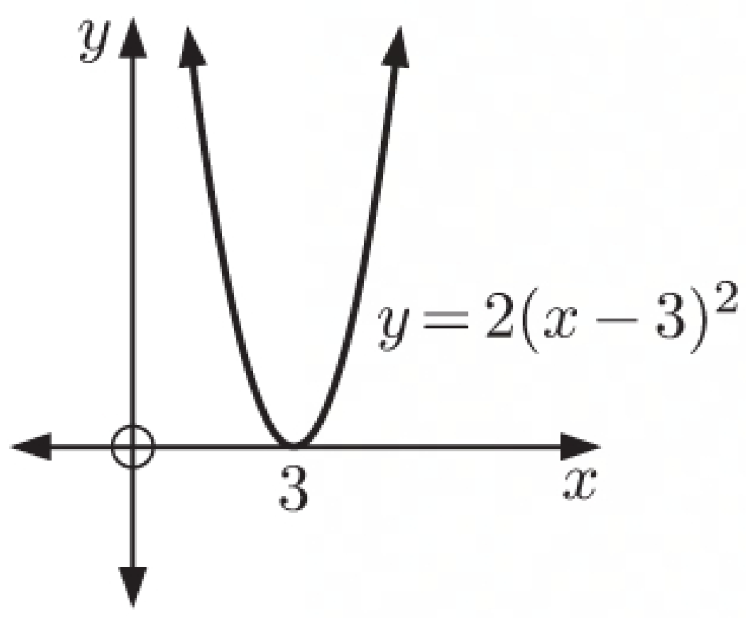
x-intercepts -2 and -4

c For $y = 2(x-p)(x-q)$, $p \neq q$, the graph has x-intercepts p and q, where it *cuts* the x-axis.

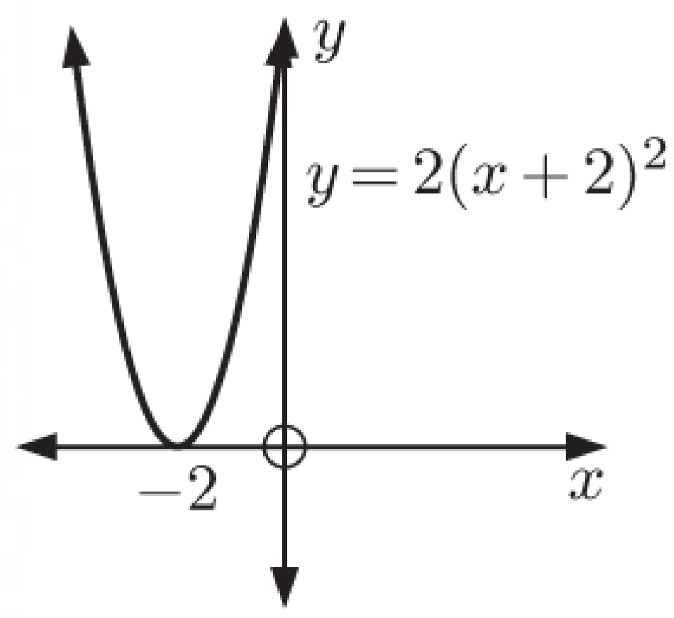
3 **a, b**



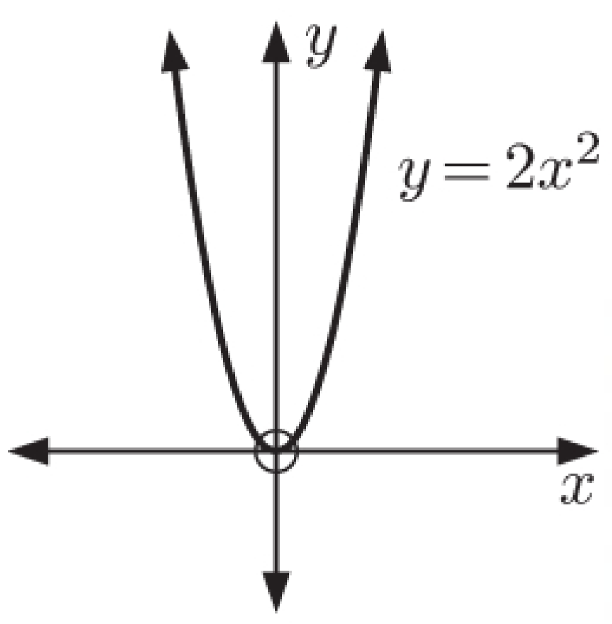
x-intercept 1



x-intercept 3



x-intercept -2



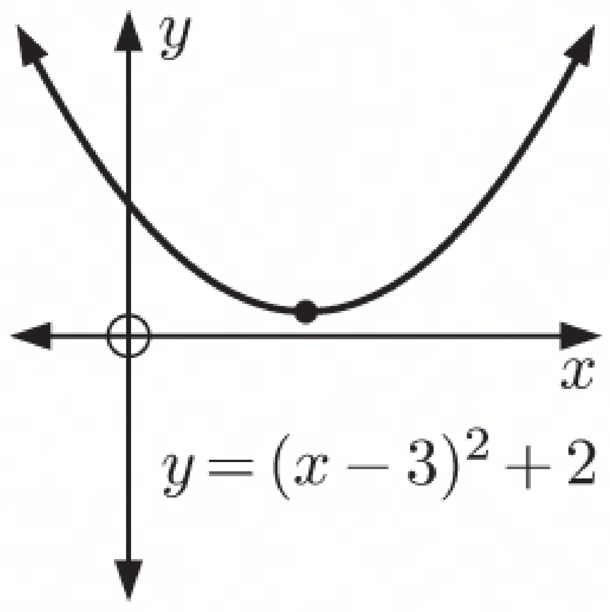
x-intercept 0

c For $y = 2(x-p)^2$, the graph has x-intercept p, where it *touches* the x-axis.

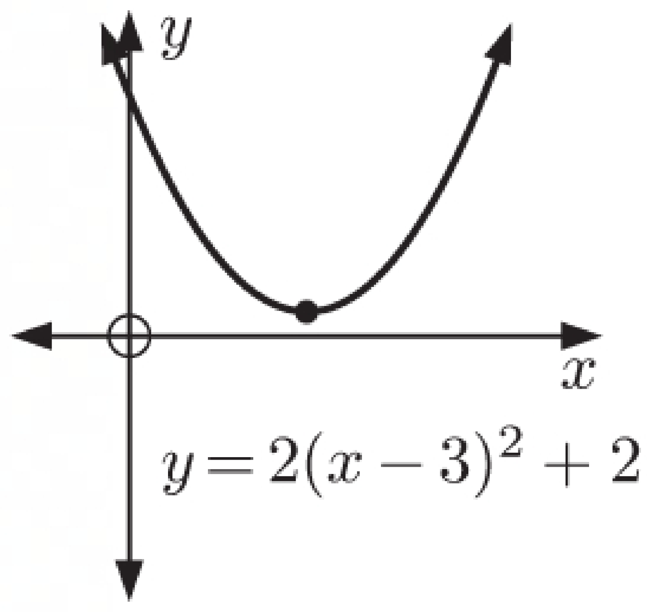
4
- If a quadratic has the form $y = a(x-p)(x-q)$ then it cuts the x-axis at p and q.
- If a quadratic has the form $y = a(x-p)^2$ then it touches the x-axis at p.

INVESTIGATION 2 GRAPHING $y = a(x-h)^2 + k$

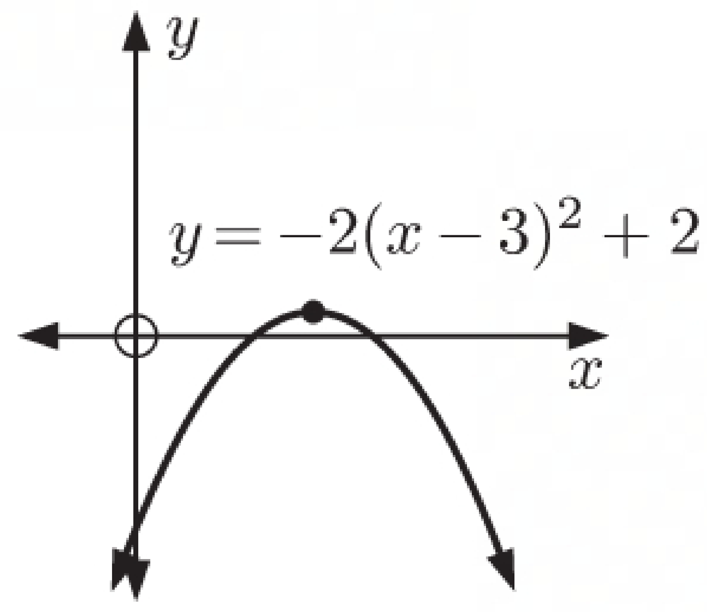
1 **a**, **b**



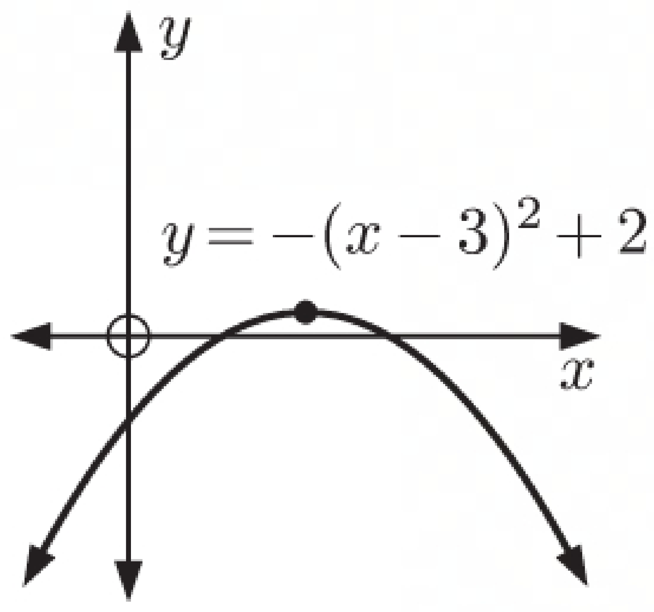
vertex is $(3, 2)$



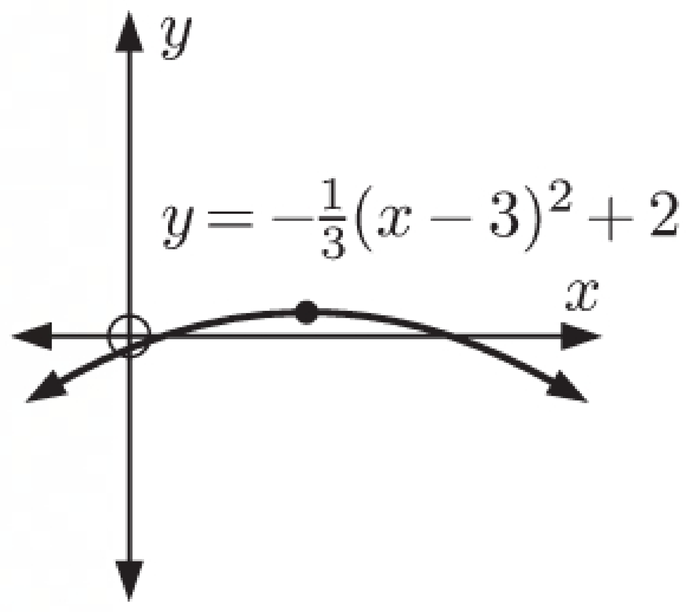
vertex is $(3, 2)$



vertex is $(3, 2)$



vertex is $(3, 2)$



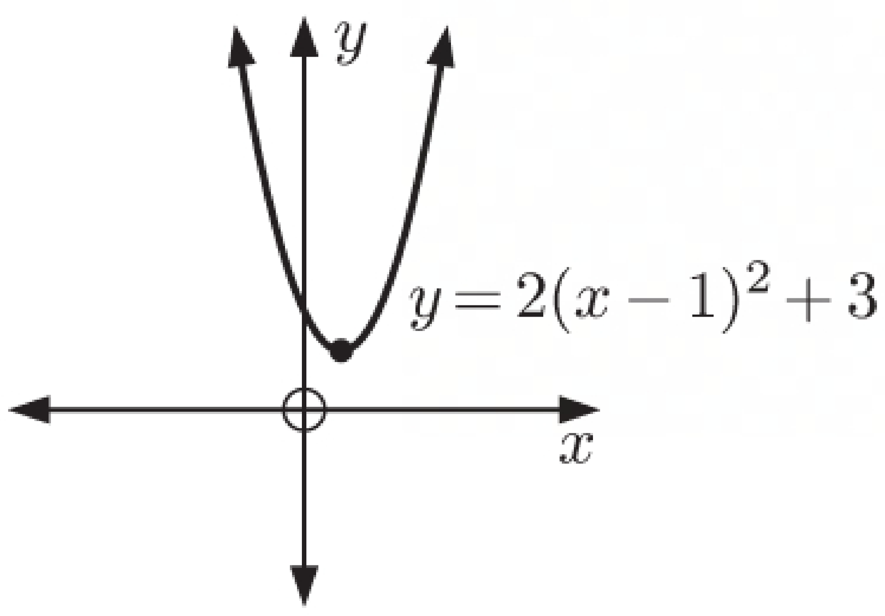
vertex is $(3, 2)$

c For $y = a(x-3)^2 + 2$: When $a > 0$ the shape is  .

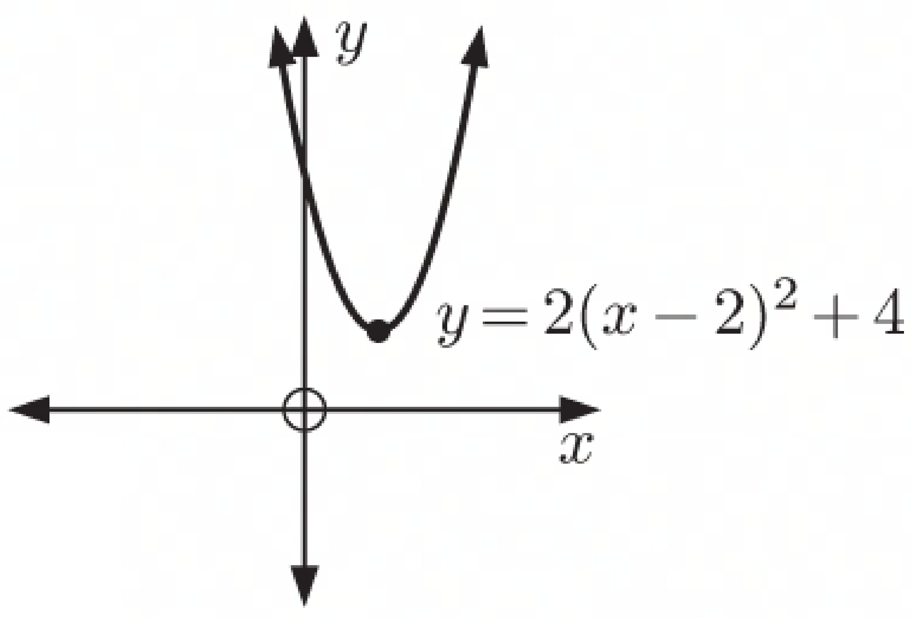
When $a < 0$ the shape is  .

As $|\,a\,|$ increases, the graph becomes narrower.

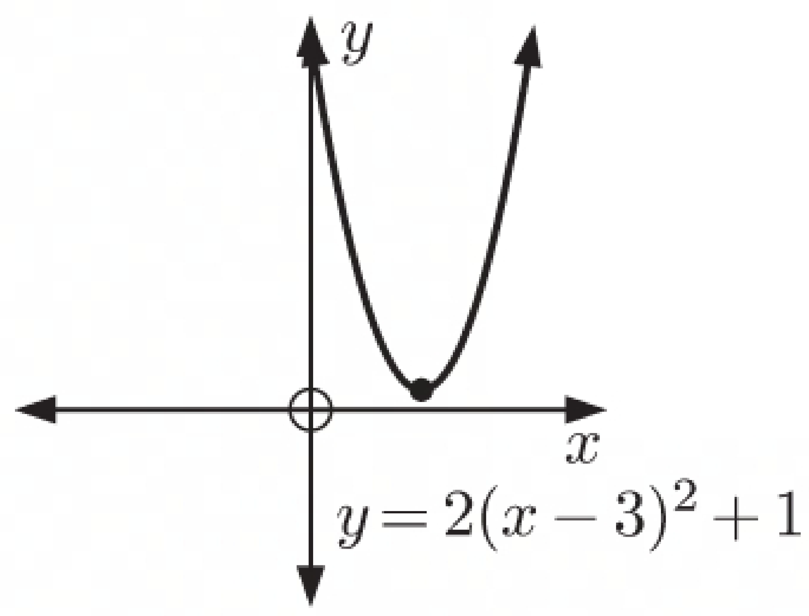
2 **a**, **b**



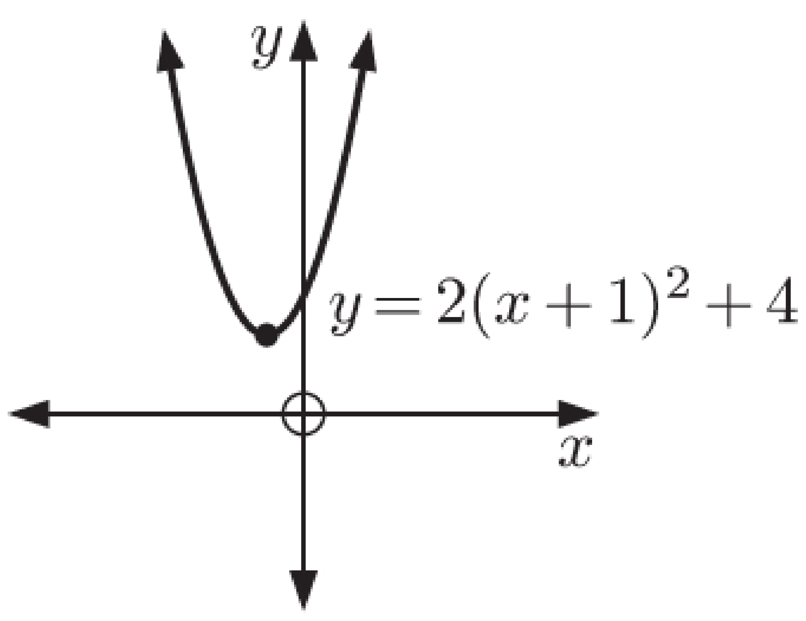
vertex is $(1, 3)$



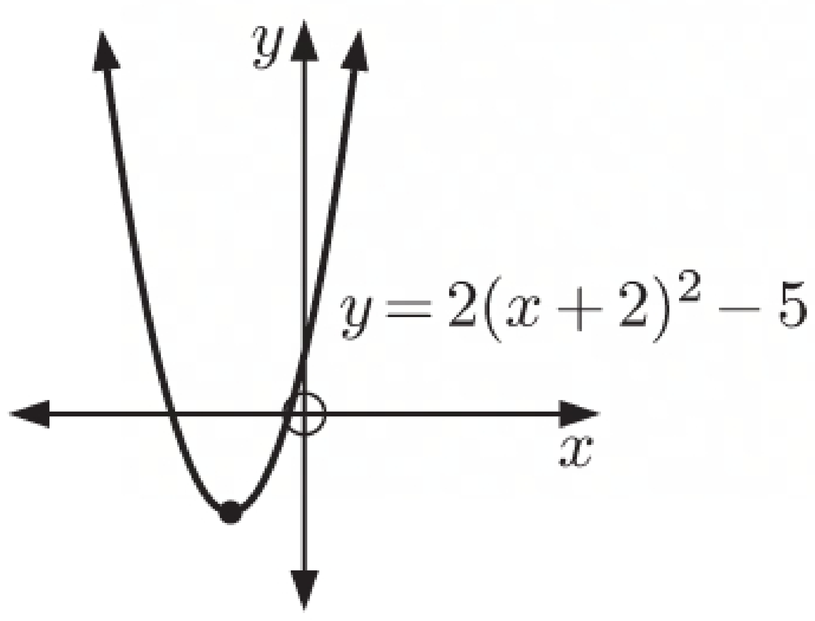
vertex is $(2, 4)$



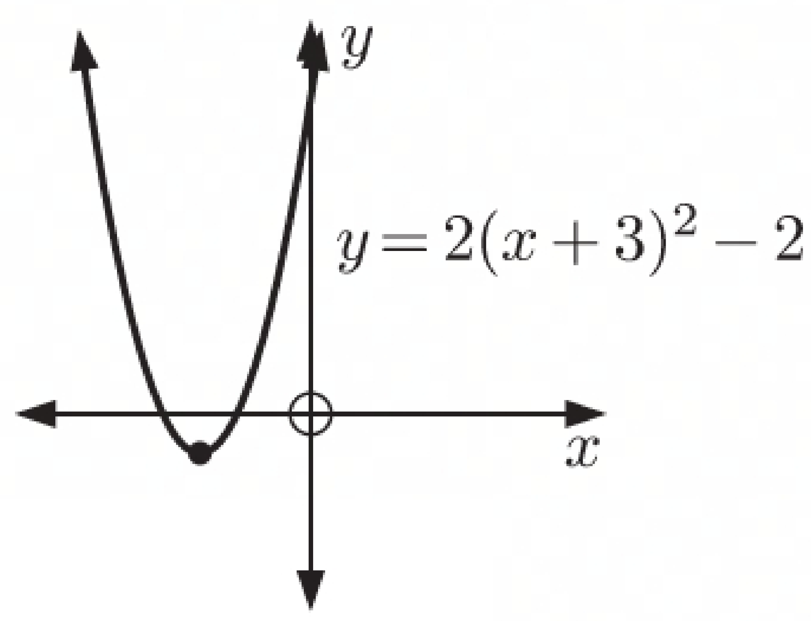
vertex is $(3, 1)$



vertex is $(-1, 4)$



vertex is $(-2, -5)$



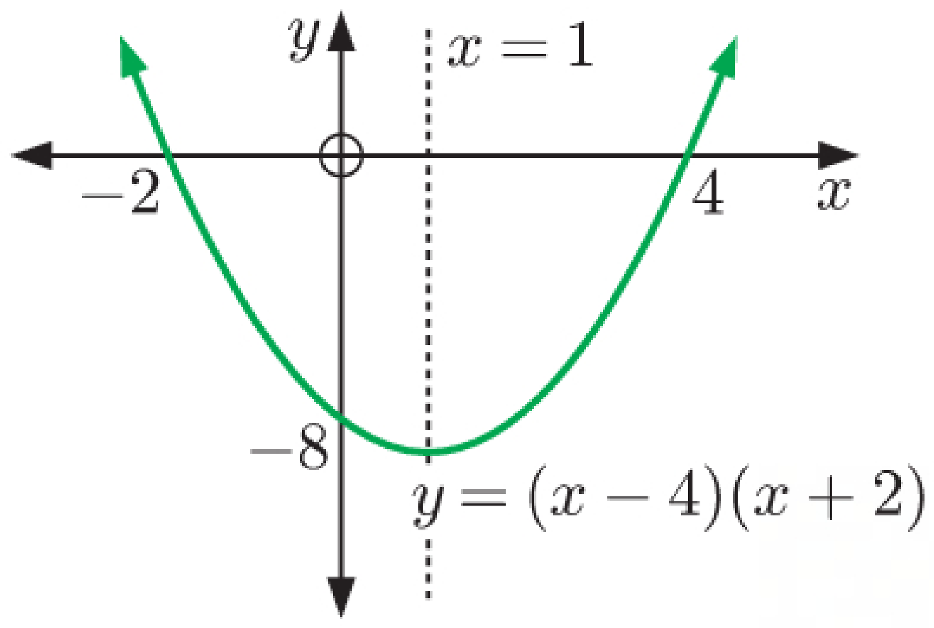
vertex is $(-3, -2)$

c For $y = 2(x-h)^2 + k$, the vertex is at (h, k).

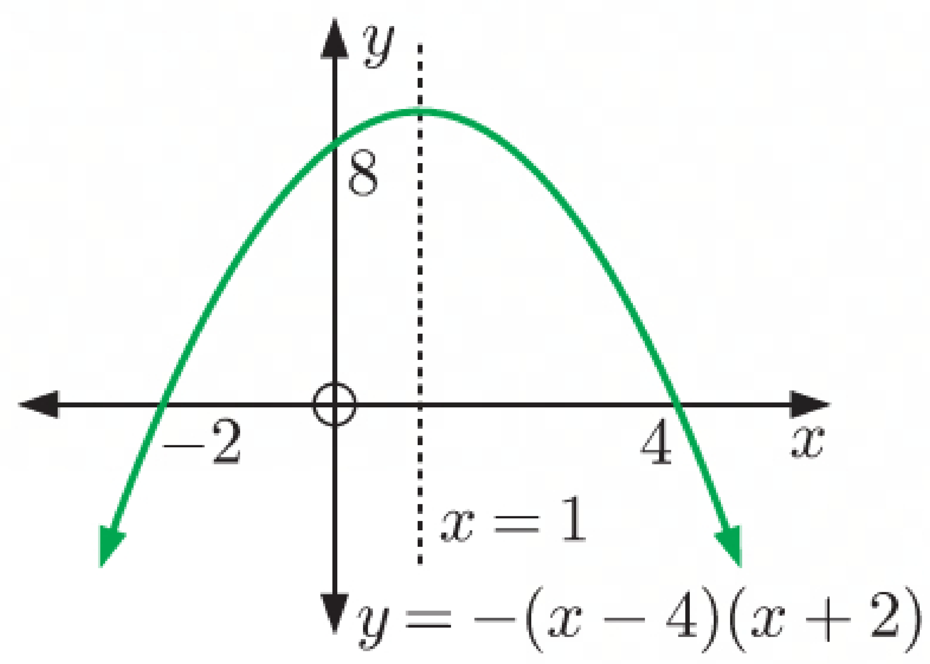
3 If a quadratic has the form $y = a(x-h)^2 + k$ then its vertex has coordinates (h, k).

EXERCISE 14B.1

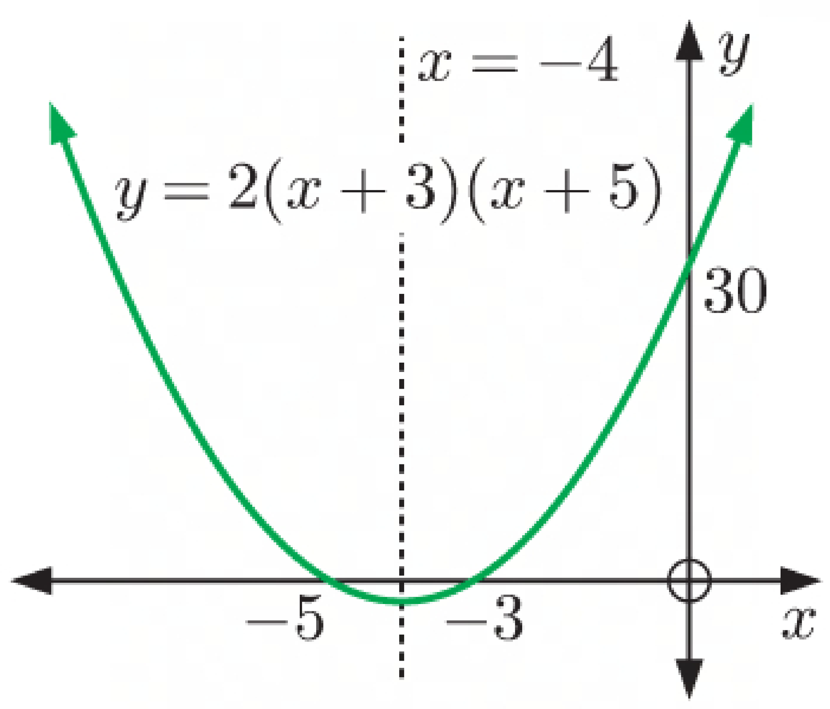
1 **a** $y = (x-4)(x+2)$
has x-intercepts 4 and -2
When $x = 0$, $y = (-4)(2)$
$= -8$
$\therefore$ the y-intercept is -8.



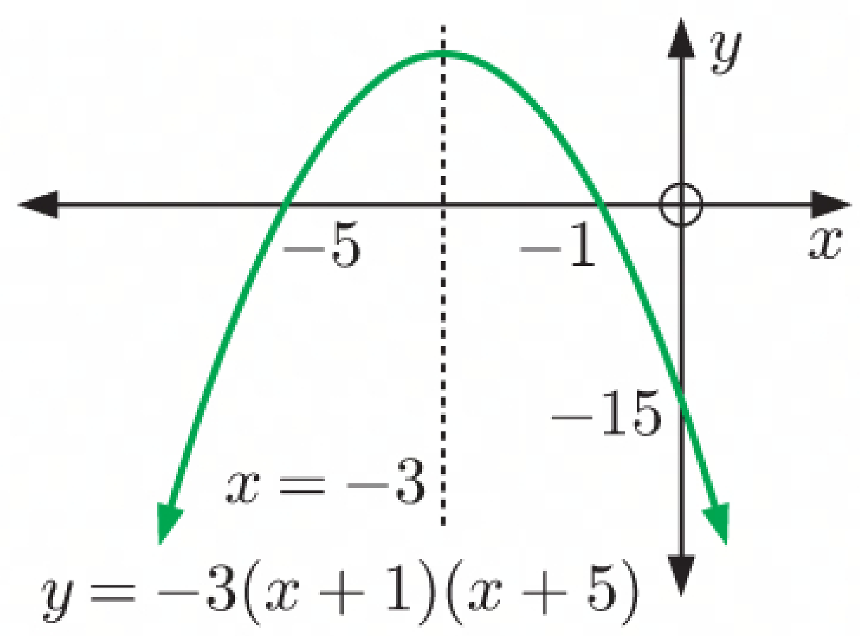
b $y = -(x-4)(x+2)$
has x-intercepts 4 and -2
When $x = 0$, $y = -(-4)(2)$
$= 8$
$\therefore$ the y-intercept is 8.



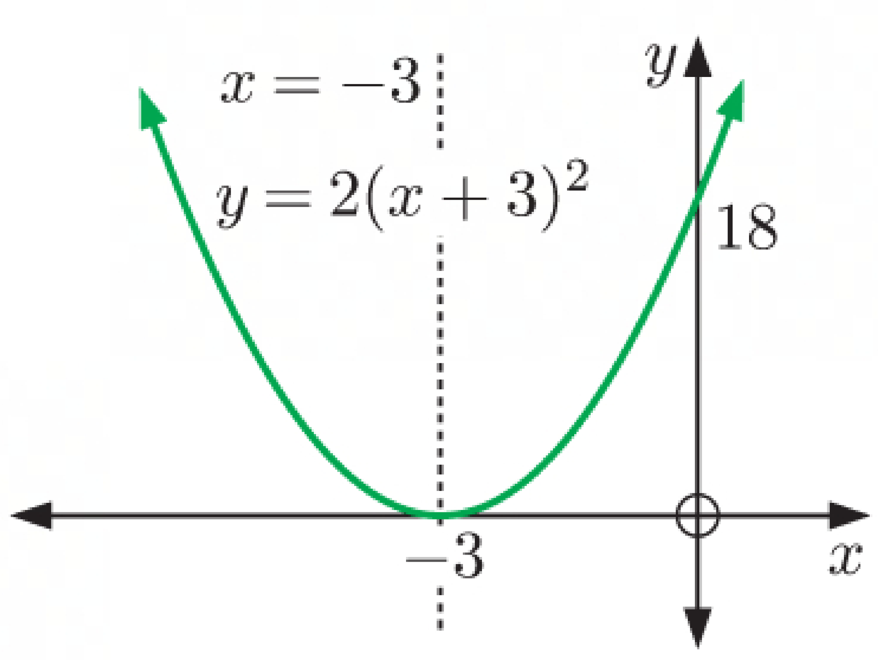
c $y = 2(x+3)(x+5)$
has x-intercepts -3 and -5
When $x = 0$, $y = 2(3)(5)$
$= 30$
$\therefore$ the y-intercept is 30.



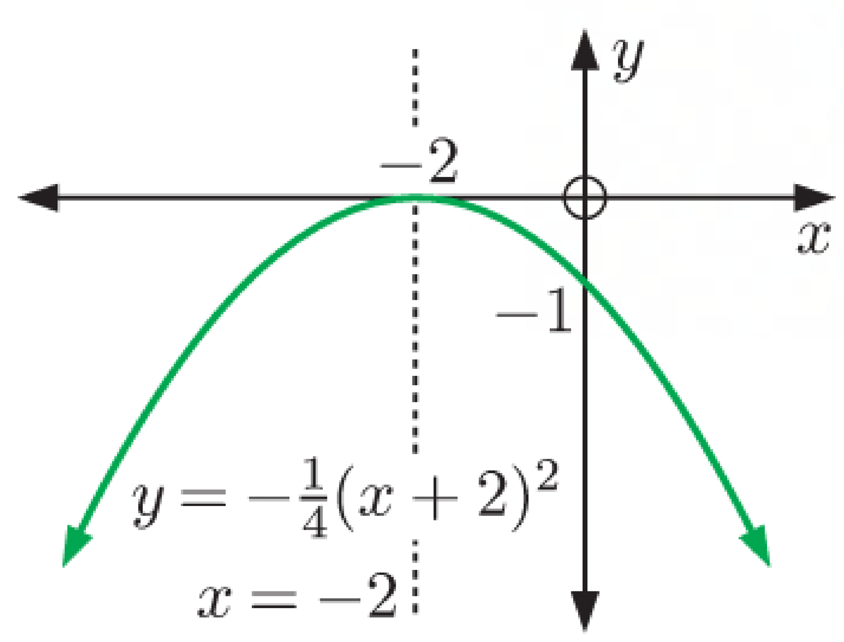
d $y = -3(x+1)(x+5)$
has x-intercepts -1 and -5
When $x = 0$, $y = -3(1)(5)$
$= -15$
$\therefore$ the y-intercept is -15.



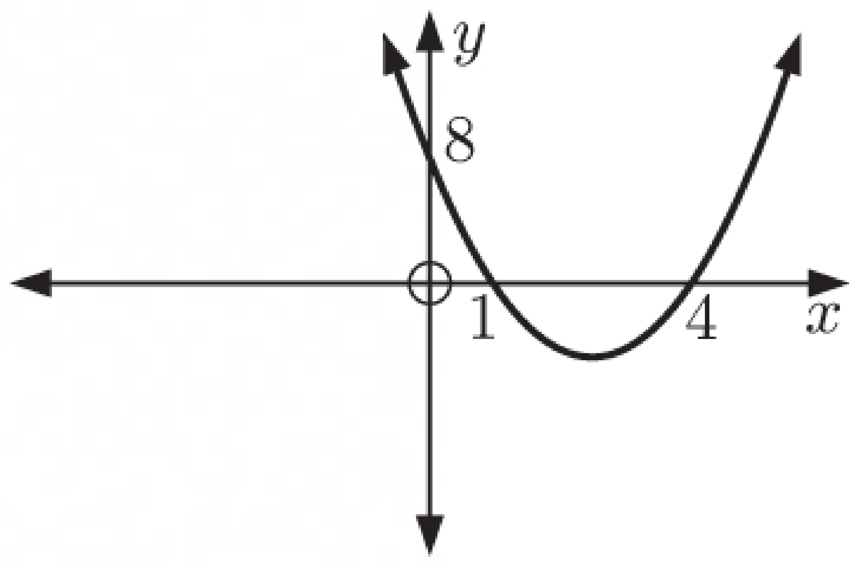
e $y = 2(x+3)^2$
touches the x-axis at -3
When $x = 0$, $y = 2(3)^2$
$= 18$
$\therefore$ the y-intercept is 18.



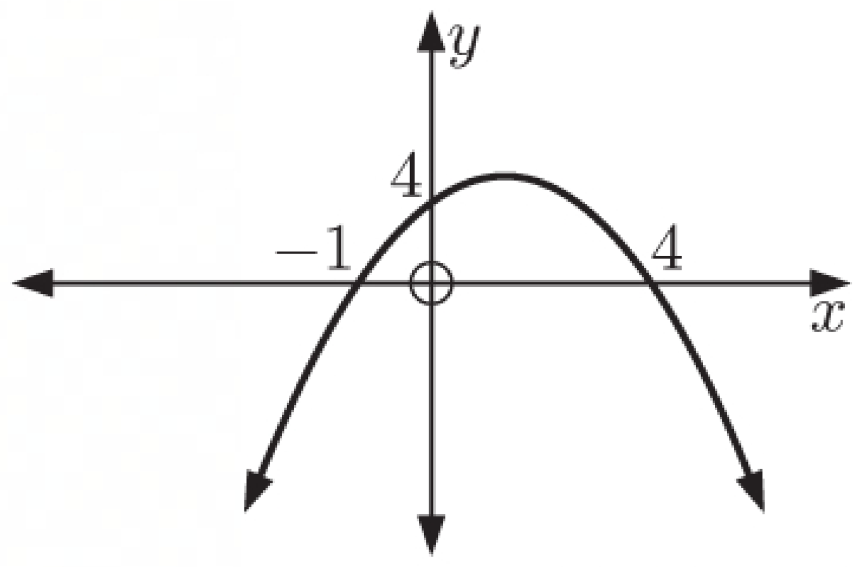
f $y = -\frac{1}{4}(x+2)^2$
touches the x-axis at -2
When $x = 0$, $y = -\frac{1}{4}(2)^2$
$= -1$
$\therefore$ the y-intercept is -1.



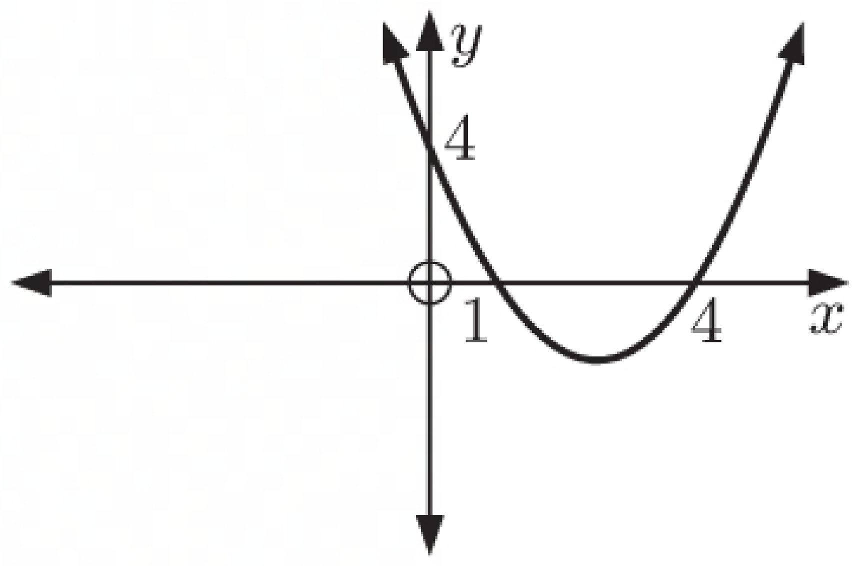
2 **a** $y = 2(x-1)(x-4)$
has x-intercepts 1 and 4
When $x = 0$, $y = 2(-1)(-4)$
$= 8$
$\therefore$ the y-intercept is 8.
The only graph with these x and y-intercepts is **C**.



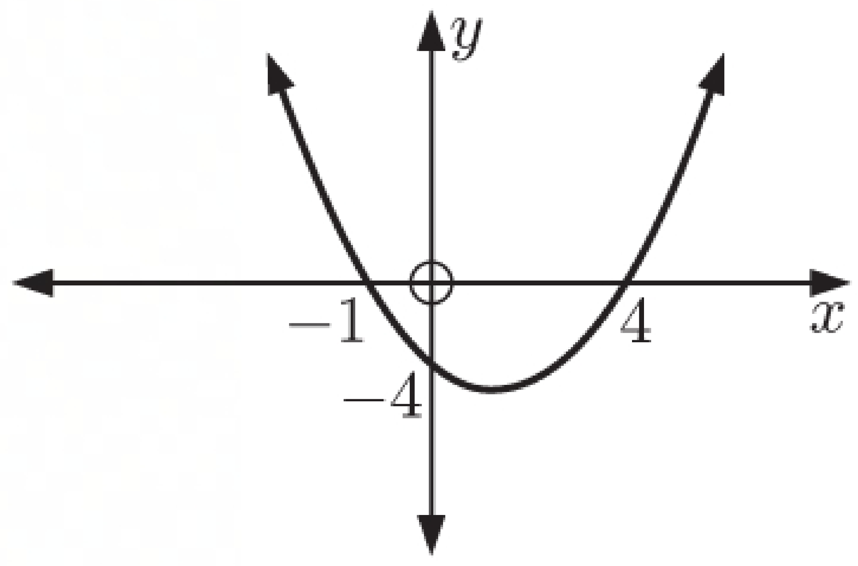
b $y = -(x+1)(x-4)$
has x-intercepts -1 and 4
When $x = 0$, $y = -(1)(-4)$
$= 4$
$\therefore$ the y-intercept is 4.
The only graph with these x and y-intercepts is **E**.



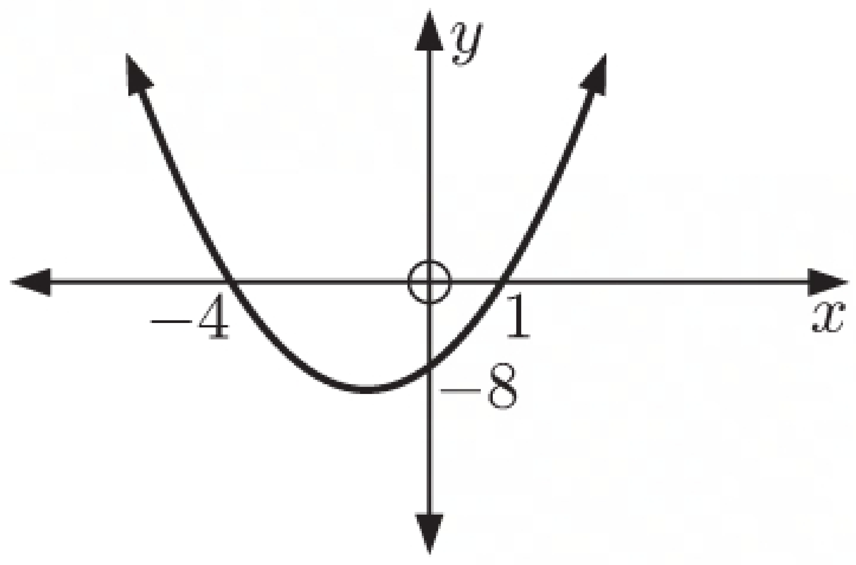
c $y = (x-1)(x-4)$
has x-intercepts 1 and 4
When $x = 0$, $y = (-1)(-4)$
$= 4$
$\therefore$ the y-intercept is 4.
The only graph with these x and y-intercepts is **B**.



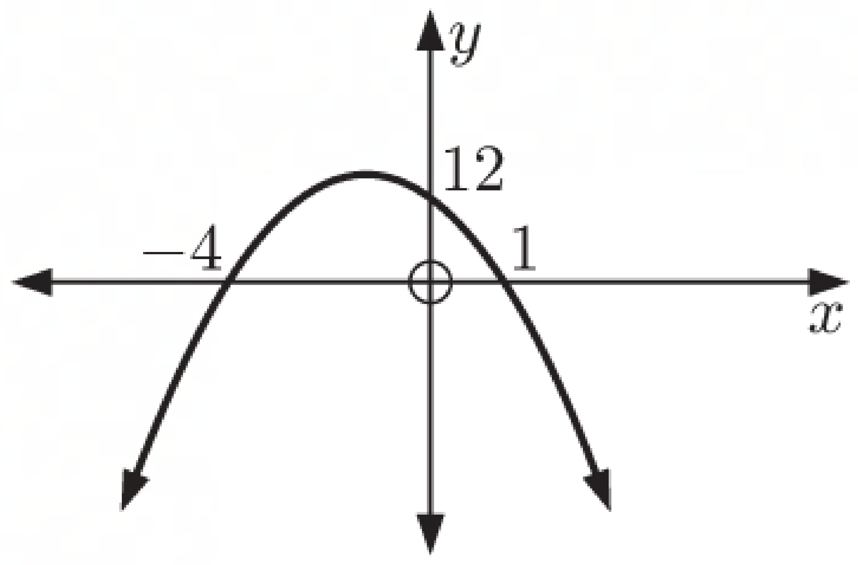
d $y = (x+1)(x-4)$
has x-intercepts -1 and 4
When $x = 0$, $y = (1)(-4)$
$= -4$
$\therefore$ the y-intercept is -4.
The only graph with these x and y-intercepts is **F**.



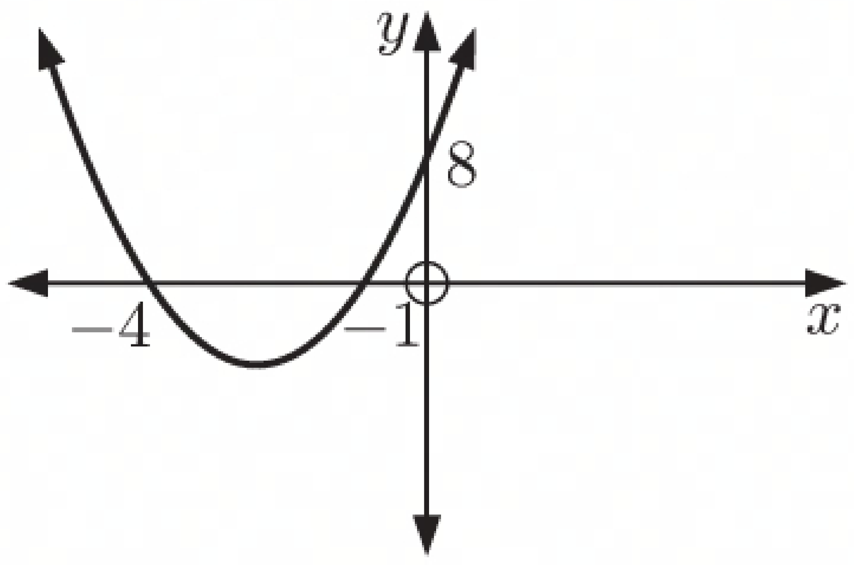
e $y = 2(x+4)(x-1)$
has x-intercepts -4 and 1
When $x = 0$, $y = 2(4)(-1)$
$= -8$
$\therefore$ the y-intercept is -8.
The only graph with these x and y-intercepts is **G**.



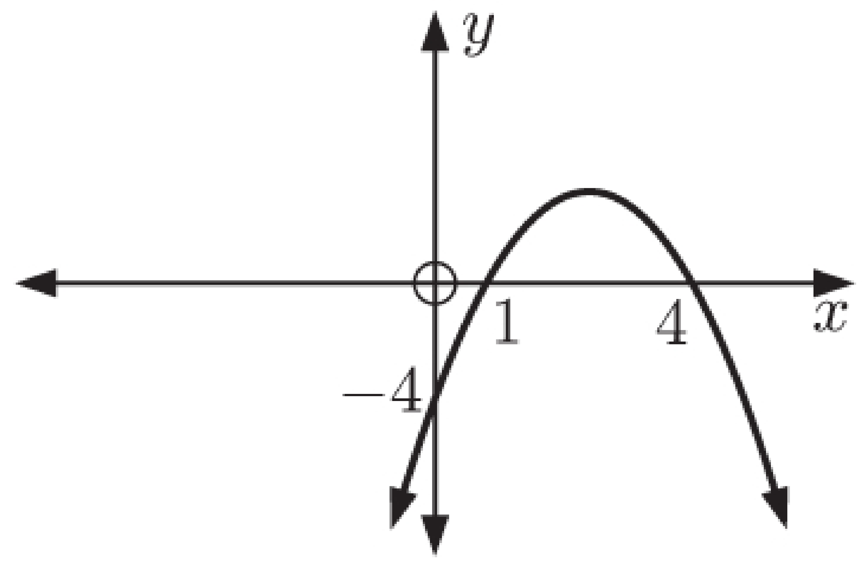
f $y = -3(x+4)(x-1)$
has x-intercepts -4 and 1
When $x = 0$, $y = -3(4)(-1)$
$= 12$
$\therefore$ the y-intercept is 12.
The only graph with these x and y-intercepts is **H**.



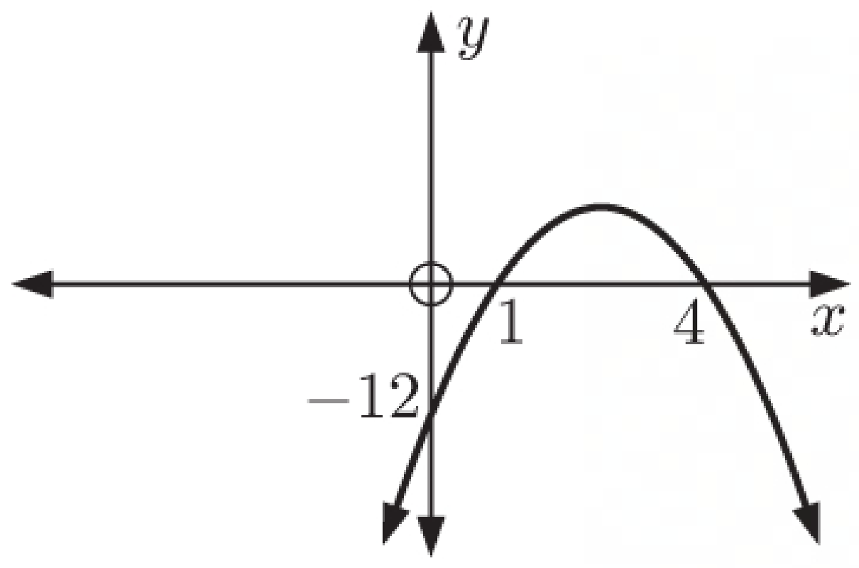
g $y = 2(x+1)(x+4)$
has x-intercepts -1 and -4
When $x = 0$, $y = 2(1)(4) = 8$
$\therefore$ the y-intercept is 8.
The only graph with these x and y-intercepts is **I**.



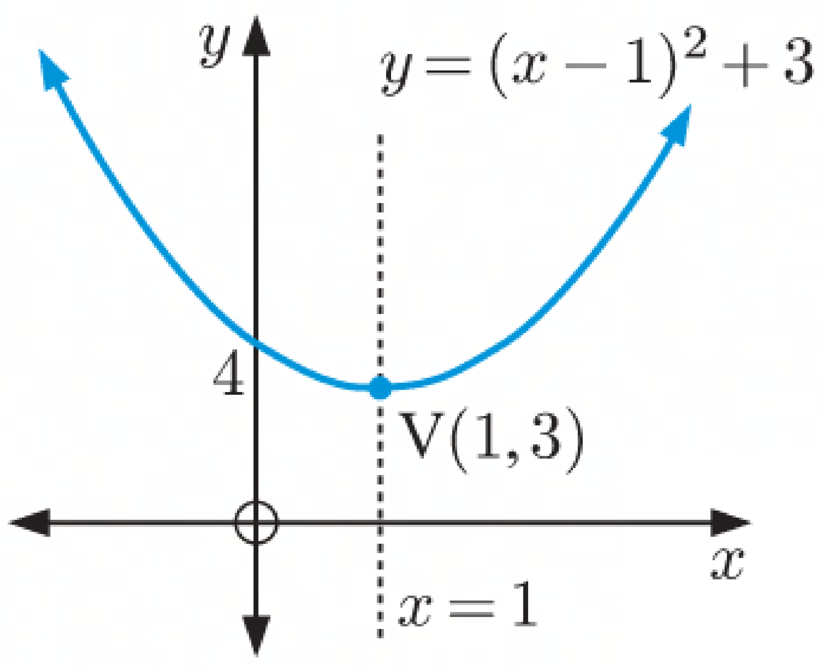
h $y = -(x-1)(x-4)$
has x-intercepts 1 and 4
When $x = 0$, $y = -(-1)(-4) = -4$
$\therefore$ the y-intercept is -4.
The only graph with these x and y-intercepts is **A**.



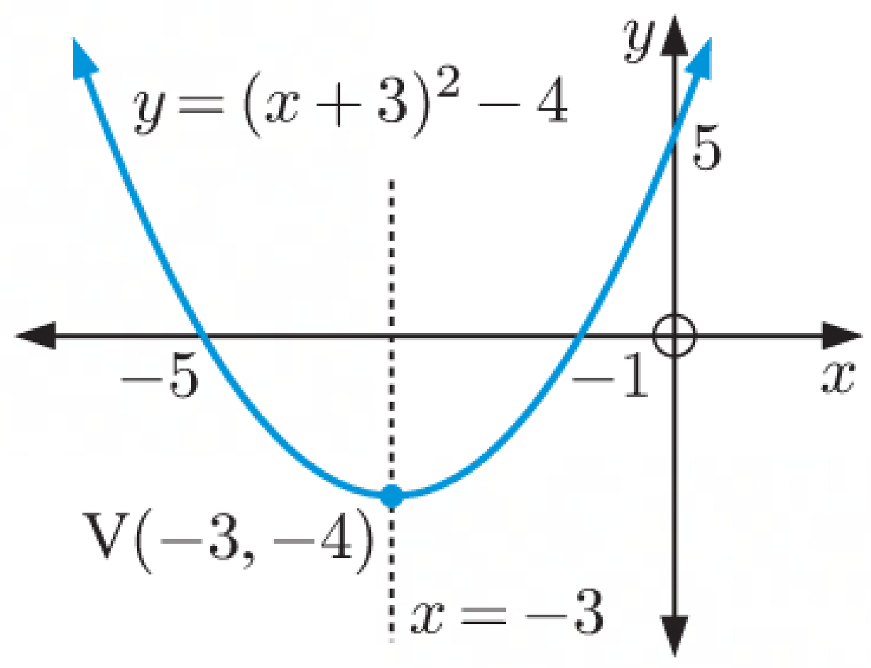
i $y = -3(x-1)(x-4)$
has x-intercepts 1 and 4
When $x = 0$, $y = -3(-1)(-4) = -12$
$\therefore$ the y-intercept is -12.
The only graph with these x and y-intercepts is **D**.



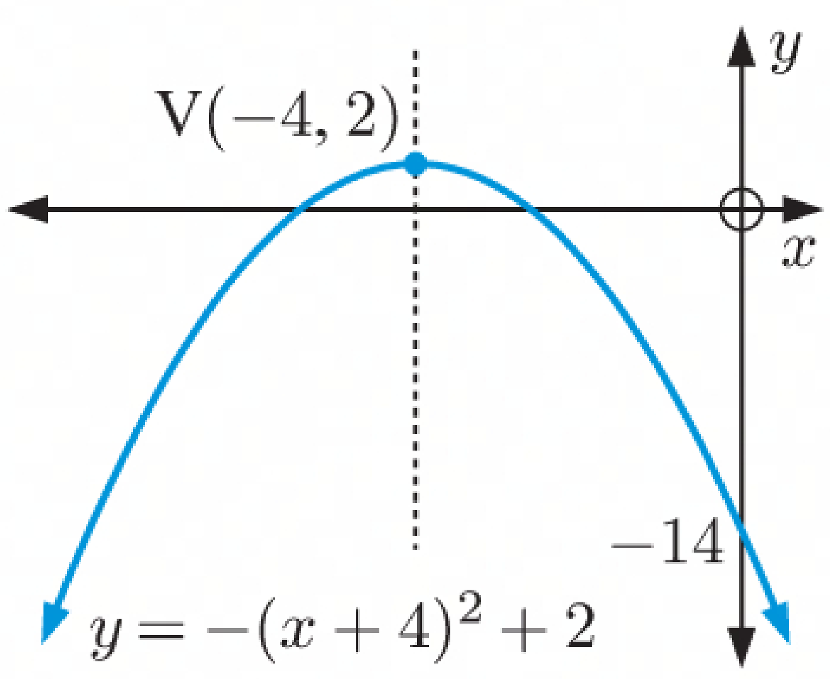
3 **a** $y = (x-1)^2 + 3$ has vertex $(1, 3)$.
The axis of symmetry is $x = 1$.
When $x = 0$, $y = (-1)^2 + 3 = 4$
$a > 0$ so the shape is 



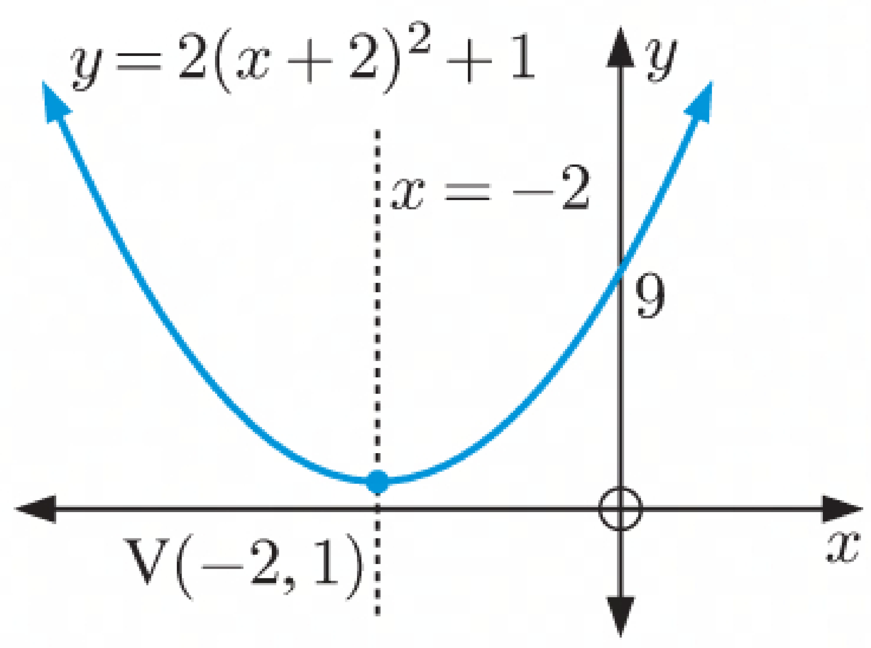
b $y = (x+3)^2 - 4$ has vertex $(-3, -4)$.
The axis of symmetry is $x = -3$.
When $x = 0$, $y = (3)^2 - 4 = 5$
$a > 0$ so the shape is 



c $y = -(x+4)^2 + 2$ has vertex $(-4, 2)$.
The axis of symmetry is $x = -4$.
When $x = 0$, $y = -(4)^2 + 2 = -14$
$a < 0$ so the shape is 



d $y = 2(x+2)^2 + 1$ has vertex $(-2, 1)$.
The axis of symmetry is $x = -2$.
When $x = 0$, $y = 2(2)^2 + 1 = 9$
$a > 0$ so the shape is 

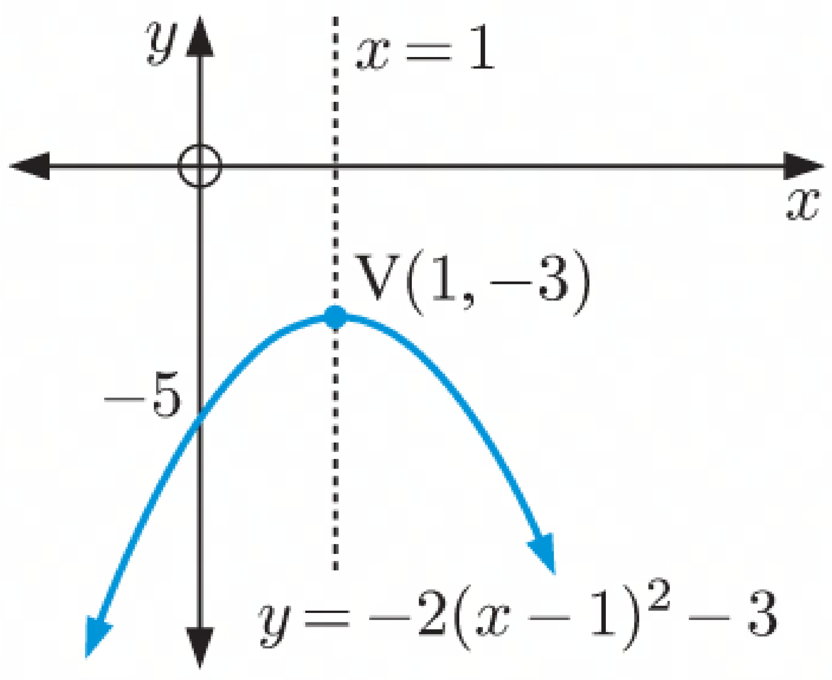


e $y = -2(x-1)^2 - 3$ has vertex $(1, -3)$.

The axis of symmetry is $x = 1$.

When $x = 0$, $y = -2(-1)^2 - 3$
$= -5$

$a < 0$ so the shape is 

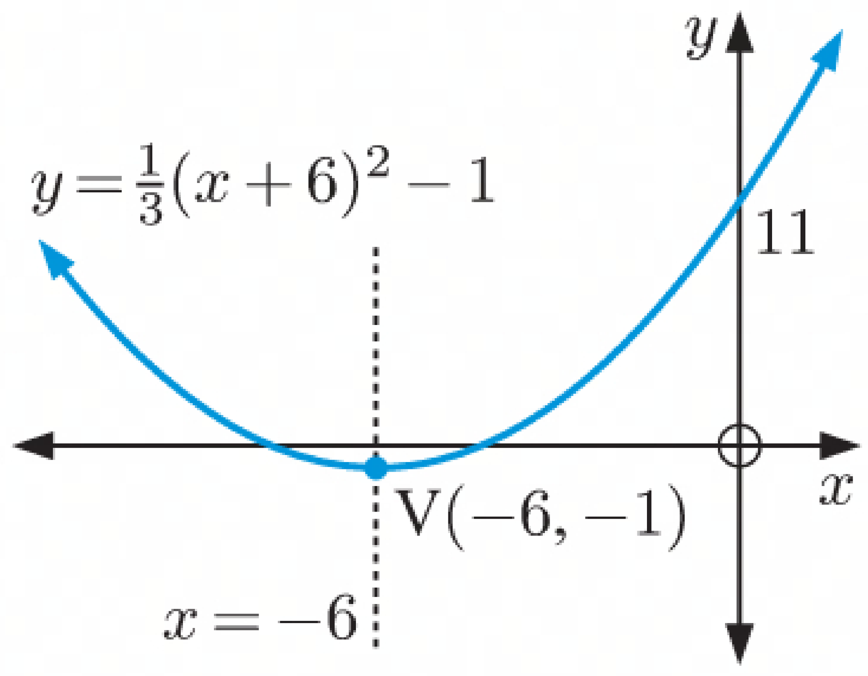


f $y = \frac{1}{3}(x+6)^2 - 1$ has vertex $(-6, -1)$.

The axis of symmetry is $x = -6$.

When $x = 0$, $y = \frac{1}{3}(6)^2 - 1$
$= 11$

$a > 0$ so the shape is 

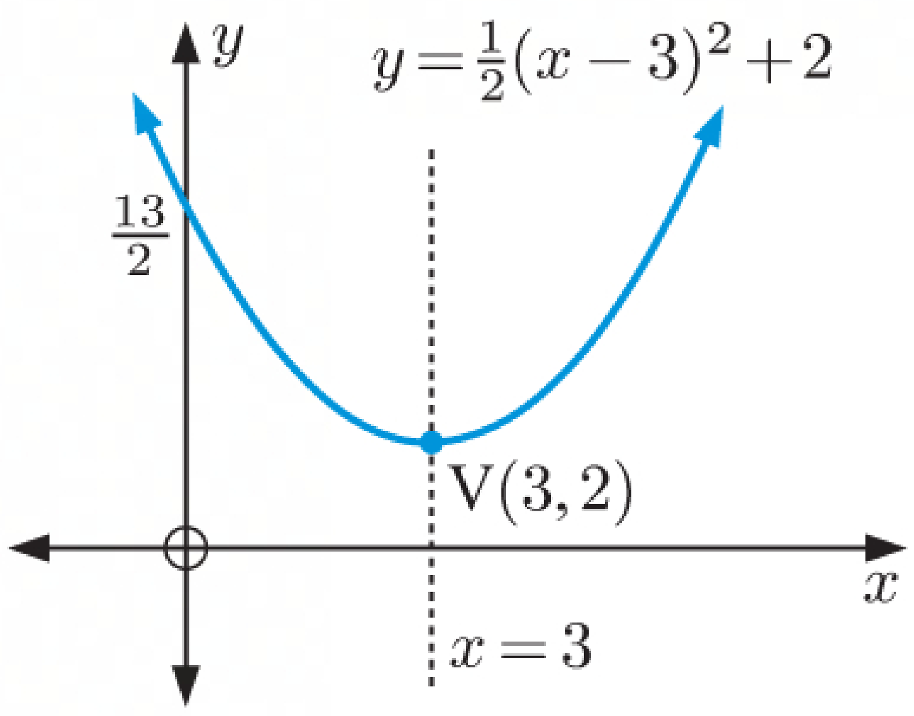


g $y = \frac{1}{2}(x-3)^2 + 2$ has vertex $(3, 2)$.

The axis of symmetry is $x = 3$.

When $x = 0$, $y = \frac{1}{2}(-3)^2 + 2$
$= \frac{13}{2}$

$a > 0$ so the shape is 

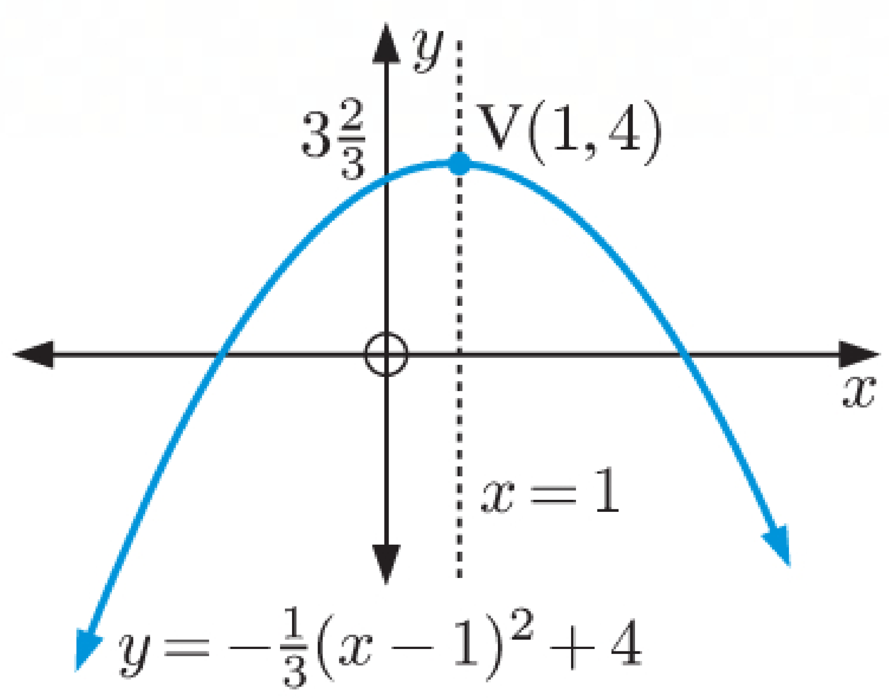


h $y = -\frac{1}{3}(x-1)^2 + 4$ has vertex $(1, 4)$.

The axis of symmetry is $x = 1$.

When $x = 0$, $y = -\frac{1}{3}(-1)^2 + 4$
$= 3\frac{2}{3}$

$a < 0$ so the shape is 

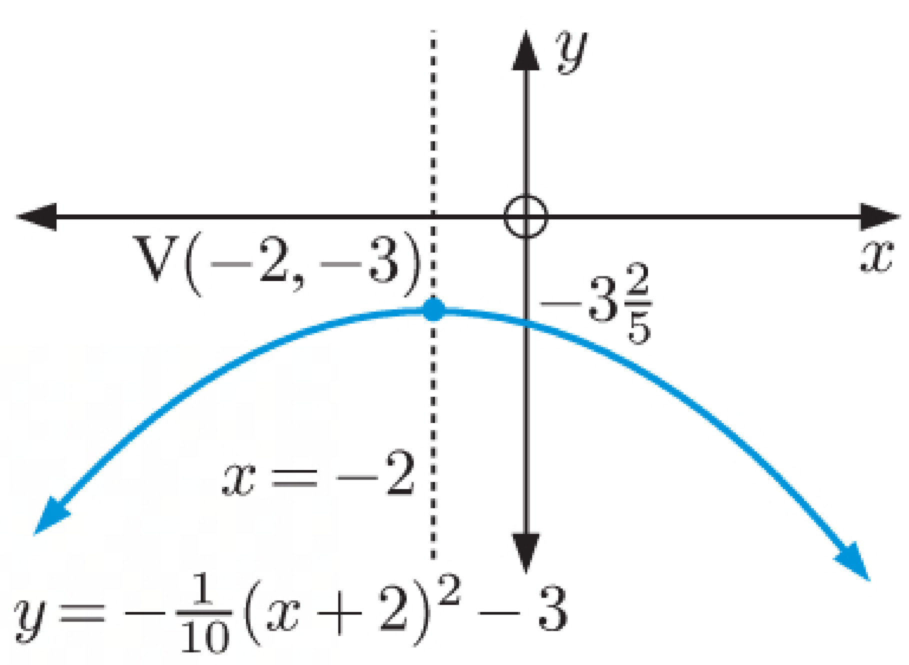


i $y = -\frac{1}{10}(x+2)^2 - 3$ has vertex $(-2, -3)$.

The axis of symmetry is $x = -2$.

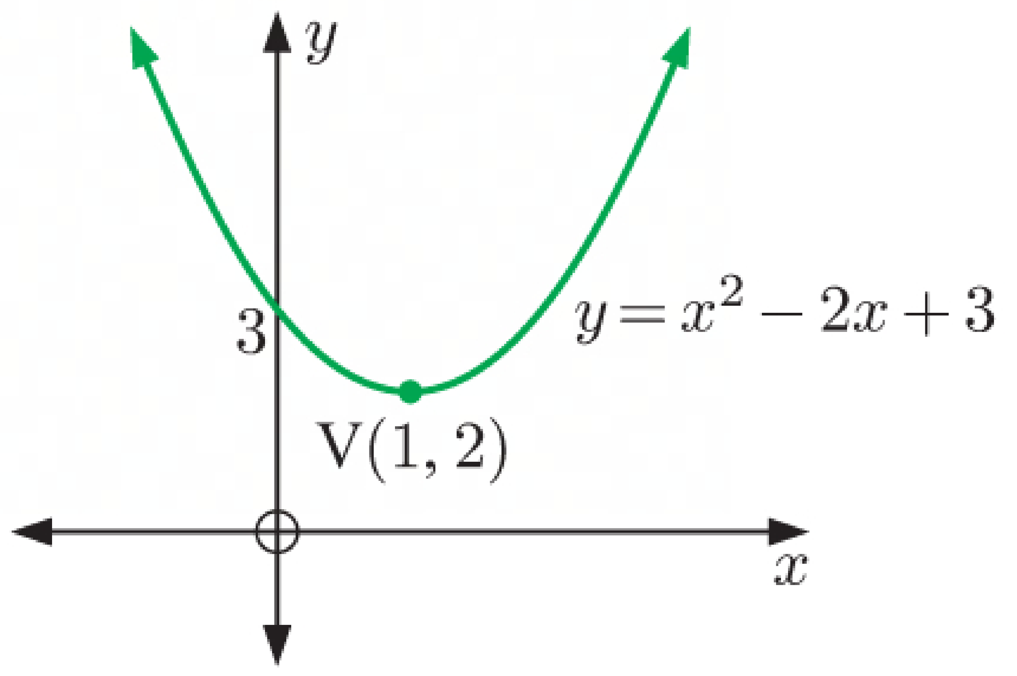
When $x = 0$, $y = -\frac{1}{10}(2)^2 - 3$
$= -3\frac{2}{5}$

$a < 0$ so the shape is 

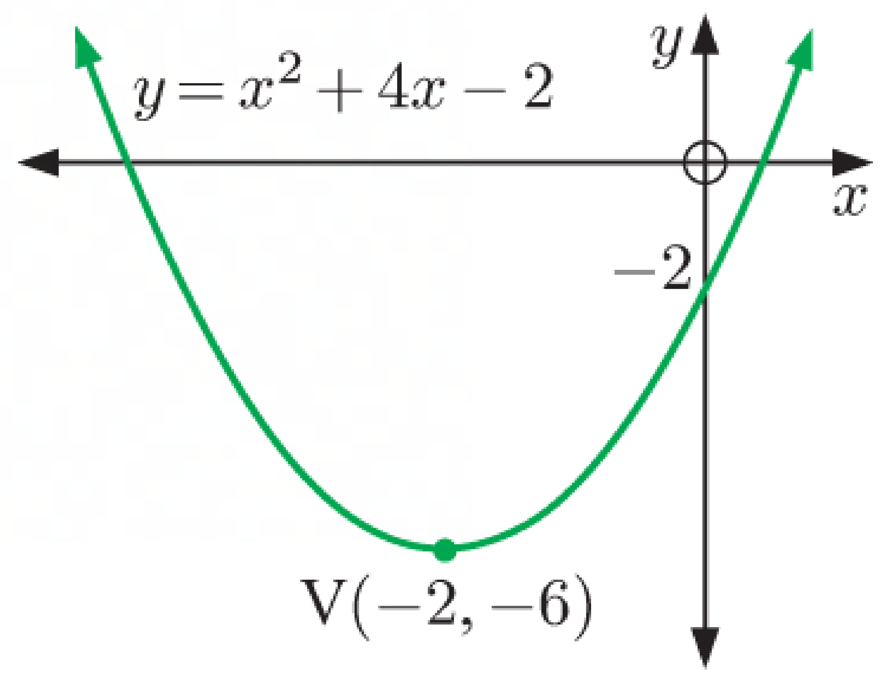


EXERCISE 14B.2

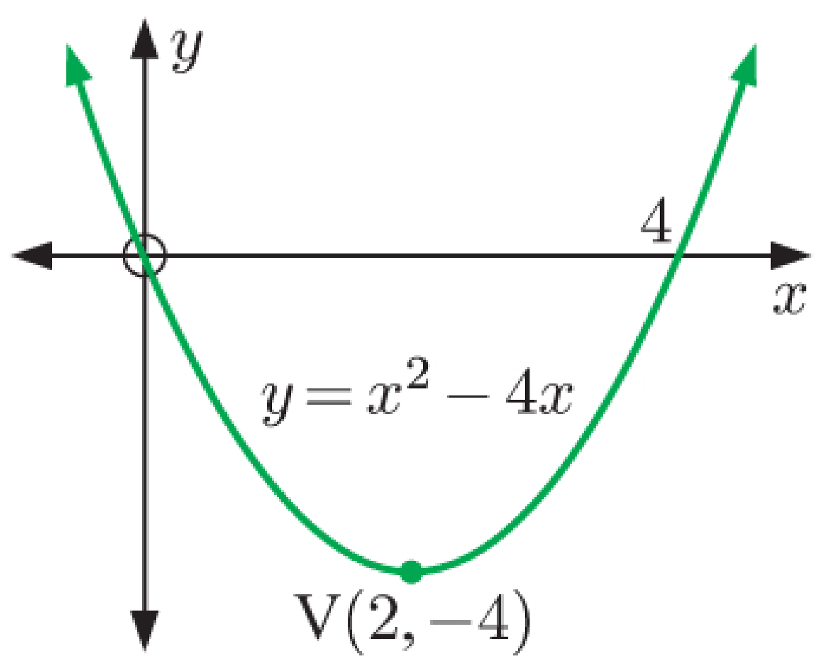
1 **a**
$$y = x^2 - 2x + 3$$
$$\therefore \quad y = x^2 - 2x + (-1)^2 + 3 - (-1)^2$$
$$\therefore \quad y = (x-1)^2 + 2$$
The vertex is $(1, 2)$, and the y-intercept is 3.



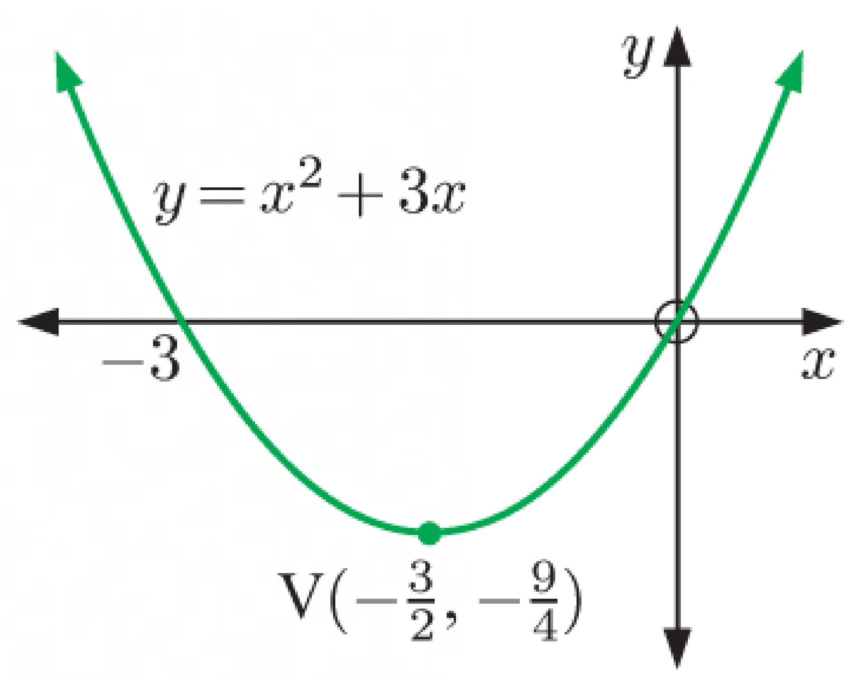
b
$$y = x^2 + 4x - 2$$
$$\therefore \quad y = x^2 + 4x + 2^2 - 2 - 2^2$$
$$\therefore \quad y = (x+2)^2 - 6$$
The vertex is $(-2, -6)$, and the y-intercept is -2.



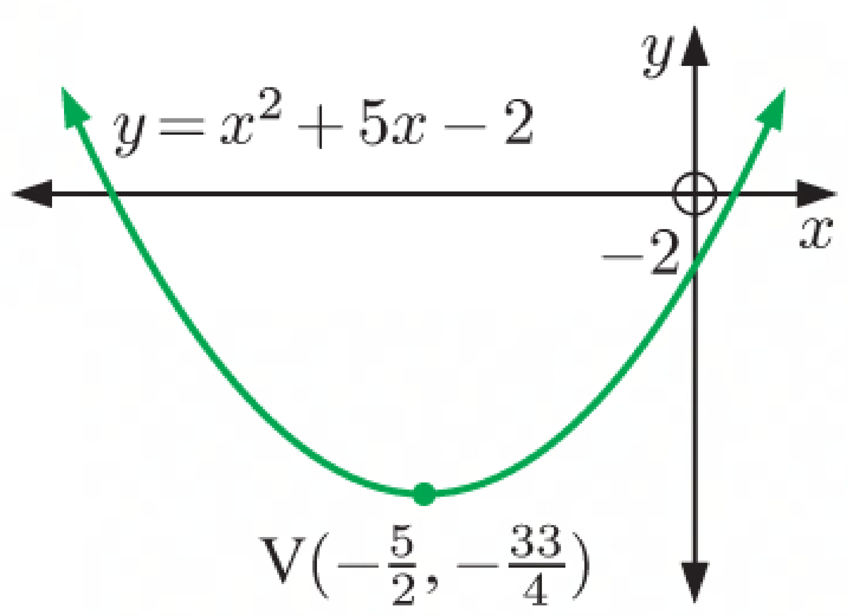
c
$$y = x^2 - 4x$$
$$\therefore \quad y = x^2 - 4x + (-2)^2 - (-2)^2$$
$$\therefore \quad y = (x-2)^2 - 4$$
The vertex is $(2, -4)$, and the y-intercept is 0.



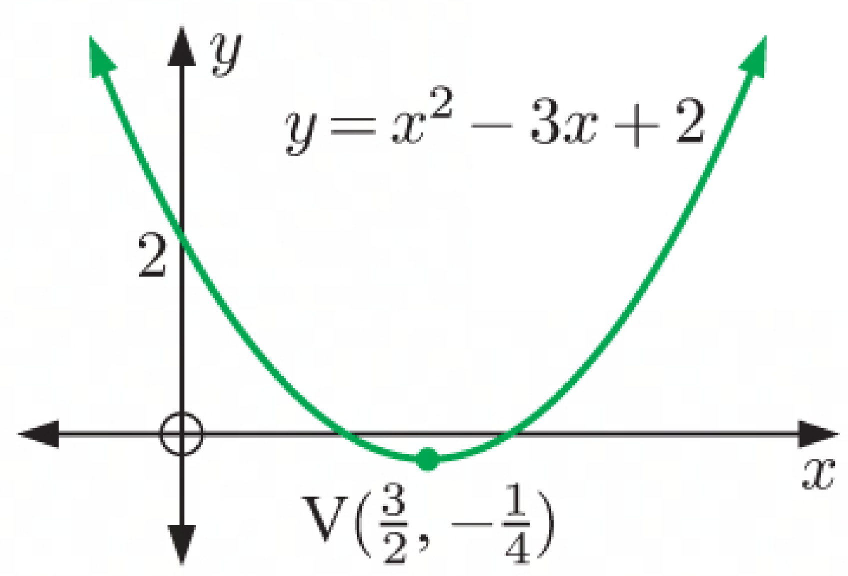
d
$$y = x^2 + 3x$$
$$\therefore \quad y = x^2 + 3x + (\tfrac{3}{2})^2 - (\tfrac{3}{2})^2$$
$$\therefore \quad y = (x + \tfrac{3}{2})^2 - \tfrac{9}{4}$$
The vertex is $(-\frac{3}{2}, -\frac{9}{4})$, and the y-intercept is 0.



e
$$y = x^2 + 5x - 2$$
$$\therefore \quad y = x^2 + 5x + (\tfrac{5}{2})^2 - 2 - (\tfrac{5}{2})^2$$
$$\therefore \quad y = (x + \tfrac{5}{2})^2 - \tfrac{33}{4}$$
The vertex is $(-\frac{5}{2}, -\frac{33}{4})$, and the y-intercept is -2.



f
$$y = x^2 - 3x + 2$$
$$\therefore \quad y = x^2 - 3x + (-\tfrac{3}{2})^2 + 2 - (-\tfrac{3}{2})^2$$
$$\therefore \quad y = (x - \tfrac{3}{2})^2 - \tfrac{1}{4}$$
The vertex is $(\frac{3}{2}, -\frac{1}{4})$, and the y-intercept is 2.

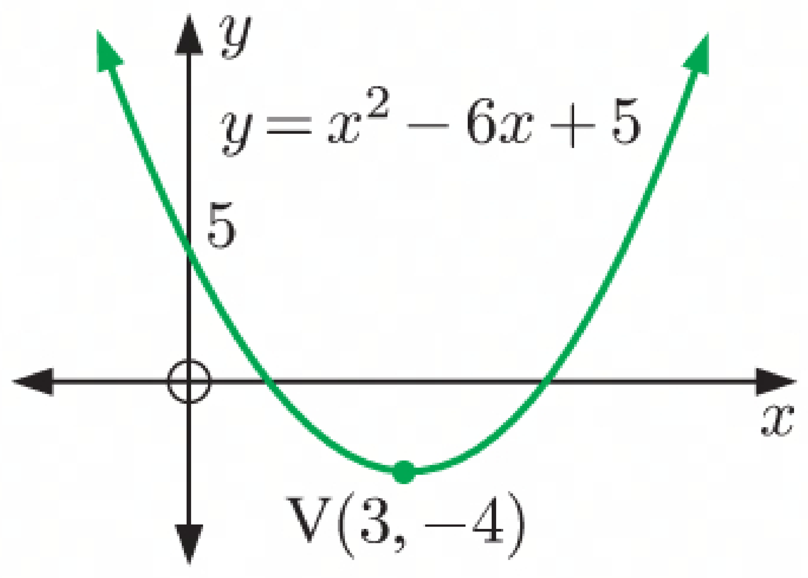


g $y = x^2 - 6x + 5$

$\therefore\ y = x^2 - 6x + (-3)^2 + 5 - (-3)^2$

$\therefore\ y = (x-3)^2 - 4$

The vertex is $(3, -4)$, and the y-intercept is 5.

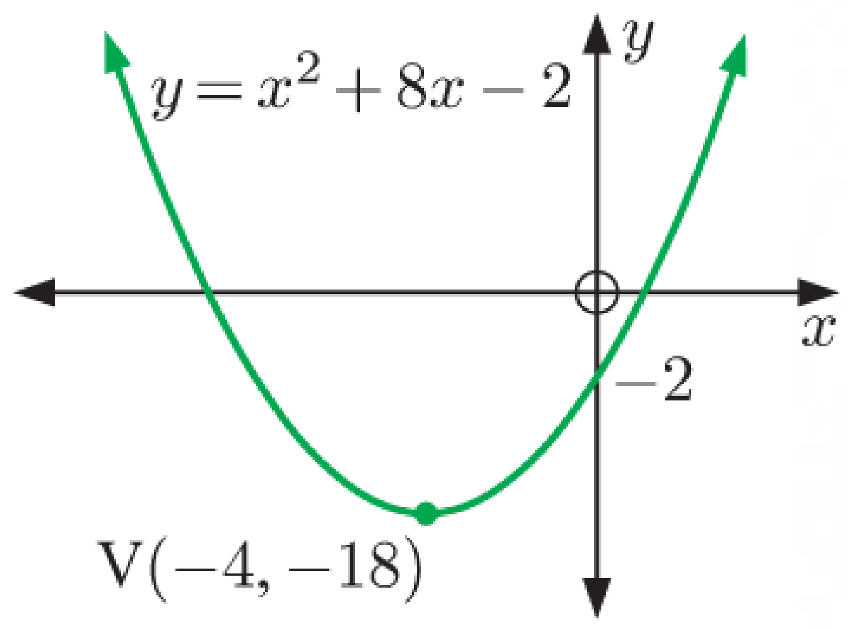


h $y = x^2 + 8x - 2$

$\therefore\ y = x^2 + 8x + 4^2 - 2 - 4^2$

$\therefore\ y = (x+4)^2 - 18$

The vertex is $(-4, -18)$, and the y-intercept is -2.

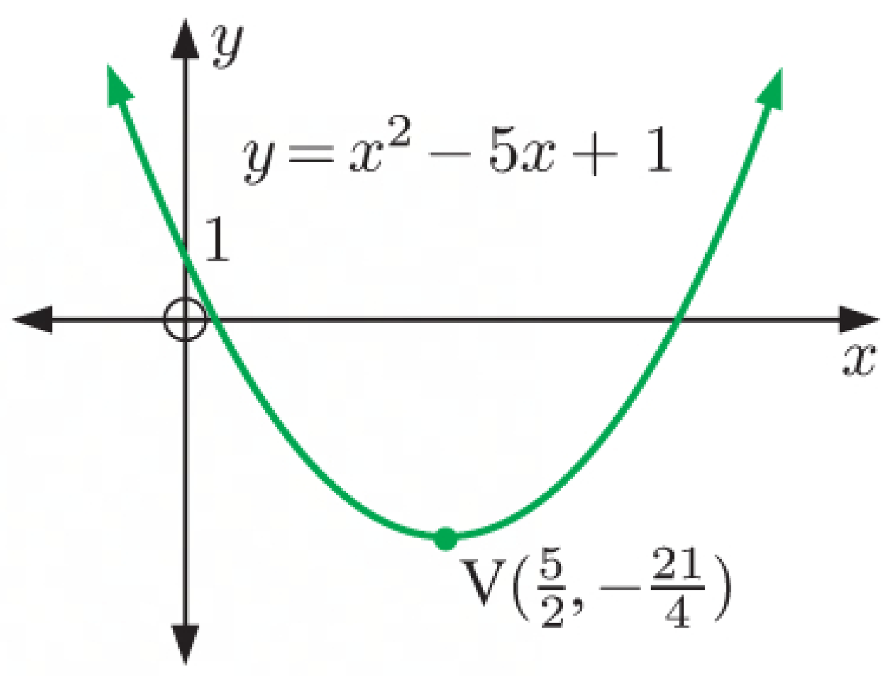


i $y = x^2 - 5x + 1$

$\therefore\ y = x^2 - 5x + (-\frac{5}{2})^2 + 1 - (-\frac{5}{2})^2$

$\therefore\ y = (x - \frac{5}{2})^2 - \frac{21}{4}$

The vertex is $(\frac{5}{2}, -\frac{21}{4})$, and the y-intercept is 1.



2 **a** **i** $y = 2x^2 + 4x + 5$

$= 2[x^2 + 2x + \frac{5}{2}]$

$= 2[x^2 + 2x + 1^2 + \frac{5}{2} - 1^2]$

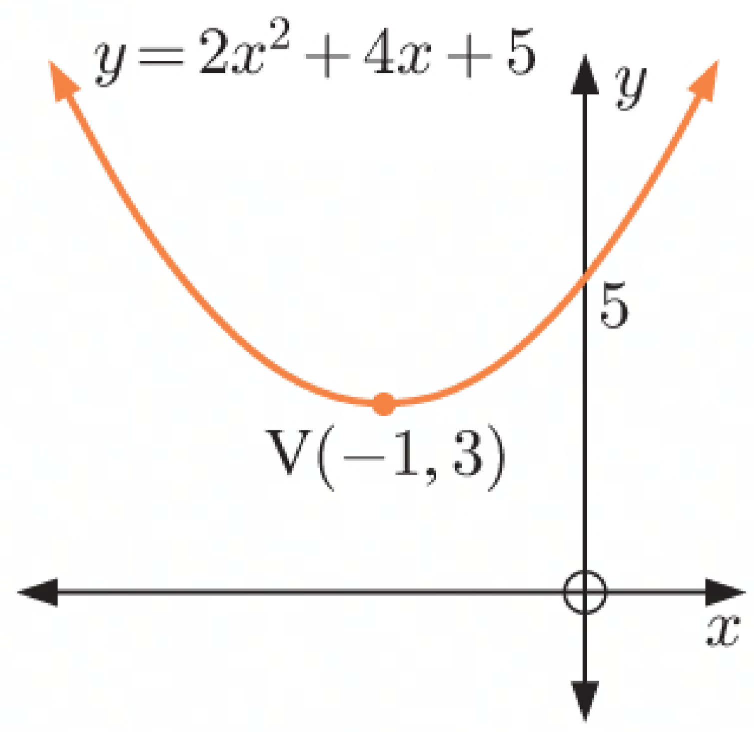
$= 2[(x+1)^2 + \frac{3}{2}]$

$\therefore\ y = 2(x+1)^2 + 3$

ii The vertex is $(-1, 3)$.

iii The y-intercept is 5.

iv



b **i** $y = 2x^2 - 8x + 3$

$= 2[x^2 - 4x + \frac{3}{2}]$

$= 2[x^2 - 4x + (-2)^2 + \frac{3}{2} - (-2)^2]$

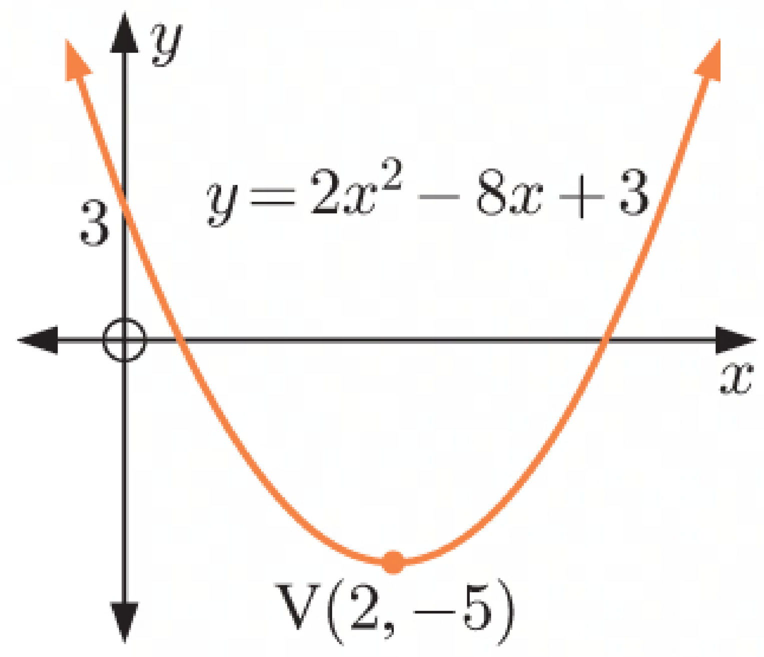
$= 2[(x-2)^2 - \frac{5}{2}]$

$\therefore\ y = 2(x-2)^2 - 5$

ii The vertex is $(2, -5)$.

iii The y-intercept is 3.

iv



c **i** $y = 2x^2 - 6x + 1$

$= 2[x^2 - 3x + \frac{1}{2}]$

$= 2[x^2 - 3x + (-\frac{3}{2})^2 + \frac{1}{2} - (-\frac{3}{2})^2]$

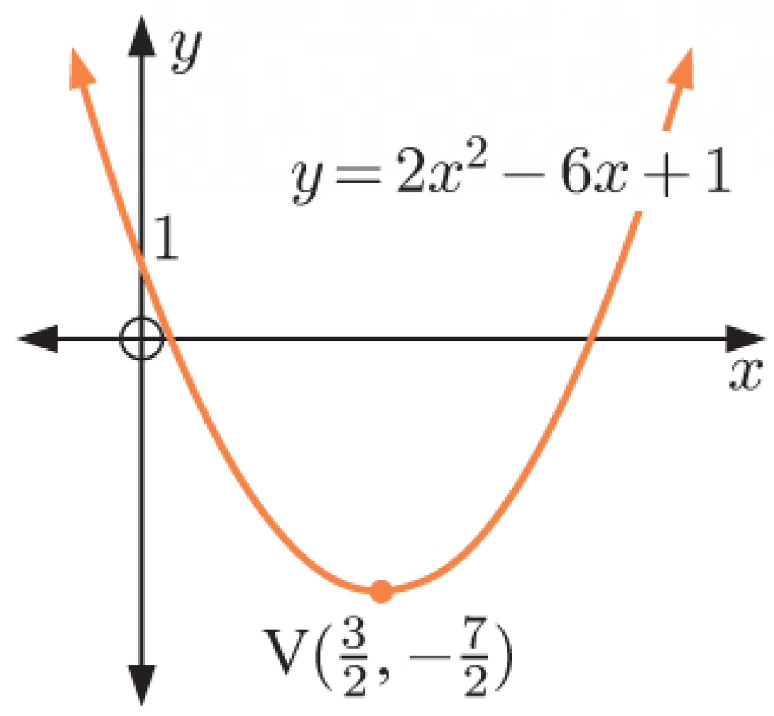
$= 2[(x - \frac{3}{2})^2 - \frac{7}{4}]$

$\therefore\ y = 2(x - \frac{3}{2})^2 - \frac{7}{2}$

ii The vertex is $(\frac{3}{2}, -\frac{7}{2})$.

iii The y-intercept is 1.

iv



d **i** $y = 3x^2 - 6x + 5$

$= 3[x^2 - 2x + \frac{5}{3}]$

$= 3[x^2 - 2x + (-1)^2 + \frac{5}{3} - (-1)^2]$

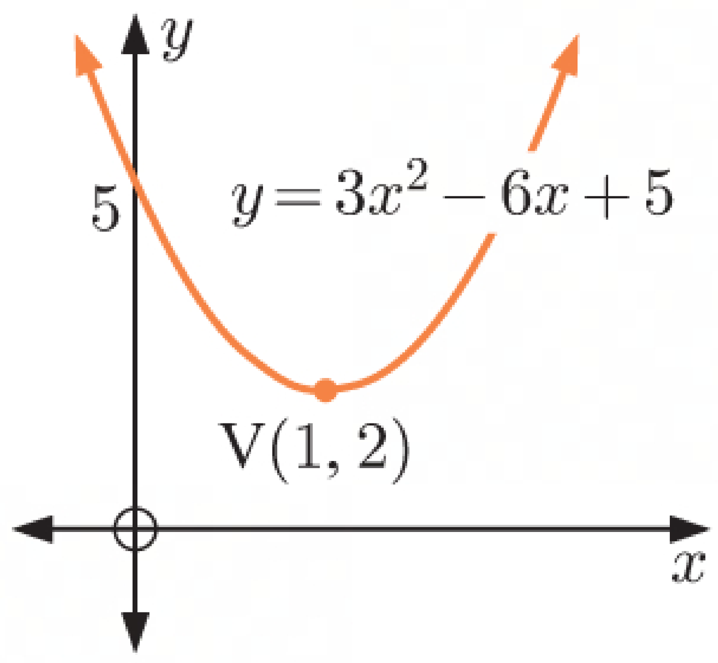
$= 3[(x-1)^2 + \frac{2}{3}]$

$\therefore\ y = 3(x-1)^2 + 2$

ii The vertex is $(1, 2)$.

iii The y-intercept is 5.

iv



e **i** $y = -x^2 + 4x + 2$

$= -[x^2 - 4x - 2]$

$= -[x^2 - 4x + (-2)^2 - 2 - (-2)^2]$

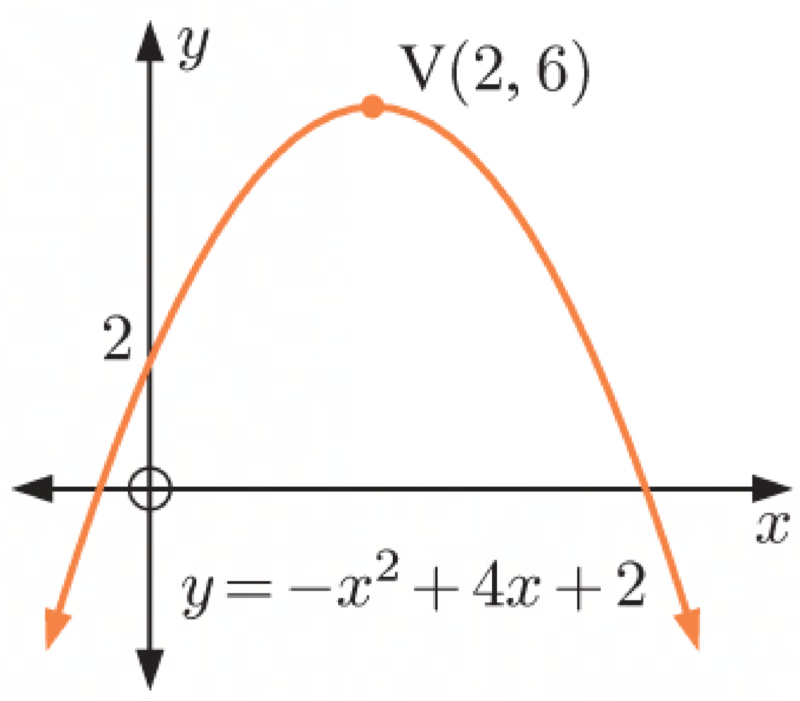
$= -[(x-2)^2 - 6]$

$\therefore\ y = -(x-2)^2 + 6$

ii The vertex is $(2, 6)$.

iii The y-intercept is 2.

iv



f **i** $y = -2x^2 - 5x + 3$

$= -2[x^2 + \frac{5}{2}x - \frac{3}{2}]$

$= -2[x^2 + \frac{5}{2}x + (\frac{5}{4})^2 - \frac{3}{2} - (\frac{5}{4})^2]$

$= -2[(x + \frac{5}{4})^2 - \frac{24}{16} - \frac{25}{16}]$

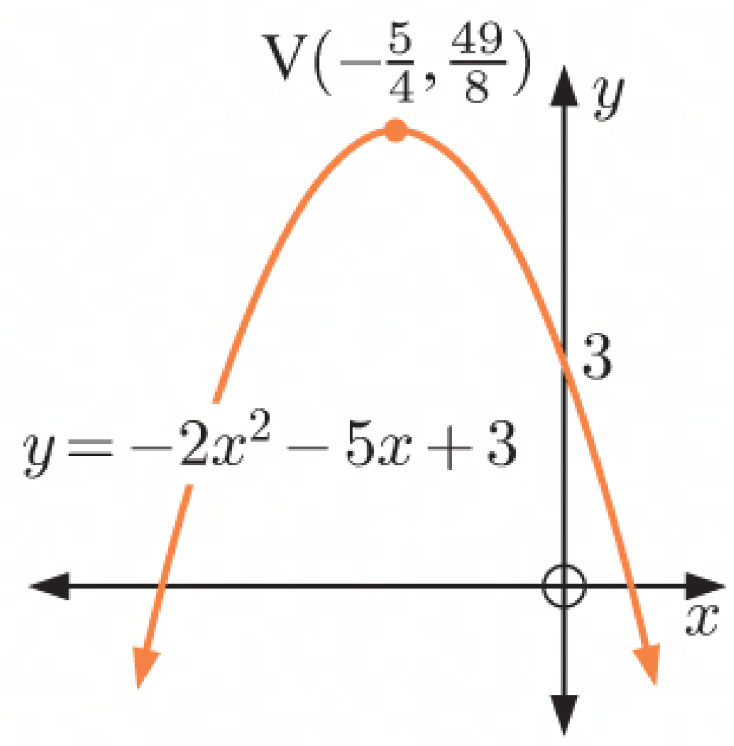
$= -2[(x + \frac{5}{4})^2 - \frac{49}{16}]$

$\therefore\ y = -2(x + \frac{5}{4})^2 + \frac{49}{8}$

ii The vertex is $(-\frac{5}{4}, \frac{49}{8})$.

iii The y-intercept is 3.

iv



g **i** $y = -\frac{1}{3}x^2 + 2x - 3$

$= -\frac{1}{3}[x^2 - 6x + 9]$

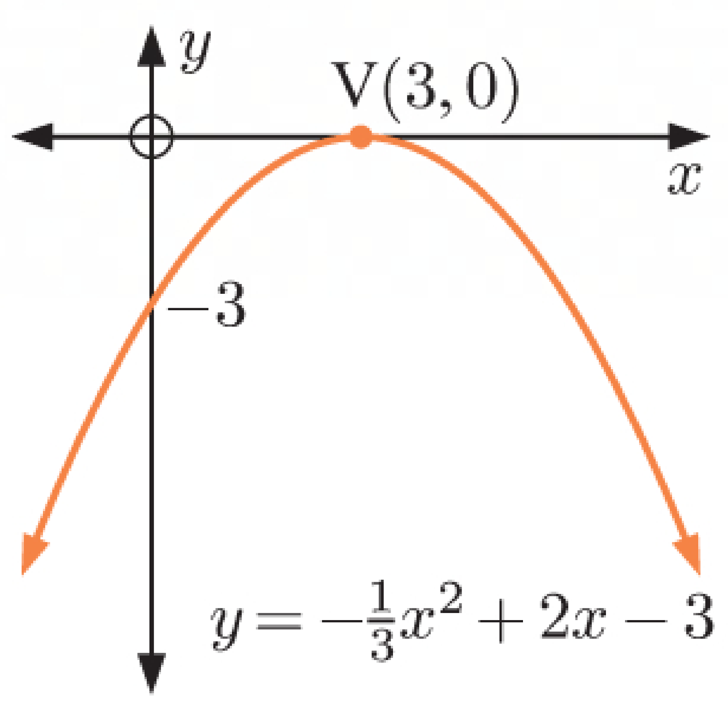
$= -\frac{1}{3}[x^2 - 6x + 3^2]$

$\therefore\ y = -\frac{1}{3}(x-3)^2$

ii The vertex is $(3, 0)$.

iii The y-intercept is -3.

iv



h **i** $y = \frac{1}{2}x^2 + 3x - 4$

$= \frac{1}{2}[x^2 + 6x - 8]$

$= \frac{1}{2}[x^2 + 6x + 3^2 - 8 - 3^2]$

$= \frac{1}{2}[(x+3)^2 - 8 - 9]$

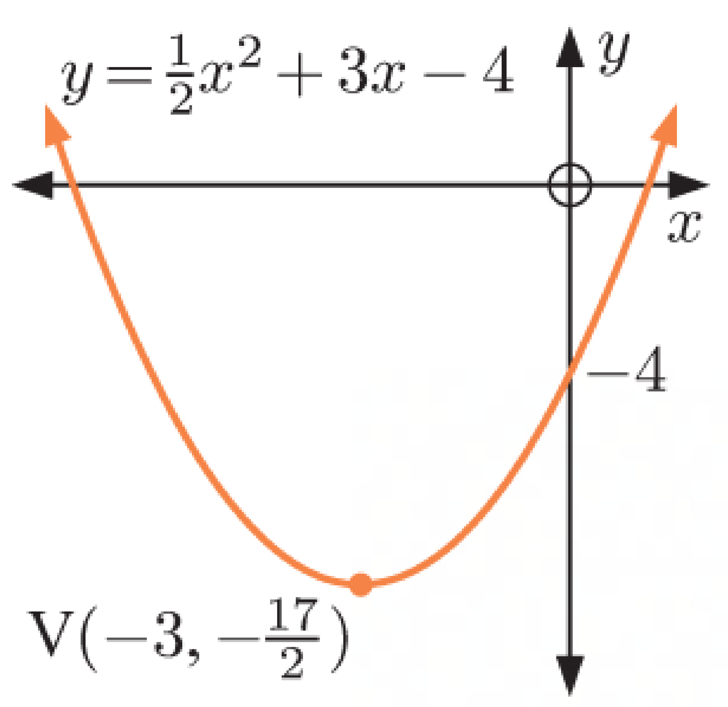
$= \frac{1}{2}[(x+3)^2 - 17]$

$\therefore\ y = \frac{1}{2}(x+3)^2 - \frac{17}{2}$

ii The vertex is $(-3, -\frac{17}{2})$.

iii The y-intercept is -4.

iv



EXERCISE 14B.3

1 **a** **i** $y = x^2 - 4x + 2$
has $a = 1,\ b = -4,\ c = 2$

$\dfrac{-b}{2a} = \dfrac{-(-4)}{2(1)} = 2$

The axis of symmetry is $x = 2$.

When $x = 2$,
$y = 2^2 - 4(2) + 2 = -2$

$\therefore$ the vertex is $(2, -2)$.

ii $a > 0$, so the shape is 

$\therefore$ the vertex $(2, -2)$ is a minimum turning point.

b **i** $y = x^2 + 2x - 3$
has $a = 1,\ b = 2,\ c = -3$

$\dfrac{-b}{2a} = \dfrac{-2}{2(1)} = -1$

The axis of symmetry is $x = -1$.

When $x = -1$,
$y = (-1)^2 + 2(-1) - 3 = -4$

$\therefore$ the vertex is $(-1, -4)$.

ii $a > 0$, so the shape is 

$\therefore$ the vertex $(-1, -4)$ is a minimum turning point.

c **i** $y = 2x^2 + 4$
has $a = 2,\ b = 0,\ c = 4$

$\dfrac{-b}{2a} = \dfrac{-0}{2(2)} = 0$

The axis of symmetry is $x = 0$.

When $x = 0$, $y = 4$

$\therefore$ the vertex is $(0, 4)$.

ii $a > 0$, so the shape is 

$\therefore$ the vertex $(0, 4)$ is a minimum turning point.

d **i** $y = -3x^2 + 1$
has $a = -3,\ b = 0,\ c = 1$

$\dfrac{-b}{2a} = \dfrac{-0}{2(-3)} = 0$

The axis of symmetry is $x = 0$.

When $x = 0$, $y = 1$

$\therefore$ the vertex is $(0, 1)$.

ii $a < 0$, so the shape is 

$\therefore$ the vertex $(0, 1)$ is a maximum turning point.

e **i** $y = 2x^2 + 8x - 7$
has $a = 2,\ b = 8,\ c = -7$

$\dfrac{-b}{2a} = \dfrac{-8}{2(2)} = -2$

The axis of symmetry is $x = -2$.

When $x = -2$,
$y = 2(-2)^2 + 8(-2) - 7$
$= -15$

$\therefore$ the vertex is $(-2, -15)$.

ii $a > 0$, so the shape is 

$\therefore$ the vertex $(-2, -15)$ is a minimum turning point.

f **i** $y = -x^2 - 4x - 9$
has $a = -1,\ b = -4,\ c = -9$

$\dfrac{-b}{2a} = \dfrac{-(-4)}{2(-1)} = -2$

The axis of symmetry is $x = -2$.

When $x = -2$,
$y = -(-2)^2 - 4(-2) - 9$
$= -4 + 8 - 9$
$= -5$

$\therefore$ the vertex is $(-2, -5)$.

ii $a < 0$, so the shape is 

$\therefore$ the vertex $(-2, -5)$ is a maximum turning point.

g **i** $y = 2x^2 + 6x - 1$
has $a = 2,\ b = 6,\ c = -1$

$\dfrac{-b}{2a} = \dfrac{-6}{2(2)} = -\dfrac{3}{2}$

The axis of symmetry is $x = -\frac{3}{2}$.

When $x = -\frac{3}{2}$,

$$\begin{aligned} y &= 2(-\tfrac{3}{2})^2 + 6(-\tfrac{3}{2}) - 1 \\ &= \tfrac{9}{2} - 9 - 1 \\ &= -\tfrac{11}{2} \end{aligned}$$

$\therefore$ the vertex is $(-\frac{3}{2}, -\frac{11}{2})$.

ii $a > 0$, so the shape is 

$\therefore$ the vertex $(-\frac{3}{2}, -\frac{11}{2})$ is a minimum turning point.

h **i** $y = 2x^2 - 10x + 3$
has $a = 2,\ b = -10,\ c = 3$

$\dfrac{-b}{2a} = \dfrac{-(-10)}{2(2)} = \dfrac{5}{2}$

The axis of symmetry is $x = \frac{5}{2}$.

When $x = \frac{5}{2}$,

$$\begin{aligned} y &= 2(\tfrac{5}{2})^2 - 10(\tfrac{5}{2}) + 3 \\ &= \tfrac{25}{2} - \tfrac{50}{2} + 3 \\ &= -\tfrac{19}{2} \end{aligned}$$

$\therefore$ the vertex is $(\frac{5}{2}, -\frac{19}{2})$.

ii $a > 0$, so the shape is 

$\therefore$ the vertex $(\frac{5}{2}, -\frac{19}{2})$ is a minimum turning point.

i **i** $y = -\frac{1}{2}x^2 + x - 5$
has $a = -\frac{1}{2},\ b = 1,\ c = -5$

$\dfrac{-b}{2a} = \dfrac{-1}{2(-\frac{1}{2})} = 1$

The axis of symmetry is $x = 1$.

When $x = 1$,

$$\begin{aligned} y &= -\tfrac{1}{2}(1)^2 + 1 - 5 \\ &= -\tfrac{9}{2} \end{aligned}$$

$\therefore$ the vertex is $(1, -\frac{9}{2})$.

ii $a < 0$, so the shape is 

$\therefore$ the vertex $(1, -\frac{9}{2})$ is a maximum turning point.

j **i** $y = \frac{1}{4}x^2 - 7x + 6$
has $a = \frac{1}{4},\ b = -7,\ c = 6$

$\dfrac{-b}{2a} = \dfrac{-(-7)}{2(\frac{1}{4})} = 14$

The axis of symmetry is $x = 14$.

When $x = 14$,

$$\begin{aligned} y &= \tfrac{1}{4}(14)^2 - 7(14) + 6 \\ &= -43 \end{aligned}$$

$\therefore$ the vertex is $(14, -43)$.

ii $a > 0$, so the shape is 

$\therefore$ the vertex $(14, -43)$ is a minimum turning point.

2 **a** $y = x^2 - 8x + 7$ has $a = 1,\ b = -8,\ c = 7$. Since $a > 0$, the shape is 

i $\dfrac{-b}{2a} = \dfrac{-(-8)}{2(1)} = 4$

The axis of symmetry is $x = 4$.

ii When $x = 4$, $y = 4^2 - 8(4) + 7$
$= -9$

$\therefore$ the vertex is $(4, -9)$.

iii The y-intercept is 7.

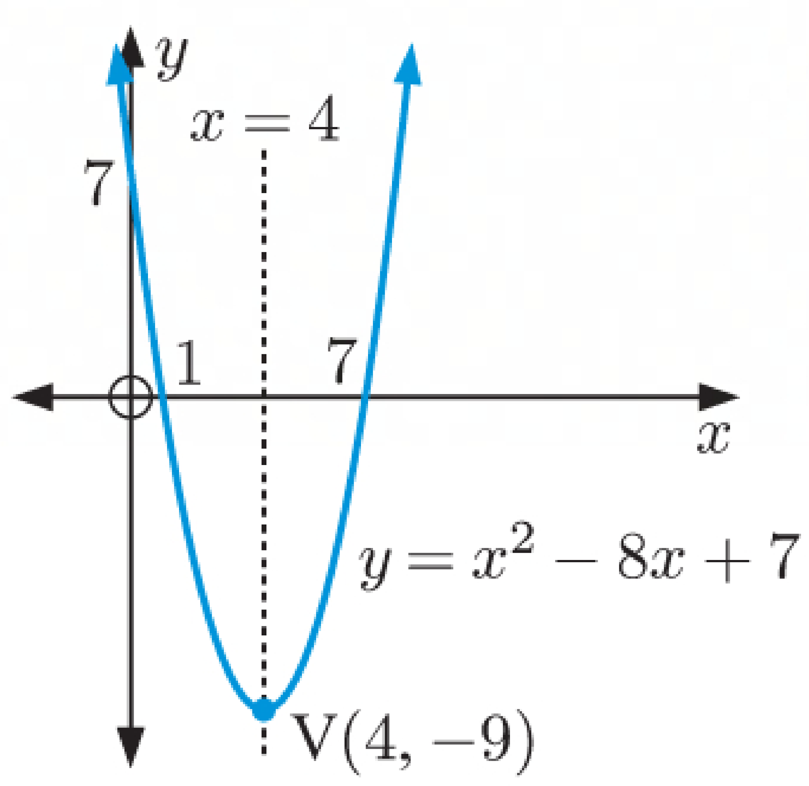
When $y = 0$, $x^2 - 8x + 7 = 0$

$\therefore (x - 1)(x - 7) = 0$

$\therefore x = 1$ or 7

$\therefore$ the x-intercepts are 1 and 7.

iv



b $y = -x^2 - 6x - 8$ has $a = -1,\ b = -6,\ c = -8.$

Since $a < 0$, the shape is 

i $\dfrac{-b}{2a} = \dfrac{-(-6)}{2(-1)} = -3$

The axis of symmetry is $x = -3$.

ii When $x = -3$,

$$y = -(-3)^2 - 6(-3) - 8$$
$$= 1$$

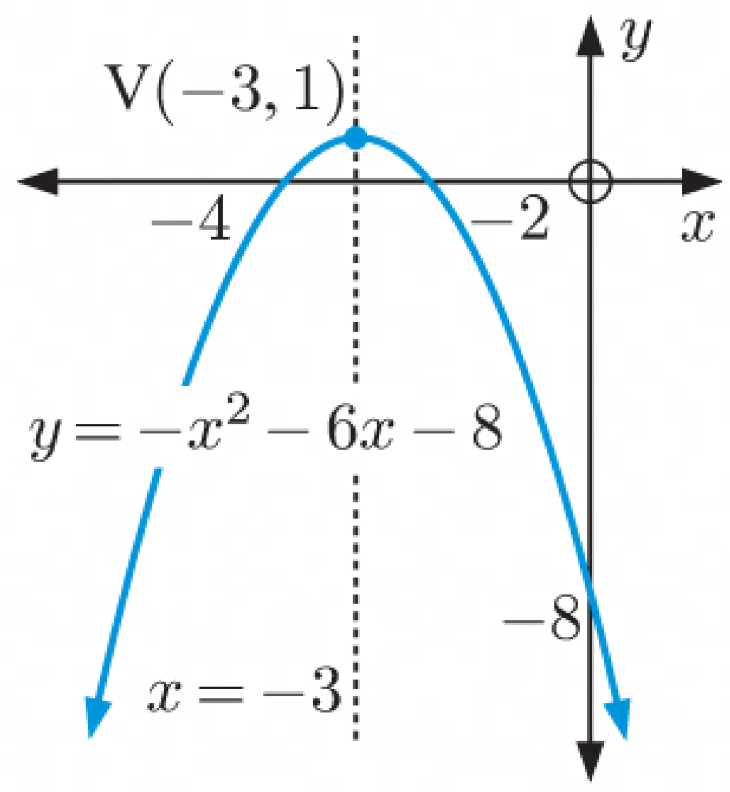
$\therefore$ the vertex is $(-3,\ 1)$.

iii The y-intercept is -8.

When $y = 0$,

$$-x^2 - 6x - 8 = 0$$
$$\therefore\ -(x^2 + 6x + 8) = 0$$
$$\therefore\ -(x+2)(x+4) = 0$$
$$\therefore\ x = -2 \text{ or } -4$$

$\therefore$ the x-intercepts are -2 and -4.

iv 

c $y = 6x - x^2$ has $a = -1,\ b = 6,\ c = 0.$ Since $a < 0$, the shape is 

i $\dfrac{-b}{2a} = \dfrac{-6}{2(-1)} = 3$

The axis of symmetry is $x = 3$.

ii When $x = 3$, $y = 6(3) - 3^2$

$$= 9$$

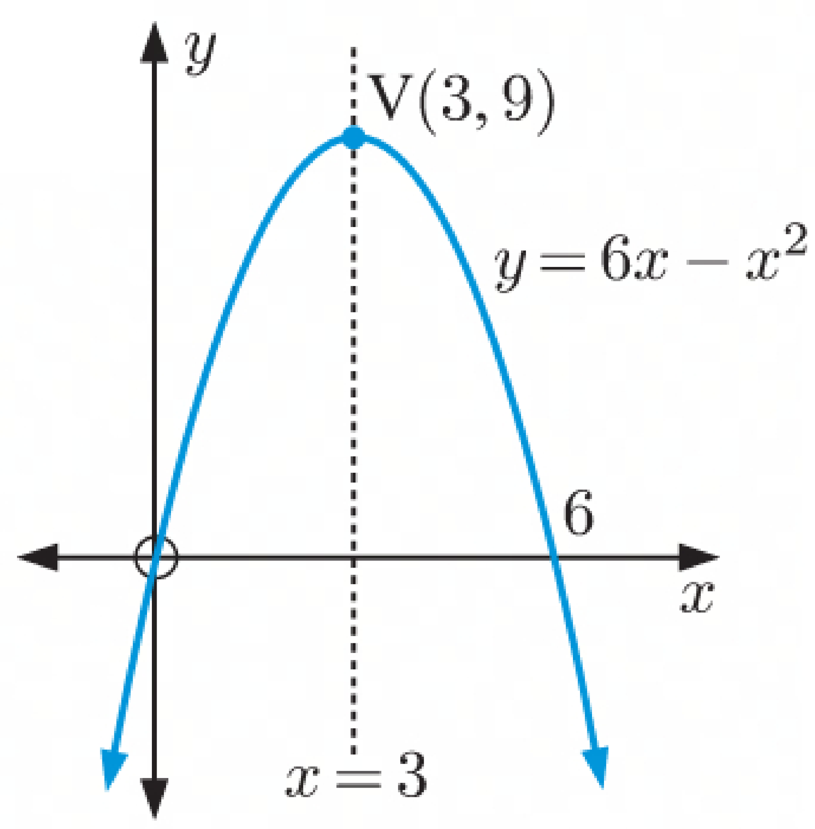
$\therefore$ the vertex is $(3,\ 9)$.

iii The y-intercept is 0.

When $y = 0$, $6x - x^2 = 0$

$$\therefore\ x(6 - x) = 0$$
$$\therefore\ x = 0 \text{ or } 6$$

$\therefore$ the x-intercepts are 0 and 6.

iv 

d $y = -x^2 + 3x - 2$ has $a = -1$, $b = 3$, $c = -2$. Since $a < 0$, the shape is 

i $\dfrac{-b}{2a} = \dfrac{-3}{2(-1)} = \dfrac{3}{2}$

The axis of symmetry is $x = \frac{3}{2}$.

ii When $x = \frac{3}{2}$,

$$\begin{aligned} y &= -(\tfrac{3}{2})^2 + 3(\tfrac{3}{2}) - 2 \\ &= -\tfrac{9}{4} + \tfrac{18}{4} - 2 \\ &= \tfrac{1}{4} \end{aligned}$$

$\therefore$ the vertex is $(\frac{3}{2}, \frac{1}{4})$.

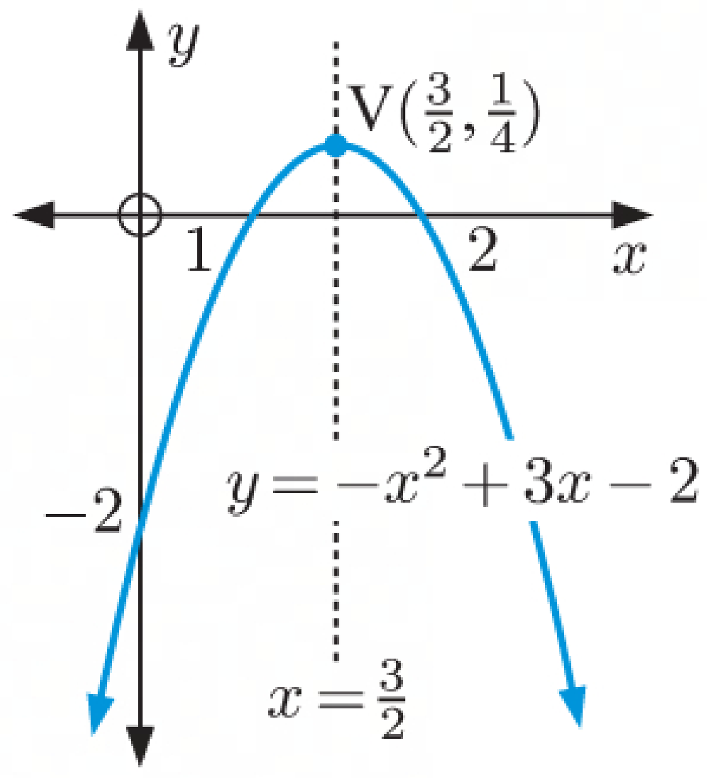
iii The y-intercept is -2.

When $y = 0$,

$$\begin{aligned} -x^2 + 3x - 2 &= 0 \\ \therefore\ -(x^2 - 3x + 2) &= 0 \\ \therefore\ -(x-1)(x-2) &= 0 \\ \therefore\ x &= 1 \text{ or } 2 \end{aligned}$$

$\therefore$ the x-intercepts are 1 and 2.

iv



e $y = 2x^2 + 4x - 24$ has $a = 2$, $b = 4$, $c = -24$. Since $a > 0$, the shape is 

i $\dfrac{-b}{2a} = \dfrac{-4}{2(2)} = -1$

The axis of symmetry is $x = -1$.

ii When $x = -1$,

$$\begin{aligned} y &= 2(-1)^2 + 4(-1) - 24 \\ &= -26 \end{aligned}$$

$\therefore$ the vertex is $(-1, -26)$.

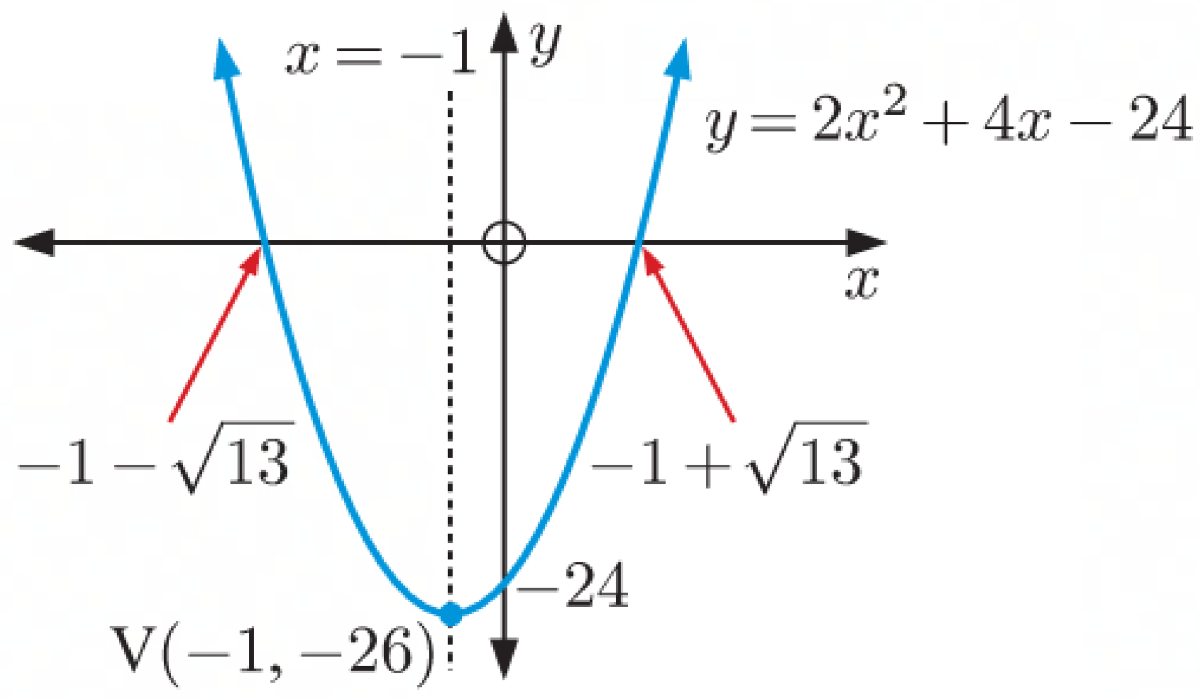
iii The y-intercept is -24.

When $y = 0$,

$$\begin{aligned} 2x^2 + 4x - 24 &= 0 \\ \therefore\ x^2 + 2x - 12 &= 0 \\ \therefore\ x^2 + 2x &= 12 \\ \therefore\ x^2 + 2x + 1^2 &= 12 + 1^2 \\ \therefore\ (x+1)^2 &= 13 \\ \therefore\ x + 1 &= \pm\sqrt{13} \\ \therefore\ x &= -1 \pm \sqrt{13} \end{aligned}$$

$\therefore$ the x-intercepts are $-1 \pm \sqrt{13}$.

iv



f $y = -3x^2 + 4x - 1$ has $a = -3$, $b = 4$, $c = -1$.

Since $a < 0$, the shape is 

i $\dfrac{-b}{2a} = \dfrac{-4}{2(-3)} = \dfrac{2}{3}$

The axis of symmetry is $x = \frac{2}{3}$.

ii When $x = \frac{2}{3}$,

$$y = -3(\tfrac{2}{3})^2 + 4(\tfrac{2}{3}) - 1$$
$$= -\tfrac{4}{3} + \tfrac{8}{3} - 1$$
$$= \tfrac{1}{3}$$

$\therefore$ the vertex is $(\frac{2}{3}, \frac{1}{3})$.

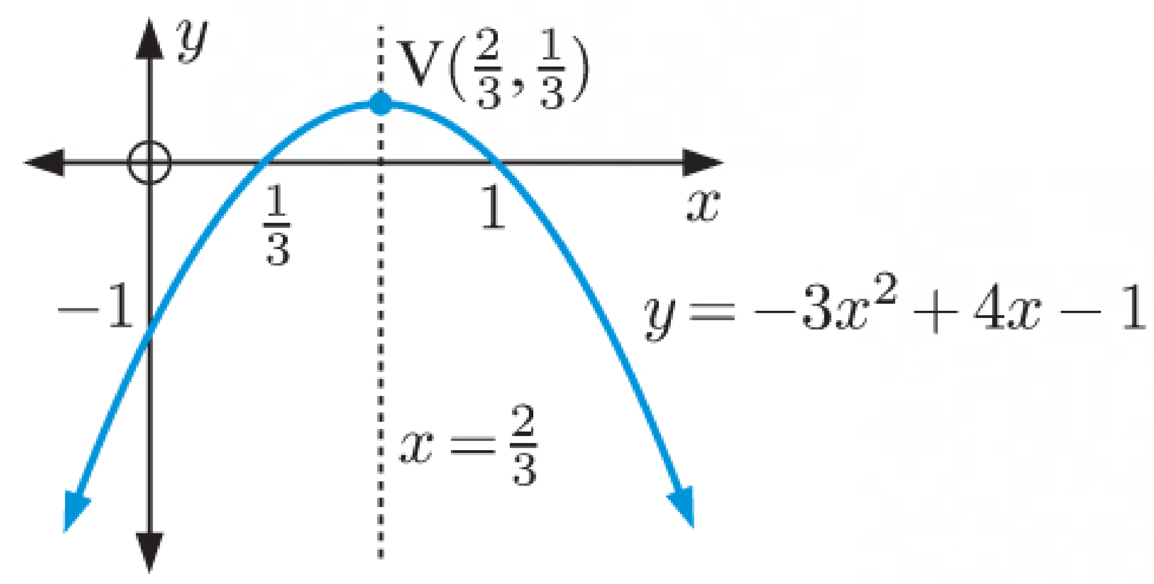
iii The y-intercept is -1.

When $y = 0$,

$$-3x^2 + 4x - 1 = 0$$
$$\therefore \; -(3x^2 - 4x + 1) = 0$$
$$\therefore \; -(3x - 1)(x - 1) = 0$$
$$\therefore \; x = \tfrac{1}{3} \text{ or } 1$$

$\therefore$ the x-intercepts are $\frac{1}{3}$ and 1.

iv



g $y = 2x^2 - 5x + 2$ has $a = 2$, $b = -5$, $c = 2$. Since $a > 0$, the shape is 

i $\dfrac{-b}{2a} = \dfrac{-(-5)}{2(2)} = \dfrac{5}{4}$

The axis of symmetry is $x = \frac{5}{4}$.

ii When $x = \frac{5}{4}$,

$$y = 2(\tfrac{5}{4})^2 - 5(\tfrac{5}{4}) + 2$$
$$= -\tfrac{9}{8}$$

$\therefore$ the vertex is $(\frac{5}{4}, -\frac{9}{8})$.

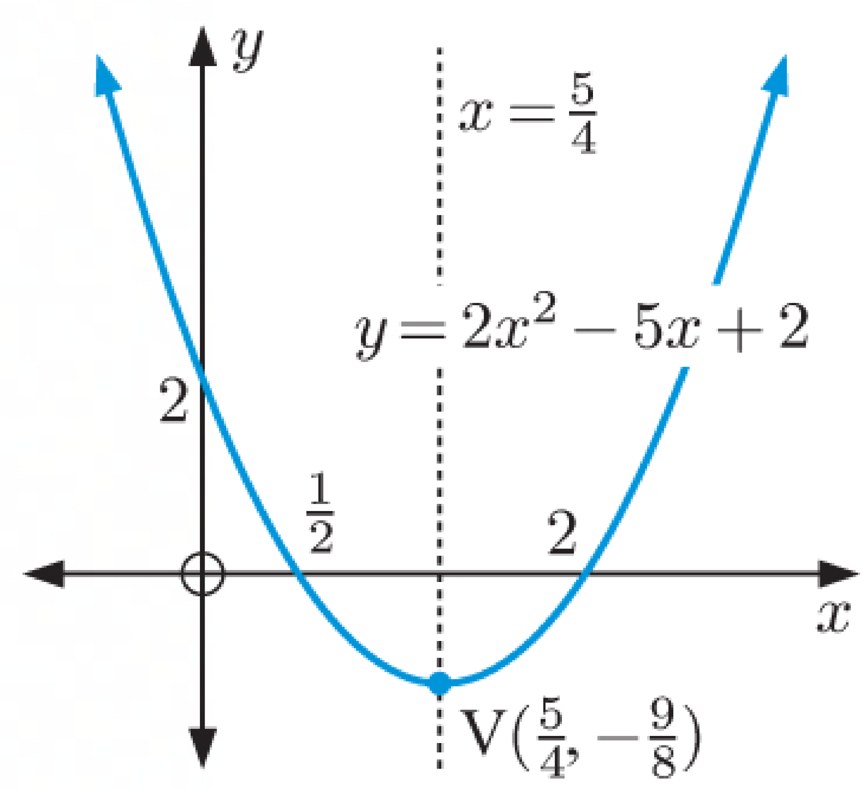
iii The y-intercept is 2.

When $y = 0$,

$$2x^2 - 5x + 2 = 0$$
$$\therefore \; (2x - 1)(x - 2) = 0$$
$$\therefore \; x = \tfrac{1}{2} \text{ or } 2$$

$\therefore$ the x-intercepts are $\frac{1}{2}$ and 2.

iv



h $y = 4x^2 - 8x - 5$ has $a = 4$, $b = -8$, $c = -5$. Since $a > 0$, the shape is 

i $\dfrac{-b}{2a} = \dfrac{-(-8)}{2(4)} = 1$

The axis of symmetry is $x = 1$.

ii When $x = 1$, $y = 4(1)^2 - 8(1) - 5$
$= -9$

$\therefore$ the vertex is $(1, -9)$.

iii The y-intercept is -5.

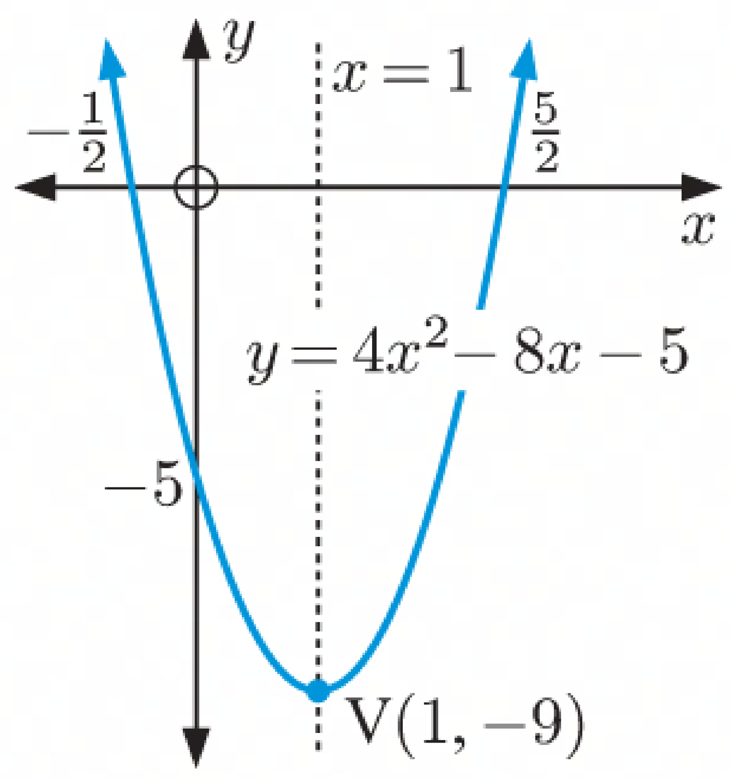
When $y = 0$, $4x^2 - 8x - 5 = 0$

$\therefore$ $(2x+1)(2x-5) = 0$

$\therefore$ $x = -\frac{1}{2}$ or $\frac{5}{2}$

$\therefore$ the x-intercepts are $-\frac{1}{2}$ and $\frac{5}{2}$.

iv



i $y = -\frac{1}{4}x^2 + 2x - 3$ has $a = -\frac{1}{4}$, $b = 2$, $c = -3$.

Since $a < 0$, the shape is 

i $\dfrac{-b}{2a} = \dfrac{-2}{2(-\frac{1}{4})} = 4$

The axis of symmetry is $x = 4$.

ii When $x = 4$, $y = -\frac{1}{4}(4)^2 + 2(4) - 3$
$= 1$

$\therefore$ the vertex is $(4, 1)$.

iii The y-intercept is -3.

When $y = 0$, $-\frac{1}{4}x^2 + 2x - 3 = 0$

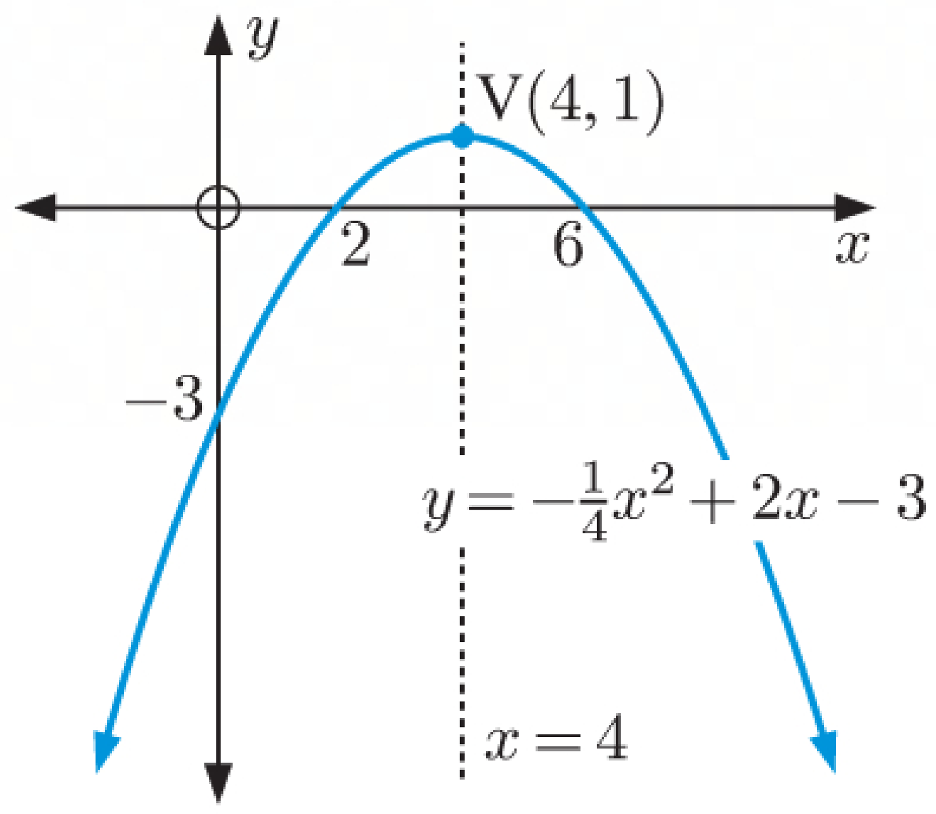
$\therefore$ $x^2 - 8x + 12 = 0$

$\therefore$ $(x-2)(x-6) = 0$

$\therefore$ $x = 2$ or 6

$\therefore$ the x-intercepts are 2 and 6.

iv



3 $y = ax^2 + bx + c$ has vertex with x-coordinate $\dfrac{-b}{2a} = X$

$$\text{and } y\text{-coordinate} \quad a\left(\frac{-b}{2a}\right)^2 + b\left(\frac{-b}{2a}\right) + c$$
$$= \frac{b^2}{4a} - \frac{b^2}{2a} + c$$
$$= c - \frac{b^2}{4a}$$
$$= c - a\left(\frac{-b}{2a}\right)^2$$
$$= c - aX^2$$

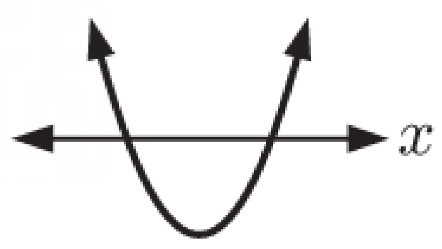
$\therefore$ the quadratic function traced out by the vertex is $y = c - ax^2$, which itself has vertex $(0, c)$.

EXERCISE 14C

1 **a** $y = x^2 + x - 2$
has $a = 1,\ b = 1,\ c = -2$

$$\therefore\ \Delta = b^2 - 4ac$$
$$= 1^2 - 4(1)(-2)$$
$$= 9$$

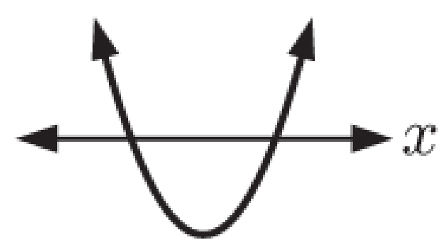
Since $\Delta > 0$, the graph cuts the x-axis twice.
Since $a > 0$, the graph is concave up.



b $y = x^2 - 4x + 1$
has $a = 1,\ b = -4,\ c = 1$

$$\therefore\ \Delta = b^2 - 4ac$$
$$= (-4)^2 - 4(1)(1)$$
$$= 12$$

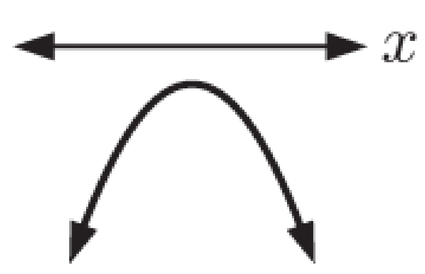
Since $\Delta > 0$, the graph cuts the x-axis twice.
Since $a > 0$, the graph is concave up.



c $y = -x^2 - 3$
has $a = -1,\ b = 0,\ c = -3$

$$\therefore\ \Delta = b^2 - 4ac$$
$$= 0^2 - 4(-1)(-3)$$
$$= -12$$

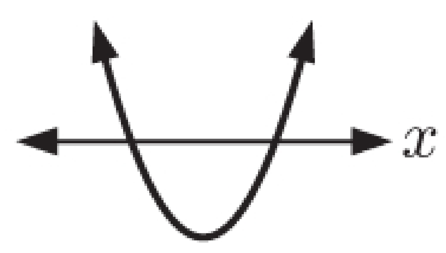
Since $\Delta < 0$, the graph does not cut the x-axis.
Since $a < 0$, the graph is concave down.
The graph is negative definite.



d $y = x^2 + 7x - 2$
has $a = 1,\ b = 7,\ c = -2$

$$\therefore\ \Delta = b^2 - 4ac$$
$$= 7^2 - 4(1)(-2)$$
$$= 57$$

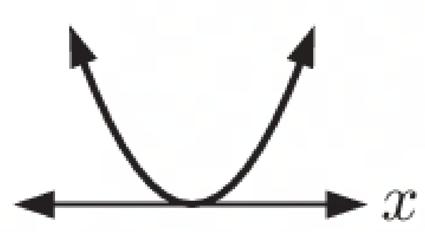
Since $\Delta > 0$, the graph cuts the x-axis twice.
Since $a > 0$, the graph is concave up.



e $y = x^2 + 8x + 16$
has $a = 1,\ b = 8,\ c = 16$

$$\therefore\ \Delta = b^2 - 4ac$$
$$= 8^2 - 4(1)(16)$$
$$= 0$$

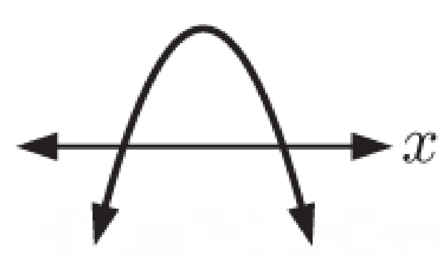
Since $\Delta = 0$, the graph touches the x-axis.
Since $a > 0$, the graph is concave up.



f $y = -2x^2 + 3x + 1$
has $a = -2,\ b = 3,\ c = 1$

$$\therefore\ \Delta = b^2 - 4ac$$
$$= 3^2 - 4(-2)(1)$$
$$= 17$$

Since $\Delta > 0$, the graph cuts the x-axis twice.
Since $a < 0$, the graph is concave down.



g $y = 6x^2 + 5x - 4$

has $a = 6,\ b = 5,\ c = -4$

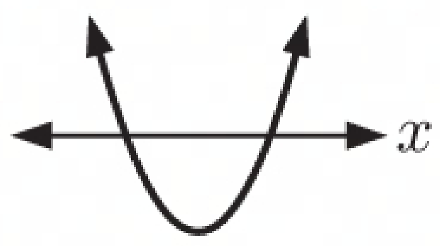
$\therefore\ \Delta = b^2 - 4ac$

$= 5^2 - 4(6)(-4)$

$= 121$

Since $\Delta > 0$, the graph cuts the x-axis twice.

Since $a > 0$, the graph is concave up.



h $y = -x^2 + x + 6$

has $a = -1,\ b = 1,\ c = 6$

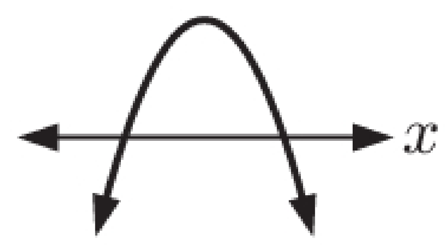
$\therefore\ \Delta = b^2 - 4ac$

$= 1^2 - 4(-1)(6)$

$= 25$

Since $\Delta > 0$, the graph cuts the x-axis twice.

Since $a < 0$, the graph is concave down.



i $y = 9x^2 + 6x + 1$

has $a = 9,\ b = 6,\ c = 1$

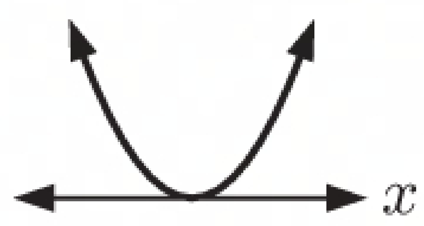
$\therefore\ \Delta = b^2 - 4ac$

$= 6^2 - 4(9)(1)$

$= 0$

Since $\Delta = 0$, the graph touches the x-axis.

Since $a > 0$, the graph is concave up.



2 $y = 2x^2 - 5x + 1$ has $a = 2,\ b = -5,\ c = 1$

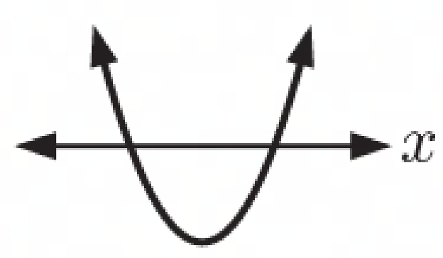
a Since $a > 0$, the graph is concave up.

b $\Delta = b^2 - 4ac$

$= (-5)^2 - 4(2)(1)$

$= 17$

Since $\Delta > 0$, the graph cuts the x-axis twice.



c When $y = 0$, $2x^2 - 5x + 1 = 0$

$$x = \frac{-b \pm \sqrt{\Delta}}{2a}$$

$$\therefore\ x = \frac{-(-5) \pm \sqrt{17}}{2(2)}$$

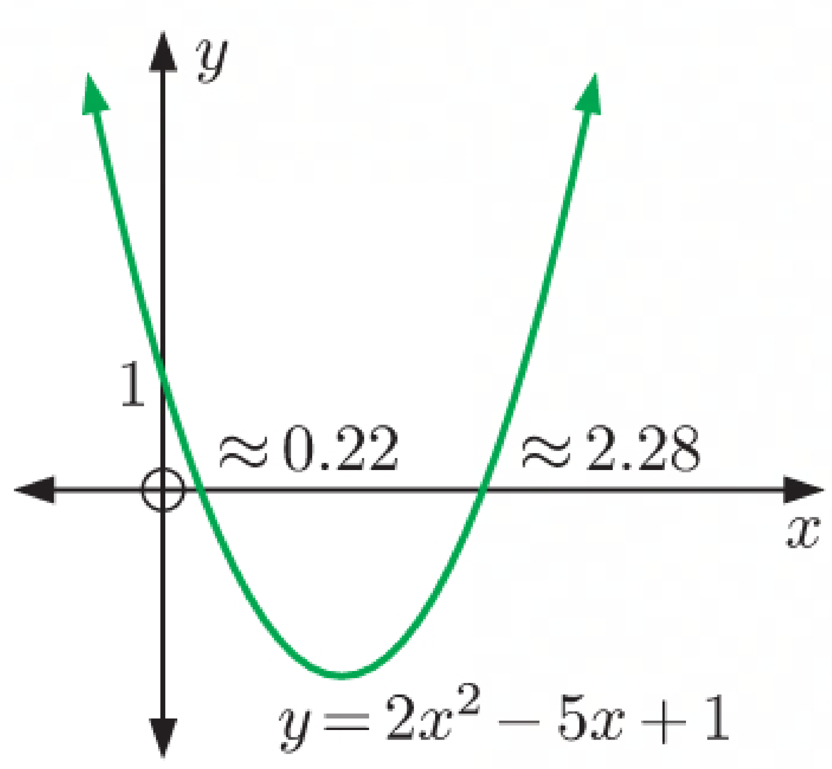
$$\therefore\ x = \frac{5}{4} \pm \frac{\sqrt{17}}{4}$$

$\therefore\ x \approx 2.28$ or 0.22

$\therefore$ the x-intercepts are ≈ 2.28 and ≈ 0.22.

d The y-intercept is 1.

e



3 **a** $y = -x^2 + 4x - 7$ has $a = -1,\ b = 4,\ c = -7$

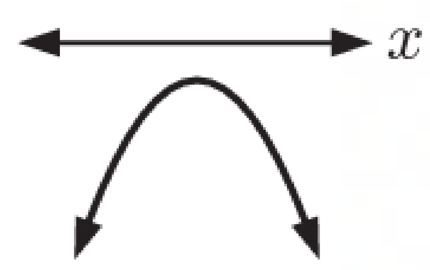
$\therefore\ \Delta = b^2 - 4ac$

$= 4^2 - 4(-1)(-7)$

$= -12$

Since $\Delta < 0$, the graph does not cut the x-axis.

b The graph is negative definite, since $a < 0$ and $\Delta < 0$. This means that it lies entirely below the x-axis.



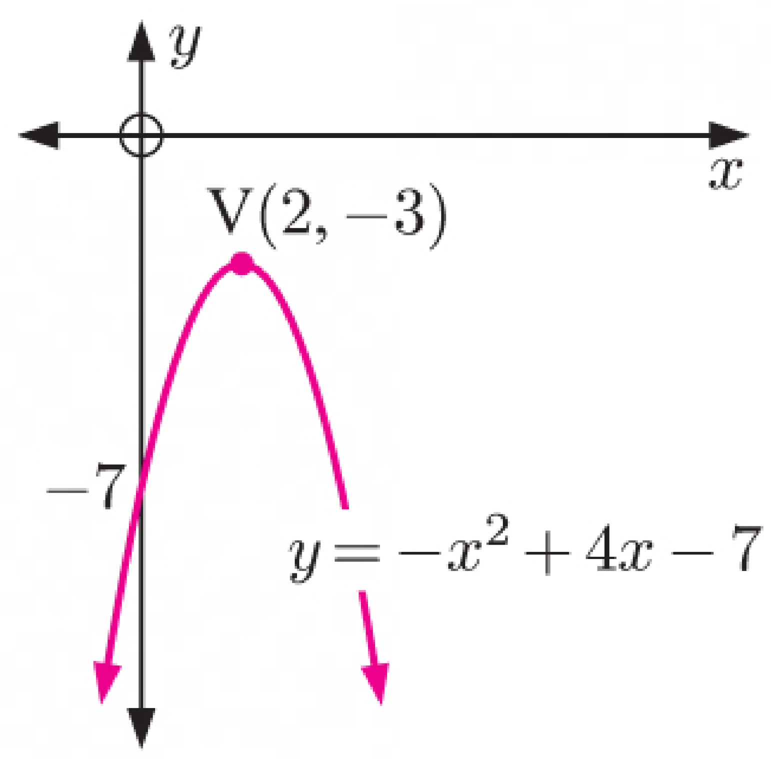
c $x = \dfrac{-b}{2a} = \dfrac{-4}{2(-1)} = 2$

When $x = 2$, $y = -2^2 + 4(2) - 7$
$= -3$

$\therefore$ the vertex is $(2, -3)$.

The y-intercept is -7.

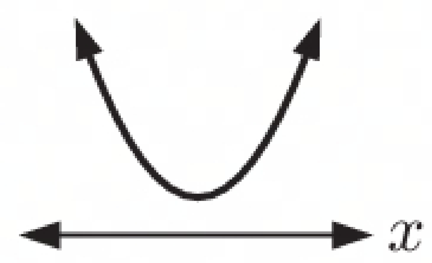
d



4 **a** $2x^2 - 4x + 7$ has $a = 2$, $b = -4$, $c = 7$

$$\begin{aligned}\therefore\ \Delta &= b^2 - 4ac \\ &= (-4)^2 - 4(2)(7) \\ &= -40\end{aligned}$$

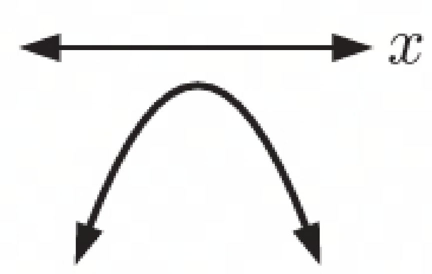
Since $\Delta < 0$, the graph does not cut the x-axis.
Since $a > 0$, the graph is concave up.
The graph is positive definite, which means that it lies entirely above the x-axis.



b $-2x^2 + 3x - 4$ has $a = -2$, $b = 3$, $c = -4$

$$\begin{aligned}\therefore\ \Delta &= b^2 - 4ac \\ &= 3^2 - 4(-2)(-4) \\ &= -23\end{aligned}$$

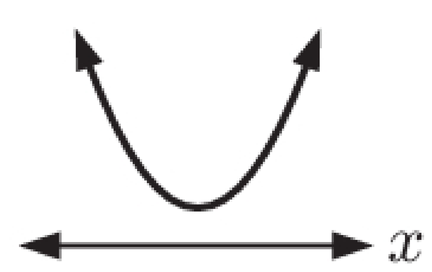
Since $\Delta < 0$, the graph does not cut the x-axis.
Since $a < 0$, the graph is concave down.
The graph is negative definite, which means that it lies entirely below the x-axis.



c $x^2 - 3x + 6$ has $a = 1$, $b = -3$, $c = 6$

$$\begin{aligned}\therefore\ \Delta &= b^2 - 4ac \\ &= (-3)^2 - 4(1)(6) \\ &= -15\end{aligned}$$

Since $\Delta < 0$, the graph does not cut the x-axis.
Since $a > 0$, the graph is concave up.
The graph is positive definite, which means that it lies entirely above the x-axis.

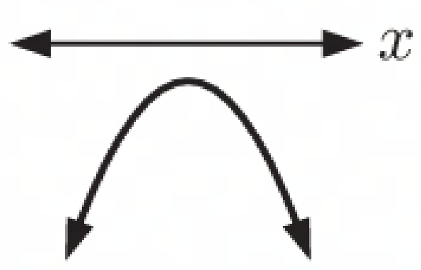


$\therefore$ $x^2 - 3x + 6 > 0$ for all x.

d $4x - x^2 - 6$ has $a = -1,\ b = 4,\ c = -6$

$\therefore\ \Delta = b^2 - 4ac$
$= 4^2 - 4(-1)(-6)$
$= -8$

Since $\Delta < 0$, the graph does not cut the x-axis.
Since $a < 0$, the graph is concave down.
The graph is negative definite, which means that it lies entirely below the x-axis.



$\therefore\ 4x - x^2 - 6 < 0$ for all x.

5 Consider $y = ax^2 + bx + c$.
The graph is concave up, so $a > 0$.
The y-intercept is positive, so $c > 0$.
The axis of symmetry is to the right of the y-axis, so $\dfrac{-b}{2a} > 0$

$\therefore\ b < 0 \quad \{a > 0\}$

The graph does not cut the x-axis, so $\Delta_1 < 0$.

Consider $y = dx^2 + ex + f$.
The graph is concave down, so $d < 0$.
The y-intercept is 0, so $f = 0$.
The axis of symmetry is to the right of the y-axis, so $\dfrac{-e}{2d} > 0$

$\therefore\ e > 0 \quad \{d < 0\}$

The graph cuts the x-axis twice, so $\Delta_2 > 0$.

Constant	a	b	c	d	e	f	Δ_1	Δ_2
Sign	+	−	+	−	+	0	−	+

6 **a** $a = 1,\ b = 3,\ c = k$

$\therefore\ \Delta = b^2 - 4ac$
$= (3)^2 - 4(1)(k)$
$= 9 - 4k$

i The graph cuts the x-axis twice if $\Delta > 0$.
$\therefore\ 9 - 4k > 0$
$\therefore\ 4k < 9$
$\therefore\ k < \frac{9}{4}$

ii The graph touches the x-axis twice if $\Delta = 0$.
$\therefore\ 9 - 4k = 0$
$\therefore\ 4k = 9$
$\therefore\ k = \frac{9}{4}$

iii The graph does not cut the x-axis if $\Delta < 0$.
$\therefore\ 9 - 4k < 0$
$\therefore\ 4k > 9$
$\therefore\ k > \frac{9}{4}$

b $a = k,\ b = -4,\ c = 1$

$\therefore\ \Delta = b^2 - 4ac$
$= (-4)^2 - 4(k)(1)$
$= 16 - 4k$

Also, if $k = 0$ then the function is the line $y = -4x + 1$, in which case it cuts the x-axis only *once* at $x = \frac{1}{4}$.

i The graph cuts the x-axis twice if $\Delta > 0$.
$\therefore\ 16 - 4k > 0$
$\therefore\ 4k < 16$
$\therefore\ k < 4,\ k \neq 0$

ii The graph touches the x-axis if $\Delta = 0$.
$\therefore\ 16 - 4k = 0$
$\therefore\ 4k = 16$
$\therefore\ k = 4$

iii The graph does not cut the x-axis if $\Delta < 0$.
$\therefore\ 16 - 4k < 0$
$\therefore\ 4k > 16$
$\therefore\ k > 4$

c $a = k+1,\ b = -2k,\ c = k-4$

$\therefore\ \Delta = b^2 - 4ac$
$= (-2k)^2 - 4(k+1)(k-4)$
$= 4k^2 - 4(k^2 - 3k - 4)$
$= 4k^2 - 4k^2 + 12k + 16$
$= 12k + 16$

Also, if $k = -1$ then the function is the line $y = 2x - 5$, in which case it cuts the x-axis only *once* at $x = \frac{5}{2}$.

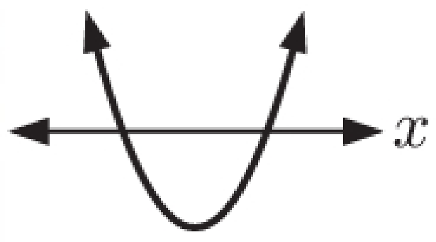
i The graph cuts the x-axis twice if $\Delta > 0$.
$\therefore\ 12k + 16 > 0$
$\therefore\ 12k > -16$
$\therefore\ k > -\frac{4}{3},\ k \neq -1$

ii The graph touches the x-axis if $\Delta = 0$.
$\therefore\ 12k + 16 = 0$
$\therefore\ 12k = -16$
$\therefore\ k = -\frac{4}{3}$

iii The graph does not cut the x-axis if $\Delta < 0$.
$\therefore\ 12k + 16 < 0$
$\therefore\ 12k < -16$
$\therefore\ k < -\frac{4}{3}$

7 $3x^2 + kx - 1$ has $a = 3,\ b = k,\ c = -1$

Since $a > 0$, the graph is concave up.



$\Delta = b^2 - 4ac$
$= k^2 - 4(3)(-1)$
$= k^2 + 12$

Now, $k^2 + 12 > 0$ for all k as $k^2 \geqslant 0$ for all real values of k.

$\therefore$ the graph cuts the x-axis twice for all k.

$\therefore$ $3x^2 + kx - 1$ is never positive definite for any value of k.

8 $y = \frac{1}{2}x^2 + (k-2)x + k^2 + 4$ has $a = \frac{1}{2},\ b = k - 2,\ c = k^2 + 4$

$\therefore\ \Delta = b^2 - 4ac$
$= (k-2)^2 - 4(\frac{1}{2})(k^2 + 4)$
$= k^2 - 4k + 4 - 2k^2 - 8$
$= -k^2 - 4k - 4$
$= -(k^2 + 4k + 4)$
$= -(k+2)^2$

So, $a > 0$, and $\Delta < 0$ for all $k \neq -2$

$\therefore$ the graph is positive definite for all $k \neq -2$

$\therefore$ the graph is *not* positive definite if $k = -2$, the graph touches the x-axis in this case.

9 Suppose one of the quadratics, say $y = x^2 + b_1x + c_1$, does not meet the x-axis.

$$\begin{aligned}
\therefore \text{ its discriminant } \Delta &< 0\\
\therefore b_1^2 - 4(1)(c_1) &< 0\\
\therefore b_1^2 &< 4c_1 \quad .... (1)\\
\therefore b_1^2b_2^2 &< 4c_1b_2^2 && \{\text{multiplying both sides by } b_2^2,\ b_2^2 > 0\}\\
\therefore [2(c_1+c_2)]^2 &< 4c_1b_2^2 && \{b_1b_2 = 2(c_1+c_2)\}\\
\therefore 4(c_1+c_2)^2 &< 4c_1b_2^2\\
\therefore c_1^2 - 2c_1c_2 + c_2^2 &< c_1b_2^2 - 4c_1c_2 && \{\text{subtracting } 4c_1c_2 \text{ from both sides}\}\\
\therefore (c_1-c_2)^2 &< c_1(b_2^2 - 4c_2)\\
\therefore c_1(b_2^2-4c_2) &> (c_1-c_2)^2
\end{aligned}$$

Now $(c_1 - c_2)^2 \geqslant 0$ $\therefore$ $c_1(b_2^2 - 4c_2) > 0$

We know $c_1 > 0$ $\{4c_1 > b_1^2 > 0$ using (1)$\}$

$\therefore$ $b_2^2 - 4c_2 > 0$

$\therefore$ $y = x^2 + b_2x + c_2$ meets the x-axis $\{b_2^2 - 4c_2$ is its discriminant$\}$

$\therefore$ if one of the quadratics does not meet the x-axis, the other quadratic must meet the x-axis.

$\therefore$ at least one of the quadratics meets the x-axis.

EXERCISE 14D

1 **a** Since the x-intercepts are 1 and 2, $y = a(x-1)(x-2)$.

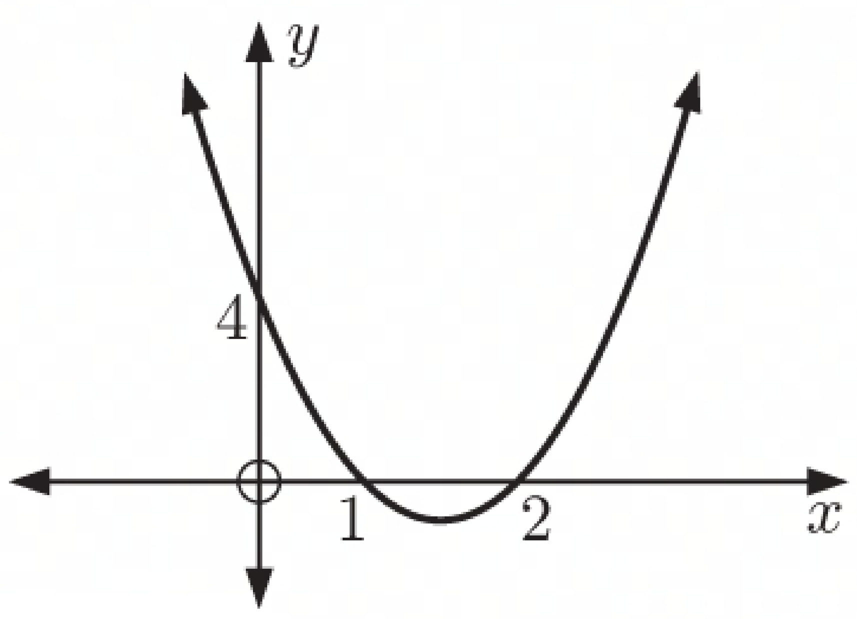
The graph is concave up, so $a > 0$.

When $x = 0$, $y = 4$

$\therefore$ $4 = a(-1)(-2)$

$\therefore$ $a = 2$

The quadratic is $y = 2(x-1)(x-2)$.



b The graph touches the x-axis at $x = 2$, so $y = a(x-2)^2$.

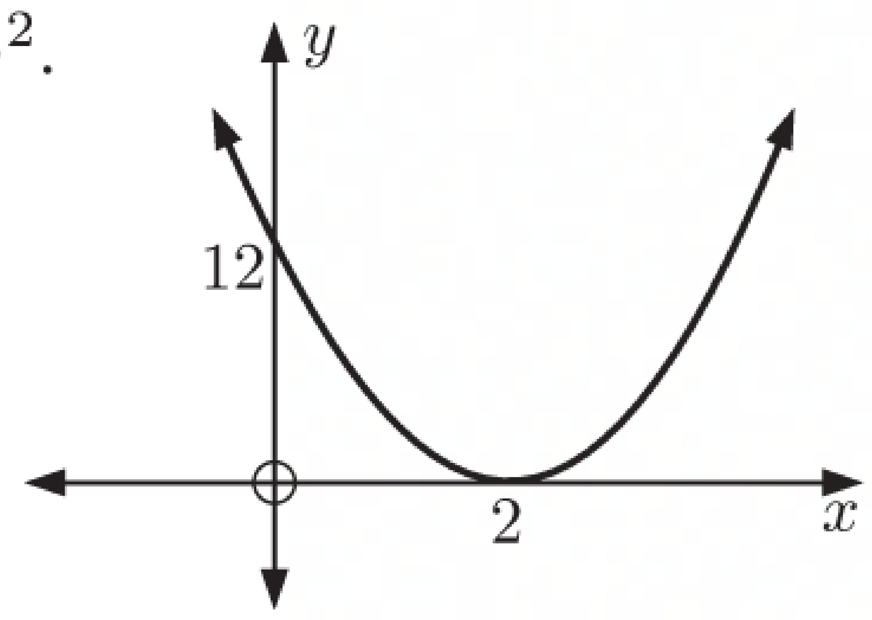
The graph is concave up, so $a > 0$.

When $x = 0$, $y = 12$

$\therefore$ $12 = a(-2)^2$

$\therefore$ $a = 3$

The quadratic is $y = 3(x-2)^2$.



c Since the x-intercepts are 1 and 3, $y = a(x-1)(x-3)$.

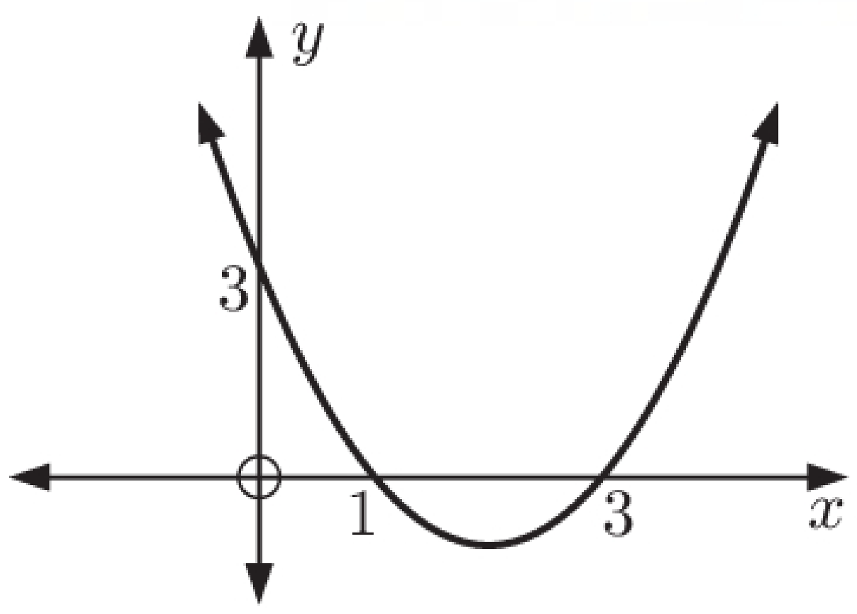
The graph is concave up, so $a > 0$.

When $x = 0$, $y = 3$

$\therefore$ $3 = a(-1)(-3)$

$\therefore$ $a = 1$

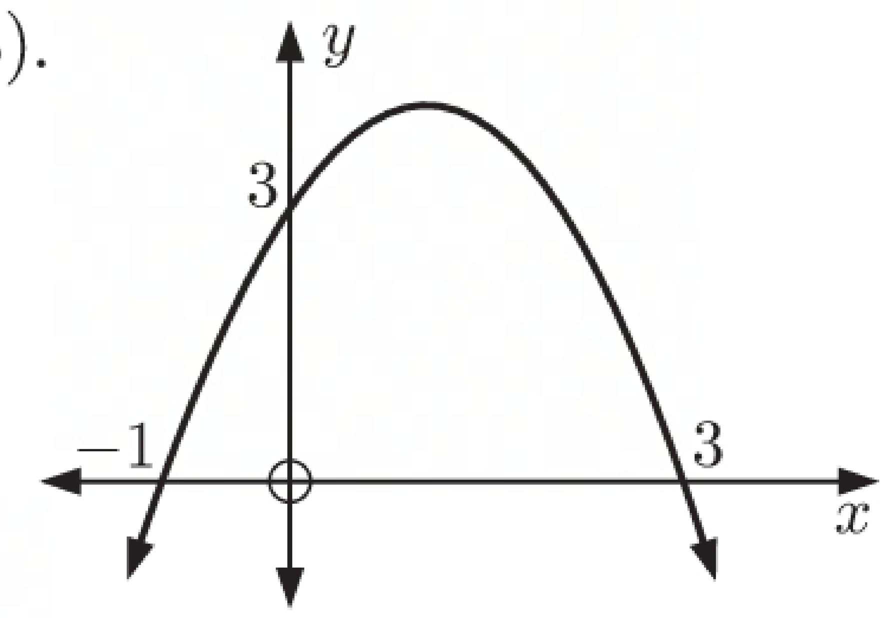
The quadratic is $y = (x-1)(x-3)$.



d Since the x-intercepts are -1 and 3, $y = a(x+1)(x-3)$.
The graph is concave down, so $a < 0$.
When $x = 0$, $y = 3$

$$\therefore \; 3 = a(1)(-3)$$
$$\therefore \; a = -1$$

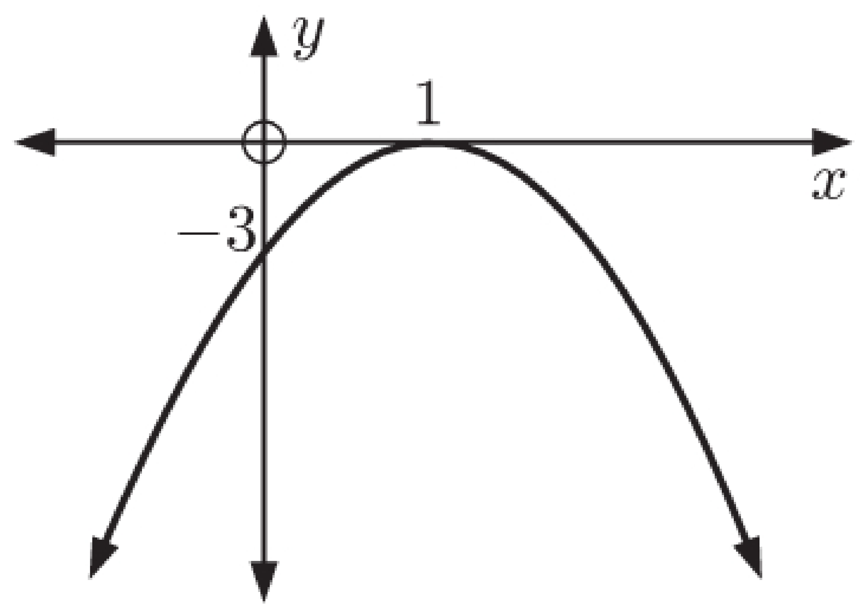
The quadratic is $y = -(x+1)(x-3)$.



e The graph touches the x-axis at $x = 1$, so $y = a(x-1)^2$.
The graph is concave down, so $a < 0$.
When $x = 0$, $y = -3$

$$\therefore \; -3 = a(-1)^2$$
$$\therefore \; a = -3$$

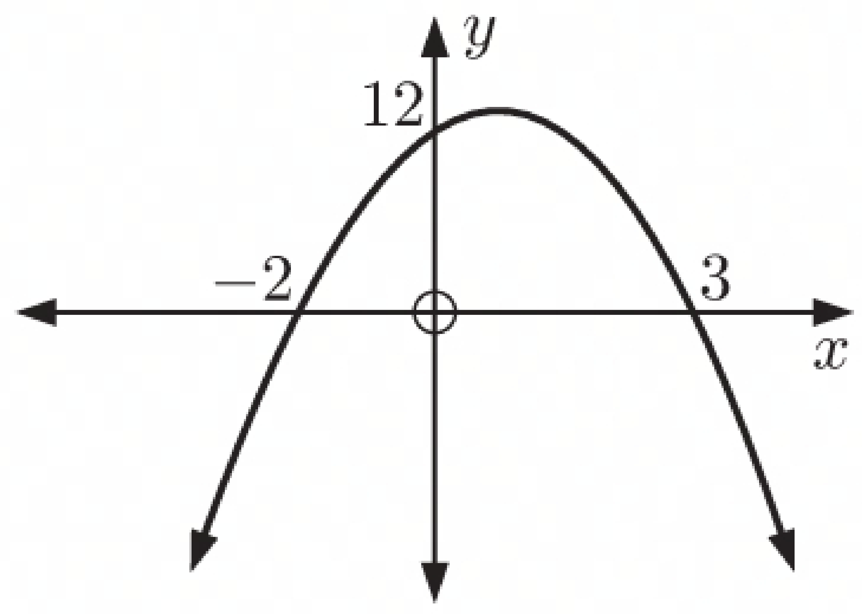
The quadratic is $y = -3(x-1)^2$.



f Since the x-intercepts are -2 and 3, $y = a(x+2)(x-3)$.
The graph is concave down, so $a < 0$.
When $x = 0$, $y = 12$

$$\therefore \; 12 = a(2)(-3)$$
$$\therefore \; a = -2$$

The quadratic is $y = -2(x+2)(x-3)$.



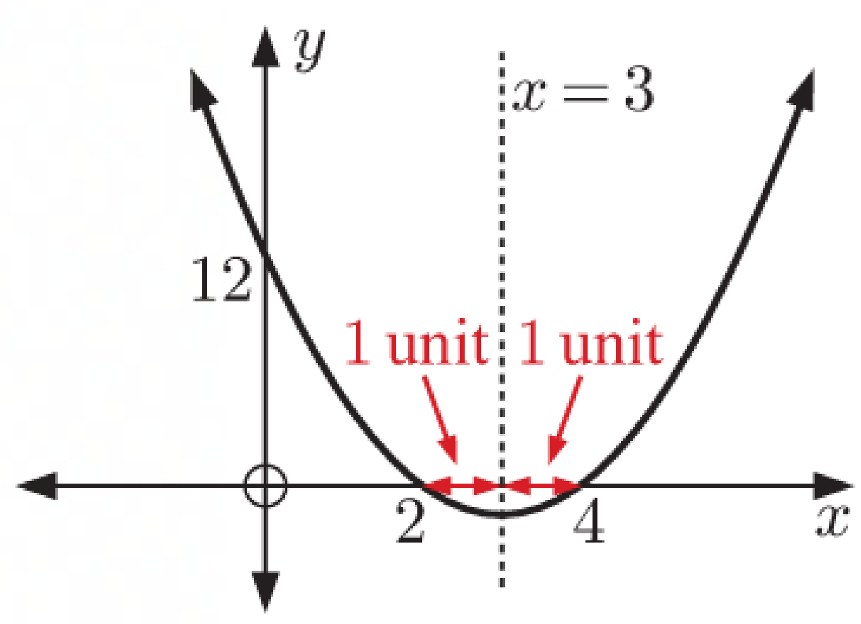
2 **a** The axis of symmetry $x = 3$ lies midway between the x-intercepts.
$\therefore$ the other x-intercept is 4.
$\therefore$ the quadratic has the form

$$y = a(x-2)(x-4) \quad \text{where} \quad a > 0$$

But when $x = 0$, $y = 12$

$$\therefore \; 12 = a(-2)(-4)$$
$$\therefore \; a = \tfrac{3}{2}$$

The quadratic is $y = \frac{3}{2}(x-2)(x-4)$.



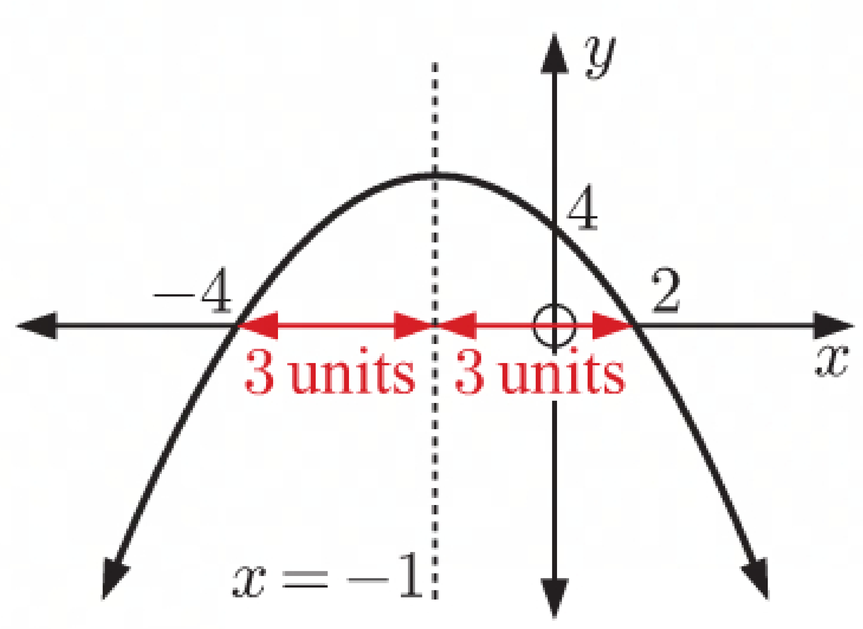
b The axis of symmetry $x = -1$ lies midway between the x-intercepts.
$\therefore$ the other x-intercept is 2.
$\therefore$ the quadratic has the form

$$y = a(x+4)(x-2) \quad \text{where} \quad a < 0$$

But when $x = 0$, $y = 4$

$$\therefore \; 4 = a(4)(-2)$$
$$\therefore \; a = -\tfrac{1}{2}$$

The quadratic is $y = -\frac{1}{2}(x+4)(x-2)$.



c The graph touches the x-axis at $x = -3$.

$\therefore$ the quadratic has the form

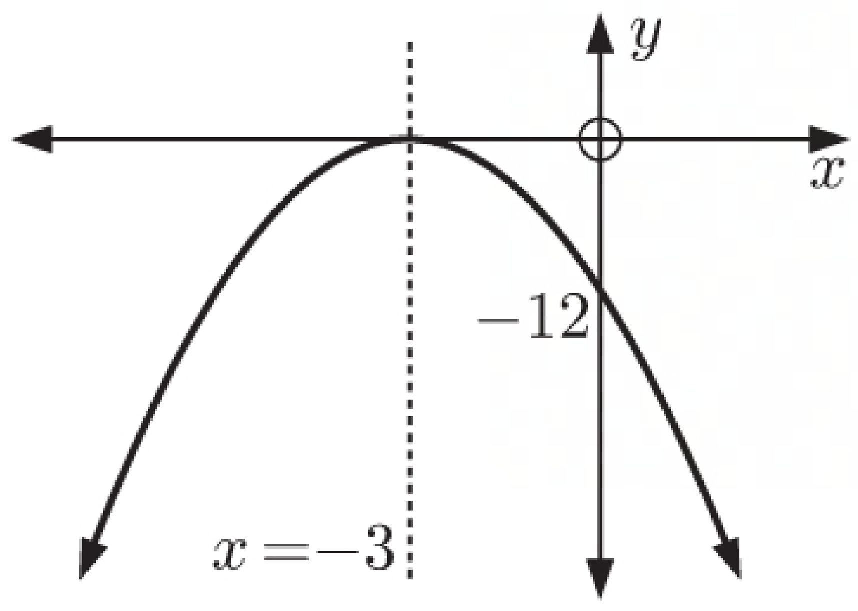
$$y = a(x+3)^2 \quad \text{where} \quad a < 0$$

But when $x = 0$, $y = -12$

$\therefore$ $-12 = a(3)^2$

$\therefore$ $a = -\frac{4}{3}$

The quadratic is $y = -\frac{4}{3}(x+3)^2$.



3 **a** Since the x-intercepts are 5 and 1, the quadratic has the form

$y = a(x-5)(x-1)$, $a \neq 0$.

When $x = 2$, $y = -9$

$\therefore$ $-9 = a(2-5)(2-1)$

$\therefore$ $-9 = a(-3)(1)$

$\therefore$ $a = 3$

The quadratic is $y = 3(x-5)(x-1)$

$= 3(x^2 - 6x + 5)$

$\therefore$ $y = 3x^2 - 18x + 15$

b Since the x-intercepts are 2 and $-\frac{1}{2}$, the quadratic has the form

$y = a(x-2)(2x+1)$, $a \neq 0$.

When $x = 3$, $y = -14$

$\therefore$ $-14 = a(3-2)(2(3)+1)$

$\therefore$ $-14 = a(1)(7)$

$\therefore$ $a = -2$

The quadratic is $y = -2(x-2)(2x+1)$

$= -2(2x^2 - 3x - 2)$

$\therefore$ $y = -4x^2 + 6x + 4$

c Since the graph touches the x-axis at 3, the quadratic has the form

$y = a(x-3)^2$, $a \neq 0$.

When $x = -2$, $y = -25$

$\therefore$ $-25 = a(-2-3)^2$

$\therefore$ $-25 = a(-5)^2$

$\therefore$ $a = -1$

The quadratic is $y = -(x-3)^2$

$= -(x^2 - 6x + 9)$

$\therefore$ $y = -x^2 + 6x - 9$

d Since the graph touches the x-axis at -2, the quadratic has the form $y = a(x+2)^2$, $a \neq 0$.

When $x = -1$, $y = 4$

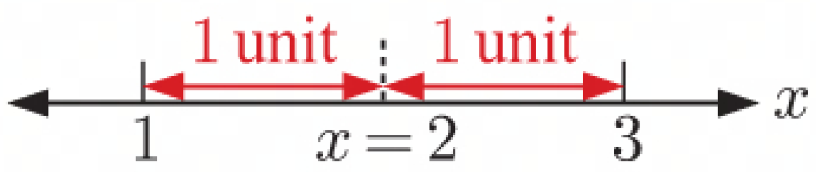
$$\therefore\ 4 = a(-1+2)^2$$
$$\therefore\ 4 = a(1)^2$$
$$\therefore\ a = 4$$

The quadratic is $y = 4(x+2)^2$

$$= 4(x^2 + 4x + 4)$$
$$\therefore\ y = 4x^2 + 16x + 16$$

e The axis of symmetry $x = 2$ lies midway between the x-intercepts.

$\therefore$ the other x-intercept is 1.



Since the x-intercepts are 3 and 1, the quadratic has the form $y = a(x-3)(x-1)$, $a \neq 0$.

When $x = 5$, $y = 12$

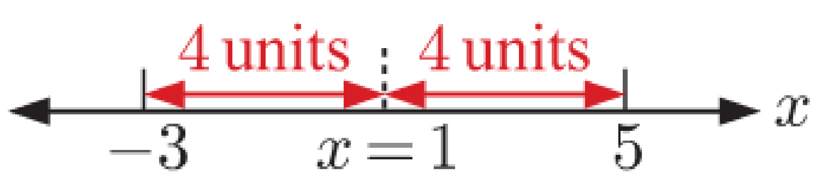
$$\therefore\ 12 = a(5-3)(5-1)$$
$$\therefore\ 12 = a(2)(4)$$
$$\therefore\ a = \tfrac{3}{2}$$

The quadratic is $y = \frac{3}{2}(x-3)(x-1)$

$$= \tfrac{3}{2}(x^2 - 4x + 3)$$
$$\therefore\ y = \tfrac{3}{2}x^2 - 6x + \tfrac{9}{2}$$

f The axis of symmetry $x = 1$ lies midway between the x-intercepts.

$\therefore$ the other x-intercept is -3.



Since the x-intercepts are 5 and -3, the quadratic has the form $y = a(x-5)(x+3)$, $a \neq 0$.

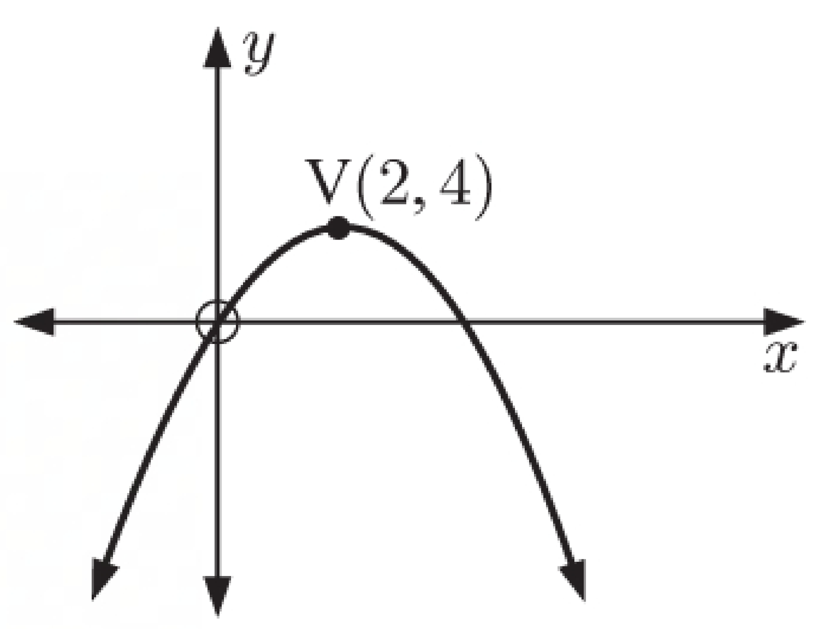
When $x = 2$, $y = 5$

$$\therefore\ 5 = a(2-5)(2+3)$$
$$\therefore\ 5 = a(-3)(5)$$
$$\therefore\ a = -\tfrac{1}{3}$$

The quadratic is $y = -\frac{1}{3}(x-5)(x+3)$

$$= -\tfrac{1}{3}(x^2 - 2x - 15)$$
$$\therefore\ y = -\tfrac{1}{3}x^2 + \tfrac{2}{3}x + 5$$

4 **a** Since the vertex is $(2, 4)$, the quadratic has the form $y = a(x-2)^2 + 4$, where $a < 0$.

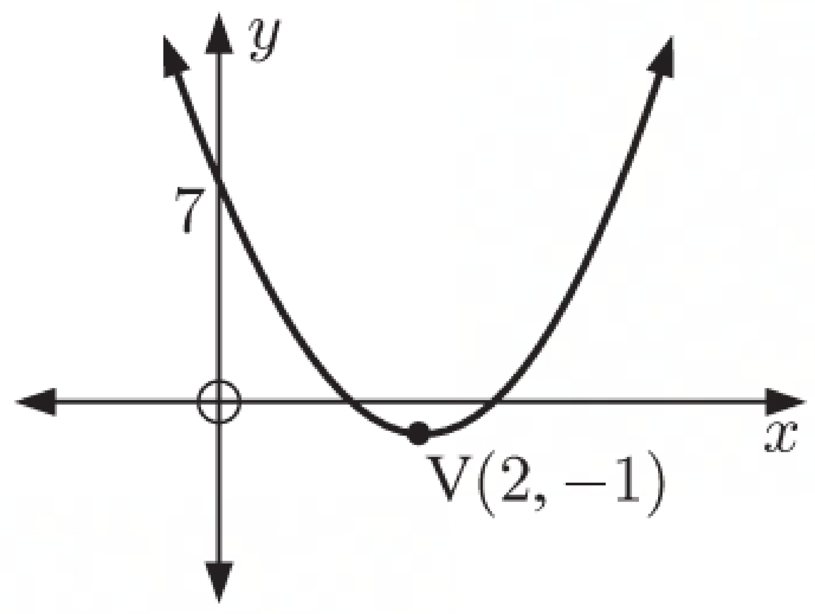


When $x = 0$, $y = 0$

$$\therefore\ 0 = a(-2)^2 + 4$$
$$\therefore\ 0 = 4a + 4$$
$$\therefore\ -4 = 4a$$
$$\therefore\ a = -1$$

The quadratic is $y = -(x-2)^2 + 4$.

b Since the vertex is $(2, -1)$, the quadratic has the form $y = a(x-2)^2 - 1$, where $a > 0$.

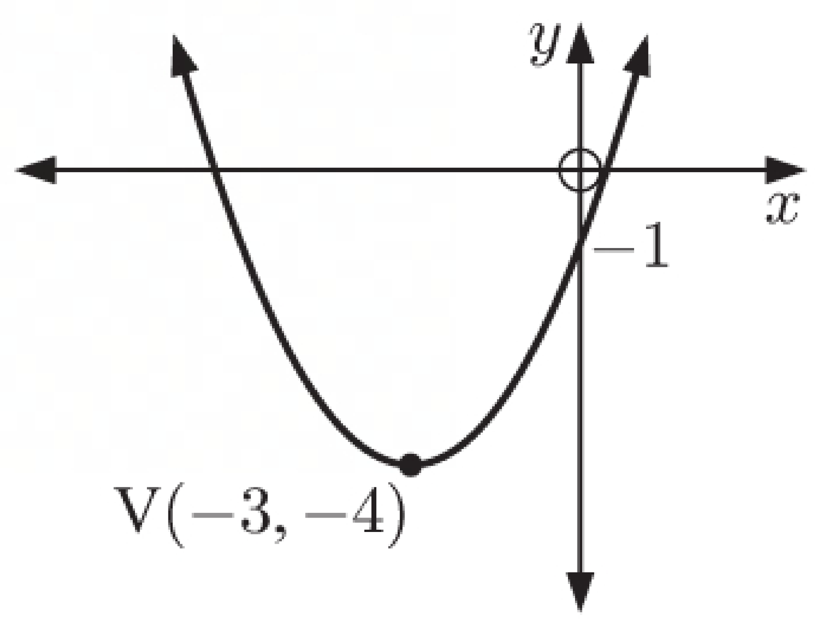


When $x = 0$, $y = 7$

$$\begin{aligned} \therefore\ 7 &= a(-2)^2 - 1 \\ \therefore\ 7 &= 4a - 1 \\ \therefore\ 8 &= 4a \\ \therefore\ a &= 2 \end{aligned}$$

The quadratic is $y = 2(x-2)^2 - 1$.

c Since the vertex is $(-3, -4)$, the quadratic has the form $y = a(x+3)^2 - 4$, where $a > 0$.

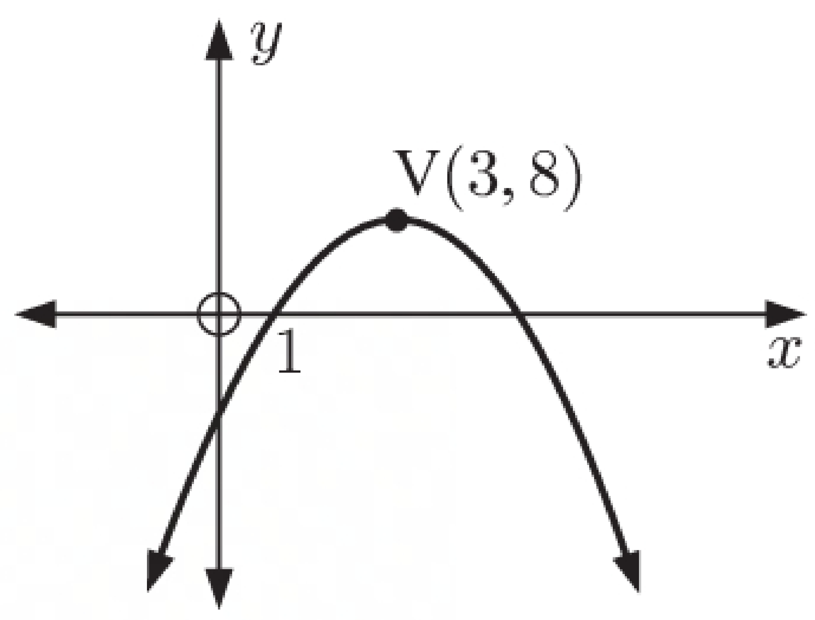


When $x = 0$, $y = -1$

$$\begin{aligned} \therefore\ -1 &= a(0+3)^2 - 4 \\ \therefore\ -1 &= a(3)^2 - 4 \\ \therefore\ -1 &= 9a - 4 \\ \therefore\ 3 &= 9a \\ \therefore\ a &= \tfrac{1}{3} \end{aligned}$$

The quadratic is $y = \frac{1}{3}(x+3)^2 - 4$.

d Since the vertex is $(3, 8)$, the quadratic has the form $y = a(x-3)^2 + 8$, where $a < 0$.

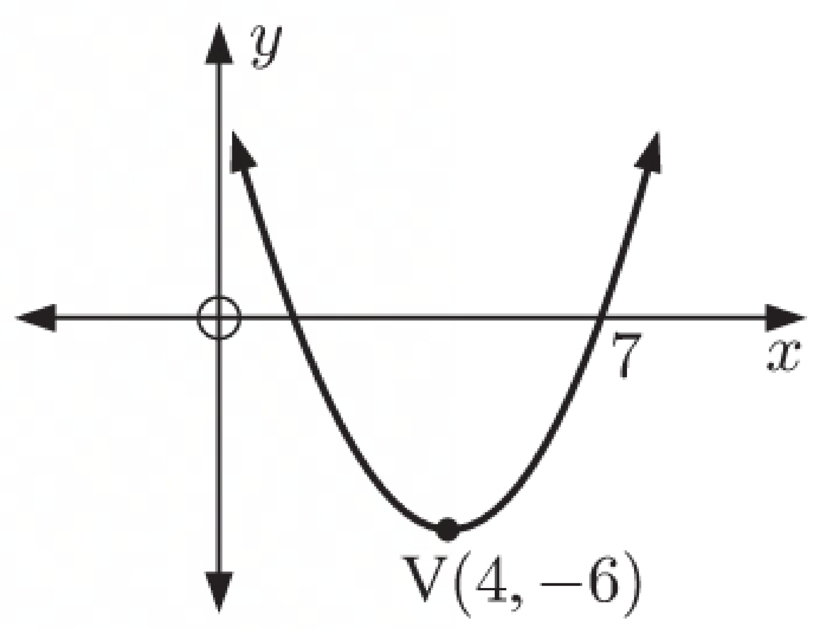


When $x = 1$, $y = 0$

$$\begin{aligned} \therefore\ 0 &= a(1-3)^2 + 8 \\ \therefore\ 0 &= a(-2)^2 + 8 \\ \therefore\ 0 &= 4a + 8 \\ \therefore\ -8 &= 4a \\ \therefore\ a &= -2 \end{aligned}$$

The quadratic is $y = -2(x-3)^2 + 8$.

e Since the vertex is $(4, -6)$, the quadratic has the form $y = a(x-4)^2 - 6$, where $a > 0$.



When $x = 7$, $y = 0$

$$\begin{aligned} \therefore\ 0 &= a(7-4)^2 - 6 \\ \therefore\ 0 &= a(3)^2 - 6 \\ \therefore\ 0 &= 9a - 6 \\ \therefore\ 6 &= 9a \\ \therefore\ a &= \tfrac{2}{3} \end{aligned}$$

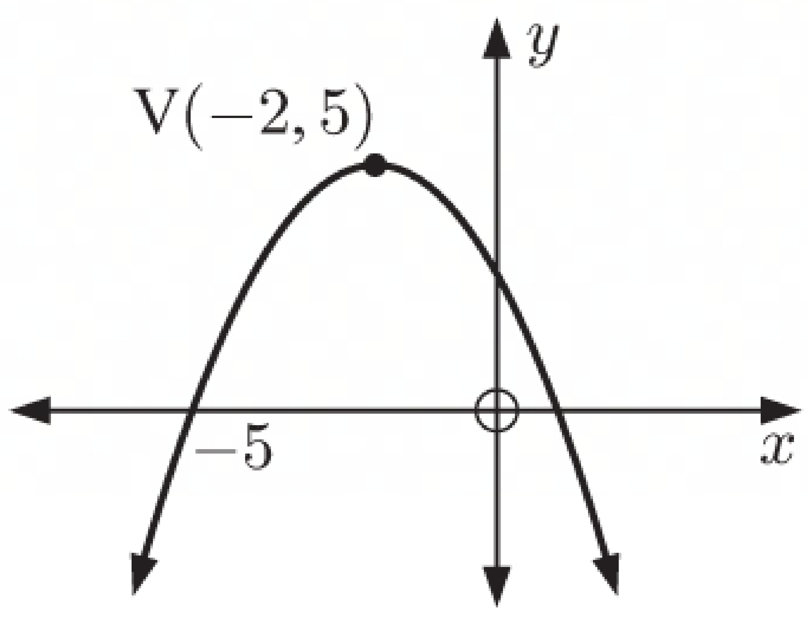
The quadratic is $y = \frac{2}{3}(x-4)^2 - 6$.

f Since the vertex is $(-2,\ 5)$, the quadratic has the form $y = a(x+2)^2 + 5$, where $a < 0$.

When $x = -5$, $y = 0$

$$\therefore\ 0 = a(-5+2)^2 + 5$$
$$\therefore\ 0 = a(-3)^2 + 5$$
$$\therefore\ 0 = 9a + 5$$
$$\therefore\ 9a = -5$$
$$\therefore\ a = -\tfrac{5}{9}$$

The quadratic is $y = -\frac{5}{9}(x+2)^2 + 5$.

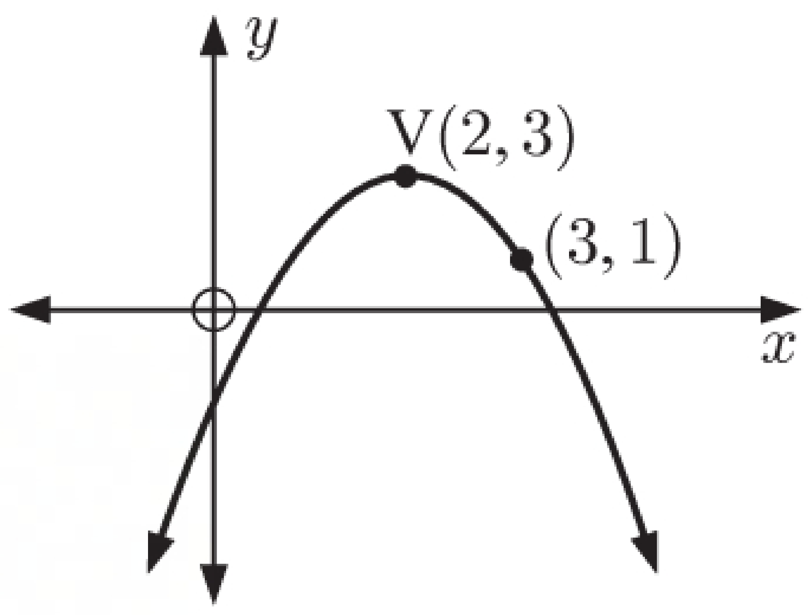


g Since the vertex is $(2,\ 3)$, the quadratic has the form $y = a(x-2)^2 + 3$, where $a < 0$.

When $x = 3$, $y = 1$

$$\therefore\ 1 = a(3-2)^2 + 3$$
$$\therefore\ 1 = a(1)^2 + 3$$
$$\therefore\ 1 = a + 3$$
$$\therefore\ a = -2$$

The quadratic is $y = -2(x-2)^2 + 3$.

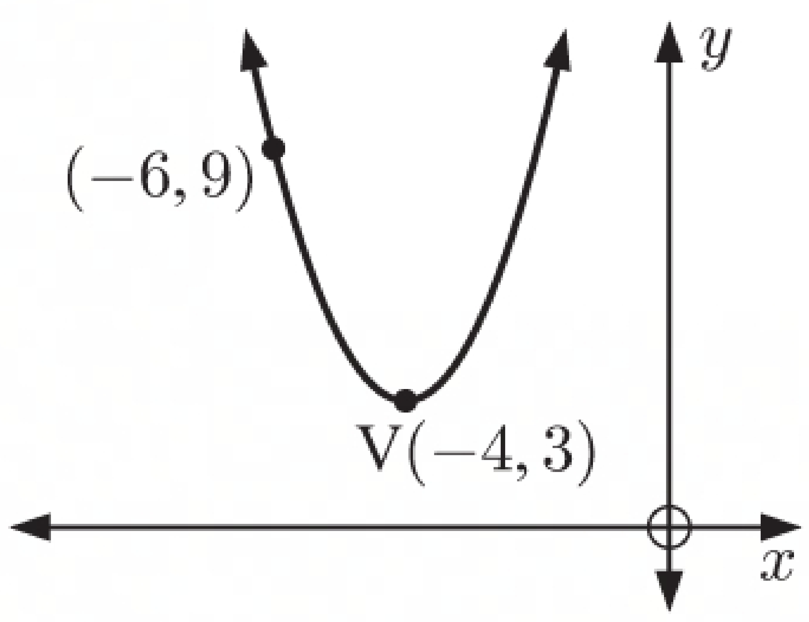


h Since the vertex is $(-4,\ 3)$, the quadratic has the form $y = a(x+4)^2 + 3$, where $a > 0$.

When $x = -6$, $y = 9$

$$\therefore\ 9 = a(-6+4)^2 + 3$$
$$\therefore\ 9 = a(-2)^2 + 3$$
$$\therefore\ 9 = 4a + 3$$
$$\therefore\ 6 = 4a$$
$$\therefore\ a = \tfrac{3}{2}$$

The quadratic is $y = \frac{3}{2}(x+4)^2 + 3$.

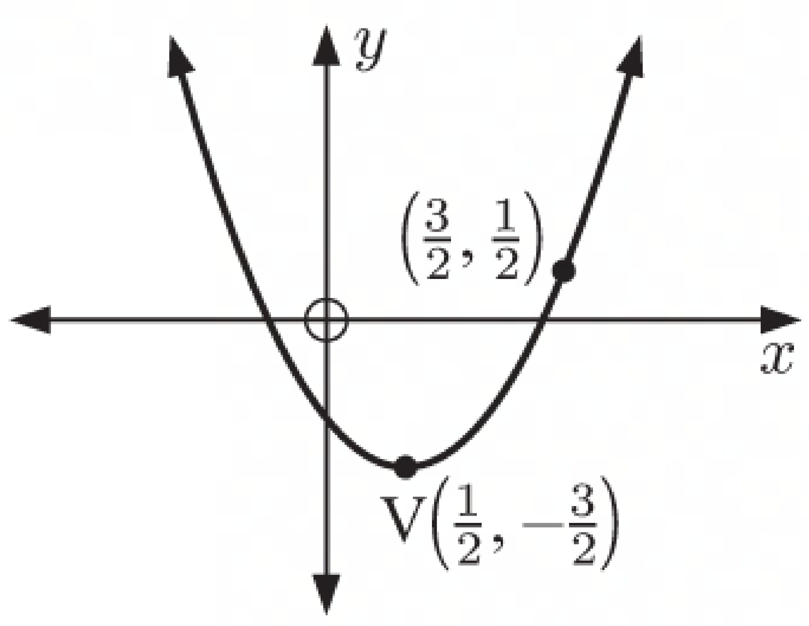


i Since the vertex is $(\frac{1}{2},\ -\frac{3}{2})$, the quadratic has the form $y = a(x-\frac{1}{2})^2 - \frac{3}{2}$, where $a > 0$.

When $x = \frac{3}{2}$, $y = \frac{1}{2}$

$$\therefore\ \tfrac{1}{2} = a(\tfrac{3}{2} - \tfrac{1}{2})^2 - \tfrac{3}{2}$$
$$\therefore\ \tfrac{1}{2} = a(1)^2 - \tfrac{3}{2}$$
$$\therefore\ \tfrac{1}{2} = a - \tfrac{3}{2}$$
$$\therefore\ a = 2$$

The quadratic is $y = 2(x-\frac{1}{2})^2 - \frac{3}{2}$.



5 Since the vertex is $(2, -5)$, the quadratic has the form $y = a(x-2)^2 - 5$, $a \neq 0$.

When $x = -1$, $y = 13$

$\therefore\ 13 = a(-1-2)^2 - 5$

$\therefore\ 13 = a(-3)^2 - 5$

$\therefore\ 18 = 9a$

$\therefore\ a = 2$

The quadratic is $y = 2(x-2)^2 - 5$.

When $x = 4$, $y = 2(4-2)^2 - 5$

$= 2(2)^2 - 5$

$\therefore\ y = 3$

INVESTIGATION 3 FINDING QUADRATICS

1 **a** $y = x^2 + 4x + 3$

x	0	1	2	3	4	5
y	3	8	15	24	35	48
Δ_1	5	7	9	11	13	
Δ_2	2	2	2	2		

b $y = 3x^2 - 4x$

x	0	1	2	3	4	5
y	0	-1	4	15	32	55
Δ_1	-1	5	11	17	23	
Δ_2	6	6	6	6		

c $y = 5x - x^2$

x	0	1	2	3	4	5
y	0	4	6	6	4	0
Δ_1	4	2	0	-2	-4	
Δ_2	-2	-2	-2	-2		

d $y = 4x^2 - 5x + 2$

x	0	1	2	3	4	5
y	2	1	8	23	46	77
Δ_1	-1	7	15	23	31	
Δ_2	8	8	8	8		

2 For each quadratic in **1**, every value in the Δ_2 row is a single constant.

3 **a** $y = ax^2 + bx + c$

x	0	1	2	3	4	5
y	c	$a+b+c$	$4a+2b+c$	$9a+3b+c$	$16a+4b+c$	$25a+5b+c$
Δ_1	$a+b$	$3a+b$	$5a+b$	$7a+b$	$9a+b$	
Δ_2	$2a$	$2a$	$2a$	$2a$		

b Every value in the Δ_2 row is the constant $2a$.

c We can use the circled numbers to find the constants a, b, and c in $y = ax^2 + bx + c$ if we are given a quadratic as a table of values.

4 **a**

x	0	1	2	3	4
y	6	5	8	15	26
Δ_1	-1	3	7	11	
Δ_2	4	4	4		

$c = 6, \quad 2a = 4$
$\therefore \quad a = 2$
$a + b = -1$
$\therefore \quad 2 + b = -1$
$\therefore \quad b = -3$
$\therefore$ the quadratic with this table of values is $y = 2x^2 - 3x + 6$.

b

x	0	1	2	3	4
y	8	10	18	32	52
Δ_1	2	8	14	20	
Δ_2	6	6	6		

$c = 8, \quad 2a = 6$
$\therefore \quad a = 3$
$a + b = 2$
$\therefore \quad 3 + b = 2$
$\therefore \quad b = -1$
$\therefore$ the quadratic with this table of values is $y = 3x^2 - x + 8$.

c

x	0	1	2	3	4
y	1	2	-1	-8	-19
Δ_1	1	-3	-7	-11	
Δ_2	-4	-4	-4		

$c = 1, \quad 2a = -4$
$\therefore \quad a = -2$
$a + b = -1$
$\therefore \quad -2 + b = 1$
$\therefore \quad b = 3$
$\therefore$ the quadratic with this table of values is $y = -2x^2 + 3x + 1$.

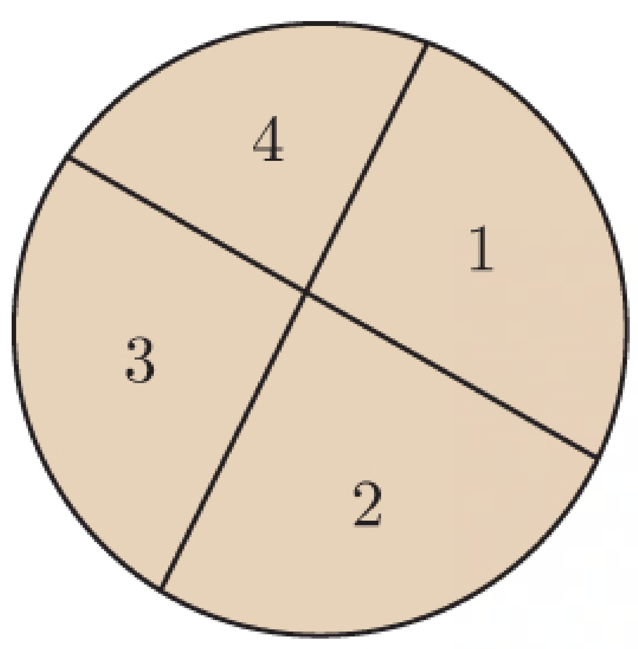
d

x	0	1	2	3	4
y	5	3	-1	-7	-15
Δ_1	-2	-4	-6	-8	
Δ_2	-2	-2	-2		

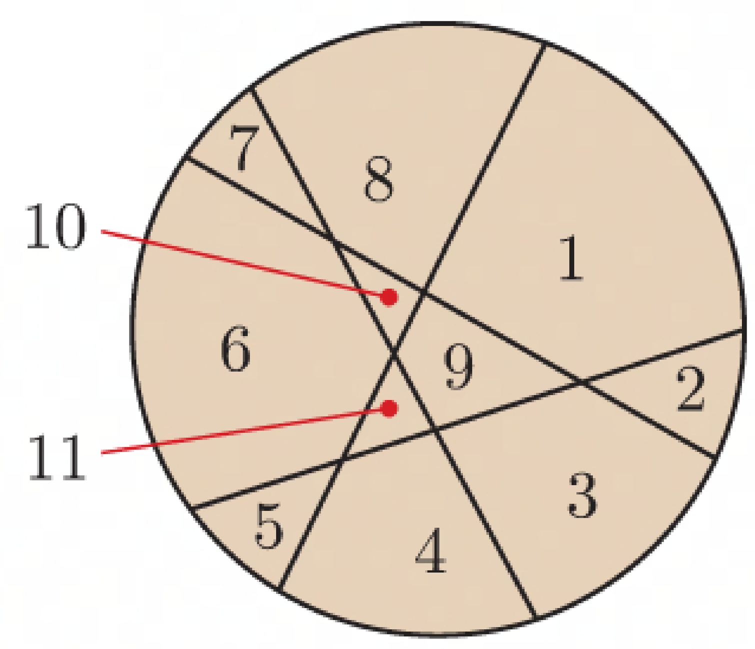
$c = 5, \quad 2a = -2$
$\therefore \quad a = -1$
$a + b = -2$
$\therefore \quad -1 + b = -2$
$\therefore \quad b = -1$
$\therefore$ the quadratic with this table of values is $y = -x^2 - x + 5$.

5

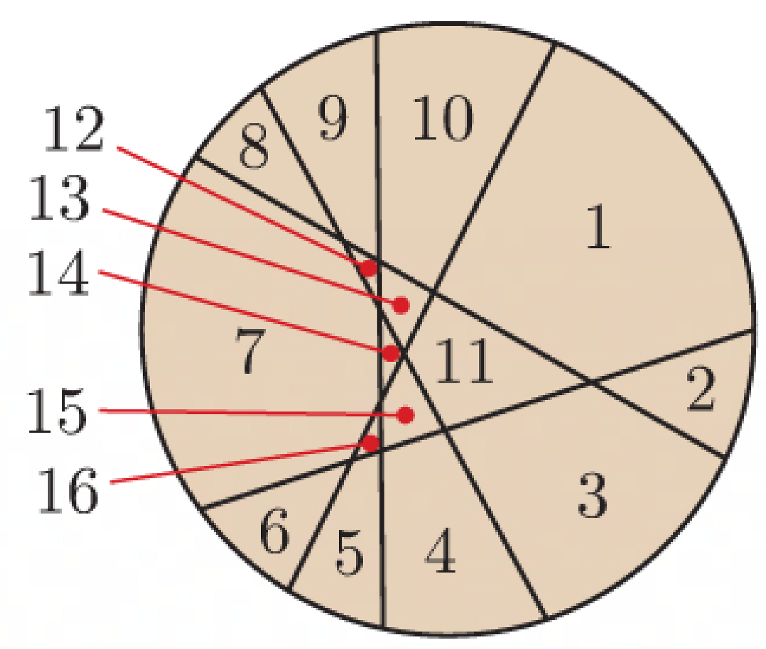
- for $n = 2$ we can make 4 pieces



- for $n = 4$ we can make 11 pieces.



- for $n = 5$ we can make 16 pieces.



a **b**

Number of cuts, n	0	1	2	3	4	5
Maximum number of pieces, P_n	1	2	4	7	11	16
Δ_1	1	2	3	4	5	
Δ_2	1	1	1	1		

$c = 1, \quad 2a = 1$
$\therefore \quad a = \frac{1}{2}$

$a + b = 1$
$\therefore \quad \frac{1}{2} + b = 1$
$\therefore \quad b = \frac{1}{2}$

$\therefore \quad P_n = \frac{1}{2}n^2 + \frac{1}{2}n + 1$

c If $n = 12$, $P_n = \frac{1}{2}(12)^2 + \frac{1}{2}(12) + 1$

$= 72 + 6 + 1$

$= 79$

$\therefore$ the maximum number of pieces for a pizza with 12 cuts is 79.

EXERCISE 14E

1 **a** $y = x^2 - 2x + 8$ meets $y = x + 6$ where

$x^2 - 2x + 8 = x + 6$

$\therefore x^2 - 3x + 2 = 0$

$\therefore (x-1)(x-2) = 0$

$\therefore x = 1$ or 2

Substituting into $y = x + 6$, when $x = 1$, $y = 7$ and when $x = 2$, $y = 8$.

$\therefore$ the graphs meet at $(1, 7)$ and $(2, 8)$.

b $y = -x^2 + 3x + 9$ meets $y = 2x - 3$ where

$-x^2 + 3x + 9 = 2x - 3$

$\therefore x^2 - x - 12 = 0$

$\therefore (x-4)(x+3) = 0$

$\therefore x = 4$ or -3

Substituting into $y = 2x - 3$, when $x = 4$, $y = 2(4) - 3 = 5$ and when $x = -3$, $y = 2(-3) - 3 = -9$.

$\therefore$ the graphs meet at $(4, 5)$ and $(-3, -9)$.

c $y = x^2 - 4x + 3$ meets $y = 2x - 6$ where

$x^2 - 4x + 3 = 2x - 6$

$\therefore x^2 - 6x + 9 = 0$

$\therefore (x-3)^2 = 0$

$\therefore x = 3$

Substituting into $y = 2x - 6$, when $x = 3$, $y = 0$.

$\therefore$ the graphs touch at $(3, 0)$.

d $y = -x^2 + 4x - 7$ meets $y = 5x - 4$ where

$-x^2 + 4x - 7 = 5x - 4$

$\therefore x^2 + x + 3 = 0$

which has $a = 1$, $b = 1$, $c = 3$

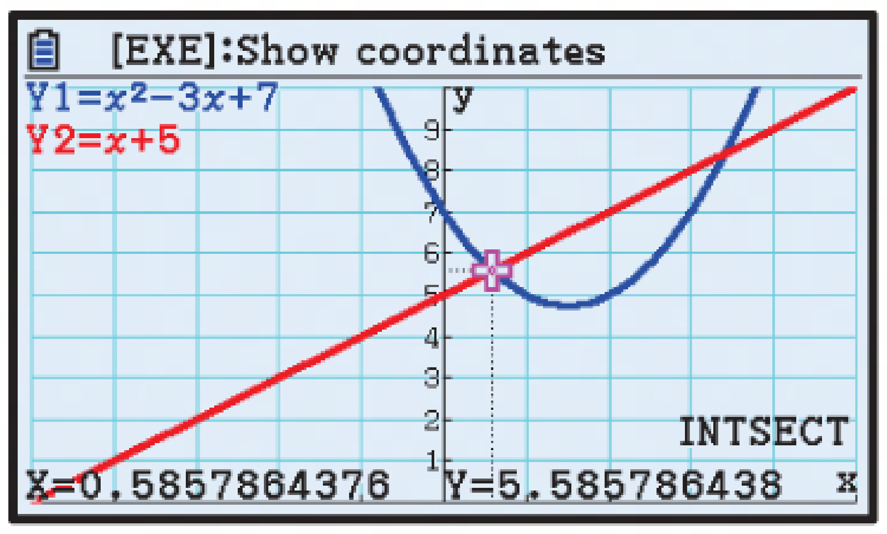
$$\therefore x = \frac{-1 \pm \sqrt{1^2 - 4(1)(3)}}{2(1)}$$

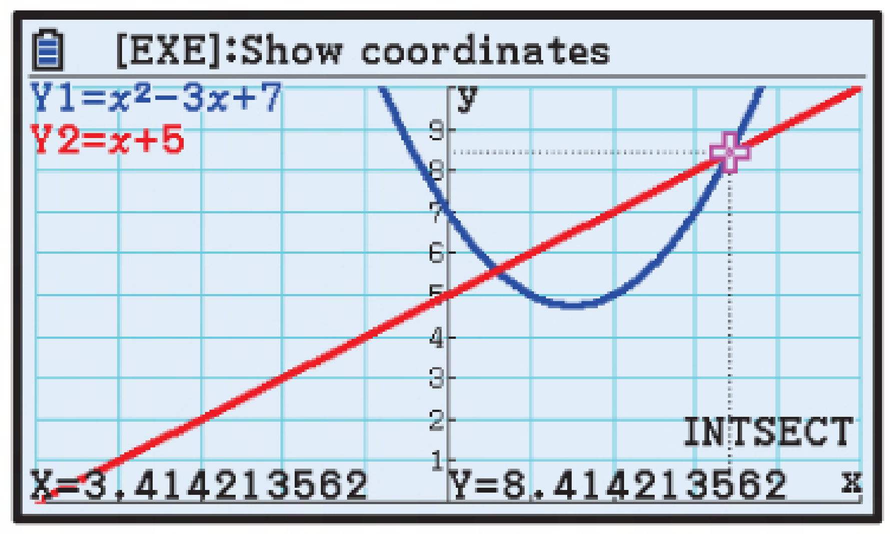
$$\therefore x = \frac{-1 \pm \sqrt{-11}}{2}$$

$\therefore$ there are no real solutions

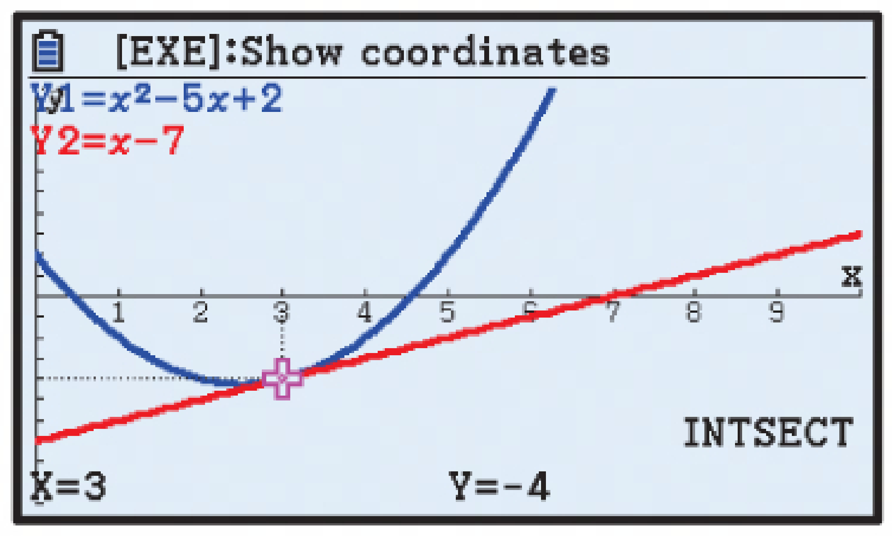
$\therefore$ the graphs do not meet.

2 **a** $y = x^2 - 3x + 7$ and $y = x + 5$ intersect at $(0.586,\ 5.59)$ and $(3.41,\ 8.41)$.

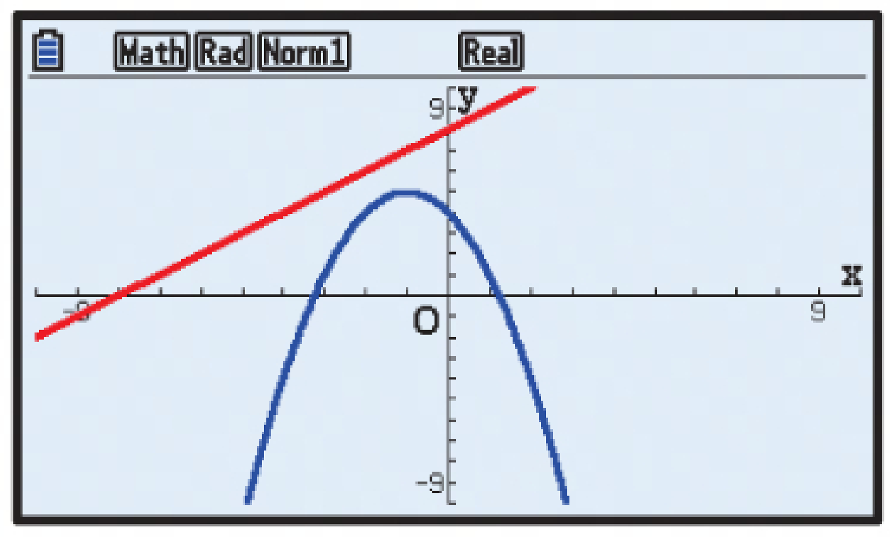




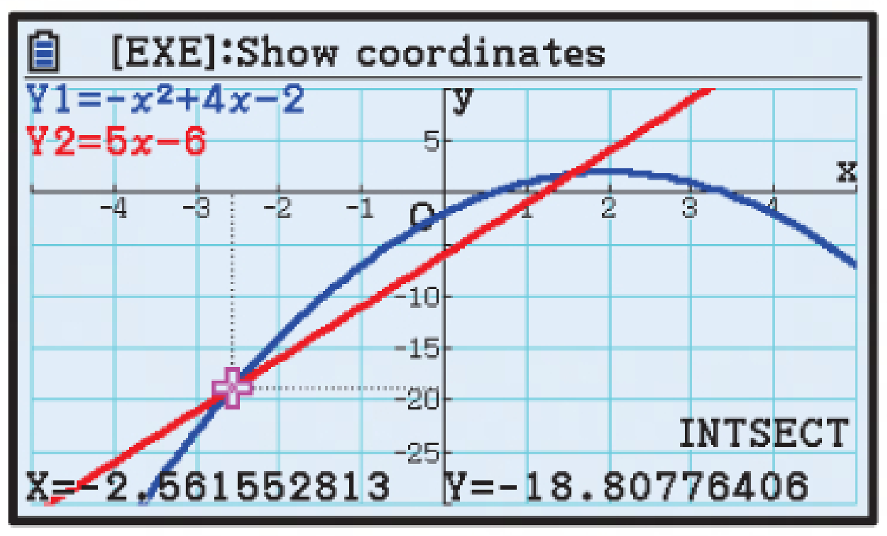
b $y = x^2 - 5x + 2$ and $y = x - 7$ intersect at $(3,\ -4)$ (touching).

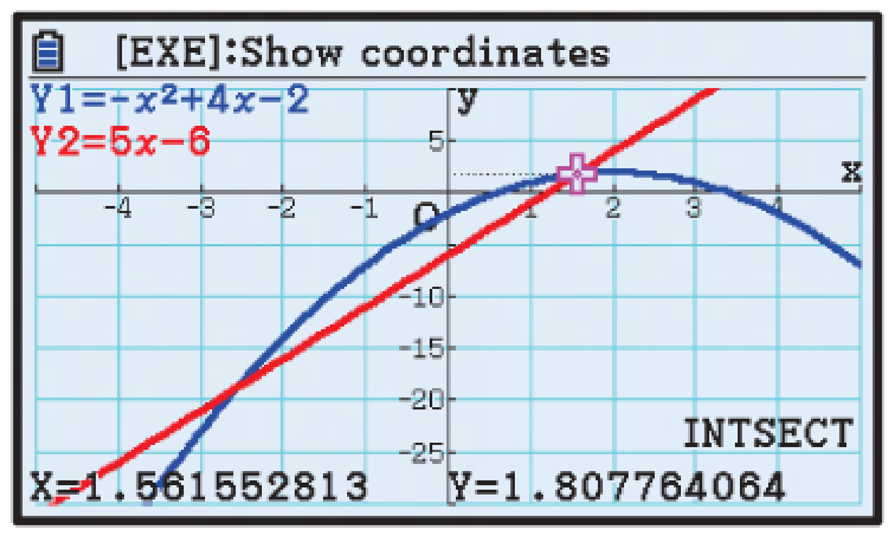


c $y = -x^2 - 2x + 4$ and $y = x + 8$ do not intersect.



d $y = -x^2 + 4x - 2$ and $y = 5x - 6$ intersect at $(-2.56,\ -18.8)$ and $(1.56,\ 1.81)$.





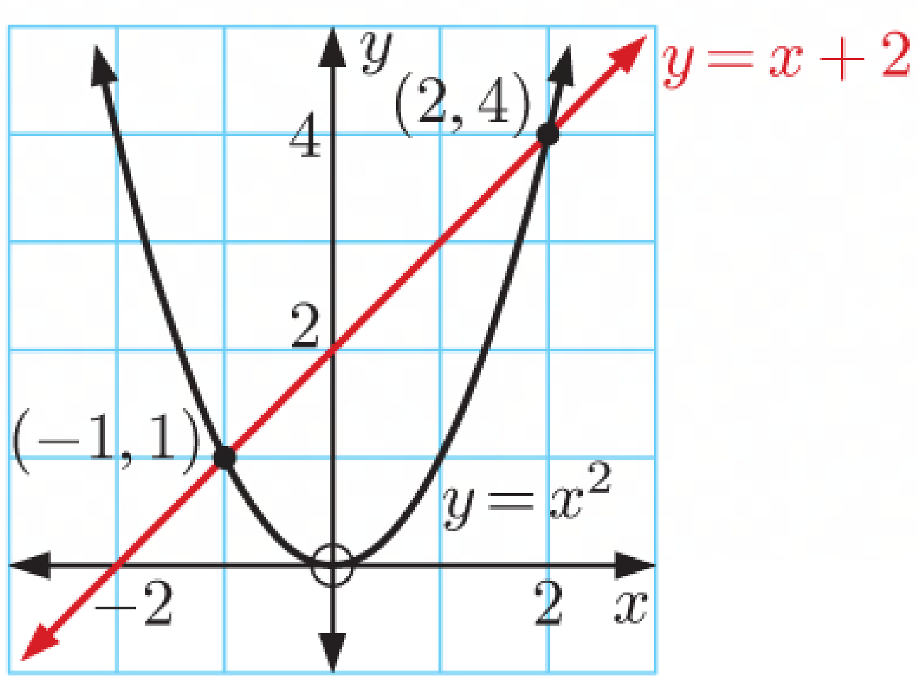
3 **a** $y = x^2$ meets $y = x + 2$ where

$$x^2 = x + 2$$
$$\therefore\ x^2 - x - 2 = 0$$
$$\therefore\ (x + 1)(x - 2) = 0$$
$$\therefore\ x = -1 \text{ or } 2$$

Substituting into $y = x + 2$, when $x = -1$, $y = 1$ and when $x = 2$, $y = 4$.

$\therefore$ the graphs meet at $(-1,\ 1)$ and $(2,\ 4)$.

b



c If $x^2 > x + 2$, the graph of $y = x^2$ is above the graph of $y = x + 2$. This occurs when $x < -1$ or $x > 2$.

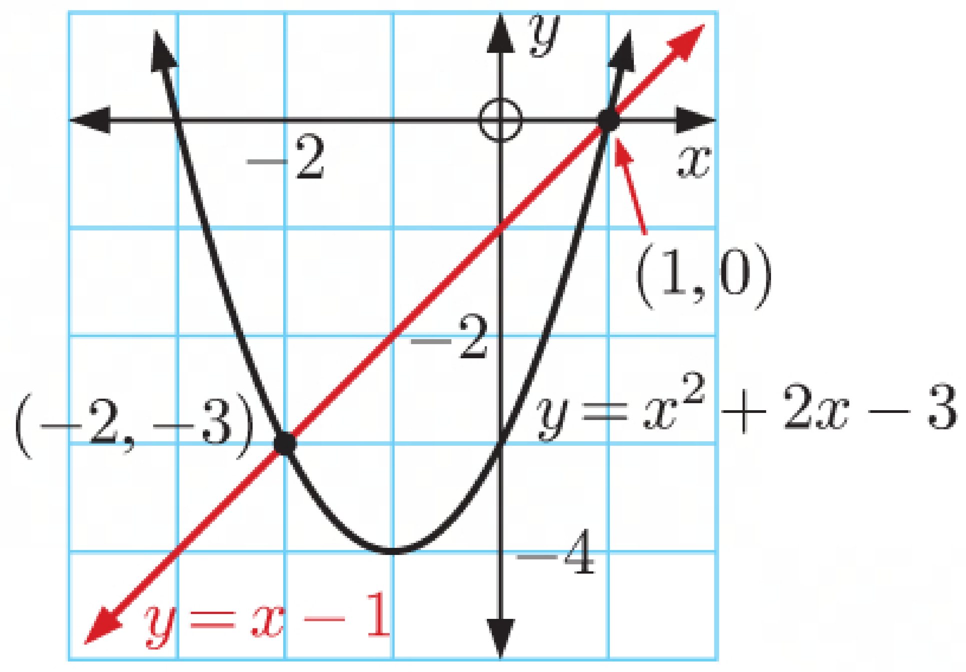
4 **a** $y = x^2 + 2x - 3$ meets $y = x - 1$ where

$$x^2 + 2x - 3 = x - 1$$
$$\therefore\ x^2 + x - 2 = 0$$
$$\therefore\ (x+2)(x-1) = 0$$
$$\therefore\ x = -2 \text{ or } 1$$

Substituting into $y = x - 1$, when $x = -2$, $y = -3$ and when $x = 1$, $y = 0$.

$\therefore$ the graphs meet at $(-2, -3)$ and $(1, 0)$.

b



c If $x^2 + 2x - 3 > x - 1$, the graph of $y = x^2 + 2x - 3$ is above the graph of $y = x - 1$.
This occurs when $x < -2$ or $x > 1$.

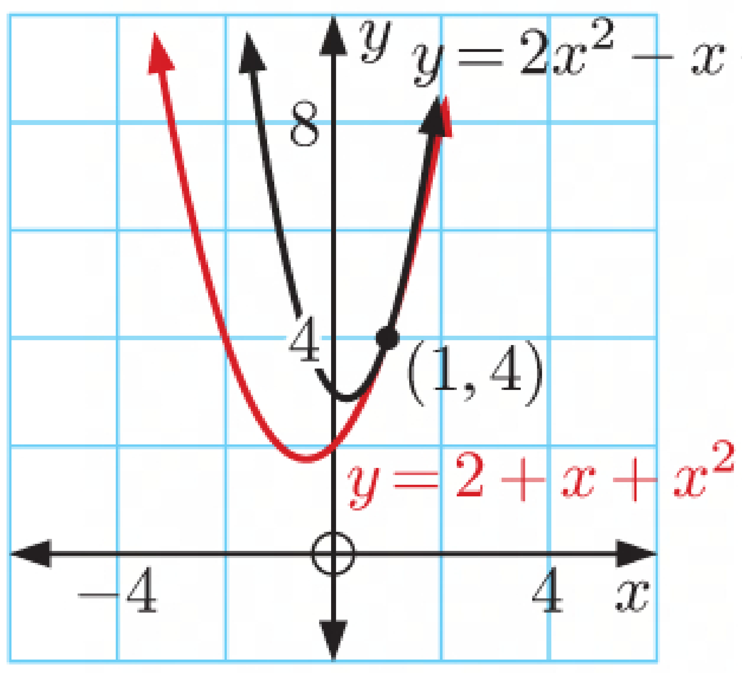
5 **a** $y = 2x^2 - x + 3$ meets $y = 2 + x + x^2$ where

$$2x^2 - x + 3 = 2 + x + x^2$$
$$\therefore\ x^2 - 2x + 1 = 0$$
$$\therefore\ (x-1)^2 = 0$$
$$\therefore\ x = 1$$

Substituting into $y = 2 + x + x^2$, when $x = 1$, $y = 4$.

$\therefore$ the graphs meet at $(1, 4)$.

b

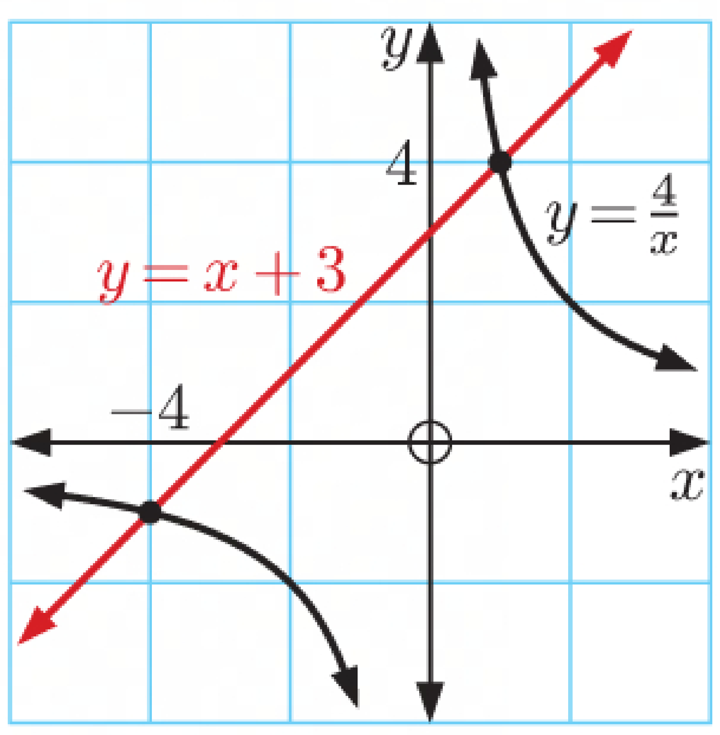


c If $2x^2 - x + 3 > 2 + x + x^2$, the graph of $y = 2x^2 - x + 3$ is above the graph of $y = 2 + x + x^2$.
This occurs for all $x \in \mathbb{R}$, $x \neq 1$.

6 **a**

$$\frac{4}{x} = x + 3$$
$$\therefore\ 4 = x^2 + 3x$$
$$\therefore\ x^2 + 3x - 4 = 0$$
$$\therefore\ (x+4)(x-1) = 0$$
$$\therefore\ x = -4 \text{ or } 1$$

b



c If $\frac{4}{x} > x + 3$, the graph of $y = \frac{4}{x}$ is above the graph of $y = x + 3$.
This occurs when $x < -4$ or $0 < x < 1$.

7 $y = 3x + c$ is a tangent to $y = x^2 - 5x + 7$ if they meet at exactly one point (touch).
$y = x^2 - 5x + 7$ meets $y = 3x + c$ where $x^2 - 5x + 7 = 3x + c$
$\therefore\ x^2 - 8x + (7 - c) = 0$
The graphs meet exactly once when this equation has a repeated root $\therefore\ \Delta = 0$
$\therefore\ (-8)^2 - 4(1)(7 - c) = 0$
$\therefore\ 64 - 28 + 4c = 0$
$\therefore\ 4c = -36$
$\therefore\ c = -9$

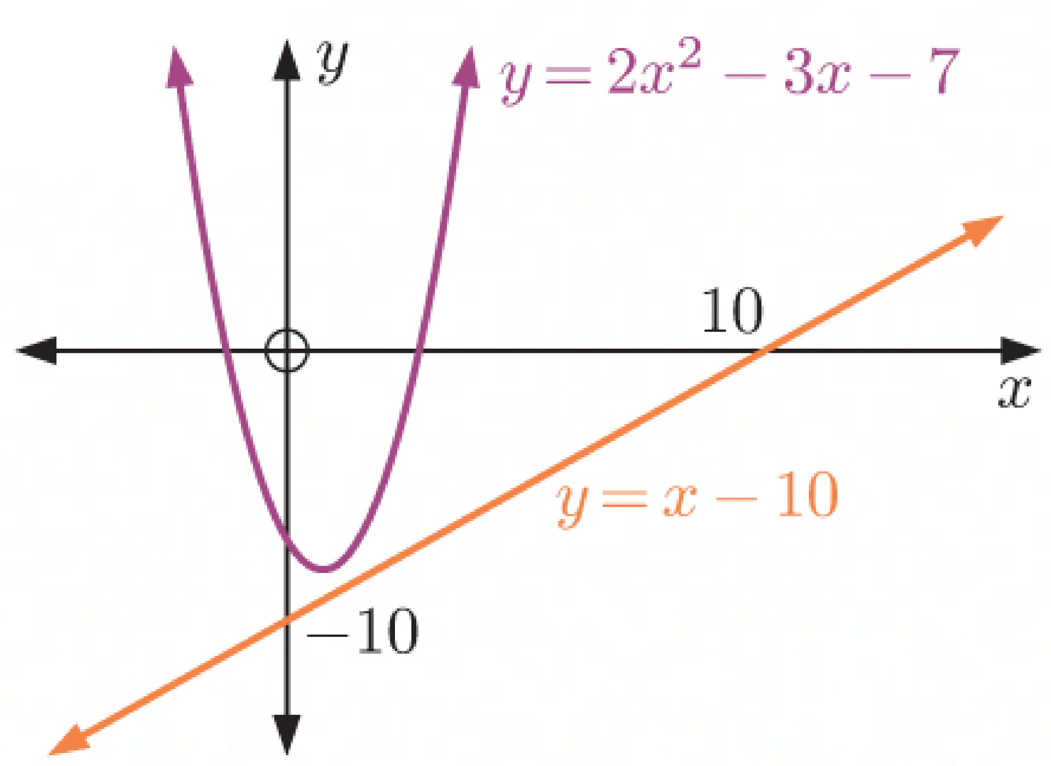
8 $y = mx - 2$ is a tangent to $y = x^2 - 4x + 2$ if they meet at exactly one point (touch).
$y = x^2 - 4x + 2$ meets $y = mx - 2$ where $x^2 - 4x + 2 = mx - 2$
$\therefore\ x^2 - (m + 4)x + 4 = 0$
The graphs meet exactly once when this equation has a repeated root $\therefore\ \Delta = 0$
$\therefore\ (-(m + 4))^2 - 4(1)(4) = 0$
$\therefore\ m^2 + 8m + 16 - 16 = 0$
$\therefore\ m(m + 8) = 0$
$\therefore\ m = 0$ or -8

9 Lines with y-intercept 1 have the form $y = mx + 1$.
$y = mx + 1$ is a tangent to $y = 3x^2 + 5x + 4$ if they meet at exactly one point (touch).
$y = 3x^2 + 5x + 4$ meets $y = mx + 1$ where $3x^2 + 5x + 4 = mx + 1$
$\therefore\ 3x^2 + (5 - m)x + 3 = 0$
The graphs meet exactly once when this equation has a repeated root $\therefore\ \Delta = 0$
$\therefore\ (5 - m)^2 - 4(3)(3) = 0$
$\therefore\ 25 - 10m + m^2 - 36 = 0$
$\therefore\ m^2 - 10m - 11 = 0$
$\therefore\ (m + 1)(m - 11) = 0$
$\therefore\ m = -1$ or 11

$\therefore$ the required lines have gradient -1 or 11.

10 **a** $y = x + c$ meets $y = 2x^2 - 3x - 7$
where $2x^2 - 3x - 7 = x + c$
$\therefore\ 2x^2 - 4x - (7 + c) = 0$
The graphs will never meet if this equation has no real roots.
$\therefore\ \Delta < 0$
$\therefore\ (-4)^2 - 4(2)(-(7 + c)) < 0$
$\therefore\ 16 + 56 + 8c < 0$
$\therefore\ 8c < -72$
$\therefore\ c < -9$

b **Note:** Other solutions are possible.
Choose c such that $c < -9$,
for example $c = -10$:



11 Let the two quadratics be $y = a_1x^2 + b_1x + c_1$ and $y = a_2x^2 + b_2x + c_2$.

The graphs meet where $a_1x^2 + b_1x + c_1 = a_2x^2 + b_2x + c_2$

$\therefore\ (a_1 - a_2)x^2 + (b_1 - b_2)x + (c_1 - c_2) = 0$

which is a quadratic with at most two solutions (when $\Delta > 0$).

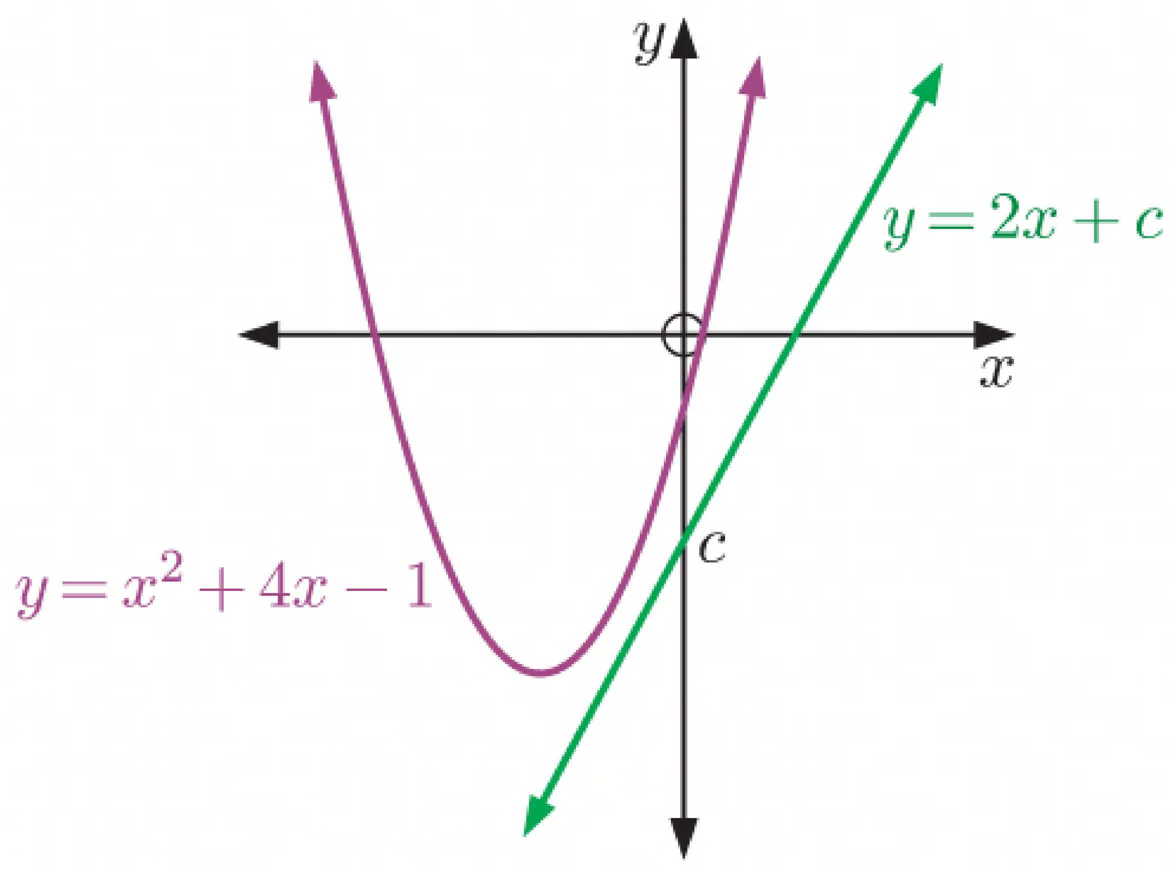
$\therefore$ two quadratic functions can intersect at most twice.

12 $y = 2x + c$ meets $y = x^2 + 4x - 1$ where

$$x^2 + 4x - 1 = 2x + c$$

$\therefore\ x^2 + 2x - (1 + c) = 0$

Now $\Delta = b^2 - 4ac$
$= 2^2 - 4(1)(-(1 + c))$
$= 4 + 4(1 + c)$
$= 4 + 4 + 4c$
$= 4c + 8$



a The graphs meet twice if $\Delta > 0$
$\therefore\ 4c + 8 > 0$
$\therefore\ 4c > -8$
$\therefore\ c > -2$

b The graphs touch if $\Delta = 0$
$\therefore\ 4c + 8 = 0$
$\therefore\ 4c = -8$
$\therefore\ c = -2$

c The graphs do not meet if $\Delta < 0$
$\therefore\ 4c + 8 < 0$
$\therefore\ 4c < -8$
$\therefore\ c < -2$

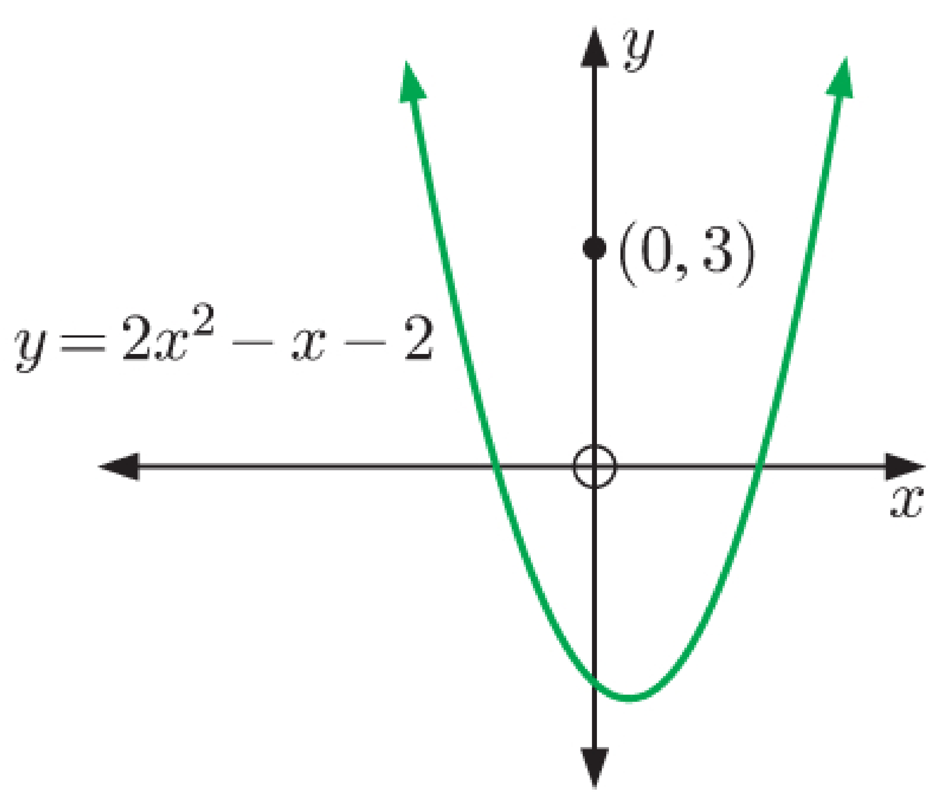
13 Let the linear function have equation $y = mx + c$.

Since the y-intercept is 3, then $y = mx + 3$.

$y = mx + 3$ meets $y = 2x^2 - x - 2$ where

$$2x^2 - x - 2 = mx + 3$$

$\therefore\ 2x^2 - (1 + m)x - 5 = 0$



Now $\Delta = b^2 - 4ac$
$= (-(1 + m))^2 - 4(2)(-5)$
$= 1 + 2m + m^2 + 40$
$= m^2 + 2m + 41$
$= m^2 + 2m + 1^2 + 41 - 1^2$
$= (m + 1)^2 + 40$
which is always > 0 for any value of m

$\therefore\ \Delta > 0$ for any value of m

$\therefore$ the linear function $y = mx + 3$ will always meet the curve $y = 2x^2 - x - 2$ twice.

14 $y = (x-2)^2$ and $y = -x^2 + bx + c$ touch when $x = 3$

$\therefore\ (3-2)^2 = -(3)^2 + b(3) + c$

$\therefore\ 1 = -9 + 3b + c$

$\therefore\ 3b + c = 10$

$\therefore\ c = 10 - 3b \quad .... (1)$

Now, consider $(x-2)^2 = -x^2 + bx + c$

$\therefore\ x^2 - 4x + 4 = -x^2 + bx + c$

$\therefore\ -2x^2 + (b+4)x + c - 4 = 0$

This quadratic has $\Delta = 0$ since the graphs *touch*.

$\therefore\ (b+4)^2 - 4(-2)(c-4) = 0$

$\therefore\ b^2 + 8b + 16 + 8c - 32 = 0$

$\therefore\ b^2 + 8b - 16 + 8(10 - 3b) = 0 \qquad$ {using (1)}

$\therefore\ b^2 + 8b - 16 + 80 - 24b = 0$

$\therefore\ b^2 - 16b + 64 = 0$

$\therefore\ (b-8)^2 = 0$

$\therefore\ b = 8$

Substituting into (1) gives $c = 10 - 3(8)$

$= -14$

$\therefore\ b = 8,\ c = -14$

15 **a** $y = m(x-a) + c$ passes through $P(a, a^2)$

$\therefore\ a^2 = m(a-a) + c$

$\therefore\ c = a^2,\ m \in \mathbb{R}$

b Consider $m(x-a) + a^2 = x^2$

$\therefore\ mx - ma + a^2 = x^2$

$\therefore\ x^2 - mx + (ma - a^2) = 0$

This quadratic has $\Delta = 0$ since the graphs *touch*.

$\therefore\ (-m)^2 - 4(1)(ma - a^2) = 0$

$\therefore\ m^2 - 4am + 4a^2 = 0$

$\therefore\ (m - 2a)^2 = 0$

$\therefore\ m = 2a$

EXERCISE 14F

1 Let the smaller of the integers be x. The other integer is $(x+12)$.

$\therefore$ the sum of their squares is $x^2 + (x+12)^2 = 74$

$\therefore\ x^2 + x^2 + 24x + 144 = 74$

$\therefore\ 2x^2 + 24x + 70 = 0$

$\therefore\ x^2 + 12x + 35 = 0$

$\therefore\ (x+7)(x+5) = 0$

$\therefore\ x = -7$ or -5

So, the integers are 7 and -5, or -7 and 5.

2 Let the number be x, so its reciprocal is $\frac{1}{x}$.

They have sum $x + \frac{1}{x} = \frac{26}{5}$

$$\therefore \quad x^2 + 1 = \tfrac{26}{5}x$$
$$\therefore \quad x^2 - \tfrac{26}{5}x + 1 = 0$$
$$\therefore \quad 5x^2 - 26x + 5 = 0$$
$$\therefore \quad (5x-1)(x-5) = 0$$
$$\therefore \quad x = \tfrac{1}{5} \text{ or } 5$$

So, the number is either 5 or $\frac{1}{5}$.

3 Let the number be x.
The sum of the number and its square is 210.

$$\therefore \quad x + x^2 = 210$$
$$\therefore \quad x^2 + x - 210 = 0$$
$$\therefore \quad (x+15)(x-14) = 0$$
$$\therefore \quad x = -15 \text{ or } 14$$

However, the number is a natural number.
$\therefore$ the number is 14.

4 Suppose the numbers are x and $(x+2)$.

Then $x(x+2) = 360$

$$\therefore \quad x^2 + 2x - 360 = 0$$
$$\therefore \quad (x+20)(x-18) = 0$$
$$\therefore \quad x = -20 \text{ or } 18$$

$\therefore$ the numbers are 18 and 20, or -20 and -18.

5 Suppose the numbers are x and $(x+2)$.

Then $x(x+2) = 255$

$$\therefore \quad x^2 + 2x - 255 = 0$$
$$\therefore \quad (x-15)(x+17) = 0$$
$$\therefore \quad x = 15 \text{ or } -17$$

$\therefore$ the numbers are 15 and 17, or -17 and -15.

6 If the polygon has 90 diagonals, then $\frac{n}{2}(n-3) = 90$

$$\therefore \quad \tfrac{1}{2}n^2 - \tfrac{3}{2}n = 90$$
$$\therefore \quad n^2 - 3n - 180 = 0$$
$$\therefore \quad (n-15)(n+12) = 0$$
$$\therefore \quad n = -12 \text{ or } 15$$

We reject the negative solution, as a polygon must have a positive number of sides.
$\therefore$ the polygon has 15 sides.

7 If the width of the rectangle is w cm, then its length is $(w+4)$ cm.

$\therefore$ the area is $w(w+4) = 26$

$\therefore w^2 + 4w - 26 = 0$

which has $a = 1,\ b = 4,\ c = -26$

$$\therefore \quad w = \frac{-4 \pm \sqrt{4^2 - 4(1)(-26)}}{2(1)}$$

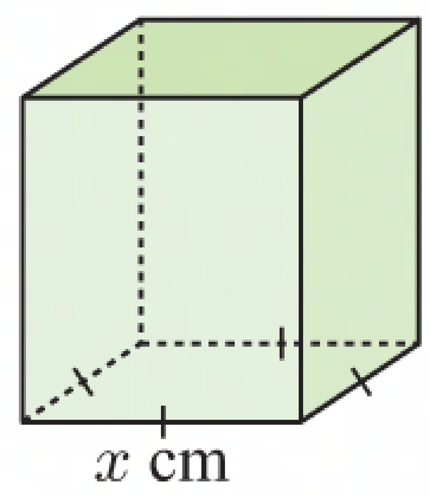
$$\therefore \quad w = \frac{-4 \pm \sqrt{120}}{2}$$

$$\therefore \quad w = -2 \pm \sqrt{30}$$

But $w > 0$, so $w = -2 + \sqrt{30}$
≈ 3.477

So, the width is approximately 3.48 cm.

8 **a** The base has sides of length x cm, so the areas of the top and bottom surfaces are both x^2 cm^2.
The box has height $(x+1)$ cm, so the area of each of the side faces is $x(x+1)$ cm^2.



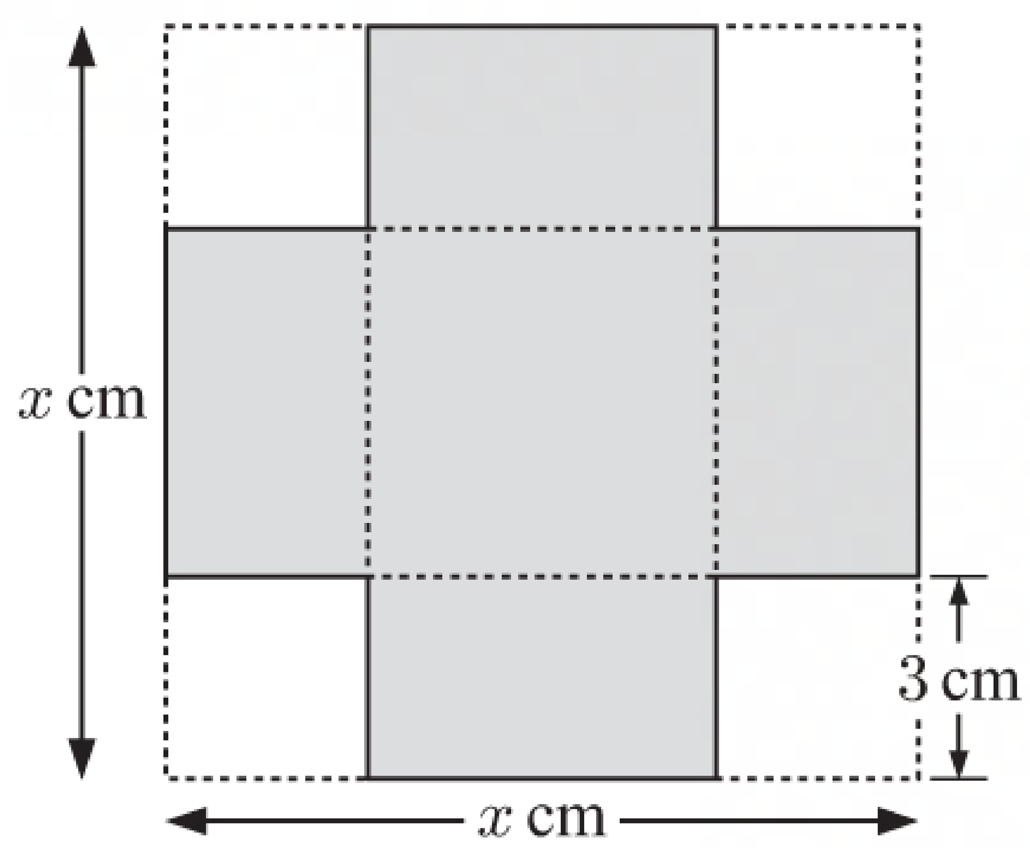
$\therefore$ the total surface area is
$$\begin{aligned} A &= 2x^2 + 4x(x+1) \\ &= 2x^2 + 4x^2 + 4x \\ \therefore \quad A &= 6x^2 + 4x \ \text{cm}^2 \end{aligned}$$

b
$$\begin{aligned} 6x^2 + 4x &= 240 \\ \therefore \quad 3x^2 + 2x - 120 &= 0 \\ \therefore \quad (3x+20)(x-6) &= 0 \\ \therefore \quad x &= -\tfrac{20}{3} \text{ or } 6 \end{aligned}$$

but $x > 0$, so $x = 6$

$\therefore$ the box is 6 cm by 6 cm by 7 cm.

9 Suppose the tinplate was x cm $\times$ x cm.
When 3 cm $\times$ 3 cm squares are cut from the corners, the base of the open box formed is $(x-6)$ cm $\times$ $(x-6)$ cm.
The open box has height 3 cm, so its volume is



$3 \times (x-6) \times (x-6) = 80$

$\therefore \quad 3(x^2 - 12x + 36) = 80$

$\therefore \quad 3x^2 - 36x + 108 = 80$

$\therefore \quad 3x^2 - 36x + 28 = 0$

which has $a = 3,\ b = -36,\ c = 28$

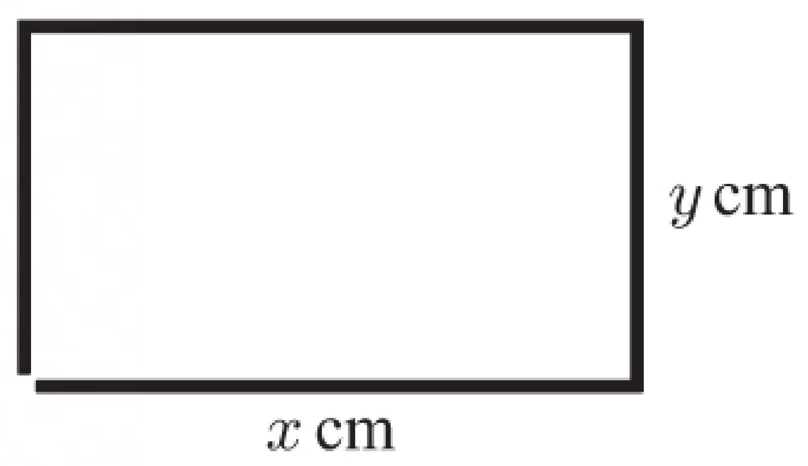
$$\therefore \quad x = \frac{-(-36) \pm \sqrt{(-36)^2 - 4(3)(28)}}{2(3)}$$

$$= \frac{36 \pm \sqrt{960}}{6} \quad \text{and since } x > 6,$$

$$x = 6 + \frac{\sqrt{960}}{6} \approx 11.16$$

$\therefore$ the original piece of tinplate was about 11.2 cm square.

10 Suppose one side of the rectangle has length x cm and the other has length y cm.



The perimeter is $(2x+2y)$ cm, so

$$\begin{aligned} 2x+2y &= 20 \\ \therefore\ 2y &= 20-2x \\ \therefore\ y &= 10-x \end{aligned}$$

The area $A = x(10-x)$ cm^2.

If the area is 30 cm^2, then

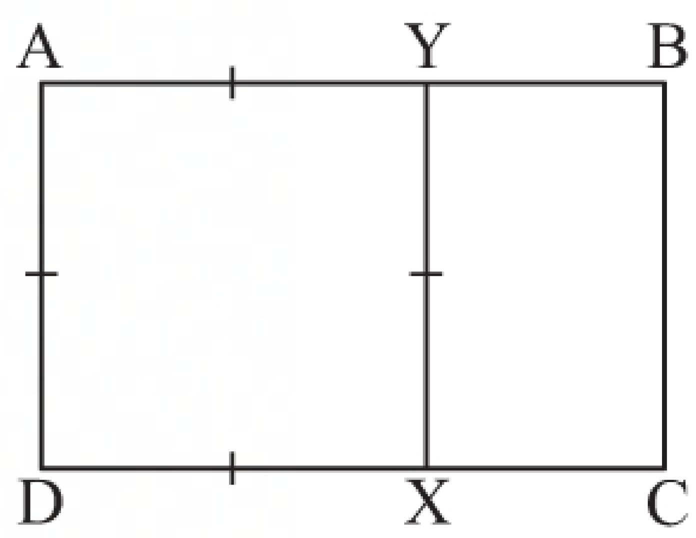
$$\begin{aligned} x(10-x) &= 30 \\ \therefore\ 10x - x^2 &= 30 \\ \therefore\ x^2 - 10x + 30 &= 0 \end{aligned}$$

Now

$$\begin{aligned} \Delta &= (-10)^2 - 4(1)(30) \\ &= 100 - 120 \\ &= -20 \quad \text{which is } < 0 \end{aligned}$$

There are no real solutions, indicating that this situation is **impossible**.

11 The smaller rectangle is similar to the original rectangle.



$$\therefore\ \frac{\text{AB}}{\text{AD}} = \frac{\text{BC}}{\text{BY}}$$

Suppose AB $= x$ units, and AD $=$ BC $= 1$ unit

$$\therefore\ \frac{x}{1} = \frac{1}{x-1}$$

$$\therefore\ x(x-1) = 1$$

$$\therefore\ x^2 - x - 1 = 0$$

which has $a = 1$, $b = -1$, $c = -1$

$$\begin{aligned} \therefore\ x &= \frac{-(-1) \pm \sqrt{(-1)^2 - 4(1)(-1)}}{2(1)} \\ &= \frac{1 \pm \sqrt{1+4}}{2} \\ &= \frac{1 \pm \sqrt{5}}{2} \end{aligned}$$

$$\therefore\ x = \frac{1+\sqrt{5}}{2} \quad \{\text{since } x > 0\}$$

But $x = \dfrac{\text{AB}}{\text{AD}}$, which is the golden ratio

$\therefore$ the golden ratio is $\dfrac{1+\sqrt{5}}{2}$.

12 Let the speed of the normal train be x km h^{-1}.

$\therefore$ the speed of the express train is $(x+10)$ km h^{-1}.

Time $= \dfrac{\text{distance}}{\text{speed}}$, so the normal train takes $\dfrac{160}{x}$ hours, and the express train takes $\dfrac{160}{x+10}$ hours to travel 160 km.

The express train takes half an hour less time than the normal train to travel 160 km.

$$\therefore \quad \frac{160}{x} - \frac{160}{x+10} = \frac{1}{2}$$

$$\therefore \quad 160(x+10) - 160x = \tfrac{1}{2}x(x+10)$$

$$\therefore \quad 160x + 1600 - 160x = \tfrac{1}{2}x^2 + 5x$$

$$\therefore \quad \tfrac{1}{2}x^2 + 5x - 1600 = 0$$

$$\therefore \quad x \approx 51.8 \quad \text{or} \quad -61.8 \qquad \{\text{using technology}\}$$

But $x > 0$, so $x \approx 51.8$

$\therefore$ the speed of the express train is $x + 10 \approx 51.8 + 10 \approx 61.8$ km h^{-1}.

13 Let the number of elderly citizens in the original group be x.

The amount paid by each elderly citizen was originally $\dfrac{160}{x}$ dollars.

Now, 8 elderly citizens fell ill, so the amount paid by each elderly citizen is $\dfrac{160}{x-8}$ dollars.

These elderly citizens had to pay \$1 more than before.

$$\therefore \quad \frac{160}{x-8} = \frac{160}{x} + 1$$

$$\therefore \quad 160x = 160(x-8) + x(x-8)$$

$$\therefore \quad 160x = 160x - 1280 + x^2 - 8x$$

$$\therefore \quad x^2 - 8x - 1280 = 0$$

$$\therefore \quad (x+32)(x-40) = 0$$

$$\therefore \quad x = -32 \text{ or } 40$$

But $x > 0$, so $x = 40$

$\therefore$ $x - 8 = 40 - 8 = 32$ elderly citizens went on the trip.

14 **a** The parabola has vertex $(0, 8)$, so it has equation

$$y = a(x-0)^2 + 8$$

$$\therefore \quad y = ax^2 + 8$$

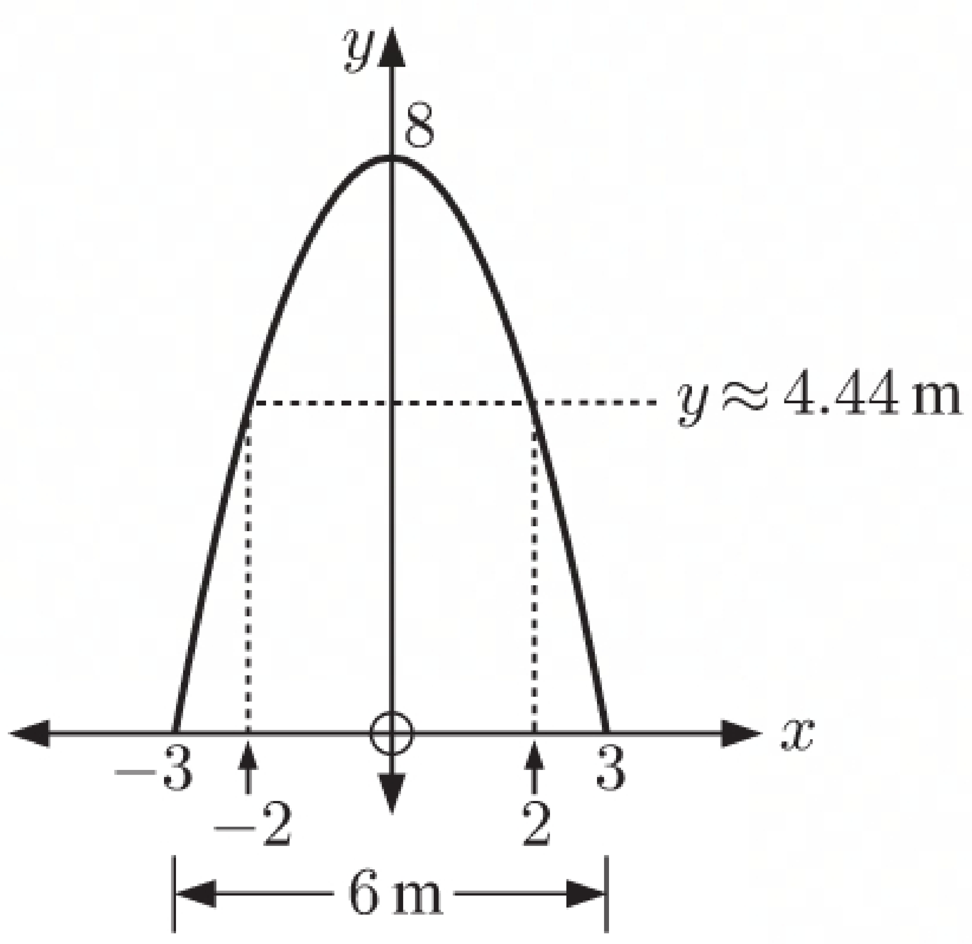
When $x = 3$, $y = 0$, so

$$0 = a(3^2) + 8$$

$$\therefore \quad 9a = -8$$

$$\therefore \quad a = -\tfrac{8}{9}$$

$\therefore$ the equation of the parabola is $y = -\frac{8}{9}x^2 + 8$.



b The truck is 4 m wide, so we use the equation in **a** to find the height of the tunnel when it is 4 m wide.

When $x = \pm 2$, $y = -\frac{8}{9}(2)^2 + 8$

$= -\frac{32}{9} + 8$

$= \frac{40}{9} \approx 4.44$ m

For heights greater than 4.44 m, the tunnel is less than 4 m wide. But the truck is 5 m high.

$\therefore$ the truck will not fit through the tunnel.

15 **a** The position of the stone above sea level is plotted on the graph as shown, where the maximum height reached is 80 m, when $t = 2$ s.

So, the vertex is $(2, 80)$, and the h-intercept is 60, as the height is 60 m when $t = 0$.

The quadratic has the form

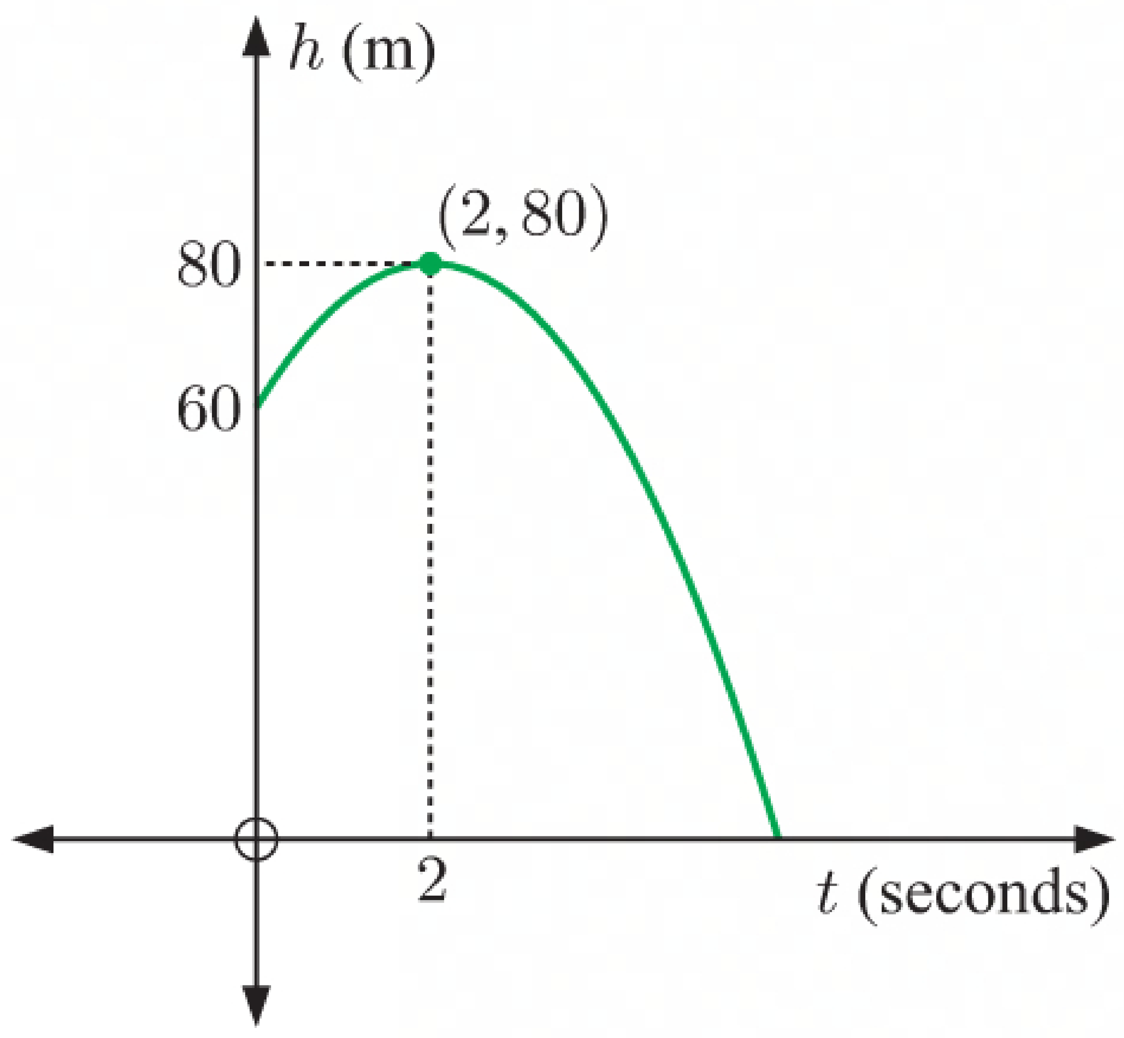
$$h = a(t-2)^2 + 80, \quad \text{where} \quad a < 0$$

When $t = 0$, $h = 60$

$\therefore$ $60 = a(-2)^2 + 80$

$\therefore$ $4a = -20$

$\therefore$ $a = -5$



The quadratic is $h = -5(t-2)^2 + 80$, which gives the stone's height above sea level for any $t \geqslant 0$.

b When $t = 3$, $h = -5(3-2)^2 + 80$

$= -5(1) + 80$

$= 75$

The stone is 75 m above sea level after 3 seconds.

c The stone will hit the water when $h = 0$.

$\therefore$ $-5(t-2)^2 + 80 = 0$

$\therefore$ $5(t-2)^2 = 80$

$\therefore$ $(t-2)^2 = 16$

$\therefore$ $t - 2 = \pm\sqrt{16}$

$\therefore$ $t = 2 \pm 4$

but $t \geqslant 0$, $\therefore$ $t = 2 + 4 = 6$

It will take 6 seconds for the stone to hit the water.

EXERCISE 14G

1 **a** $y = x^2 - 2x$

has $a = 1$, $b = -2$, $c = 0$.

Since $a > 0$, the shape is 

The minimum value occurs when

$$x = \frac{-b}{2a} = \frac{2}{2(1)} = 1$$

and $y = 1^2 - 2(1) = -1$

So, the minimum value of $y = x^2 - 2x$ is -1, occurring when $x = 1$.

b $y = 7 - 2x - x^2$

has $a = -1$, $b = -2$, $c = 7$.

Since $a < 0$, the shape is 

The maximum value occurs when

$$x = \frac{-b}{2a} = \frac{-(-2)}{2(-1)} = -1$$

and $y = 7 - 2(-1) - (-1)^2 = 8$

So, the maximum value of $y = 7 - 2x - x^2$ is 8, occurring when $x = -1$.

c $y = 8 + 2x - 3x^2$
has $a = -3$, $b = 2$, $c = 8$.

Since $a < 0$, the shape is 

The maximum value occurs when

$$x = \frac{-b}{2a} = \frac{-2}{2(-3)} = \tfrac{1}{3}$$

and $y = 8 + 2(\tfrac{1}{3}) - 3(\tfrac{1}{3})^2 = 8\tfrac{1}{3}$

So, the maximum value of
$y = 8 + 2x - 3x^2$ is $8\tfrac{1}{3}$, occurring
when $x = \tfrac{1}{3}$.

d $y = 2x^2 + x - 1$
has $a = 2$, $b = 1$, $c = -1$.

Since $a > 0$, the shape is 

The minimum value occurs when

$$x = \frac{-b}{2a} = \frac{-1}{2(2)} = -\tfrac{1}{4}$$

and $y = 2(-\tfrac{1}{4})^2 + (-\tfrac{1}{4}) - 1$
$= \tfrac{1}{8} - \tfrac{1}{4} - 1 = -1\tfrac{1}{8}$

So, the minimum value of
$y = 2x^2 + x - 1$ is $-1\tfrac{1}{8}$, occurring
when $x = -\tfrac{1}{4}$.

e $y = 4x^2 - x + 5$
has $a = 4$, $b = -1$, $c = 5$.

Since $a > 0$, the shape is 

The minimum value occurs when

$$x = \frac{-b}{2a} = \frac{-(-1)}{2(4)} = \tfrac{1}{8}$$

and $y = 4(\tfrac{1}{8})^2 - \tfrac{1}{8} + 5$
$= \tfrac{1}{16} - \tfrac{1}{8} + 5$
$= 4\tfrac{15}{16}$

So, the minimum value of
$y = 4x^2 - x + 5$ is $4\tfrac{15}{16}$, occurring
when $x = \tfrac{1}{8}$.

f $y = 7x - 2x^2$
has $a = -2$, $b = 7$, $c = 0$.

Since $a < 0$, the shape is 

The maximum value occurs when

$$x = \frac{-b}{2a} = \frac{-7}{2(-2)} = \tfrac{7}{4}$$

and $y = 7(\tfrac{7}{4}) - 2(\tfrac{7}{4})^2$
$= \tfrac{49}{4} - \tfrac{49}{8}$
$= \tfrac{49}{8}$ or $6\tfrac{1}{8}$

So, the maximum value of
$y = 7x - 2x^2$ is $6\tfrac{1}{8}$, occurring
when $x = \tfrac{7}{4}$.

2 **a** $P = -3x^2 + 240x - 800$ has $a = -3$, $b = 240$, $c = -800$.

Since $a < 0$, the shape is 

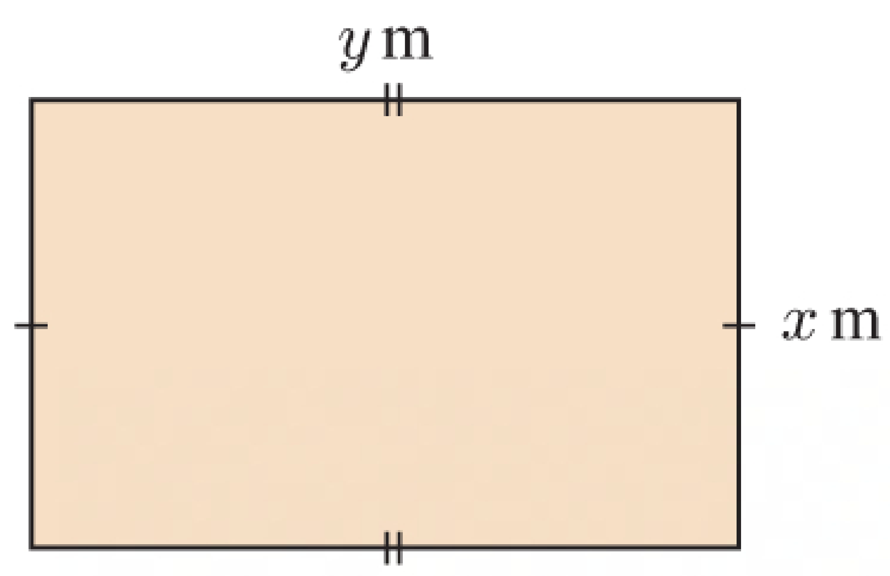
The maximum profit occurs when $x = \dfrac{-b}{2a} = \dfrac{-240}{2(-3)} = 40$

So, 40 refrigerators should be made each day to maximise the total profit.

b When $x = 40$, $P = -3(40)^2 + 240(40) - 800$
$= 4000$

So, the maximum profit is €4000.

3 **a** Let the other side be y m long.



The perimeter is 200 m.

$\therefore\ 2x + 2y = 200$

$\therefore\ x + y = 100$

$\therefore\ y = 100 - x$

The area $A = xy$

$\therefore\ A = x(100 - x)$

$\therefore\ A = 100x - x^2$

b $A = 100x - x^2$ is a quadratic function with $a = -1$, $b = 100$, $c = 0$.

Since $a < 0$, the shape is $\frown$

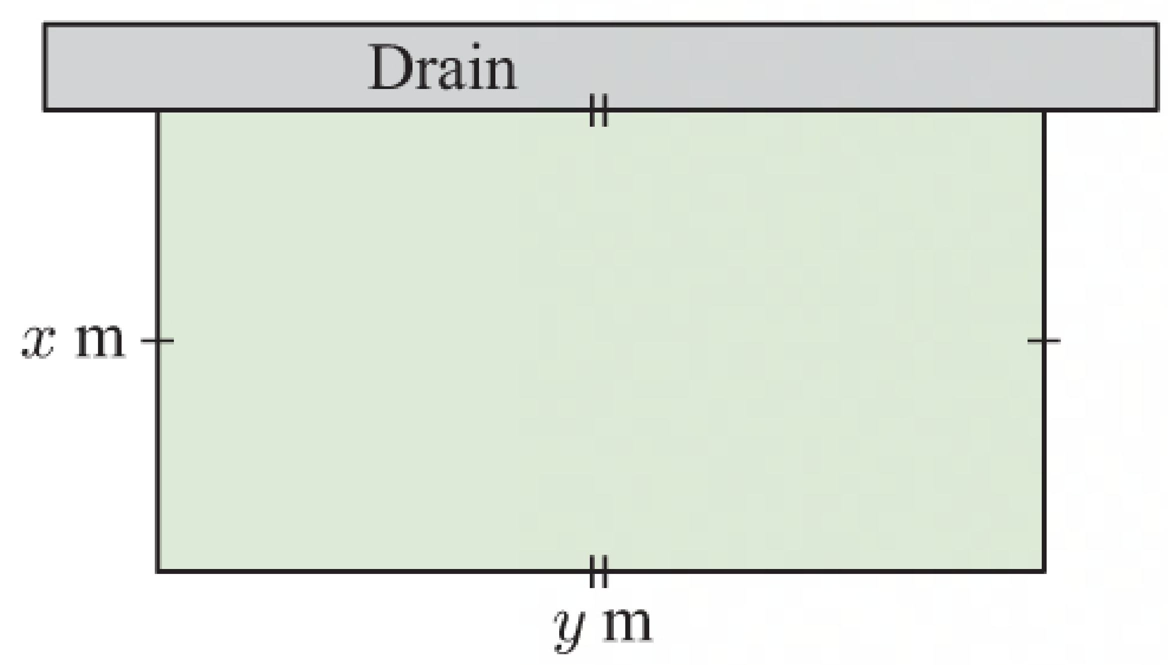
The area is maximised when

$$x = \frac{-b}{2a} = \frac{-100}{2(-1)} = 50$$

and $y = 100 - 50 = 50$

So, the area of the rectangle is maximised when $x = y = 50$, which is when the rectangle is a square.

4 Let the dimensions of the paddock be x m $\times$ y m.



If 1000 m of fence is available, then

$2x + y = 1000$ {perimeter}

$\therefore\ y = 1000 - 2x$ .... (1)

The area of the enclosure $A = xy$

Since $y = 1000 - 2x$, $A = x(1000 - 2x)$

$= 1000x - 2x^2$

$\therefore\ A = -2x^2 + 1000x$

A is a quadratic and $a < 0$, so its shape is $\frown$

So, the area is maximised when $x = \dfrac{-b}{2a} = \dfrac{-1000}{2(-2)} = 250$

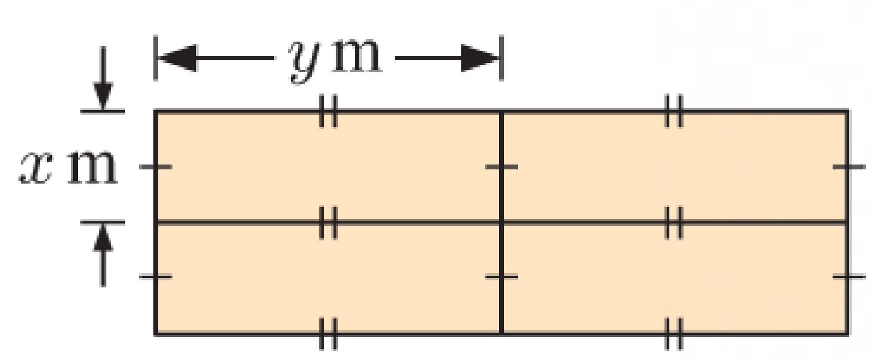
and when $x = 250$, $y = 1000 - 2(250) = 500$

So, the paddock has a maximum area when the dimensions are 250 m $\times$ 500 m.

5 **a** Let x m $\times$ y m be the dimensions of a single pen as shown below.

So, the total length of fencing required is $6x + 6y$.

If there is 500 m of fencing available, then



$6x + 6y = 500$

$\therefore\ x + y = 83\frac{1}{3}$

$\therefore\ y = 83\frac{1}{3} - x$ .... (1)

The area of each pen will be $A = xy$ and substituting equation (1), we have

$A = x(83\frac{1}{3} - x)$

$\therefore\ A = -x^2 + 83\frac{1}{3}x$

which is a quadratic with $a < 0$, so its shape is $\frown$

So, at $x = \dfrac{-b}{2a}$ we have a maximum value of A.

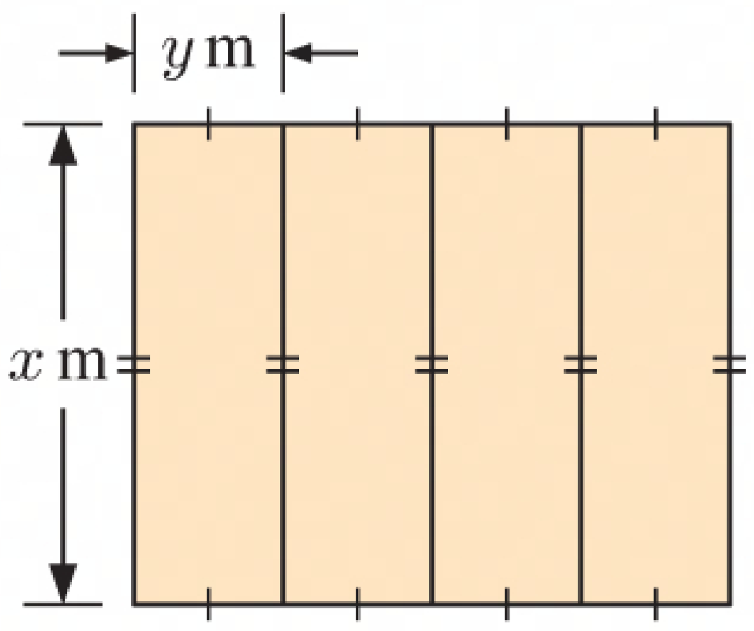
$\therefore\ x = \dfrac{-83\frac{1}{3}}{2(-1)} = 41\frac{2}{3}$ and so $y = 83\frac{1}{3} - 41\frac{2}{3} = 41\frac{2}{3}$

So, the dimensions that maximise the area are $41\frac{2}{3}$ m $\times$ $41\frac{2}{3}$ m.

b Let x m $\times$ y m be the dimensions of a single pen as shown below.

So, the total length of fencing required is $5x + 8y$.

If there is 500 m of fencing available, then



$$5x + 8y = 500$$
$$\therefore\ 8y = 500 - 5x$$
$$\therefore\ y = \frac{500 - 5x}{8}$$
$$\therefore\ y = 62\tfrac{1}{2} - \tfrac{5}{8}x \quad .... (1)$$

The area of each pen will be $A = xy$ and substituting equation (1), we have

$$A = x(62\tfrac{1}{2} - \tfrac{5}{8}x)$$

$\therefore\ A = -\frac{5}{8}x^2 + 62\frac{1}{2}x$ which is a quadratic with $a < 0$, so its shape is 

So, when $x = \dfrac{-b}{2a}$ we have a maximum value of A.

$\therefore\ x = \dfrac{-62\frac{1}{2}}{2(-\frac{5}{8})} = 50$, and substituting $x = 50$ into $y = 62\frac{1}{2} - \frac{5}{8}x$, we have

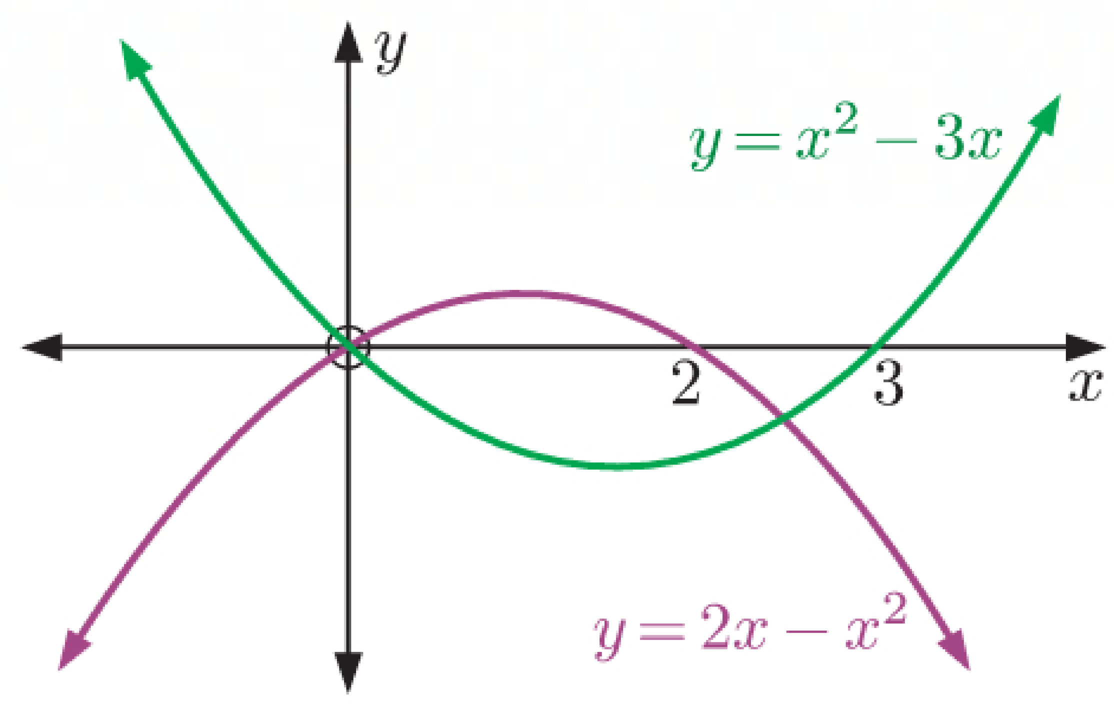
$y = 31\frac{1}{4}$

So, the dimensions that maximise the area are 50 m $\times$ $31\frac{1}{4}$ m.

6 **a** The graphs of $y = x^2 - 3x$ and $y = 2x - x^2$

meet where $x^2 - 3x = 2x - x^2$

$$\therefore\ 2x^2 - 5x = 0$$
$$\therefore\ x(2x - 5) = 0$$
$$\therefore\ x = 0 \text{ or } 2\tfrac{1}{2}$$



b The vertical separation between the curves is given by

$S = (2x - x^2) - (x^2 - 3x)$ $\{y = 2x - x^2$ is above $y = x^2 - 3x$ for $0 \leqslant x \leqslant 2\frac{1}{2}\}$

$$\therefore\ S = 2x - x^2 - x^2 + 3x$$
$$\therefore\ S = -2x^2 + 5x$$

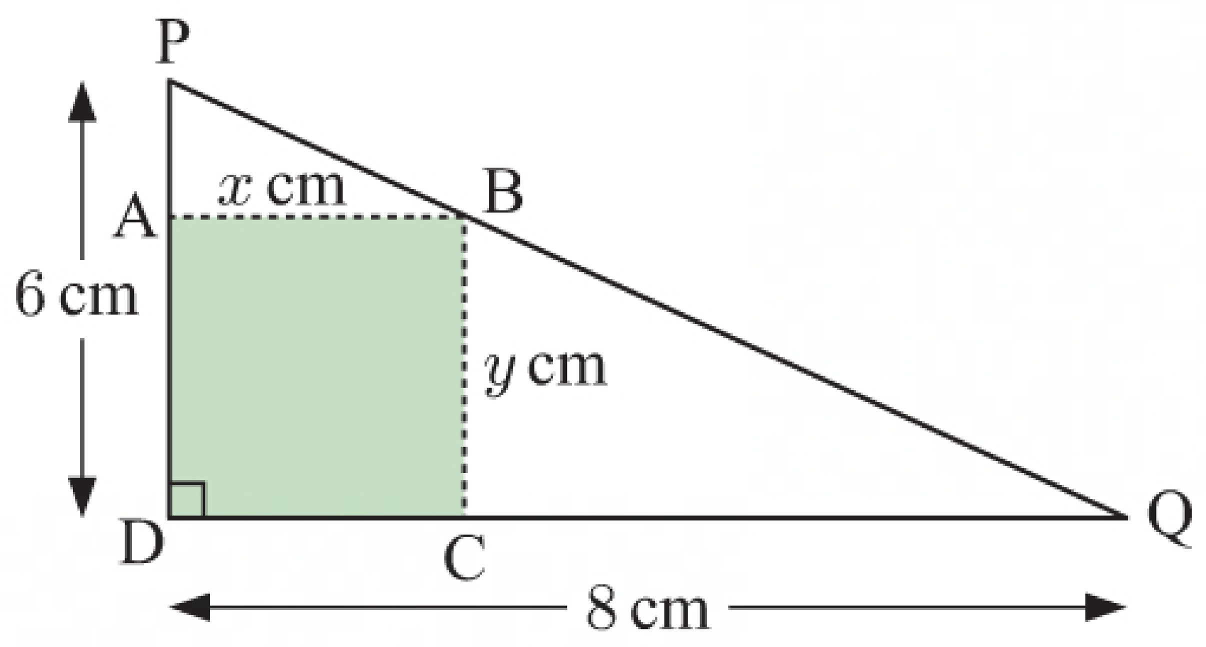
Thus S is a quadratic with $a < 0$, so its shape is 

The maximum separation occurs when $x = \dfrac{-b}{2a} = \dfrac{-5}{2(-2)} = \frac{5}{4}$

$$\text{and}\quad S = -2(\tfrac{5}{4})^2 + 5(\tfrac{5}{4})$$
$$= -\tfrac{25}{8} + \tfrac{25}{4}$$
$$= \tfrac{25}{8} \text{ or } 3\tfrac{1}{8}$$

So, the maximum vertical separation between the curves for $0 \leqslant x \leqslant 2\frac{1}{2}$ is $3\frac{1}{8}$ units.

7 **a**



Triangles PAB and PDQ are similar

$\{A\widehat{P}B$ is common,

$A\widehat{B}P = D\widehat{Q}P$ as $[AB] \parallel [DQ]\}$

$$\therefore \frac{PA}{PD} = \frac{AB}{DQ}$$

$$\therefore \frac{6-y}{6} = \frac{x}{8}$$

$$\therefore 6 - y = \tfrac{3}{4}x$$

$$\therefore y = 6 - \tfrac{3}{4}x$$

b Rectangle ABCD has area $A = xy$

$$= x(6 - \tfrac{3}{4}x)$$

$$= -\tfrac{3}{4}x^2 + 6x$$

which is a quadratic with $a < 0$, so its shape is $\frown$

The area is maximised when $x = \dfrac{-b}{2a} = \dfrac{-6}{2(-\frac{3}{4})} = 4$

and when $x = 4$, $y = 6 - \frac{3}{4}(4) = 3$

So, the dimensions of rectangle ABCD of maximum area are 4 cm × 3 cm.

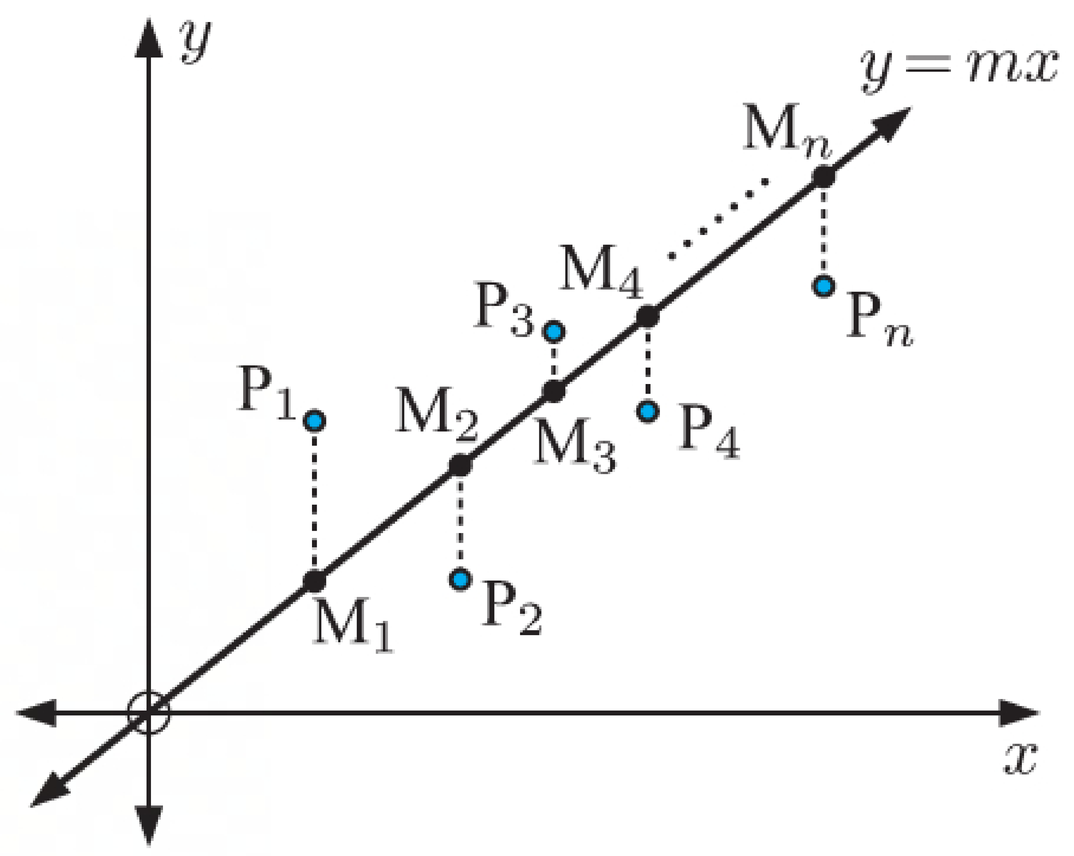
8 Let the "line of best fit" through $(0, 0)$ have slope m.

$\therefore$ the line has equation $y = mx$.

$\therefore$ for $P_1(a_1, b_1)$, the coordinates of M_1 are (a_1, ma_1).

$\therefore$ $P_1M_1 = ma_1 - b_1$

In general, $P_iM_i = |\, ma_i - b_i \,|$, $i = 1, 2, ..., n$



$$\therefore \sum_{i=1}^{n} (P_iM_i)^2 = \sum_{i=1}^{n} |\, ma_i - b_i \,|^2$$

$$= \sum_{i=1}^{n} (ma_i - b_i)^2 \qquad \{|\, z \,|^2 = z^2\}$$

$$= \sum_{i=1}^{n} (m^2 a_i{}^2 - 2ma_ib_i + b_i{}^2)$$

$$= m^2 \sum_{i=1}^{n} a_i{}^2 - m \sum_{i=1}^{n} 2a_ib_i + \sum_{i=1}^{n} b_i{}^2$$

which is a quadratic in m, with $a = \sum_{i=1}^{n} a_i{}^2$, $b = -\sum_{i=1}^{n} 2a_ib_i$, $c = \sum_{i=1}^{n} b_i{}^2$

$a = \sum_{i=1}^{n} a_i{}^2 > 0$, so the quadratic has shape $\smile$

$$\therefore \text{ the sum } \sum_{i=1}^{n} (P_iM_i)^2 \text{ is minimised when } m = \frac{-b}{2a} = \frac{\sum_{i=1}^{n} 2a_ib_i}{\sum_{i=1}^{n} 2a_i{}^2}$$

$$= \frac{\sum_{i=1}^{n} a_ib_i}{\sum_{i=1}^{n} a_i{}^2}$$

9 $y = (x - a - b)(x - a + b)(x + a - b)(x + a + b)$

$= [x - (a + b)][x + (a + b)][x - (a - b)][x + (a - b)]$ {rearranging}

$= [x^2 - (a + b)^2][x^2 - (a - b)^2]$

$= x^4 - x^2(a - b)^2 - x^2(a + b)^2 + (a + b)^2(a - b)^2$

$= x^4 - x^2[(a - b)^2 + (a + b)^2] + [(a + b)(a - b)]^2$

$= x^4 - x^2(a^2 - 2ab + b^2 + a^2 + 2ab + b^2) + (a^2 - b^2)^2$

$= x^4 - 2(a^2 + b^2)x^2 + (a^2 - b^2)^2$

which is a quadratic in x^2 with "a" $= 1$, "b" $= -2(a^2 + b^2)$, "c" $= (a^2 - b^2)^2$

"a" > 0, so the quadratic has shape 

$\therefore$ y is minimised when $x^2 = \dfrac{2(a^2 + b^2)}{2} = a^2 + b^2$

When $x^2 = a^2 + b^2$, $y = (a^2 + b^2)^2 - 2(a^2 + b^2)(a^2 + b^2) + (a^2 - b^2)^2$

$= (a^2 + b^2)^2 - 2(a^2 + b^2)^2 + (a^2 - b^2)^2$

$= (a^2 - b^2)^2 - (a^2 + b^2)^2$

$= a^4 - 2a^2b^2 + b^4 - a^4 - 2a^2b^2 - b^4$

$= -4a^2b^2$

$\therefore$ the least value of y is $-4a^2b^2$.

10 $y = (a_1x - b_1)^2 + (a_2x - b_2)^2$

$= a_1^2x^2 - 2a_1b_1x + b_1^2 + a_2^2x^2 - 2a_2b_2x + b_2^2$

$= (a_1^2 + a_2^2)x^2 - 2(a_1b_1 + a_2b_2)x + (b_1^2 + b_2^2)$

which is a quadratic in x with $a = a_1^2 + a_2^2 > 0$, so it has shape 

$\therefore$ y is minimised when $x = \dfrac{-b}{2a} = \dfrac{2(a_1b_1 + a_2b_2)}{2(a_1^2 + a_2^2)} = \dfrac{a_1b_1 + a_2b_2}{a_1^2 + a_2^2}$

When $x = \dfrac{a_1b_1 + a_2b_2}{a_1^2 + a_2^2}$,

$$y = (a_1^2 + a_2^2)\left(\frac{a_1b_1 + a_2b_2}{a_1^2 + a_2^2}\right)^2 - 2(a_1b_1 + a_2b_2)\left(\frac{a_1b_1 + a_2b_2}{a_1^2 + a_2^2}\right) + b_1^2 + b_2^2$$

$$= \frac{(a_1b_1 + a_2b_2)^2}{a_1^2 + a_2^2} - \frac{2(a_1b_1 + a_2b_2)^2}{a_1^2 + a_2^2} + b_1^2 + b_2^2$$

$$= b_1^2 + b_2^2 - \frac{(a_1b_1 + a_2b_2)^2}{a_1^2 + a_2^2}$$

But since $y = (a_1x - b_1)^2 + (a_2x - b_2)^2$, $y \geqslant 0$ for all x {sum of two squared terms}

$\therefore$ $b_1^2 + b_2^2 - \dfrac{(a_1b_1 + a_2b_2)^2}{a_1^2 + a_2^2} \geqslant 0$

$\therefore$ $b_1^2 + b_2^2 \geqslant \dfrac{(a_1b_1 + a_2b_2)^2}{a_1^2 + a_2^2}$

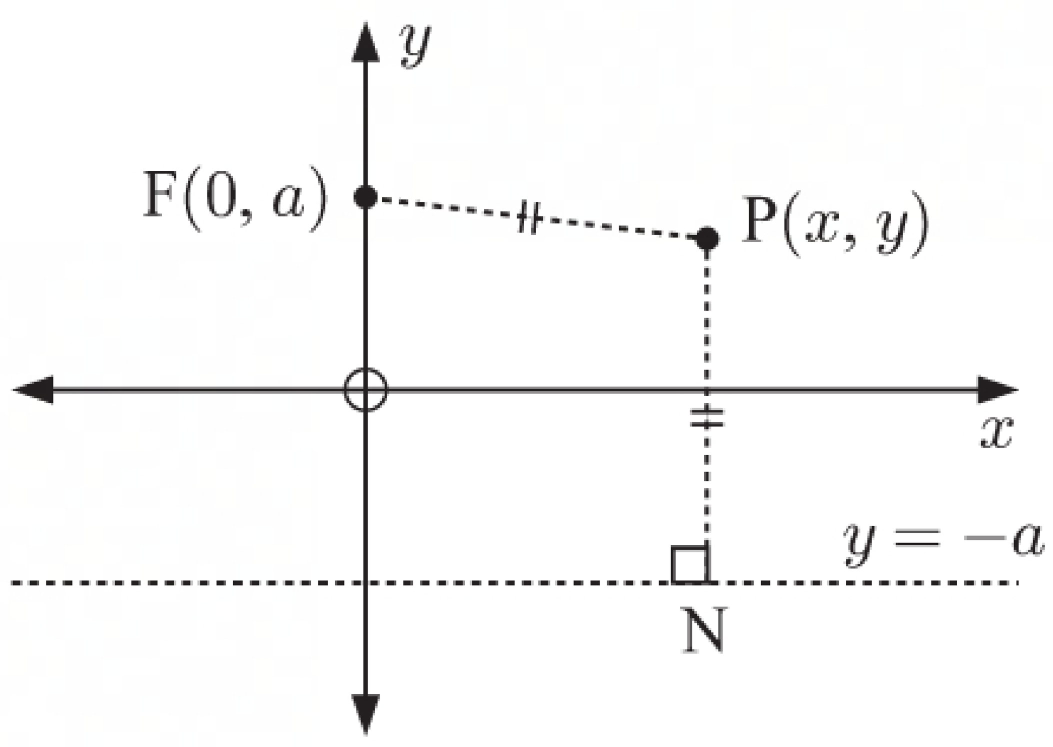
$\therefore$ $(a_1^2 + a_2^2)(b_1^2 + b_2^2) \geqslant (a_1b_1 + a_2b_2)^2$ $\{a_1^2 + a_2^2 \geqslant 0\}$

$\therefore$ $\sqrt{a_1^2 + a_2^2}\sqrt{b_1^2 + b_2^2} \geqslant \sqrt{(a_1b_1 + a_2b_2)^2}$

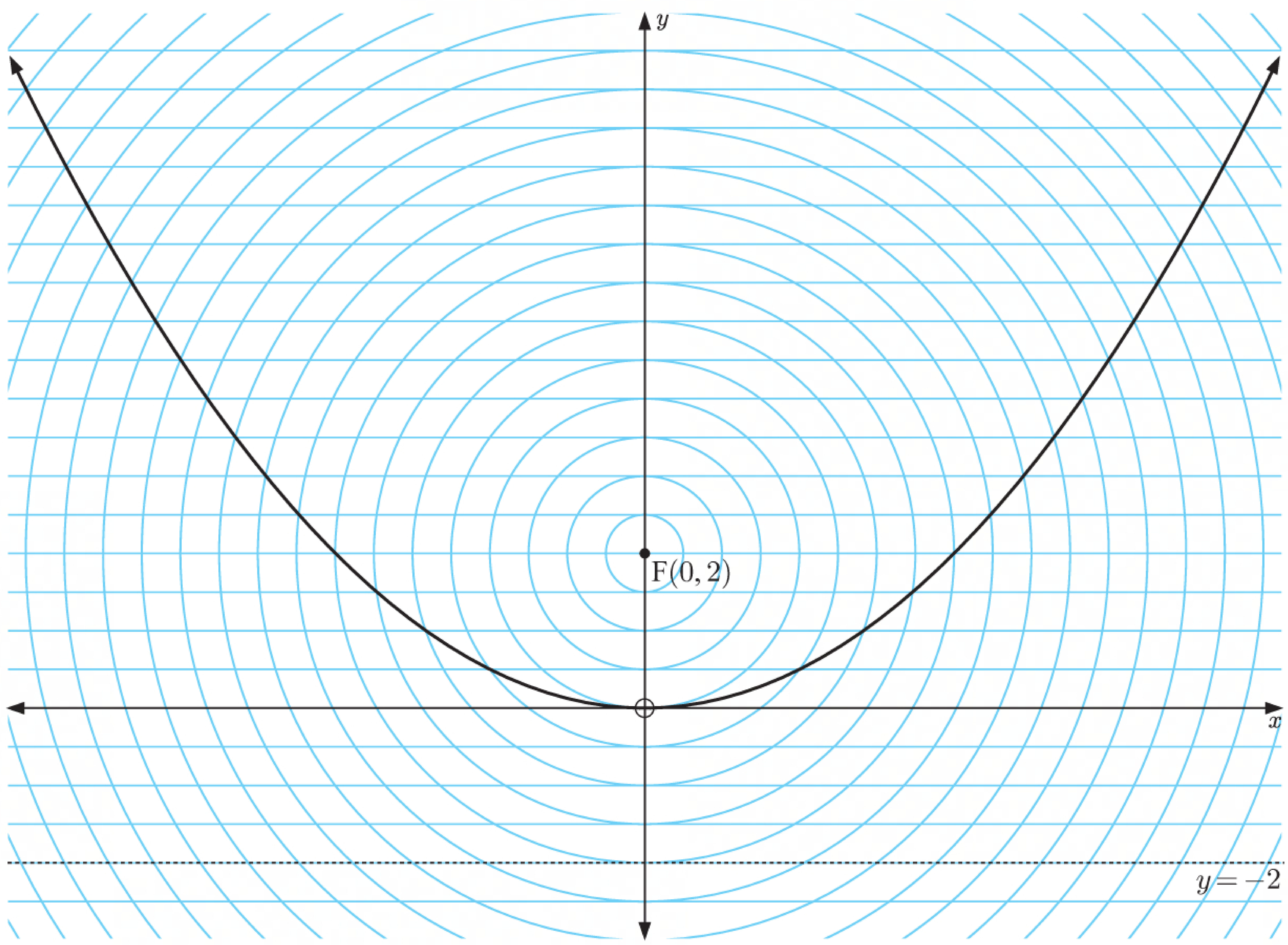
$\therefore$ $|\, a_1b_1 + a_2b_2 \,| \leqslant \sqrt{a_1^2 + a_2^2}\sqrt{b_1^2 + b_2^2}$

INVESTIGATION 4 THE GEOMETRIC DEFINITION OF A PARABOLA

1 If we let the focus be at $(0, a)$ and the directrix be $y = -a$, then the origin $(0, 0)$ is always the vertex of the parabola.



2



3 **a** N is $(x, -a)$

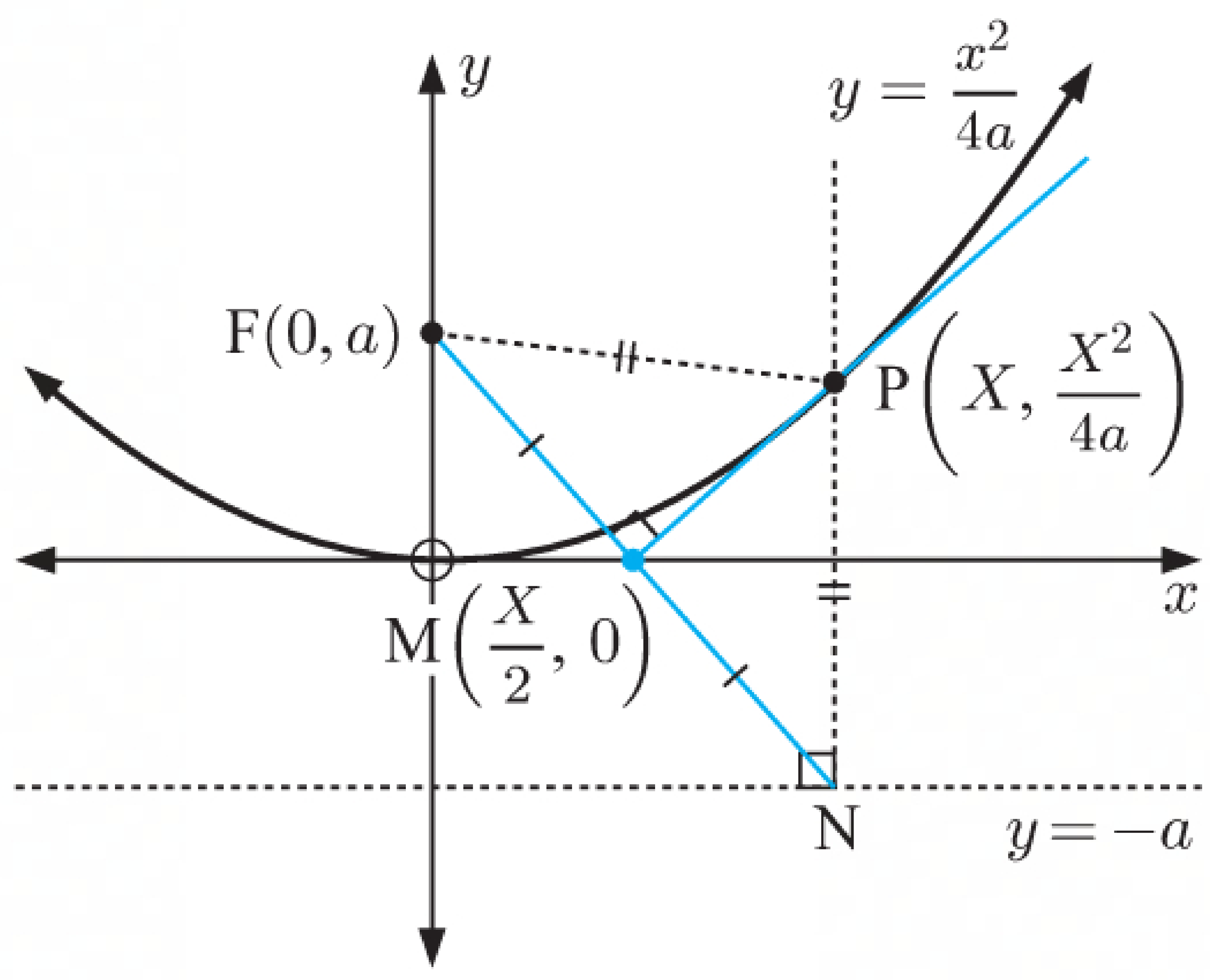
b $\text{FP} = \sqrt{(x-0)^2 + (y-a)^2}$ $\quad\quad$ $\text{NP} = y + a$

$= \sqrt{x^2 + y^2 - 2ay + a^2}$

c The parabola is the set of all points P such that $\text{FP} = \text{NP}$

$$\therefore \sqrt{x^2 + y^2 - 2ay + a^2} = y + a$$
$$\therefore x^2 + y^2 - 2ay + a^2 = (y+a)^2$$
$$\therefore x^2 + y^2 - 2ay + a^2 = y^2 + 2ay + a^2$$
$$\therefore x^2 = 4ay$$
$$\therefore y = \frac{x^2}{4a}$$

4 **a** **i** M has coordinates $\left(\frac{X}{2}, 0\right)$.



$\therefore$ [MP] has gradient $\dfrac{\frac{X^2}{4a} - 0}{X - \frac{X}{2}} = \dfrac{\frac{X^2}{4a}}{\frac{X}{2}}$

$= \dfrac{X}{2a}$

[MP] passes through $\left(\frac{X}{2}, 0\right)$, so it has

equation $y = \dfrac{X}{2a}\left(x - \dfrac{X}{2}\right) + 0$

$\therefore\ y = \dfrac{X}{2a}\left(x - \dfrac{X}{2}\right)$

ii The parabola and (MP) meet where $\dfrac{x^2}{4a} = \dfrac{X}{2a}\left(x - \dfrac{X}{2}\right)$

$$\therefore\ \frac{1}{4a}x^2 = \frac{X}{2a}x - \frac{X^2}{4a}$$

$$\therefore\ \frac{1}{4a}x^2 - \frac{X}{2a}x + \frac{X^2}{4a} = 0$$

which is a quadratic in x with $\Delta = \left(-\dfrac{X}{2a}\right)^2 - 4\left(\dfrac{1}{4a}\right)\left(\dfrac{X^2}{4a}\right)$

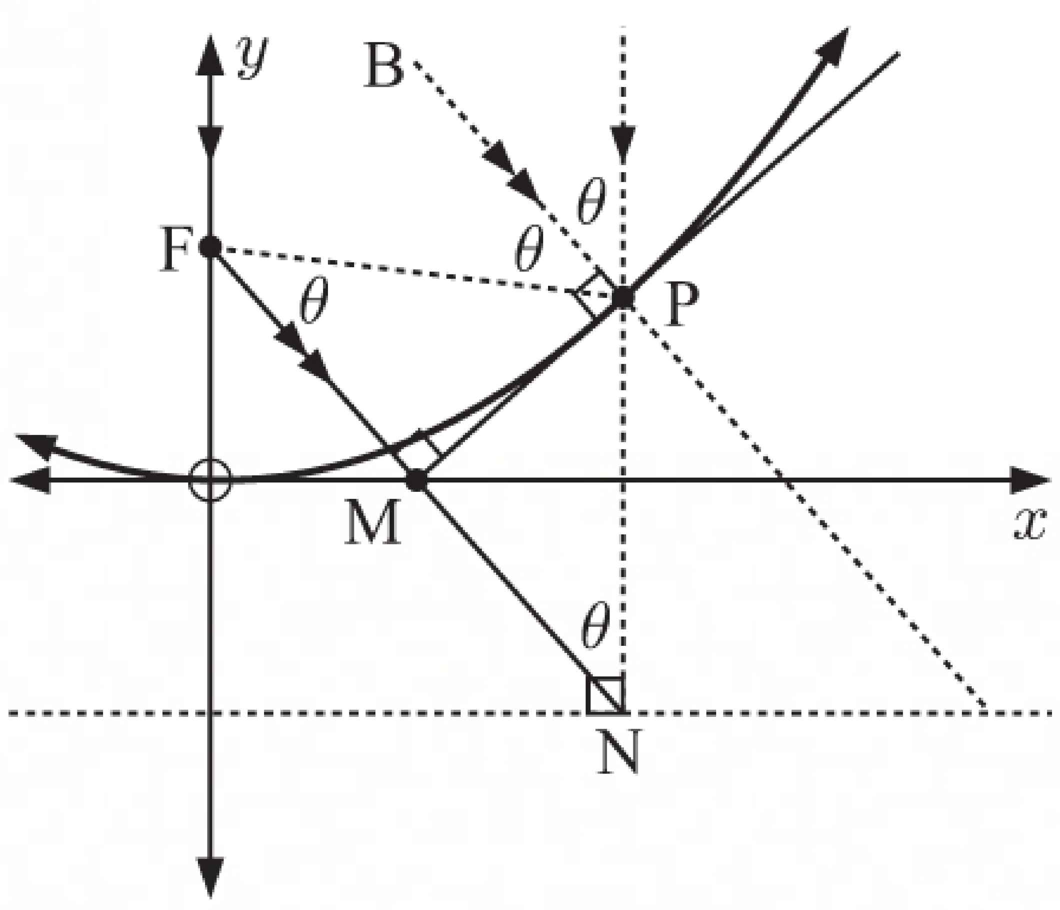
$$= \frac{X^2}{4a^2} - \frac{X^2}{4a^2}$$

$$= 0$$

So, the graphs *touch* at one point.

$\therefore$ (MP) is a tangent to the parabola.

b **i** $\mathrm{M\widehat{N}P} = \theta$ {corresponding angles}



ii Triangle FPM is isosceles

$\therefore\ \mathrm{M\widehat{F}P} = \mathrm{M\widehat{N}P}$ {base angles of an isosceles △}

$= \theta$

iii $\mathrm{F\widehat{P}B} = \mathrm{M\widehat{F}P}$ {alternate angles}

$= \theta$

iv angle of incidence = angle of reflection {from the **Opening Problem**}

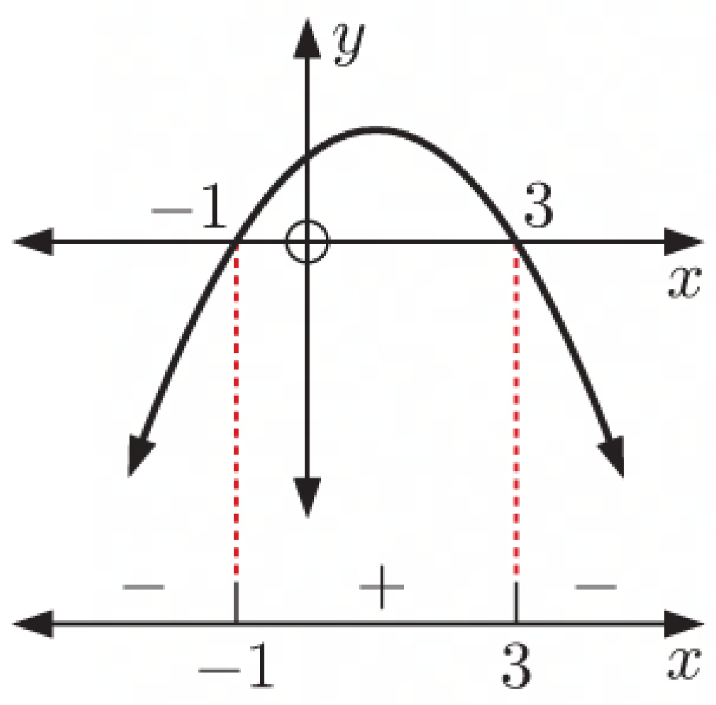
$\therefore$ angle of reflection $= \theta$

But the angle between the normal and the focus $\mathrm{F\widehat{P}B} = \theta$, so a vertical ray of light is always reflected to the focus.

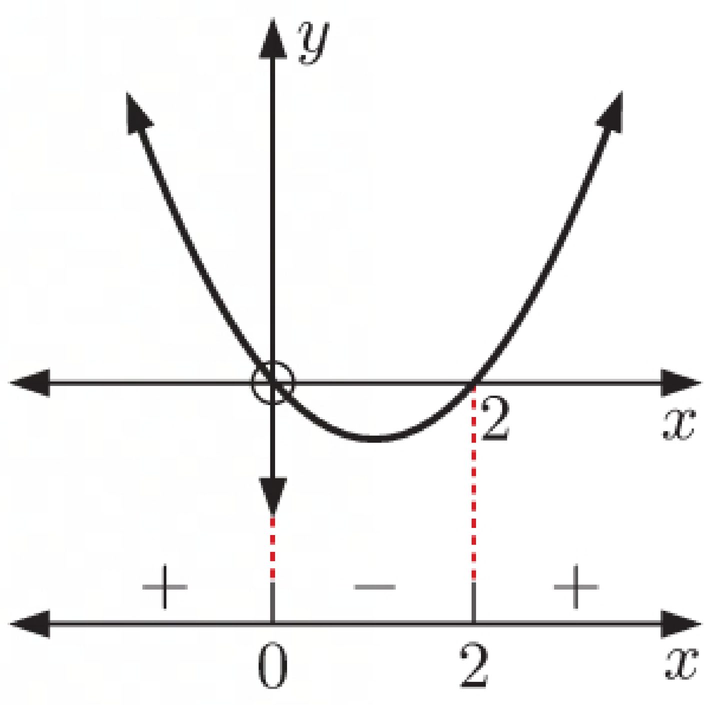
v Misha needs to use a *parabolic* mirror, and place his cup at the *focus* of the parabola.

EXERCISE 14H.1

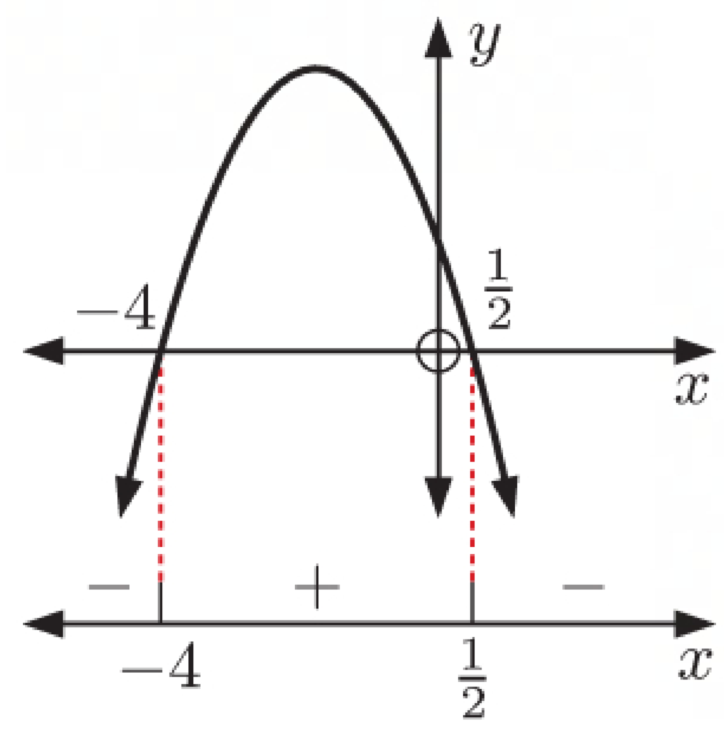
1 **a**



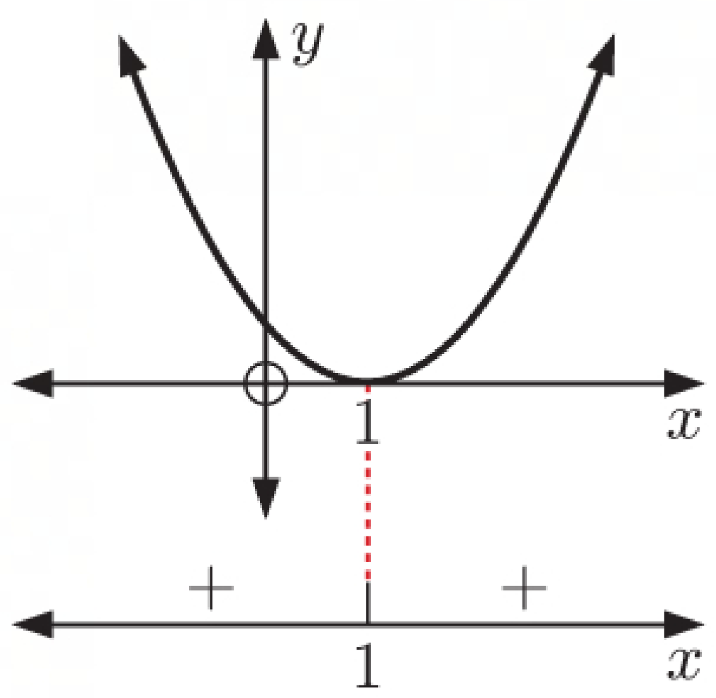
b



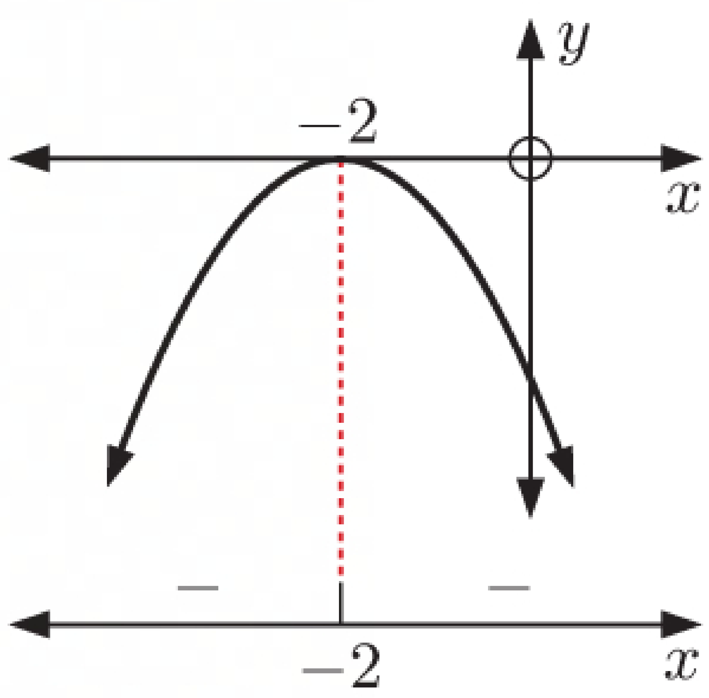
c



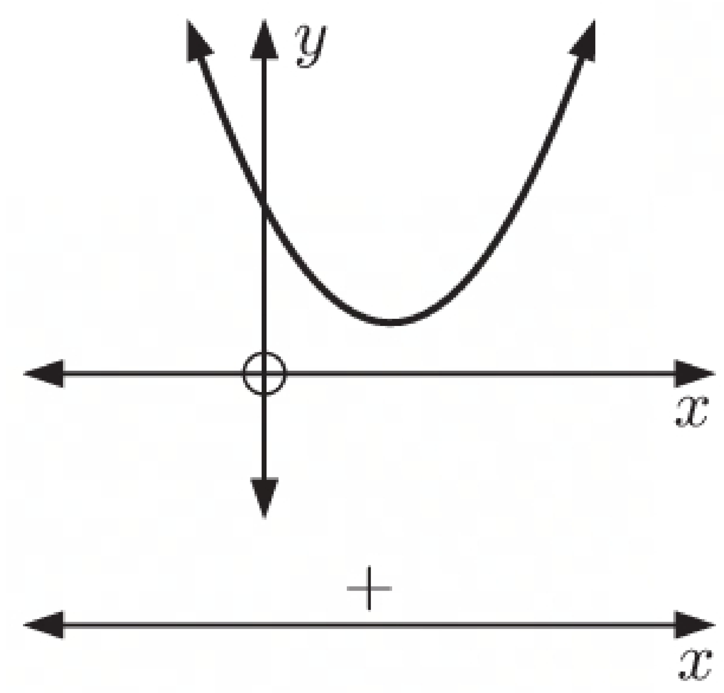
d



e



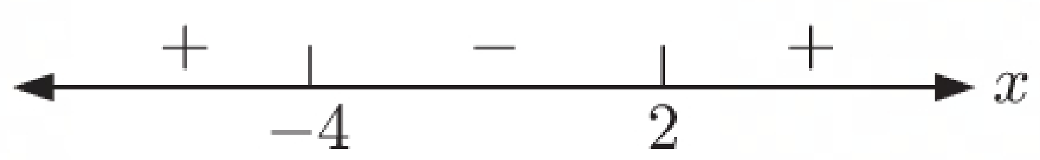
f



2 **a** $(x+4)(x-2)$ has zeros -4 and 2.

When $x = 3$ we have $(7)(1) > 0$,
so we put a $+$ sign here.

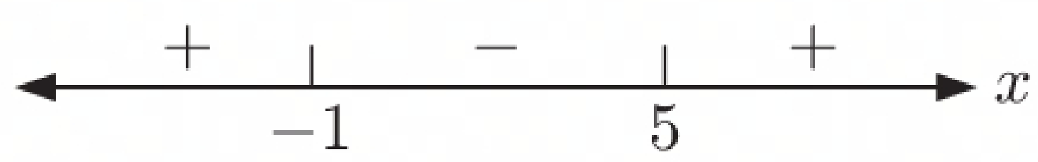
As the factors are single, the signs alternate.



b $(x+1)(x-5)$ has zeros -1 and 5.

When $x = 6$ we have $(7)(1) > 0$,
so we put a $+$ sign here.

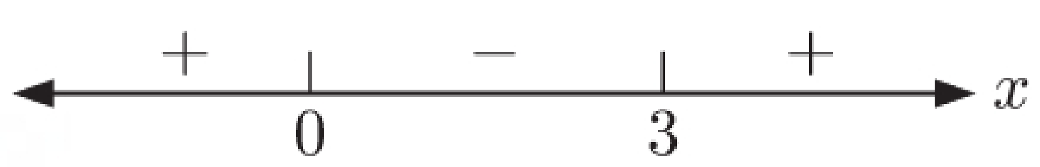
As the factors are single, the signs alternate.



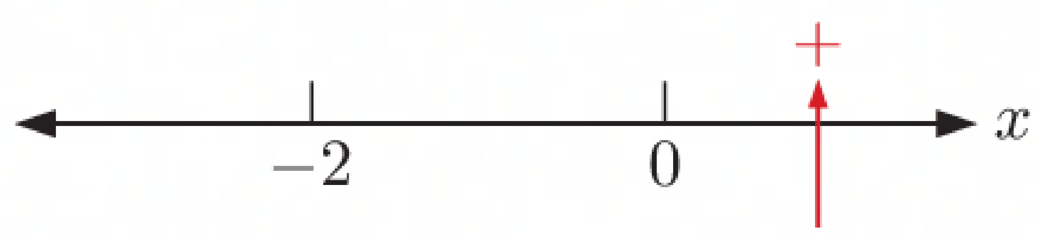
c $x(x-3)$ has zeros 0 and 3.

When $x = 4$ we have $(4)(1) > 0$,
so we put a $+$ sign here.

As the factors are single, the signs alternate.

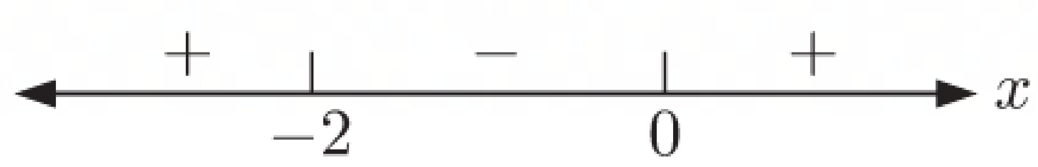


d $x(x+2)$ has zeros 0 and -2.



When $x = 1$ we have $(1)(3) > 0$,
so we put a $+$ sign here.

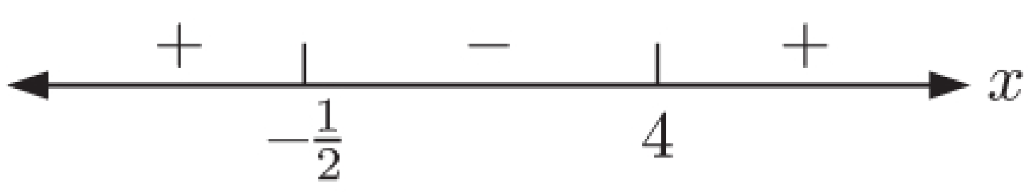
As the factors are single, the signs alternate.



e $(2x+1)(x-4)$ has zeros $-\frac{1}{2}$ and 4.

When $x = 5$ we have $(11)(1) > 0$,
so we put a $+$ sign here.

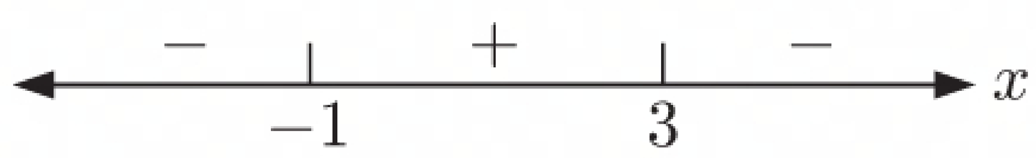
As the factors are single, the signs alternate.



f $-(x+1)(x-3)$ has zeros -1 and 3.

When $x = 4$ we have $-(5)(1) < 0$,
so we put a $-$ sign here.

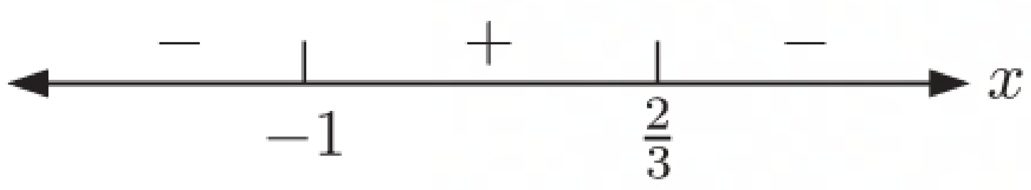
As the factors are single, the signs alternate.



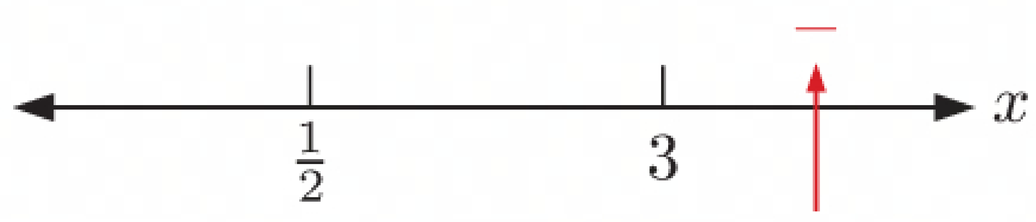
g $-(3x-2)(x+1)$ has zeros $\frac{2}{3}$ and -1.

When $x = 1$ we have $-(1)(2) < 0$,
so we put a $-$ sign here.

As the factors are single, the signs alternate.

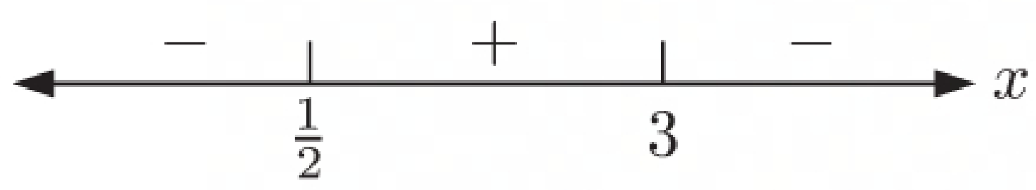


h $(2x-1)(3-x)$ has zeros $\frac{1}{2}$ and 3.



When $x = 4$ we have $(7)(-1) < 0$,
so we put a $-$ sign here.

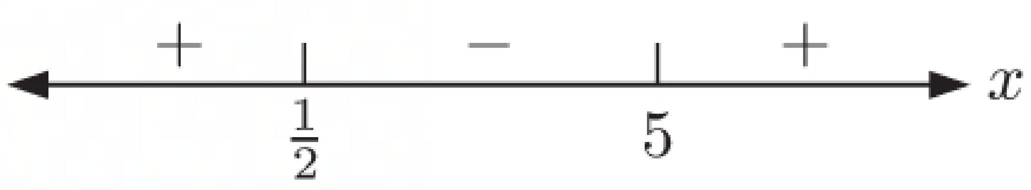
As the factors are single, the signs alternate.



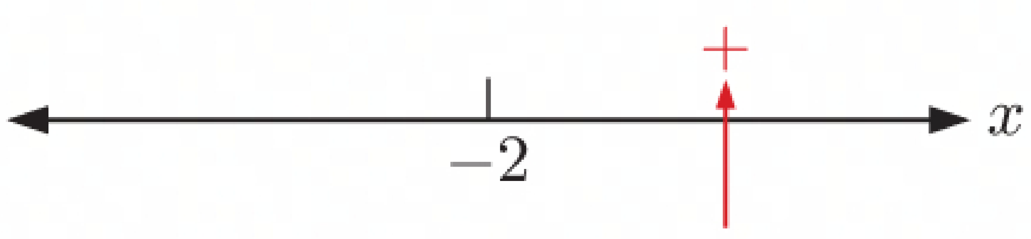
i $(5-x)(1-2x)$ has zeros 5 and $\frac{1}{2}$.

When $x = 6$ we have $(-1)(-11) > 0$,
so we put a $+$ sign here.

As the factors are single, the signs alternate.

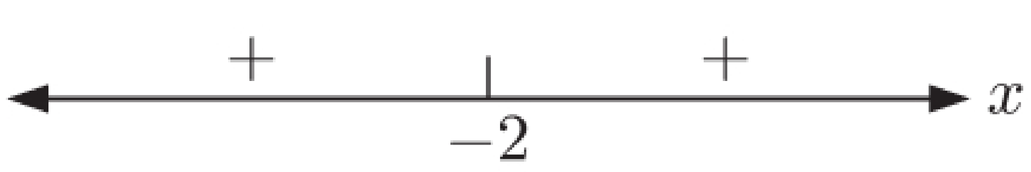


3 **a** $(x+2)^2$ has zero -2.

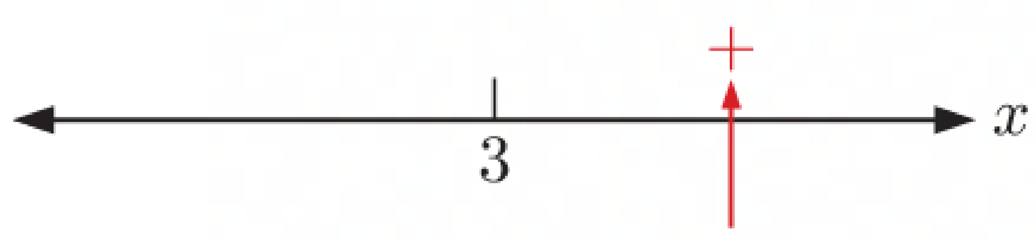


When $x = 0$ we have $(2)^2 > 0$,
so we put a $+$ sign here.

As the factor is squared, the signs do not change.

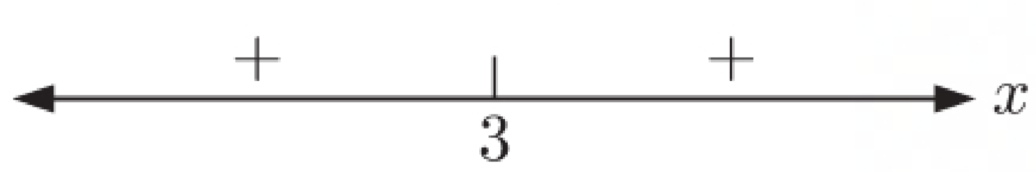


b $(x-3)^2$ has zero 3.

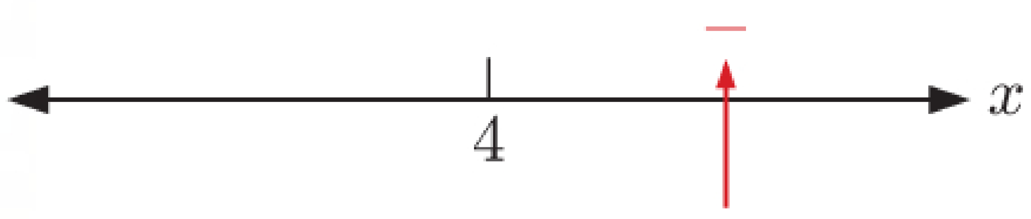


When $x = 4$ we have $(1)^2 > 0$,
so we put a $+$ sign here.

As the factor is squared, the signs do not change.

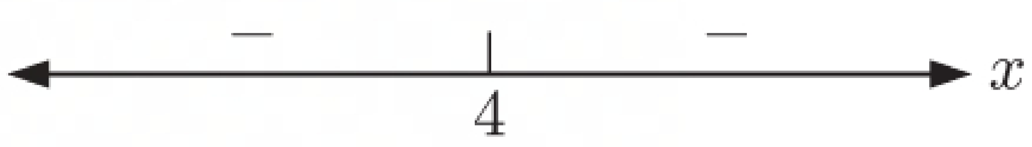


c $-(x-4)^2$ has zero 4.

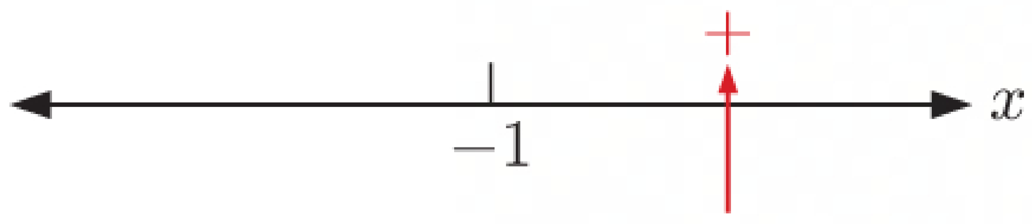


When $x = 5$ we have $-(1)^2 < 0$,
so we put a $-$ sign here.

As the factor is squared, the signs do not change.

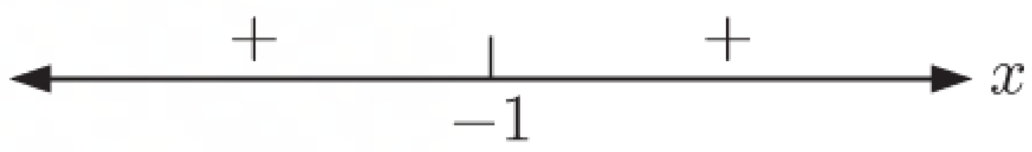


d $2(x+1)^2$ has zero -1.

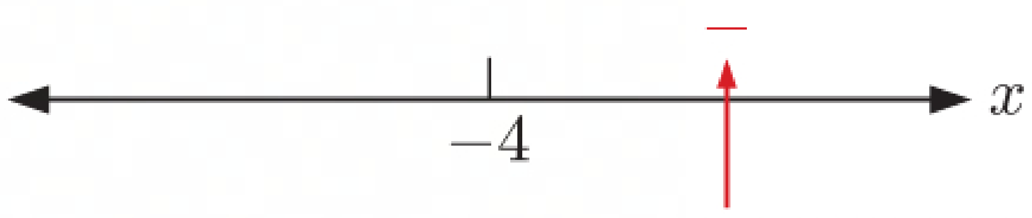


When $x = 0$ we have $2(1)^2 > 0$,
so we put a $+$ sign here.

As the factor is squared, the signs do not change.

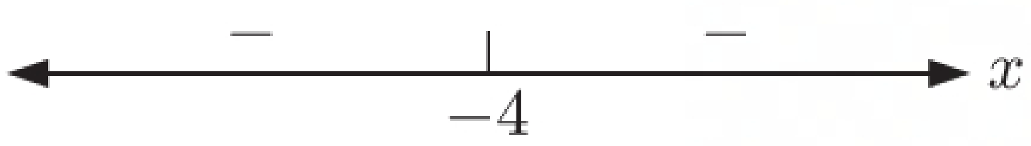


e $-3(x+4)^2$ has zero -4.

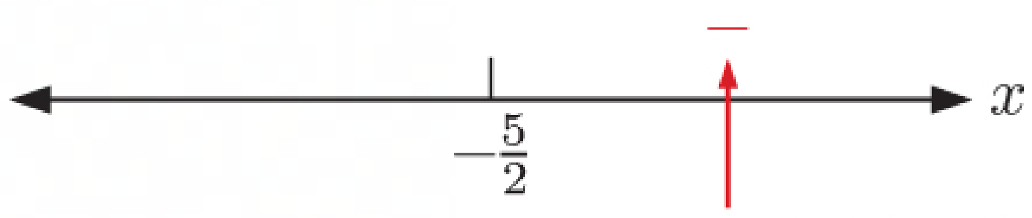


When $x = 0$ we have $-3(4)^2 < 0$,
so we put a $-$ sign here.

As the factor is squared, the signs do not change.

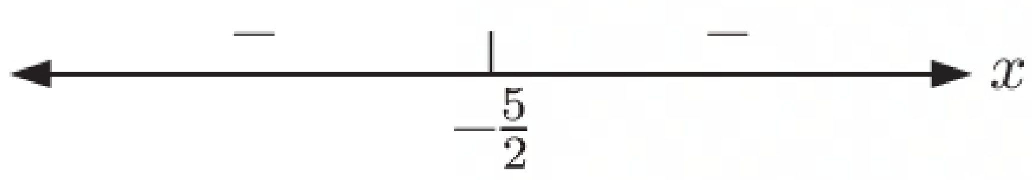


f $-\frac{1}{2}(2x+5)^2$ has zero $-\frac{5}{2}$.

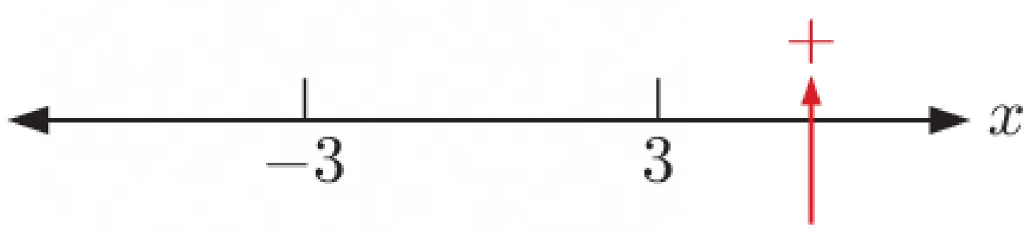


When $x = 0$ we have $-\frac{1}{2}(5)^2 < 0$,
so we put a $-$ sign here.

As the factor is squared, the signs do not change.

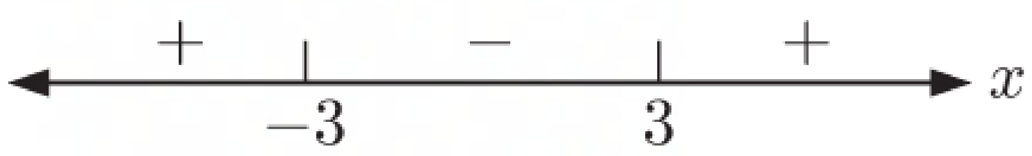


4 **a** $x^2 - 9 = (x+3)(x-3)$
has zeros -3 and 3.



When $x = 4$ we have $(7)(1) > 0$,
so we put a $+$ sign here.

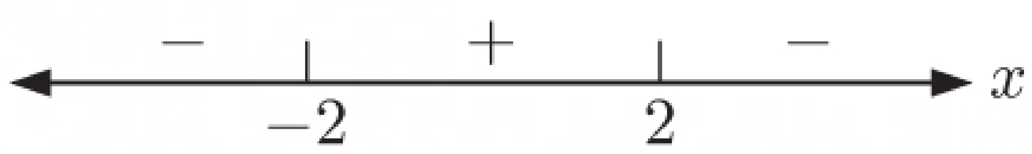
As the factors are single, the signs alternate.



b $4 - x^2 = (2+x)(2-x)$
has zeros -2 and 2.

When $x = 3$ we have $(5)(-1) < 0$,
so we put a $-$ sign here.

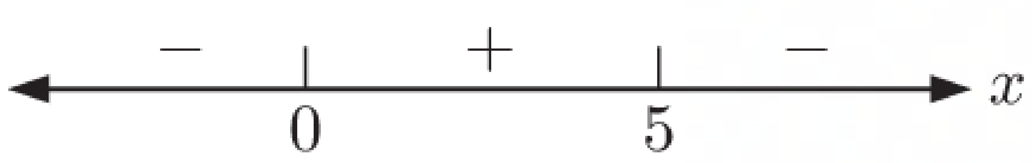
As the factors are single, the signs alternate.



c $5x - x^2 = x(5-x)$ has zeros 0 and 5.

When $x = 6$ we have $(6)(-1) < 0$,
so we put a $-$ sign here.

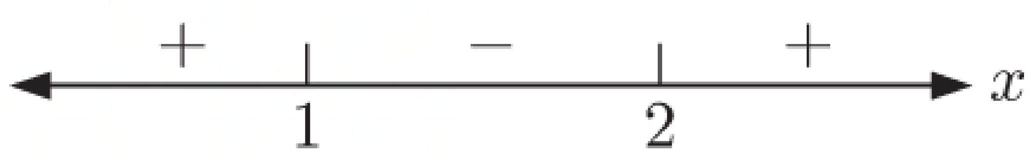
As the factors are single, the signs alternate.



d $x^2 - 3x + 2 = (x-1)(x-2)$
has zeros 1 and 2.

When $x = 3$ we have $(2)(1) > 0$,
so we put a $+$ sign here.

As the factors are single, the signs alternate.



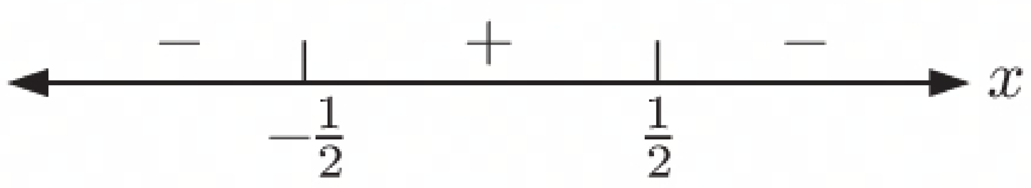
e $2 - 8x^2 = 2(1+2x)(1-2x)$

has zeros $-\frac{1}{2}$ and $\frac{1}{2}$.

When $x = 1$ we have $2(3)(-1) < 0$,

so we put a $-$ sign here.

As the factors are single, the signs alternate.



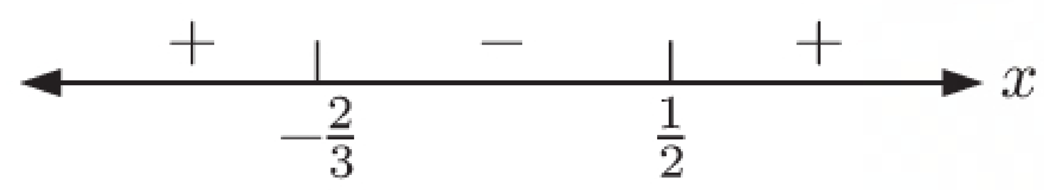
f $6x^2 + x - 2 = (3x+2)(2x-1)$

has zeros $-\frac{2}{3}$ and $\frac{1}{2}$.

When $x = 1$ we have $(5)(1) > 0$,

so we put a $+$ sign here.

As the factors are single, the signs alternate.



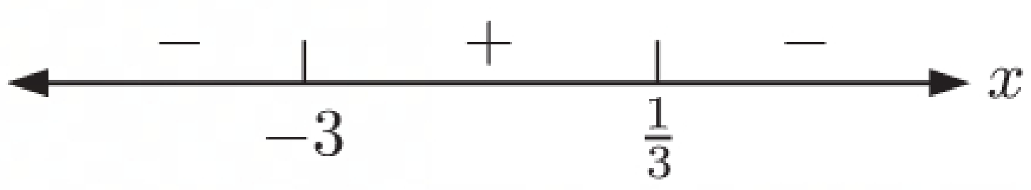
g $6 - 16x - 6x^2 = (6+2x)(1-3x)$

has zeros -3 and $\frac{1}{3}$.

When $x = 1$ we have $(8)(-2) < 0$,

so we put a $-$ sign here.

As the factors are single, the signs alternate.



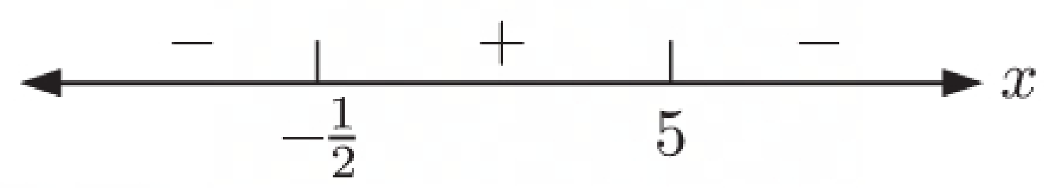
h $-2x^2 + 9x + 5 = (2x+1)(5-x)$

has zeros $-\frac{1}{2}$ and 5.

When $x = 6$ we have $(13)(-1) < 0$,

so we put a $-$ sign here.

As the factors are single, the signs alternate.



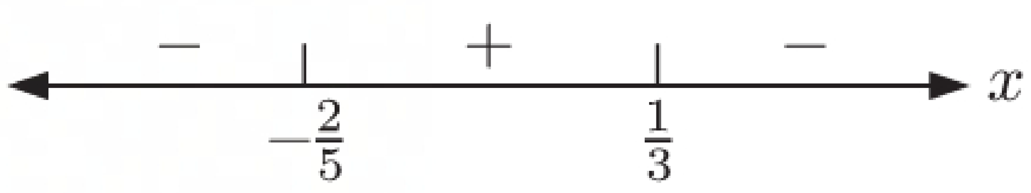
i $-15x^2 - x + 2 = (5x+2)(-3x+1)$

has zeros $-\frac{2}{5}$ and $\frac{1}{3}$.

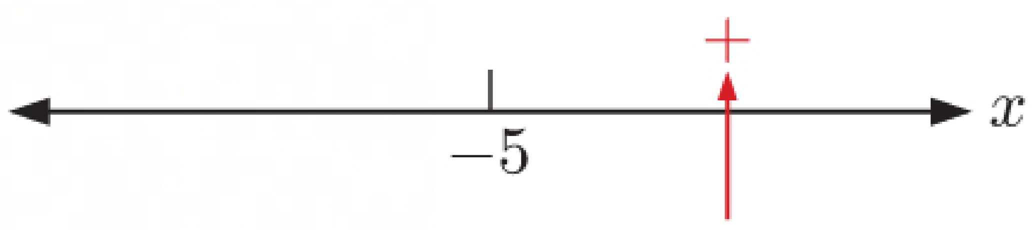
When $x = 1$ we have $(7)(-2) < 0$,

so we put a $-$ sign here.

As the factors are single, the signs alternate.



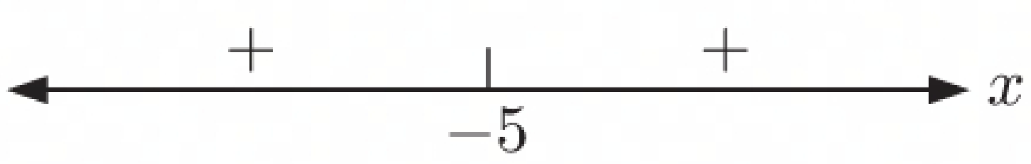
5 **a** $x^2 + 10x + 25 = (x+5)^2$ has zero -5.



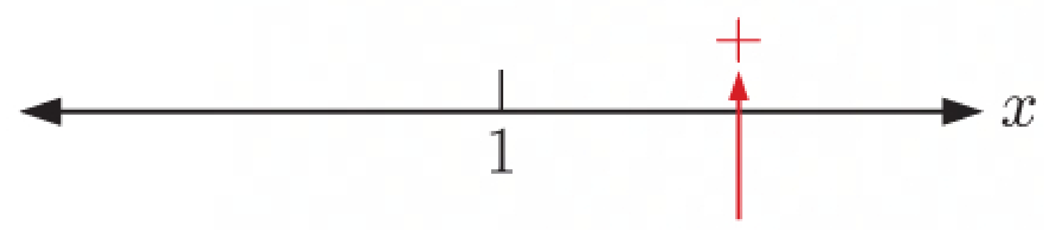
When $x = 0$ we have $(5)^2 > 0$,

so we put a $+$ sign here.

As the factor is squared, the signs do not change.



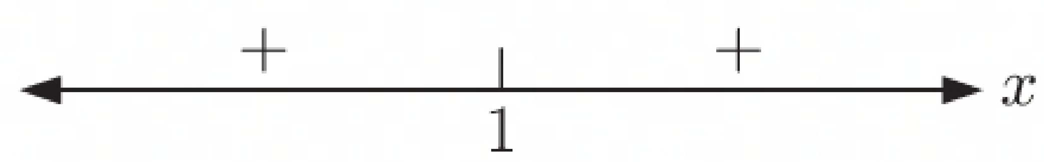
b $x^2 - 2x + 1 = (x-1)^2$ has zero 1.



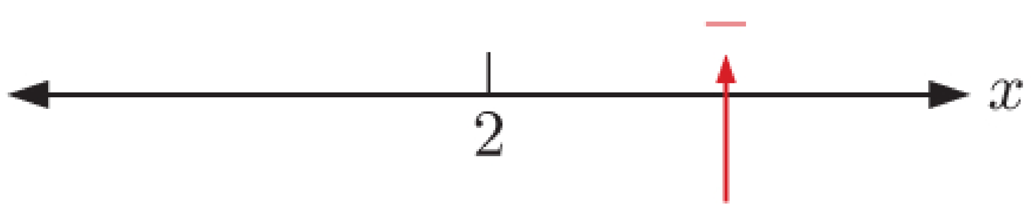
When $x = 2$ we have $(1)^2 > 0$,

so we put a $+$ sign here.

As the factor is squared, the signs do not change.

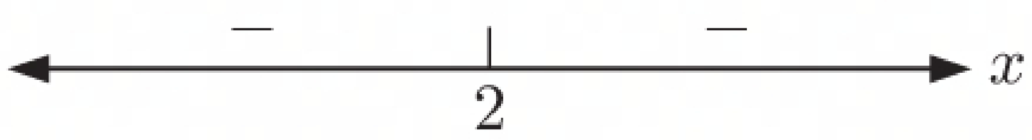


c $-x^2 + 4x - 4 = -(x-2)^2$ has zero 2.

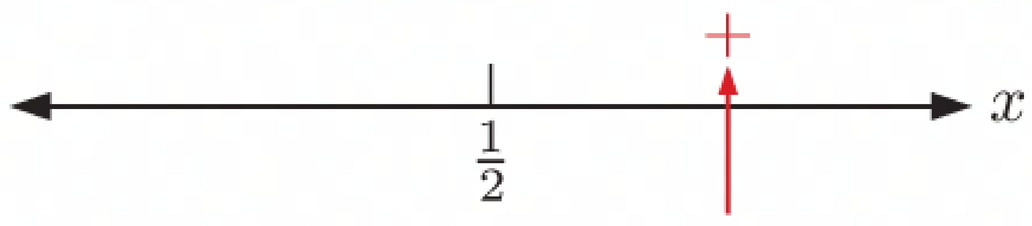


When $x = 3$ we have $-(1)^2 < 0$,
so we put a $-$ sign here.

As the factor is squared, the signs do not change.

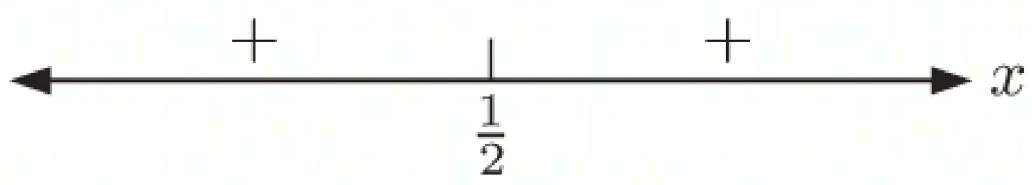


d $4x^2 - 4x + 1 = (2x-1)^2$ has zero $\frac{1}{2}$.

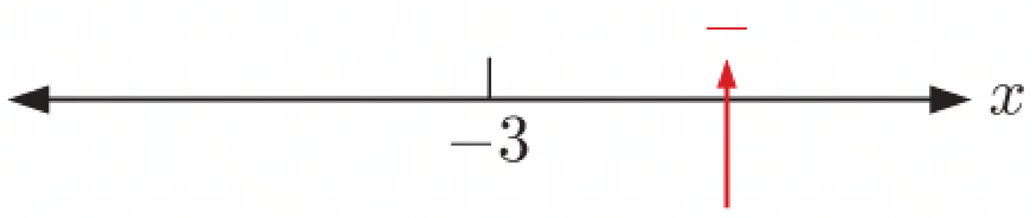


When $x = 1$ we have $(1)^2 > 0$,
so we put a $+$ sign here.

As the factor is squared, the signs do not change.

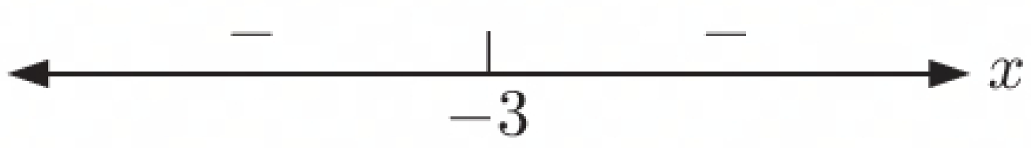


e $-x^2 - 6x - 9 = -(x+3)^2$ has zero -3.

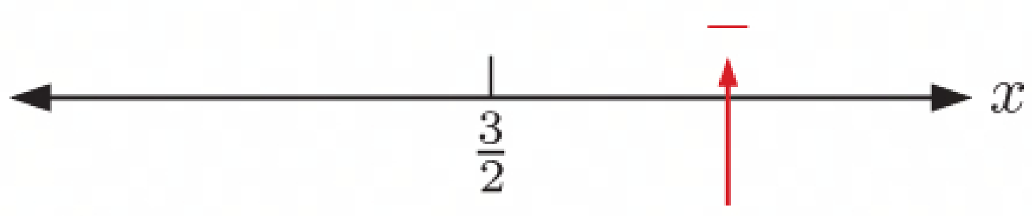


When $x = 0$ we have $-(3)^2 < 0$,
so we put a $-$ sign here.

As the factor is squared, the signs do not change.

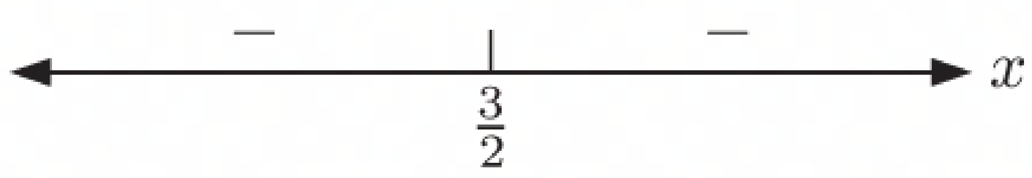


f $-4x^2 + 12x - 9 = -(2x-3)^2$
has zero $\frac{3}{2}$.



When $x = 2$ we have $-(1)^2 < 0$,
so we put a $-$ sign here.

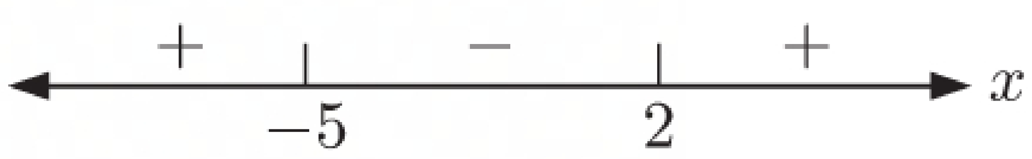
As the factor is squared, the signs do not change.



EXERCISE 14H.2

1 **a** $(x-2)(x+5) \leqslant 0$

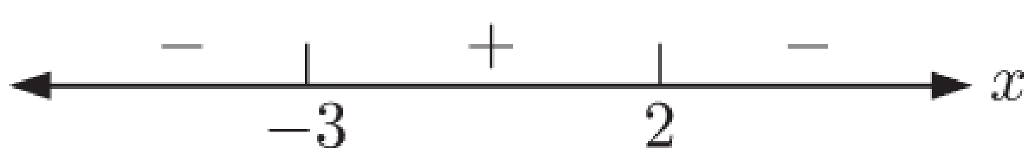
Sign diagram of LHS is



$\therefore \quad -5 \leqslant x \leqslant 2$

b $(2-x)(x+3) \geqslant 0$

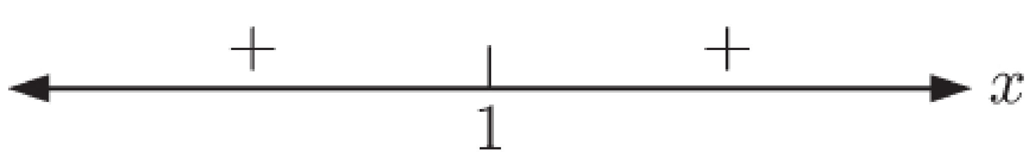
Sign diagram of LHS is



$\therefore \quad -3 \leqslant x \leqslant 2$

c $(x-1)^2 < 0$

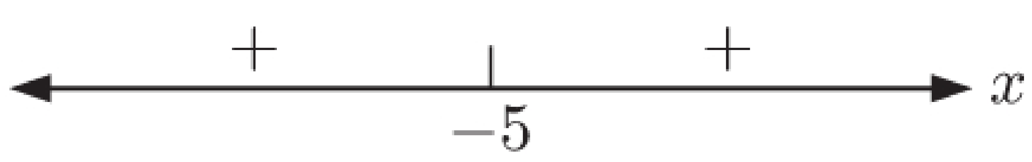
Sign diagram of LHS is



$\therefore$ the inequality is not true for any real x.

d $(x+5)^2 \geqslant 0$

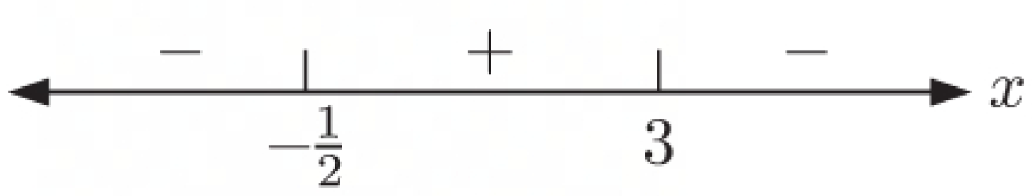
Sign diagram of LHS is



$\therefore$ the inequality is true for all real x.

e $(2x+1)(3-x) > 0$

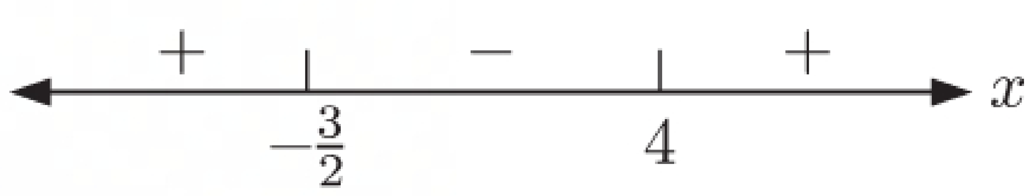
Sign diagram of LHS is



$\therefore \quad -\frac{1}{2} < x < 3$

f $(x-4)(2x+3) < 0$

Sign diagram of LHS is

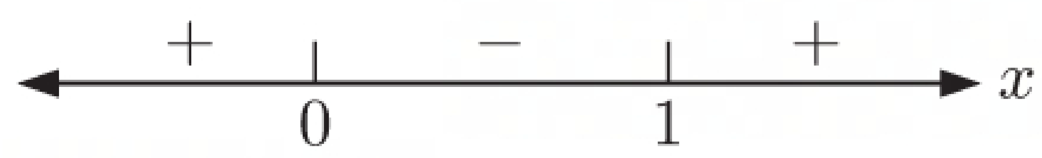


$\therefore \quad -\frac{3}{2} < x < 4$

2 **a** $x^2 - x \geqslant 0$

$\therefore\ x(x-1) \geqslant 0$

Sign diagram of LHS is

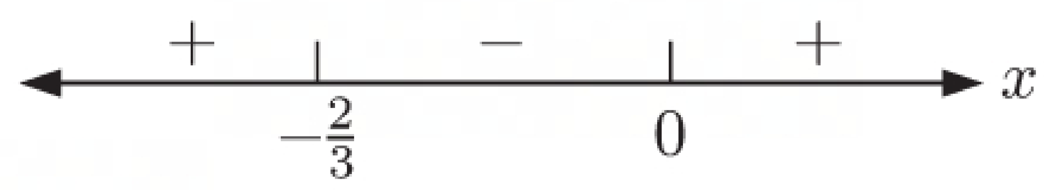


$\therefore\ x \leqslant 0$ or $x \geqslant 1$

b $3x^2 + 2x < 0$

$\therefore\ x(3x+2) < 0$

Sign diagram of LHS is

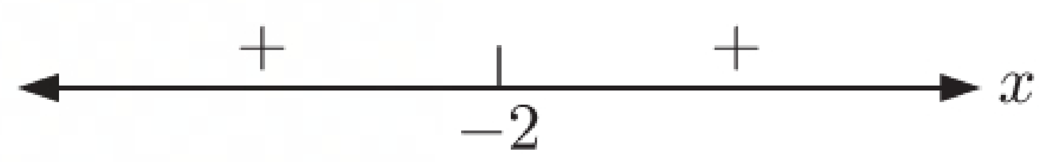


$\therefore\ -\frac{2}{3} < x < 0$

c $x^2 + 4x + 4 > 0$

$\therefore\ (x+2)^2 > 0$

Sign diagram of LHS is

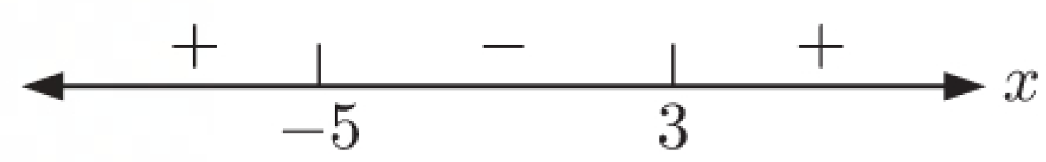


$\therefore$ the inequality is true for all $x \neq -2$.

d $x^2 + 2x - 15 \leqslant 0$

$\therefore\ (x+5)(x-3) \leqslant 0$

Sign diagram of LHS is

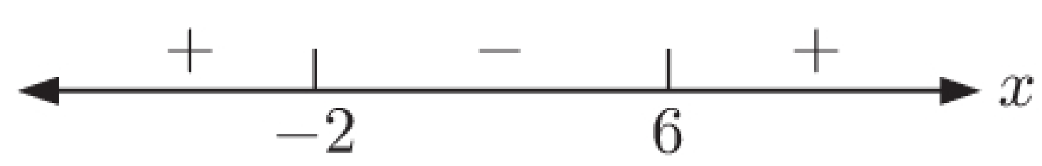


$\therefore\ -5 \leqslant x \leqslant 3$

e $x^2 - 4x - 12 > 0$

$\therefore\ (x+2)(x-6) > 0$

Sign diagram of LHS is

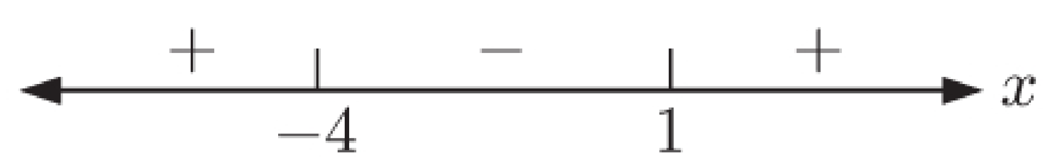


$\therefore\ x < -2$ or $x > 6$

f $3x^2 + 9x - 12 < 0$

$\therefore\ 3(x+4)(x-1) < 0$

Sign diagram of LHS is



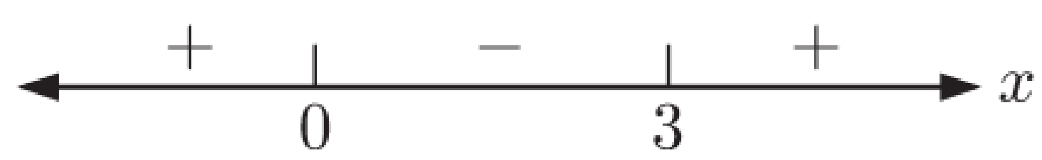
$\therefore\ -4 < x < 1$

3 **a** $x^2 \geqslant 3x$

$\therefore\ x^2 - 3x \geqslant 0$

$\therefore\ x(x-3) \geqslant 0$

Sign diagram of LHS is



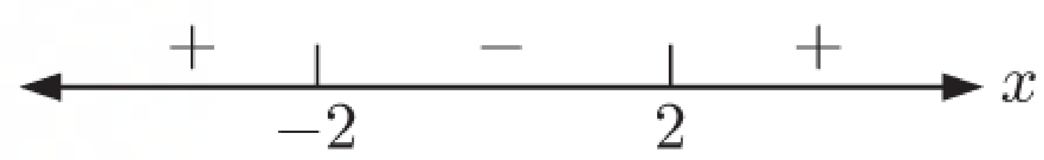
$\therefore\ x \leqslant 0$ or $x \geqslant 3$

b $x^2 < 4$

$\therefore\ x^2 - 4 < 0$

$\therefore\ (x+2)(x-2) < 0$

Sign diagram of LHS is



$\therefore\ -2 < x < 2$

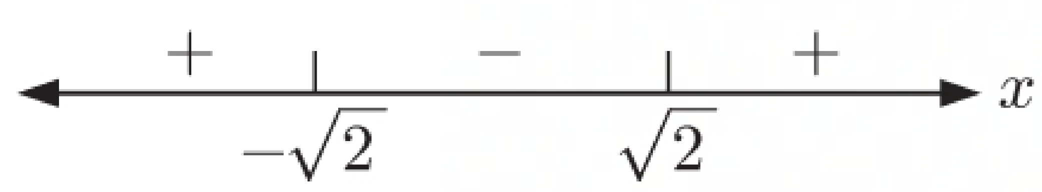
c $2x^2 \geqslant 4$

$\therefore\ 2x^2 - 4 \geqslant 0$

$\therefore\ 2(x^2 - 2) \geqslant 0$

$\therefore\ 2(x+\sqrt{2})(x-\sqrt{2}) \geqslant 0$

Sign diagram of LHS is



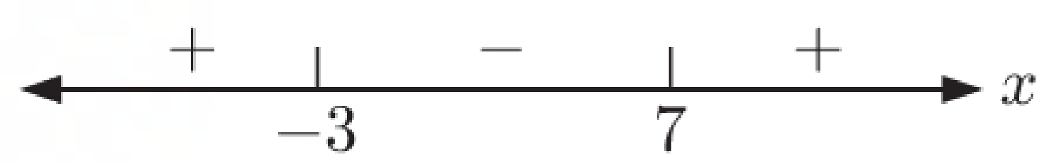
$\therefore\ x \leqslant -\sqrt{2}$ or $x \geqslant \sqrt{2}$

d $x^2 - 21 \leqslant 4x$

$\therefore\ x^2 - 4x - 21 \leqslant 0$

$\therefore\ (x-7)(x+3) \leqslant 0$

Sign diagram of LHS is



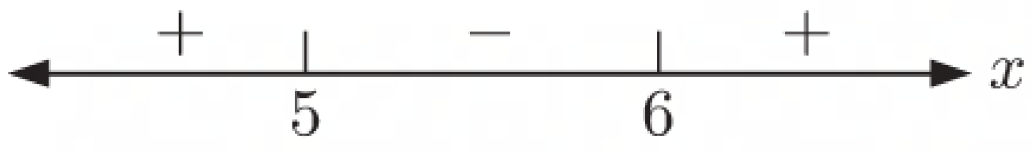
$\therefore\ -3 \leqslant x \leqslant 7$

e $x^2 + 30 > 11x$

$\therefore\ x^2 - 11x + 30 > 0$

$\therefore\ (x-5)(x-6) > 0$

Sign diagram of LHS is



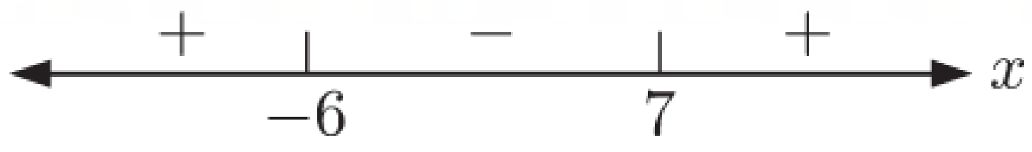
$\therefore\ x < 5$ or $x > 6$

f $x + 42 < x^2$

$\therefore\ x^2 - x - 42 > 0$

$\therefore\ (x+6)(x-7) > 0$

Sign diagram of LHS is



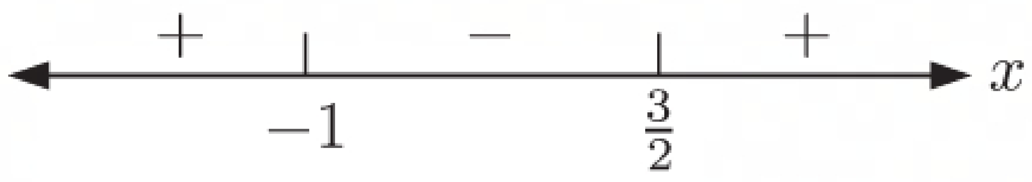
$\therefore\ x < -6$ or $x > 7$

g $2x^2 \geqslant x + 3$

$\therefore\ 2x^2 - x - 3 \geqslant 0$

$\therefore\ (2x-3)(x+1) \geqslant 0$

Sign diagram of LHS is

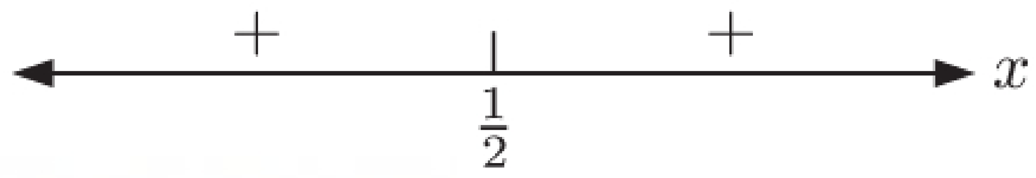


$\therefore\ x \leqslant -1$ or $x \geqslant \frac{3}{2}$

h $4x^2 - 4x + 1 < 0$

$\therefore\ (2x-1)^2 < 0$

Sign diagram of LHS is



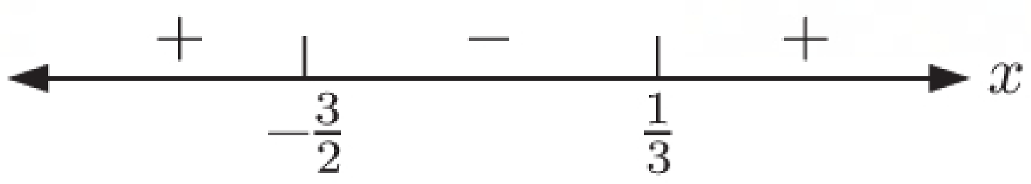
$\therefore$ the inequality is not true for any real x.

i $6x^2 + 7x < 3$

$\therefore\ 6x^2 + 7x - 3 < 0$

$\therefore\ (3x-1)(2x+3) < 0$

Sign diagram of LHS is



$\therefore\ -\frac{3}{2} < x < \frac{1}{3}$

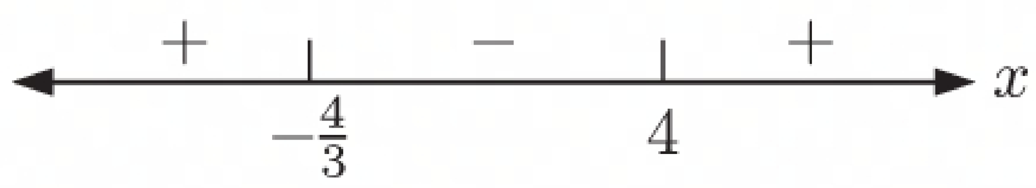
j $3x^2 > 8(x+2)$

$\therefore\ 3x^2 > 8x + 16$

$\therefore\ 3x^2 - 8x - 16 > 0$

$\therefore\ (3x+4)(x-4) > 0$

Sign diagram of LHS is



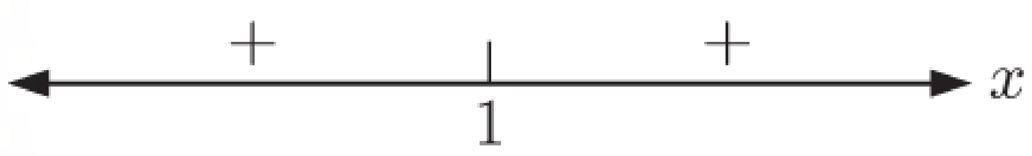
$\therefore\ x < -\frac{4}{3}$ or $x > 4$

k $2x^2 - 4x + 2 > 0$

$\therefore\ 2(x^2 - 2x + 1) > 0$

$\therefore\ 2(x-1)^2 > 0$

Sign diagram of LHS is



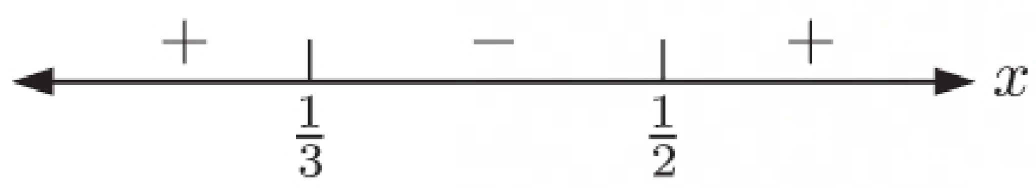
$\therefore$ the inequality is true for all $x \neq 1$.

l $6x^2 + 1 \leqslant 5x$

$\therefore\ 6x^2 - 5x + 1 \leqslant 0$

$\therefore\ (3x-1)(2x-1) \leqslant 0$

Sign diagram of LHS is



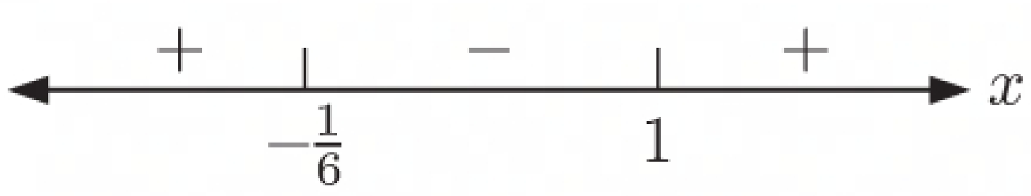
$\therefore\ \frac{1}{3} \leqslant x \leqslant \frac{1}{2}$

m $1 + 5x < 6x^2$

$\therefore\ 6x^2 - 5x - 1 > 0$

$\therefore\ (6x+1)(x-1) > 0$

Sign diagram of LHS is



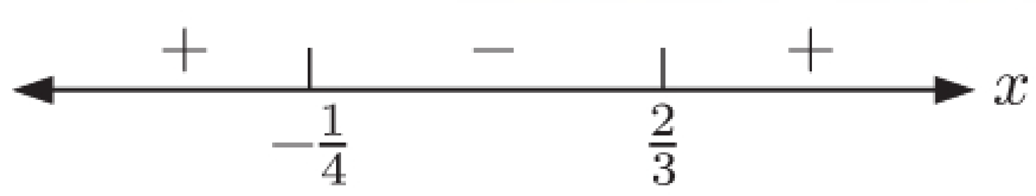
$\therefore\ x < -\frac{1}{6}$ or $x > 1$

n $12x^2 \geqslant 5x + 2$

$\therefore\ 12x^2 - 5x - 2 \geqslant 0$

$\therefore\ (4x+1)(3x-2) \geqslant 0$

Sign diagram of LHS is



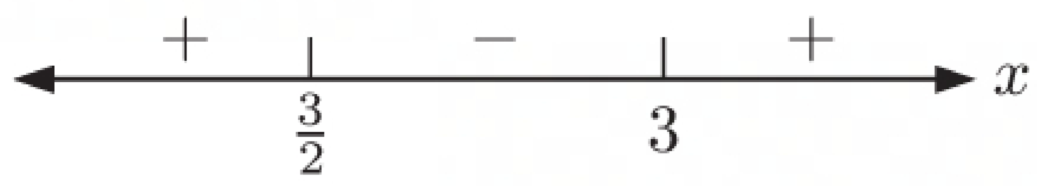
$\therefore\ x \leqslant -\frac{1}{4}$ or $x \geqslant \frac{2}{3}$

o $2x^2 + 9 > 9x$

$\therefore\ 2x^2 - 9x + 9 > 0$

$\therefore\ (2x-3)(x-3) > 0$

Sign diagram of LHS is

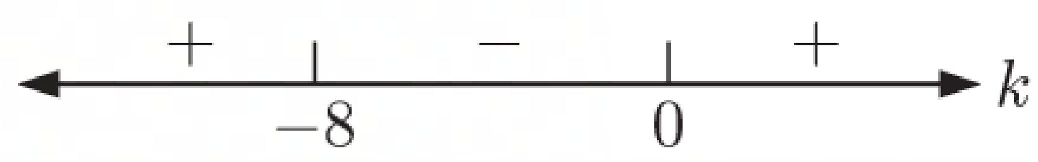


$\therefore\ x < \frac{3}{2}$ or $x > 3$

4 **a** $y = 2x^2 + kx - k$ has $a = 2,\ b = k,\ c = -k$

$$\begin{aligned}\therefore\ \Delta &= b^2 - 4ac \\ &= (k)^2 - 4(2)(-k) \\ &= k^2 + 8k \\ &= k(k+8)\end{aligned}$$

So, Δ has sign diagram:



i The graph cuts the x-axis twice if $\Delta > 0$
$\therefore\ k < -8$ or $k > 0$.

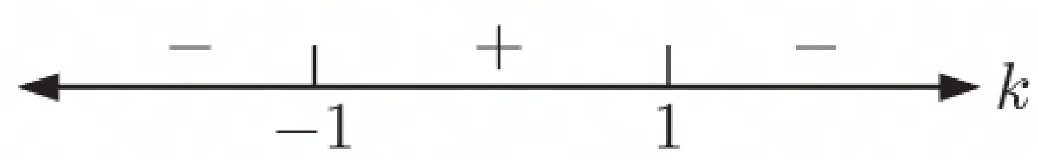
ii The graph touches the x-axis if $\Delta = 0$
$\therefore\ k = -8$ or $k = 0$.

iii The graph misses the x-axis if $\Delta < 0$
$\therefore\ -8 < k < 0$.

b $y = kx^2 - 2x + k$ has $a = k,\ b = -2,\ c = k$

$$\begin{aligned}\therefore\ \Delta &= b^2 - 4ac \\ &= (-2)^2 - 4(k)(k) \\ &= 4 - 4k^2 \\ &= 4(1-k^2) \\ &= 4(1+k)(1-k)\end{aligned}$$

So, Δ has sign diagram:



i The graph cuts the x-axis twice if $\Delta > 0$
$\therefore\ -1 < k < 1,\ k \neq 0$.

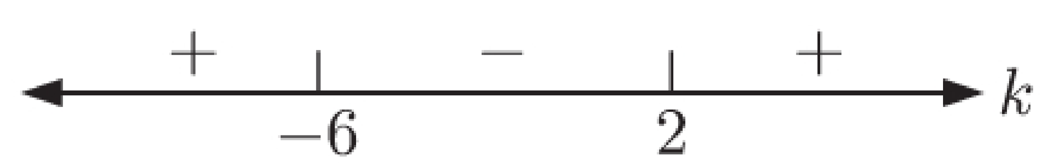
ii The graph touches the x-axis if $\Delta = 0$
$\therefore\ k = -1$ or $k = 1$.

iii The graph misses the x-axis if $\Delta < 0$
$\therefore\ k < -1$ or $k > 1$.

c $y = x^2 + (k+2)x + 4$ has $a = 1,\ b = k+2,\ c = 4$

$$\begin{aligned}\therefore\ \Delta &= b^2 - 4ac \\ &= (k+2)^2 - 4(1)(4) \\ &= k^2 + 4k + 4 - 16 \\ &= k^2 + 4k - 12 \\ &= (k+6)(k-2)\end{aligned}$$

So, Δ has sign diagram:



i The graph cuts the x-axis twice if $\Delta > 0$
$\therefore\ k < -6$ or $k > 2$.

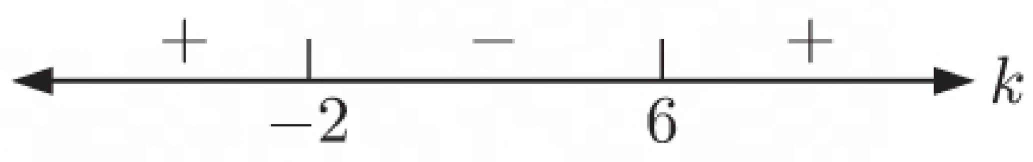
ii The graph touches the x-axis if $\Delta = 0$
$\therefore\ k = -6$ or $k = 2$.

iii The graph misses the x-axis if $\Delta < 0$
$\therefore\ -6 < k < 2$.

5 **a** $2x^2 + (k-2)x + 2$ has $a = 2,\ b = k-2,\ c = 2$

$$\begin{aligned}\therefore\ \Delta &= b^2 - 4ac\\ &= (k-2)^2 - 4(2)(2)\\ &= k^2 - 4k + 4 - 16\\ &= k^2 - 4k - 12\\ &= (k+2)(k-6)\end{aligned}$$

So, Δ has sign diagram:



$2x^2 + (k-2)x + 2 = 0$ has:

i two real roots if $\Delta > 0$
$\therefore\ k < -2$ or $k > 6$

ii a repeated real root if $\Delta = 0$
$\therefore\ k = -2$ or $k = 6$

iii no real roots if $\Delta < 0$
$\therefore\ -2 < k < 6.$

b $x^2 + (3k-1)x + (2k+10)$ has $a = 1,\ b = 3k-1,\ c = 2k+10$

$$\begin{aligned}\therefore\ \Delta &= b^2 - 4ac\\ &= (3k-1)^2 - 4(1)(2k+10)\\ &= 9k^2 - 6k + 1 - 8k - 40\\ &= 9k^2 - 14k - 39\\ &= (9k+13)(k-3)\end{aligned}$$

So, Δ has sign diagram:

$x^2 + (3k-1)x + (2k+10) = 0$ has:

i two real roots if $\Delta > 0$
$\therefore\ k < -\frac{13}{9}$ or $k > 3$

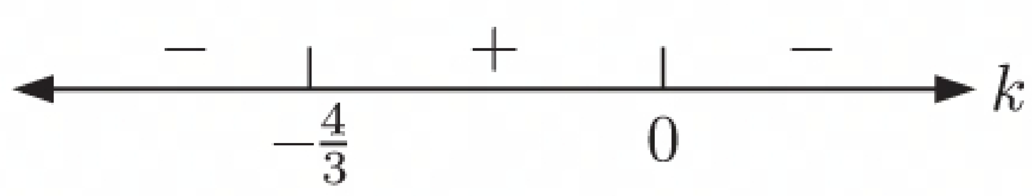
ii a repeated real root if $\Delta = 0$
$\therefore\ k = -\frac{13}{9}$ or $k = 3$

iii no real roots if $\Delta < 0$
$\therefore\ -\frac{13}{9} < k < 3.$

c $(k+1)x^2 + kx + k$ has $a = k+1,\ b = k,\ c = k$

$$\begin{aligned}\therefore\ \Delta &= b^2 - 4ac\\ &= (k)^2 - 4(k+1)(k)\\ &= k^2 - 4(k^2 + k)\\ &= k^2 - 4k^2 - 4k\\ &= -3k^2 - 4k\\ &= -k(3k+4)\end{aligned}$$

So, Δ has sign diagram:



$(k+1)x^2 + kx + k = 0$ has:

i two real roots if $\Delta > 0$
$\therefore\ -\frac{4}{3} < k < 0,\ k \neq -1$

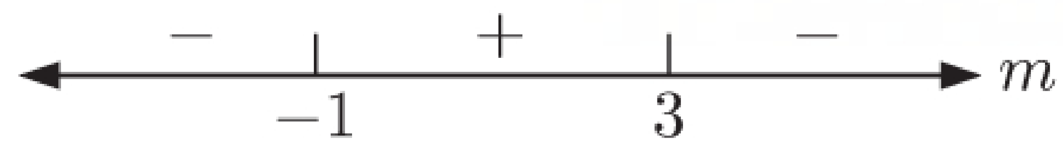
ii a repeated real root if $\Delta = 0$
$\therefore\ k = -\frac{4}{3}$ or $k = 0$

iii no real roots if $\Delta < 0$
$\therefore\ k < -\frac{4}{3}$ or $k > 0.$

6 $(m-2)x^2+6x+3m$ has $a=m-2,\ b=6,\ c=3m$

$$\begin{aligned}\therefore\ \Delta &= b^2-4ac\\ &= (6)^2-4(m-2)(3m)\\ &= 36-4(3m^2-6m)\\ &= -12m^2+24m+36\\ &= -12(m^2-2m-3)\\ &= -12(m+1)(m-3)\end{aligned}$$

So, Δ has sign diagram:



a $y=(m-2)x^2+6x+3m$ is positive definite if $a>0$ and $\Delta<0$.

Now, $a>0$ if $m-2>0$ $\therefore\ m>2$ and $\Delta<0$ if $m<-1$ or $m>3$

$\therefore$ the graph is positive definite if $m>3$.

b $y=(m-2)x^2+6x+3m$ is negative definite if $a<0$ and $\Delta<0$.

Now, $a<0$ if $m-2<0$ $\therefore\ m<2$ and $\Delta<0$ if $m<-1$ or $m>3$

$\therefore$ the graph is negative definite if $m<-1$.

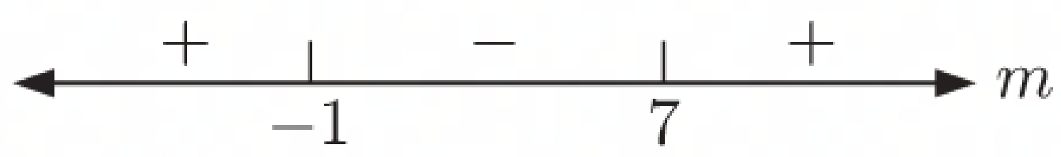
7 $y=-x^2+3x-6$ meets $y=mx-2$ where

$$-x^2+3x-6=mx-2$$

$$\therefore\ x^2+(m-3)x+4=0$$

$$\begin{aligned}\text{Now}\quad \Delta &= (m-3)^2-4(1)(4)\\ &= m^2-6m+9-16\\ &= m^2-6m-7\\ &= (m+1)(m-7)\end{aligned}$$

So, Δ has sign diagram:



a The line meets the curve twice if $\Delta>0$.

$\therefore\ m<-1$ or $m>7$

b The line is a tangent to the curve if it *touches* the curve, $\Delta=0$.

$\therefore\ m=-1$ or $m=7$

c The line does not meet the curve if $\Delta<0$.

$\therefore\ -1<m<7$

8 $y=ax^2+2x+1$ and $y=-x^2+ax-1$ meet where

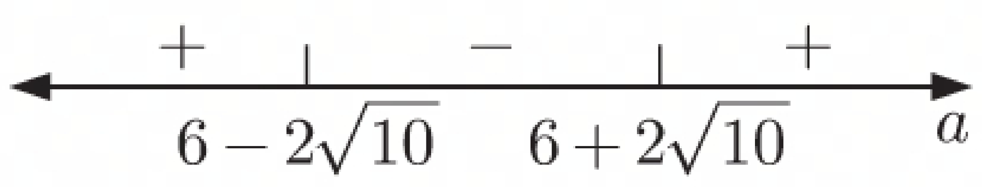
$$ax^2+2x+1=-x^2-ax-1$$

$$\therefore\ (a+1)x^2+(2-a)x+2=0$$

$$\begin{aligned}\text{Now}\quad \Delta &= (2-a)^2-4(a+1)(2)\\ &= 4-4a+a^2-4(2a+2)\\ &= 4-4a+a^2-8a-8\\ &= a^2-12a-4\end{aligned}$$

$$\Delta = 0 \text{ when } a = \frac{-(-12) \pm \sqrt{(-12)^2 - 4(1)(-4)}}{2(1)}$$
$$= \frac{12 \pm \sqrt{160}}{2}$$
$$= \frac{12 \pm 4\sqrt{10}}{2}$$
$$= 6 \pm 2\sqrt{10}$$

So, Δ has sign diagram:



a The curves meet twice if $\Delta > 0$.

$\therefore \quad a < 6 - 2\sqrt{10}$ or $a > 6 + 2\sqrt{10}$.

b The curves touch if $\Delta = 0$.

$\therefore \quad a = 6 \pm 2\sqrt{10}$

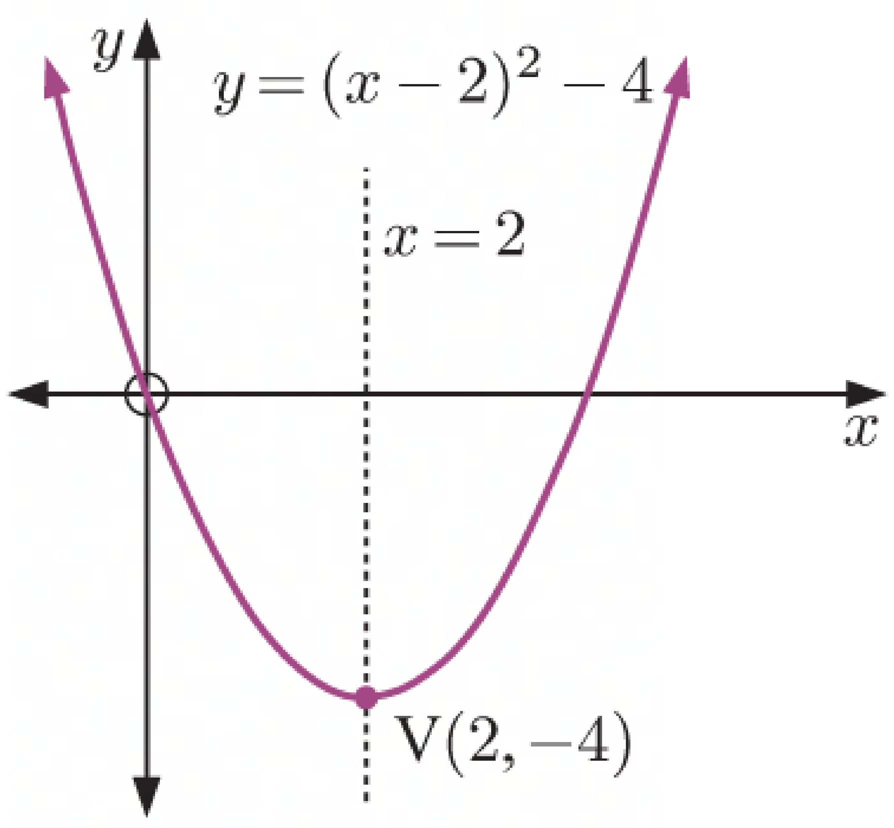
c The curves never meet if $\Delta < 0$.

$\therefore \quad 6 - 2\sqrt{10} < a < 6 + 2\sqrt{10}$.

REVIEW SET 14A

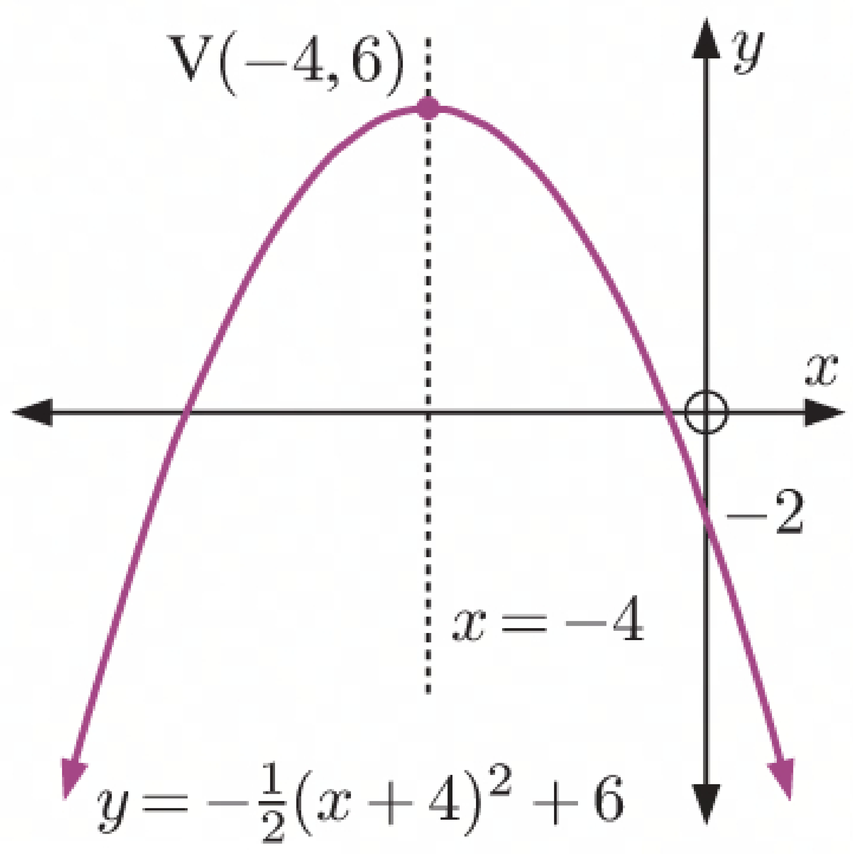
1 **a** $y = (x-2)^2 - 4$ has vertex $(2, -4)$ and axis of symmetry $x = 2$.

When $x = 0$, $y = (-2)^2 - 4 = 0$ so the y-intercept is 0.



b $y = -\frac{1}{2}(x+4)^2 + 6$ has vertex $(-4, 6)$ and axis of symmetry $x = -4$.

When $x = 0$, $y = -\frac{1}{2}(4)^2 + 6 = -2$ so the y-intercept is -2.



2 $y = x^2 - 3x$ meets $y = 3x^2 - 5x - 24$

where $x^2 - 3x = 3x^2 - 5x - 24$

$\therefore \quad 2x^2 - 2x - 24 = 0$

$\therefore \quad x^2 - x - 12 = 0$

$\therefore \quad (x-4)(x+3) = 0$

$\therefore \quad x = 4$ or -3

Substituting into $y = x^2 - 3x$,

when $x = 4$, $y = 4^2 - 3(4)$
$= 4$

and when $x = -3$, $y = (-3)^2 - 3(-3)$
$= 9 + 9$
$= 18$

$\therefore$ the graphs meet at $(4, 4)$ and $(-3, 18)$.

3 $y = -2x^2 + 5x + k$ has $a = -2$, $b = 5$, $c = k$.

$$\therefore \quad \Delta = b^2 - 4ac$$
$$= (5)^2 - 4(-2)(k)$$
$$= 25 + 8k$$

The graph does not cut the x-axis if $\Delta < 0$

$$\therefore \quad 25 + 8k < 0$$
$$\therefore \quad 8k < -25$$
$$\therefore \quad k < -\tfrac{25}{8}$$
$$\therefore \quad k < -3\tfrac{1}{8}$$

4 $2x^2 - 3x + m = 0$ has $a = 2$, $b = -3$, $c = m$.

$$\therefore \quad \Delta = b^2 - 4ac$$
$$= (-3)^2 - 4(2)(m)$$
$$= 9 - 8m$$

a There is a repeated root if $\Delta = 0$

$$\therefore \quad 9 - 8m = 0$$
$$\therefore \quad m = \tfrac{9}{8}$$

b There are two distinct real roots if $\Delta > 0$

$$\therefore \quad 9 - 8m > 0$$
$$\therefore \quad 8m < 9$$
$$\therefore \quad m < \tfrac{9}{8}$$

c There are no real roots if $\Delta < 0$

$$\therefore \quad 9 - 8m < 0$$
$$\therefore \quad 8m > 9$$
$$\therefore \quad m > \tfrac{9}{8}$$

5 Let the number be x, so its reciprocal is $\dfrac{1}{x}$.

$$\therefore \quad x + \frac{1}{x} = 2\tfrac{1}{30} = \tfrac{61}{30}$$
$$\therefore \quad x^2 + 1 = \tfrac{61}{30}x$$
$$\therefore \quad 30x^2 + 30 = 61x$$
$$\therefore \quad 30x^2 - 61x + 30 = 0$$
$$\therefore \quad (5x - 6)(6x - 5) = 0$$
$$\therefore \quad x = \tfrac{6}{5} \text{ or } \tfrac{5}{6}$$

So, the number is $\tfrac{6}{5}$ or $\tfrac{5}{6}$.

6 A line with y-intercept 10 will have an equation of the form $y = mx + 10$.

$y = 3x^2 + 7x - 2$ meets this line where $3x^2 + 7x - 2 = mx + 10$

$$\therefore\ 3x^2 + (7 - m)x - 12 = 0$$

For $y = mx + 10$ to be tangential to $y = 3x^2 + 7x - 2$, this equation must have exactly one solution, so there is a repeated root.

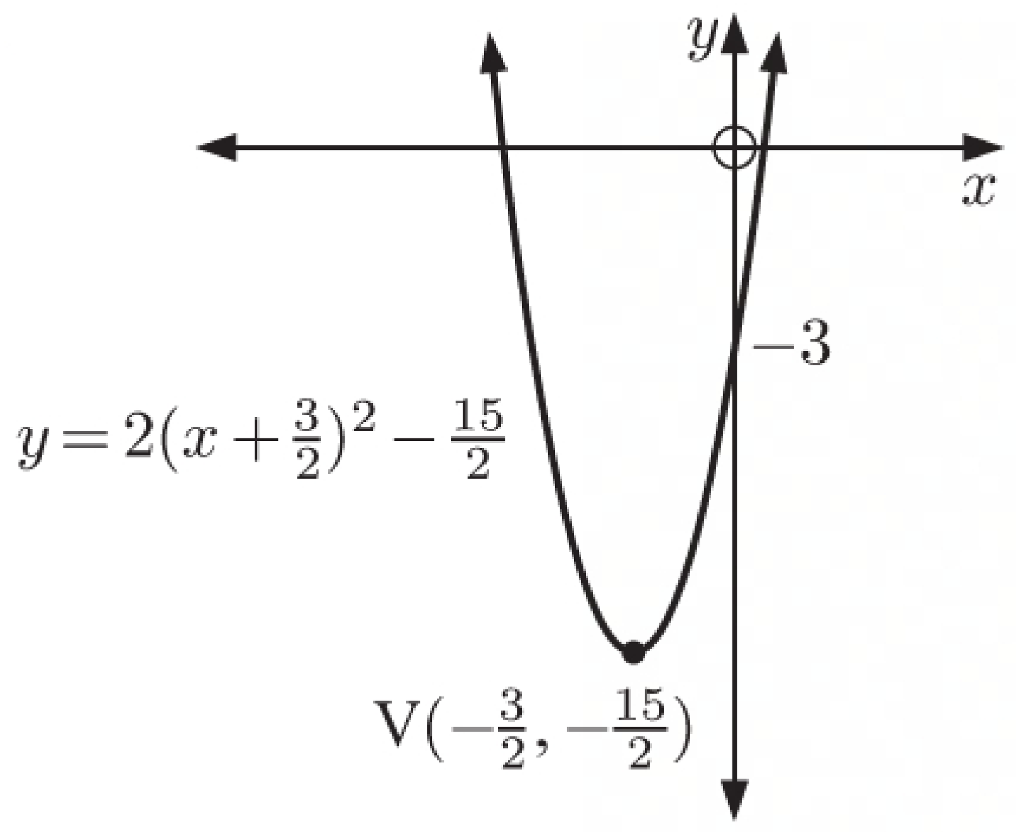
$$\therefore\ \Delta = 0$$
$$\therefore\ (7 - m)^2 - 4(3)(-12) = 0$$
$$\therefore\ 49 - 14m + m^2 + 144 = 0$$
$$\therefore\ m^2 - 14m + 193 = 0 \quad \text{which has discriminant} \quad (-14)^2 - 4(1)(193) < 0$$

So, there are no real solutions for m.

$\therefore$ no line with y-intercept 10 can be tangential to $y = 3x^2 + 7x - 2$.

7 **a**
$$\begin{aligned} y &= 2x^2 + 6x - 3 \\ &= 2\left[x^2 + 3x - \tfrac{3}{2}\right] \\ &= 2\left[x^2 + 3x + \left(\tfrac{3}{2}\right)^2 - \tfrac{3}{2} - \left(\tfrac{3}{2}\right)^2\right] \\ &= 2\left[\left(x + \tfrac{3}{2}\right)^2 - \tfrac{15}{4}\right] \\ \therefore\ y &= 2\left(x + \tfrac{3}{2}\right)^2 - \tfrac{15}{2} \end{aligned}$$

b



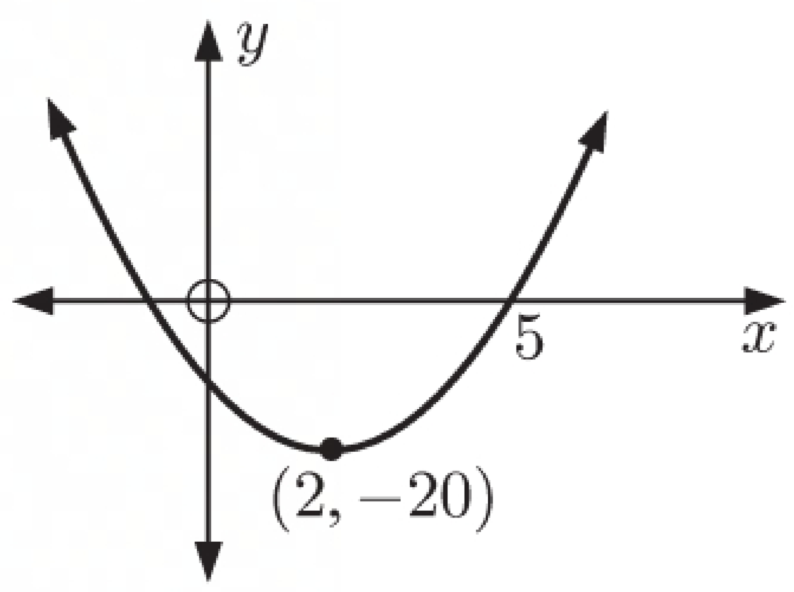
8 **a** Since the vertex is $(2, -20)$, the quadratic has the form

$$y = a(x - 2)^2 - 20 \quad \text{where} \quad a > 0$$

When $x = 5$, $y = 0$

$$\begin{aligned} \therefore\ 0 &= a(5 - 2)^2 - 20 \\ &= a(3)^2 - 20 \\ \therefore\ 9a &= 20 \\ \therefore\ a &= \tfrac{20}{9} \end{aligned}$$

The quadratic is $y = \frac{20}{9}(x - 2)^2 - 20$.



b The axis of symmetry $x = 4$ lies midway between the x-intercepts.

$\therefore$ the other x-intercept is 1.

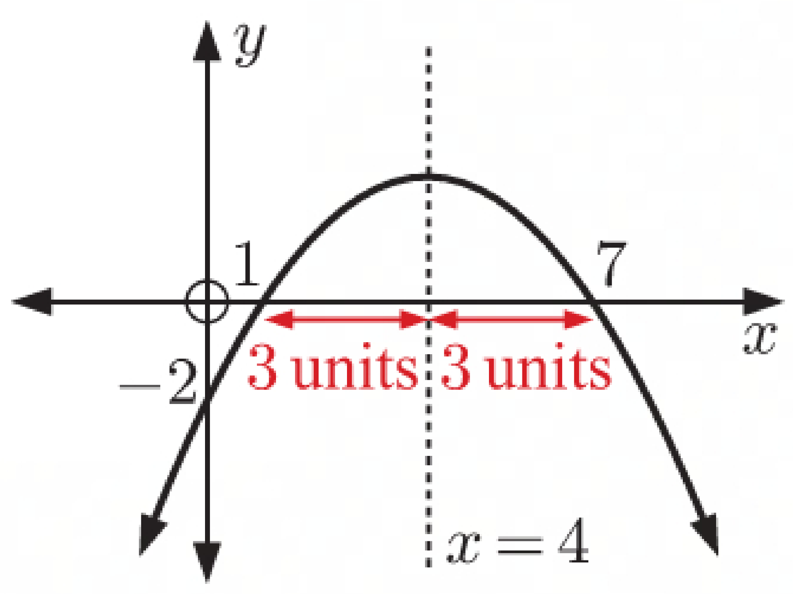
$\therefore$ the quadratic has the form

$$y = a(x - 1)(x - 7) \quad \text{where} \quad a < 0$$

When $x = 0$, $y = -2$

$$\begin{aligned} \therefore\ -2 &= a(-1)(-7) \\ \therefore\ -2 &= 7a \\ \therefore\ a &= -\tfrac{2}{7} \end{aligned}$$

The quadratic is $y = -\frac{2}{7}(x - 1)(x - 7)$.



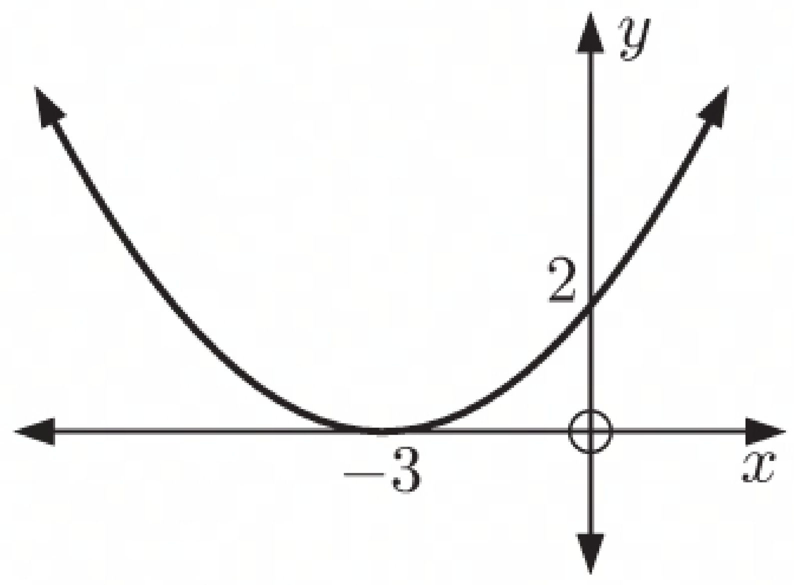
c The graph touches the x-axis at $x = -3$,
so $y = a(x+3)^2$.
The graph is concave up, so $a > 0$.
When $x = 0$, $y = 2$

$\therefore\ 2 = a(3)^2$

$\therefore\ 2 = 9a$

$\therefore\ a = \frac{2}{9}$

The quadratic is $y = \frac{2}{9}(x+3)^2$.



9 $y = -x^2 + 2x$

$\therefore\ y = x(2-x)$ has x-intercepts 0 and 2.

When $x = 0$, $y = 0(2-0)$
$= 0$

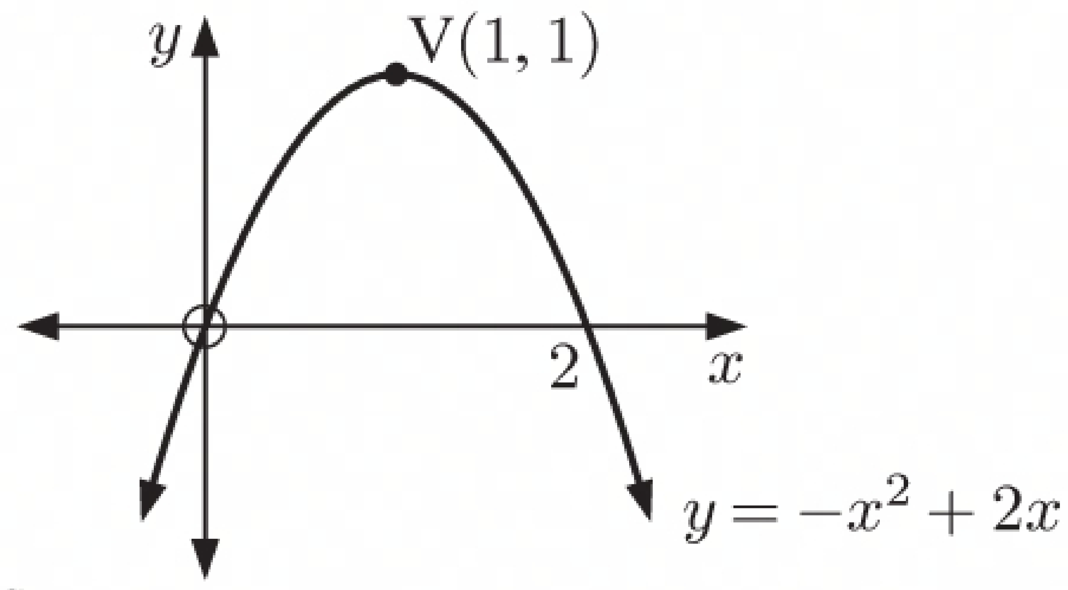
$\therefore$ y-intercept is 0.

The axis of symmetry lies midway between the x-intercepts.

$\therefore$ the axis of symmetry is $x = 1$.

When $x = 1$, $y = 1(2-1)$
$= 1$

$\therefore$ the vertex is V(1, 1).



10 A line with gradient -3 will have an equation of the form $y = -3x + c$.

$y = 2x^2 - 5x + 1$ meets this line where $2x^2 - 5x + 1 = -3x + c$

$\therefore\ 2x^2 - 2x + 1 - c = 0$

If the graphs touch, this quadratic has $\Delta = 0$

$\therefore\ (-2)^2 - 4(2)(1-c) = 0$

$\therefore\ 4 - 8 + 8c = 0$

$\therefore\ 8c - 4 = 0$

$\therefore\ 8c = 4$

$\therefore\ c = \frac{1}{2}$

$\therefore\ y = -3x + \frac{1}{2}$ is the line which is a tangent to $y = 2x^2 - 5x + 1$.

The y-intercept of the line is $\frac{1}{2}$.

11 **a** **i** The graph cuts the x-axis twice.

$\therefore\ \Delta > 0$

ii The graph is concave down.

$\therefore\ a < 0$

b **i** $y = a(x+m)(x+n)$ has x-intercepts at $x = -m$ and $x = -n$.

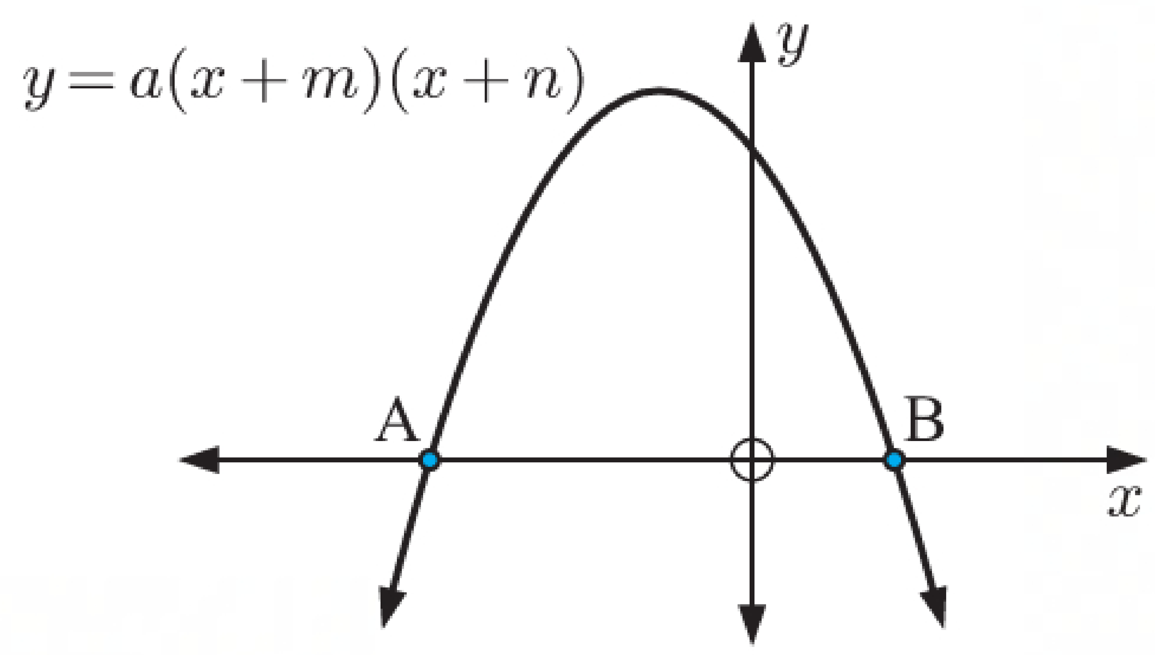
Now $m > n$

$\therefore\ -m < -n$

So, A is $(-m, 0)$ and B is $(-n, 0)$.

ii The axis of symmetry lies midway between the x-intercepts.

$\therefore$ the equation of the axis of symmetry is $x = \dfrac{-m + (-n)}{2} = \dfrac{-m-n}{2}$.



12 a, b, and c are consecutive terms in a geometric sequence with common ratio $r \neq 0$.

$\therefore \quad ax^2 + bx + c = ax^2 + (ra)x + r^2a$

which has $\Delta = (ra)^2 - 4(a)(r^2a)$

$= r^2a^2 - 4r^2a^2$

$= -3r^2a^2$

$< 0 \qquad \{r^2, a^2 > 0\}$

$\therefore$ the graph has no real roots.

$\therefore$ the graph does not cut the x-axis.

13 Since the x-intercepts are 3 and -2, $y = a(x-3)(x+2)$.

When $x = 0$, $y = 24$

$\therefore \quad 24 = a(-3)(2)$

$\therefore \quad a = -4$

The quadratic is $y = -4(x-3)(x+2)$

$\therefore \quad y = -4(x^2 - x - 6)$

$\therefore \quad y = -4x^2 + 4x + 24$

14 Consider $y = ax^2 + bx + c$. The x-intercepts of the quadratic are $\dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ (if they exist).

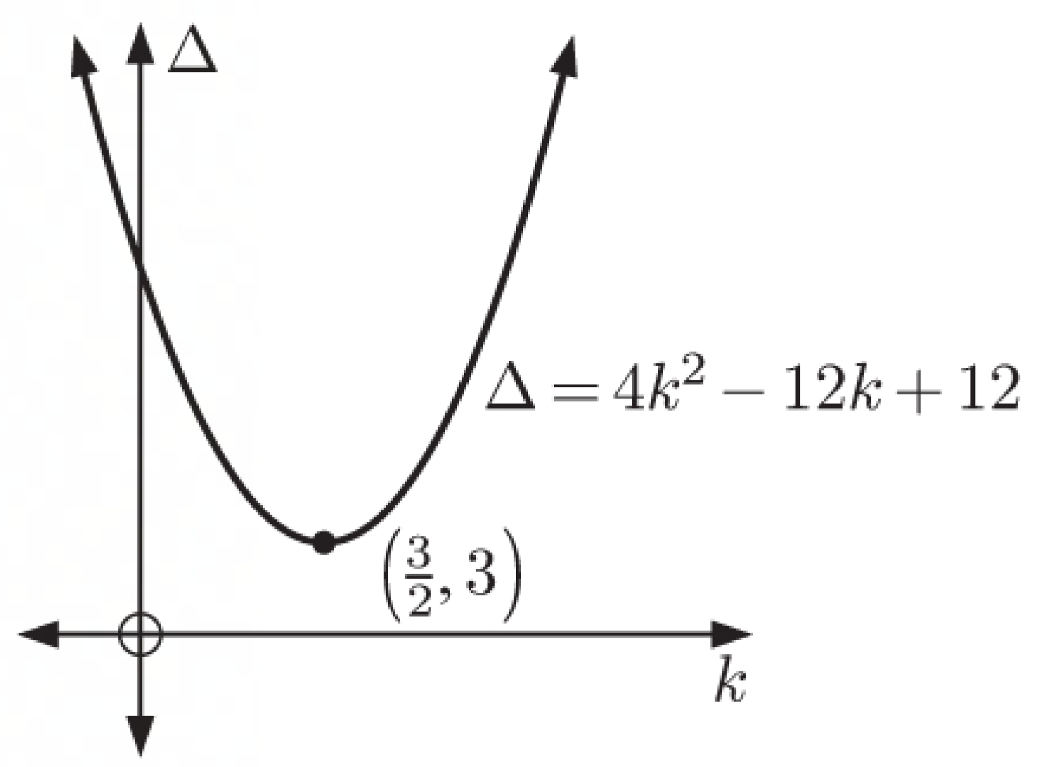
$\therefore$ the x-intercepts are closest together when $\sqrt{b^2 - 4ac}$ is as small as possible.

Now, $y = 3x^2 + 2kx + k - 1$ has $a = 3$, $b = 2k$, $c = k - 1$.

$\Delta = b^2 - 4ac$

$= (2k)^2 - 4(3)(k-1)$

$= 4k^2 - 12k + 12$



For this quadratic, $a > 0$, so its shape is 

The graph has a minimum at $k = \dfrac{-(-12)}{2(4)} = \frac{3}{2}$

$\therefore \quad \Delta = 4\left(\frac{3}{2}\right)^2 - 12\left(\frac{3}{2}\right) + 12 = 9 - 18 + 12 = 3$

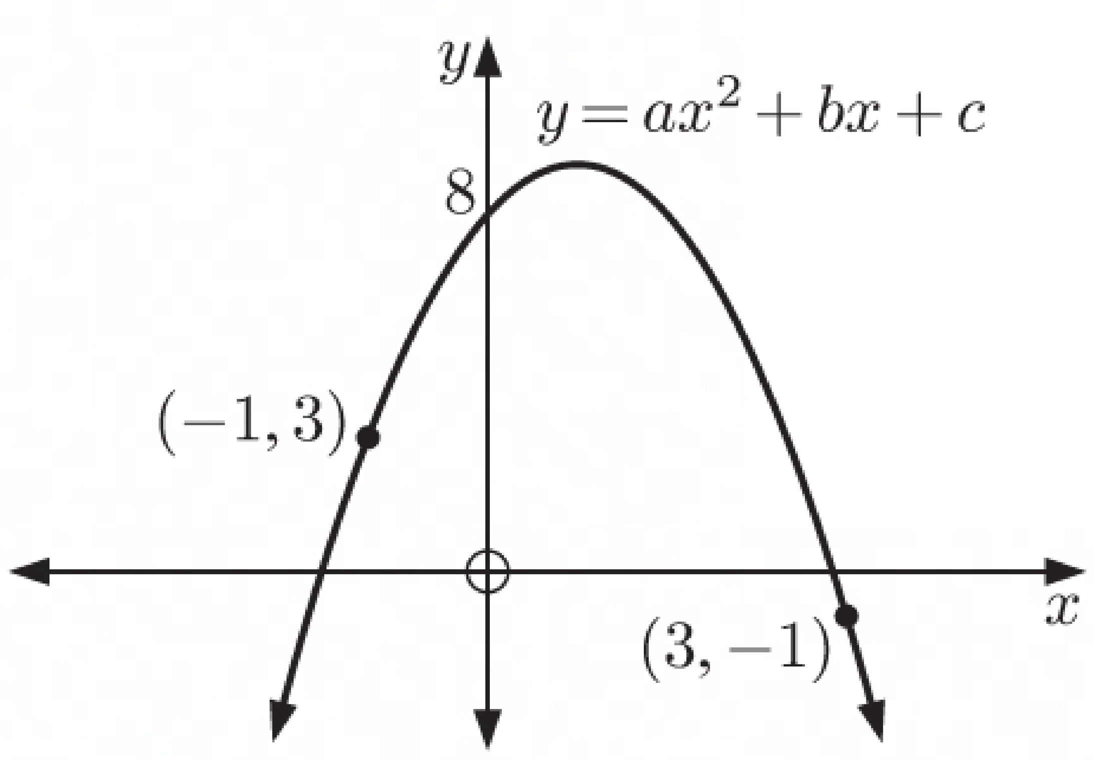
$\therefore \quad \Delta > 0$ for all real k, so $k = \frac{3}{2}$ also minimises $\sqrt{\Delta} = \sqrt{b^2 - 4ac}$.

So, the x-intercepts of $y = 3x^2 + 2kx + k - 1$ are closest together when $k = \frac{3}{2}$.

15 **a** When $x = 0$, $y = 8$

$\therefore \quad 8 = a(0)^2 + b(0) + c$

$\therefore \quad c = 8$



b When $x = 3$, $y = -1$

$\therefore \quad -1 = a(3)^2 + b(3) + 8$

$\therefore \quad 9a + 3b + 9 = 0$

$\therefore \quad 9a + 3b = -9$

$\therefore \quad 3a + b = -3 \qquad .... (1)$

When $x = -1$, $y = 3$

$\therefore \quad 3 = a(-1)^2 + b(-1) + 8$

$\therefore \quad a - b = -5 \quad .... (2)$

c $$\begin{array}{rl} 3a+b=-3 & \{(1)\} \\ a-b=-5 & \{(2)\} \\ \hline \text{Adding,}\quad 4a \qquad =-8 & \\ \therefore\ a=-2 & \end{array}$$

Substituting $a=-2$ into (1) gives
$$\begin{aligned} 3(-2)+b&=-3 \\ \therefore\ -6+b&=-3 \\ \therefore\ b&=3 \end{aligned}$$

$\therefore\ a=-2,\ b=3,$ and the quadratic is $y=-2x^2+3x+8$.

16 $y=mx-10$ and $y=3x^2+7x+2$ meet where $mx-10=3x^2+7x+2$
$$\therefore\ 3x^2+(7-m)x+12=0$$

If the graphs touch, this quadratic has $\Delta=0$
$$\begin{aligned} \therefore\ (7-m)^2-4(3)(12)&=0 \\ \therefore\ 49-14m+m^2-144&=0 \\ \therefore\ m^2-14m-95&=0 \\ \therefore\ (m+5)(m-19)&=0 \\ \therefore\ m&=-5 \text{ or } 19 \end{aligned}$$

$\therefore\ y=mx-10$ is a tangent to $y=3x^2+7x+2$ if $m=-5$ or $m=19$.

17 $h=-4.9t^2+19.6t+1.4$ is a quadratic with $a<0$, so its shape is 

So, at $t=\dfrac{-b}{2a}$ we have a maximum.

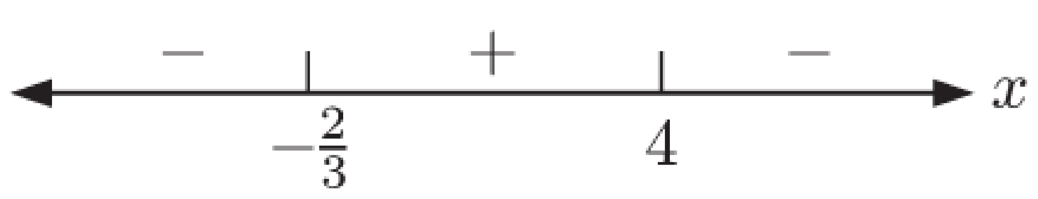
$\therefore\ t=\dfrac{-19.6}{2(-4.9)}=2,$ and when $t=2,$ $\begin{aligned} h&=-4.9(2)^2+19.6(2)+1.4 \\ &=21 \end{aligned}$

The maximum height reached by the ball is 21 m, at $t=2$ seconds.

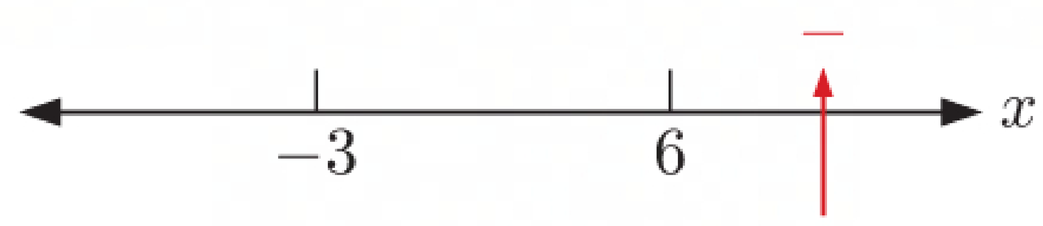
18 **a** $(3x+2)(4-x)$ has zeros $-\frac{2}{3}$ and 4.

When $x=5$ we have $(17)(-1)<0$, so we put a $-$ sign here.

As the factors are single, the signs alternate.

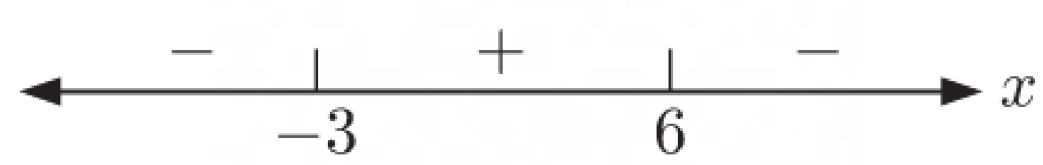


b $-x^2+3x+18=-(x+3)(x-6)$ has zeros -3 and 6.



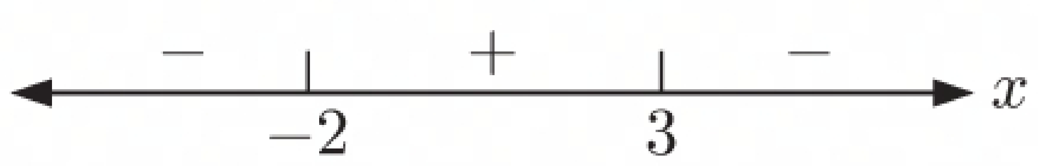
When $x=7$ we have $-(10)(1)<0$, so we put a $-$ sign here.

As the factors are single, the signs alternate.



19 **a** $(3-x)(x+2)<0$

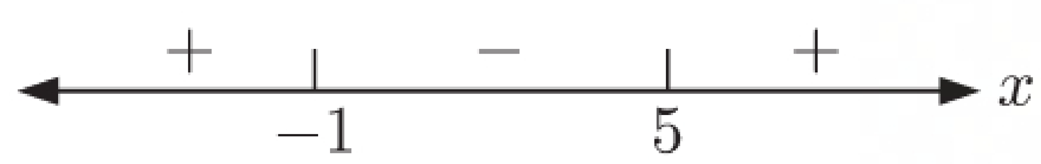
Sign diagram of LHS is



$\therefore\ x<-2$ or $x>3$

b $$\begin{aligned} x^2-4x-5&\leqslant 0 \\ \therefore\ (x+1)(x-5)&\leqslant 0 \end{aligned}$$

Sign diagram of LHS is



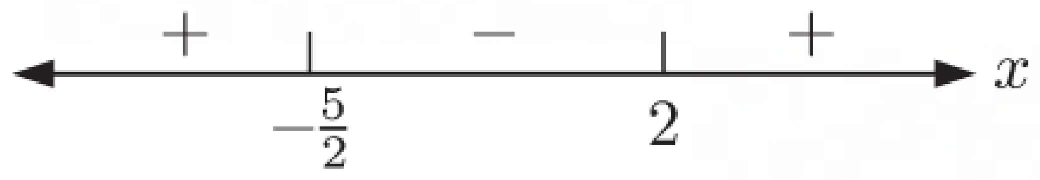
$\therefore\ -1\leqslant x\leqslant 5$

c $\quad 2x^2 + x > 10$

$\therefore \quad 2x^2 + x - 10 > 0$

$\therefore \quad (2x+5)(x-2) > 0$

Sign diagram of LHS is



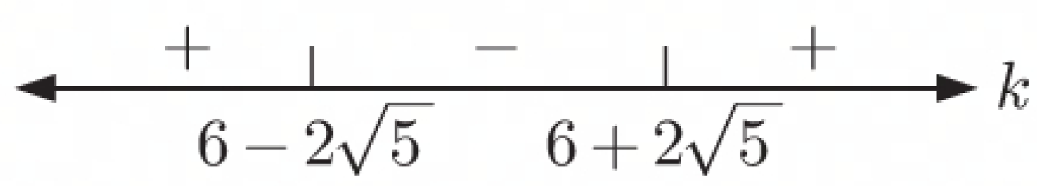
$\therefore \quad x < -\frac{5}{2}$ or $x > 2$

20 $f(x) = x^2 + kx + (3k-4)$ has $a = 1$, $b = k$, and $c = 3k - 4$.

$$\begin{aligned}\therefore \quad \Delta &= b^2 - 4ac \\ &= k^2 - 4(1)(3k-4) \\ &= k^2 - 12k + 16\end{aligned}$$

$$\begin{aligned}\Delta = 0 \text{ when } \quad k &= \frac{-(-12) \pm \sqrt{(-12)^2 - 4(1)(16)}}{2(1)} \\ &= \frac{12 \pm \sqrt{80}}{2} \\ &= \frac{12 \pm 4\sqrt{5}}{2} \\ &= 6 \pm 2\sqrt{5}\end{aligned}$$

So, Δ has sign diagram:



a The function cuts the x-axis twice if $\Delta > 0$.

$\therefore \quad k < 6 - 2\sqrt{5}$ or $k > 6 + 2\sqrt{5}$.

b The function touches the x-axis if $\Delta = 0$.

$\therefore \quad k = 6 \pm 2\sqrt{5}$.

c The function misses the x-axis if $\Delta < 0$.

$\therefore \quad 6 - 2\sqrt{5} < k < 6 + 2\sqrt{5}$.

REVIEW SET 14B

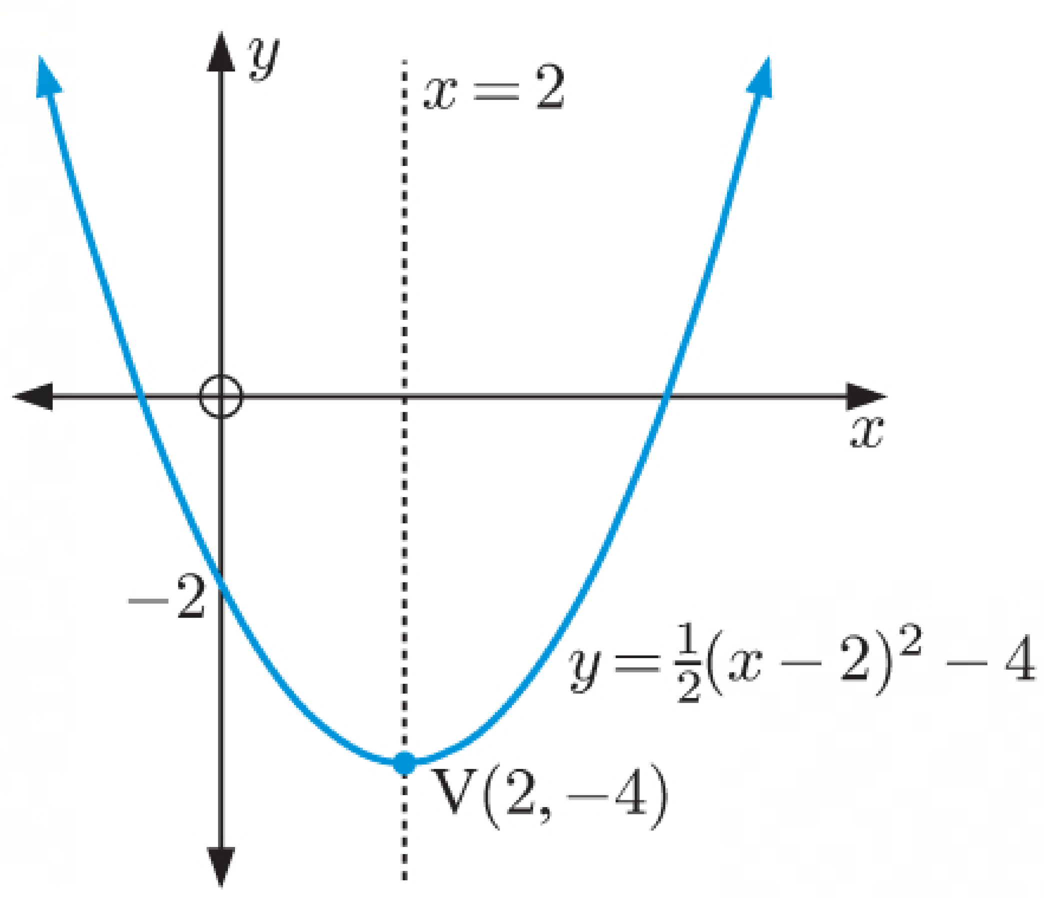
1 **a** The axis of symmetry is $x = 2$.

b The vertex is $(2, -4)$.

c When $x = 0$, $\quad y = \frac{1}{2}(-2)^2 - 4$
$\qquad\qquad\qquad\quad = -2$

$\therefore$ the y-intercept is -2.

d



2 $y = -3x^2 + 8x + 7$ has $a = -3$, $b = 8$, $c = 7$

$$\frac{-b}{2a} = \frac{-8}{2(-3)} = \frac{4}{3}$$

When $x = \frac{4}{3}$,
$$\begin{aligned} y &= -3(\tfrac{4}{3})^2 + 8(\tfrac{4}{3}) + 7 \\ &= -\tfrac{16}{3} + \tfrac{32}{3} + 7 \\ &= \tfrac{37}{3} \end{aligned}$$

The axis of symmetry is $x = \frac{4}{3}$, and the vertex is $V(\frac{4}{3}, \frac{37}{3})$ or $V(1\frac{1}{3}, 12\frac{1}{3})$.

3 **a** $y = 2x^2 + 3x - 7$
has $a = 2$, $b = 3$, $c = -7$
$$\begin{aligned} \Delta &= b^2 - 4ac \\ &= (3)^2 - 4(2)(-7) \\ &= 65 \end{aligned}$$
Since $\Delta > 0$, the graph cuts the x-axis twice.
Since $a > 0$, the graph is concave up.

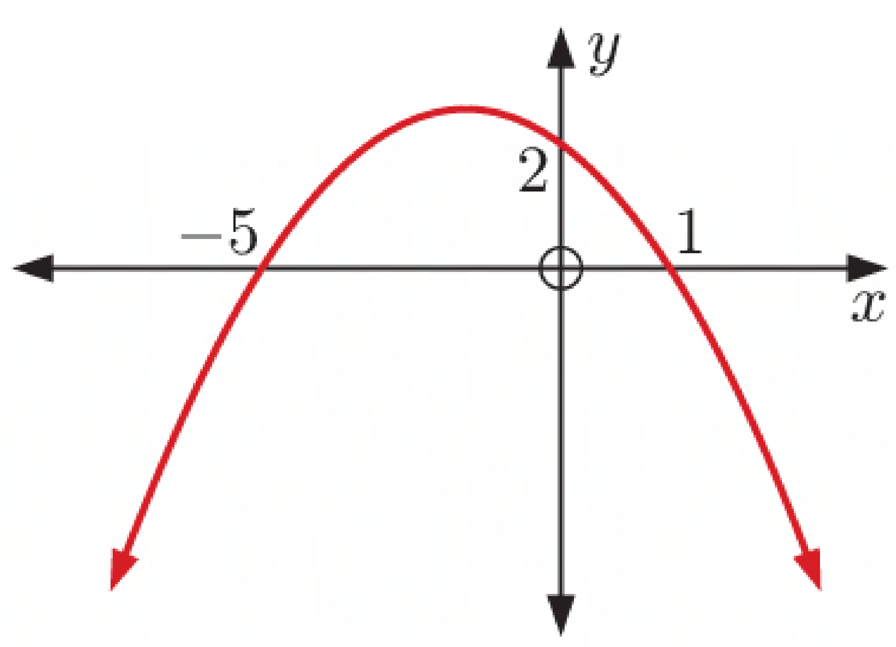
b $y = -3x^2 - 7x + 4$
has $a = -3$, $b = -7$, $c = 4$
$$\begin{aligned} \Delta &= b^2 - 4ac \\ &= (-7)^2 - 4(-3)(4) \\ &= 97 \end{aligned}$$
Since $\Delta > 0$, the graph cuts the x-axis twice.
Since $a < 0$, the graph is concave down.

4 Since the vertex is $(2, 25)$, the quadratic has the form $y = a(x-2)^2 + 25$, where $a \neq 0$.

When $x = 0$, $y = 1$
$$\begin{aligned} \therefore\ 1 &= a(-2)^2 + 25 \\ \therefore\ 1 &= 4a + 25 \\ \therefore\ 4a &= -24 \\ \therefore\ a &= -6 \end{aligned}$$

The quadratic is $y = -6(x-2)^2 + 25$.

5 **a** Since the x-intercepts are -5 and 1, $y = a(x+5)(x-1)$.
The graph is concave down, so $a < 0$.
When $x = 0$, $y = 2$
$$\begin{aligned} \therefore\ 2 &= a(5)(-1) \\ \therefore\ a &= -\tfrac{2}{5} \end{aligned}$$
The quadratic is $y = -\frac{2}{5}(x+5)(x-1)$.



b The axis of symmetry is $x = \dfrac{-5+1}{2}$
$$\therefore\ x = -2$$

When $x = -2$,
$$\begin{aligned} y &= -\tfrac{2}{5}(-2+5)(-2-1) \\ &= -\tfrac{2}{5}(3)(-3) \\ &= \tfrac{18}{5} = 3\tfrac{3}{5} \end{aligned}$$

The vertex is $(-2, 3\frac{3}{5})$, and the axis of symmetry is $x = -2$.

6 **a** $y = 2x^2 + 4x - 1$ has $a = 2$, $b = 4$, $c = -1$

$\dfrac{-b}{2a} = \dfrac{-4}{2(2)} = -1$

The axis of symmetry is $x = -1$.

b When $x = -1$, $y = 2(-1)^2 + 4(-1) - 1$

$= 2 - 4 - 1$

$= -3$

$\therefore$ the vertex is $(-1, -3)$.

c The y-intercept is -1.

When $y = 0$, $2x^2 + 4x - 1 = 0$

$\therefore\ x = \dfrac{-4 \pm \sqrt{4^2 - 4(2)(-1)}}{2(2)}$

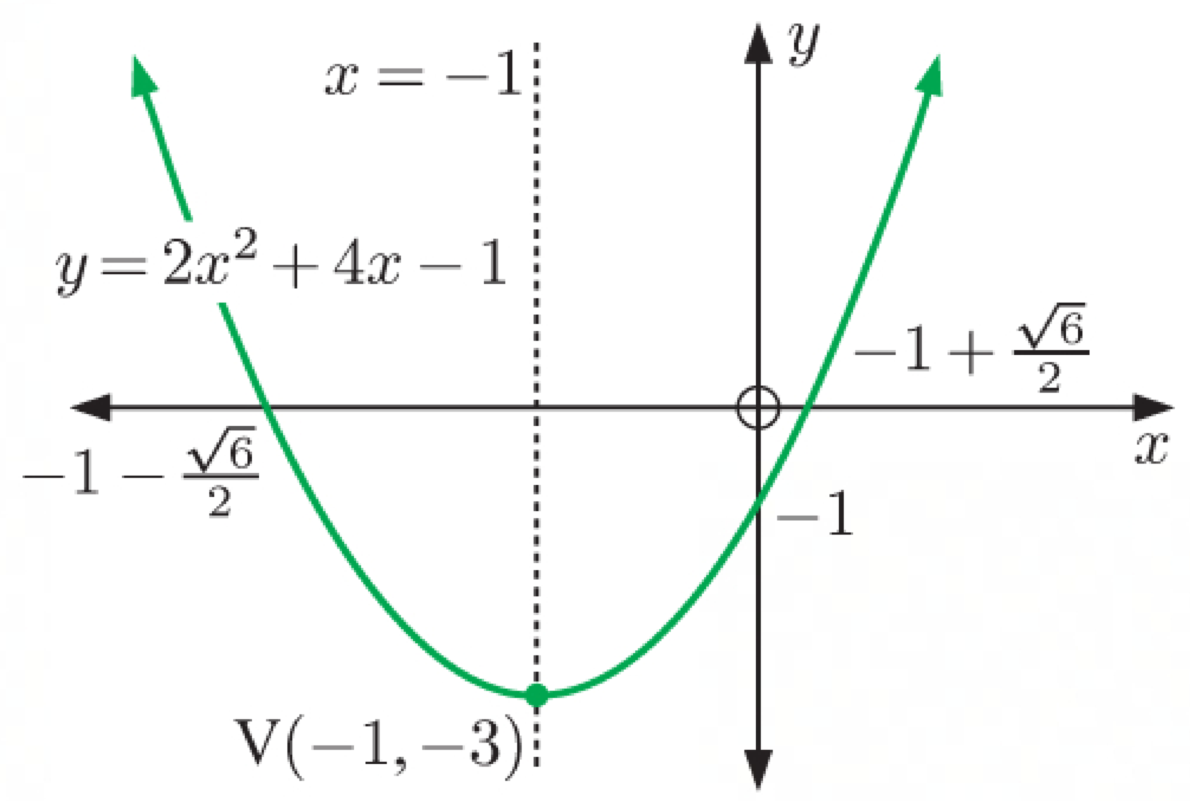
$= \dfrac{-4 \pm \sqrt{24}}{4}$

$= \dfrac{-4 \pm 2\sqrt{6}}{4}$

$= -1 \pm \frac{\sqrt{6}}{2}$

$\therefore$ the x-intercepts are $-1 \pm \frac{\sqrt{6}}{2}$.

d



7 **a** Since the graph touches the x-axis at 3, the quadratic has the form $y = a(x - 3)^2$.

When $x = 2$, $y = 2$

$\therefore\ 2 = a(2 - 3)^2$

$\therefore\ 2 = a(-1)^2$

$\therefore\ a = 2$

The quadratic is $y = 2(x - 3)^2$

$\therefore\ y = 2(x^2 - 6x + 9)$

$\therefore\ y = 2x^2 - 12x + 18$

b Since the x-intercepts are 3 and -2, $y = a(x - 3)(x + 2)$.

The y-intercept is 3, so when $x = 0$, $y = 3$

$\therefore\ 3 = a(-3)(2)$

$\therefore\ a = -\frac{1}{2}$

The quadratic is $y = -\frac{1}{2}(x - 3)(x + 2)$

$\therefore\ y = -\frac{1}{2}(x^2 - x - 6)$

$\therefore\ y = -\frac{1}{2}x^2 + \frac{1}{2}x + 3$

c Let the quadratic have the form $y = ax^2 + bx + c$.

When $x = -1$, $y = -9$

$\therefore\ -9 = a(-1)^2 + b(-1) + c$

$\therefore\ a - b + c = -9$ .... (1)

When $x = 1$, $y = 5$

$\therefore\ 5 = a(1)^2 + b(1) + c$

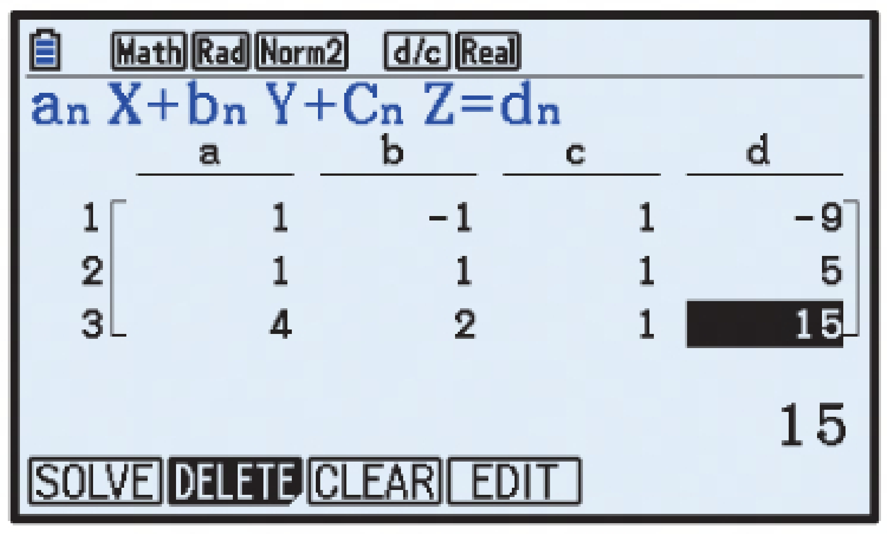
$\therefore\ a + b + c = 5$ .... (2)

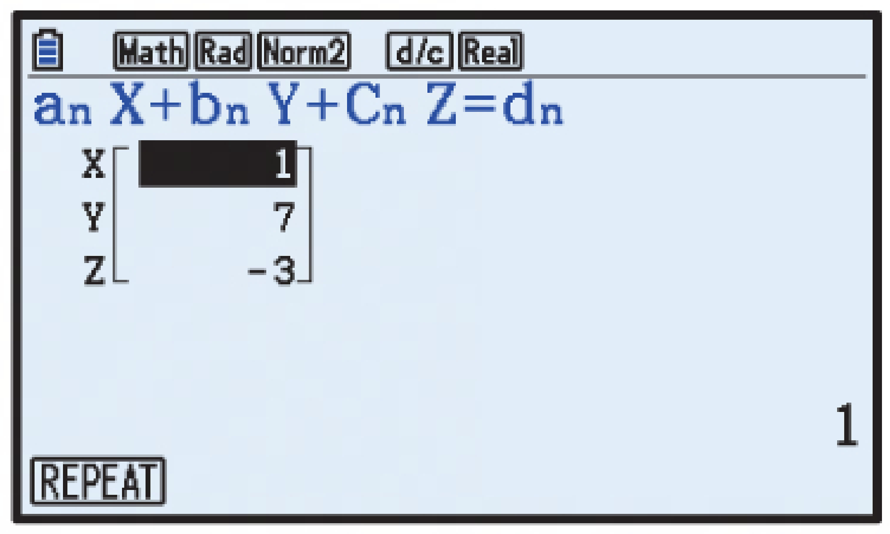
When $x = 2$, $y = 15$

$\therefore\ 15 = a(2)^2 + b(2) + c$

$\therefore\ 4a + 2b + c = 15$ .... (3)

Using technology,
$a = 1$, $b = 7$, and $c = -3$.
The quadratic is
$y = x^2 + 7x - 3$.





d Since the vertex is $(3, 15)$, the quadratic has the form $y = a(x-3)^2 + 15$.
When $x = 1$, $y = 7$

$$\therefore \; y = a(1-3)^2 + 15$$
$$\therefore \; 7 = a(-2)^2 + 15$$
$$\therefore \; 4a = -8$$
$$\therefore \; a = -2$$

The quadratic is $y = -2(x-3)^2 + 15$ or $y = -2x^2 + 12x - 3$.

8 **a** $y = 3x + c$ intersects the parabola $y = x^2 + x - 5$ where $x^2 + x - 5 = 3x + c$
$$\therefore \; x^2 - 2x - 5 - c = 0$$

The graphs meet in two distinct points when this equation has two distinct real roots.

$$\therefore \; \Delta > 0$$
$$\therefore \; (-2)^2 - 4(1)(-5-c) > 0$$
$$\therefore \; 4 + 20 + 4c > 0$$
$$\therefore \; 4c > -24$$
$$\therefore \; c > -6$$

b Choose c such that $c > -6$, for example, $c = -2$.
The graphs meet where $x^2 + x - 5 = 3x - 2$

$$\therefore \; x^2 - 2x - 3 = 0$$
$$\therefore \; (x+1)(x-3) = 0$$
$$\therefore \; x = -1 \text{ or } 3$$

Using the line $y = 3x - 2$, when $x = -1$, $y = 3(-1) - 2 = -5$
and when $x = 3$, $y = 3(3) - 2 = 7$

$\therefore$ the points of intersection are $(-1, -5)$ and $(3, 7)$.

9 **a** $y = 3x^2 + 4x + 7$
has $a = 3$, $b = 4$, $c = 7$
Since $a > 0$, the shape is 
The minimum value occurs when

$$x = \frac{-b}{2a} = \frac{-4}{2(3)} = -\frac{2}{3}$$

and $y = 3(-\frac{2}{3})^2 + 4(-\frac{2}{3}) + 7$
$= \frac{4}{3} - \frac{8}{3} + 7$
$= \frac{17}{3} = 5\frac{2}{3}$

So, the minimum value of y is $5\frac{2}{3}$, occurring when $x = -\frac{2}{3}$.

b $y = -2x^2 - 5x + 2$
has $a = -2$, $b = -5$, $c = 2$
Since $a < 0$, the shape is 
The maximum value occurs when

$$x = \frac{-b}{2a} = \frac{-(-5)}{2(-2)} = -\frac{5}{4}$$

and $y = -2(-\frac{5}{4})^2 - 5(-\frac{5}{4}) + 2$
$= -\frac{25}{8} + \frac{25}{4} + 2$
$= \frac{41}{8} = 5\frac{1}{8}$

So, the maximum value of y is $5\frac{1}{8}$, occurring when $x = -\frac{5}{4}$.

10 **a** Since the graph cuts the x-axis at -2 and 3, the quadratic has the form $y = a(x+2)(x-3)$.

When $x = -3$, $y = 18$

$$\therefore\ 18 = a(-3+2)(-3-3)$$
$$\therefore\ 18 = a(-1)(-6)$$
$$\therefore\ a = 3$$

The quadratic is $y = 3(x+2)(x-3)$

$$\therefore\ y = 3(x^2 - x - 6)$$
$$\therefore\ y = 3x^2 - 3x - 18$$

b When $x = 0$,

$$y = 3(0)^2 - 3(0) - 18$$
$$= -18$$

$\therefore$ the y-intercept is -18.

c $y = 3x^2 - 3x - 18$ has $a = 3$, $b = -3$, $c = -18$

The axis of symmetry is $x = \dfrac{-b}{2a}$

$$= \frac{-(-3)}{2(3)}$$
$$= \tfrac{1}{2}$$

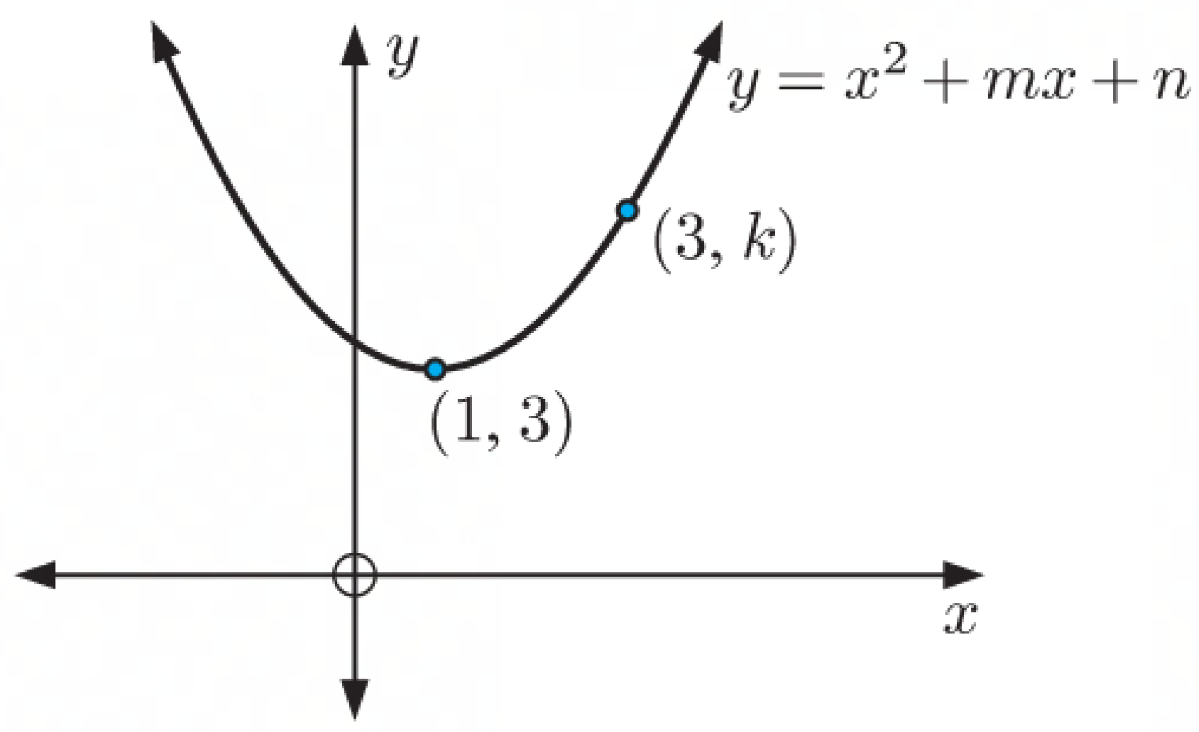
When $x = \frac{1}{2}$, $y = 3(\frac{1}{2})^2 - 3(\frac{1}{2}) - 18$

$$= \tfrac{3}{4} - \tfrac{3}{2} - 18$$
$$= -\tfrac{75}{4} = -18\tfrac{3}{4}$$

The vertex is $(\frac{1}{2}, -18\frac{3}{4})$.

11 **a** The axis of symmetry is $x = 1$.

$\therefore$ $x = \dfrac{-b}{2a} = \dfrac{-m}{2(1)} = 1$ and so $m = -2$.



Now when $x = 1$, $y = 3$

$$\therefore\ 1^2 - 2(1) + n = 3$$
$$\therefore\ n = 4$$

So, $m = -2$ and $n = 4$.

b $y = x^2 - 2x + 4$

When $x = 3$, $y = 3^2 - 2(3) + 4$

$$= 7$$
$$\therefore\ k = 7$$

12 Let the original piece of tinplate be x cm by x cm.
The volume of the folded box is 120 mL.

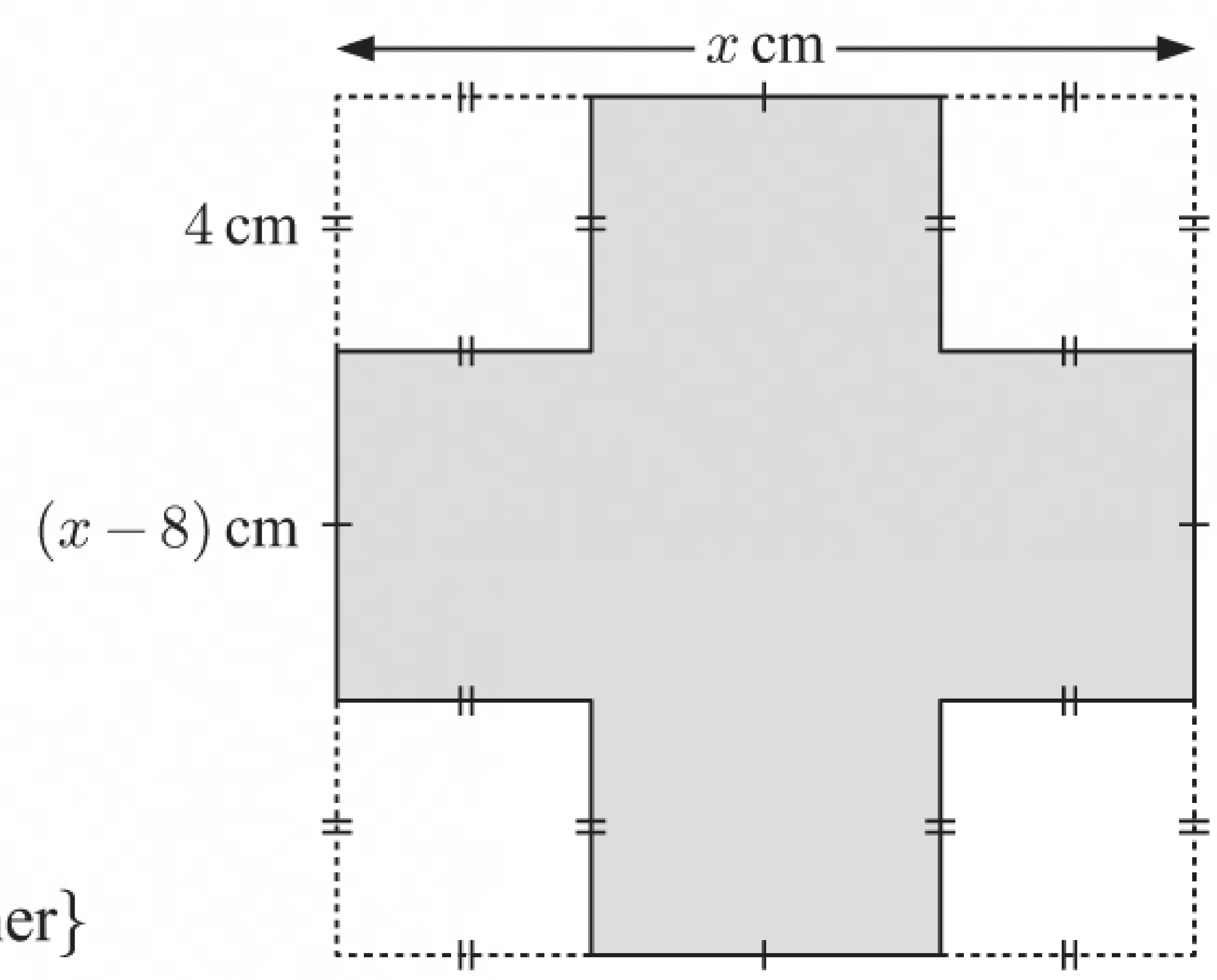
The volume $V = \text{length} \times \text{width} \times \text{height}$

$$\therefore\ 120 = (x-8) \times (x-8) \times 4$$
$$\therefore\ 120 = 4(x^2 - 16x + 64)$$
$$\therefore\ 120 = 4x^2 - 64x + 256$$

$\therefore$ $4x^2 - 64x + 136 = 0$

$$\therefore\ x \approx 13.5 \text{ or } 2.52$$

{using technology}

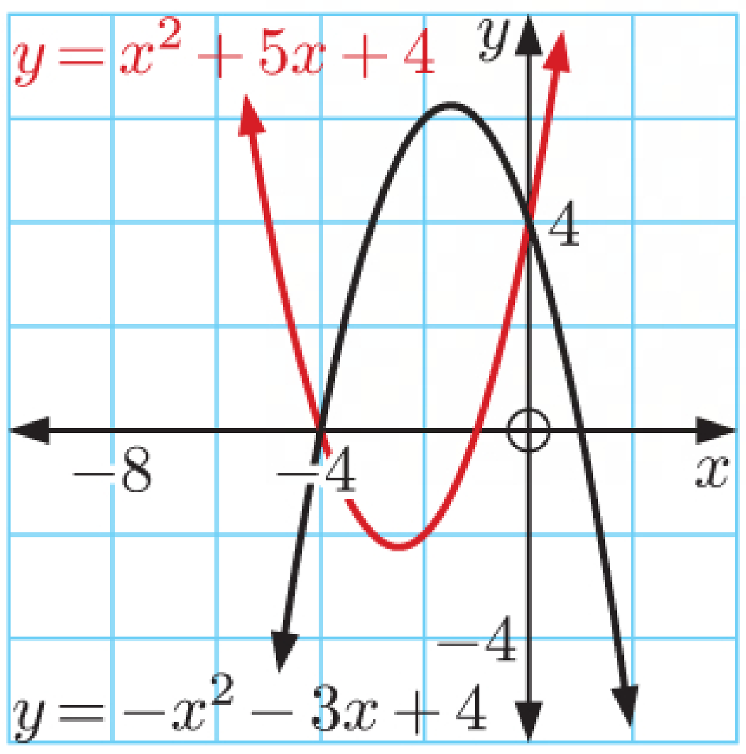


But $x > 8$ {4 cm squares are cut from each corner}

$\therefore$ $x \approx 13.5$

$\therefore$ the original piece of tinplate was about 13.5 cm square.

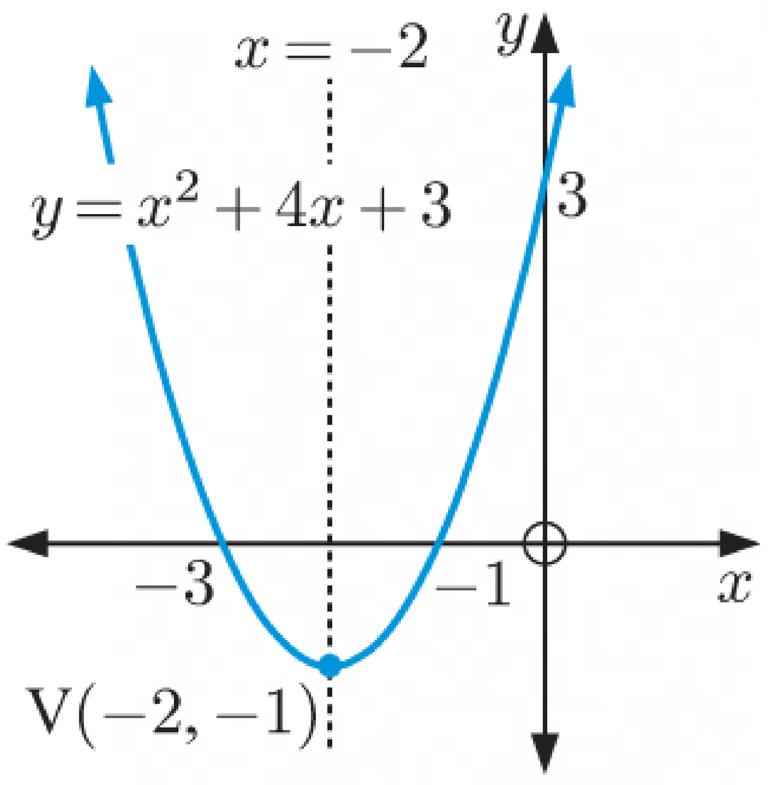
13 **a** $-x^2 - 3x + 4 = x^2 + 5x + 4$
$\therefore\ 2x^2 + 8x = 0$
$\therefore\ 2x(x+4) = 0$
$\therefore\ x = 0$ or -4

b


c $x^2 + 5x + 4 > -x^2 - 3x + 4$ where the graph of $y = x^2 + 5x + 4$ is *above* the graph of $y = -x^2 - 3x + 4$.
$\therefore\ x < -4$ or $x > 0$

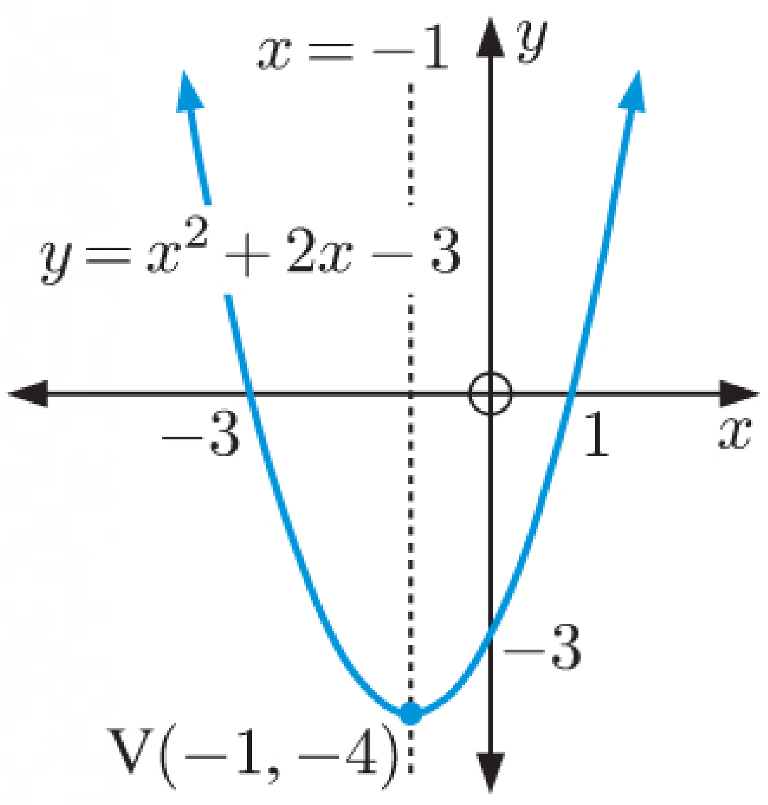
14 **a** **i** $y = x^2 + 4x + 3$
$\therefore\ y = x^2 + 4x + 2^2 + 3 - 2^2$
$\therefore\ y = (x+2)^2 - 1$

ii $y = x^2 + 4x + 3$
$\therefore\ y = (x+3)(x+1)$

iii


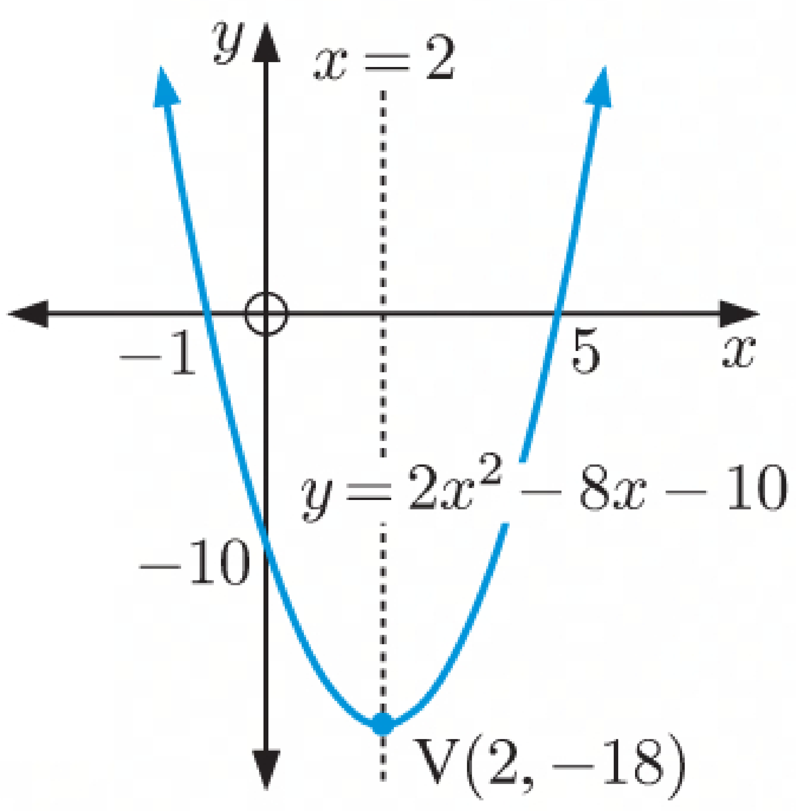
b **i** $y = x^2 + 2x - 3$
$\therefore\ y = x^2 + 2x + 1^2 - 3 - 1^2$
$\therefore\ y = (x+1)^2 - 4$

ii $y = x^2 + 2x - 3$
$\therefore\ y = (x+3)(x-1)$

iii


c **i** $y = 2x^2 - 8x - 10$
$\therefore\ y = 2(x^2 - 4x - 5)$
$\therefore\ y = 2[x^2 - 4x + (-2)^2 - 5 - (-2)^2]$
$\therefore\ y = 2[(x-2)^2 - 9]$
$\therefore\ y = 2(x-2)^2 - 18$

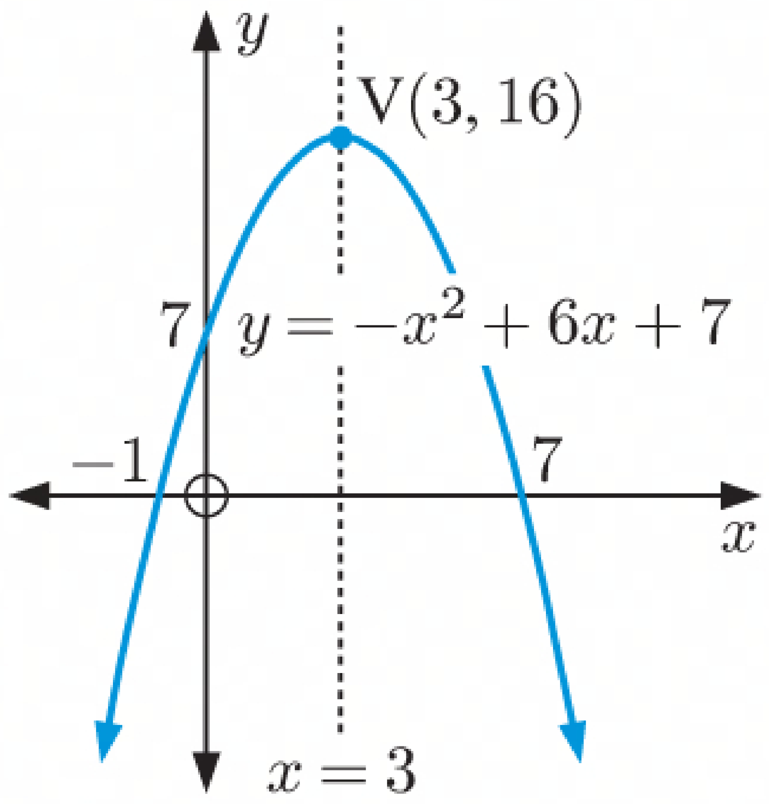
ii $y = 2x^2 - 8x - 10$
$\therefore\ y = 2(x^2 - 4x - 5)$
$\therefore\ y = 2(x-5)(x+1)$

iii


d **i**
$$y = -x^2 + 6x + 7$$
$$\therefore\ y = -(x^2 - 6x - 7)$$
$$\therefore\ y = -[x^2 - 6x + (-3)^2 - 7 - (-3)^2]$$
$$\therefore\ y = -[(x-3)^2 - 16]$$
$$\therefore\ y = -(x-3)^2 + 16$$

ii
$$y = -x^2 + 6x + 7$$
$$\therefore\ y = -(x^2 - 6x - 7)$$
$$\therefore\ y = -(x-7)(x+1)$$

iii



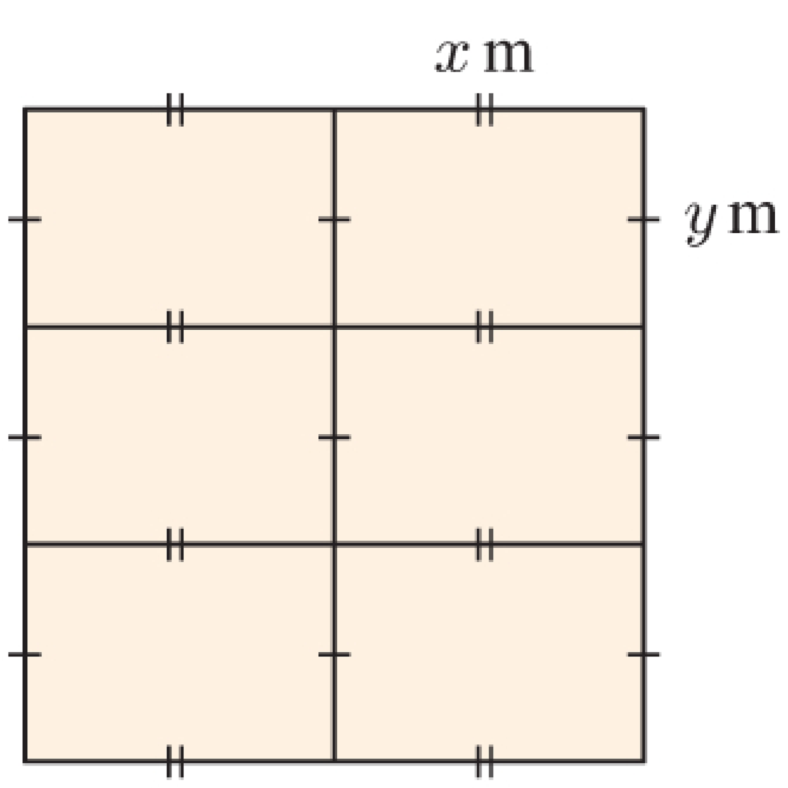
15 **a** $y = 9x^2 - kx + 4$ *touches* the x-axis.

$\therefore$ there is a repeated root, $\Delta = 0$
$$\therefore\ (-k)^2 - 4(9)(4) = 0$$
$$\therefore\ k^2 - 144 = 0$$
$$\therefore\ k^2 = 144$$
$$\therefore\ k = \pm 12$$

b $y = 9x^2 - 12x + 4$ and $y = 9x^2 + 12x + 4$ meet where
$$9x^2 - 12x + 4 = 9x^2 + 12x + 4$$
$$\therefore\ 24x = 0$$
$$\therefore\ x = 0$$
Substituting $x = 0$ into either equation, we get $y = 4$.

$\therefore$ the two quadratic functions intersect at $(0, 4)$.

16 **a** The total length of fencing is $(8x + 9y)$ m



$$\therefore\ 8x + 9y = 600$$
$$\therefore\ 9y = 600 - 8x$$
$$\therefore\ y = \frac{600 - 8x}{9}$$
The area of each pen is $A = xy$
$$= x\left(\frac{600 - 8x}{9}\right)\ \text{m}^2$$

b $A = x\left(\dfrac{600 - 8x}{9}\right)$
$$= \tfrac{600}{9}x - \tfrac{8}{9}x^2 \quad \text{which has } a = -\tfrac{8}{9},\ b = \tfrac{600}{9}$$
Since $a < 0$, the shape is 

$\therefore$ A is maximised at the axis of symmetry, which is $x = \dfrac{-b}{2a}$
$$\therefore\ x = \frac{-\frac{600}{9}}{2(-\frac{8}{9})}$$
$$= \frac{600}{16}$$
$$\therefore\ x = \tfrac{75}{2}$$

When $x = \tfrac{75}{2}$, $y = \dfrac{600 - 8(\frac{75}{2})}{9} = 33\tfrac{1}{3}$

$\therefore$ for maximum area, each pen should be $37\tfrac{1}{2}$ m $\times$ $33\tfrac{1}{3}$ m.

c The area of each pen in this case is $37\frac{1}{2} \times 33\frac{1}{3}$

$$= \tfrac{75}{2} \times \tfrac{100}{3}$$
$$= 1250 \text{ m}^2$$

17 **a** $\$x$ is the price increase of the sunglasses.

The cost of one pair of sunglasses is now $\$(45 + x)$.

By increasing the price, the retailer will lose $\dfrac{x}{1.5}$ customers per day.

$\therefore$ the total number of customers per day is $\left(50 - \dfrac{x}{1.5}\right)$.

$\therefore$ the revenue collected by the retailer each day is the cost of one pair of sunglasses multiplied by the number of customers.

$\therefore$ $R = (45 + x)\left(50 - \dfrac{x}{1.5}\right)$ dollars.

b $R = (45 + x)\left(50 - \dfrac{x}{1.5}\right)$

$\therefore$ $R = 2250 - 30x + 50x - \dfrac{x^2}{1.5}$

$\therefore$ $R = -\frac{2}{3}x^2 + 20x + 2250$ has $a = -\frac{2}{3}$, $b = 20$, $c = 2250$

Since $a < 0$, the shape is 

$\therefore$ R is maximised at the axis of symmetry, which is $x = \dfrac{-b}{2a}$

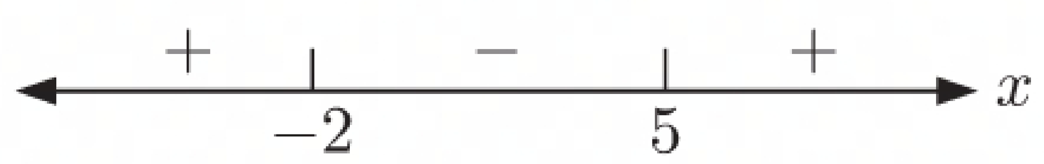
$$\therefore \quad x = \frac{-20}{2(-\frac{2}{3})}$$
$$= 15$$

$\therefore$ the retailer should set a price of $\$(45+15) = \$60$ for his sunglasses in order to maximise his daily revenue. The total revenue is $60\left(50 - \dfrac{15}{1.5}\right) = 60(50 - 10) = \$2400$ per day.

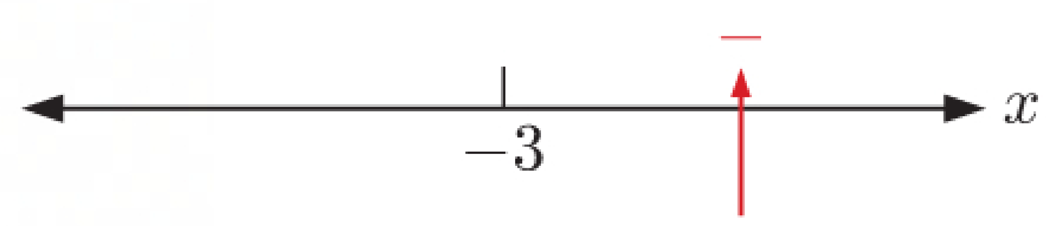
18 **a** $x^2 - 3x - 10 = (x + 2)(x - 5)$ has zeros -2 and 5.

When $x = 6$ we have $(8)(1) > 0$, so we put a $+$ sign here.

As the factors are single, the signs alternate.

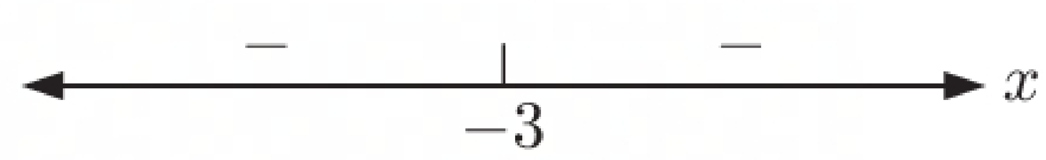


b $-(x + 3)^2$ has zero -3.



When $x = 0$ we have $-(3)^2 < 0$, so we put a $-$ sign here.

As the factor is squared, the signs do not change.



19 **a** $4x^2 - 3x < 0$

$\therefore \ x(4x-3) < 0$

LHS has sign diagram

$\therefore \ 0 < x < \frac{3}{4}$

b $2x^2 - 3x - 5 \geqslant 0$

$\therefore \ (2x-5)(x+1) \geqslant 0$

LHS has sign diagram

$\therefore \ x \leqslant -1 \ \text{ or } \ x \geqslant \frac{5}{2}$

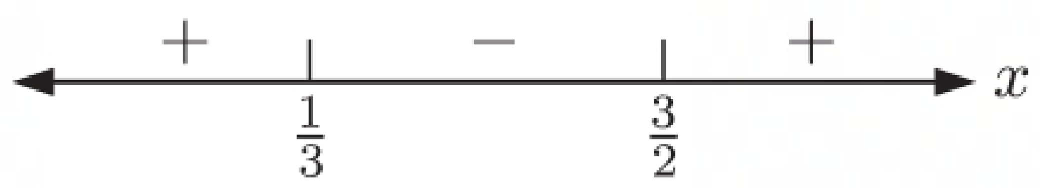
c $\frac{11}{3}x \leqslant 2x^2 + 1$

$\therefore \ 2x^2 - \frac{11}{3}x + 1 \geqslant 0$

$\therefore \ 6x^2 - 11x + 3 \geqslant 0$

$\therefore \ (3x-1)(2x-3) \geqslant 0$

LHS has sign diagram

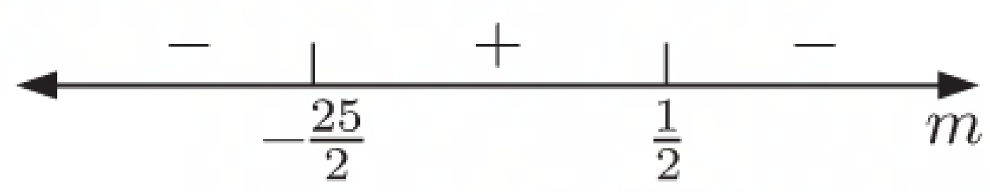


$\therefore \ x \leqslant \frac{1}{3} \ \text{ or } \ x \geqslant \frac{3}{2}$

20 $y = mx^2 + 5x + (m+12)$ has $a = m$, $b = 5$, $c = m + 12$

$$\begin{aligned}\therefore \ \Delta &= b^2 - 4ac \\ &= (5)^2 - 4(m)(m+12) \\ &= 25 - 4m^2 - 48m \\ &= -(4m^2 + 48m - 25) \\ &= -(2m-1)(2m+25)\end{aligned}$$

So, Δ has sign diagram:



a The function cuts the x-axis twice if $\Delta > 0$.

$\therefore \ -\frac{25}{2} < m < \frac{1}{2}, \ m \neq 0$

b The function touches the x-axis if $\Delta = 0$.

$\therefore \ m = -\frac{25}{2} \ \text{ or } \ m = \frac{1}{2}$

c The function misses the x-axis if $\Delta < 0$.

$\therefore \ m < -\frac{25}{2} \ \text{ or } \ m > \frac{1}{2}$

Chapter 15

FUNCTIONS

EXERCISE 15A

1 **a** $\{(1, 3), (2, 3), (3, 1), (4, 2)\}$ is a function, since for every value of x there is only one corresponding value of y.

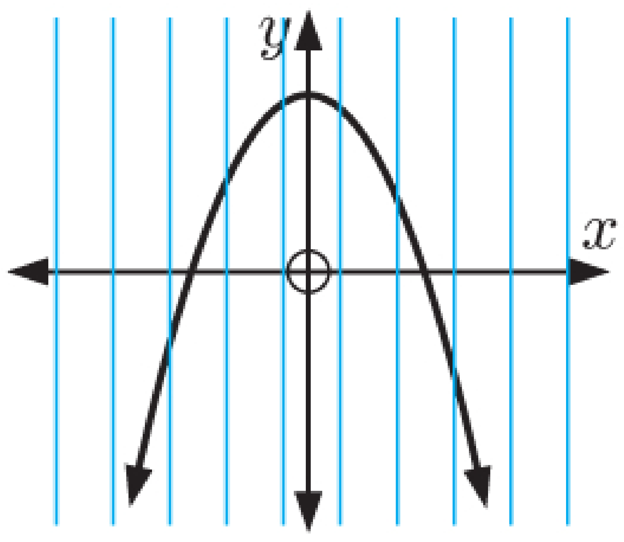
b $\{(2, 1), (3, 1), (-1, 2), (2, 0)\}$ is not a function. When $x = 2$, $y = 1$ or 0.

2 **a** $y = x^2 - 9$ is a function, since for any value of x there is at most one value of y.

b $x + y = 9$ is a function, since for any value of x there is at most one value of y.

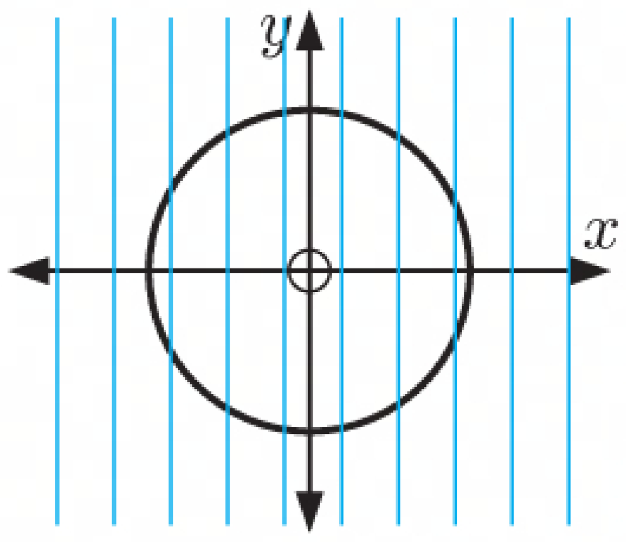
c $x^2 + y^2 = 9$ is not a function. If $x^2 + y^2 = 9$, then $y = \pm\sqrt{9 - x^2}$. So, for example, for $x = 2$, $y = \pm\sqrt{5}$.

3 **a**



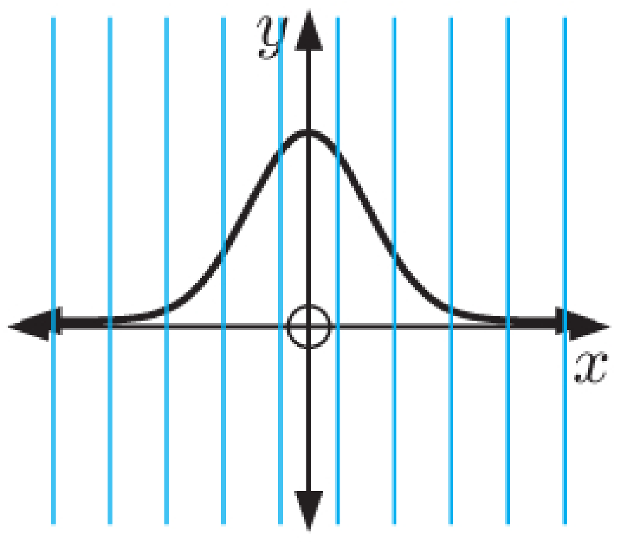
Each vertical line cuts the graph at most once, so this relation is a function.

b



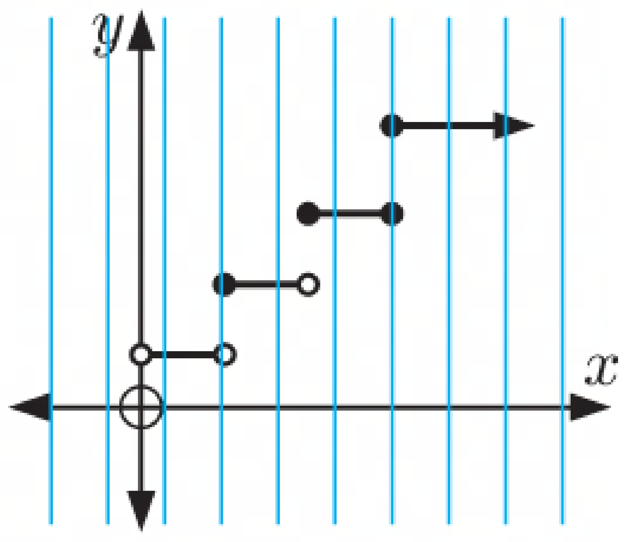
Some vertical lines cut the graph more than once, so this relation is not a function.

c



Each vertical line cuts the graph at most once, so this relation is a function.

d

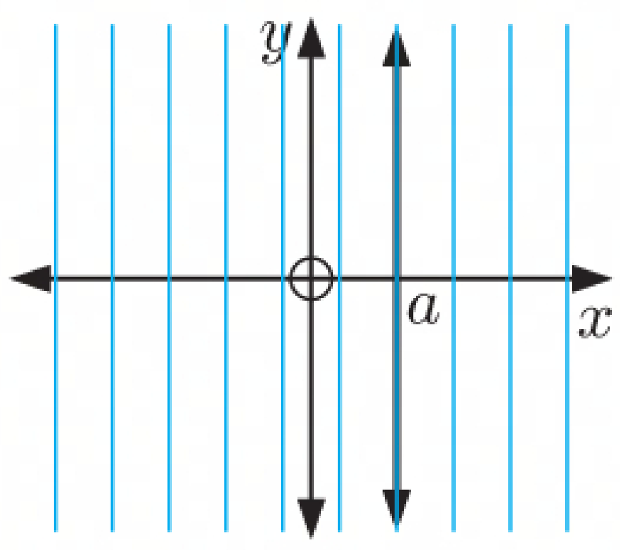


One vertical line cuts the graph more than once, so this relation is not a function.

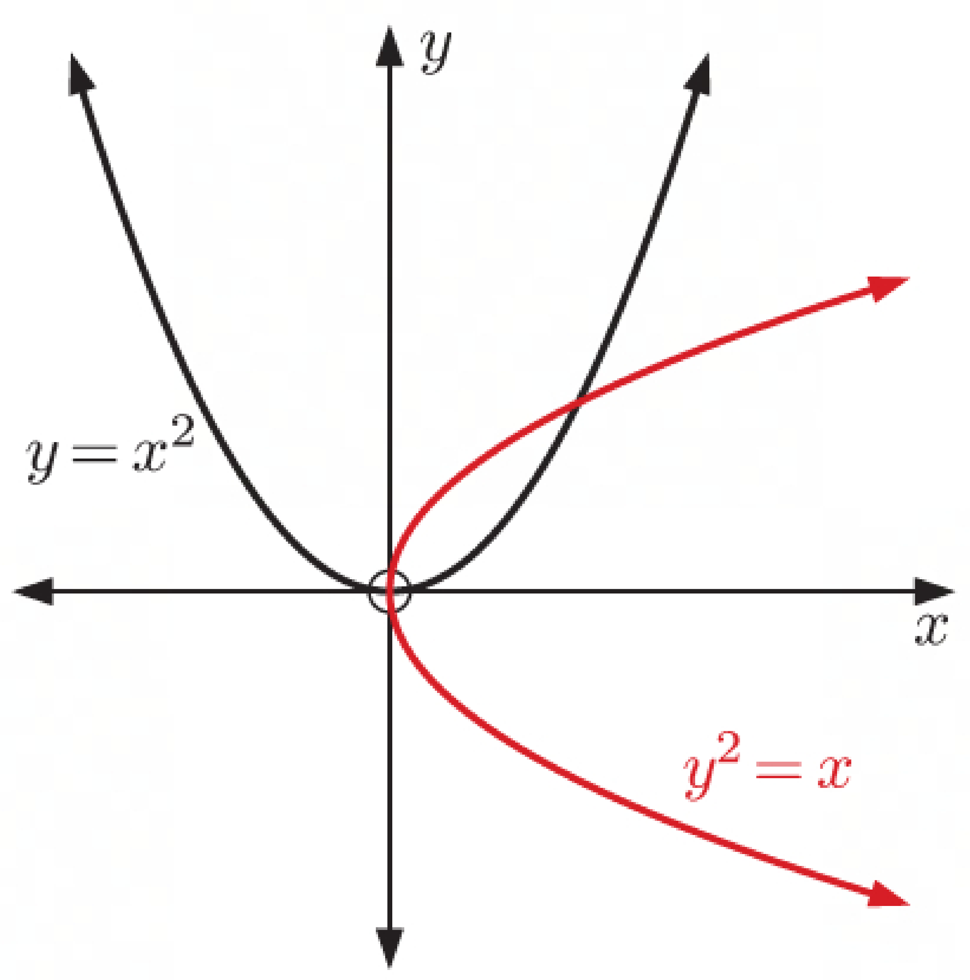
4 The relation between *age* and *cost* is not a function as there are two corresponding values of *cost* for a 2 year old child.
The categories 0 - 2 years and 2 - 16 years both include the age 2 years.
So, a ticket for a 2 year old child could cost either \$0 or \$20 according to the schedule.

5 It is not possible for a function to have more than one y-intercept. The y-axis is a vertical line, so if a relation has more than one y-intercept, then a vertical line cuts the relation more than once. So, such a relation cannot be a function.

6 No, the graph of a straight line will not be a function if it is a vertical line which has the form $x = a$ for some constant a.
The vertical line through $x = a$ cuts the graph at every point, so the straight line $x = a$ does not pass the vertical line test and hence is not a function.



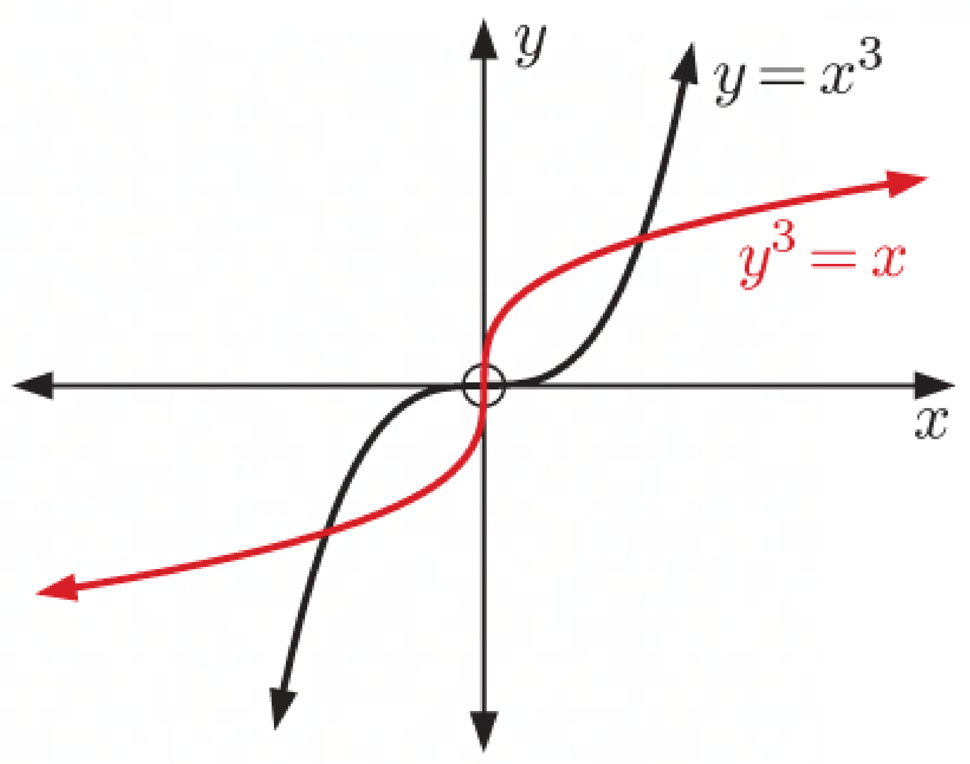
7



a $y^2 = x$ is a relation but not a function.
$y = x^2$ is a function (and a relation).
$y^2 = x$ has a horizontal axis of symmetry (the x-axis).
$y = x^2$ has a vertical axis of symmetry (the y-axis).
Both $y^2 = x$ and $y = x^2$ pass through $(0, 0)$ and $(1, 1)$.
$y^2 = x$ is a rotation of $y = x^2$ clockwise through $90°$ about the origin *or* $y^2 = x$ is a reflection of $y = x^2$ in the line $y = x$.

b **i** The part of the graph of $y^2 = x$ in the first quadrant corresponds to $y = \sqrt{x}$.
ii $y = \sqrt{x}$ is a function as any vertical line cuts the graph at most once.

8



a Both curves are functions since any vertical line will cut each curve at most once.

b $y^3 = x$

$\therefore\ y = x^{\frac{1}{3}}$

$\therefore\ y = \sqrt[3]{x}$

EXERCISE 15B

1 $f(x) = 3x - x^2 + 2$

a $f(0) = 3(0) - 0^2 + 2$
$= 0 - 0 + 2$
$= 2$

b $f(3) = 3(3) - 3^2 + 2$
$= 9 - 9 + 2$
$= 2$

c $f(-3) = 3(-3) - (-3)^2 + 2$
$= -9 - 9 + 2$
$= -16$

d $f(-7) = 3(-7) - (-7)^2 + 2$
$= -21 - 49 + 2$
$= -68$

e $f(\frac{3}{2}) = 3(\frac{3}{2}) - (\frac{3}{2})^2 + 2$
$= \frac{9}{2} - \frac{9}{4} + 2$
$= \frac{17}{4}$

2 $g : x \mapsto x - \dfrac{4}{x}$

a $g(1) = 1 - \dfrac{4}{1} = -3$

b $g(4) = 4 - \dfrac{4}{4} = 3$

c $g(-1) = -1 - \dfrac{4}{(-1)} = 3$

d $g(-4) = -4 - \dfrac{4}{(-4)} = -3$

e $g(-\frac{1}{2}) = -\dfrac{1}{2} - \dfrac{4}{(-\frac{1}{2})} = -\dfrac{1}{2} + 8 = \frac{15}{2}$

3 $G(x) = \dfrac{2x+3}{x-4}$

a **i** $G(2) = \dfrac{2(2)+3}{2-4} = \frac{7}{-2} = -\frac{7}{2}$

ii $G(0) = \dfrac{2(0)+3}{0-4} = \frac{3}{-4} = -\frac{3}{4}$

iii $G(-\frac{1}{2}) = \dfrac{2(-\frac{1}{2})+3}{-\frac{1}{2}-4} = \dfrac{-1+3}{(-\frac{9}{2})} = \dfrac{2}{(-\frac{9}{2})} = -\frac{4}{9}$

b $G(x) = \dfrac{2x+3}{x-4}$ is undefined when $x-4=0$

$\therefore\ x = 4$

So, when $x=4$, $G(x)$ does not exist.

c $G(x) = -3$, so $\dfrac{2x+3}{x-4} = -3$

$\therefore\ 2x+3 = -3(x-4)$

$\therefore\ 2x+3 = -3x+12$

$\therefore\ 5x = 9$

$\therefore\ x = \frac{9}{5}$

4 $f(x) = 7-3x$

a $f(a) = 7-3a$

b $f(-a) = 7-3(-a) = 7+3a$

c $f(a+3) = 7-3(a+3) = 7-3a-9 = -3a-2$

d $f(2a) = 7-3(2a) = 7-6a$

e $f(x+2) = 7-3(x+2) = 7-3x-6 = 1-3x$

f $f(x+h) = 7-3(x+h) = 7-3x-3h$

5 $F(x) = 2x^2+3x-1$

a $F(x+4)$
$= 2(x+4)^2+3(x+4)-1$
$= 2(x^2+8x+16)+3x+12-1$
$= 2x^2+16x+32+3x+11$
$= 2x^2+19x+43$

b $F(2-x)$
$= 2(2-x)^2+3(2-x)-1$
$= 2(4-4x+x^2)+6-3x-1$
$= 8-8x+2x^2+5-3x$
$= 2x^2-11x+13$

c $F(-x)$
$= 2(-x)^2+3(-x)-1$
$= 2x^2-3x-1$

d $F(x^2)$
$= 2(x^2)^2+3(x^2)-1$
$= 2x^4+3x^2-1$

e $F(3x)$
$= 2(3x)^2+3(3x)-1$
$= 2(9x^2)+9x-1$
$= 18x^2+9x-1$

f $F(x+h)$
$= 2(x+h)^2+3(x+h)-1$
$= 2(x^2+2xh+h^2)+3x+3h-1$
$= 2x^2+4xh+2h^2+3x+3h-1$
$= 2x^2+(4h+3)x+2h^2+3h-1$

6 $f(x) = x^2$

a $f(3x) = (3x)^2$
$= 9x^2$

b $f\left(\frac{x}{2}\right) = \left(\frac{x}{2}\right)^2$
$= \frac{x^2}{4}$

c $3f(x) = 3x^2$

d $2f(x-1)+5 = 2(x-1)^2+5$
$= 2(x^2-2x+1)+5$
$= 2x^2-4x+2+5$
$= 2x^2-4x+7$

7 $f(x) = \frac{1}{x}$

a $f(-x) = \frac{1}{(-x)}$
$= -\frac{1}{x}$

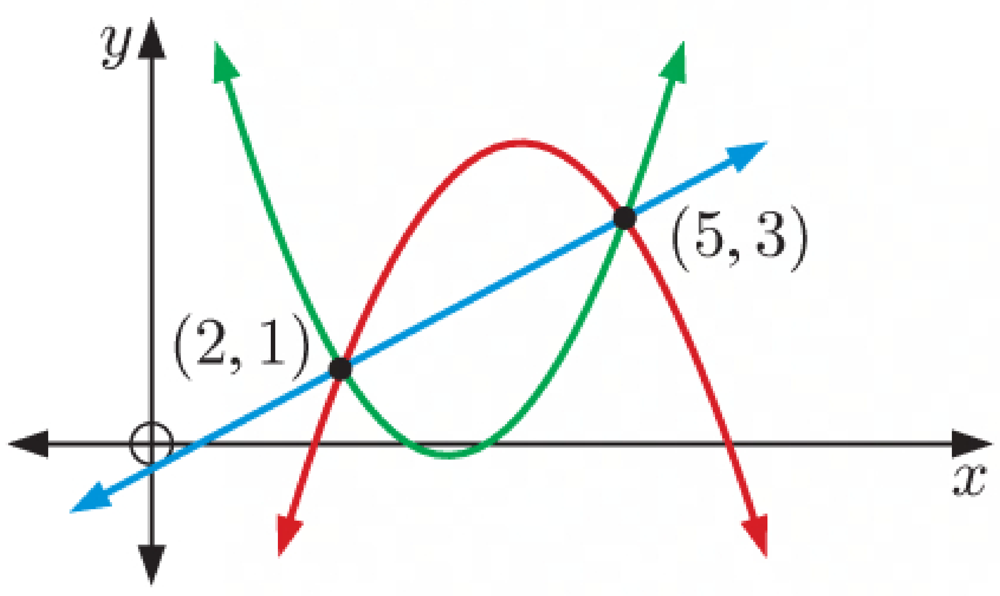
b $f(\frac{1}{2}x) = \frac{1}{\frac{1}{2}x}$
$= \frac{1}{(\frac{x}{2})}$
$= \frac{2}{x}$

c $2f(x)+3 = 2\times\frac{1}{x}+3$
$= \frac{2}{x}+\frac{3x}{x}$
$= \frac{2+3x}{x}$

d $3f(x-1)+2 = 3\times\frac{1}{x-1}+2$
$= \frac{3}{x-1}+\frac{2(x-1)}{x-1}$
$= \frac{3+2x-2}{x-1}$
$= \frac{2x+1}{x-1}$

8 f is the function which converts x into $f(x)$ whereas $f(x)$ is the value of the function at any value of x.

9



Note: Other answers are possible.
First sketch the straight line which passes through the points $(2, 1)$ and $(5, 3)$.
Then sketch two quadratic functions which also pass through the two points.

10 $f(x) = ax+b$ where $f(2) = 1$ and $f(-3) = 11$

So, $a(2)+b = 1$
$\therefore\ 2a+b = 1$
$\therefore\ b = 1-2a$ .... $(*)$

and $a(-3)+b = 11$
$\therefore\ -3a+b = 11$
$\therefore\ -3a+(1-2a) = 11$ {using $(*)$}
$\therefore\ -5a = 10$
$\therefore\ a = -2$

Substituting $a = -2$ into $(*)$ gives $b = 1-2(-2) = 5$.

So, $a = -2$, $b = 5$, and hence $f(x) = -2x+5$.

11 **a** $H(30) = 800$

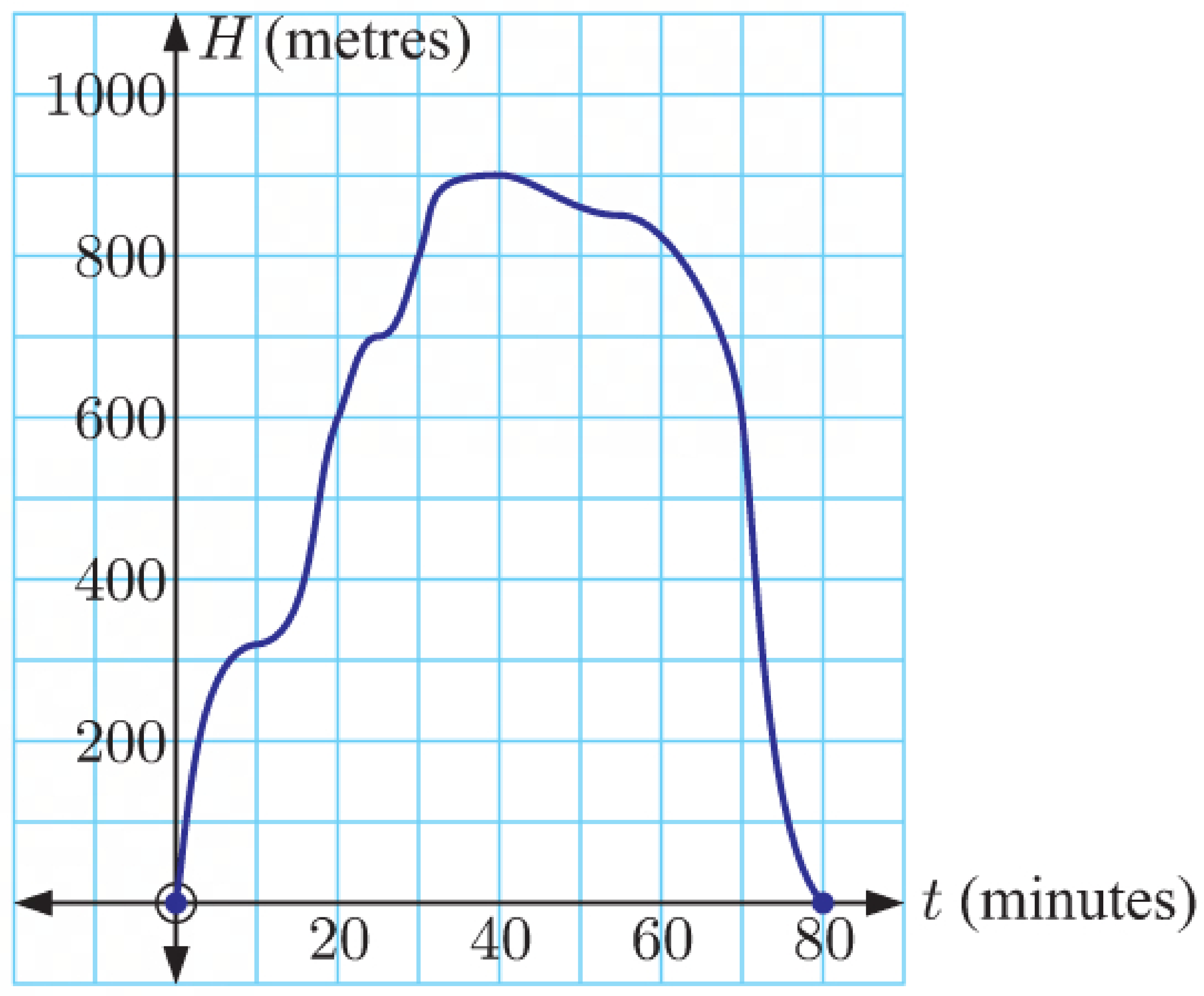
After 30 minutes, the balloon is 800 m high.

b $H(t) = 600$ when $t = 20$ or 70.

After 20 minutes and after 70 minutes the balloon is 600 m high.

c The height of the balloon was recorded for $0 \leqslant t \leqslant 80$ minutes.

d The range of heights recorded was 0 m to 900 m.



12 $f(x) = ax + \frac{b}{x}$ where $f(1) = 1$ and $f(2) = 5$

So, $a(1) + \frac{b}{(1)} = 1$

$\therefore\ a + b = 1$

$\therefore\ b = 1 - a \quad .... (*)$

and $a(2) + \frac{b}{(2)} = 5$

$\therefore\ 2a + \frac{b}{2} = 5$

$\therefore\ 4a + b = 10$

$\therefore\ 4a + (1 - a) = 10 \quad \{\text{using } (*)\}$

$\therefore\ 3a = 9$

$\therefore\ a = 3$

Substituting $a = 3$ into $(*)$ gives $b = 1 - 3 = -2$.

So, $a = 3$, $b = -2$.

13 $T(x) = ax^2 + bx + c$ where $T(0) = -4$, $T(1) = -2$, and $T(2) = 6$

So, $a(0)^2 + b(0) + c = -4$

$\therefore\ c = -4$

Now $a(1)^2 + b(1) - 4 = -2$

$\therefore\ a + b = 2$

$\therefore\ b = 2 - a \quad .... (*)$

and $a(2)^2 + b(2) - 4 = 6$

$\therefore\ 4a + 2b = 10$

$\therefore\ 2a + b = 5$

$\therefore\ 2a + (2 - a) = 5 \quad \{\text{using } (*)\}$

$\therefore\ a = 3$

Substituting $a = 3$ into $(*)$ gives $b = 2 - 3 = -1$.

So, $a = 3$, $b = -1$, and $c = -4$.

14 $V(t) = 9000 - 900t$

a $V(4) = 9000 - 900(4)$

$= 9000 - 3600$

$= 5400$

$V(4)$ is the value of the photocopier in pounds after 4 years.

$\therefore$ the value of the photocopier 4 years after purchase is £5400.

b $V(t) = 3600$, so $9000 - 900t = 3600$

$\therefore\ 900t = 5400$

$\therefore\ t = 6$

After 6 years, the value of the photocopier is £3600.

c The original purchase price is when $t = 0$.

Now, $V(0) = 9000 - 900(0)$

$= 9000$

The original purchase price was £9000.

d We have $t \geqslant 0$ and $V \geqslant 0$ since time and value are always positive.

So, $9000 - 900t \geqslant 0$

$\therefore\ 9000 \geqslant 900t$

$\therefore\ t \leqslant 10$

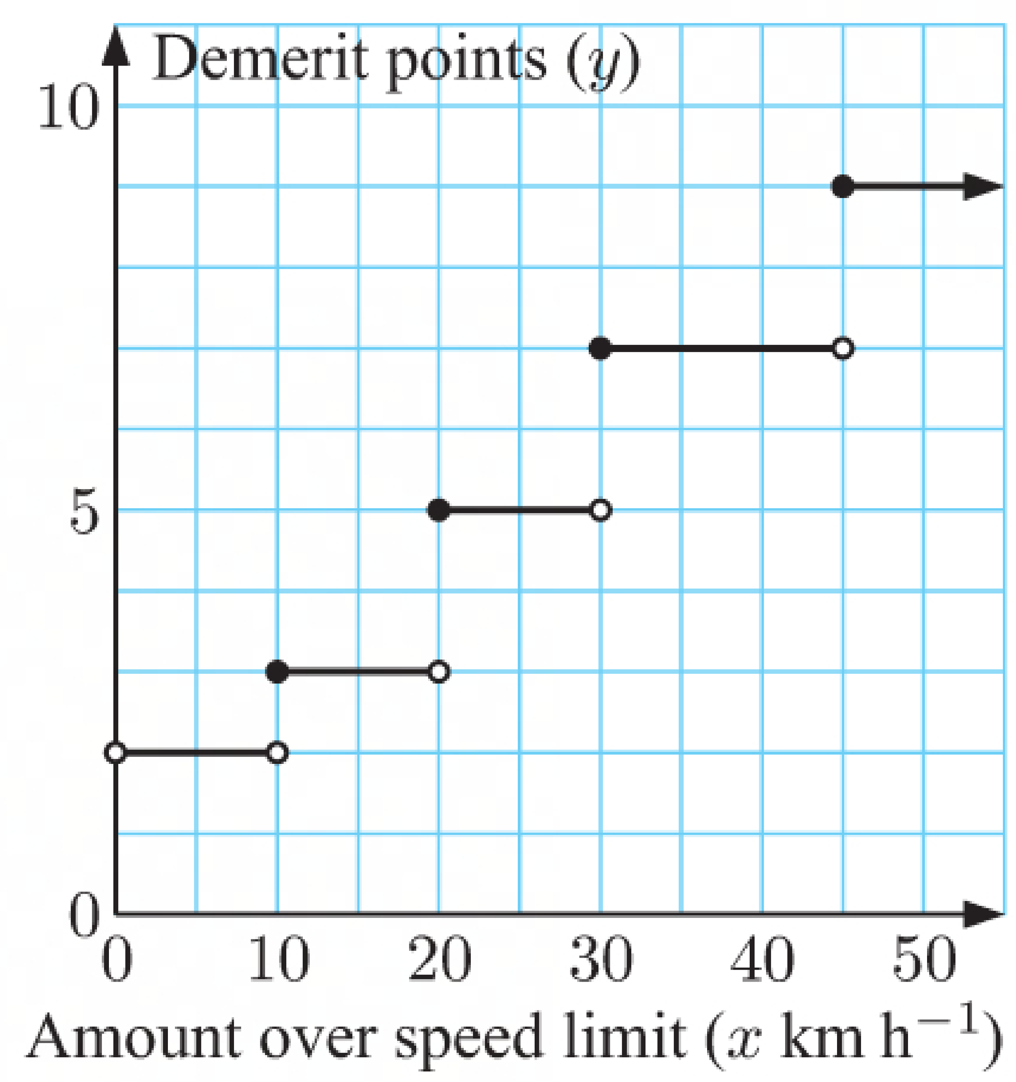
So, $0 \leqslant t \leqslant 10$ years.

EXERCISE 15C

1

Amount over speed limit (x km h^{-1})	*Demerit points* (y)
$0 < x < 10$	2
$10 \leqslant x < 20$	3
$20 \leqslant x < 30$	5
$30 \leqslant x < 45$	7
$x \geqslant 45$	9

a



b Yes, the relation is a function since for any value of x, there is at most one value of y.

c The function is defined for x such that $x > 0$.

$\therefore$ the domain is $\{x \mid x > 0\}$.

The possible demerit points are 2, 3, 5, 7, and 9.

$\therefore$ the range is $\{y \mid y = 2, 3, 5, 7, \text{ or } 9\}$.

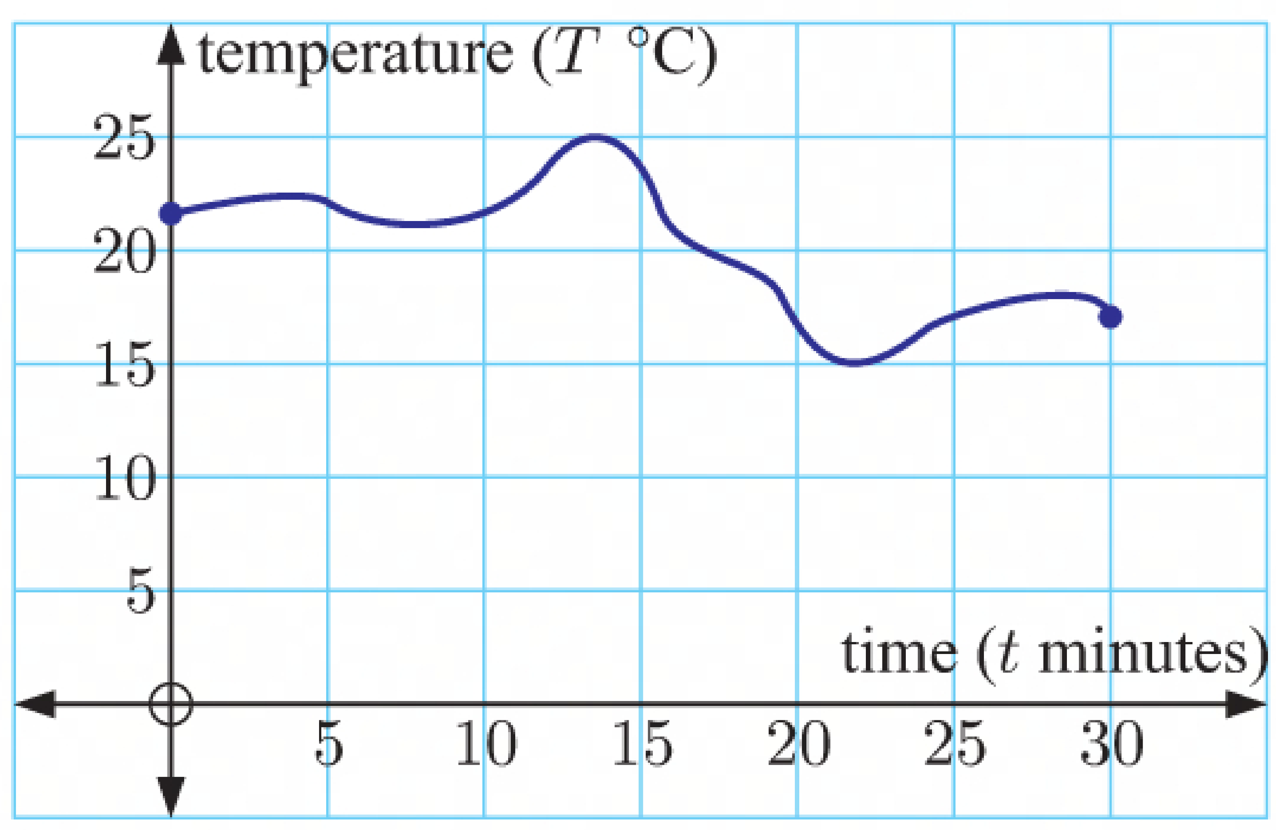
2 **a** At any moment in time there can be only one temperature, so the graph is a function.

b The temperature function is defined for all time t such that $0 \leqslant t \leqslant 30$ minutes.

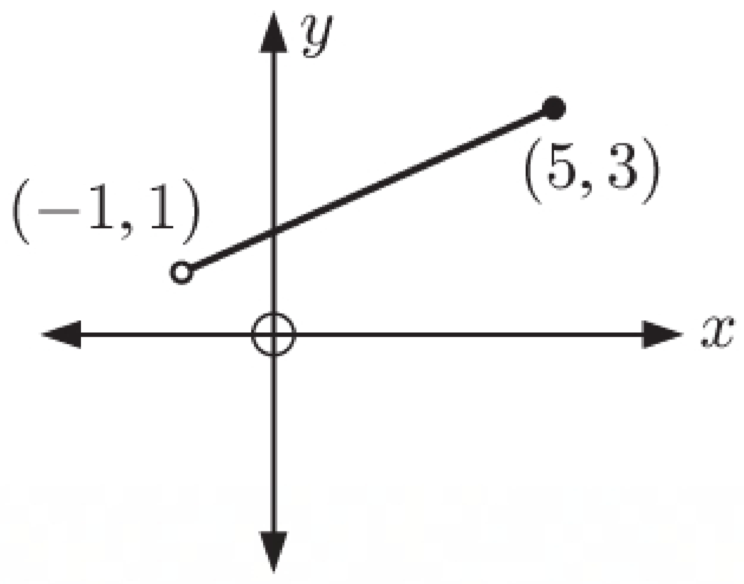
$\therefore$ the domain is $\{t \mid 0 \leqslant t \leqslant 30\}$.

The recorded temperatures lie between $T = 15°\text{C}$ and $T = 25°\text{C}$.

$\therefore$ the range is $\{T \mid 15 \leqslant T \leqslant 25\}$.

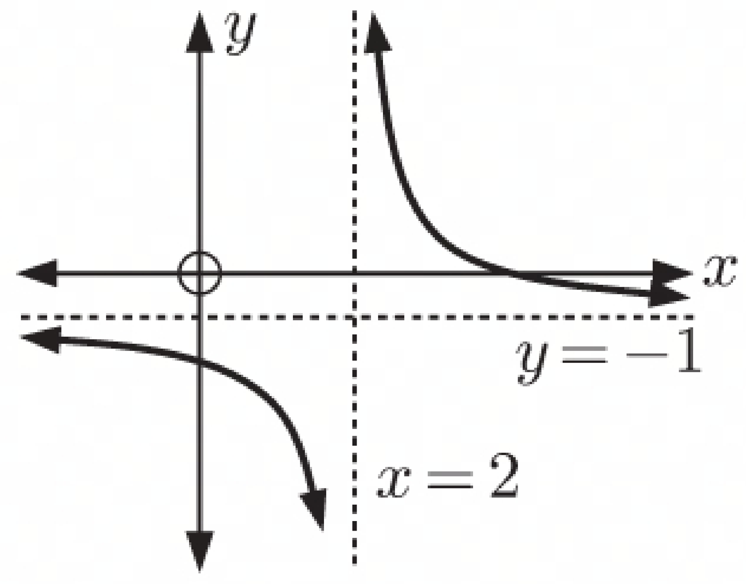


3 **a**



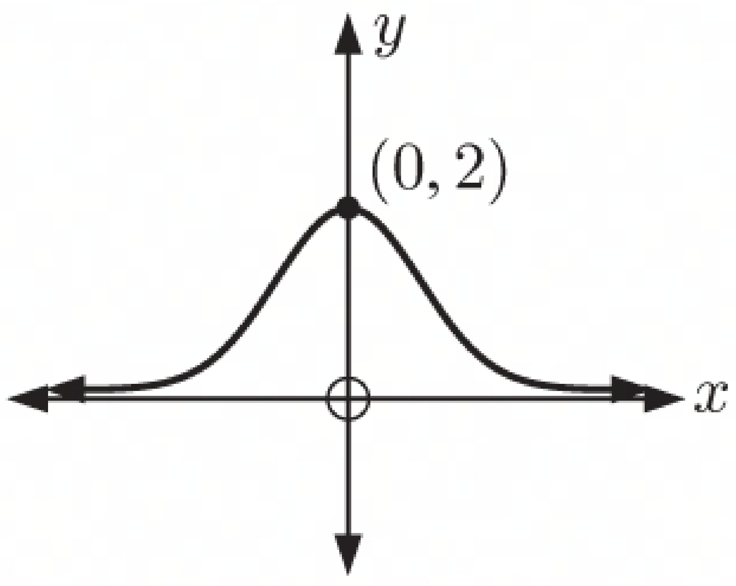
Domain is $\{x \mid -1 < x \leqslant 5\}$
Range is $\{y \mid 1 < y \leqslant 3\}$

b



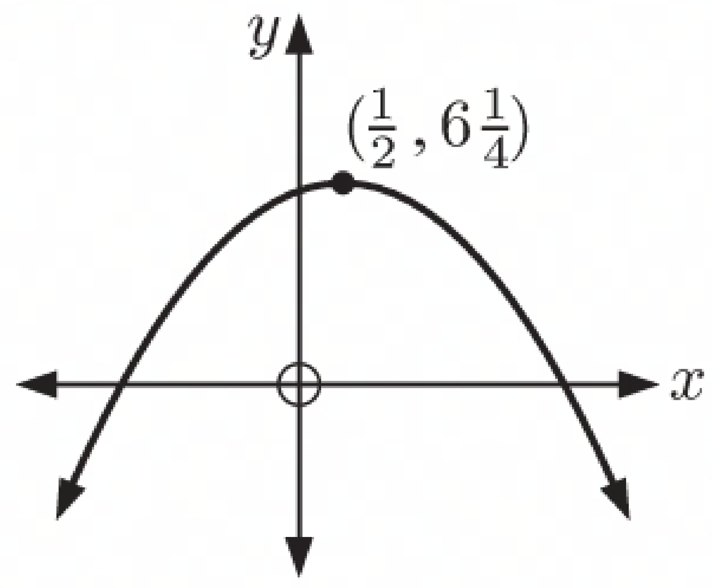
Domain is $\{x \mid x \neq 2\}$
Range is $\{y \mid y \neq -1\}$

c



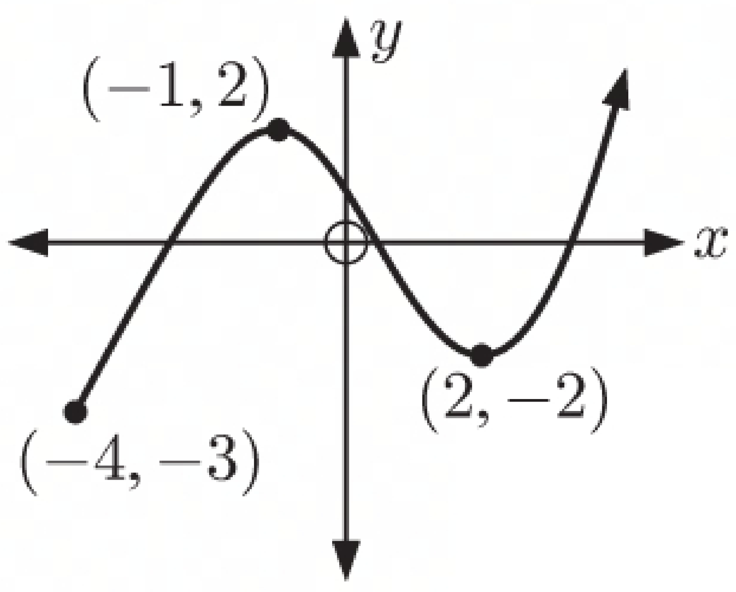
Domain is $\{x \mid x \in \mathbb{R}\}$
Range is $\{y \mid 0 < y \leqslant 2\}$

d



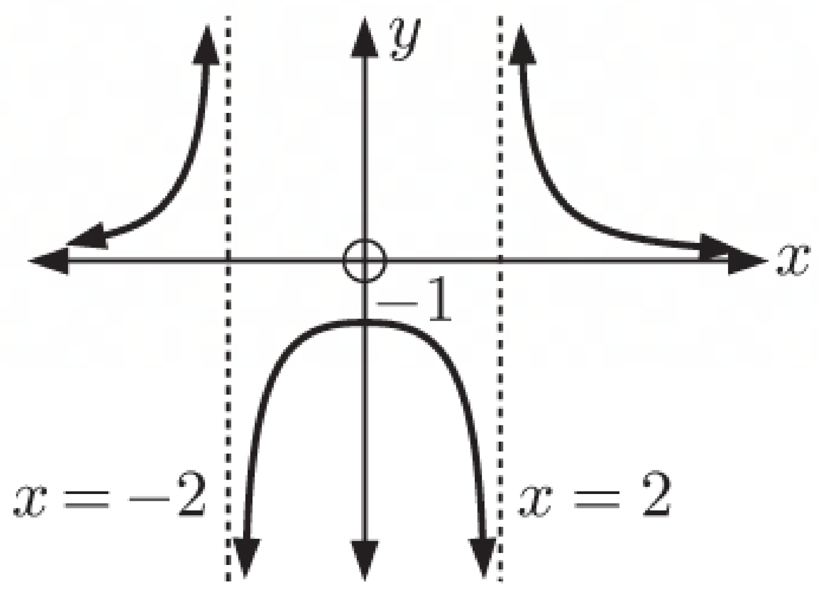
Domain is $\{x \mid x \in \mathbb{R}\}$
Range is $\{y \mid y \leqslant \frac{25}{4}\}$

e



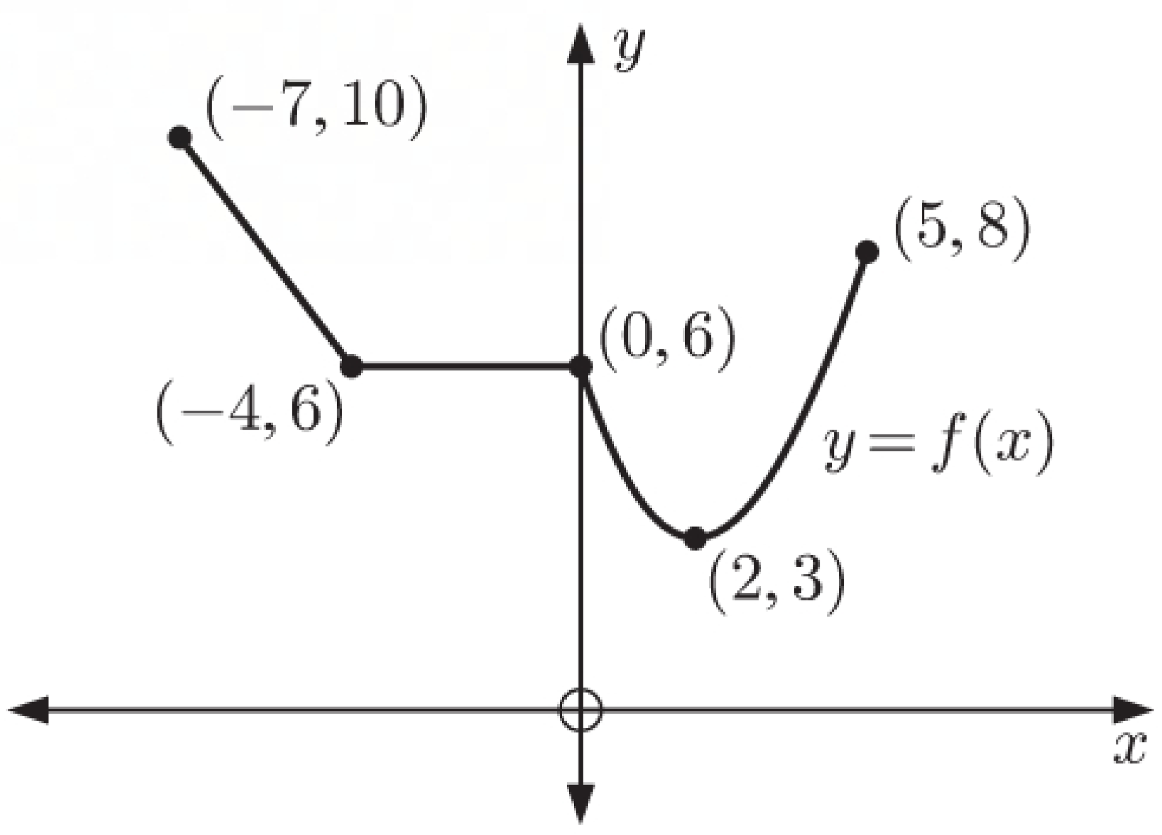
Domain is $\{x \mid x \geqslant -4\}$
Range is $\{y \mid y \geqslant -3\}$

f



Domain is $\{x \mid x \neq \pm 2\}$
Range is $\{y \mid y \leqslant -1 \text{ or } y > 0\}$

4



From the graph:
Domain is $\{x \mid -7 \leqslant x \leqslant 5\}$
Range is $\{y \mid 3 \leqslant y \leqslant 10\}$

a $x = -5$ satisfies $-7 \leqslant x \leqslant 5$.
$\therefore$ "-5 is in the domain of f" is true.

b $y = 2$ does not satisfy $3 \leqslant y \leqslant 10$.
$\therefore$ "2 is in the range of f" is false.

c $y = 9$ satisfies $3 \leqslant y \leqslant 10$.
$\therefore$ "9 is in the range of f" is true.

d $x = \sqrt{2} \approx 1.41$ satisfies $-7 \leqslant x \leqslant 5$.
$\therefore$ "$\sqrt{2}$ is in the domain of f" is true.

5 **a** $y = x^2$ has vertex $(0, 0)$ and shape  $(a > 0)$.

$\therefore$ the minimum y-value is 0 and there is no maximum y-value.
$\therefore$ the range is $\{y \mid y \geqslant 0\}$.

b $y = -x^2$ has vertex $(0,\ 0)$ and shape $(a < 0)$.

$\therefore$ the maximum y-value is 0 and there is no minimum y-value.

$\therefore$ the range is $\{y \mid y \leqslant 0\}$.

c $y = x^2 + 2$ has vertex $(0,\ 2)$ and shape $(a > 0)$.

$\therefore$ the minimum y-value is 2 and there is no maximum y-value.

$\therefore$ the range is $\{y \mid y \geqslant 2\}$.

d $y = -2(x+3)^2$ has vertex $(-3,\ 0)$ and shape $(a < 0)$.

$\therefore$ the maximum y-value is 0 and there is no minimum y-value.

$\therefore$ the range is $\{y \mid y \leqslant 0\}$.

e $y = 1 - (x-2)^2 = -(x-2)^2 + 1$ has vertex $(2,\ 1)$ and shape $(a < 0)$.

$\therefore$ the maximum y-value is 1 and there is no minimum y-value.

$\therefore$ the range is $\{y \mid y \leqslant 1\}$.

f $y = (2x+1)^2 + 3$ has vertex $(-\frac{1}{2},\ 3)$ and shape $(a > 0)$.

$\therefore$ the minimum y-value is 3 and there is no maximum y-value.

$\therefore$ the range is $\{y \mid y \geqslant 3\}$.

g $y = x^2 - 7x + 10$

$\therefore\ y = x^2 - 7x + \left(\frac{7}{2}\right)^2 + 10 - \left(\frac{7}{2}\right)^2$ {completing the square}

$\therefore\ y = \left(x - \frac{7}{2}\right)^2 - \frac{9}{4}$ which has vertex $\left(\frac{7}{2},\ -\frac{9}{4}\right)$ and shape $(a > 0)$.

$\therefore$ the minimum y-value is $-\frac{9}{4}$ and there is no maximum y-value.

$\therefore$ the range is $\{y \mid y \geqslant -\frac{9}{4}\}$.

h $y = -x^2 + 2x + 8$

$\therefore\ y = -(x^2 - 2x - 8)$

$\therefore\ y = -(x^2 - 2x + (-1)^2 - 8 - (-1)^2)$ {completing the square}

$\therefore\ y = -\left[(x-1)^2 - 9\right]$

$\therefore\ y = -(x-1)^2 + 9$ which has vertex $(1,\ 9)$ and shape $(a < 0)$.

$\therefore$ the maximum y-value is 9 and there is no minimum y-value.

$\therefore$ the range is $\{y \mid y \leqslant 9\}$.

i $y = 5x - 3x^2$

$\therefore\ y = -3\left(x^2 - \frac{5}{3}x\right)$

$\therefore\ y = -3\left(x^2 - \frac{5}{3}x + \left(-\frac{5}{6}\right)^2 - \left(-\frac{5}{6}\right)^2\right)$ {completing the square}

$\therefore\ y = -3\left[\left(x - \frac{5}{6}\right)^2 - \frac{25}{36}\right]$

$\therefore\ y = -3\left(x - \frac{5}{6}\right)^2 + \frac{25}{12}$ which has vertex $\left(\frac{5}{6},\ \frac{25}{12}\right)$ and shape $(a < 0)$.

$\therefore$ the maximum y-value is $\frac{25}{12}$ and there is no minimum y-value.

$\therefore$ the range is $\{y \mid y \leqslant \frac{25}{12}\}$.

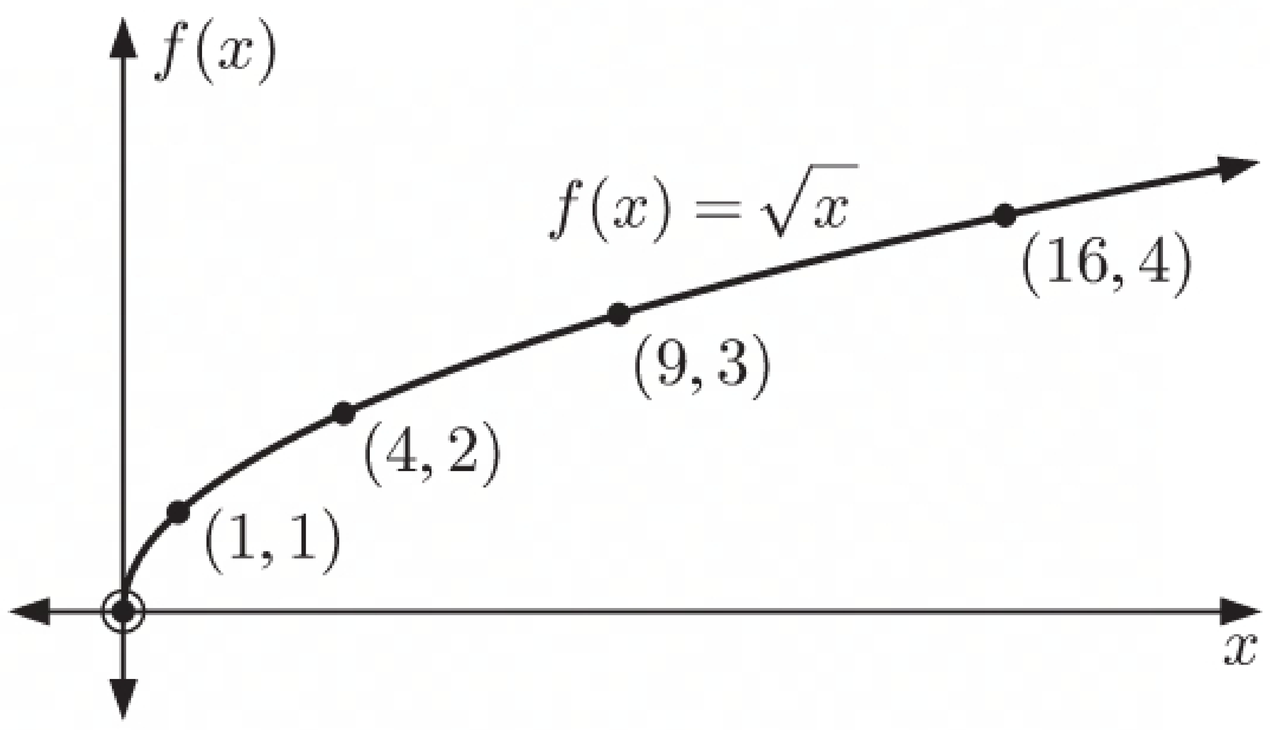
6 $f(x) = \sqrt{x}$

a $\sqrt{x}$ is defined when $x \geqslant 0$.
$\therefore$ the domain is $\{x \mid x \geqslant 0\}$.

b

x	0	1	4	9	16
$f(x)$	0	1	2	3	4

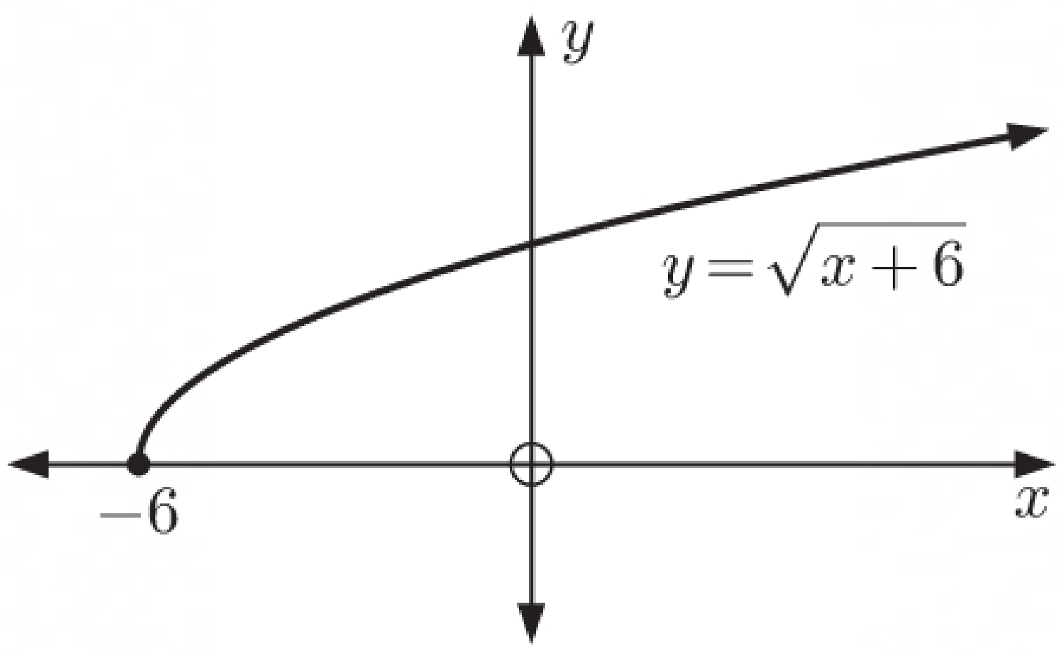
c



d A square root cannot be negative.
$\therefore$ the range is $\{y \mid y \geqslant 0\}$.

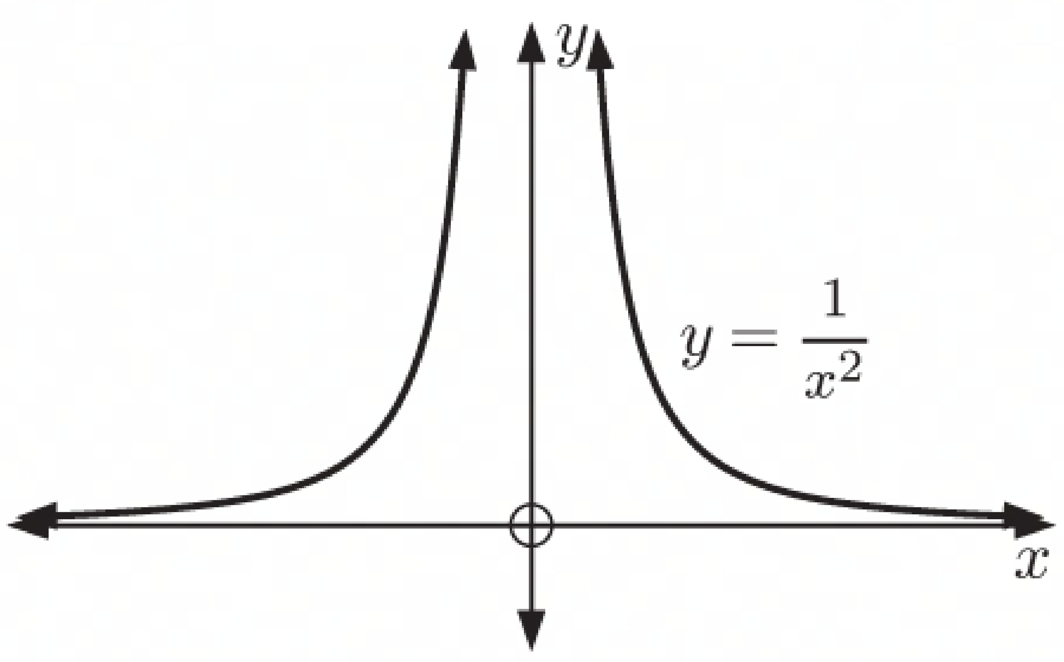
7 **a** $\sqrt{x+6}$ is defined when $x + 6 \geqslant 0$
$\therefore\ x \geqslant -6$

$\therefore$ the domain is $\{x \mid x \geqslant -6\}$.
A square root cannot be negative.
$\therefore$ the range is $\{y \mid y \geqslant 0\}$.



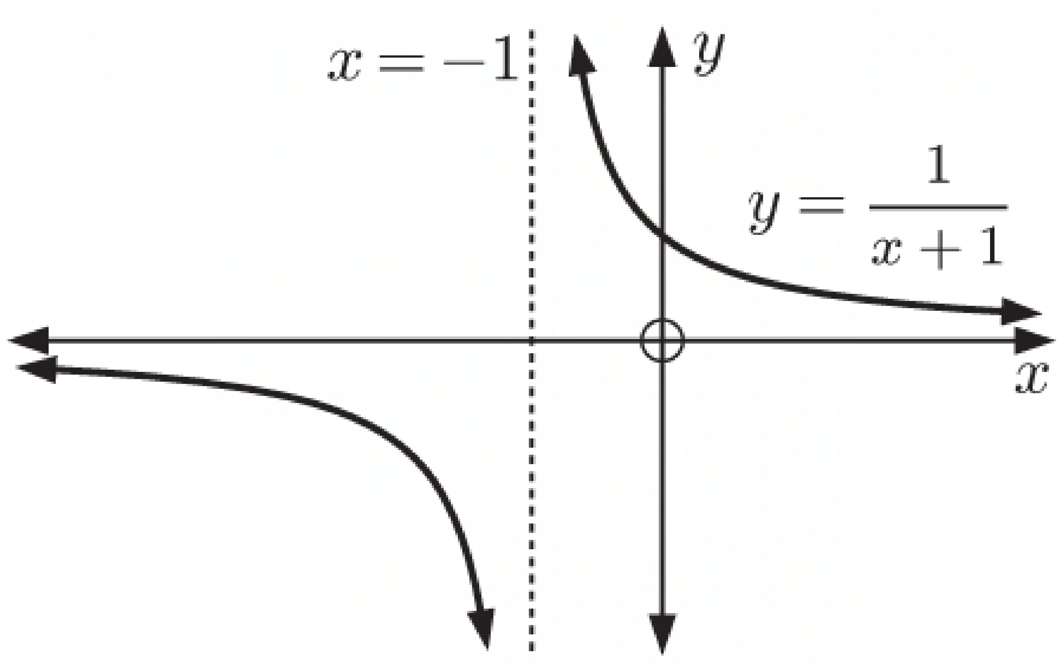
b $\dfrac{1}{x^2}$ is defined when $x^2 \neq 0$
$\therefore\ x \neq 0$

$\therefore$ the domain is $\{x \mid x \neq 0\}$.
$y = f(x)$ is always positive and never zero.
$\therefore$ the range is $\{y \mid y > 0\}$.



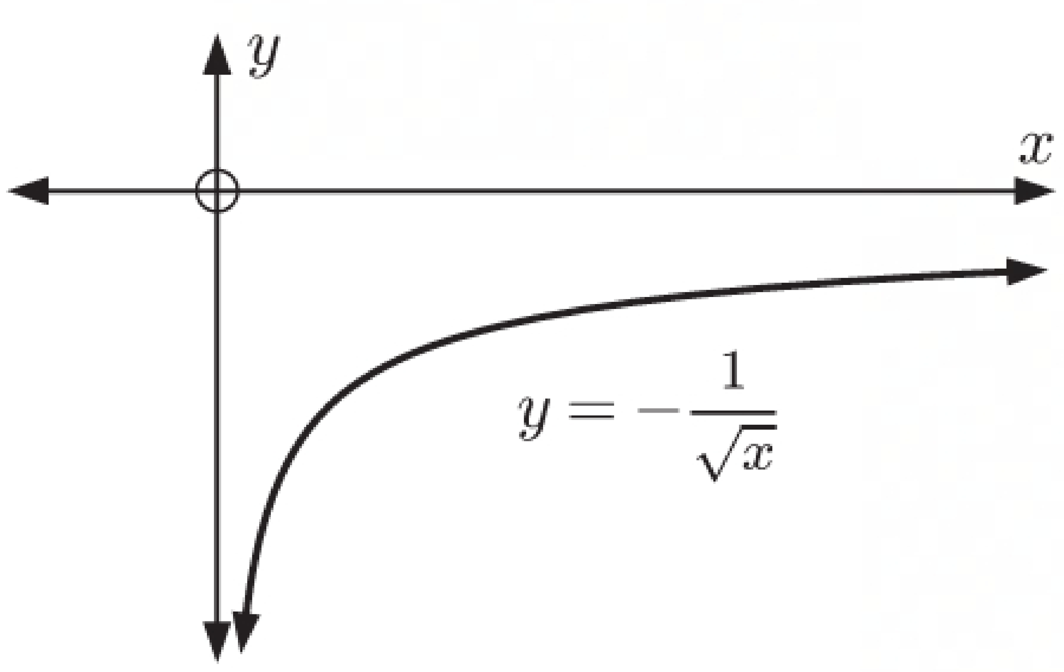
c $\dfrac{1}{x+1}$ is defined when $x + 1 \neq 0$
$\therefore\ x \neq -1$

$\therefore$ the domain is $\{x \mid x \neq -1\}$.
No matter how large or small x is,
$y = f(x)$ is never zero.
$\therefore$ the range is $\{y \mid y \neq 0\}$.



d $-\dfrac{1}{\sqrt{x}}$ is defined when $x > 0$

$\therefore$ the domain is $\{x \mid x > 0\}$.
$y = f(x)$ is always negative and never zero.
$\therefore$ the range is $\{y \mid y < 0\}$.



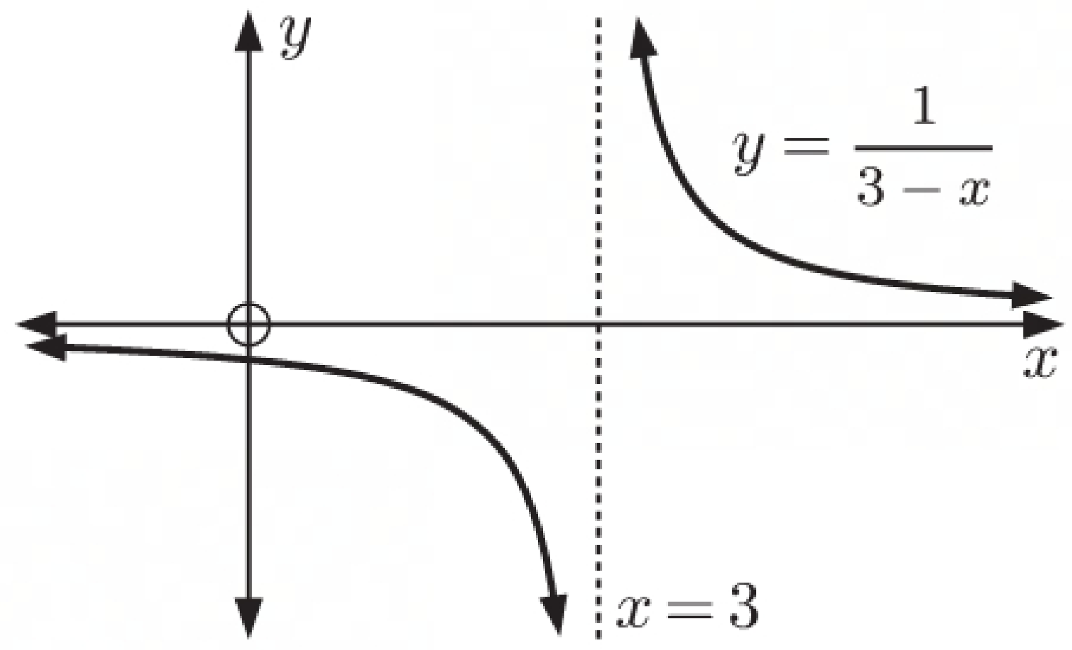
e $\dfrac{1}{3-x}$ is defined when $3 - x \neq 0$

$\therefore \quad x \neq 3$

$\therefore$ the domain is $\{x \mid x \neq 3\}$.

No matter how large or small x is, $y = f(x)$ is never zero.

$\therefore$ the range is $\{y \mid y \neq 0\}$.



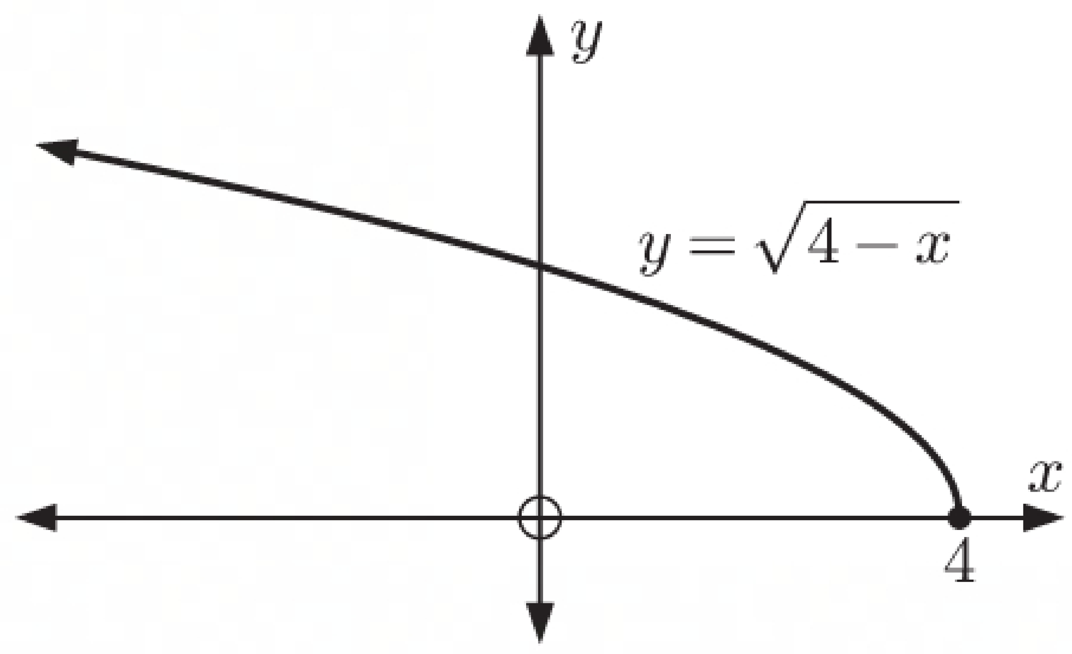
f $\sqrt{4-x}$ is defined when $4 - x \geqslant 0$

$\therefore \quad x \leqslant 4$

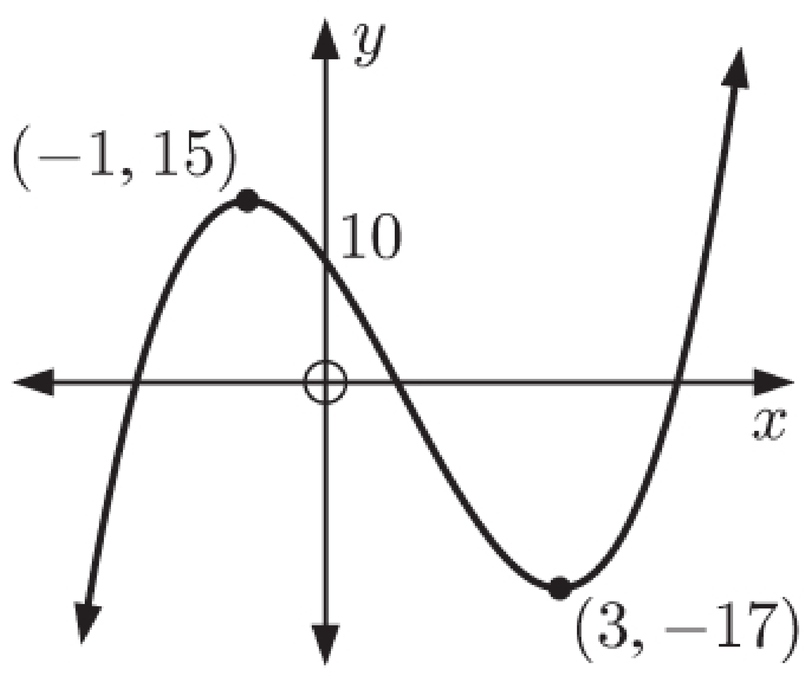
$\therefore$ the domain is $\{x \mid x \leqslant 4\}$.

A square root cannot be negative.

$\therefore$ the range is $\{y \mid y \geqslant 0\}$.



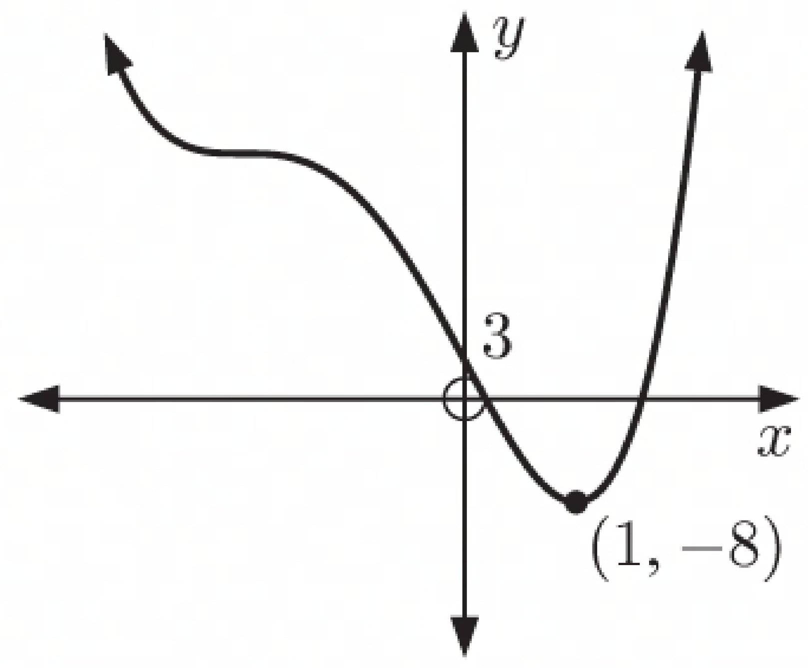
8 **a**



$f(x) = x^3 - 3x^2 - 9x + 10$

The domain is $\{x \mid x \in \mathbb{R}\}$.
The range is $\{y \mid y \in \mathbb{R}\}$.

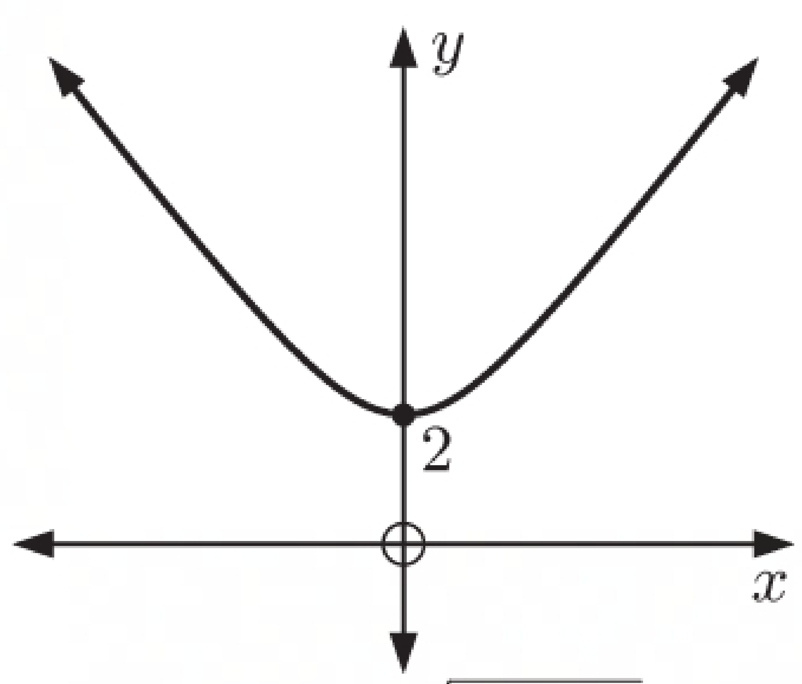
b



$f(x) = x^4 + 4x^3 - 16x + 3$

The domain is $\{x \mid x \in \mathbb{R}\}$.
The range is $\{y \mid y \geqslant -8\}$.

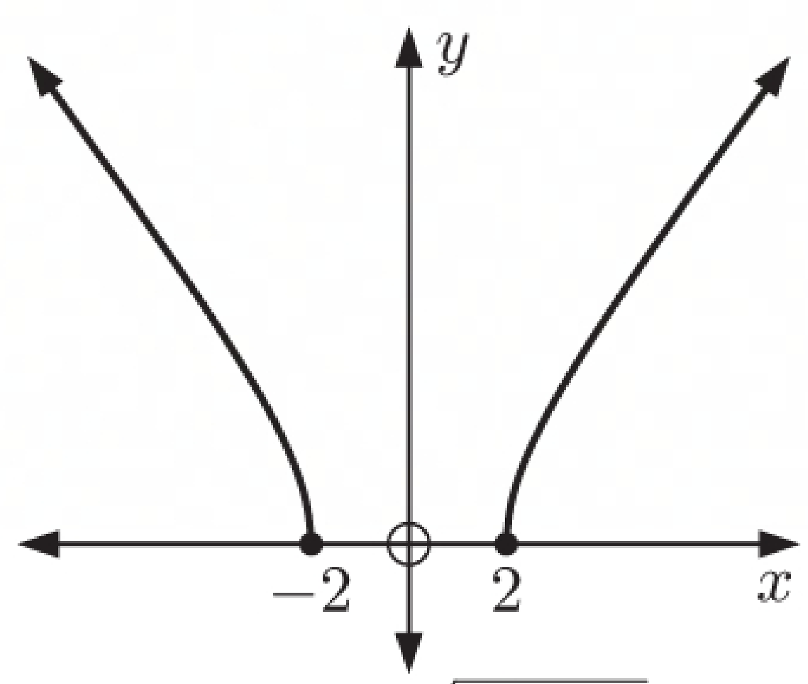
c



$f(x) = \sqrt{x^2 + 4}$

The domain is $\{x \mid x \in \mathbb{R}\}$.
The range is $\{y \mid y \geqslant 2\}$.

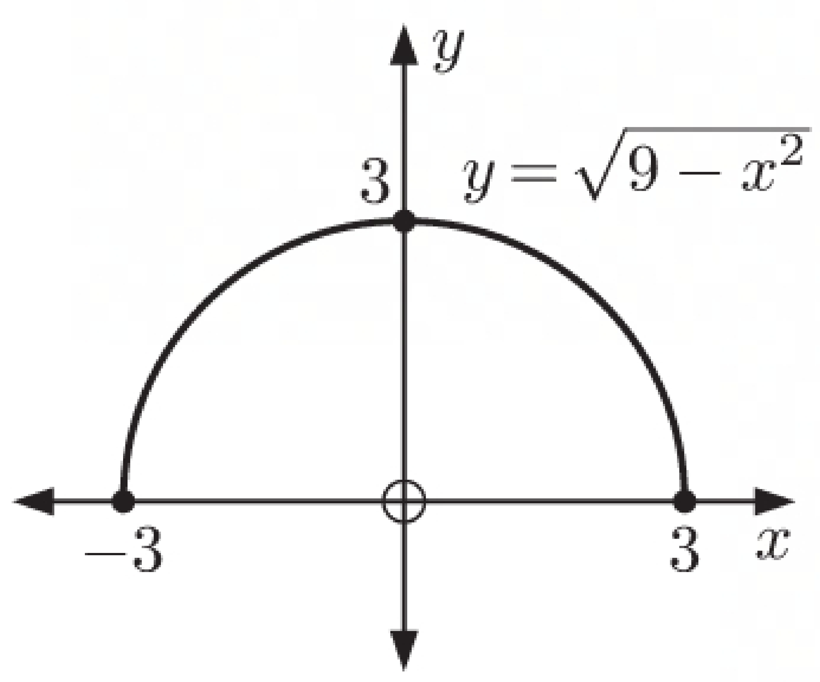
d



$f(x) = \sqrt{x^2 - 4}$

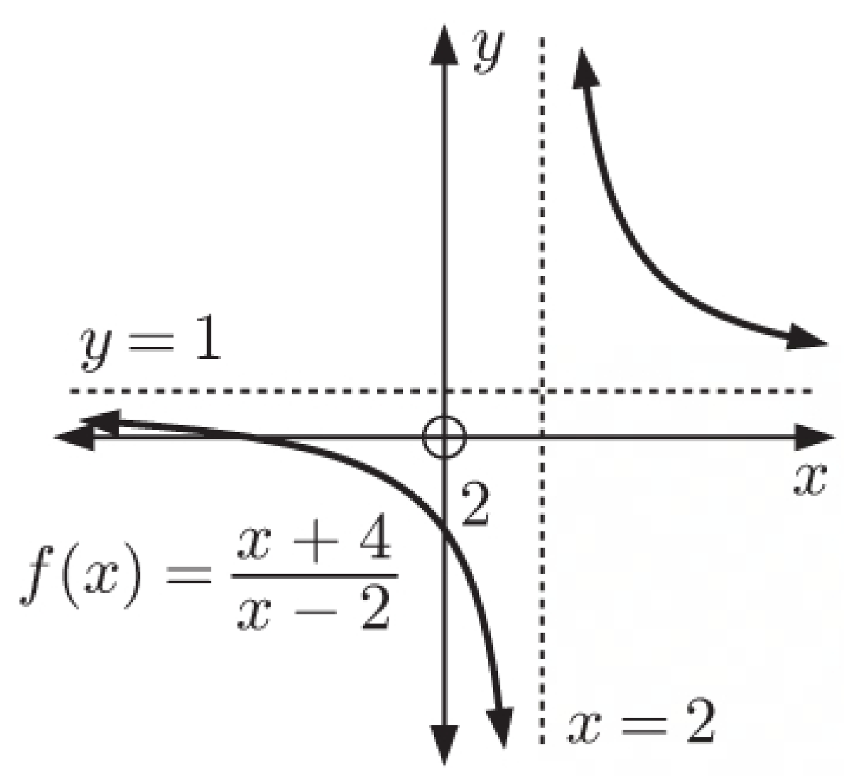
The domain is $\{x \mid x \leqslant -2 \text{ or } x \geqslant 2\}$.
The range is $\{y \mid y \geqslant 0\}$.

e



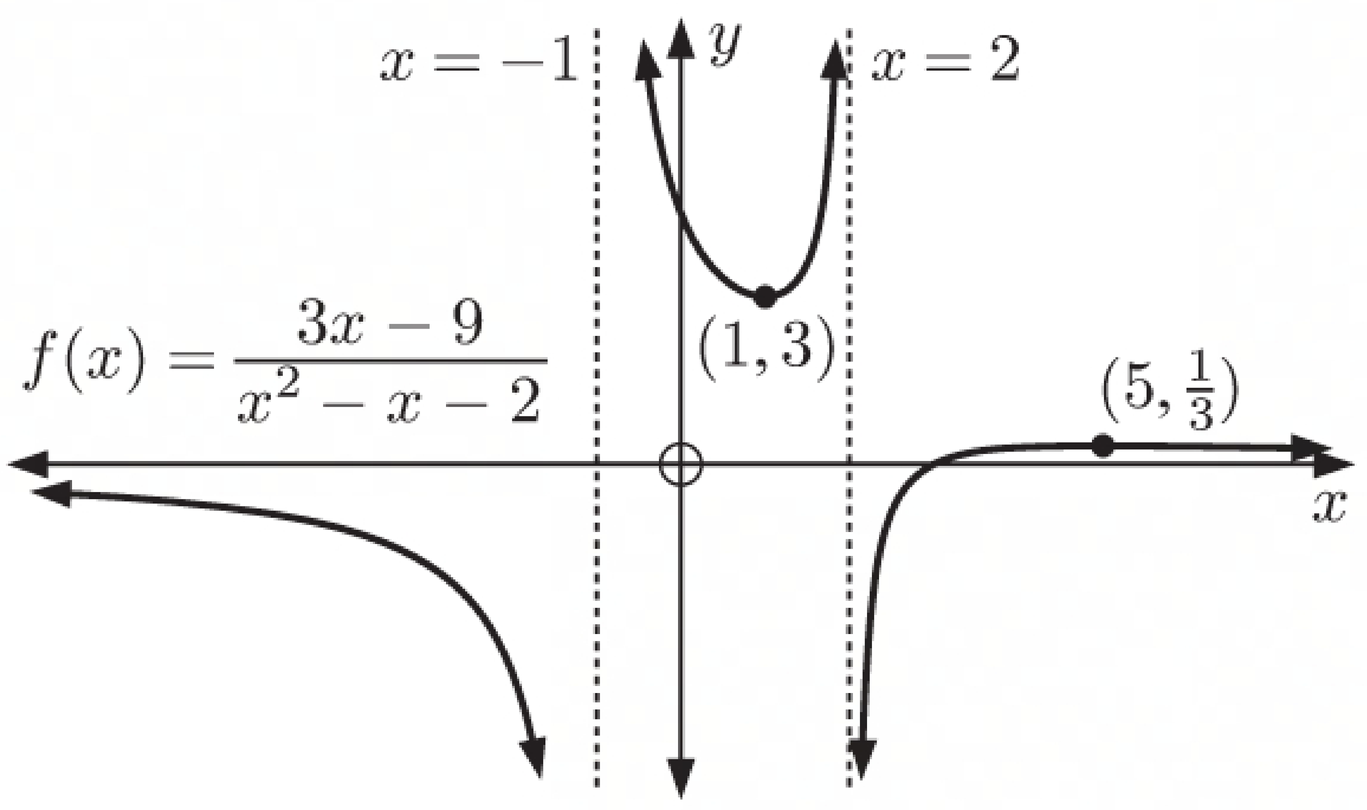
The domain is $\{x \mid -3 \leqslant x \leqslant 3\}$.
The range is $\{y \mid 0 \leqslant y \leqslant 3\}$.

f



The domain is $\{x \mid x \neq 2\}$.
The range is $\{y \mid y \neq 1\}$.

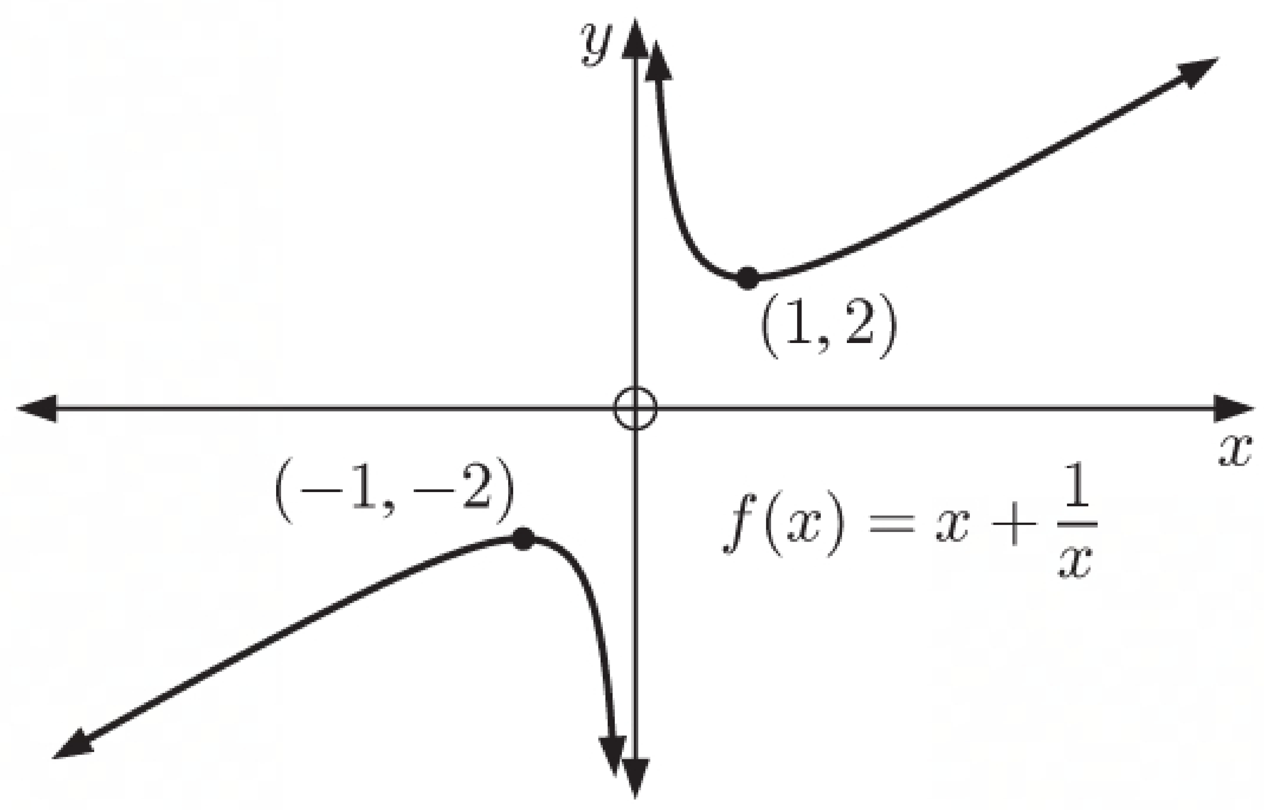
g



The domain is $\{x \mid x \neq -1 \text{ or } 2\}$.

The range is $\{y \mid y \leqslant \frac{1}{3} \text{ or } y \geqslant 3\}$.

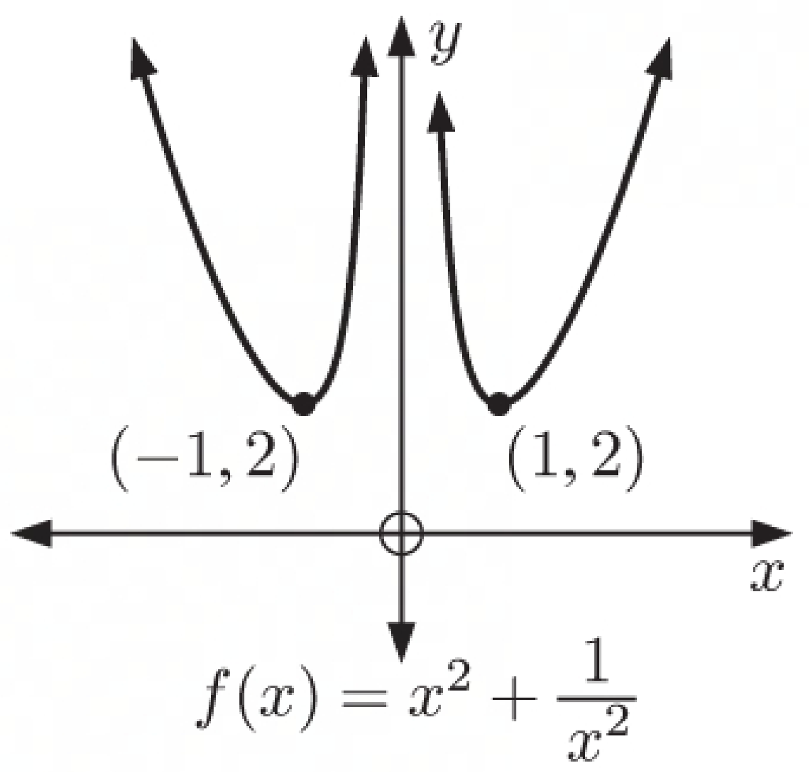
h



The domain is $\{x \mid x \neq 0\}$.

The range is $\{y \mid y \leqslant -2 \text{ or } y \geqslant 2\}$.

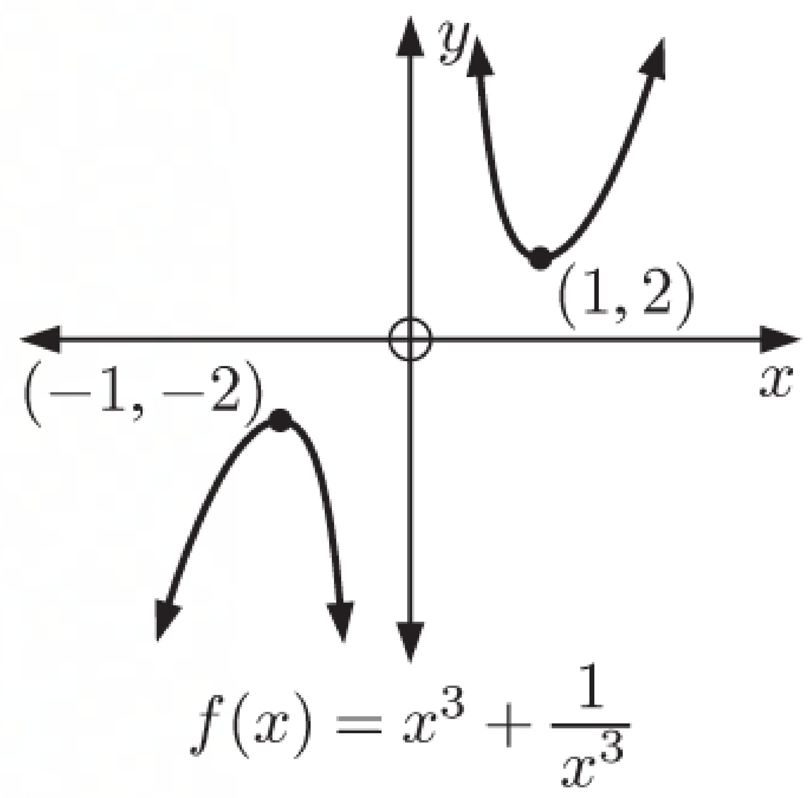
i



The domain is $\{x \mid x \neq 0\}$.

The range is $\{y \mid y \geqslant 2\}$.

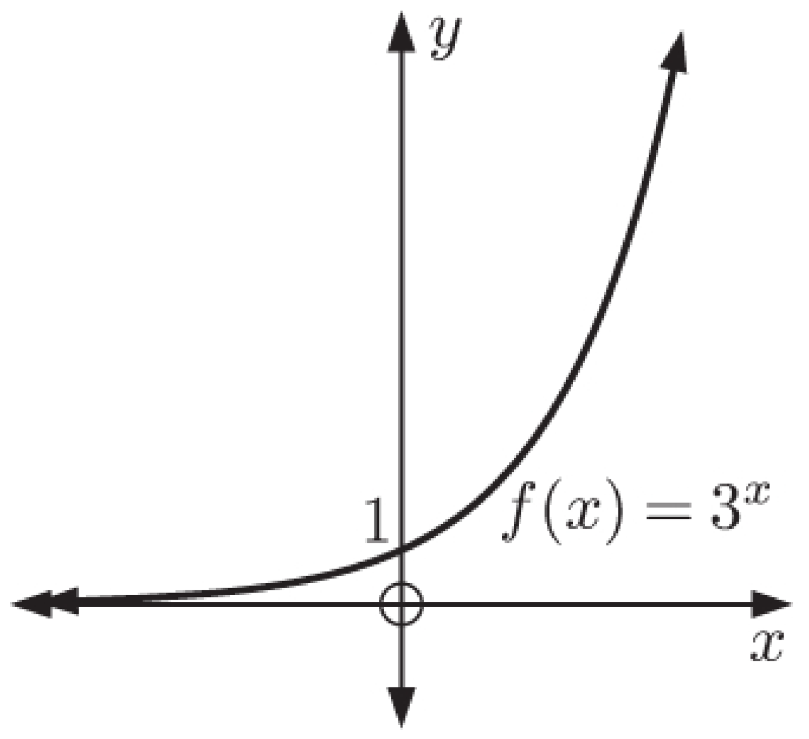
j



The domain is $\{x \mid x \neq 0\}$.

The range is $\{y \mid y \leqslant -2 \text{ or } y \geqslant 2\}$.

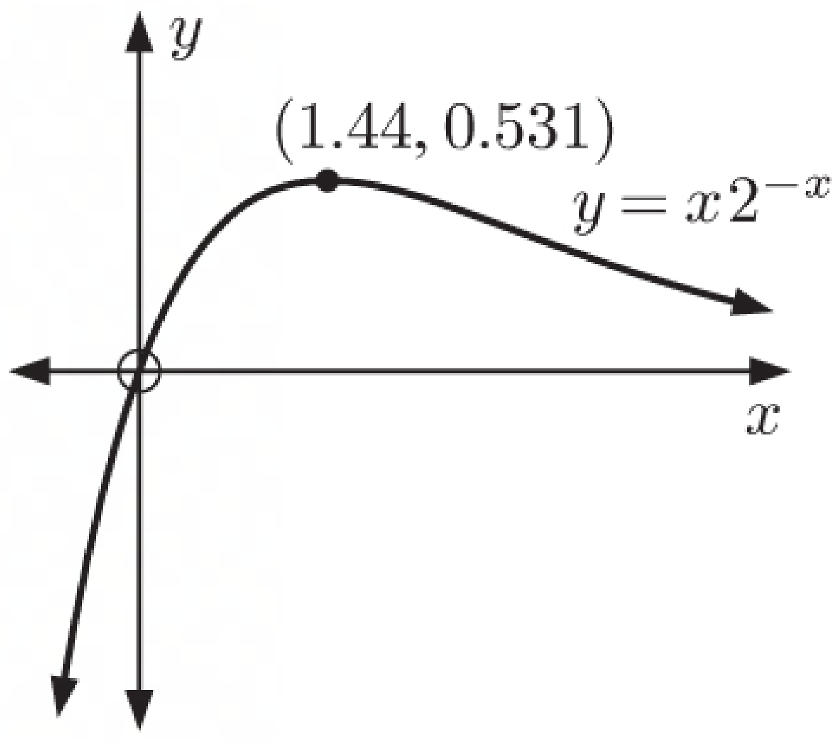
k



The domain is $\{x \mid x \in \mathbb{R}\}$.

The range is $\{y \mid y > 0\}$.

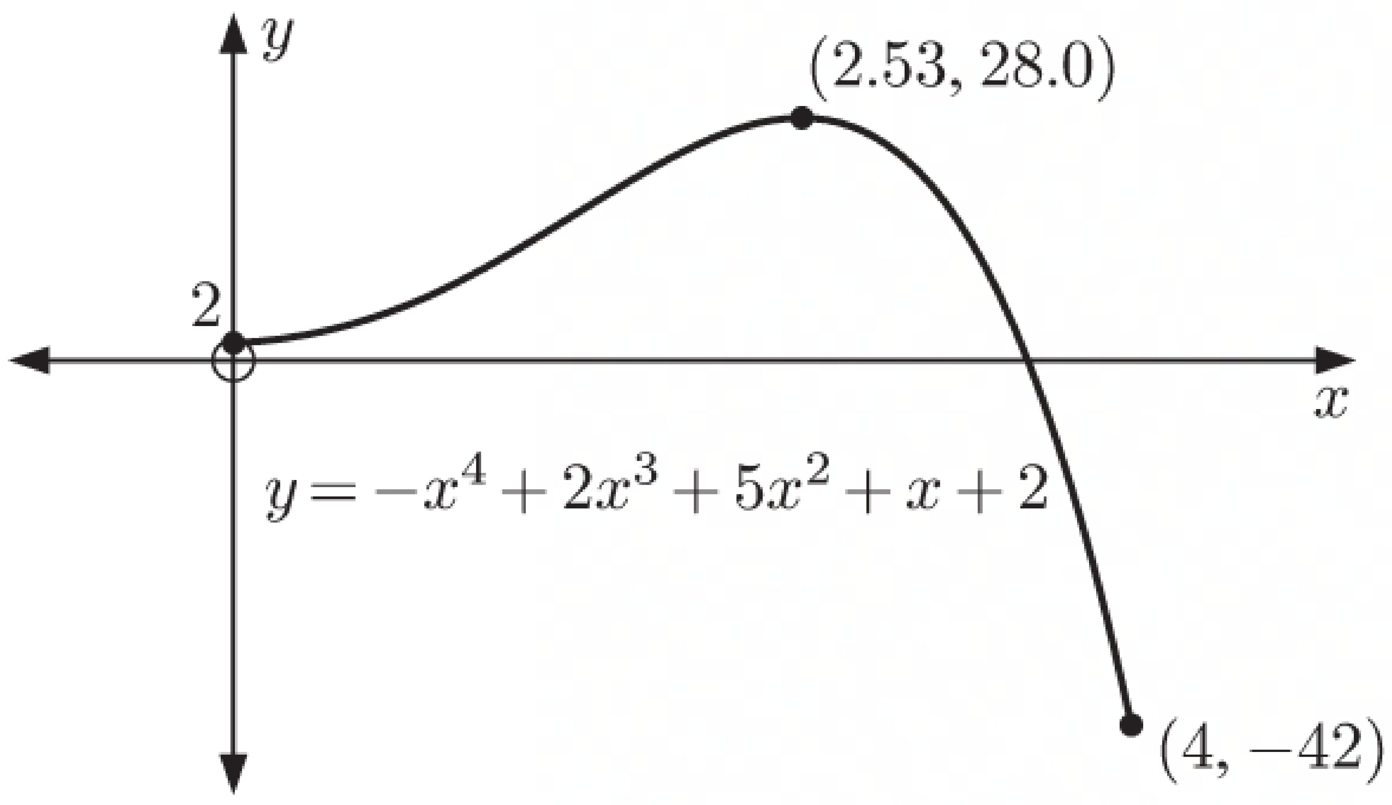
l



The domain is $\{x \mid x \in \mathbb{R}\}$.

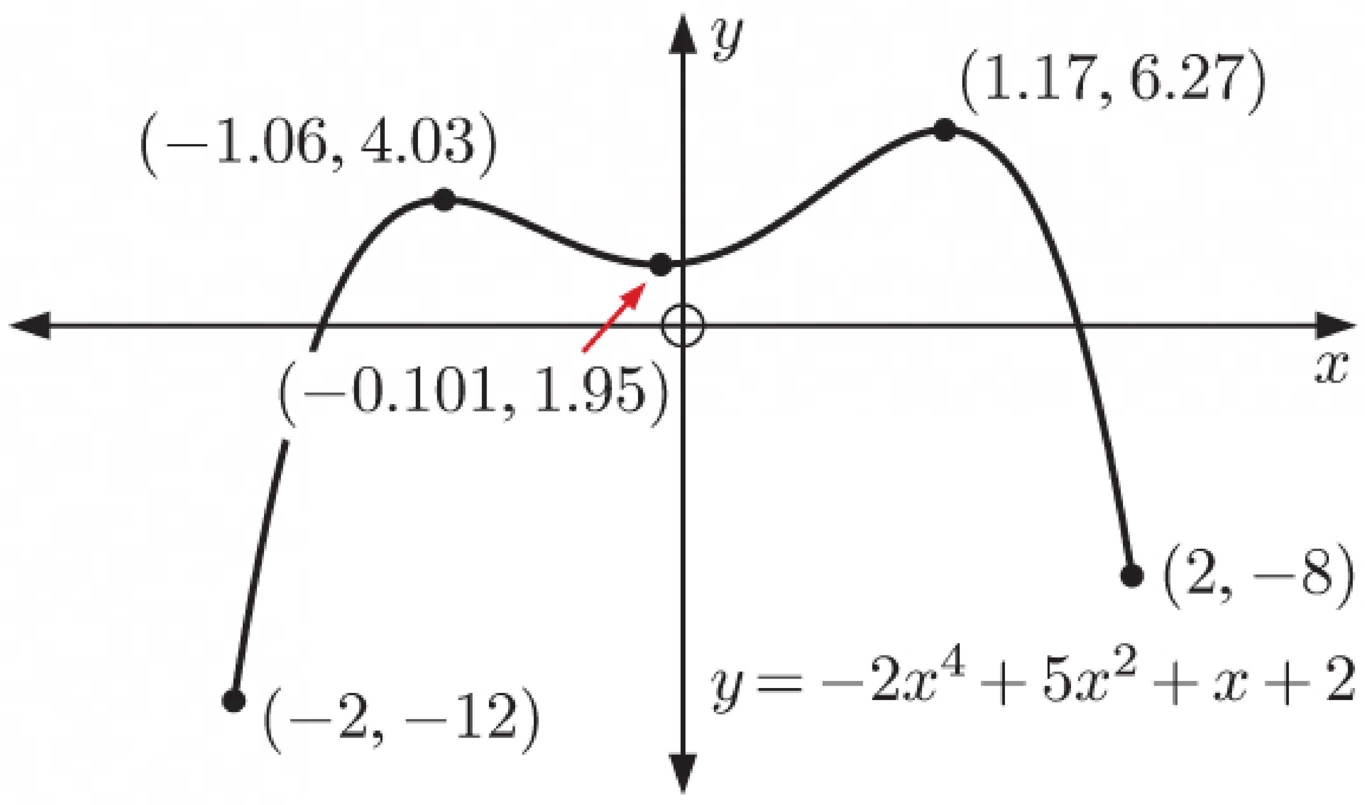
The range is $\{y \mid y \leqslant 0.531\}$.

9 **a**



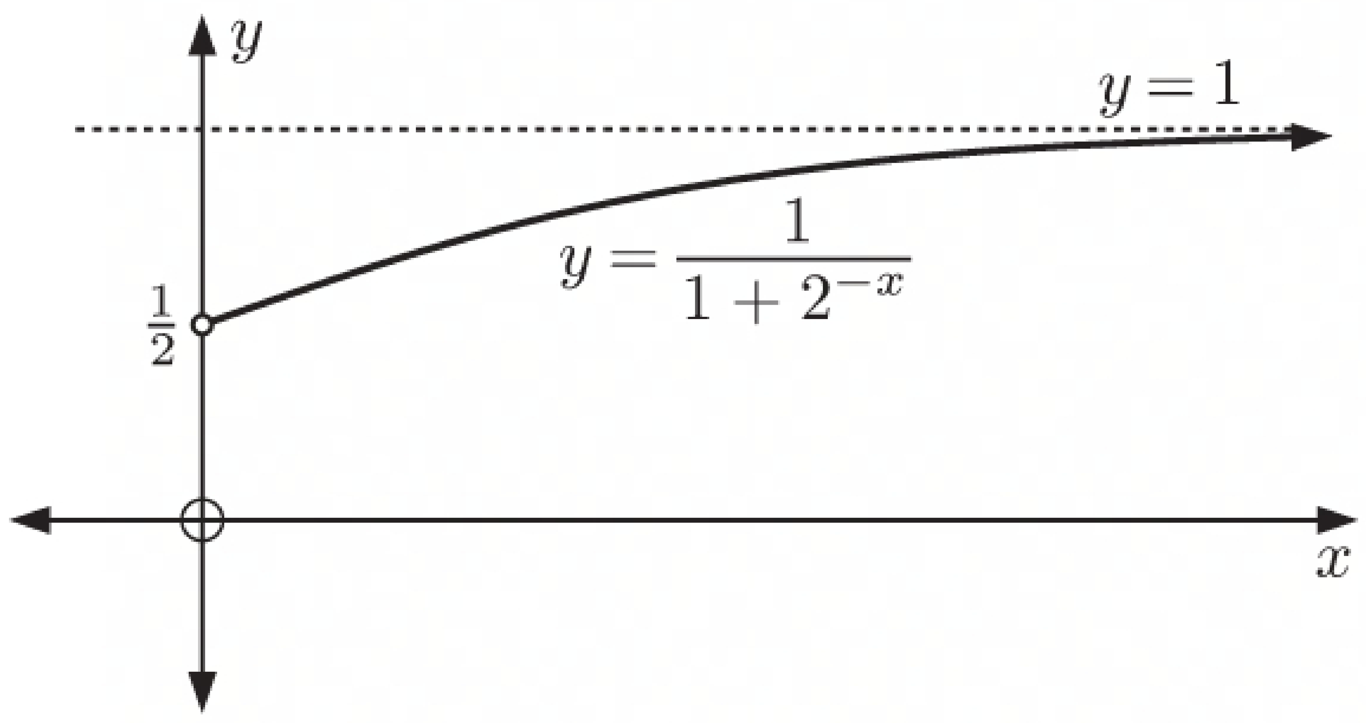
The range is $\{y \mid -42 \leqslant y \leqslant 28.0\}$.

b



The range is $\{y \mid -12 \leqslant y \leqslant 6.27\}$.

c



The range is $\{y \mid \frac{1}{2} < y < 1\}$.

10 $f(x) = \sqrt{x^2 + 5x + k}$

a $\sqrt{x^2 + 5x + k}$ is defined when $x^2 + 5x + k \geqslant 0$

$$\therefore \quad x^2 + 5x + \left(\tfrac{5}{2}\right)^2 + k - \left(\tfrac{5}{2}\right)^2 \geqslant 0$$

$$\therefore \quad \left(x + \tfrac{5}{2}\right)^2 + k - \tfrac{25}{4} \geqslant 0$$

which is true for all $x \in \mathbb{R}$ when $k - \frac{25}{4} \geqslant 0$ $\quad$ {as $\left(x + \frac{5}{2}\right)^2 \geqslant 0$ for all $x \in \mathbb{R}$}

$$\therefore \quad k \geqslant \tfrac{25}{4}$$

$\therefore$ $f(x)$ has natural domain $x \in \mathbb{R}$ when $k \geqslant \frac{25}{4}$.

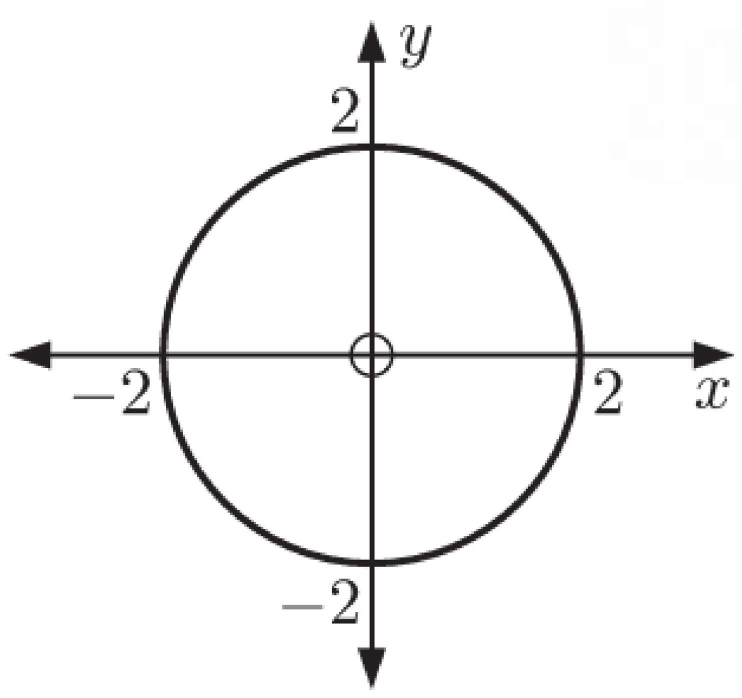
b $y = \left(x + \frac{5}{2}\right)^2 + k - \frac{25}{4}$ is a quadratic with vertex $\left(-\frac{5}{2},\ k - \frac{25}{4}\right)$ and shape  $(a > 0)$.

$\therefore$ the minimum y-value is $k - \frac{25}{4}$ and there is no maximum y-value.

So, $f(x) = \sqrt{\left(x + \frac{5}{2}\right)^2 + k - \frac{25}{4}}$ has minimum value $\sqrt{k - \frac{25}{4}}$ and no maximum value.

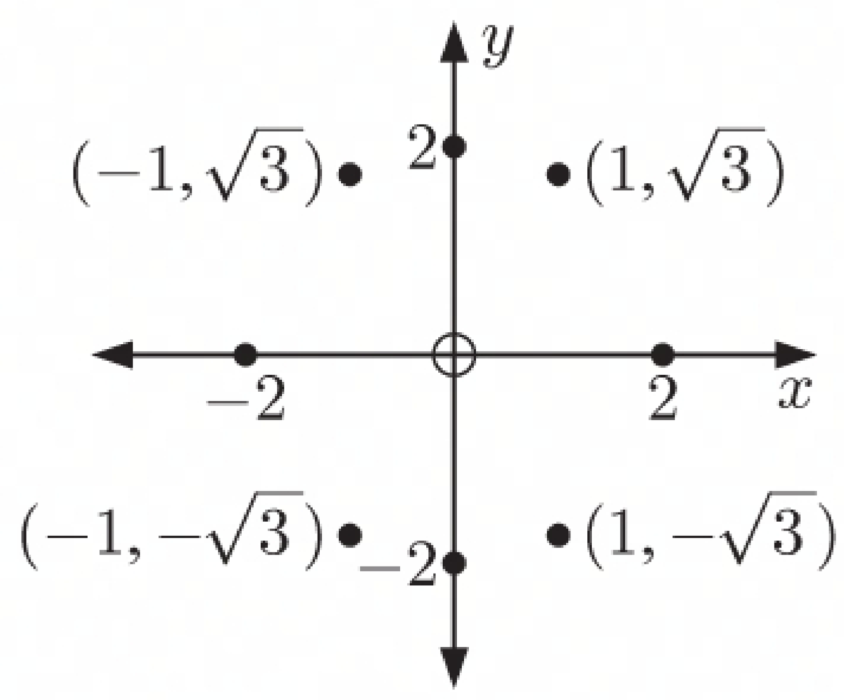
$\therefore$ the range is $\left\{y \mid y \geqslant \sqrt{k - \frac{25}{4}}\right\}$.

11 **a**



The domain is $\{x \mid -2 \leqslant x \leqslant 2\}$.
The range is $\{y \mid -2 \leqslant y \leqslant 2\}$.

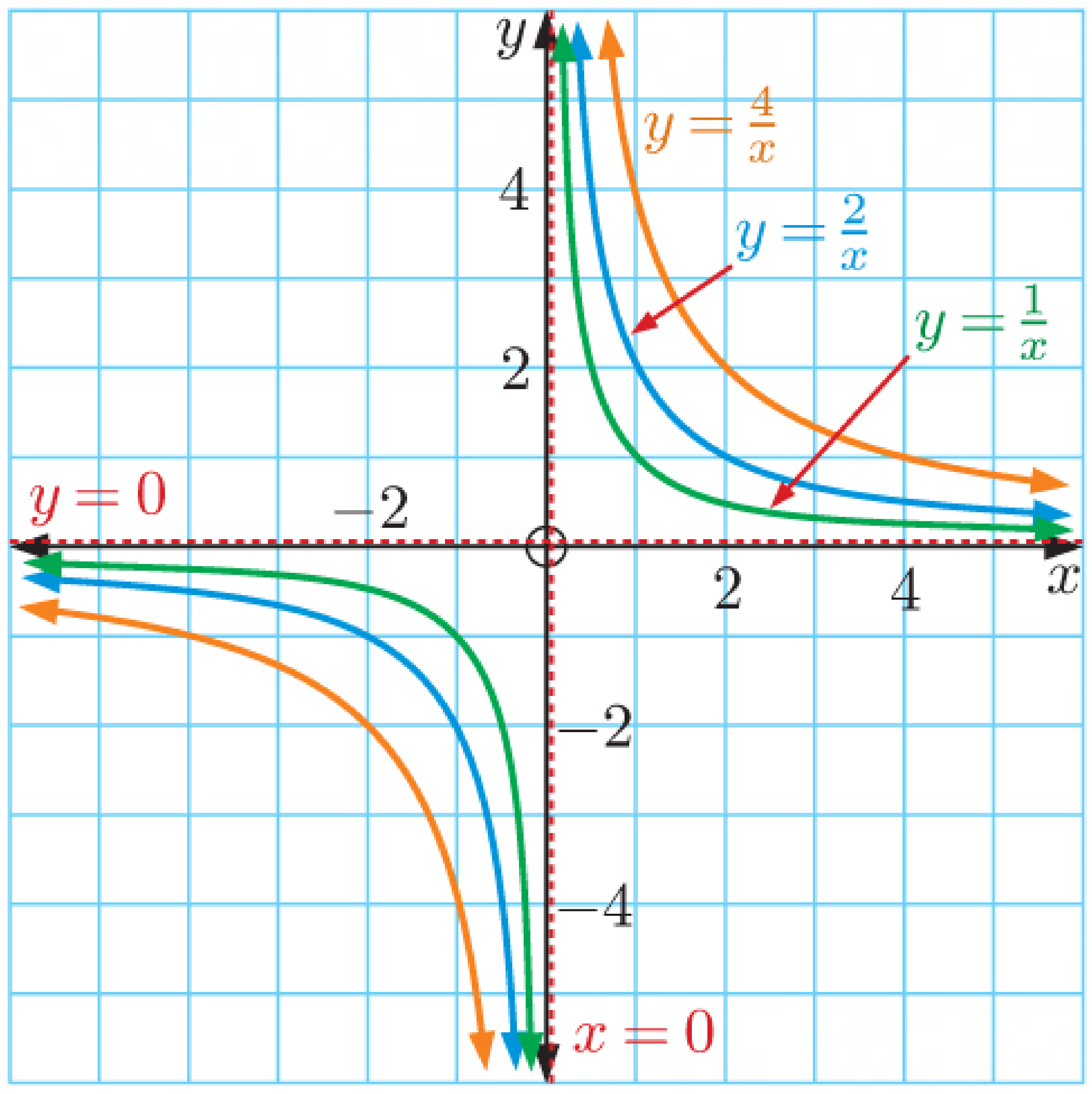
b



The domain is $\{-2,\ -1,\ 0,\ 1,\ 2\}$.
The range is $\{-2,\ -\sqrt{3},\ 0,\ \sqrt{3},\ 2\}$.

EXERCISE 15D.1

1 **a**

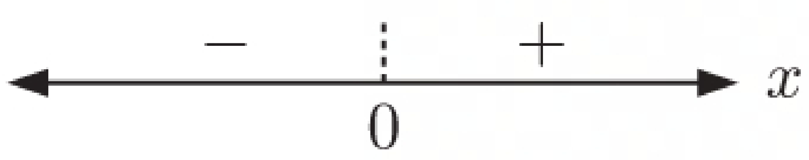


b $y = \dfrac{k}{x}, \quad k > 0$

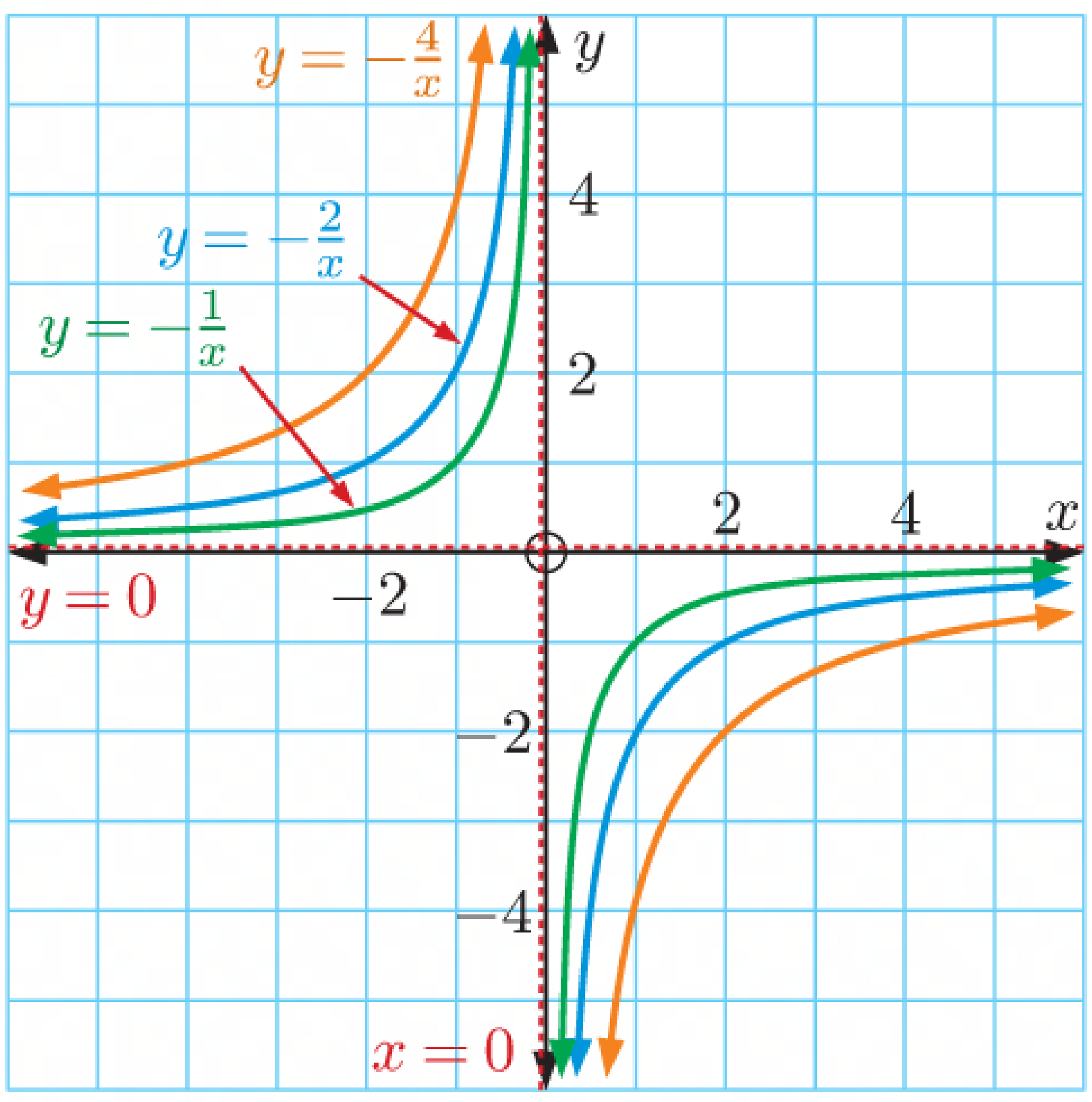
i As k becomes larger the graphs move further from the origin.

ii The graph lies in quadrants 1 and 3.

iii



2 **a**

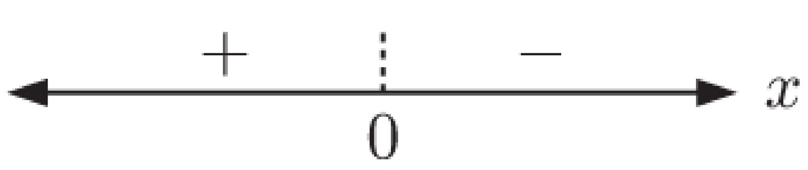


b $y = \dfrac{k}{x}, \quad k < 0$

i As $|\,k\,|$ becomes larger, the graphs move further from the origin.

ii The graph lies in quadrants 2 and 4.

iii



3 $y = \frac{k}{x}, \ k \neq 0$

a $\frac{k}{x}$ is defined when $x \neq 0$.

$\therefore$ the domain is $\{x \mid x \neq 0\}$.

b No matter how large or small x is, $y = \frac{k}{x}$ is never zero.

$\therefore$ the range is $\{y \mid y \neq 0\}$.

c For $k > 0$, as $x \to 0^-$, $\frac{k}{x} \to -\infty$

and as $x \to 0^+$, $\frac{k}{x} \to \infty$.

For $k < 0$, as $x \to 0^-$, $\frac{k}{x} \to \infty$

and as $x \to 0^+$, $\frac{k}{x} \to -\infty$.

$\therefore$ the vertical asymptote is $x = 0$.

d For $k > 0$, as $x \to \infty$, $\frac{k}{x} \to 0^+$

and as $x \to -\infty$, $\frac{k}{x} \to 0^-$.

For $k < 0$, as $x \to \infty$, $\frac{k}{x} \to 0^-$

and as $x \to -\infty$, $\frac{k}{x} \to 0^+$.

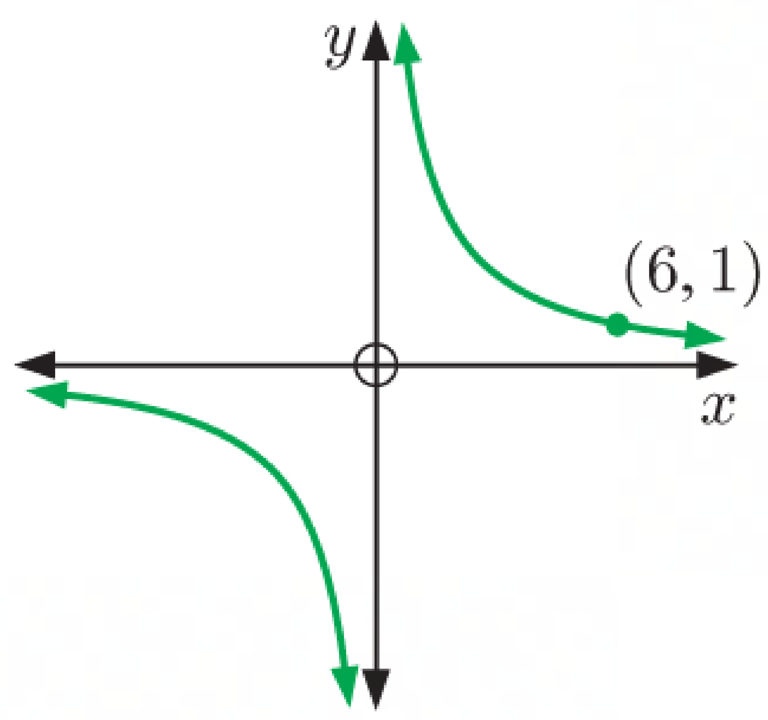
$\therefore$ the horizontal asymptote is $y = 0$.

4 **a** Let the function be $y = \frac{k}{x}$.

When $x = 6$, $y = 1$, so $1 = \frac{k}{6}$

$\therefore k = 6$

So, the function is $y = \frac{6}{x}$.

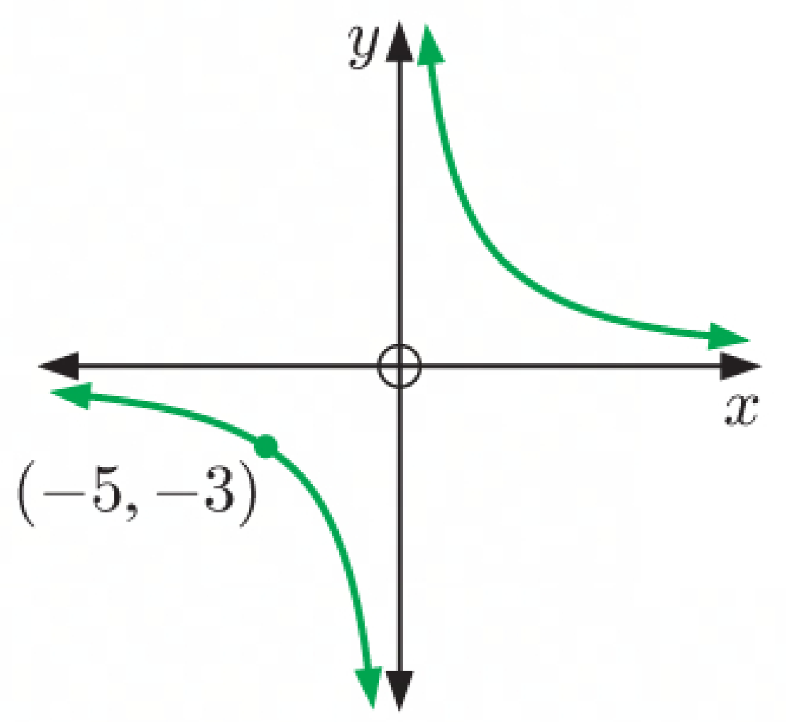


b Let the function be $y = \frac{k}{x}$.

When $x = -5$, $y = -3$, so $-3 = \frac{k}{-5}$

$\therefore k = 15$

So, the function is $y = \frac{15}{x}$.

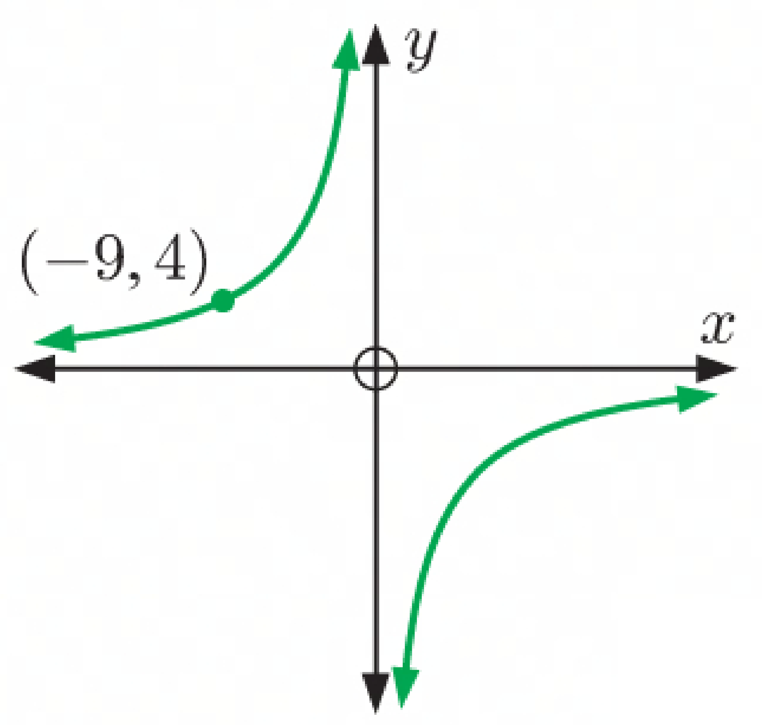


c Let the function be $y = \frac{k}{x}$.

When $x = -9$, $y = 4$, so $4 = \frac{k}{-9}$

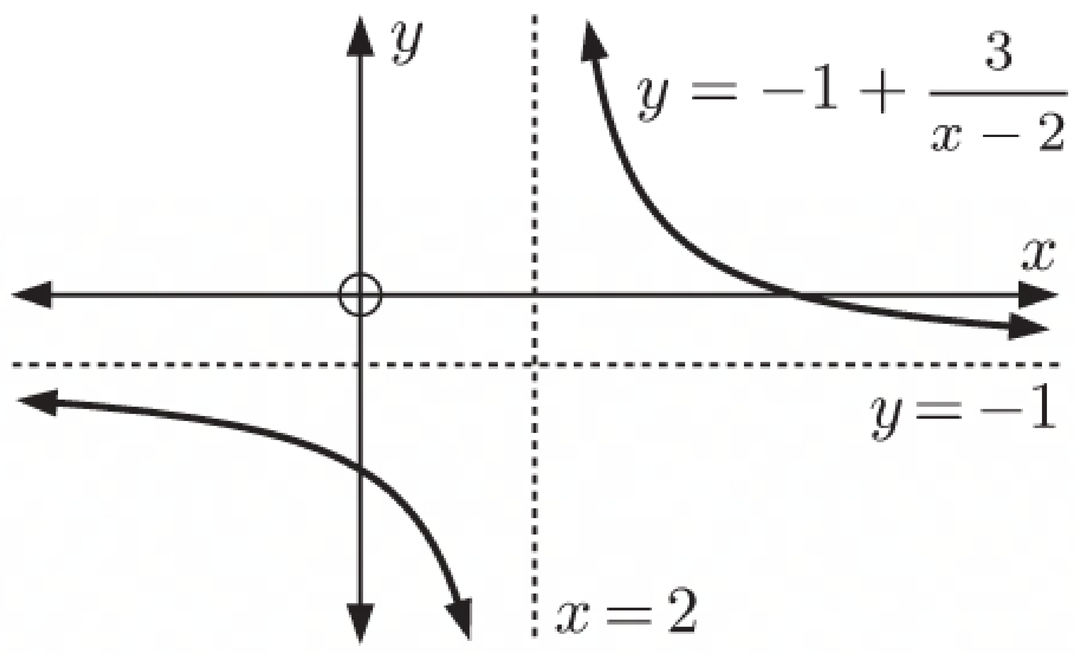
$\therefore k = -36$

So, the function is $y = -\frac{36}{x}$.



INVESTIGATION RATIONAL FUNCTIONS

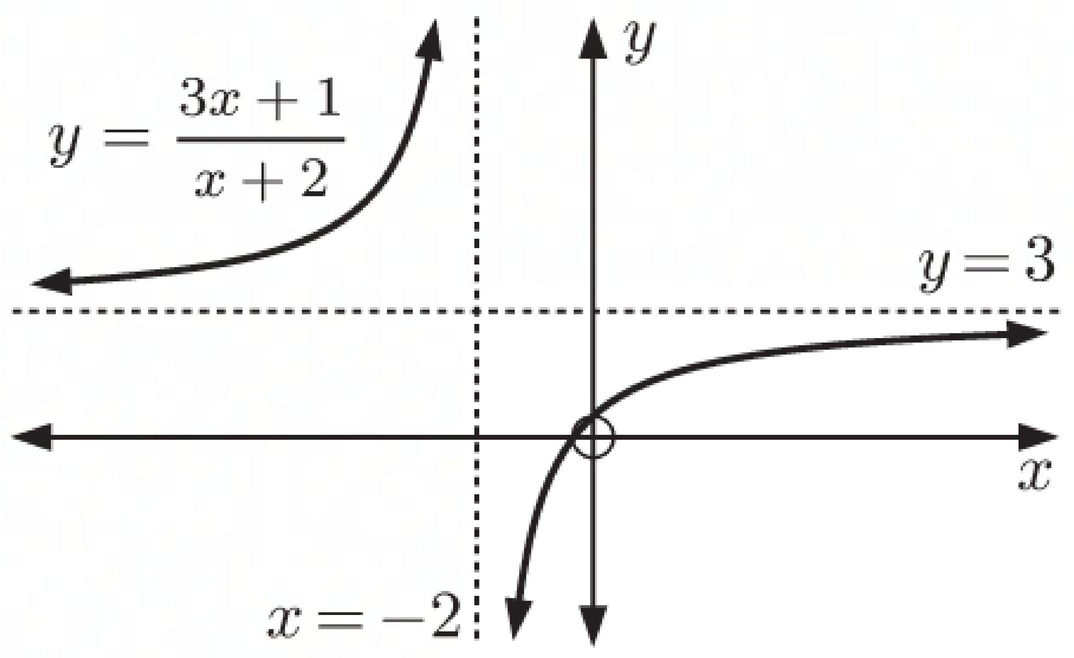
1 **a**



i The domain is $\{x \mid x \neq 2\}$.

ii The vertical asymptote is $x = 2$.
The horizontal asymptote is $y = -1$.

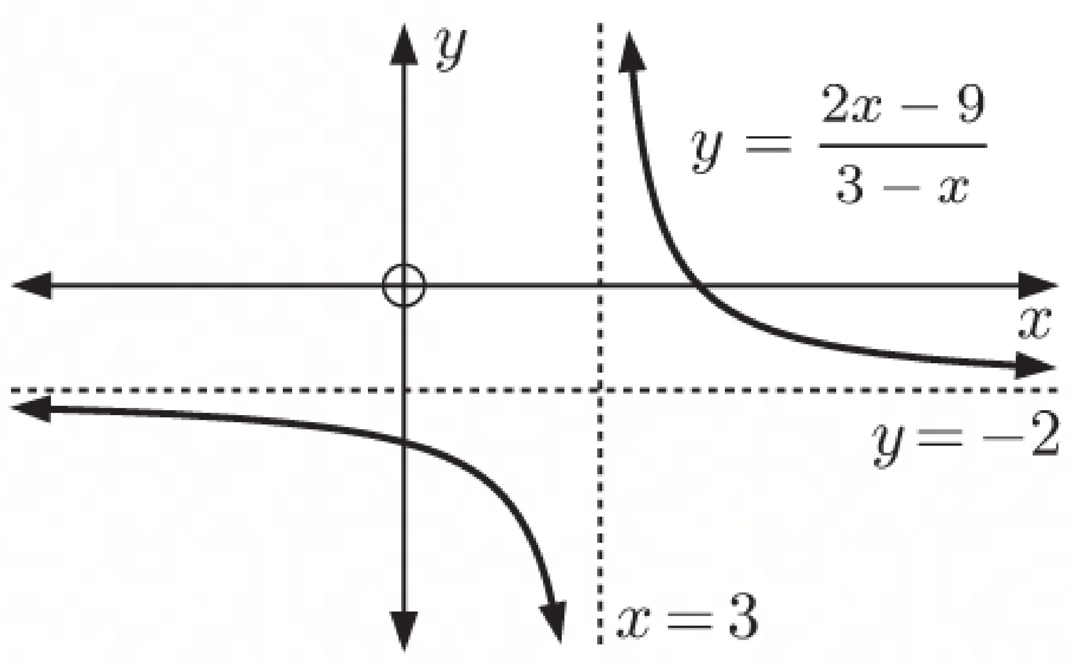
b



i The domain is $\{x \mid x \neq -2\}$.

ii The vertical asymptote is $x = -2$.
The horizontal asymptote is $y = 3$.

c



i The domain is $\{x \mid x \neq 3\}$.

ii The vertical asymptote is $x = 3$.
The horizontal asymptote is $y = -2$.

2 $y = \dfrac{b}{cx+d} + a$ where $b, c \neq 0$

a The horizontal asymptote is $y = a$.

b The vertical asymptote is $cx + d = 0$

$$\therefore \quad x = -\frac{d}{c}$$

3 $y = \dfrac{ax+b}{cx+d}$ where $c \neq 0$

a The vertical asymptote is $cx + d = 0$

$$\therefore \quad x = -\frac{d}{c}$$

b If $|x|$ is very large, then $\dfrac{ax+b}{cx+d} \approx \dfrac{ax}{cx} \approx \dfrac{a}{c}$

So, the graph approaches the line $y = \dfrac{a}{c}$, but never reaches it.

This tells us that the horizontal asymptote is $y = \dfrac{a}{c}$.

EXERCISE 15D.2

1 **a** $f(x) = \dfrac{3}{x-2}$

i The vertical asymptote is $x = 2$.
The horizontal asymptote is $y = 0$.

ii The domain is $\{x \mid x \neq 2\}$.
The range is $\{y \mid y \neq 0\}$.

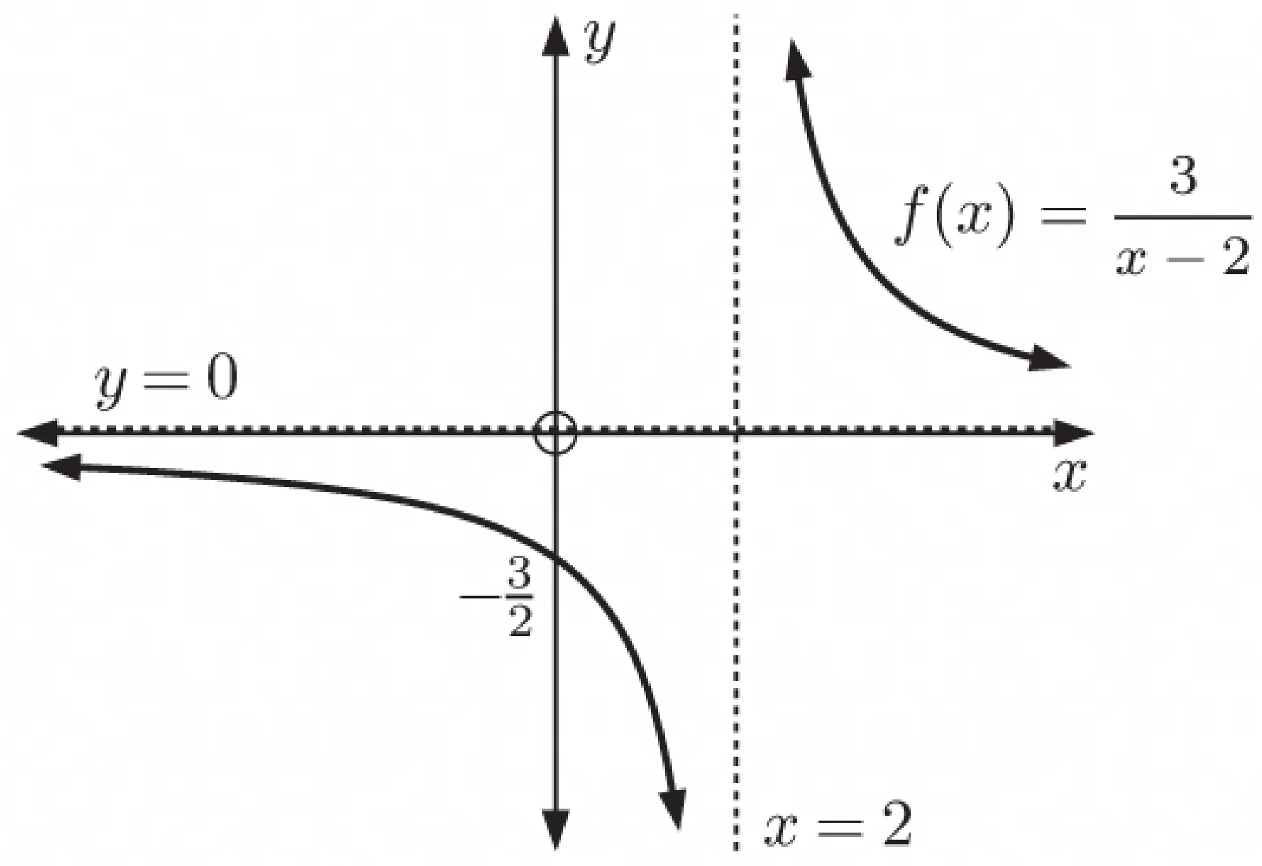
iii When $x = 0$, $y = \dfrac{3}{-2} = -\dfrac{3}{2}$

$\therefore$ the y-intercept is $-\frac{3}{2}$.

$y \neq 0$, so there is no x-intercept.

iv As $x \to 2^-$, $f(x) \to -\infty$
As $x \to 2^+$, $f(x) \to \infty$
As $x \to -\infty$, $f(x) \to 0^-$
As $x \to \infty$, $f(x) \to 0^+$

v



b $f : x \mapsto 2 + \dfrac{1}{x-3}$

i The vertical asymptote is $x = 3$.
The horizontal asymptote is $y = 2$.

ii The domain is $\{x \mid x \neq 3\}$.
The range is $\{y \mid y \neq 2\}$.

iii When $y = 0$, $2 + \dfrac{1}{x-3} = 0$

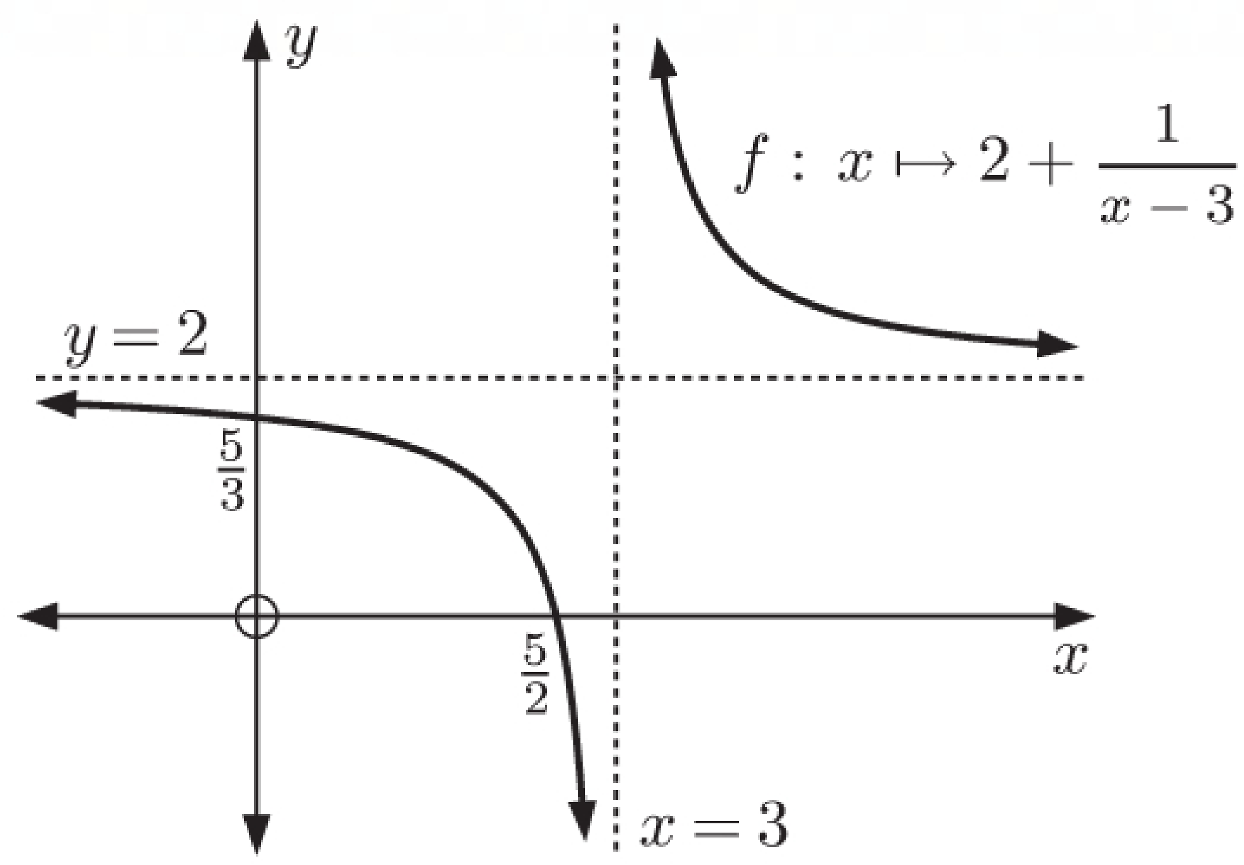
$$\therefore\ 2(x-3) + 1 = 0$$
$$\therefore\ 2x - 6 + 1 = 0$$
$$\therefore\ 2x = 5$$
$$\therefore\ x = \tfrac{5}{2}$$

When $x = 0$, $y = 2 + \dfrac{1}{-3} = \dfrac{5}{3}$

So, the x-intercept is $\frac{5}{2}$, and the y-intercept is $\frac{5}{3}$.

iv As $x \to 3^-$, $f(x) \to -\infty$
As $x \to 3^+$, $f(x) \to \infty$
As $x \to -\infty$, $f(x) \to 2^-$
As $x \to \infty$, $f(x) \to 2^+$

v



c $f(x) = 2 - \dfrac{3}{x+1}$

i The vertical asymptote is $x = -1$.
The horizontal asymptote is $y = 2$.

ii The domain is $\{x \mid x \neq -1\}$.
The range is $\{y \mid y \neq 2\}$.

iii When $y = 0$, $2 - \dfrac{3}{x+1} = 0$

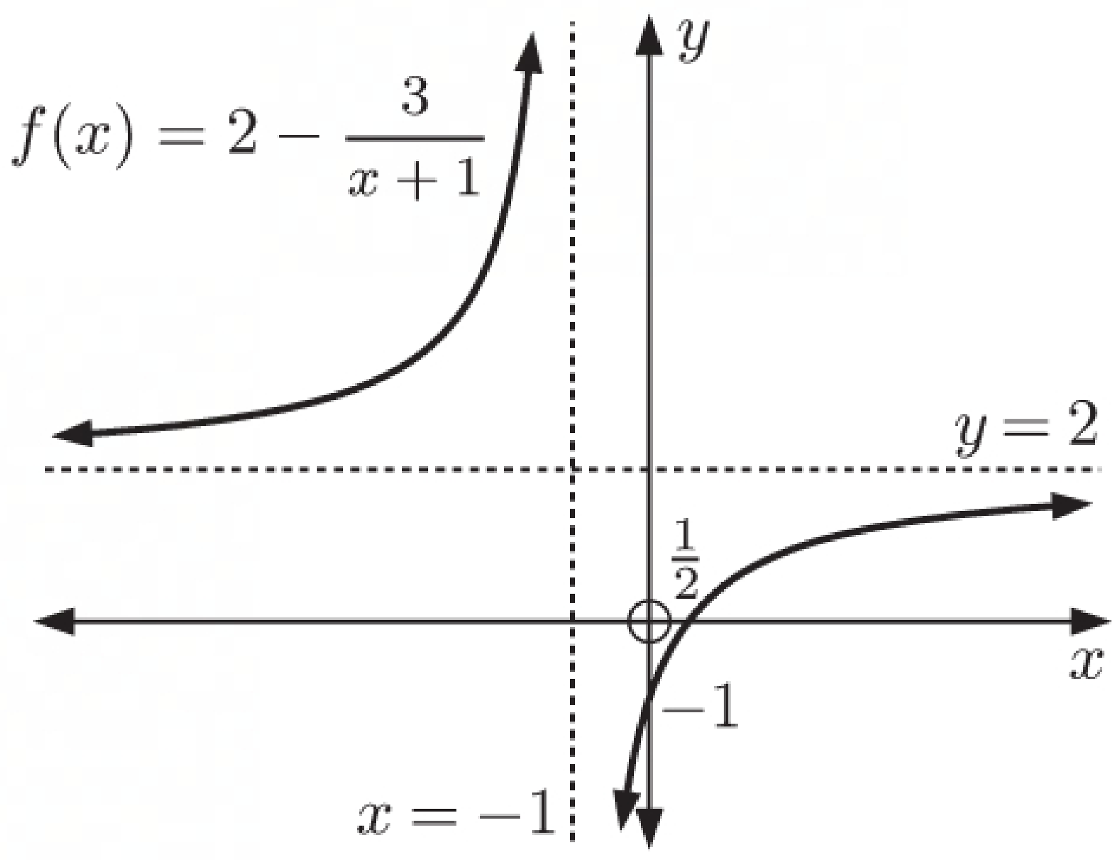
$$\therefore\ 2(x+1) - 3 = 0$$
$$\therefore\ 2x + 2 - 3 = 0$$
$$\therefore\ 2x = 1$$
$$\therefore\ x = \tfrac{1}{2}$$

When $x = 0$, $y = 2 - \dfrac{3}{1} = -1$

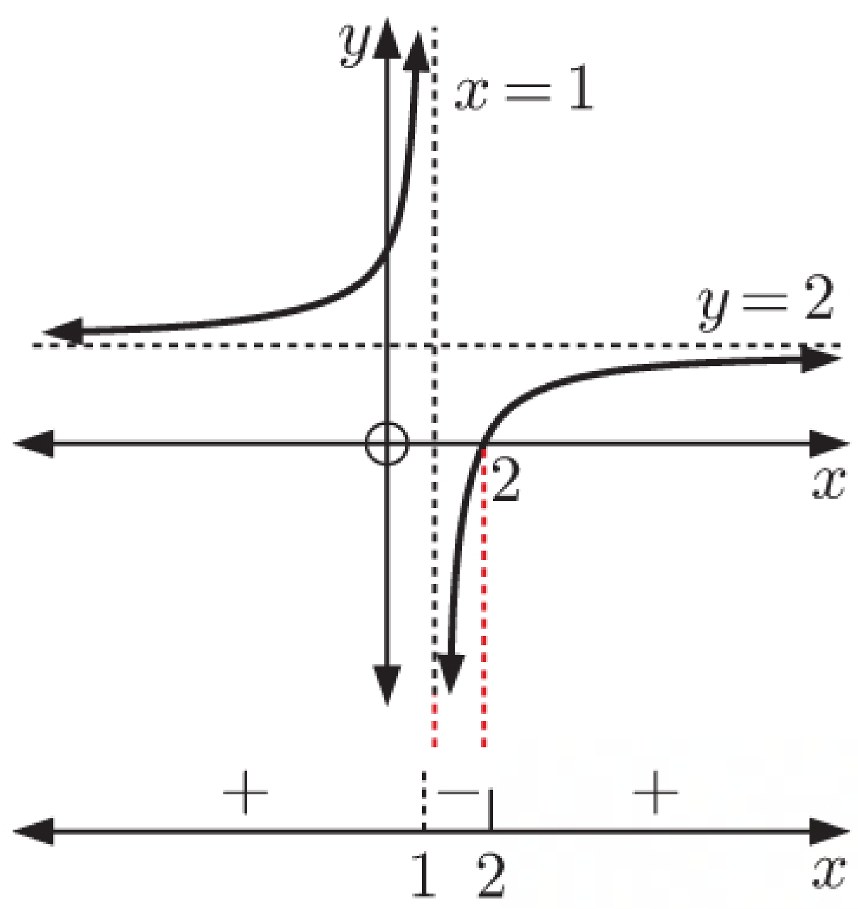
So, the x-intercept is $\frac{1}{2}$, and the y-intercept is -1.

iv As $x \to -1^-$, $f(x) \to \infty$
As $x \to -1^+$, $f(x) \to -\infty$
As $x \to -\infty$, $f(x) \to 2^+$
As $x \to \infty$, $f(x) \to 2^-$

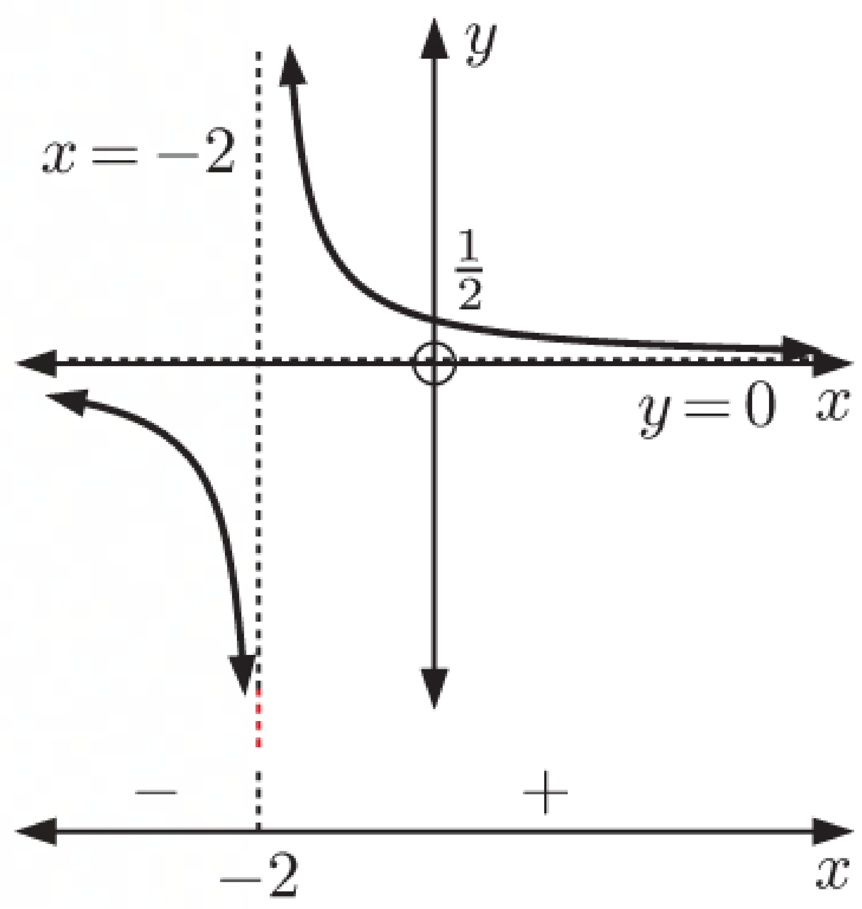
v



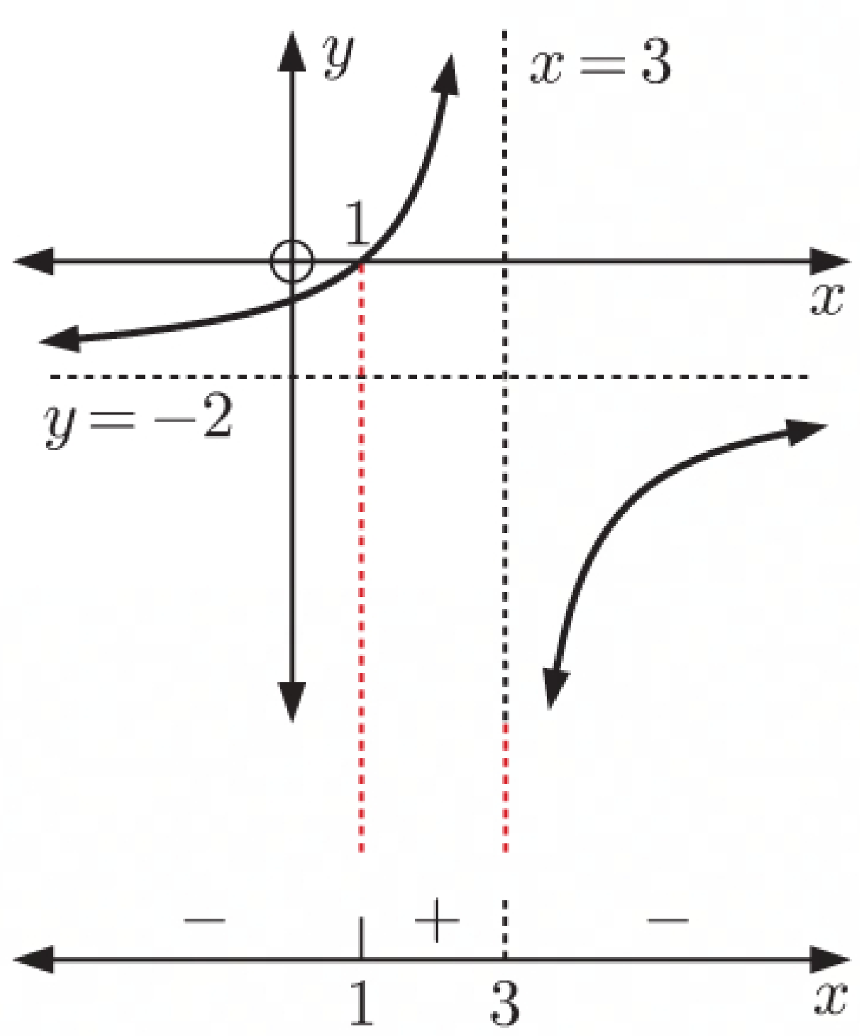
2 **a**



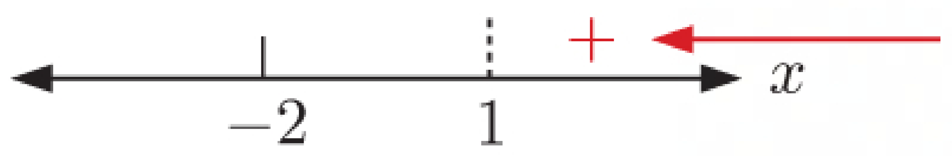
b



c

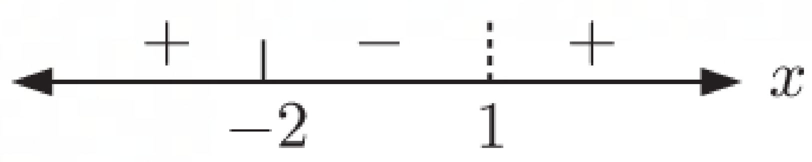


3 **a** $\dfrac{x+2}{x-1}$ is zero when $x = -2$ and undefined when $x = 1$.

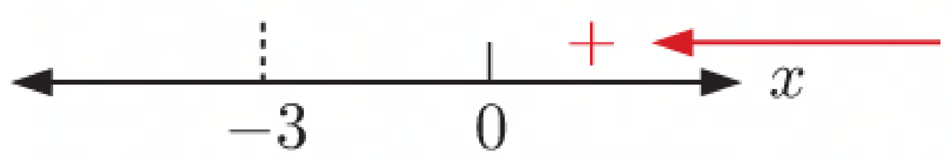


When $x = 10$, $\dfrac{x+2}{x-1} = \dfrac{12}{9} > 0$

Since $(x+2)$ and $(x-1)$ are single factors, the signs alternate.



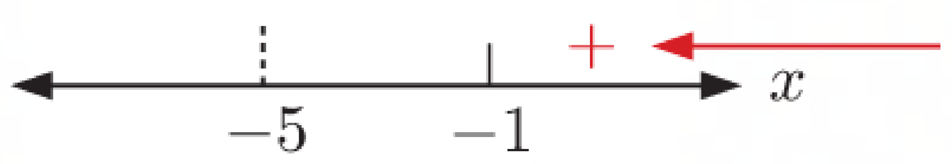
b $\dfrac{x}{x+3}$ is zero when $x = 0$ and undefined when $x = -3$.



When $x = 10$, $\dfrac{x}{x+3} = \dfrac{10}{13} > 0$

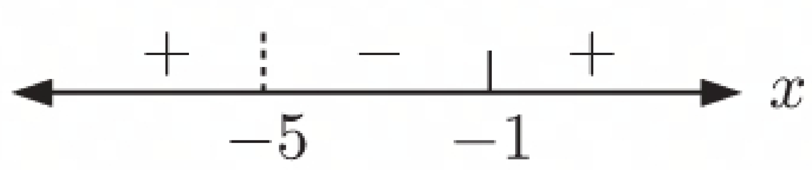
Since x and $(x+3)$ are single factors, the signs alternate.

c $\dfrac{x+1}{x+5}$ is zero when $x = -1$ and undefined when $x = -5$.

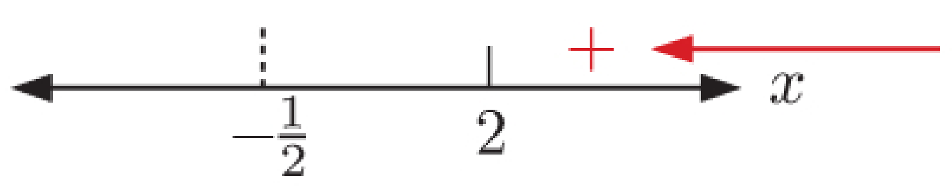


When $x = 10$, $\dfrac{x+1}{x+5} = \dfrac{11}{15} > 0$

Since $(x+1)$ and $(x+5)$ are single factors, the signs alternate.

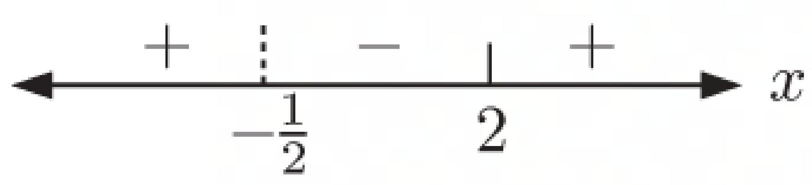


d $\dfrac{x-2}{2x+1}$ is zero when $x = 2$ and undefined when $x = -\frac{1}{2}$.

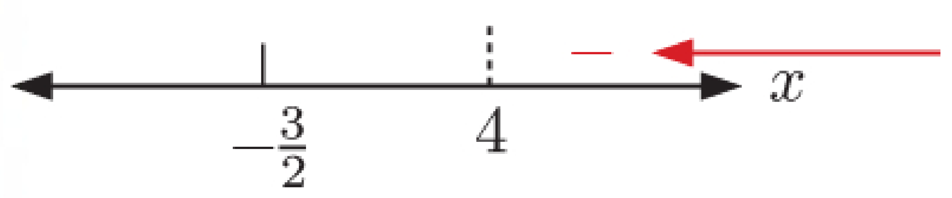


When $x = 10$, $\dfrac{x-2}{2x+1} = \dfrac{8}{21} > 0$

Since $(x-2)$ and $(2x+1)$ are single factors, the signs alternate.

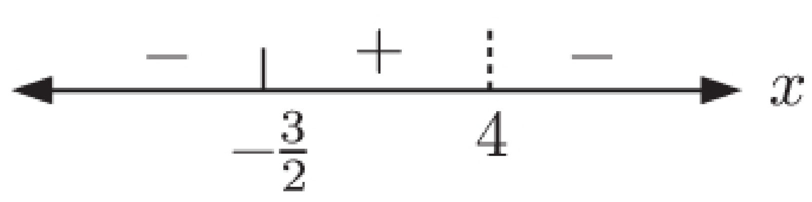


e $\dfrac{2x+3}{4-x}$ is zero when $x = -\frac{3}{2}$ and undefined when $x = 4$.

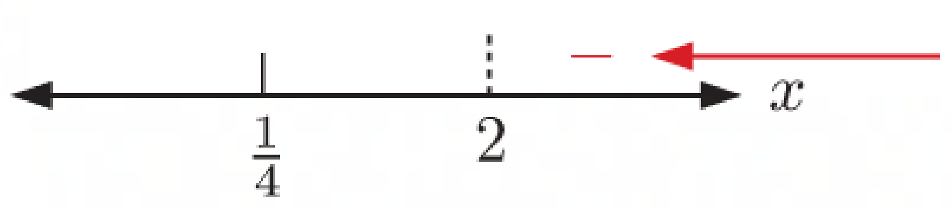


When $x = 10$, $\dfrac{2x+3}{4-x} = \dfrac{23}{-6} < 0$

Since $(2x+3)$ and $(4-x)$ are single factors, the signs alternate.

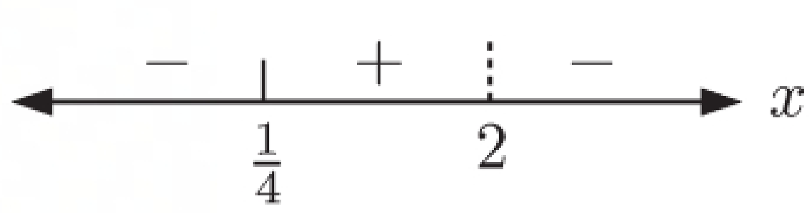


f $\dfrac{4x-1}{2-x}$ is zero when $x = \frac{1}{4}$ and undefined when $x = 2$.

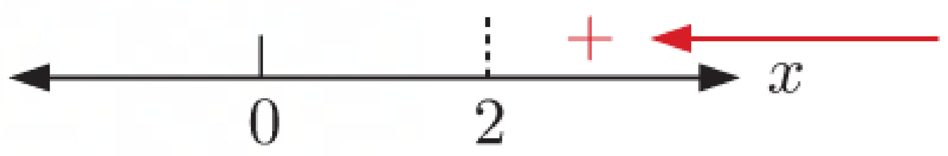


When $x = 10$, $\dfrac{4x-1}{2-x} = \dfrac{39}{-8} < 0$

Since $(4x-1)$ and $(2-x)$ are single factors, the signs alternate.

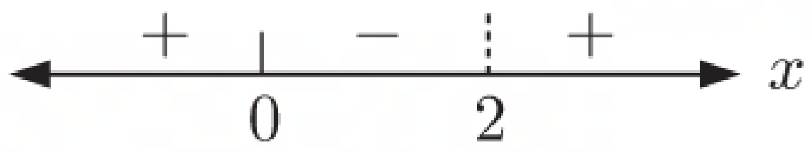


g $\dfrac{3x}{x-2}$ is zero when $x=0$ and undefined when $x=2$.



When $x=10$, $\dfrac{3x}{x-2}=\dfrac{30}{8}>0$

Since $3x$ and $(x-2)$ are single factors, the signs alternate.



h $\dfrac{-8x}{3-x}$ is zero when $x=0$ and undefined when $x=3$.

When $x=10$, $\dfrac{-8x}{3-x}=\dfrac{-80}{-7}>0$

Since $-8x$ and $(3-x)$ are single factors, the signs alternate.

4 **a** $f(x)=\dfrac{x}{x-1}$

i The vertical asymptote is $x=1$.

ii $f(0)=\dfrac{0}{-1}=0$, so the y-intercept is 0.

$f(x)=0$ when $x=0$

$\therefore$ the x-intercept is 0.

iii $f(x)=\dfrac{x}{x-1}$

$=\dfrac{(x-1)+1}{x-1}$

$=1+\dfrac{1}{x-1}$

$\therefore$ the horizontal asymptote is $y=1$.

iv

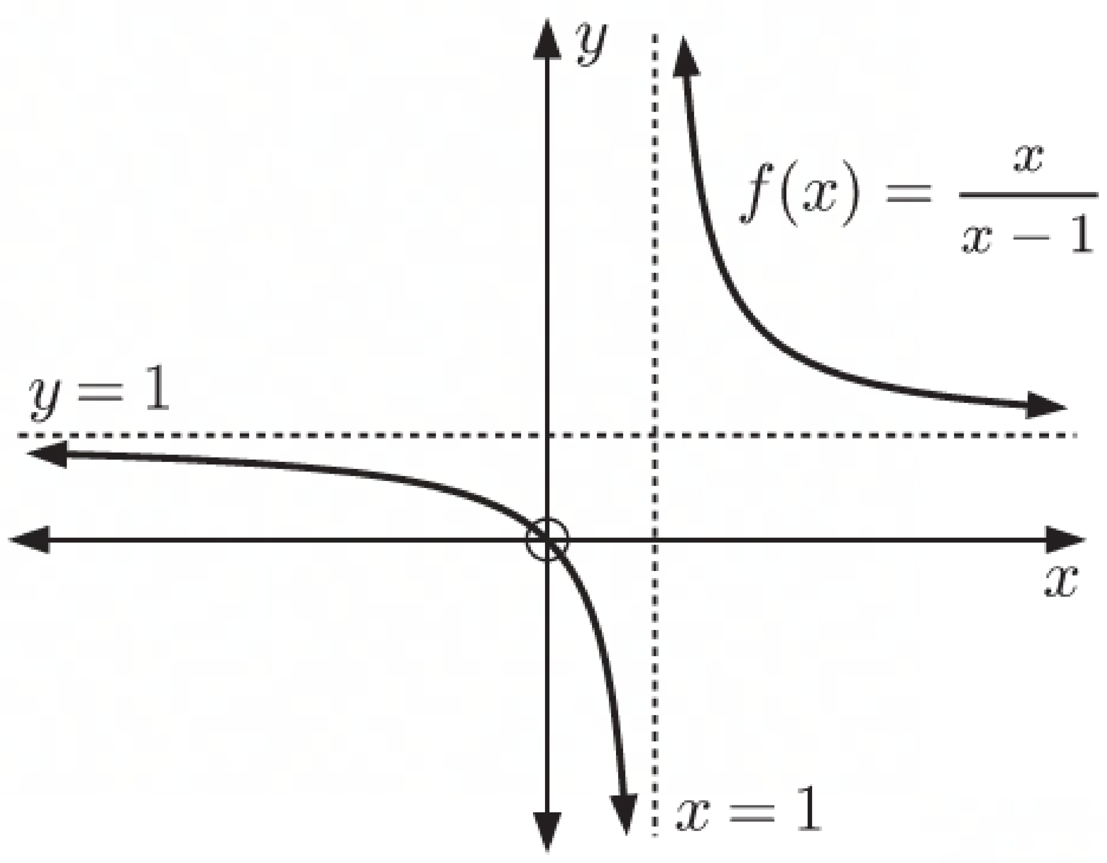
v As $x\to 1^-$, $f(x)\to -\infty$

As $x\to 1^+$, $f(x)\to \infty$

As $x\to -\infty$, $f(x)\to 1^-$

As $x\to \infty$, $f(x)\to 1^+$

vi



b $f: x\mapsto \dfrac{x+3}{x-2}$

i The vertical asymptote is $x=2$.

ii $f(0)=\dfrac{3}{-2}=-\dfrac{3}{2}$, so the y-intercept is $-\frac{3}{2}$.

$f(x)=0$ when $x+3=0$

$\therefore\ x=-3$

$\therefore$ the x-intercept is -3.

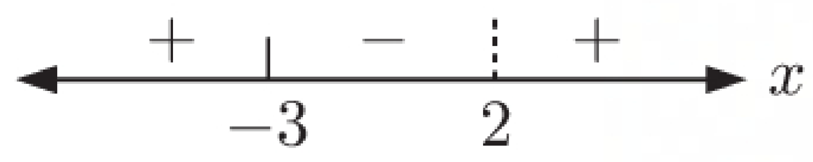
iii $f(x) = \dfrac{x+3}{x-2}$

$= \dfrac{(x-2)+5}{x-2}$

$= 1 + \dfrac{5}{x-2}$

$\therefore$ the horizontal asymptote is $y = 1$.

iv



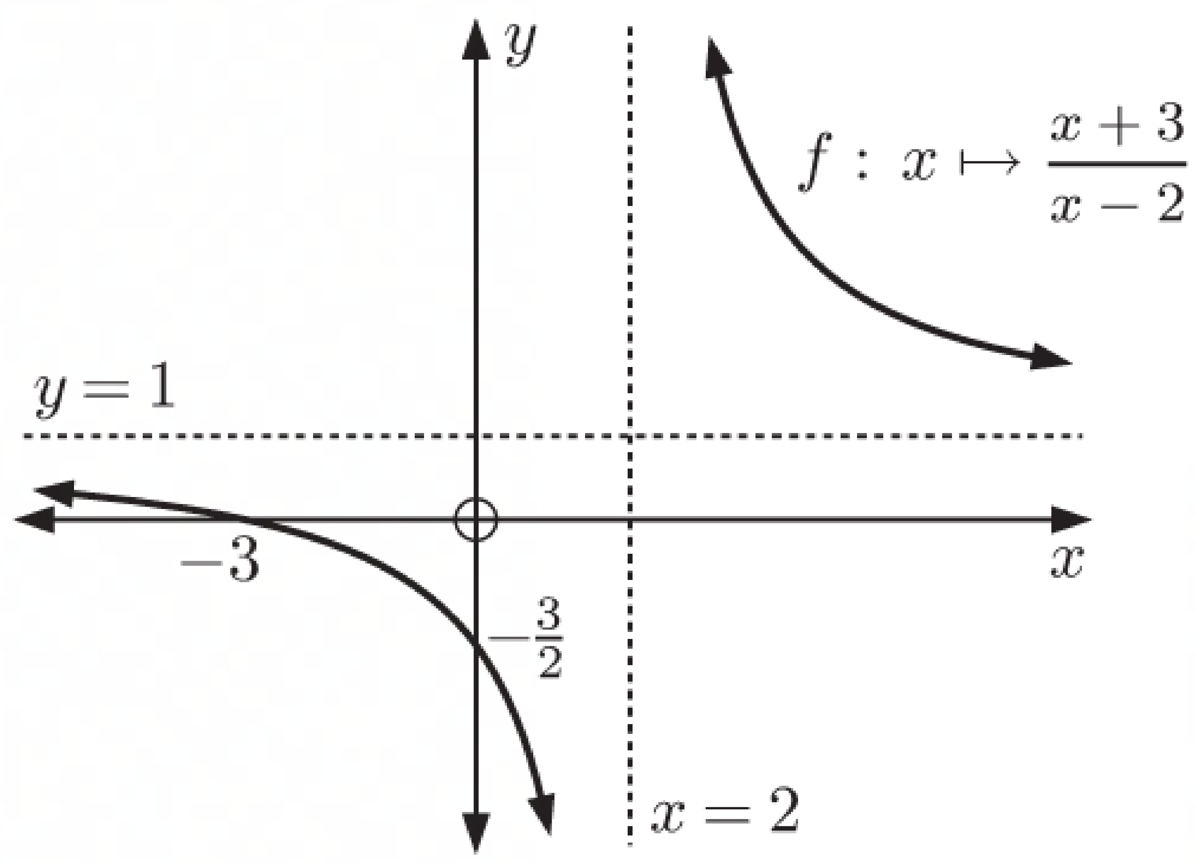
v As $x \to 2^-$, $f(x) \to -\infty$

As $x \to 2^+$, $f(x) \to \infty$

As $x \to -\infty$, $f(x) \to 1^-$

As $x \to \infty$, $f(x) \to 1^+$

vi



c $f(x) = \dfrac{3x-1}{x+2}$

i The vertical asymptote is $x = -2$.

ii $f(0) = \dfrac{-1}{2} = -\dfrac{1}{2}$, so the y-intercept is $-\frac{1}{2}$.

$f(x) = 0$ when $3x - 1 = 0$

$\therefore\ 3x = 1$

$\therefore\ x = \frac{1}{3}$

$\therefore$ the x-intercept is $\frac{1}{3}$.

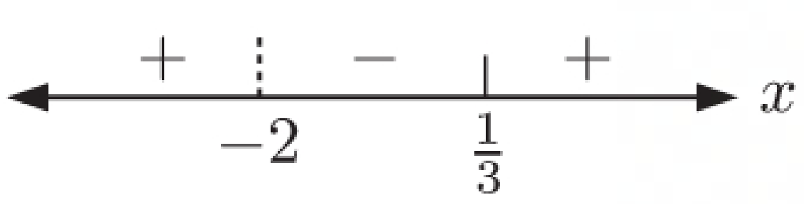
iii $f(x) = \dfrac{3x-1}{x+2}$

$= \dfrac{3(x+2)-7}{x+2}$

$= 3 - \dfrac{7}{x+2}$

$\therefore$ the horizontal asymptote is $y = 3$.

iv



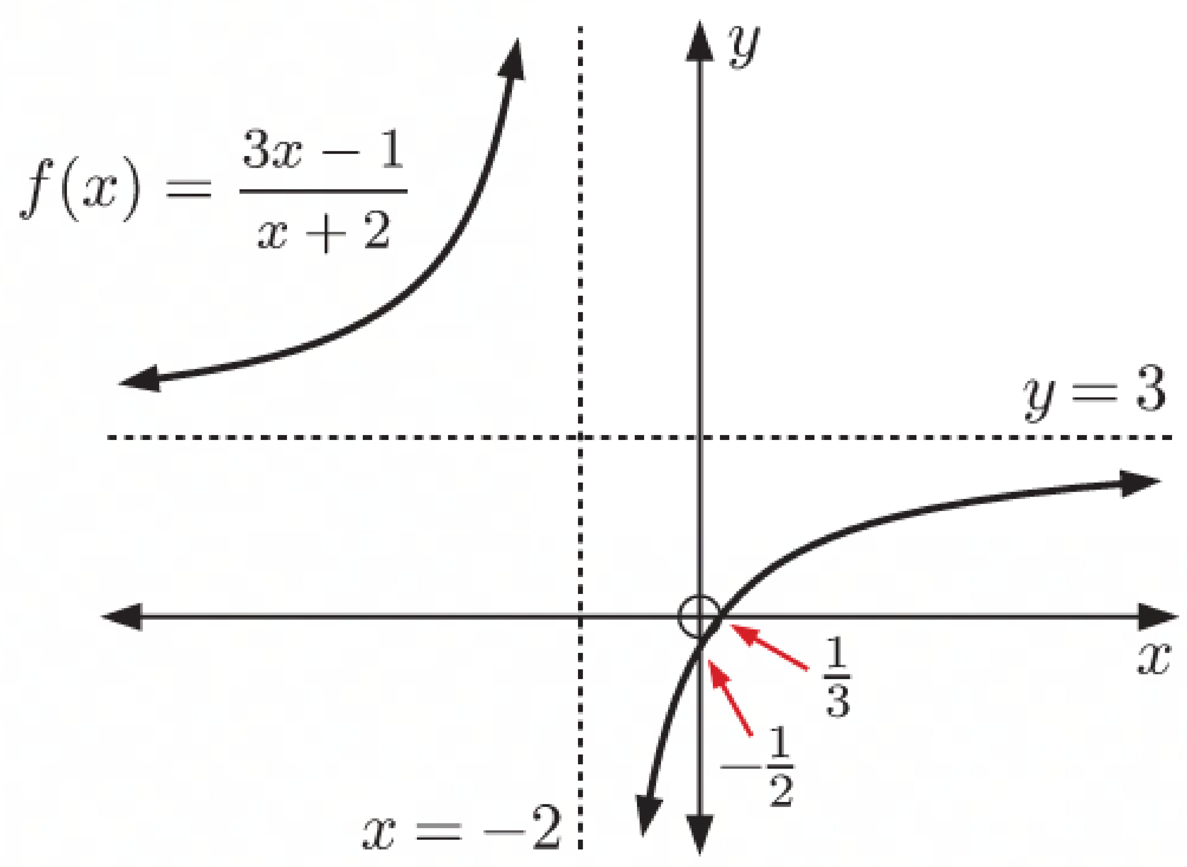
v As $x \to -2^-$, $f(x) \to \infty$

As $x \to -2^+$, $f(x) \to -\infty$

As $x \to -\infty$, $f(x) \to 3^+$

As $x \to \infty$, $f(x) \to 3^-$

vi



d $f(x) = -\dfrac{2x+1}{x-3}$

i The vertical asymptote is $x = 3$.

ii $f(0) = -\dfrac{1}{(-3)} = \dfrac{1}{3}$, so the y-intercept is $\frac{1}{3}$.

$f(x) = 0$ when $2x + 1 = 0$

$\therefore\ 2x = -1$

$\therefore\ x = -\frac{1}{2}$

$\therefore$ the x-intercept is $-\frac{1}{2}$.

iii $f(x) = -\dfrac{2x+1}{x-3}$

$= -\dfrac{2(x-3)+7}{x-3}$

$= -2 - \dfrac{7}{x-3}$

$\therefore$ the horizontal asymptote is $y = -2$.

iv

$$\begin{array}{ccccc} - & | & + & \vdots & - \\ & -\frac{1}{2} & & 3 & \end{array} \quad x$$

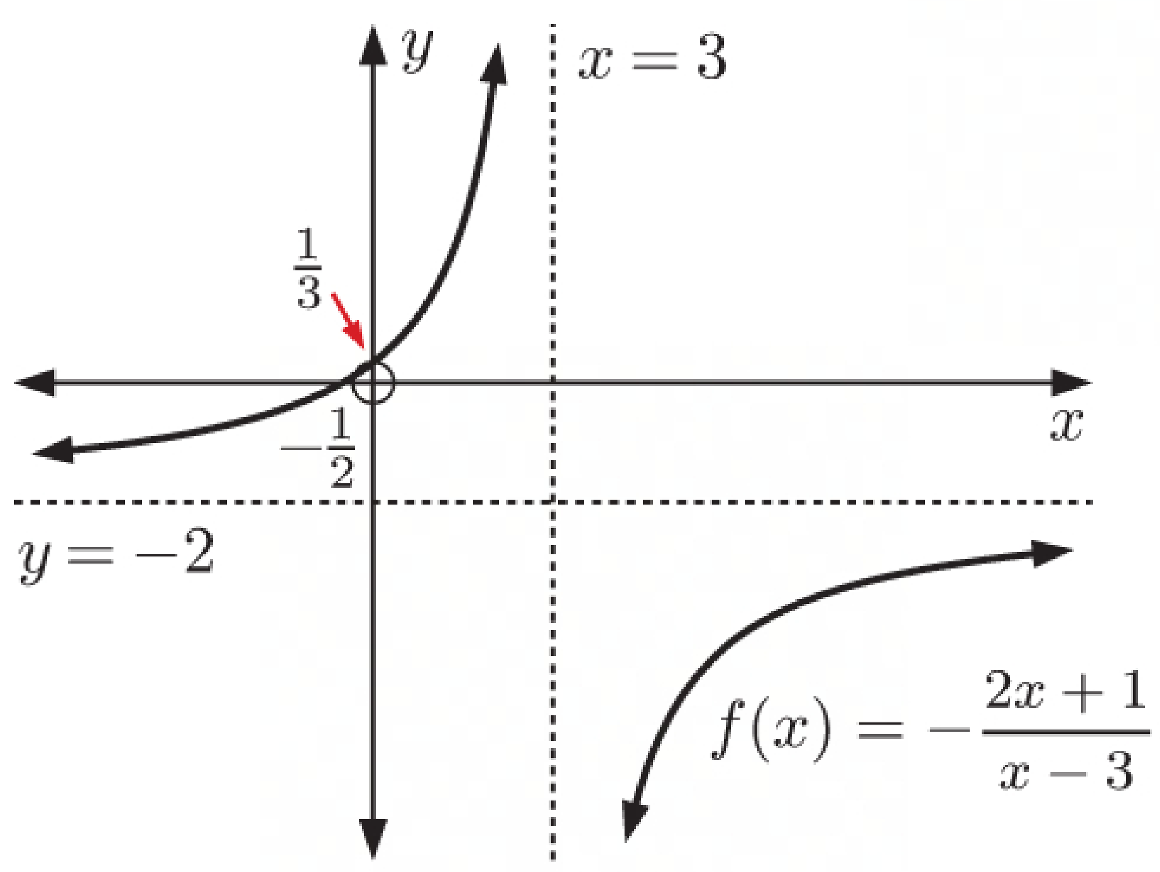
v As $x \to 3^-$, $f(x) \to \infty$

As $x \to 3^+$, $f(x) \to -\infty$

As $x \to -\infty$, $f(x) \to -2^+$

As $x \to \infty$, $f(x) \to -2^-$

vi



e $f : x \mapsto \dfrac{2x+4}{3-x}$

i The vertical asymptote has equation $x = 3$.

ii $f(0) = \dfrac{4}{3}$, so the y-intercept is $\frac{4}{3}$.

$f(x) = 0$ when $2x + 4 = 0$

$\therefore\ 2x = -4$

$\therefore\ x = -2$

$\therefore$ the x-intercept is -2.

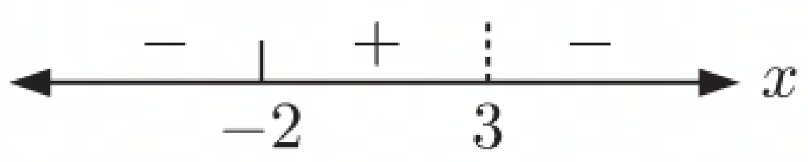
iii $f(x) = \dfrac{2x+4}{3-x}$

$= \dfrac{-2(3-x)+10}{3-x}$

$= -2 + \dfrac{10}{3-x}$

$\therefore$ the horizontal asymptote is $y = -2$.

iv



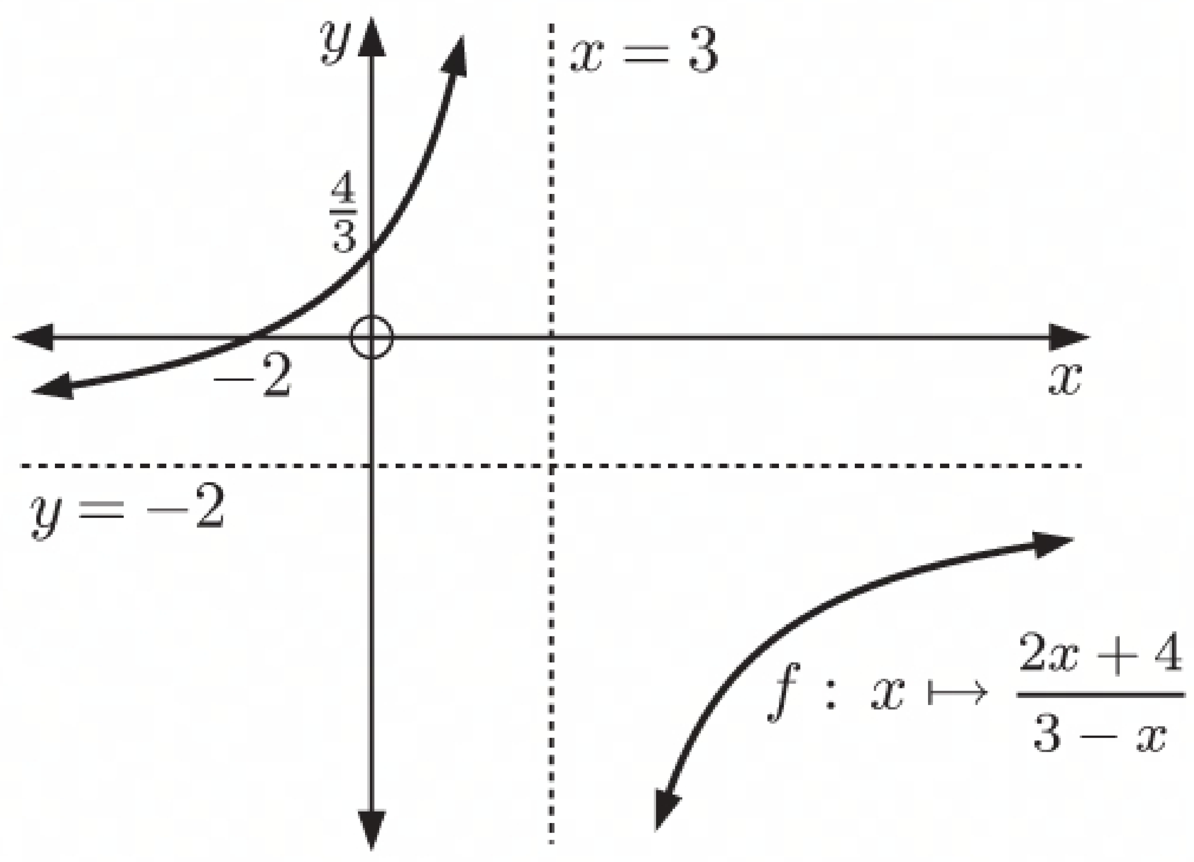
v As $x \to 3^-$, $f(x) \to \infty$

As $x \to 3^+$, $f(x) \to -\infty$

As $x \to -\infty$, $f(x) \to -2^+$

As $x \to \infty$, $f(x) \to -2^-$

vi



f $f(x) = \dfrac{x+3}{2x-1}$

i The vertical asymptote has equation $x = \frac{1}{2}$.

ii $f(0) = \dfrac{3}{-1} = -3$, so the y-intercept is -3.

$f(x) = 0$ when $x + 3 = 0$

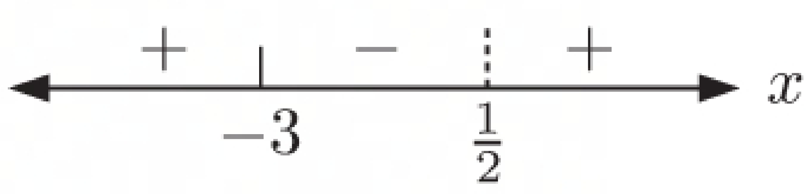
$\therefore\ x = -3$

$\therefore$ the x-intercept is -3.

iii
$$\begin{aligned} f(x) &= \frac{x+3}{2x-1} \\ &= \frac{\frac{1}{2}(2x-1)+\frac{7}{2}}{2x-1} \\ &= \frac{1}{2} + \frac{7}{2(2x-1)} \\ &= \tfrac{1}{2} + \frac{7}{4x-2} \end{aligned}$$

$\therefore$ the horizontal asymptote is $y = \frac{1}{2}$.

iv



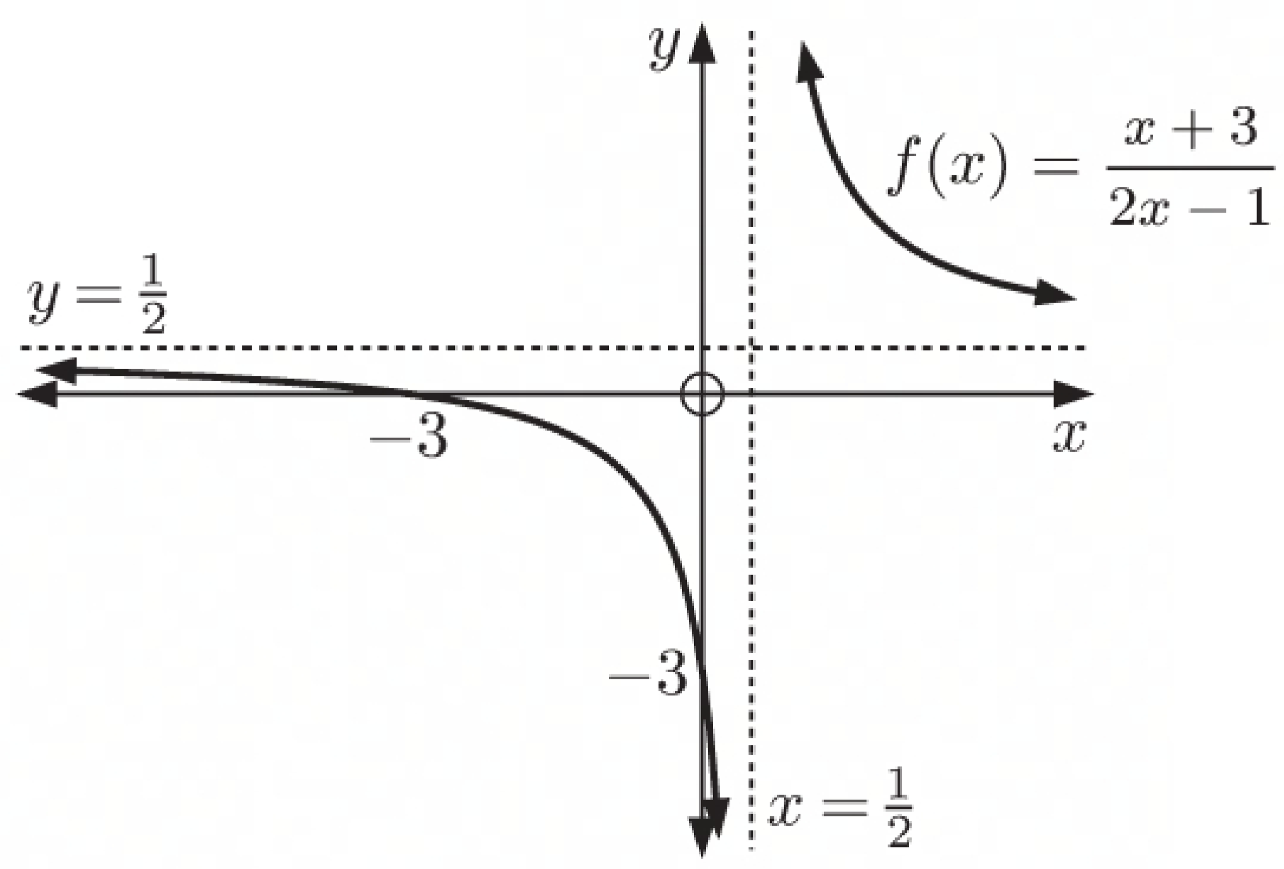
v As $x \to \frac{1}{2}^-$, $f(x) \to -\infty$

As $x \to \frac{1}{2}^+$, $f(x) \to \infty$

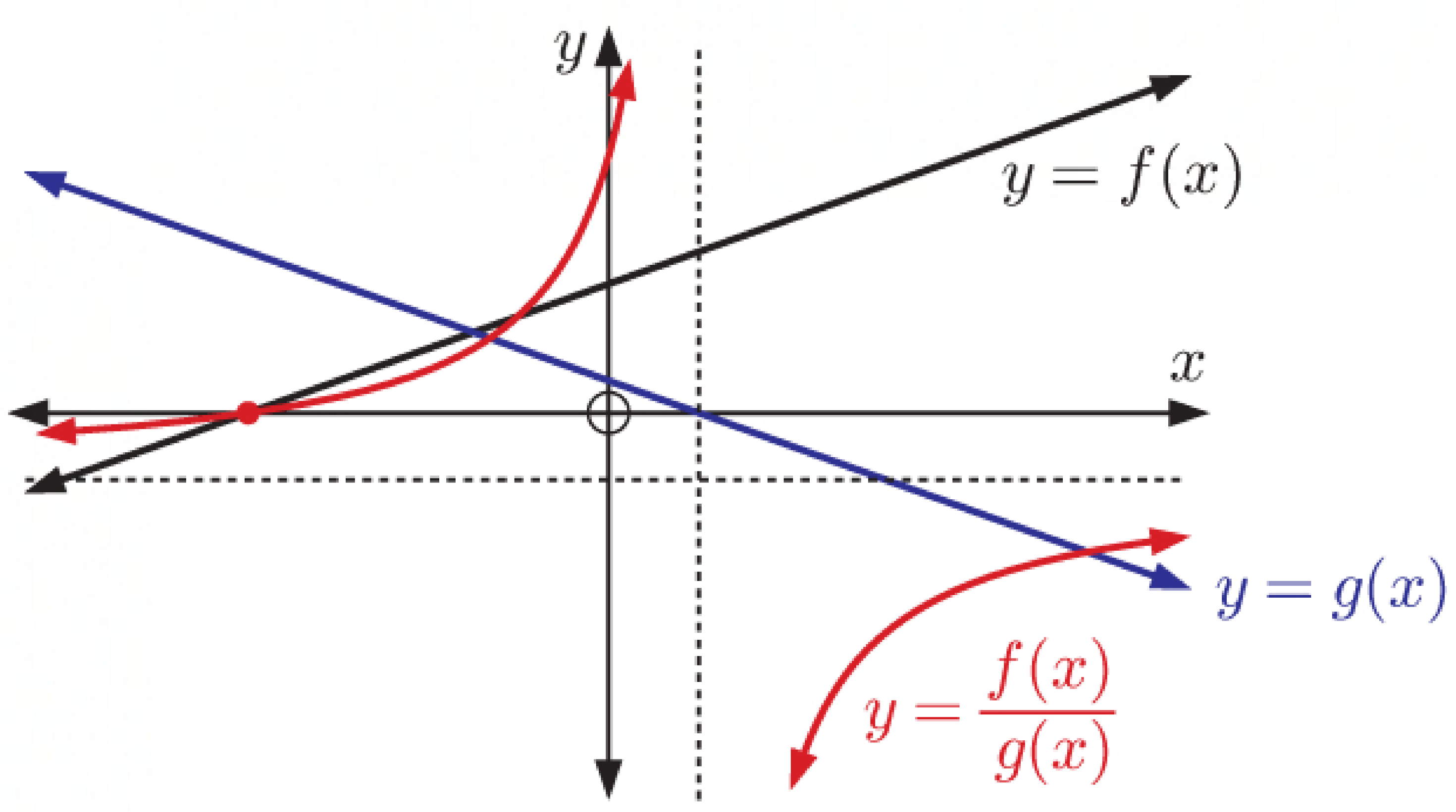
As $x \to -\infty$, $f(x) \to \frac{1}{2}^-$

As $x \to \infty$, $f(x) \to \frac{1}{2}^+$

vi



5



The graph of $y = \dfrac{f(x)}{g(x)}$ has the same x-intercept as the graph of $y = f(x)$, and has a vertical asymptote at the x-intercept of the graph of $y = g(x)$.

6 $y = \dfrac{ax+b}{cx+d}$ where a, b, c, and d are constants and $c \neq 0$.

a $\dfrac{ax+b}{cx+d}$ is undefined when $cx + d = 0$

$$\therefore \quad x = -\frac{d}{c}$$

$\therefore$ the domain is $\{x \mid x \neq -\frac{d}{c}\}$.

b The vertical asymptote is $x = -\dfrac{d}{c}$.

c When $x = 0$, $y = \dfrac{b}{d}$, $d \neq 0$

$\therefore$ the y-intercept is $\dfrac{b}{d}$, $d \neq 0$.

When $y = 0$, $ax + b = 0$

$$\therefore \quad x = -\frac{b}{a}, \quad a \neq 0$$

$\therefore$ the x-intercept is $-\dfrac{b}{a}$, $a \neq 0$.

d
$$\begin{aligned}\frac{ax+b}{cx+d} &= \frac{\frac{a}{c}(cx+d)+b-\frac{ad}{c}}{cx+d}\\ &= \frac{a}{c} + \frac{b-\frac{ad}{c}}{cx+d}\end{aligned}$$

As $|\,x\,| \to \infty$, $\dfrac{a}{c} + \dfrac{b-\frac{ad}{c}}{cx+d} \to \dfrac{a}{c} + 0 = \dfrac{a}{c}$

$\therefore$ the horizontal asymptote is $y = \dfrac{a}{c}$.

EXERCISE 15E

1 $f : x \mapsto -2x$ and $g : x \mapsto 1 + x^2$

a
$$\begin{aligned}(f \circ g)(x) &= f(g(x))\\ &= f(1+x^2)\\ &= -2(1+x^2)\\ &= -2-2x^2\end{aligned}$$

b
$$\begin{aligned}(g \circ f)(x) &= g(f(x))\\ &= g(-2x)\\ &= 1+(-2x)^2\\ &= 1+4x^2\end{aligned}$$

c
$$\begin{aligned}(f \circ g)(x) &= -2-2x^2\\ \therefore \quad (f \circ g)(2) &= -2-2(2)^2\\ &= -10\end{aligned}$$

d
$$\begin{aligned}(f \circ f)(-1) &= f(f(-1))\\ &= f(-2(-1))\\ &= f(2)\\ &= -2(2)\\ &= -4\end{aligned}$$

2 $f(x) = 3 - x^2$ and $g(x) = 2x + 4$

a $$\begin{aligned}(f \circ g)(x) &= f(g(x))\\ &= f(2x+4)\\ &= 3 - (2x+4)^2\\ &= 3 - 4x^2 - 16x - 16\\ &= -4x^2 - 16x - 13\end{aligned}$$

b $$\begin{aligned}(g \circ f)(x) &= g(f(x))\\ &= g(3 - x^2)\\ &= 2(3 - x^2) + 4\\ &= 6 - 2x^2 + 4\\ &= 10 - 2x^2\end{aligned}$$

c $$\begin{aligned}(g \circ g)(\tfrac{1}{2}) &= g(g(\tfrac{1}{2}))\\ &= g(2(\tfrac{1}{2}) + 4)\\ &= g(5)\\ &= 2(5) + 4\\ &= 14\end{aligned}$$

d $$\begin{aligned}(f \circ f)(-\tfrac{1}{2}) &= f(f(-\tfrac{1}{2}))\\ &= f(3 - (-\tfrac{1}{2})^2)\\ &= f(\tfrac{11}{4})\\ &= 3 - (\tfrac{11}{4})^2\\ &= 3 - \tfrac{121}{16}\\ &= -\tfrac{73}{16}\end{aligned}$$

3 $f(x) = \sqrt{6 - x}$ and $g(x) = 5x - 7$

a $$\begin{aligned}(g \circ g)(x) &= g(g(x))\\ &= g(5x - 7)\\ &= 5(5x - 7) - 7\\ &= 25x - 35 - 7\\ &= 25x - 42\end{aligned}$$

b $$\begin{aligned}(f \circ g)(1) &= f(g(1))\\ &= f(5(1) - 7)\\ &= f(-2)\\ &= \sqrt{6 - (-2)}\\ &= \sqrt{8}\end{aligned}$$

c $$\begin{aligned}(g \circ f)(6) &= g(f(6))\\ &= g(\sqrt{6 - 6})\\ &= g(0)\\ &= 5(0) - 7\\ &= -7\end{aligned}$$

d $$\begin{aligned}(f \circ f)(2) &= f(f(2))\\ &= f(\sqrt{6 - 2})\\ &= f(2)\\ &= \sqrt{6 - 2}\\ &= 2\end{aligned}$$

4 $f : x \mapsto x^2 + 1$ and $g : x \mapsto 3 - x$

a **i** $$\begin{aligned}(f \circ g)(x) &= f(g(x))\\ &= f(3 - x)\\ &= (3 - x)^2 + 1\\ &= 9 - 6x + x^2 + 1\\ &= x^2 - 6x + 10\end{aligned}$$

ii $$\begin{aligned}(g \circ f)(x) &= g(f(x))\\ &= g(x^2 + 1)\\ &= 3 - (x^2 + 1)\\ &= 3 - x^2 - 1\\ &= 2 - x^2\end{aligned}$$

b $$\begin{aligned}(g \circ f)(x) &= f(x)\\ \therefore\ 2 - x^2 &= x^2 + 1\\ \therefore\ 2x^2 &= 1\\ \therefore\ x^2 &= \tfrac{1}{2}\\ \therefore\ x &= \pm\tfrac{1}{\sqrt{2}}\end{aligned}$$

5 $f(x) = 9 - \sqrt{x}$ and $g(x) = x^2 + 4$

a $(f \circ g)(x) = f(g(x))$
$= f(x^2 + 4)$
$= 9 - \sqrt{x^2 + 4}$

$x^2 + 4 \geqslant 0$, so $\sqrt{x^2+4}$ is defined for every value of x.

$\therefore$ domain is $\{x \mid x \in \mathbb{R}\}$

$-\sqrt{x^2+4} \leqslant -2$

$\therefore$ $9 - \sqrt{x^2+4} \leqslant 7$

$\therefore$ range is $\{y \mid y \leqslant 7\}$

b $(g \circ f)(4) = g(f(4))$
$= g(9 - \sqrt{4})$
$= g(7)$
$= (7)^2 + 4$
$= 53$

c $(f \circ f)(x) = f(f(x))$
$= f(9 - \sqrt{x})$
$= 9 - \sqrt{9 - \sqrt{x}}$

$\sqrt{x}$ is defined when $x \geqslant 0$,

and $\sqrt{9 - \sqrt{x}}$ is defined when $\sqrt{x} \leqslant 9$

$\therefore$ $x \leqslant 81$

$\therefore$ domain is $\{x \mid 0 \leqslant x \leqslant 81\}$

$-3 \leqslant -\sqrt{9 - \sqrt{x}} \leqslant 0$

$\therefore$ $6 \leqslant 9 - \sqrt{9 - \sqrt{x}} \leqslant 9$

$\therefore$ range is $\{y \mid 6 \leqslant y \leqslant 9\}$

6 $f(x) = 1 - 2x$ and $g(x) = 3x + 5$

a $f(g(x)) = f(3x + 5)$
$= 1 - 2(3x + 5)$
$= 1 - 6x - 10$
$= -6x - 9$

b $(f \circ g)(x) = f(x + 3)$
$\therefore$ $f(g(x)) = 1 - 2(x + 3)$
$\therefore$ $-6x - 9 = 1 - 2x - 6$
$\therefore$ $-4x = 4$
$\therefore$ $x = -1$

7 $f : x \mapsto 2x - x^2$ and $g : x \mapsto 1 + 3x$

a **i** $(f \circ g)(x) = f(g(x))$
$= f(1 + 3x)$
$= 2(1 + 3x) - (1 + 3x)^2$
$= 2 + 6x - 1 - 6x - 9x^2$
$= 1 - 9x^2$

ii $(g \circ f)(x) = g(f(x))$
$= g(2x - x^2)$
$= 1 + 3(2x - x^2)$
$= 1 + 6x - 3x^2$

b $(f \circ g)(x) = 3(g \circ f)(x)$
$\therefore$ $1 - 9x^2 = 3(1 + 6x - 3x^2)$
$\therefore$ $1 - 9x^2 = 3 + 18x - 9x^2$
$\therefore$ $-2 = 18x$
$\therefore$ $x = -\frac{1}{9}$

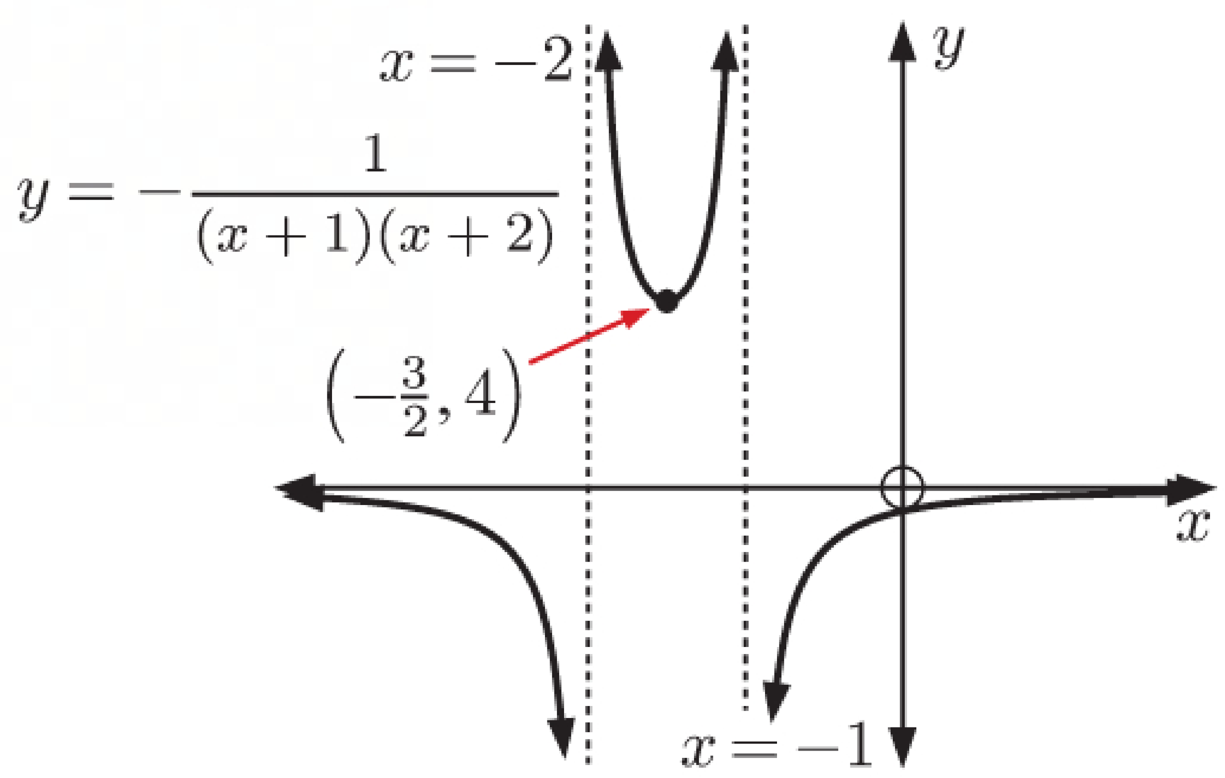
8 **a** $f(x) = \dfrac{1}{x}$ and $g(x) = x - 3$

$$\therefore \quad (f \circ g)(x) = f(g(x)) = f(x-3) = \frac{1}{x-3}$$

$\therefore$ domain is $\{x \mid x \neq 3\}$ and range is $\{y \mid y \neq 0\}$.

b $f(x) = -\dfrac{1}{x}$ and $g(x) = x^2 + 3x + 2$

$$\therefore \quad (f \circ g)(x) = f(g(x)) = f(x^2+3x+2) = -\frac{1}{x^2+3x+2} = -\frac{1}{(x+1)(x+2)}$$



$\therefore$ domain is $\{x \mid x \neq -1 \text{ or } -2\}$

From the graph, the range is $\{y \mid y \geqslant 4 \text{ or } y < 0\}$.

9 $f = \{(0, 2), (1, 5), (2, 7), (3, 9)\}$ and $g = \{(2, 2), (5, 0), (7, 1), (9, 3)\}$

a $(f \circ g)(2) = f(g(2)) = f(2) = 7$ $\quad (f \circ g)(5) = f(g(5)) = f(0) = 2$ $\quad (f \circ g)(7) = f(g(7)) = f(1) = 5$ $\quad (f \circ g)(9) = f(g(9)) = f(3) = 9$

$\therefore \quad f \circ g = \{(2, 7), (5, 2), (7, 5), (9, 9)\}$

b $(g \circ f)(0) = g(f(0)) = g(2) = 2$ $\quad (g \circ f)(1) = g(f(1)) = g(5) = 0$ $\quad (g \circ f)(2) = g(f(2)) = g(7) = 1$ $\quad (g \circ f)(3) = g(f(3)) = g(9) = 3$

$\therefore \quad g \circ f = \{(0, 2), (1, 0), (2, 1), (3, 3)\}$

10 $f(x) = \dfrac{x+3}{x+2}$ and $g(x) = \dfrac{x+1}{x-1}$

a
$$(f \circ g)(x) = f(g(x)) = f\left(\frac{x+1}{x-1}\right) = \frac{\frac{x+1}{x-1}+3}{\frac{x+1}{x-1}+2} \times \frac{x-1}{x-1} = \frac{x+1+3(x-1)}{x+1+2(x-1)} = \frac{x+1+3x-3}{x+1+2x-2} = \frac{4x-2}{3x-1}$$

The domain is $\{x \mid x \neq \frac{1}{3} \text{ or } 1\}$ since $(f \circ g)(x)$ is undefined when $x = \frac{1}{3}$, and $g(x)$ is undefined when $x = 1$.

b
$$(g \circ f)(x) = g(f(x)) = g\left(\frac{x+3}{x+2}\right) = \frac{\frac{x+3}{x+2}+1}{\frac{x+3}{x+2}-1} \times \frac{x+2}{x+2} = \frac{x+3+(x+2)}{x+3-(x+2)} = \frac{x+3+x+2}{x+3-x-2} = \frac{2x+5}{1} = 2x+5$$

The domain is $\{x \mid x \neq -2\}$ since $g(x)$ is undefined when $x = -2$.

c $(g \circ g)(x) = g(g(x))$

$$= g\left(\frac{x+1}{x-1}\right)$$

$$= \frac{\frac{x+1}{x-1}+1}{\frac{x+1}{x-1}-1} \times \frac{x-1}{x-1}$$

$$= \frac{x+1+(x-1)}{x+1-(x-1)}$$

$$= \frac{x+1+x-1}{x+1-x+1}$$

$$= \frac{2x}{2}$$

$$= x$$

The domain is $\{x \mid x \neq 1\}$ since $g(x)$ is undefined when $x = 1$.

11 **a** $ax + b = cx + d$ is true for all x

When $x = 0$, $a(0) + b = c(0) + d$

$\therefore \; b = d \quad .... \; (*)$

When $x = 1$, $a(1) + b = c(1) + d$

$\therefore \; a + b = c + d$

$\therefore \; a + d = c + d \quad$ {using $(*)$}

$\therefore \; a = c$

b $f(x) = 2x + 3$ and $g(x) = ax + b$

$(f \circ g)(x) = x$ for all x

$\therefore \; f(g(x)) = x$

$\therefore \; f(ax + b) = x$

$\therefore \; 2(ax + b) + 3 = x$

$\therefore \; 2ax + (2b + 3) = x$

$\therefore \; 2a = 1$ and $2b + 3 = 0$ {using **a**}

$\therefore \; a = \frac{1}{2}$ and $2b = -3$

So, $a = \frac{1}{2}$ and $b = -\frac{3}{2}$ as required.

c If $(g \circ f)(x) = x$ for all x

then $g(f(x)) = x$

$\therefore \; g(2x + 3) = x$

$\therefore \; a(2x + 3) + b = x$

$\therefore \; 2ax + (3a + b) = x$

$\therefore \; 2a = 1$ and $3a + b = 0$ {using **a**}

$\therefore \; a = \frac{1}{2}$ and $b = -3a$

So, $a = \frac{1}{2}$ and $b = -\frac{3}{2}$

$\therefore$ the result in **b** is also true if $(g \circ f)(x) = x$ for all x.

12 $f(x) = \sqrt{1 - x}$ and $g(x) = x^2$

a $(f \circ g)(x) = f(g(x))$

$$= f(x^2)$$

$$= \sqrt{1 - x^2}$$

b $(f \circ g)(x) = \sqrt{1 - x^2}$ is defined when $1 - x^2 \geqslant 0$

$\therefore \; x^2 \leqslant 1$

$\therefore \; -1 \leqslant x \leqslant 1$

$\therefore$ the domain is $\{x \mid -1 \leqslant x \leqslant 1\}$

$y = (f \circ g)(x)$ is always positive and $\leqslant 1$ as $-1 \leqslant x \leqslant 1$.

$\therefore$ the range is $\{y \mid 0 \leqslant y \leqslant 1\}$.

c $$\begin{aligned}(g \circ f)(x) &= g(f(x)) \\ &= g(\sqrt{1-x}) \\ &= (\sqrt{1-x})^2 \\ &= 1-x\end{aligned}$$

d $f(x)=\sqrt{1-x}$ is defined when $1-x \geqslant 0$

$\therefore\ x \leqslant 1$

$\therefore$ $(g \circ f)(x) = 1-x$ is defined when $x \leqslant 1$.

$\therefore$ the domain is $\{x \mid x \leqslant 1\}$

$y=(g \circ f)(x)$ is always positive as $x \leqslant 1$.

$\therefore$ the range is $\{y \mid y \geqslant 0\}$.

13 **a** Since $(f \circ g)(x) = f(g(x))$, x will only be in the domain of $(f \circ g)$ if:

- it is in the domain of g, so $g(x)$ exists, and
- the value $g(x)$ is in the domain of f, so $f(g(x))$ exists.

Now $g(x)$ is in the range of g, so $(f \circ g)(x)$ will only be defined when $R_g \cap D_f \neq \varnothing$.

b The domain of $(f \circ g)(x)$ is $\{x \mid x \in D_g \text{ and } g(x) \in D_f\}$.

14 $V(D) = 10\,000 - 40D$ and $D(t) = 80 + 10t$

a $$\begin{aligned}V \circ D &= V(D(t)) \\ &= V(80+10t) \\ &= 10\,000 - 40(80+10t) \\ &= 10\,000 - 3200 - 400t \\ &= 6800 - 400t\end{aligned}$$

The function $V \circ D$ gives the value of Mila's car after t years.

b $$\begin{aligned}(V \circ D)(6) &= 6800 - 400(6) \\ &= 6800 - 2400 \\ &= 4400\end{aligned}$$

The value of Mila's car 6 years after purchase is \$4400.

15 $S(x) = x + 50$ and $T(x) = 1.1x$

a **i** If the tax must be paid on both the marked price and shipping fee, we need to apply S on the marked price, and *then* T. This is given by $T \circ S$.

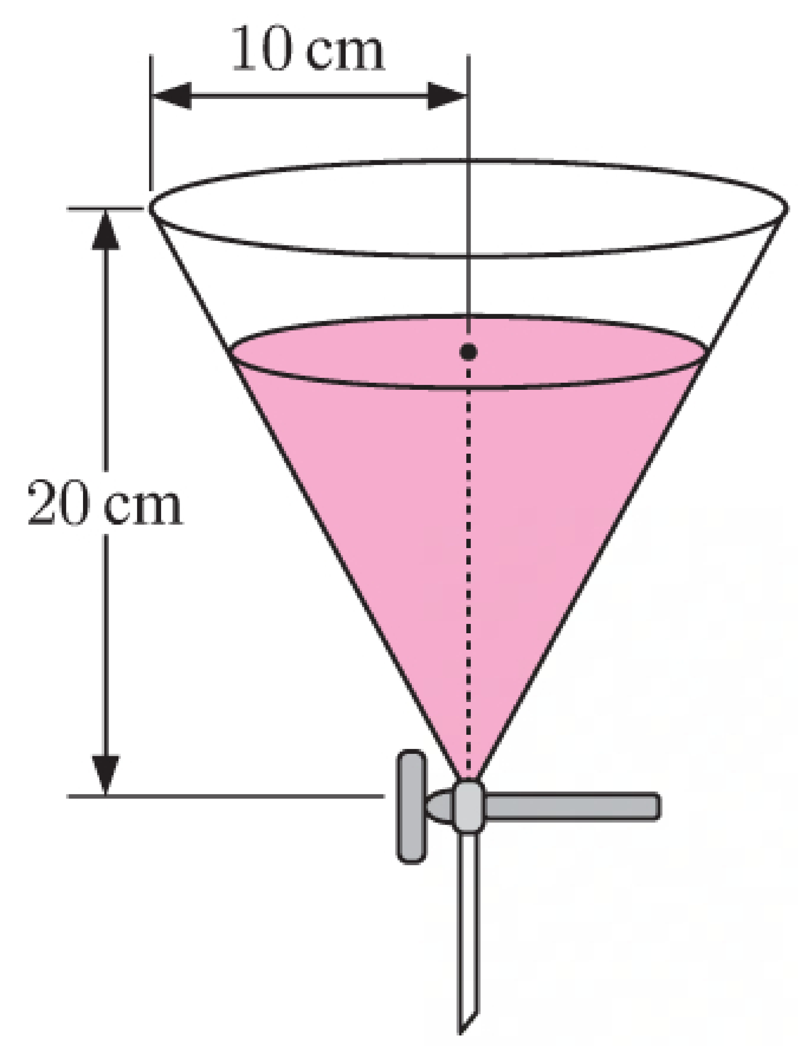
ii If the tax must be paid on the marked price but not the shipping fee, we apply T on the marked price, and *then* S. This is given by $S \circ T$.

b The marked price of the sculpture is €600, so $x = 600$.
The tax must be paid on both the marked price and the shipping fee, so we use the composite function $T \circ S$.

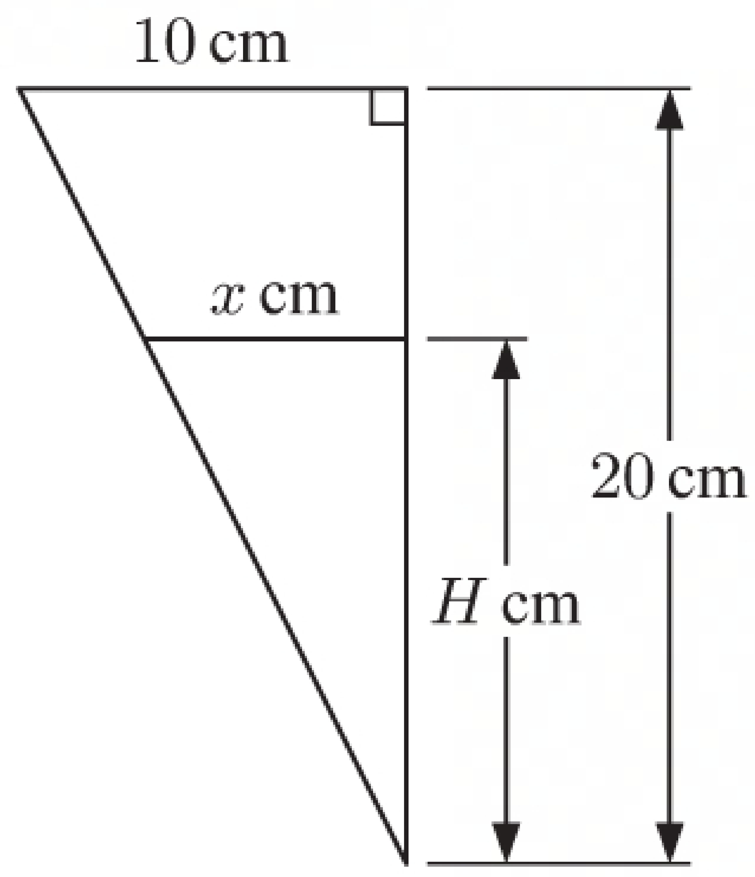
$$\begin{aligned}(T \circ S)(600) &= T(S(600)) \\ &= T(600+50) \\ &= T(650) \\ &= 1.1(650) \\ &= 715\end{aligned}$$

$\therefore$ the total cost is €715.

16 **a** The funnel initially contains $2\text{ L} = 2000\text{ mL}$ of solution.
The solution flows from the funnel at 20 mL per minute.
$\therefore$ the volume of solution in the funnel after t minutes is $V = 2000 - 20t$ mL.



b



Using similar triangles,

$$\frac{x}{10} = \frac{H}{20}$$

$$\therefore\ x = \frac{H}{2}$$

Now, $V = \frac{1}{3} \times \text{base} \times \text{height}$

$$\therefore\ V = \frac{1}{3} \times \pi \left(\frac{H}{2}\right)^2 \times H$$

$$\therefore\ V = \frac{1}{3} \times \pi \frac{H^2}{4} \times H$$

$$\therefore\ V = \frac{\pi H^3}{12}$$

$$\therefore\ 12V = \pi H^3$$

$$\therefore\ H^3 = \frac{12V}{\pi}$$

$$\therefore\ H = \sqrt[3]{\frac{12V}{\pi}}$$

So, the height of the solution in the funnel is $H(V) = \sqrt[3]{\frac{12V}{\pi}}$ cm, as required.

c
$$\begin{aligned} H \circ V &= H(V(t)) \\ &= H(2000 - 20t) \\ &= \sqrt[3]{\frac{12(2000 - 20t)}{\pi}} \\ &= \sqrt[3]{\frac{24\,000 - 240t}{\pi}} \end{aligned}$$

The function $H \circ V$ gives the height of the solution after t minutes.

d
$$\begin{aligned} (H \circ V)(30) &= \sqrt[3]{\frac{24\,000 - 240(30)}{\pi}} \\ &= \sqrt[3]{\frac{16\,800}{\pi}} \\ &\approx 17.5 \end{aligned}$$

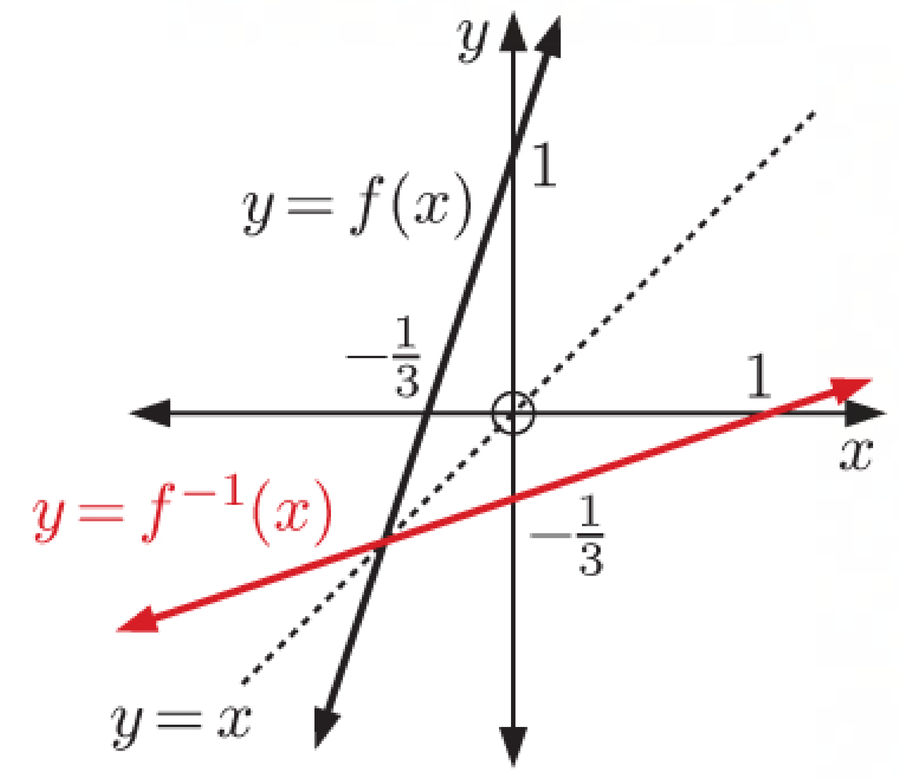
The height of the solution after 30 minutes is about 17.5 cm.

EXERCISE 15F

1 **a** $f : x \mapsto 3x + 1$

i $f(x) = 3x + 1$ passes through $(0, 1)$ and $(-\frac{1}{3}, 0)$.

$\therefore$ $f^{-1}(x)$ passes through $(1, 0)$ and $(0, -\frac{1}{3})$.



ii $y = f^{-1}(x)$ has gradient $\dfrac{-\frac{1}{3} - 0}{0 - 1} = \dfrac{1}{3}$

Its equation is $\dfrac{y - 0}{x - 1} = \dfrac{1}{3}$

$\therefore \; y = \dfrac{x - 1}{3}$

$\therefore \; f^{-1}(x) = \dfrac{x - 1}{3}$

iii f is $y = 3x + 1,$

$\therefore$ f^{-1} is $x = 3y + 1$

$\therefore \; x - 1 = 3y$

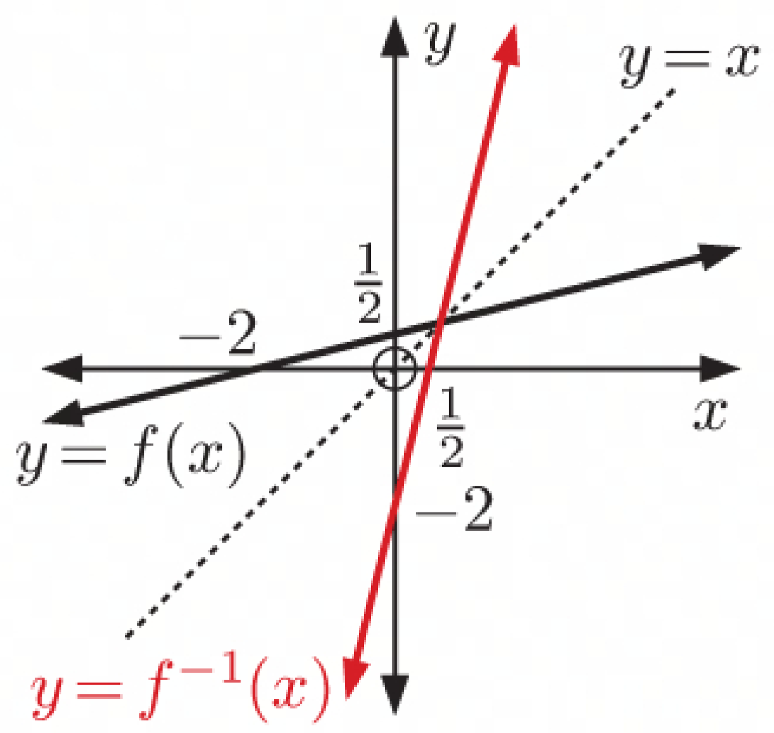
$\therefore \; \dfrac{x - 1}{3} = y$

$\therefore \; f^{-1}(x) = \dfrac{x - 1}{3}$

b $f : x \mapsto \dfrac{x + 2}{4}$

i $f(x) = \dfrac{x + 2}{4}$ passes through $(0, \frac{1}{2})$ and $(-2, 0)$.

$\therefore$ $f^{-1}(x)$ passes through $(\frac{1}{2}, 0)$ and $(0, -2)$.



ii $y = f^{-1}(x)$ has gradient $\dfrac{-2 - 0}{0 - \frac{1}{2}} = 4$

Its equation is $\dfrac{y - 0}{x - \frac{1}{2}} = 4$

$\therefore \; y = 4x - 2$

$\therefore \; f^{-1}(x) = 4x - 2$

iii f is $y = \dfrac{x + 2}{4},$

$\therefore$ f^{-1} is $x = \dfrac{y + 2}{4}$

$\therefore \; 4x = y + 2$

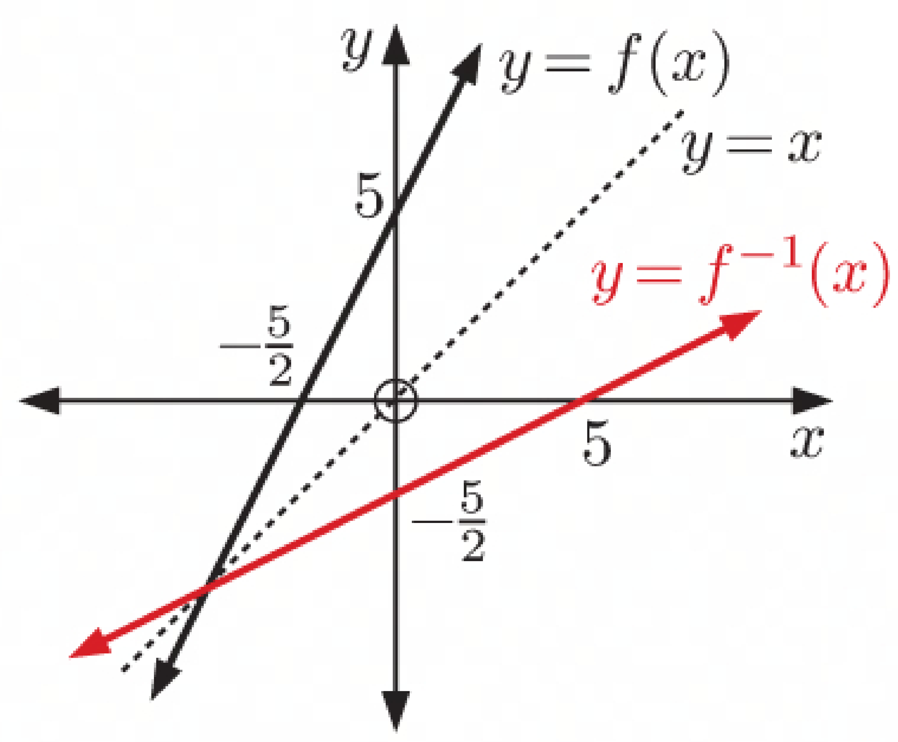
$\therefore \; 4x - 2 = y$

$\therefore \; f^{-1}(x) = 4x - 2$

2 **a** $f : x \mapsto 2x + 5$

i
$$\begin{aligned} f \text{ is } \quad & y = 2x + 5 \\ \therefore\ f^{-1} \text{ is } \quad & x = 2y + 5 \\ \therefore\ & x - 5 = 2y \\ \therefore\ & \frac{x-5}{2} = y \\ \therefore\ & f^{-1}(x) = \frac{x-5}{2} \end{aligned}$$

ii



iii
$$\begin{aligned} (f^{-1} \circ f)(x) &= f^{-1}(f(x)) \\ &= f^{-1}(2x+5) \\ &= \frac{2x+5-5}{2} \\ &= \frac{2x}{2} \\ &= x \end{aligned}$$

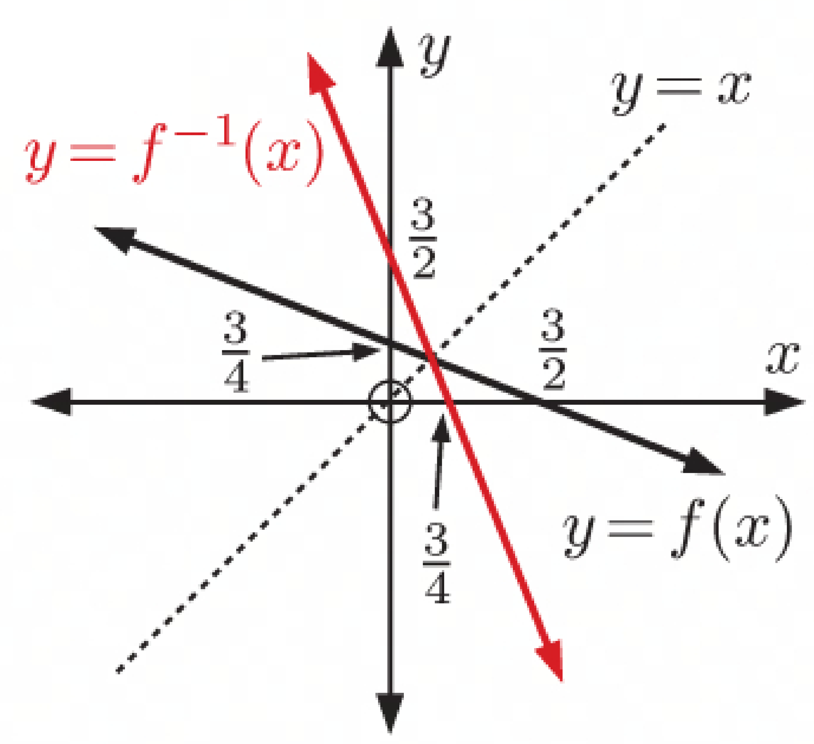
$$\begin{aligned} (f \circ f^{-1})(x) &= f(f^{-1}(x)) \\ &= f\left(\frac{x-5}{2}\right) \\ &= 2\left(\frac{x-5}{2}\right) + 5 \\ &= x - 5 + 5 \\ &= x \end{aligned}$$

$\therefore\ (f^{-1} \circ f)(x) = (f \circ f^{-1})(x) = x$ as required

b $f : x \mapsto \dfrac{3-2x}{4}$

i
$$\begin{aligned} f \text{ is } \quad & y = \frac{3-2x}{4} \\ \therefore\ f^{-1} \text{ is } \quad & x = \frac{3-2y}{4} \\ \therefore\ & 4x = 3 - 2y \\ \therefore\ & 4x - 3 = -2y \\ \therefore\ & -2x + \tfrac{3}{2} = y \\ \therefore\ & f^{-1}(x) = -2x + \tfrac{3}{2} \end{aligned}$$

ii



iii
$$\begin{aligned} (f^{-1} \circ f)(x) &= f^{-1}(f(x)) \\ &= f^{-1}\left(\frac{3-2x}{4}\right) \\ &= -2\left(\frac{3-2x}{4}\right) + \frac{3}{2} \\ &= -\tfrac{3}{2} + x + \tfrac{3}{2} \\ &= x \end{aligned}$$

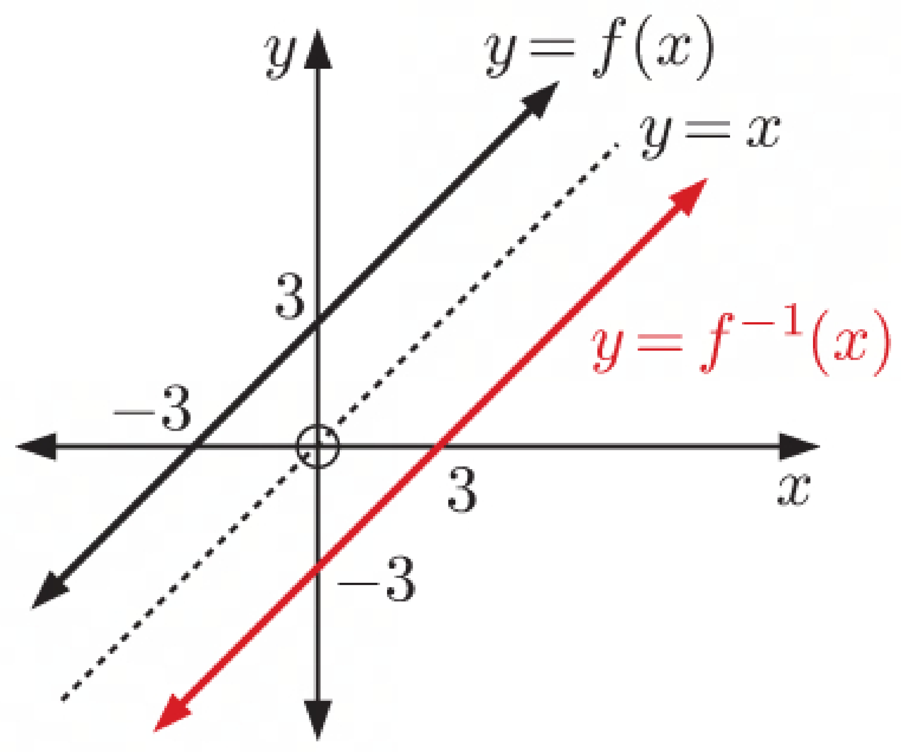
$$\begin{aligned} (f \circ f^{-1})(x) &= f(f^{-1}(x)) \\ &= f(-2x + \tfrac{3}{2}) \\ &= \frac{3 - 2(-2x + \frac{3}{2})}{4} \\ &= \frac{3 + 4x - 3}{4} \\ &= \frac{4x}{4} \\ &= x \end{aligned}$$

$\therefore\ (f^{-1} \circ f)(x) = (f \circ f^{-1})(x) = x$ as required

c $f: x \mapsto x+3$

i
$$\begin{aligned} f \text{ is } \quad & y = x+3 \\ \therefore\ f^{-1} \text{ is } \quad & x = y+3 \\ \therefore\ & x-3 = y \\ \therefore\ & f^{-1}(x) = x-3 \end{aligned}$$

ii



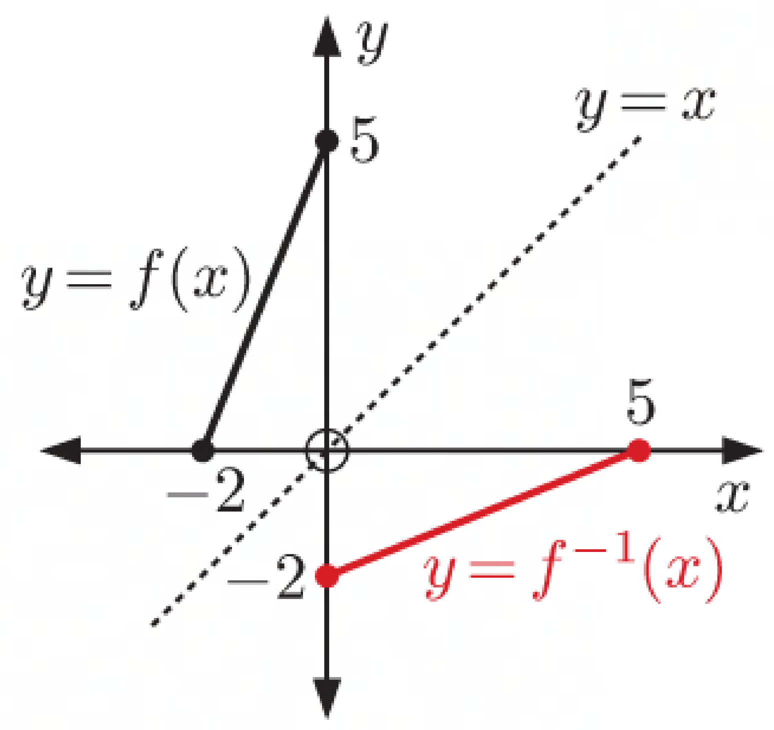
iii
$$\begin{aligned} (f^{-1} \circ f)(x) &= f^{-1}(f(x)) \\ &= f^{-1}(x+3) \\ &= x+3-3 \\ &= x \end{aligned}$$

$$\begin{aligned} (f \circ f^{-1})(x) &= f(f^{-1}(x)) \\ &= f(x-3) \\ &= x-3+3 \\ &= x \end{aligned}$$

$\therefore\ (f^{-1} \circ f)(x) = (f \circ f^{-1})(x) = x$ as required

3 **a** $f(x)$ passes through $(0, 5)$ and $(-2, 0)$.

$\therefore\ f^{-1}(x)$ passes through $(5, 0)$ and $(0, -2)$.

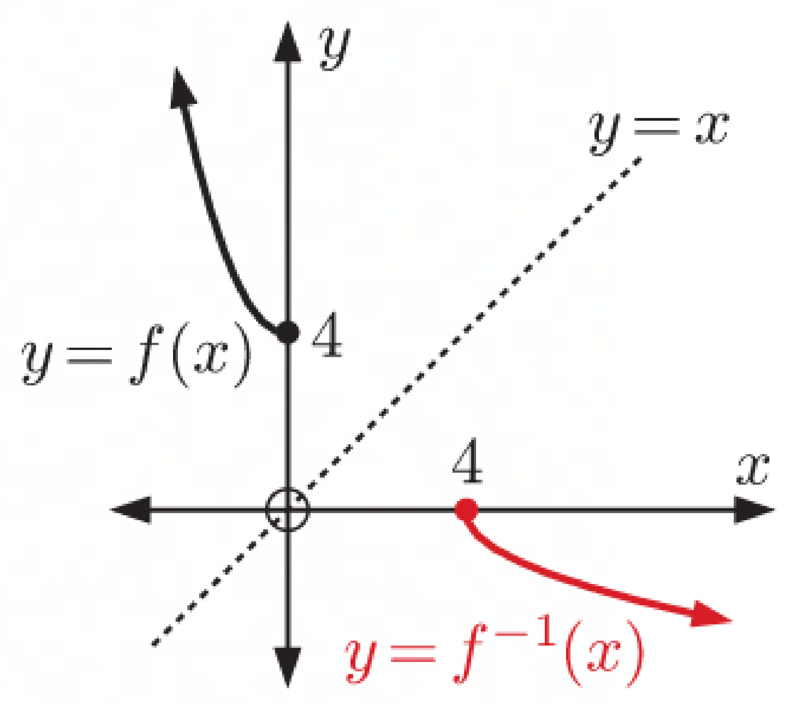


f: Domain is $\{x \mid -2 \leqslant x \leqslant 0\}$
Range is $\{y \mid 0 \leqslant y \leqslant 5\}$

f^{-1}: Domain is $\{x \mid 0 \leqslant x \leqslant 5\}$
Range is $\{y \mid -2 \leqslant y \leqslant 0\}$

b $f(x)$ passes through $(0, 4)$.

$\therefore\ f^{-1}(x)$ passes through $(4, 0)$.

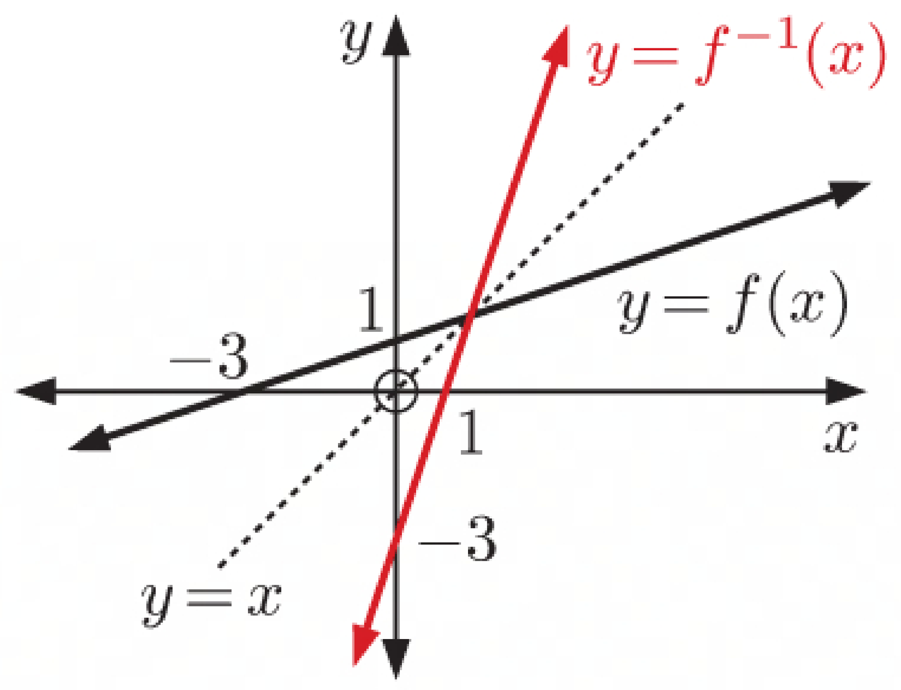


f: Domain is $\{x \mid x \leqslant 0\}$
Range is $\{y \mid y \geqslant 4\}$

f^{-1}: Domain is $\{x \mid x \geqslant 4\}$
Range is $\{y \mid y \leqslant 0\}$

c $f(x)$ passes through $(0,\ 1)$ and $(-3,\ 0)$.

$\therefore$ $f^{-1}(x)$ passes through $(1,\ 0)$ and $(0,\ -3)$.

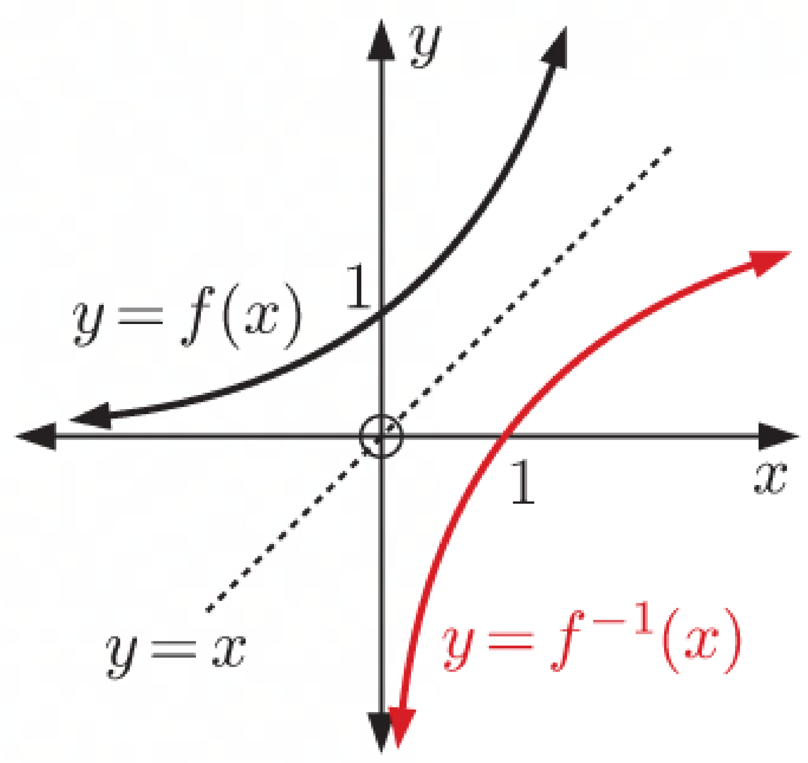


f: Domain is $\{x \mid x \in \mathbb{R}\}$
Range is $\{y \mid y \in \mathbb{R}\}$

f^{-1}: Domain is $\{x \mid x \in \mathbb{R}\}$
Range is $\{y \mid y \in \mathbb{R}\}$

d $f(x)$ passes through $(0,\ 1)$.

$\therefore$ $f^{-1}(x)$ passes through $(1,\ 0)$.

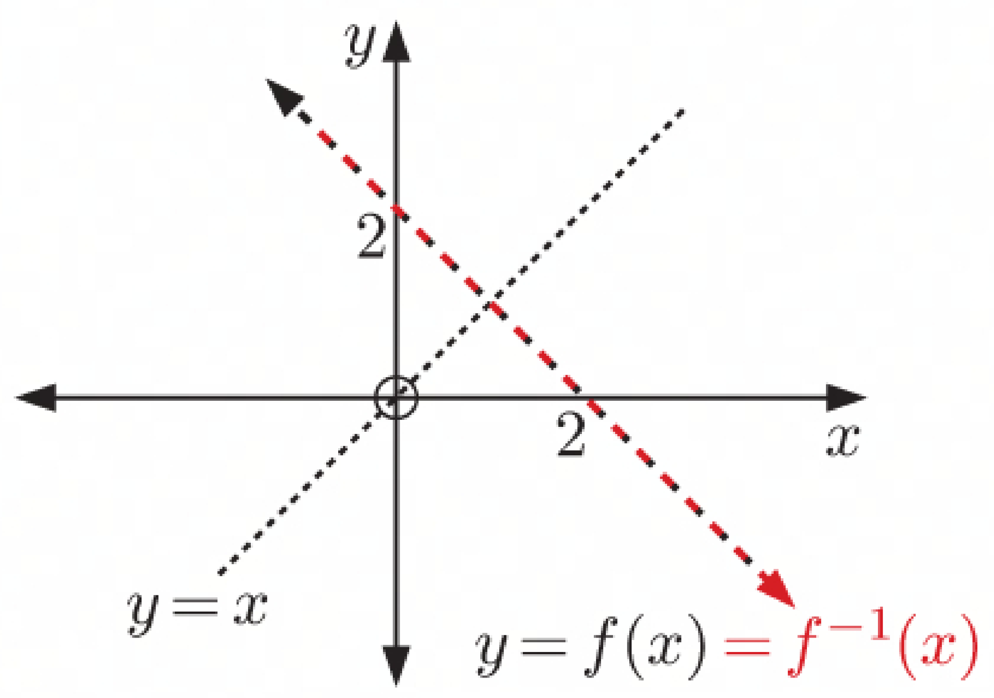


f: Domain is $\{x \mid x \in \mathbb{R}\}$
Range is $\{y \mid y > 0\}$

f^{-1}: Domain is $\{x \mid x > 0\}$
Range is $\{y \mid y \in \mathbb{R}\}$

e $f(x)$ passes through $(0,\ 2)$ and $(2,\ 0)$.

$\therefore$ $f^{-1}(x)$ passes through $(2,\ 0)$ and $(0,\ 2)$.

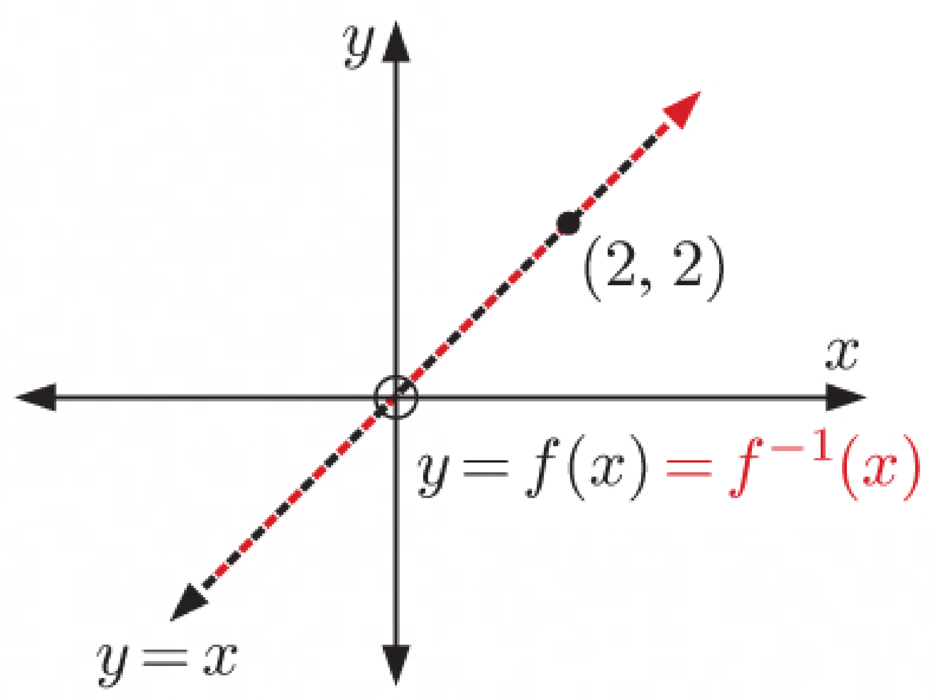


f: Domain is $\{x \mid x \in \mathbb{R}\}$
Range is $\{y \mid y \in \mathbb{R}\}$

f^{-1}: Domain is $\{x \mid x \in \mathbb{R}\}$
Range is $\{y \mid y \in \mathbb{R}\}$

f $f(x)$ passes through $(0,\ 0)$ and $(2,\ 2)$.

$\therefore$ $f^{-1}(x)$ passes through $(0,\ 0)$ and $(2,\ 2)$.



f: Domain is $\{x \mid x \in \mathbb{R}\}$
Range is $\{y \mid y \in \mathbb{R}\}$

f^{-1}: Domain is $\{x \mid x \in \mathbb{R}\}$
Range is $\{y \mid y \in \mathbb{R}\}$

4

$$\begin{aligned} f \text{ is } \quad y &= 2x-5 \\ \therefore\ f^{-1} \text{ is } \quad x &= 2y-5 \\ \therefore\ x+5 &= 2y \\ \therefore\ \frac{x+5}{2} &= y \\ \therefore\ f^{-1}(x) &= \frac{x+5}{2} \end{aligned}$$

$$\begin{aligned} f^{-1} \text{ is } \quad y &= \frac{x+5}{2} \\ \therefore\ (f^{-1})^{-1} \text{ is } \quad x &= \frac{y+5}{2} \\ \therefore\ 2x &= y+5 \\ \therefore\ 2x-5 &= y \\ \therefore\ (f^{-1})^{-1}(x) &= 2x-5 = f(x) \end{aligned}$$

5 **a** For $\{(1, 2), (2, 4), (3, 5)\}$, there is at most one x-value corresponding to each y-value. So, the function is one-to-one and hence has an inverse.
The inverse function is $\{(2, 1), (4, 2), (5, 3)\}$.

b For $\{(-1, 3), (0, 2), (1, 3)\}$, there are two x-values corresponding to the y-value of 3. So, the function is many-to-one and hence does not have an inverse.

c For $\{(2, 1), (-1, 0), (0, 2), (1, 3)\}$, there is at most one x-value corresponding to each y-value. So, the function is one-to-one and hence has an inverse.
The inverse function is $\{(1, 2), (0, -1), (2, 0), (3, 1)\}$.

d For $\{(-1, -1), (0, 0), (1, 1)\}$, there is at most one x-value corresponding to each y-value. So, the function is one-to-one and hence has an inverse.
The inverse function is $\{(-1, -1), (0, 0), (1, 1)\}$.

6 A linear function is of the form $f(x) = mx + c$ where m and c are constants.

$$\begin{aligned} \text{So, } f \text{ is } \quad y &= mx+c \\ \therefore\ f^{-1} \text{ is } \quad x &= my+c \\ \therefore\ x-c &= my \\ \therefore\ \frac{x-c}{m} &= y, \quad m \neq 0 \\ \therefore\ f^{-1}(x) &= \frac{x-c}{m} \end{aligned}$$

For the function f to be self-inverse, $f^{-1}(x) = f(x)$ for all x

$$\begin{aligned} \therefore\ \frac{x-c}{m} &= mx+c \\ \therefore\ \frac{x}{m} - \frac{c}{m} &= mx+c \\ \therefore\ \frac{1}{m} = m \quad &\text{and} \quad -\frac{c}{m} = c \\ \therefore\ m^2 = 1 \quad &\text{and} \quad -c = cm \\ \therefore\ m = \pm 1 \quad &\text{and} \quad c + cm = 0 \end{aligned}$$

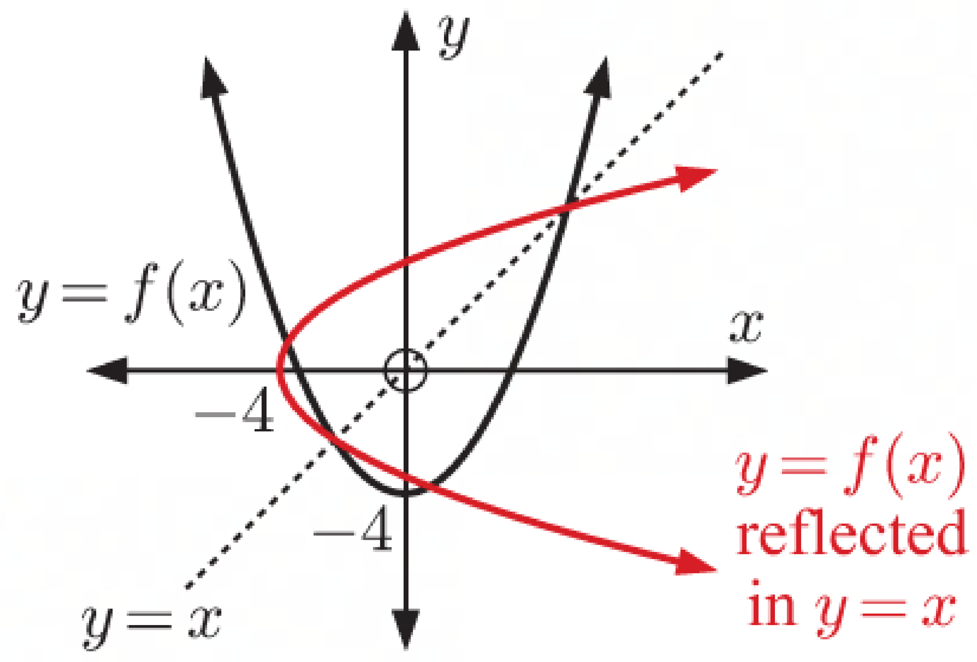
If $m = 1$, $c + c(1) = 0$
$\therefore\ 2c = 0$
$\therefore\ c = 0$

If $m = -1$, $c + c(-1) = 0$
$\therefore\ 0 = 0$ which is true for all $c \in \mathbb{R}$

$\therefore$ all linear functions which are self-inverse are of the form $f(x) = x$ or $f(x) = -x + c$, $c \in \mathbb{R}$.

7 The range of $H^{-1}(x)$ is the domain of $H(x)$.
$\therefore$ the range of $H^{-1}(x)$ is $\{y \mid -2 \leqslant y < 3\}$.

8 **a**



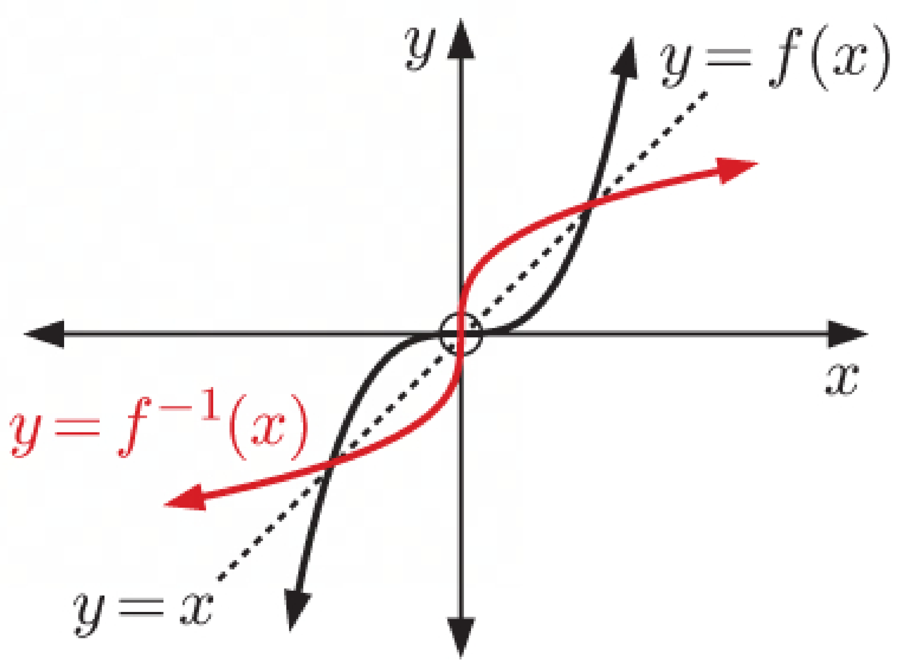
b f does not have an inverse function as the reflection of f in $y = x$ fails the vertical line test.

c For $x \geqslant 0$, the reflection of f in $y = x$ passes the vertical line test.

$\therefore$ f does have an inverse function for $x \geqslant 0$;

$f^{-1}(x) : x \mapsto \sqrt{x+4}$.

9



10

$$f \text{ is } y = \frac{1}{x}, \quad x \neq 0$$

$$\therefore \quad f^{-1} \text{ is } x = \frac{1}{y}, \quad y \neq 0$$

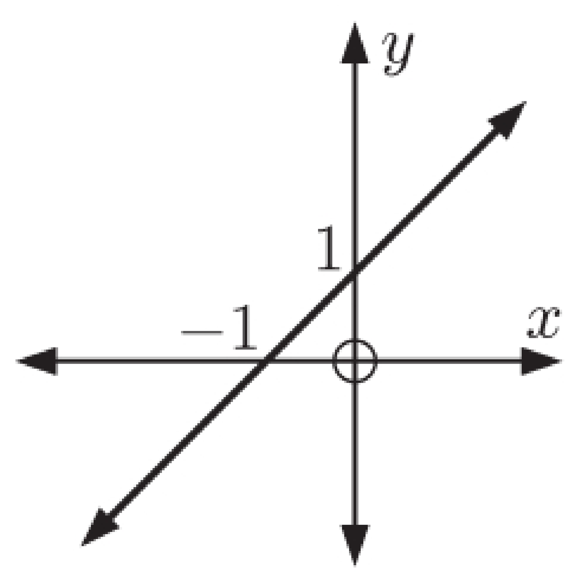
$$\therefore \quad y = \frac{1}{x}$$

So, $f^{-1}(x) = \dfrac{1}{x} = f(x)$

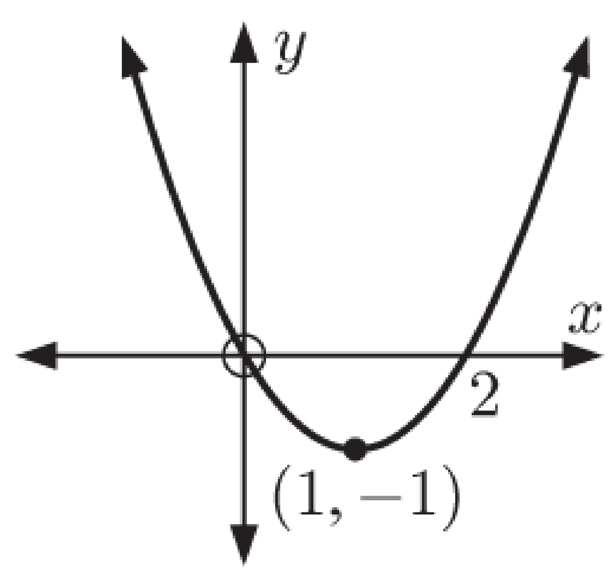
$\therefore$ f is self-inverse, as required.

11 **a** The inverse function must also be a function and must therefore satisfy the vertical line test, which it can only do if the original function satisfies the horizontal line test.

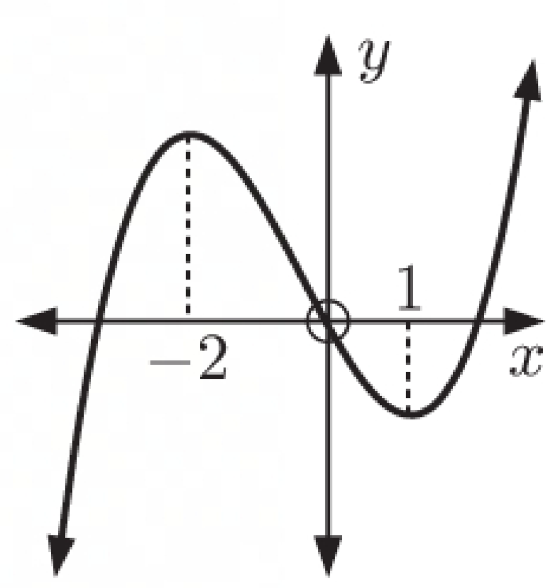
b **i** This graph satisfies the horizontal line test and therefore has an inverse function.



ii This graph fails the horizontal line test and therefore does not have an inverse function.



iii This graph fails the horizontal line test and therefore does not have an inverse function.



c **ii** Domain is $\{x \mid x \geqslant 1\}$ or $\{x \mid x \leqslant 1\}$

iii Domain is $\{x \mid x \geqslant 1\}$ or $\{x \mid -2 \leqslant x \leqslant 1\}$ or $\{x \mid x \leqslant -2\}$

12 $f : x \mapsto x^2, \quad x \leqslant 0$

a f is $y = x^2, \quad x \leqslant 0$

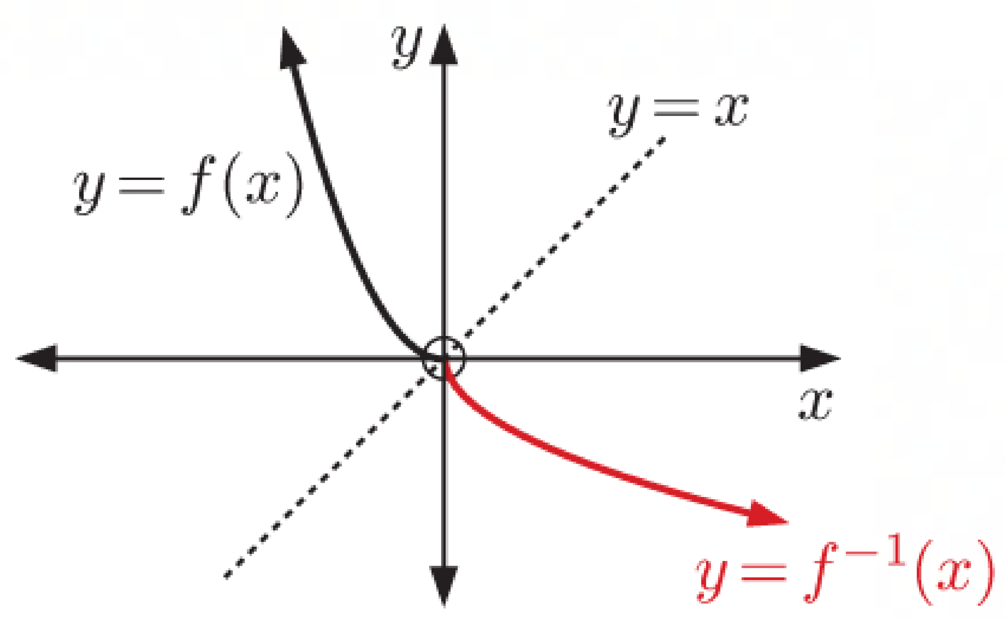
$\therefore$ f^{-1} is $x = y^2, \quad y \leqslant 0$

$\therefore \quad y = \pm\sqrt{x}, \quad y \leqslant 0$

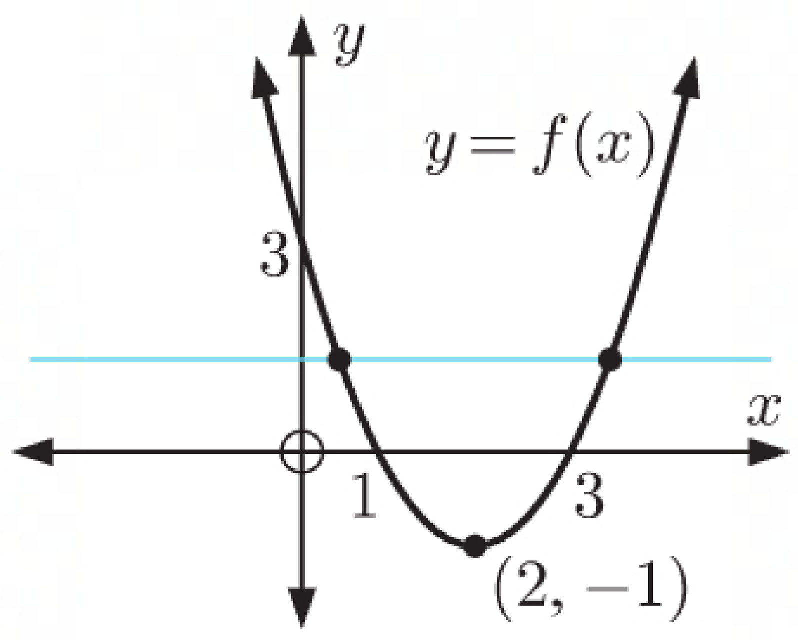
$\therefore \quad y = -\sqrt{x} \quad \{\text{as } y \leqslant 0 \text{ and } -\sqrt{x} \leqslant 0\}$

So, $f^{-1}(x) = -\sqrt{x}$

b



13 **a**



$f(x) = x^2 - 4x + 3$ satisfies the vertical line test so is therefore a function. It does not however satisfy the horizontal line test as any horizontal line above the vertex cuts the graph twice. Therefore it does not have an inverse function.

b For $g(x) = x^2 - 4x + 3$ where $x \geqslant 2$, all horizontal lines cut the graph at most once. Therefore $g(x)$ has an inverse function for $x \geqslant 2$.

g is $y = x^2 - 4x + 3, \quad x \geqslant 2$

$\therefore$ g^{-1} is $x = y^2 - 4y + 3, \quad y \geqslant 2$

$\therefore \quad x = y^2 - 4y + (-2)^2 + 3 - (-2)^2$

$\therefore \quad x = (y-2)^2 - 1$

$\therefore \quad x + 1 = (y-2)^2$

$\therefore \quad y - 2 = \pm\sqrt{x+1}, \quad y \geqslant 2, \quad x \geqslant -1$

$\therefore \quad y - 2 = \sqrt{x+1} \quad \{\text{as } y - 2 \geqslant 0 \text{ and } \sqrt{x+1} \geqslant 0\}$

$\therefore \quad y = 2 + \sqrt{1+x}, \quad y \geqslant 2, \quad x \geqslant -1$

So, $g^{-1}(x) = 2 + \sqrt{1+x}$ as required.

c g:

Domain is $\{x \mid x \geqslant 2\}$

Range is $\{y \mid y \geqslant -1\}$

g^{-1}:

Domain is $\{x \mid x \geqslant -1\}$

Range is $\{y \mid y \geqslant 2\}$

d
$$\begin{aligned}(g \circ g^{-1})(x) &= g(g^{-1}(x)) \\ &= \left(2 + \sqrt{1+x}\right)^2 - 4\left(2 + \sqrt{1+x}\right) + 3 \\ &= 4 + 4\sqrt{1+x} + 1 + x - 8 - 4\sqrt{1+x} + 3 \\ &= x\end{aligned}$$

$$\begin{aligned}(g^{-1} \circ g)(x) &= g^{-1}(g(x)) \\ &= 2 + \sqrt{1 + x^2 - 4x + 3} \\ &= 2 + \sqrt{(x-2)^2} \\ &= 2 + |x - 2| \\ &= 2 + x - 2, \quad x \geqslant 2 \\ &= x\end{aligned}$$

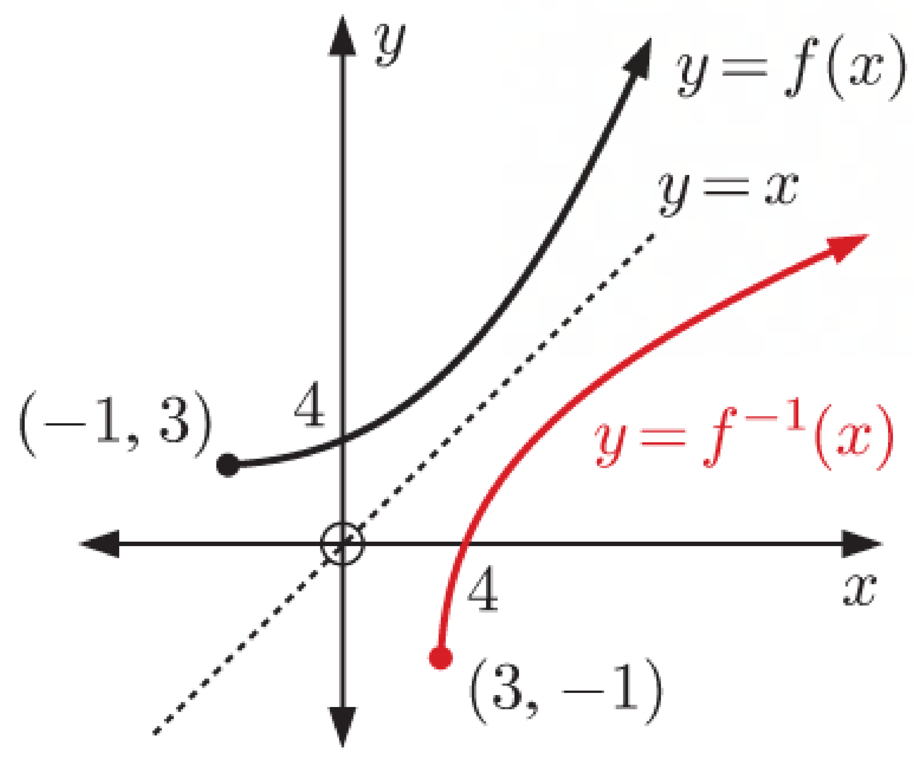
$\therefore \quad (g \circ g^{-1})(x) = (g^{-1} \circ g)(x) = x$ as required

14 $f : x \mapsto (x+1)^2 + 3, \quad x \geqslant -1$

a
$$\begin{aligned}
f \text{ is } \quad & y = (x+1)^2 + 3, \quad x \geqslant -1 \\
\therefore\ f^{-1} \text{ is } \quad & x = (y+1)^2 + 3, \quad y \geqslant -1 \\
\therefore\ & x - 3 = (y+1)^2 \\
\therefore\ & y + 1 = \pm\sqrt{x-3}, \quad y \geqslant -1, \quad x \geqslant 3 \\
\therefore\ & y + 1 = \sqrt{x-3} \qquad \{\text{as } y+1 \geqslant 0 \text{ and } \sqrt{x-3} \geqslant 0\} \\
\therefore\ & y = \sqrt{x-3} - 1, \quad y \geqslant -1, \quad x \geqslant 3
\end{aligned}$$

So, $f^{-1}(x) = \sqrt{x-3} - 1$

b



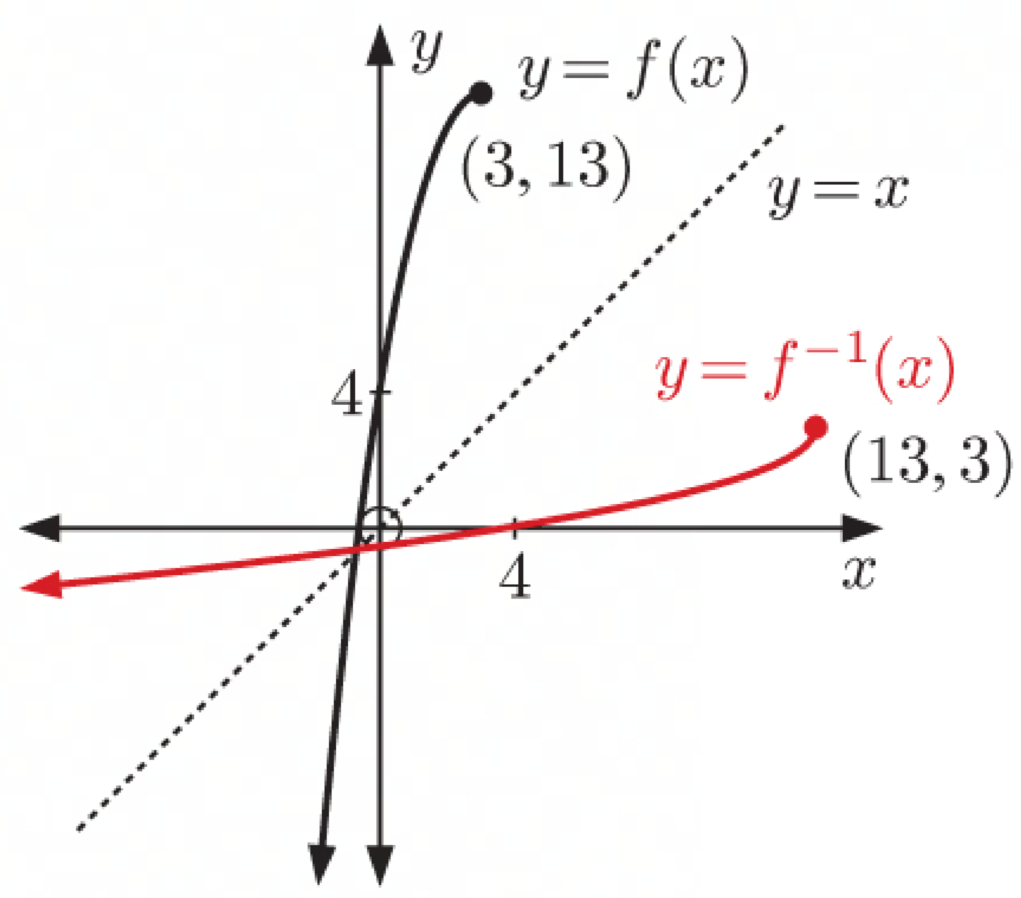
c f: Domain is $\{x \mid x \geqslant -1\}$
Range is $\{y \mid y \geqslant 3\}$

f^{-1}: Domain is $\{x \mid x \geqslant 3\}$
Range is $\{y \mid y \geqslant -1\}$

15 $f(x) = 4 + 6x - x^2, \quad x \leqslant 3$

a
$$\begin{aligned}
f \text{ is } \quad & y = 4 + 6x - x^2, \quad x \leqslant 3 \\
\therefore\ f^{-1} \text{ is } \quad & x = 4 + 6y - y^2, \quad y \leqslant 3 \\
\therefore\ & -x = y^2 - 6y - 4 \\
\therefore\ & -x = y^2 - 6y + (-3)^2 - 4 - (-3)^2 \\
\therefore\ & -x = (y-3)^2 - 13 \\
\therefore\ & (y-3)^2 = 13 - x \\
\therefore\ & y - 3 = \pm\sqrt{13-x}, \quad y \leqslant 3, \quad x \leqslant 13 \\
\therefore\ & y - 3 = -\sqrt{13-x} \\
& \{\text{as } y-3 \leqslant 0 \text{ and } -\sqrt{13-x} \leqslant 0\} \\
\therefore\ & y = 3 - \sqrt{13-x}, \quad y \leqslant 3, \quad x \leqslant 13
\end{aligned}$$

So, $f^{-1}(x) = 3 - \sqrt{13-x}$



b f: Domain is $\{x \mid x \leqslant 3\}$
Range is $\{y \mid y \leqslant 13\}$

f^{-1}: Domain is $\{x \mid x \leqslant 13\}$
Range is $\{y \mid y \leqslant 3\}$

16 $f(x) = 2x^2 - 10x + 6, \quad x \leqslant k$

a The x-coordinate of the vertex of the graph of $y = 2x^2 - 10x + 6$ is
$$\frac{-b}{2a} = \frac{-(-10)}{2(2)} = \frac{10}{4} = \frac{5}{2}$$

Since $y = 2x^2 - 10x + 6$ is a quadratic, the part of the graph where $x \leqslant \frac{5}{2}$ will satisfy the horizontal line test and hence have an inverse.

$\therefore$ the largest value of k such that $f^{-1}(x)$ exists is $k = \frac{5}{2}$.

b **i**

$$
\begin{aligned}
f \text{ is } \quad & y = 2x^2 - 10x + 6, \quad x \leqslant \tfrac{5}{2} \\
\therefore\ f^{-1} \text{ is } \quad & x = 2y^2 - 10y + 6, \quad y \leqslant \tfrac{5}{2} \\
\therefore\ & x = 2(y^2 - 5y + 3) \\
\therefore\ & x = 2\left(y^2 - 5y + \left(-\tfrac{5}{2}\right)^2 + 3 - \left(-\tfrac{5}{2}\right)^2\right) \\
\therefore\ & x = 2\left[\left(y - \tfrac{5}{2}\right)^2 - \tfrac{13}{4}\right] \\
\therefore\ & x = 2\left(y - \tfrac{5}{2}\right)^2 - \tfrac{13}{2} \\
\therefore\ & 2x = 4\left(y - \tfrac{5}{2}\right)^2 - 13 \\
\therefore\ & 2x + 13 = 4\left(y - \tfrac{5}{2}\right)^2 \\
\therefore\ & \left(y - \tfrac{5}{2}\right)^2 = \frac{2x+13}{4} \\
\therefore\ & y - \tfrac{5}{2} = \pm\sqrt{\frac{2x+13}{4}}, \quad y \leqslant \tfrac{5}{2}, \quad x \geqslant -\tfrac{13}{2} \\
\therefore\ & y - \tfrac{5}{2} = \frac{-\sqrt{2x+13}}{2} \quad \left\{\text{as } y - \tfrac{5}{2} \leqslant 0 \text{ and } -\frac{\sqrt{2x+13}}{2} \leqslant 0\right\} \\
\therefore\ & y = \frac{5 - \sqrt{2x+13}}{2}
\end{aligned}
$$

So, $f^{-1}(x) = \dfrac{5 - \sqrt{2x+13}}{2}$

ii f^{-1}: Domain is $\{x \mid x \geqslant -\frac{13}{2}\}$

Range is $\{y \mid y \leqslant \frac{5}{2}\}$

17 $f : x \mapsto 2x + 5$ and $g : x \mapsto \dfrac{8-x}{2}$

a

$$
\begin{aligned}
g \text{ is } \quad & y = \frac{8-x}{2} \\
\therefore\ g^{-1} \text{ is } \quad & x = \frac{8-y}{2} \\
\therefore\ & 2x = 8 - y \\
\therefore\ & y = 8 - 2x
\end{aligned}
$$

So, $g^{-1}(x) = 8 - 2x$

b

$$
\begin{aligned}
& g(x) = -1 \\
\therefore\ & g^{-1}(g(x)) = g^{-1}(-1) \\
\therefore\ & x = 8 - 2(-1) \\
\therefore\ & x = 10
\end{aligned}
$$

c

$$
\begin{aligned}
f \text{ is } \quad & y = 2x + 5 \\
\therefore\ f^{-1} \text{ is } \quad & x = 2y + 5 \\
\therefore\ & 2y = x - 5 \\
\therefore\ & y = \frac{x-5}{2}
\end{aligned}
$$

So, $f^{-1}(x) = \dfrac{x-5}{2}$

$$
\therefore\ f^{-1}(-3) = \frac{(-3)-5}{2} = \frac{-8}{2} = -4 \quad \text{and} \quad g^{-1}(6) = 8 - 2(6) = 8 - 12 = -4
$$

$$
\begin{aligned}
\therefore\ f^{-1}(-3) - g^{-1}(6) &= -4 - (-4) \\
&= 0 \quad \text{as required}
\end{aligned}
$$

d

$$
\begin{aligned}
& (f \circ g^{-1})(x) = 9 \\
\therefore\ & f(g^{-1}(x)) = 9 \\
\therefore\ & f(8 - 2x) = 9 \\
\therefore\ & 2(8 - 2x) + 5 = 9 \\
\therefore\ & 16 - 4x + 5 = 9 \\
\therefore\ & -4x = -12 \\
\therefore\ & x = 3
\end{aligned}
$$

18 $f : x \mapsto 2x$ and $g : x \mapsto 4x - 3$

a
$$\begin{aligned} (f \circ g)(x) &= f(g(x)) \\ &= f(4x-3) \\ &= 2(4x-3) \\ &= 8x - 6 \end{aligned}$$
$\therefore\ (f \circ g)(x) = 8x - 6$

b
$$(f \circ g)^{-1}(k) = 2$$
$$\therefore\ (f \circ g)\big((f \circ g)^{-1}(k)\big) = (f \circ g)(2)$$
$$\therefore\ k = 8(2) - 6$$
$$\therefore\ k = 10$$

c
f is $y = 2x$
$\therefore\ f^{-1}$ is $x = 2y$
$\therefore\ y = \dfrac{x}{2}$
$\therefore\ f^{-1}(x) = \dfrac{x}{2}$

g is $y = 4x - 3$
$\therefore\ g^{-1}$ is $x = 4y - 3$
$\therefore\ 4y = x + 3$
$\therefore\ y = \dfrac{x+3}{4}$
$\therefore\ g^{-1}(x) = \dfrac{x+3}{4}$

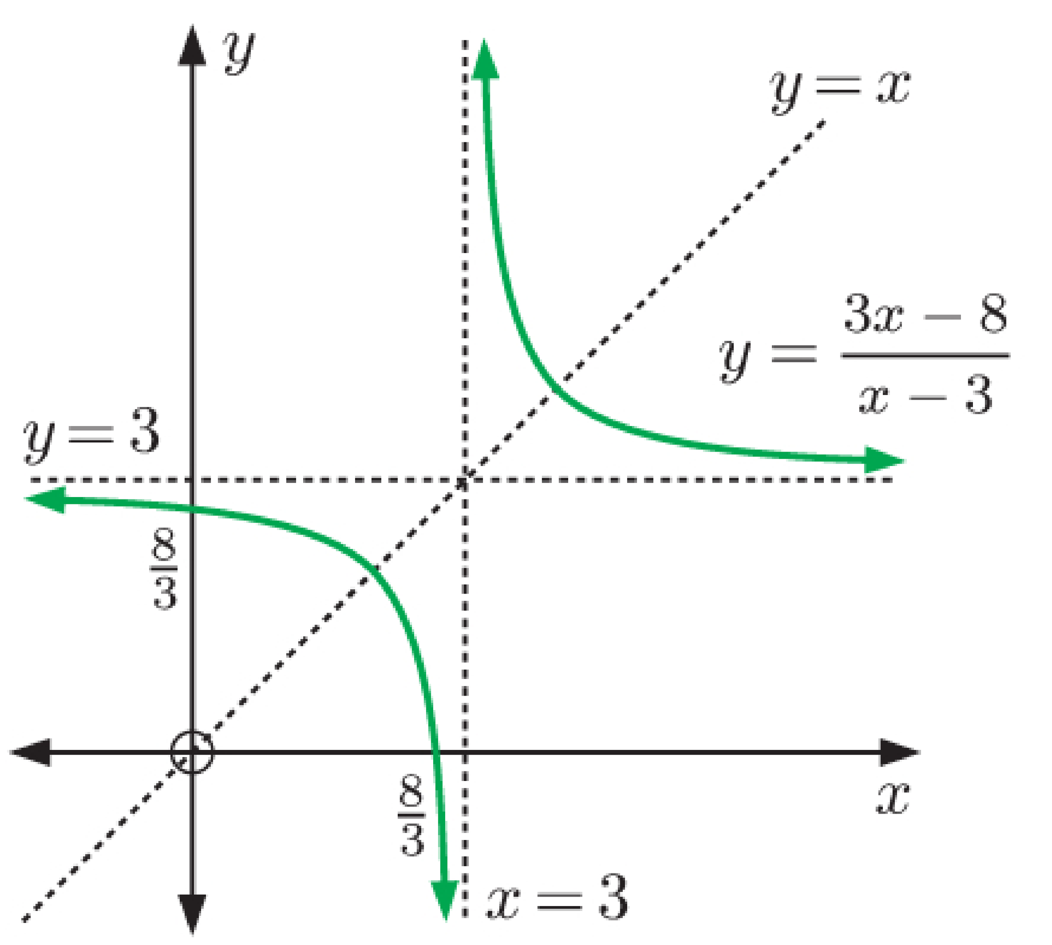
$$\begin{aligned} (g \circ f)(x) &= g(f(x)) \\ &= g(2x) \\ &= 4(2x) - 3 \end{aligned}$$
$\therefore\ (g \circ f)(x) = 8x - 3$ .... $(*)$

$g \circ f$ is $y = 8x - 3$ {using $(*)$}
$\therefore\ (g \circ f)^{-1}$ is $x = 8y - 3$
$\therefore\ y = \dfrac{x+3}{8}$
$\therefore\ (g \circ f)^{-1}(x) = \dfrac{x+3}{8}$

Now
$$\begin{aligned} (f^{-1} \circ g^{-1})(x) &= f^{-1}(g^{-1}(x)) \\ &= f^{-1}\left(\frac{x+3}{4}\right) \\ &= \frac{\left(\frac{x+3}{4}\right)}{2} \\ &= \frac{x+3}{8} \\ &= (g \circ f)^{-1}(x) \quad \text{as required} \end{aligned}$$

19 $f : x \mapsto \dfrac{3x-8}{x-3}$, $x \neq 3$

a



$y = \dfrac{3x-8}{x-3}$ is symmetrical about $y = x$

$\therefore\ f$ is a self-inverse function.

b
f is $y = \dfrac{3x-8}{x-3}$
$\therefore\ f^{-1}$ is $x = \dfrac{3y-8}{y-3}$
$\therefore\ x(y-3) = 3y - 8$
$\therefore\ xy - 3x = 3y - 8$
$\therefore\ y(x-3) = 3x - 8$
$\therefore\ y = \dfrac{3x-8}{x-3}$

So, $f^{-1}(x) = \dfrac{3x-8}{x-3} = f(x)$

$\therefore\ f$ is a self-inverse function.

20 Let $f(x)=\dfrac{ax+b}{cx+d}, \quad a,\ c\neq 0$

$$\therefore\ f(x)=\frac{\frac{a}{c}(cx+d)+b-\frac{ad}{c}}{cx+d}$$
$$=\frac{a}{c}+\frac{b-\frac{ad}{c}}{cx+d}$$

If $b-\dfrac{ad}{c}=0$, then $f(x)=\dfrac{a}{c}$ which is not invertible.

So, for the inverse to exist, we require $b-\dfrac{ad}{c}\neq 0$ or $ad\neq bc$.

Now f is $y=\dfrac{ax+b}{cx+d}$

$\therefore\ f^{-1}$ is $x=\dfrac{ay+b}{cy+d}$

$$\therefore\ x(cy+d)=ay+b$$
$$\therefore\ cxy+dx=ay+b$$
$$\therefore\ cxy-ay=b-dx$$
$$\therefore\ y(cx-a)=b-dx$$
$$\therefore\ y=\frac{b-dx}{cx-a}$$

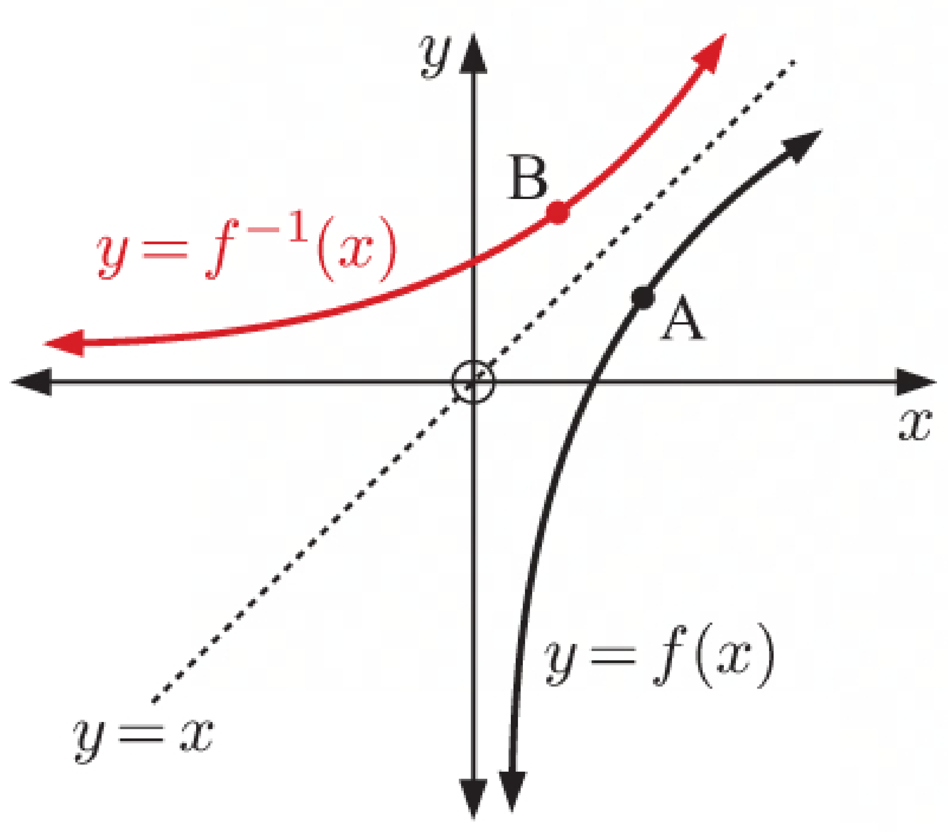
So, $f^{-1}(x)=\dfrac{b-dx}{cx-a}$

f is self-inverse is $f(x)=f^{-1}(x)$

$$\therefore\ \frac{ax+b}{cx+d}=\frac{b-dx}{cx-a}$$
$$\therefore\ (ax+b)(cx-a)=(b-dx)(cx+d)$$
$$\therefore\ acx^2-a^2x+bcx-ab=bcx+bd-cdx^2-d^2x$$
$$\therefore\ c(a+d)x^2+(d^2-a^2)x-b(a+d)=0$$
$$\therefore\ c(a+d)=0 \qquad \{\text{equating coefficients of } x^2\}$$
$$\therefore\ a+d=0 \qquad \{c\neq 0\}$$
$$\therefore\ d=-a$$

So f is self-inverse if $d=-a$, $ad\neq bc$.

21 **a** $f(x)$ passes through $A(x,\ f(x))$, so $f^{-1}(x)$ passes through $B(f(x),\ x)$.



b Substituting the coordinates of $B(f(x),\ x)$ into $y=f^{-1}(x)$ gives $x=f^{-1}(f(x))$.

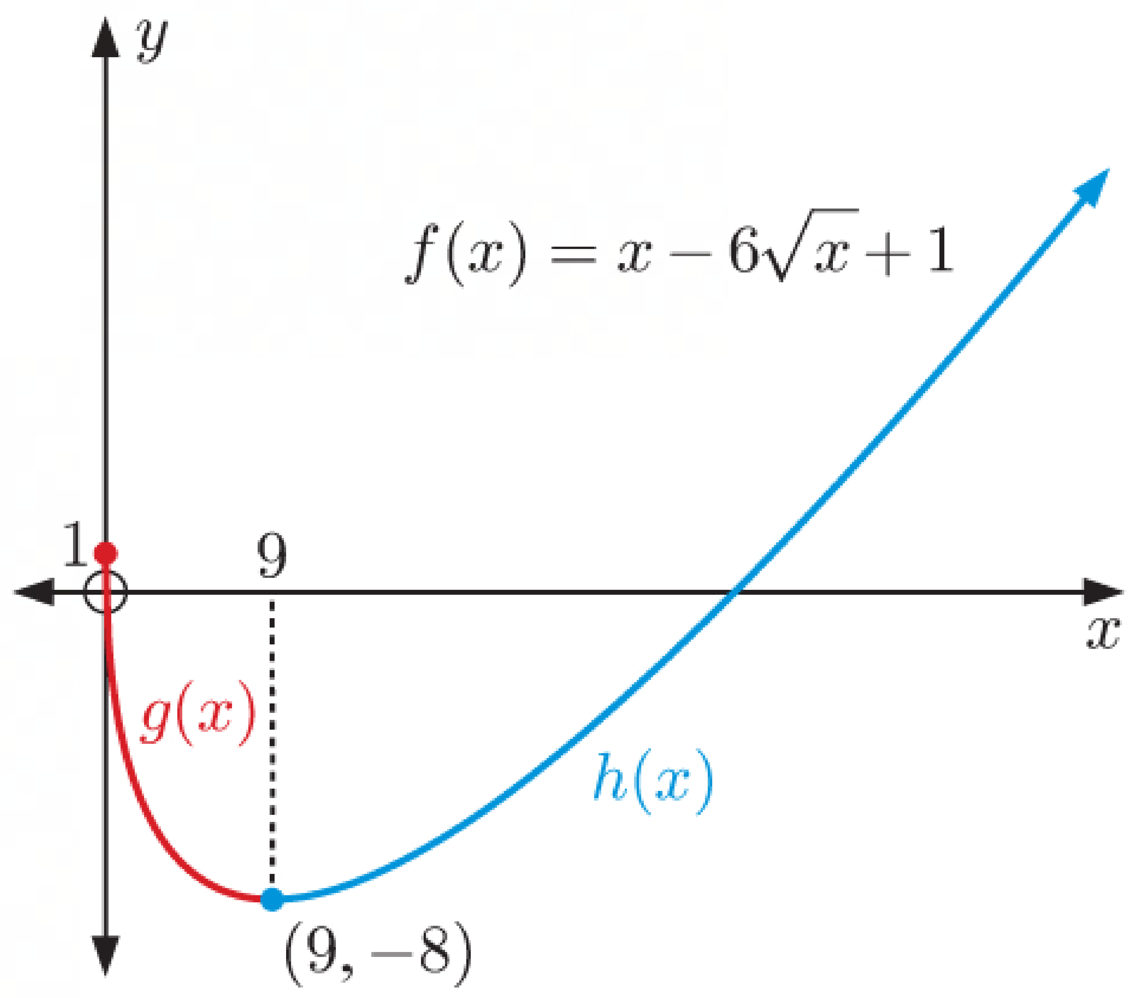
$\therefore\ f^{-1}(f(x))=x$ as required

c B has coordinates $(x,\ f^{-1}(x))$ since it lies on $y=f^{-1}(x)$, so A has coordinates $(f^{-1}(x),\ x)$ as $f(x)$ is the inverse of $f^{-1}(x)$.

Substituting the coordinates of $A(f^{-1}(x),\ x)$ into $y=f(x)$ gives $x=f(f^{-1}(x))$.

$\therefore\ f(f^{-1}(x))=x$ as required

22 $f(x) = x - 6\sqrt{x} + 1$



a The natural domain of $f(x)$ is $\{x \mid x \geqslant 0\}$.

b On its natural domain, the graph does not pass the horizontal line test.

$\therefore$ $f(x)$ does not have an inverse.

c $g(x) = x - 6\sqrt{x} + 1, \;\; 0 \leqslant x \leqslant 9$

i

$$\begin{aligned}
g \text{ is } \quad & y = x - 6\sqrt{x} + 1, \quad 0 \leqslant x \leqslant 9 \\
\therefore\; g^{-1} \text{ is } \quad & x = y - 6\sqrt{y} + 1, \quad 0 \leqslant y \leqslant 9 \\
\therefore\; & x = y - 6\sqrt{y} + (-3)^2 + 1 - (-3)^2 \\
\therefore\; & x = (\sqrt{y} - 3)^2 - 8 \\
\therefore\; & (\sqrt{y} - 3)^2 = x + 8 \\
\therefore\; & \sqrt{y} - 3 = -\sqrt{x+8}, \quad 0 \leqslant y \leqslant 9 \\
\therefore\; & \sqrt{y} = 3 - \sqrt{x+8} \\
\therefore\; & y = (3 - \sqrt{x+8})^2
\end{aligned}$$

So, $\quad g^{-1}(x) = (3 - \sqrt{x+8})^2$

$$\begin{aligned}
\text{Now,} \quad (g \circ g^{-1})(0) &= g(g^{-1}(0)) \\
&= g((3 - \sqrt{8})^2) \\
&= (3 - \sqrt{8})^2 - 6\sqrt{(3 - \sqrt{8})^2} + 1 \\
&= 9 - 6\sqrt{8} + 8 - 6(3 - \sqrt{8}) + 1 \\
&= 18 - 6\sqrt{8} - 18 + 6\sqrt{8} \\
&= 0 \quad \text{as required}
\end{aligned}$$

ii $g(0) = 1$ and $g(9) = 9 - 6\sqrt{9} + 1 = -8$

g: Domain is $\{x \mid 0 \leqslant x \leqslant 9\}$
Range is $\{y \mid -8 \leqslant y \leqslant 1\}$

g^{-1}: Domain is $\{x \mid -8 \leqslant x \leqslant 1\}$
Range is $\{y \mid 0 \leqslant y \leqslant 9\}$

d $h(x) = x - 6\sqrt{x} + 1, \;\; x \geqslant 9$

i

$$\begin{aligned}
h \text{ is } \quad & y = x - 6\sqrt{x} + 1, \quad x \geqslant 9 \\
\therefore\; h^{-1} \text{ is } \quad & x = y - 6\sqrt{y} + 1, \quad y \geqslant 9 \\
\therefore\; & x = y - 6\sqrt{y} + (-3)^2 + 1 - (-3)^2 \\
\therefore\; & x = (\sqrt{y} - 3)^2 - 8 \\
\therefore\; & (\sqrt{y} - 3)^2 = x + 8 \\
\therefore\; & \sqrt{y} - 3 = \sqrt{x+8}, \quad y \geqslant 9 \\
\therefore\; & \sqrt{y} = 3 + \sqrt{x+8} \\
\therefore\; & y = (3 + \sqrt{x+8})^2
\end{aligned}$$

So, $\quad h^{-1}(x) = (3 + \sqrt{x+8})^2$

$$\begin{aligned}\text{Now,}\quad (h \circ h^{-1})(16) &= h(h^{-1}(16)) \\ &= h((3+\sqrt{16+8})^2) \\ &= h((3+\sqrt{24})^2) \\ &= (3+\sqrt{24})^2 - 6\sqrt{(3+\sqrt{24})^2} + 1 \\ &= 9 + 6\sqrt{24} + 24 - 6(3+\sqrt{24}) + 1 \\ &= 34 + 6\sqrt{24} - 18 - 6\sqrt{24} \\ &= 16 \quad \text{as required}\end{aligned}$$

ii h: Domain is $\{x \mid x \geqslant 9\}$, Range is $\{y \mid y \geqslant -8\}$

h^{-1}: Domain is $\{x \mid x \geqslant -8\}$, Range is $\{y \mid y \geqslant 9\}$

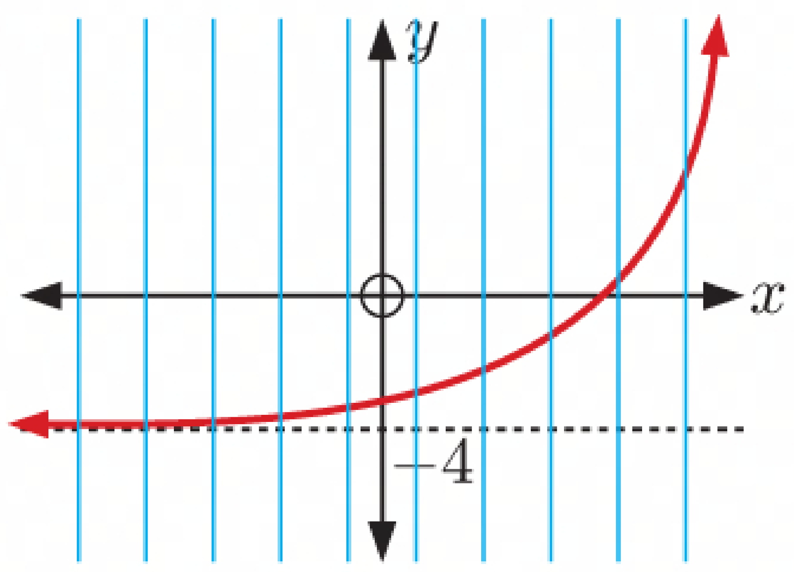
iii For $g^{-1}(x)$ and $h^{-1}(x)$ to be defined, x must be in the domain of g^{-1} and h^{-1}

$\therefore \quad -8 \leqslant x \leqslant 1$

$$\begin{aligned}\text{If}\quad g^{-1}(x) &= h^{-1}(x) \\ \text{then}\quad (3-\sqrt{x+8})^2 &= (3+\sqrt{x+8})^2 \\ \therefore\quad 3-\sqrt{x+8} &= 3+\sqrt{x+8} \qquad \{-8 \leqslant x \leqslant 1\} \\ \therefore\quad -2\sqrt{x+8} &= 0 \\ \therefore\quad x &= -8\end{aligned}$$

REVIEW SET 15A

1 **a**

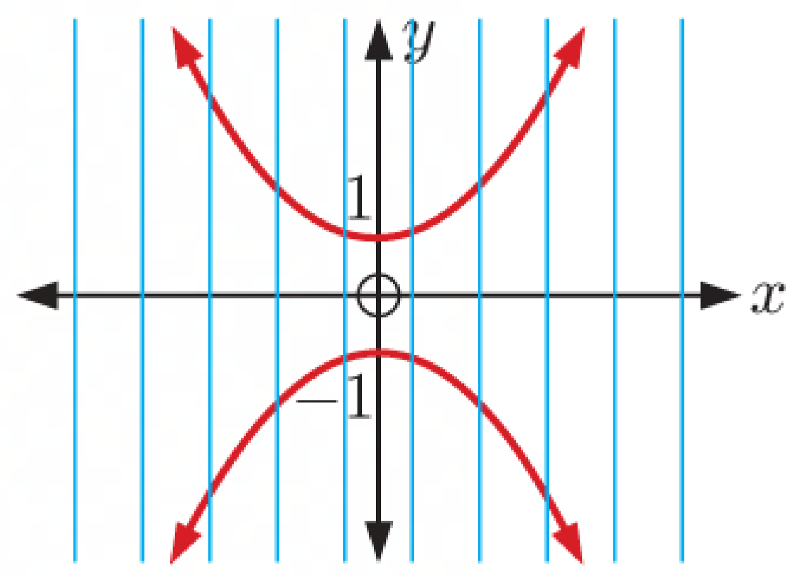


i Domain is $\{x \mid x \in \mathbb{R}\}$.

ii Range is $\{y \mid y > -4\}$.

iii Each vertical line cuts the graph at most once, so the graph shows a function.

b

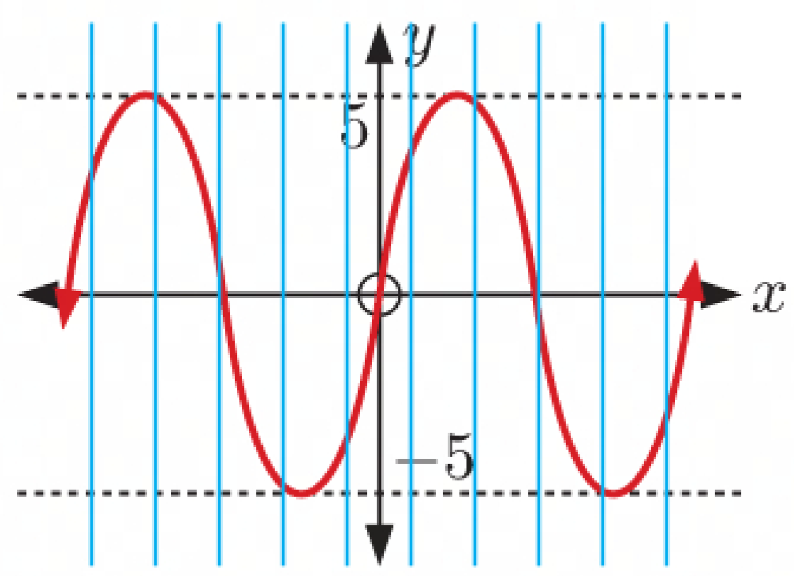


i Domain is $\{x \mid x \in \mathbb{R}\}$.

ii Range is $\{y \mid y \leqslant -1 \text{ or } y \geqslant 1\}$.

iii All vertical lines cut the graph more than once, so the graph does not show a function.

c



i Domain is $\{x \mid x \in \mathbb{R}\}$.

ii Range is $\{y \mid -5 \leqslant y \leqslant 5\}$.

iii Each vertical line cuts the graph at most once, so the graph shows a function.

2 $f(x) = 2x - x^2$

a $f(2) = 2(2) - 2^2$
$= 4 - 4$
$= 0$

b $f(-3) = 2(-3) - (-3)^2$
$= -6 - 9$
$= -15$

c $f(-\frac{1}{2}) = 2(-\frac{1}{2}) - (-\frac{1}{2})^2$
$= -1 - \frac{1}{4}$
$= -\frac{5}{4}$

3 $f(x) = ax + b$ where $f(1) = 7$ and $f(3) = -5$

So, $a(1) + b = 7$
$\therefore\ a + b = 7$
$\therefore\ b = 7 - a$ .... $(*)$

and $a(3) + b = -5$
$\therefore\ 3a + b = -5$
$\therefore\ 3a + (7 - a) = -5$ {using $(*)$}
$\therefore\ 2a = -12$
$\therefore\ a = -6$

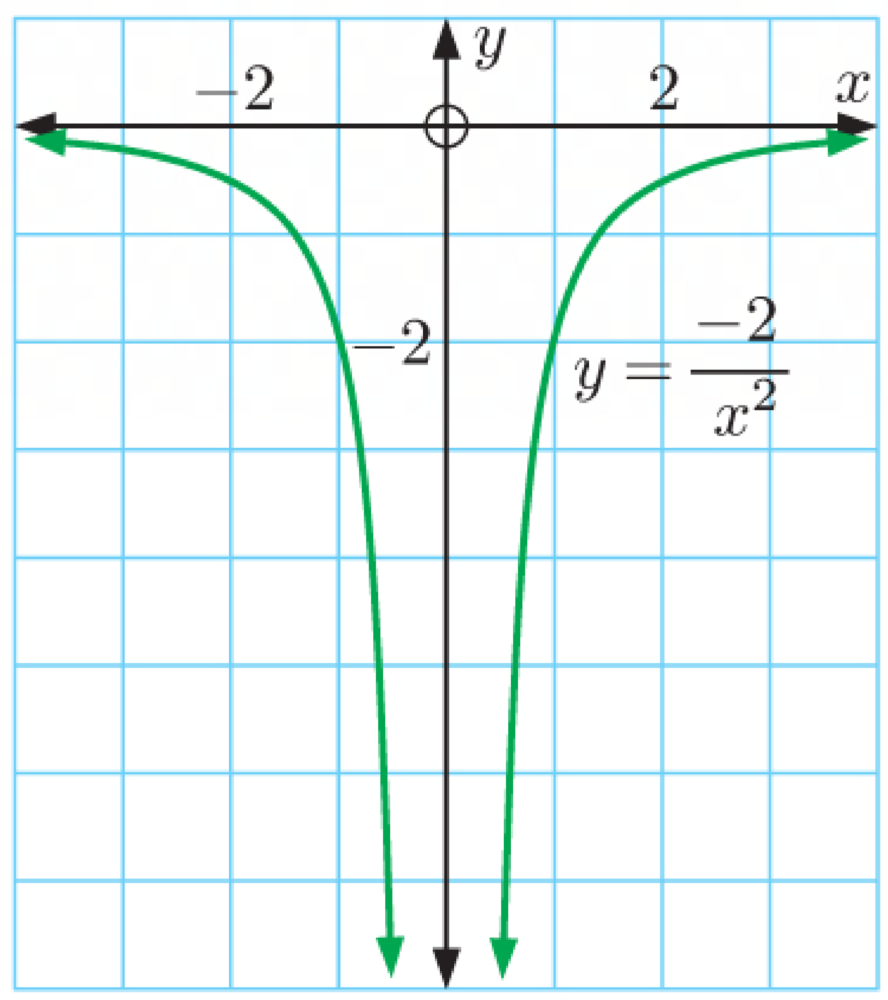
Substituting $a = -6$ into $(*)$ gives $b = 7 - (-6) = 13$

So, $a = -6$, $b = 13$.

4 $f(x) = \dfrac{-2}{x^2}$

a $f(x) = \dfrac{-2}{x^2}$ is undefined when $x^2 = 0$
$\therefore\ x = 0$

b



c The domain is $\{x \mid x \neq 0\}$.
The range is $\{y \mid y < 0\}$.

5 $f(x) = x^2$ and $g(x) = 1 - 6x$

a $f(-3) = (-3)^2$
$= 9$

and $g(-\frac{4}{3}) = 1 - 6(-\frac{4}{3})$
$= 1 + 8$
$= 9$

$\therefore\ f(-3) = g(-\frac{4}{3})$ as required

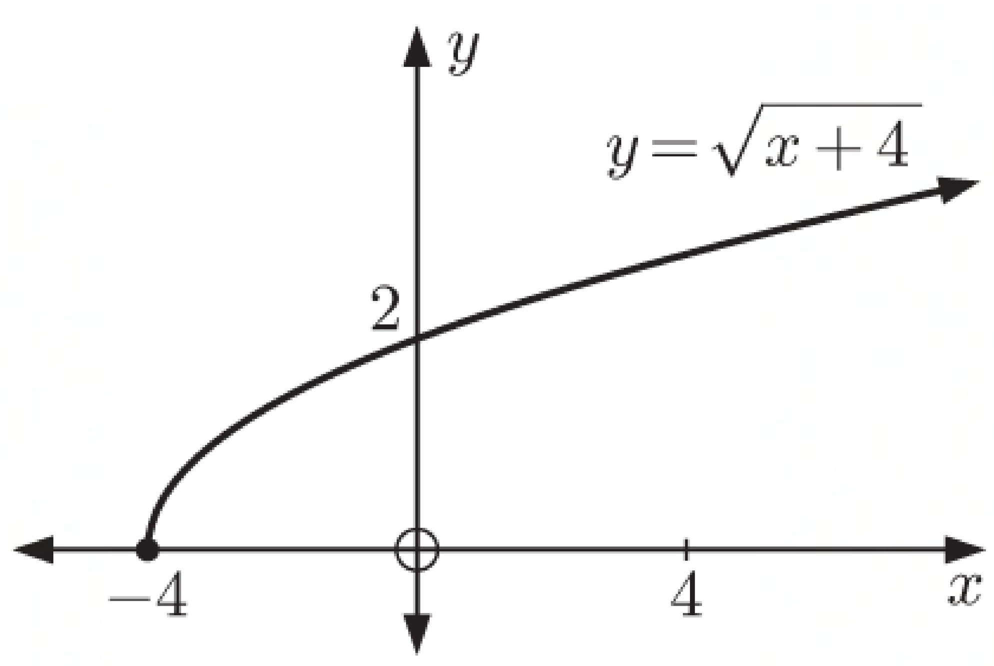
b $g(x) = f(5)$
$\therefore\ 1 - 6x = 5^2$
$\therefore\ 1 - 6x = 25$
$\therefore\ -6x = 24$
$\therefore\ x = -4$

6 **a** $y = \sqrt{x+4}$ is defined when $x + 4 \geqslant 0$
$\therefore\ x \geqslant -4$

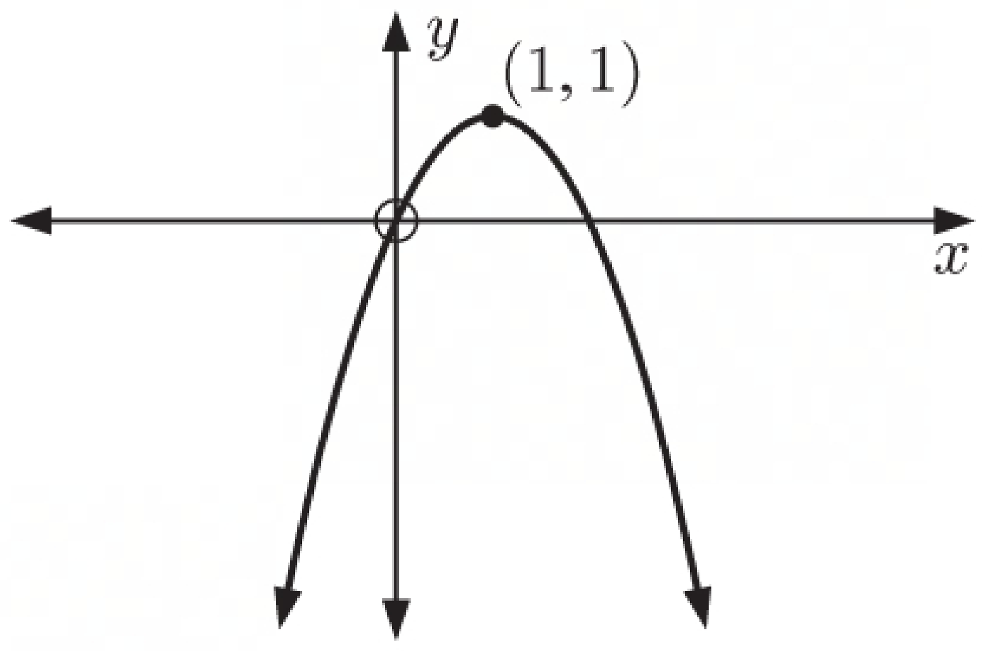
$\therefore$ the domain is $\{x \mid x \geqslant -4\}$.

A square root cannot be negative.

$\therefore$ the range is $\{y \mid y \geqslant 0\}$.



b $y = -(1-x)^2 + 1$ is a quadratic with vertex $(1,\ 1)$ and shape  $(a < 0)$.



$\therefore$ the domain is $\{x \mid x \in \mathbb{R}\}$.

$\therefore$ the maximum y-value is 1 and there is no minimum y-value.

$\therefore$ the range is $\{y \mid y \leqslant 1\}$.

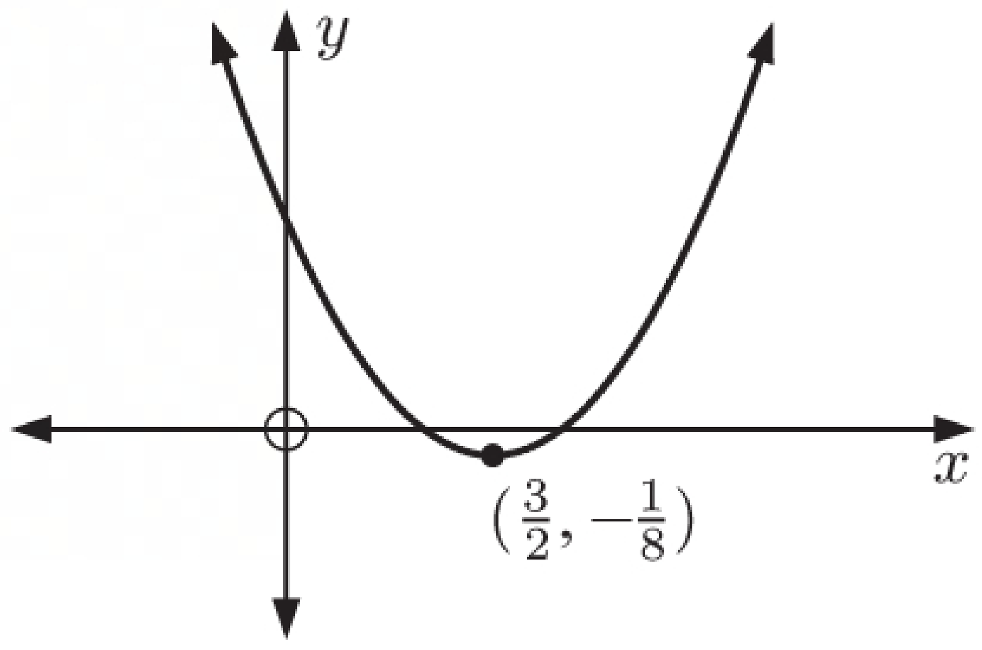
c $y = 2x^2 - 3x + 1$

$\therefore\ y = 2(x^2 - \frac{3}{2}x + \frac{1}{2})$

$\therefore\ y = 2\left(x^2 - \frac{3}{2}x + (-\frac{3}{4})^2 + \frac{1}{2} - (-\frac{3}{4})^2\right)$

$\therefore\ y = 2\left[(x - \frac{3}{2})^2 - \frac{1}{16}\right]$

$\therefore\ y = 2(x - \frac{3}{2})^2 - \frac{1}{8}$



which is a quadratic with vertex $(\frac{3}{2},\ -\frac{1}{8})$ and shape  $(a > 0)$.

$\therefore$ the domain is $\{x \mid x \in \mathbb{R}\}$

$\therefore$ the minimum y-value is $-\frac{1}{8}$ and there is no maximum y-value.

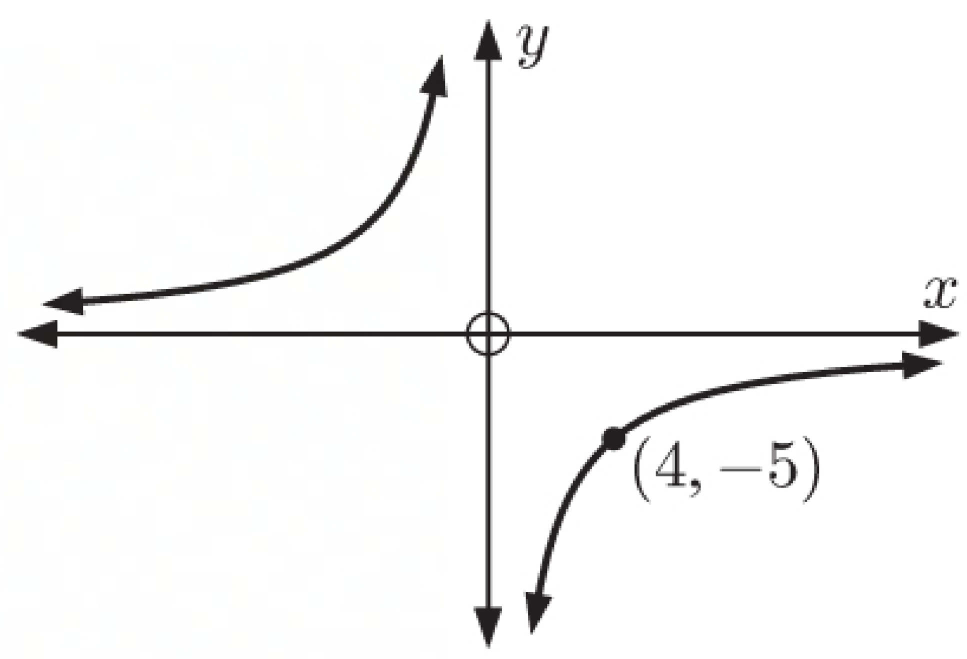
$\therefore$ the range is $\{y \mid y \geqslant -\frac{1}{8}\}$.

7 **a** Let the function be $y = \dfrac{k}{x}$.

When $x = 4$, $y = -5$, so $-5 = \dfrac{k}{4}$

$\therefore\ k = -20$

$\therefore$ the function is $y = -\dfrac{20}{x}$.

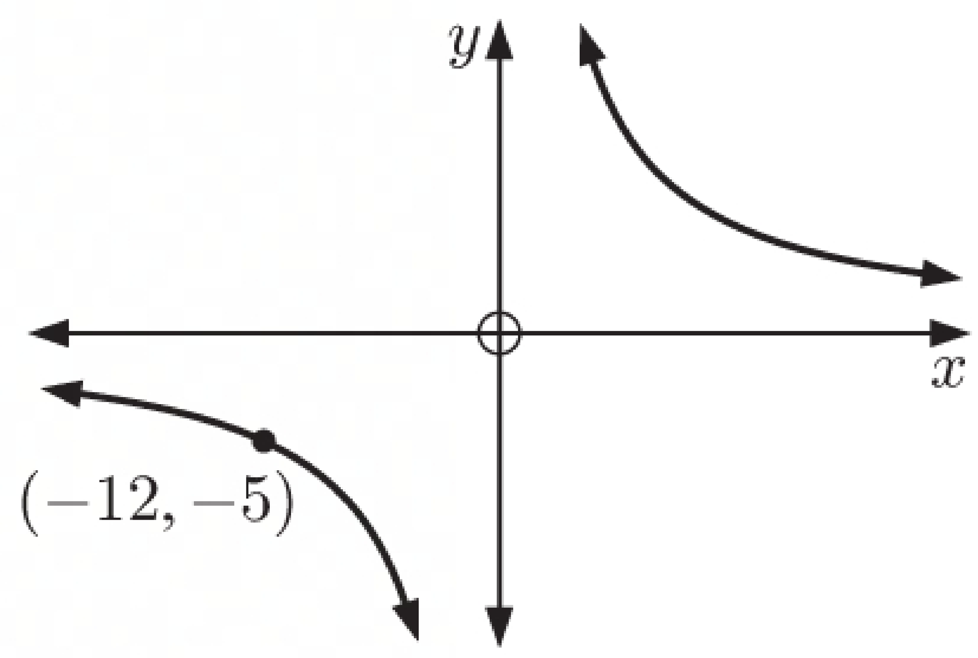


b Let the function be $y = \dfrac{k}{x}$.

When $x = -12$, $y = -5$, so $-5 = \dfrac{k}{-12}$

$\therefore\ k = 60$

$\therefore$ the function is $y = \dfrac{60}{x}$.



8 $f : x \mapsto \dfrac{4x+1}{2-x}$

a
$$f(x) = \frac{4x+1}{2-x} = \frac{-4(2-x)+9}{2-x} = -4 + \frac{9}{2-x}$$

$\therefore$ the vertical asymptote is $x = 2$ and the horizontal asymptote is $y = -4$.

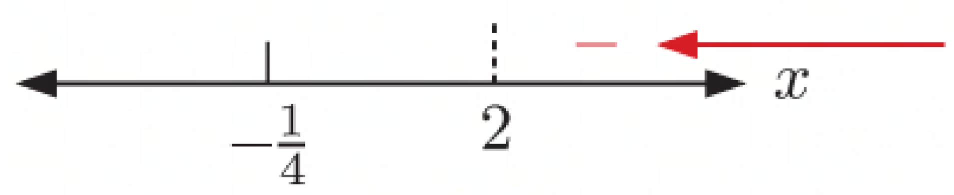
b The domain is $\{x \mid x \neq 2\}$. The range is $\{y \mid y \neq -4\}$.

c $f(x) = 0$ when $4x + 1 = 0$

$\therefore \quad 4x = -1$

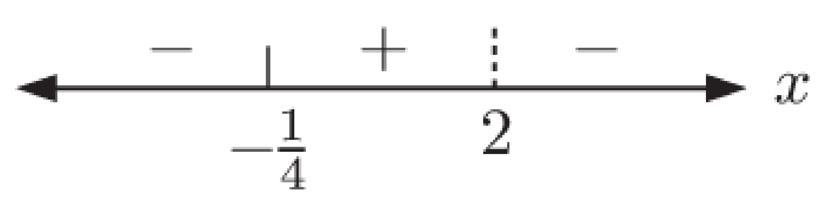
$\therefore \quad x = -\frac{1}{4}$

$\therefore$ the x-intercept is $-\frac{1}{4}$.



When $x = 10$, $\dfrac{4(10)+1}{2-10} = \dfrac{41}{-8} < 0$

Since $(4x+1)$ and $(2-x)$ are single factors, the signs alternate.



As $x \to 2^-$, $y \to \infty$

As $x \to 2^+$, $y \to -\infty$

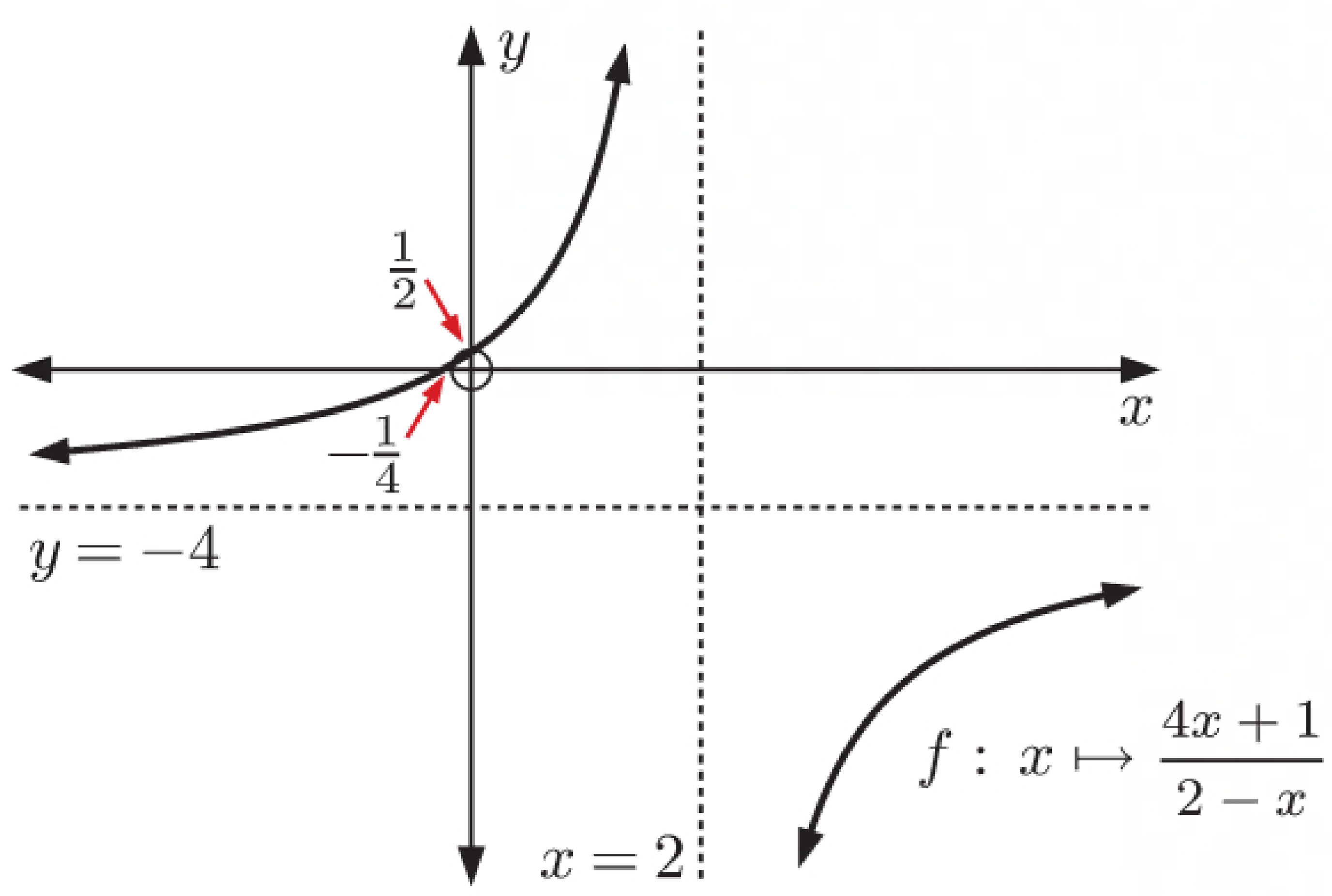
As $x \to -\infty$, $y \to -4^+$

As $x \to \infty$, $y \to -4^-$

d $f(0) = \frac{1}{2}$ $\quad\therefore$ the y-intercept is $\frac{1}{2}$.

$\therefore$ the x-intercept is $-\frac{1}{4}$ {from **b**} and the y-intercept is $\frac{1}{2}$.

e



9 $f(x) = 2x - 5$ and $g(x) = 3x + 1$

a $(f \circ g)(x) = f(g(x))$
$= f(3x+1)$
$= 2(3x+1) - 5$
$= 6x + 2 - 5$
$= 6x - 3$

b $(f \circ g)(x) = f(x+3)$
$\therefore \quad 6x - 3 = 2(x+3) - 5$
$\therefore \quad 6x - 3 = 2x + 6 - 5$
$\therefore \quad 4x = 4$
$\therefore \quad x = 1$

10 $f(x) = 1 - 2x$ and $g(x) = \sqrt{x}$

a $(f \circ g)(x) = f(g(x))$
$= f(\sqrt{x})$
$= 1 - 2\sqrt{x}$

b $(g \circ f)(x) = g(f(x))$
$= g(1 - 2x)$
$= \sqrt{1 - 2x}$

c $(g \circ g)(81) = g(g(81))$
$= g(\sqrt{81})$
$= g(9)$
$= \sqrt{9}$
$= 3$

11 $f(x)=\sqrt{x+2}$ and $g(x)=x^2-3$

a $(f\circ g)(x)=f(g(x))$
$=f(x^2-3)$
$=\sqrt{(x^2-3)+2}$
$=\sqrt{x^2-1}$

$\sqrt{x^2-1}$ is defined when
$x^2-1\geqslant 0$
$\therefore\ x^2\geqslant 1$
$\therefore\ x\leqslant -1$ or $x\geqslant 1$
$\therefore$ the domain is
$\{x \mid x\leqslant -1 \text{ or } x\geqslant 1\}$.

A square root cannot be negative.
$\therefore$ the range is $\{y \mid y\geqslant 0\}$.

b $(g\circ f)(x)=g(f(x))$
$=g(\sqrt{x+2})$
$=(\sqrt{x+2})^2-3$
$=x+2-3$
$=x-1$

$\sqrt{x+2}$ is defined when $x+2\geqslant 0$
$\therefore\ x\geqslant -2$
$\therefore$ the domain is $\{x \mid x\geqslant -2\}$.

$y=(g\circ f)(x)$ is always $\geqslant -3$ as $x\geqslant -2$.
$\therefore$ the range is $\{y \mid y\geqslant -3\}$.

12 $f(x)=ax+b$ where $f(2)=1$ and $f^{-1}(3)=4$

f is $y=ax+b$
$\therefore\ f^{-1}$ is $x=ay+b$
$\therefore\ ay=x-b$
$\therefore\ y=\dfrac{x-b}{a}$

So, $f^{-1}(x)=\dfrac{x-b}{a}$

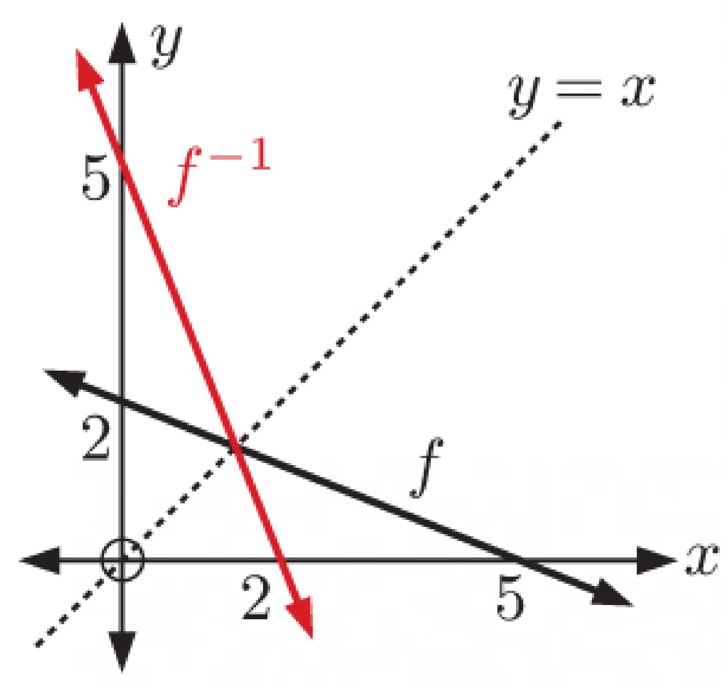
$\therefore\ a(2)+b=1$
$\therefore\ 2a+b=1$
$\therefore\ b=1-2a \quad .... \ (*)$

and

$\dfrac{3-b}{a}=4$
$\therefore\ 3-b=4a$
$\therefore\ 3-(1-2a)=4a \quad \{\text{using } (*)\}$
$\therefore\ 3-1+2a=4a$
$\therefore\ 2a=2$
$\therefore\ a=1$

Substituting $a=1$ into $(*)$ gives $b=1-2(1)=-1$.
So, $a=1$, $b=-1$.

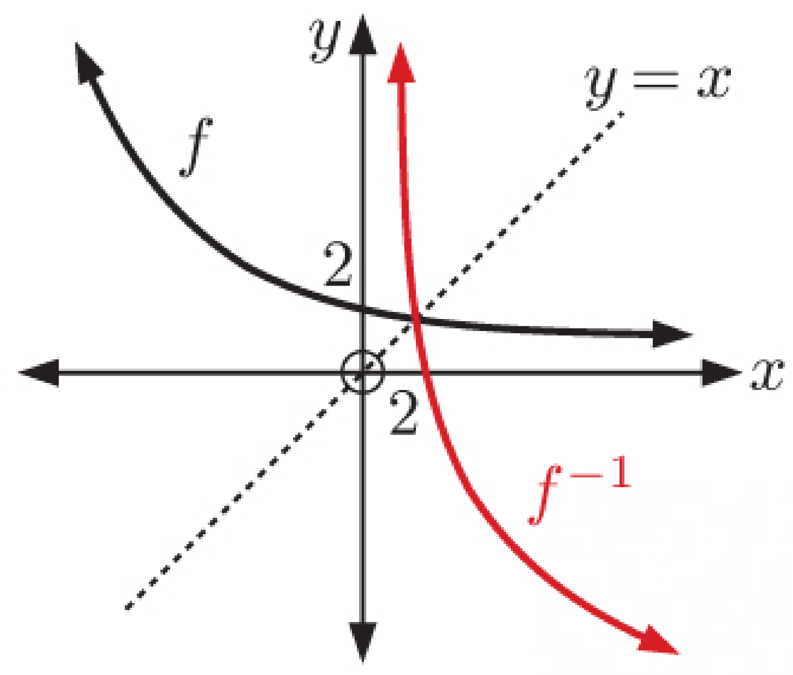
13 **a** f passes through $(0, 2)$ and $(5, 0)$.
$\therefore\ f^{-1}$ passes through $(2, 0)$ and $(0, 5)$.
The graphs are reflections of each other in the line $y=x$.



b f passes through $(0, 2)$.

$\therefore$ f^{-1} passes through $(2, 0)$.

The graphs are reflections of each other in the line $y = x$.



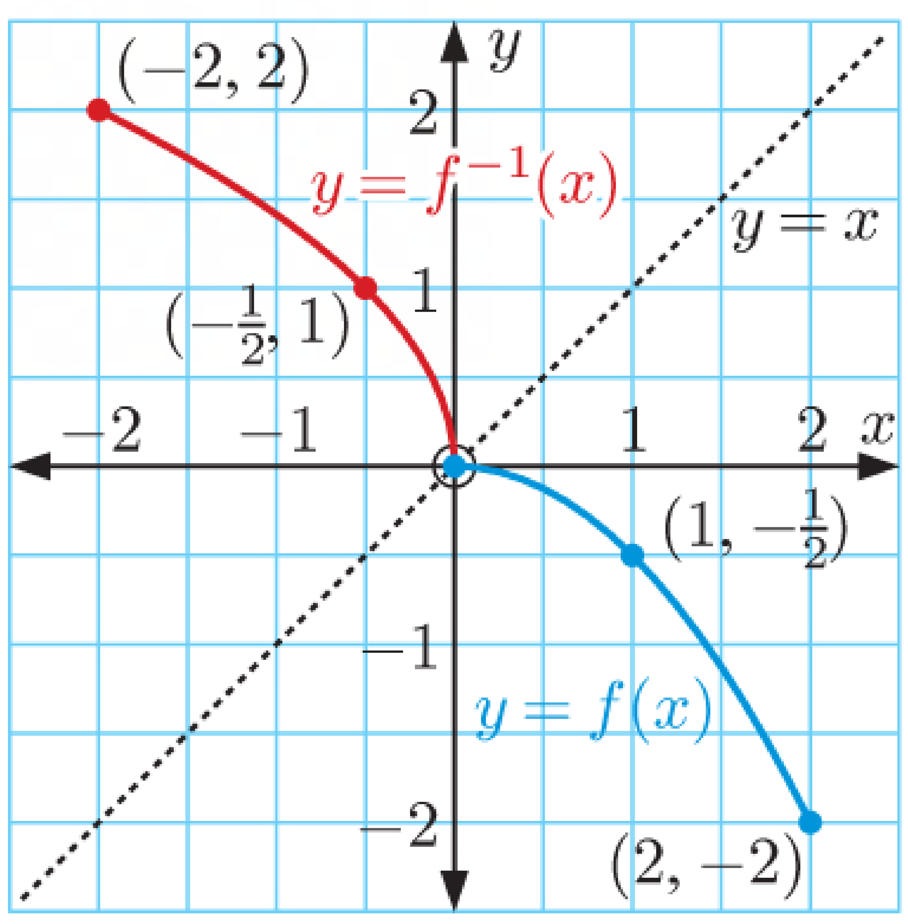
14 **a**

$$f \text{ is } y = 4x + 2$$
$$\therefore f^{-1} \text{ is } x = 4y + 2$$
$$\therefore y = \frac{x-2}{4}$$
$$\therefore f^{-1}(x) = \frac{x-2}{4}$$

b

$$f \text{ is } y = \frac{3-5x}{4}$$
$$\therefore f^{-1} \text{ is } x = \frac{3-5y}{4}$$
$$\therefore 4x = 3 - 5y$$
$$\therefore y = \frac{3-4x}{5}$$
$$\therefore f^{-1}(x) = \frac{3-4x}{5}$$

15 **a**



b The domain of f is $\{x \mid 0 \leqslant x \leqslant 2\}$.

$\therefore$ the range of f^{-1} is $\{y \mid 0 \leqslant y \leqslant 2\}$.

c **i**

$$f(x) = -\tfrac{3}{2}$$
$$\therefore -\tfrac{1}{2}x^2 = -\tfrac{3}{2}$$
$$\therefore x^2 = 3$$
$$\therefore x = \sqrt{3} \qquad \{0 \leqslant x \leqslant 2\}$$

ii

$$f^{-1}(x) = 1$$
$$\therefore f(f^{-1}(x)) = f(1)$$
$$\therefore x = -\tfrac{1}{2}(1)^2$$
$$\therefore x = -\tfrac{1}{2}$$

16

$$f \text{ is } y = 3x + 6$$
$$\therefore f^{-1} \text{ is } x = 3y + 6$$
$$\therefore y = \frac{x-6}{3}$$
$$\therefore f^{-1}(x) = \frac{x-6}{3}$$

$$h \text{ is } y = \frac{x}{3}$$
$$\therefore h^{-1} \text{ is } x = \frac{y}{3}$$
$$\therefore y = 3x$$
$$\therefore h^{-1}(x) = 3x$$

$$(h \circ f)(x) = h(f(x))$$
$$= h(3x + 6)$$
$$= \frac{3x+6}{3}$$
$$= x + 2$$
$$\therefore (h \circ f)(x) = x + 2$$

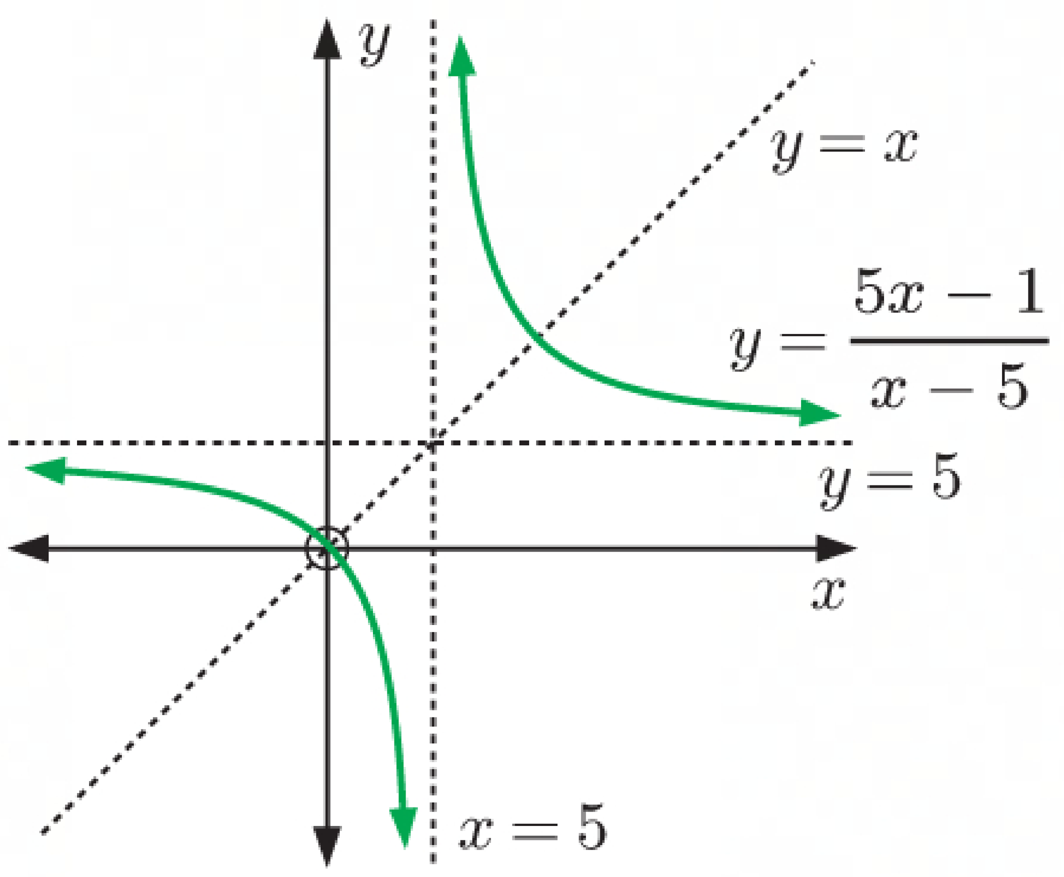
$$h \circ f \text{ is } y = x + 2$$
$$\therefore (h \circ f)^{-1} \text{ is } x = y + 2$$
$$\therefore y = x - 2$$
$$\therefore (h \circ f)^{-1}(x) = x - 2$$

$$\text{Now} \quad (f^{-1} \circ h^{-1})(x) = f^{-1}(h^{-1}(x))$$
$$= f^{-1}(3x)$$
$$= \frac{3x-6}{3}$$
$$= x - 2$$

$\therefore \quad (f^{-1} \circ h^{-1})(x) = (h \circ f)^{-1}(x)$ as required

17 $f : x \mapsto \dfrac{5x-1}{x-5}, \quad x \neq 5$

a



$y = \dfrac{5x-1}{x-5}$ is symmetrical about $y = x$

$\therefore \quad f$ is a self-inverse function.

b

f is $y = \dfrac{5x-1}{x-5}$

$\therefore \quad f^{-1}$ is $x = \dfrac{5y-1}{y-5}$

$\therefore \quad xy - 5x = 5y - 1$

$\therefore \quad xy - 5y = 5x - 1$

$\therefore \quad y(x-5) = 5x - 1$

$\therefore \quad y = \dfrac{5x-1}{x-5}$

$\therefore \quad f^{-1}(x) = \dfrac{5x-1}{x-5} = f(x)$

$\therefore \quad f$ is a self-inverse function.

18 $f : x \mapsto \sqrt{x}$ and $g : x \mapsto 3 + x$

a

f is $y = \sqrt{x}, \quad x \geqslant 0$

$\therefore \quad f^{-1}$ is $x = \sqrt{y}, \quad y \geqslant 0$

$\therefore \quad y = x^2$

So, $f^{-1}(x) = x^2, \quad x \geqslant 0$

g is $y = 3 + x$

$\therefore \quad g^{-1}$ is $x = 3 + y$

$\therefore \quad y = x - 3$

So, $g^{-1}(x) = x - 3$

$$f^{-1}(2) \times g^{-1}(2) = (2)^2 \times (2-3)$$
$$= 4 \times -1$$
$$= -4$$

b

$$(f \circ g)(x) = f(g(x))$$
$$= f(3 + x)$$
$$= \sqrt{3+x}$$

$\therefore \quad (f \circ g)(x) = \sqrt{3+x} \quad .... \ (*)$

$f \circ g$ is $y = \sqrt{3+x}, \quad x \geqslant -3$ {using $(*)$}

$\therefore \quad (f \circ g)^{-1}$ is $x = \sqrt{3+y}, \quad y \geqslant -3$

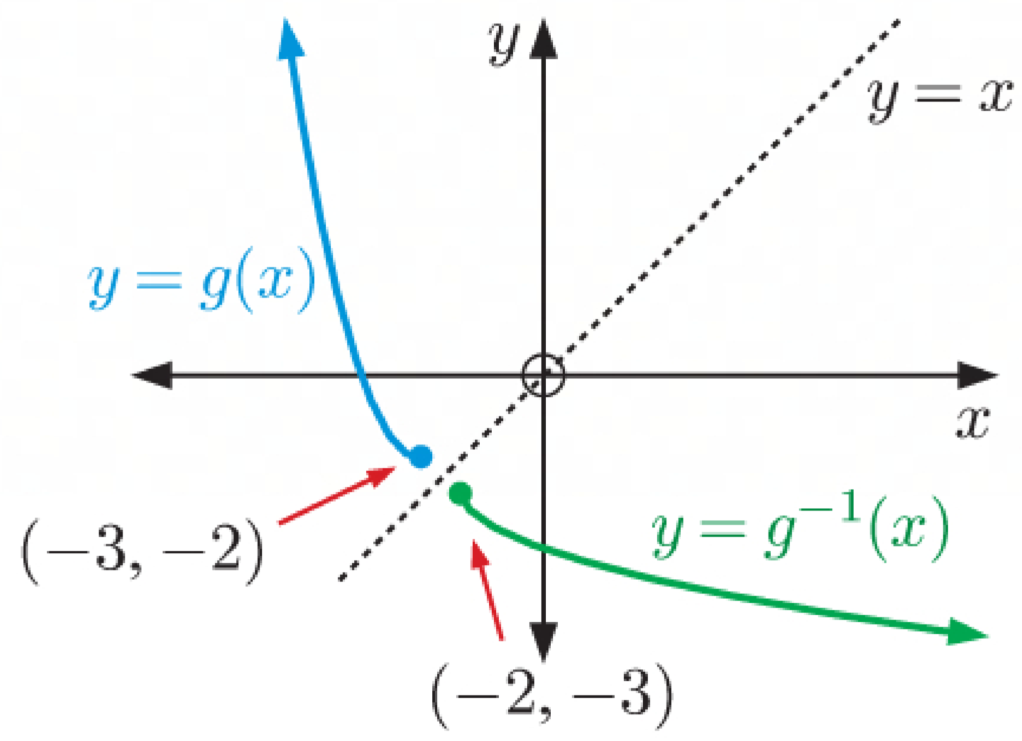
$\therefore \quad x^2 = 3 + y$

$\therefore \quad y = x^2 - 3, \quad x \geqslant 0$

So, $(f \circ g)^{-1}(x) = x^2 - 3$

$$\therefore \quad (f \circ g)^{-1}(2) = (2)^2 - 3$$
$$= 4 - 3$$
$$= 1$$

19 **a, d**



b Any horizontal line cuts the graph at most once. Therefore it has an inverse function.

c
$$\begin{aligned}
g \text{ is } & \quad y = x^2 + 6x + 7, \quad x \leqslant -3 \\
\therefore\ g^{-1} \text{ is } & \quad x = y^2 + 6y + 7, \quad y \leqslant -3 \\
\therefore\ & \quad x = y^2 + 6y + 3^2 + 7 - 3^2 \\
\therefore\ & \quad x = (y+3)^2 - 2 \\
\therefore\ & \quad x + 2 = (y+3)^2 \\
\therefore\ & \quad y + 3 = \pm\sqrt{x+2}, \quad y \leqslant -3 \\
\therefore\ & \quad y + 3 = -\sqrt{x+2} \qquad \{\text{as } y+3 \leqslant 0 \text{ and } -\sqrt{x+2} \leqslant 0\} \\
\therefore\ & \quad y = -3 - \sqrt{x+2}
\end{aligned}$$

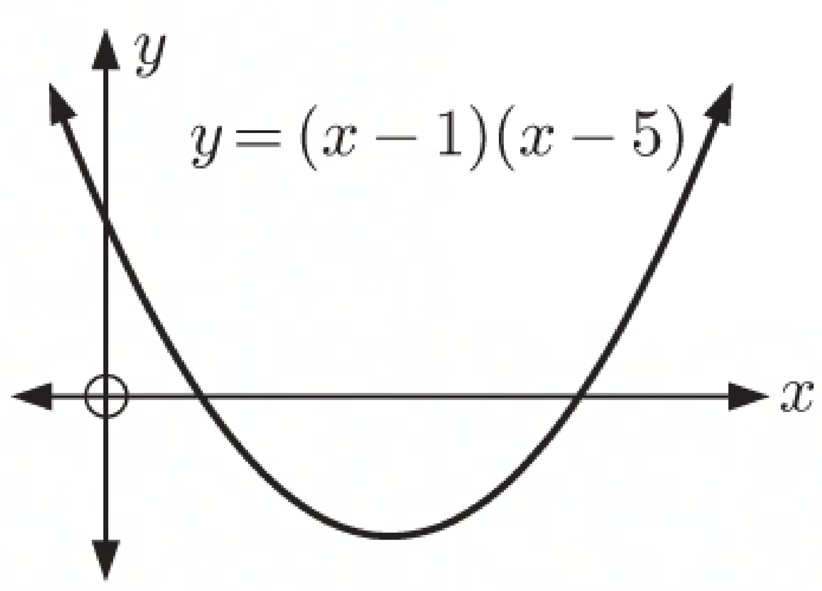
So, $g^{-1}(x) = -3 - \sqrt{x+2}$

e The range of g is $\{y \mid y \geqslant -2\}$.

f The domain of g^{-1} is $\{x \mid x \geqslant -2\}$, and the range is $\{y \mid y \leqslant -3\}$.

REVIEW SET 15B

1 **a**



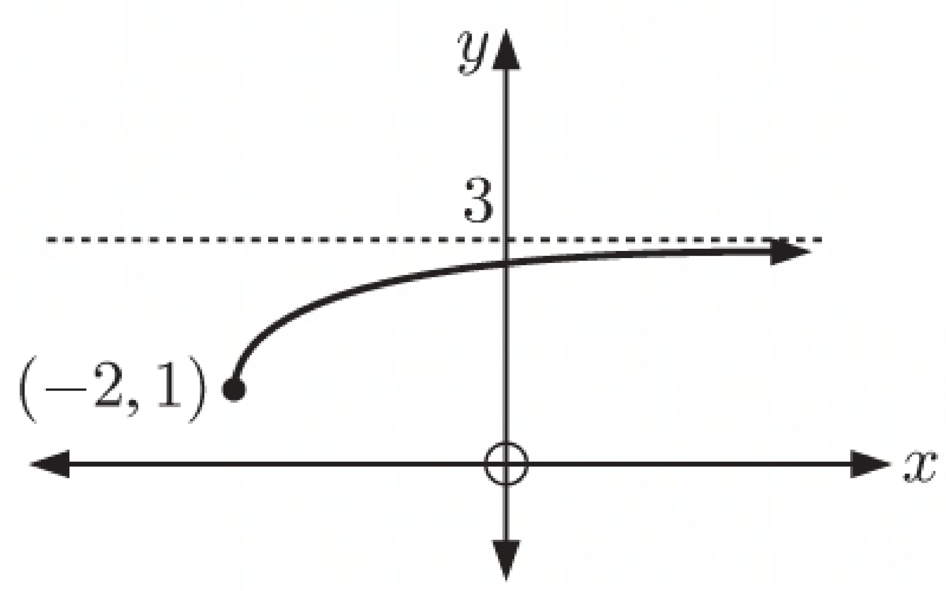
The minimum y-value occurs at $x = 3$.

When $x = 3$, $y = (3-1)(3-5)$
$= (2)(-2)$
$= -4$

So, the minimum y-value is -4 and there is no maximum y-value.

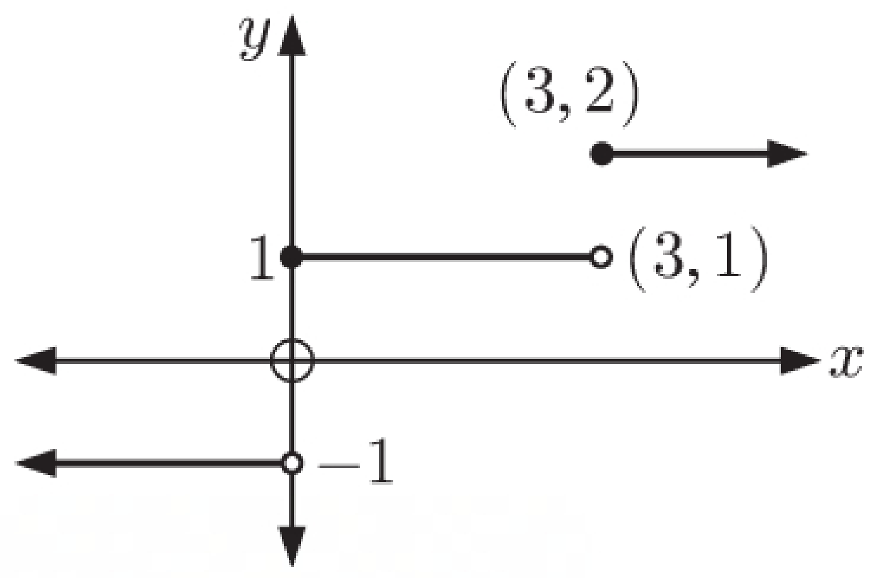
$\therefore$ the domain is $\{x \mid x \in \mathbb{R}\}$,
and the range is $\{y \mid y \geqslant -4\}$.

b



The domain is $\{x \mid x \geqslant -2\}$,
and the range is $\{y \mid 1 \leqslant y < 3\}$.

c



x can take any value.

$\therefore$ the domain is $\{x \mid x \in \mathbb{R}\}$.

The possible values of y are -1, 1, or 2.

$\therefore$ the range is $\{y \mid y = -1, 1, \text{ or } 2\}$.

2 $g(x) = x^2 - 3x$

a $g(x+1) = (x+1)^2 - 3(x+1)$
$= x^2 + 2x + 1 - 3x - 3$
$= x^2 - x - 2$

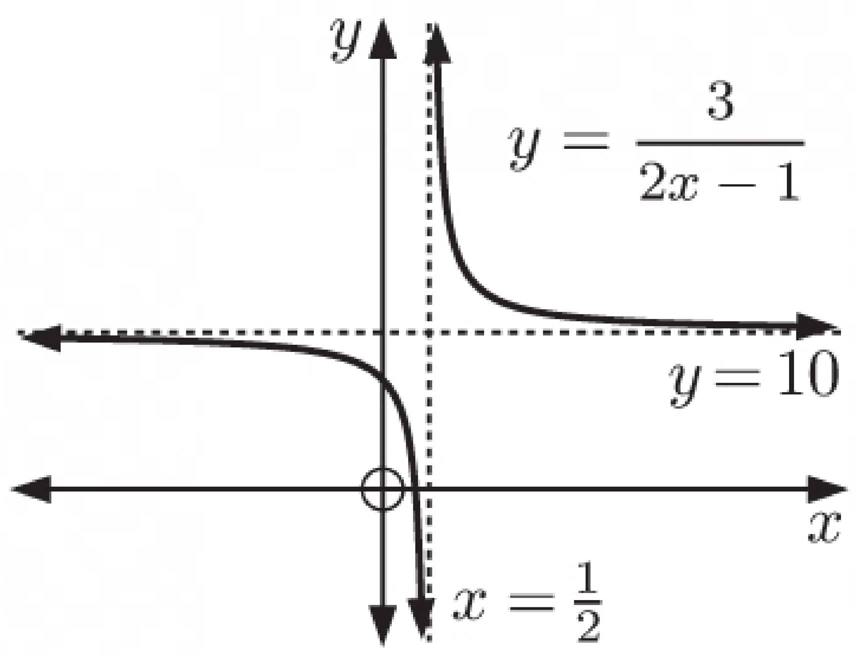
b $g(4x) = (4x)^2 - 3(4x)$
$= 16x^2 - 12x$

3 **a** $x + 2y = 10$ is a function, since for any value of x there is at most one value of y.

b $x + y^2 = 10$ is not a function.

If $x + y^2 = 10$, then $y = \pm\sqrt{10 - x}$. So, for example, for $x = 1$, $y = \pm 3$.

4 **a** $f(x) = 10 + \dfrac{3}{2x-1}$ is undefined when $2x - 1 = 0$
$\therefore 2x = 1$
$\therefore x = \frac{1}{2}$



$\therefore$ the domain is $\{x \mid x \neq \frac{1}{2}\}$.

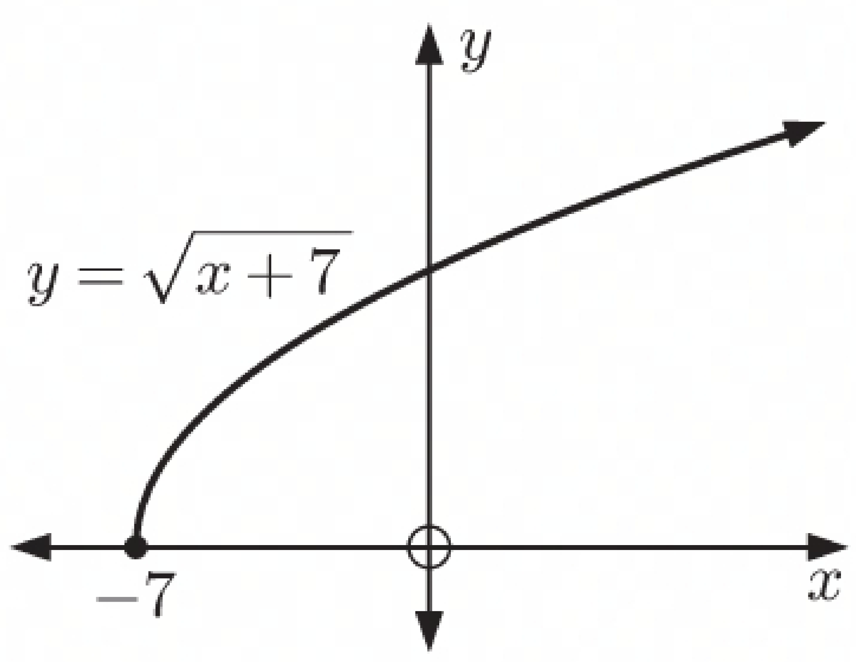
No matter how large or small x is,

$y = \dfrac{3}{2x-1}$ is never zero.

$\therefore$ $y = f(x)$ is never 10.

$\therefore$ the range is $\{y \mid y \neq 10\}$.

b $f(x) = \sqrt{x+7}$ is defined when $x + 7 \geqslant 0$
$\therefore x \geqslant -7$

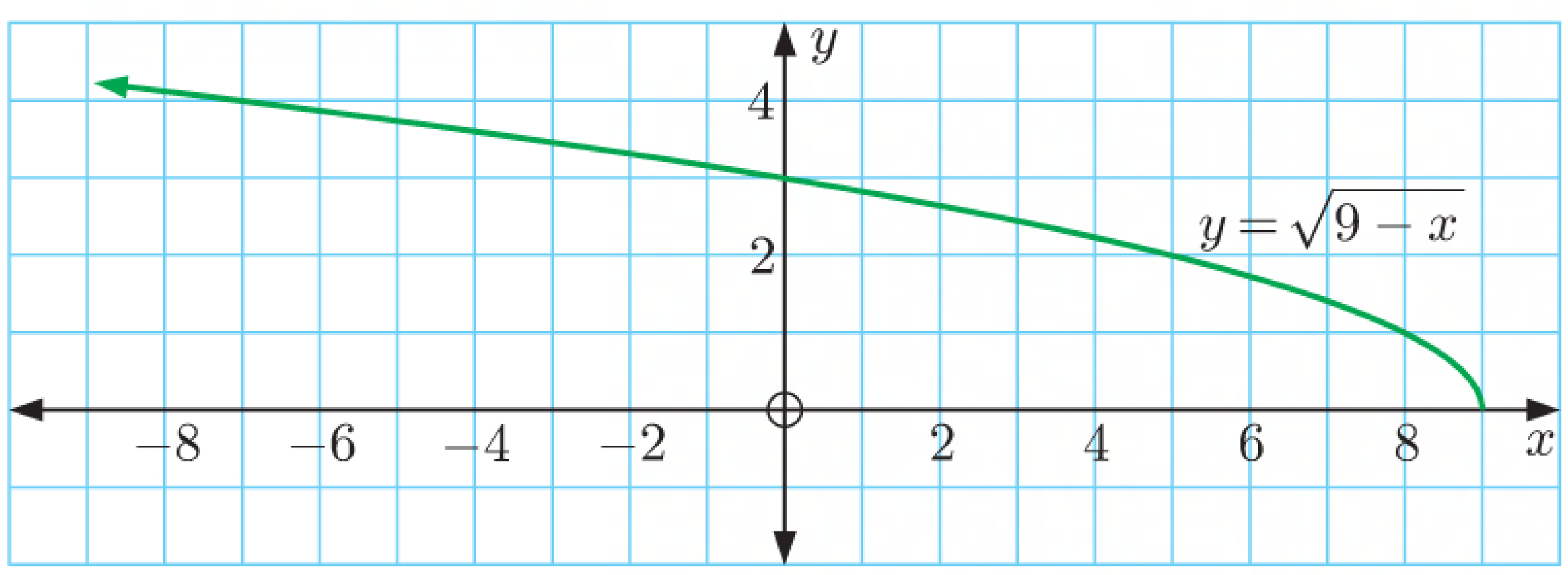


$\therefore$ the domain is $\{x \mid x \geqslant -7\}$.

A square root cannot be negative.

$\therefore$ the range is $\{y \mid y \geqslant 0\}$.

5 **a**



b The graph passes the vertical line test. There is only one value of y for any one value of x.

$\therefore$ the relation is a function.

c $\sqrt{9-x}$ is defined when $9-x \geqslant 0$

$\therefore\ x \leqslant 9$

$\therefore$ the domain is $\{x \mid x \leqslant 9\}$.

A square root cannot be negative.

$\therefore$ the range is $\{y \mid y \geqslant 0\}$.

6 $y = ax^2 + 12x - 13$ is a quadratic with maximum value 5 since the range is $\{y \mid y \leqslant 5\}$.

Now, the vertex of the quadratic is located at $x = \dfrac{-12}{2a} = \dfrac{-6}{a}$.

When $x = \dfrac{-6}{a}$, $y = 5$

$$\begin{aligned}
\therefore\ 5 &= a\left(\frac{-6}{a}\right)^2 + 12\left(\frac{-6}{a}\right) - 13 \\
\therefore\ 5 &= a\left(\frac{36}{a^2}\right) - \frac{72}{a} - 13 \\
\therefore\ 5 &= \frac{36}{a} - \frac{72}{a} - 13 \\
\therefore\ 5 &= -\frac{36}{a} - 13 \\
\therefore\ 5a &= -36 - 13a \\
\therefore\ 18a &= -36 \\
\therefore\ a &= -2
\end{aligned}$$

7 $f(x) = ax^2 + bx + c$ where $f(0) = 5$, $f(-2) = 21$, and $f(3) = -4$

So, $a(0)^2 + b(0) + c = 5$

$\therefore\ c = 5$

Now $a(-2)^2 + b(-2) + 5 = 21$

$\therefore\ 4a - 2b + 5 = 21$

$\therefore\ 4a - 2b = 16$

$\therefore\ 2a - b = 8$

$\therefore\ b = 2a - 8$ .... $(*)$

and $a(3)^2 + b(3) + 5 = -4$

$\therefore\ 9a + 3b = -9$

$\therefore\ 3a + b = -3$

$\therefore\ 3a + (2a - 8) = -3$ {using $(*)$}

$\therefore\ 5a = 5$

$\therefore\ a = 1$

Substituting $a = 1$ into $(*)$ gives $b = 2(1) - 8 = -6$.

So, $a = 1$, $b = -6$, $c = 5$.

8 $f(x) = -1 + \dfrac{3}{x+2}$

a The vertical asymptote is $x = -2$.

The horizontal asymptote is $y = -1$.

b The domain is $\{x \mid x \neq -2\}$.

The range is $\{y \mid y \neq -1\}$.

c $f(0) = -1 + \frac{3}{2} = \frac{1}{2}$

$\therefore$ the y-intercept is $\frac{1}{2}$.

$f(x) = 0$ when $-1 + \dfrac{3}{x+2} = 0$

$\therefore \quad \dfrac{3}{x+2} = 1$

$\therefore \quad 3 = x + 2$

$\therefore \quad x = 1$

$\therefore$ the x-intercept is 1.

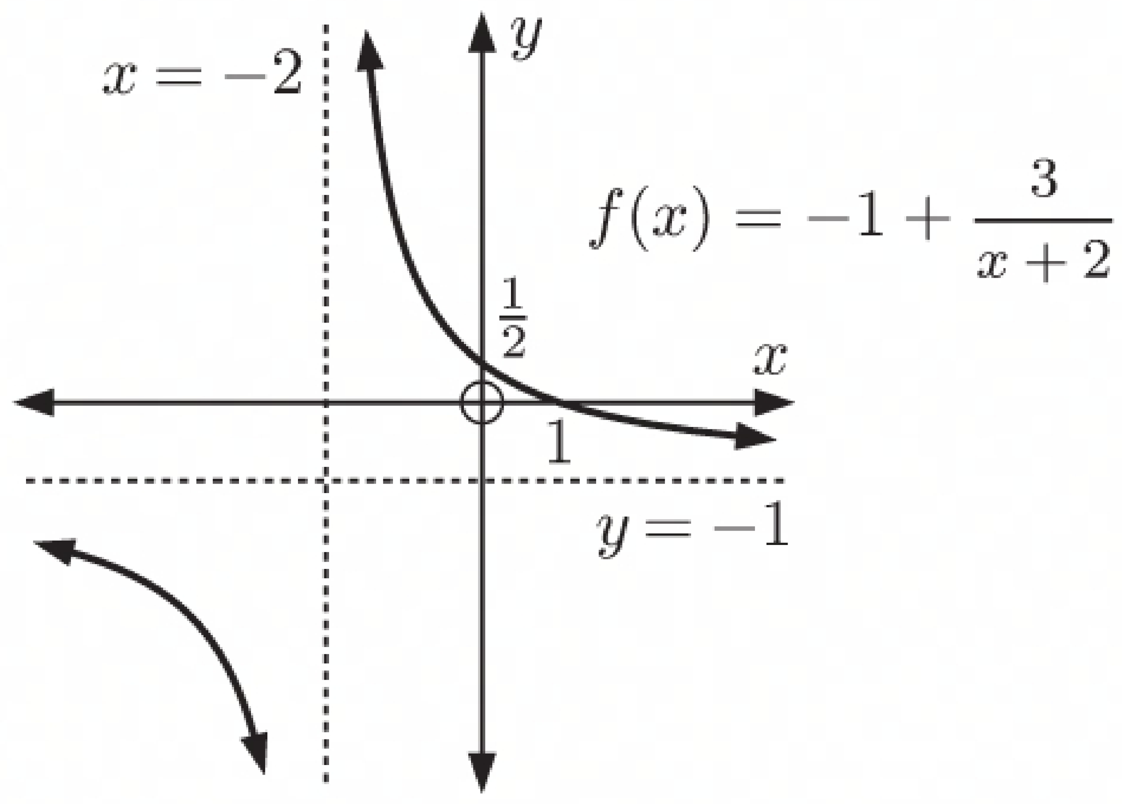
d As $x \to -2^-$, $f(x) \to -\infty$

As $x \to -2^+$, $f(x) \to \infty$

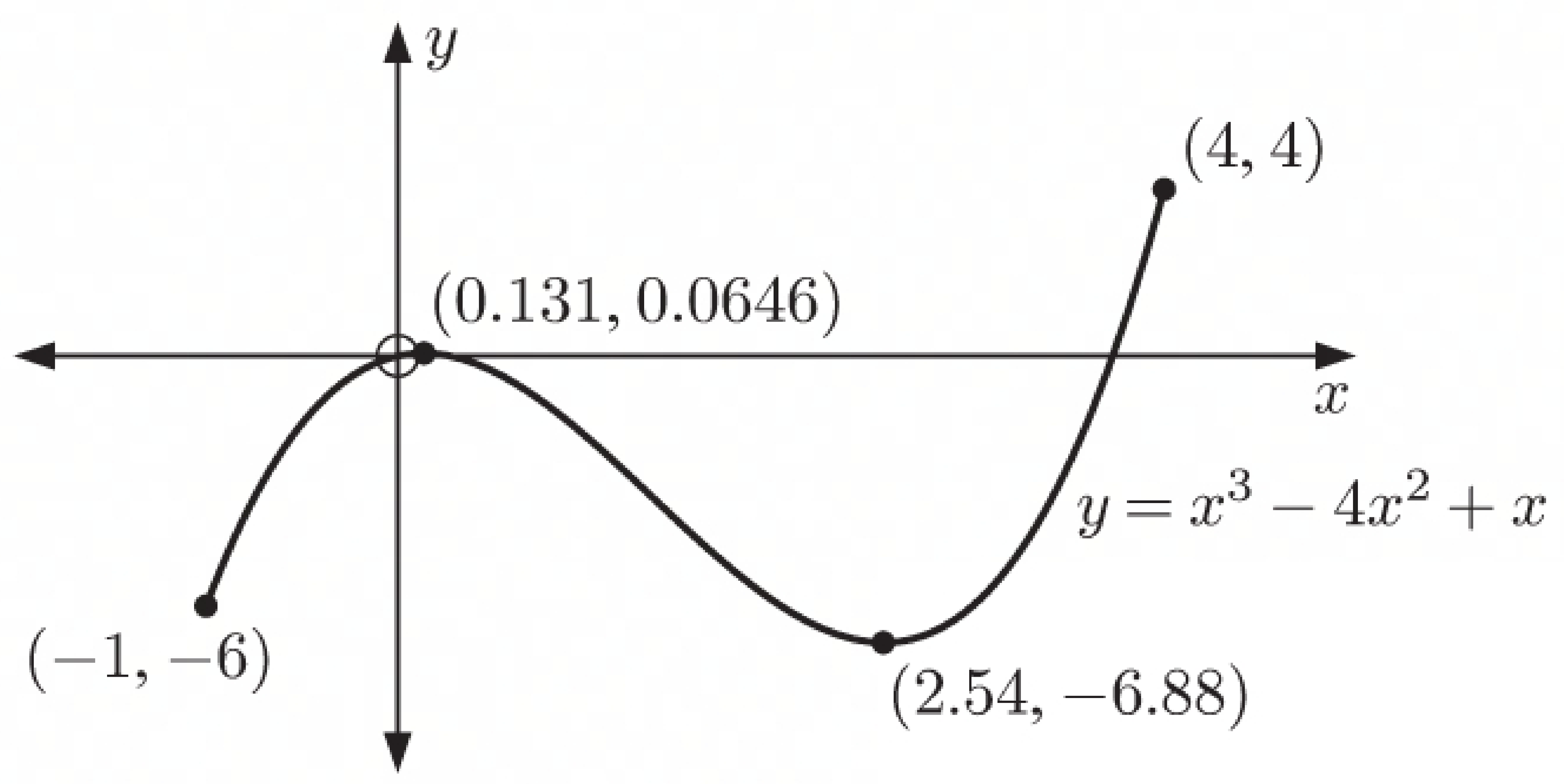
As $x \to -\infty$, $f(x) \to -1^-$

As $x \to \infty$, $f(x) \to -1^+$

e



9



The range is $\{y \mid -6.88 \leqslant y \leqslant 4\}$.

10 $f(x) = 3 - x^2$ and $g(x) = 2x - 1$

a
$$\begin{aligned}(f \circ g)(x) &= f(g(x)) \\ &= f(2x-1) \\ &= 3 - (2x-1)^2 \\ &= 3 - 4x^2 + 4x - 1 \\ &= -4x^2 + 4x + 2\end{aligned}$$

b
$$\begin{aligned}(g \circ f)(x) &= g(f(x)) \\ &= g(3 - x^2) \\ &= 2(3 - x^2) - 1 \\ &= 6 - 2x^2 - 1 \\ &= 5 - 2x^2\end{aligned}$$

c
$$\begin{aligned}(f \circ f)(-2) &= f(f(-2)) \\ &= f(3 - (-2)^2) \\ &= f(-1) \\ &= 3 - (-1)^2 \\ &= 2\end{aligned}$$

11 $f(x) = \dfrac{1}{x^2}$ and $g(x) = x^2 - 4x + 3$

$$\begin{aligned}(f \circ g)(x) &= f(g(x))\\ &= f(x^2 - 4x + 3)\\ &= \frac{1}{(x^2-4x+3)^2}\\ &= \frac{1}{[(x-3)(x-1)]^2}\end{aligned}$$

$\therefore$ the domain is $\{x \mid x \neq 3 \text{ or } 1\}$ and the range is $\{y \mid y > 0\}$.

12 $f(x) = 3x + 5$ and $g(x) = 2x^2 - x$

a **i** $$\begin{aligned}(f \circ g)(x) &= f(g(x))\\ &= f(2x^2 - x)\\ &= 3(2x^2 - x) + 5\\ &= 6x^2 - 3x + 5\end{aligned}$$

ii $$\begin{aligned}(g \circ f)(x) &= g(f(x))\\ &= g(3x+5)\\ &= 2(3x+5)^2 - (3x+5)\\ &= 2(9x^2 + 30x + 25) - 3x - 5\\ &= 18x^2 + 60x + 50 - 3x - 5\\ &= 18x^2 + 57x + 45\end{aligned}$$

b $$\begin{aligned}3(f \circ g)(x) &= (g \circ f)(x)\\ \therefore\ 3(6x^2 - 3x + 5) &= 18x^2 + 57x + 45\\ \therefore\ 18x^2 - 9x + 15 &= 18x^2 + 57x + 45\\ \therefore\ -30 &= 66x\\ \therefore\ x &= -\tfrac{5}{11}\end{aligned}$$

13 **a** $$\begin{aligned}D \circ S &= D(S(t))\\ &= D(9.8t)\\ &= \frac{(9.8t)^2}{19.6}\\ &= \frac{96.04t^2}{19.6}\\ &= 4.9t^2\end{aligned}$$

The function $D \circ S$ is the distance travelled by the object after t seconds.

b $$\begin{aligned}(D \circ S)(5) &= 4.9(5)^2\\ &= 4.9 \times 25\\ &= 122.5\end{aligned}$$

The object has travelled 122.5 m after 5 seconds.

14 $f(2x+3) = 5x - 7$

Let $z = 2x + 3$

$$\begin{aligned}\therefore\ x &= \frac{z-3}{2}\\ \therefore\ f(z) &= 5\left(\frac{z-3}{2}\right) - 7\\ &= \frac{5z-15}{2} - \frac{14}{2}\\ &= \frac{5z-29}{2}\\ \therefore\ f(x) &= \frac{5x-29}{2}\end{aligned}$$

So, f is $y = \dfrac{5x-29}{2}$

$\therefore$ f^{-1} is $x = \dfrac{5y-29}{2}$

$$\begin{aligned}\therefore\ 2x &= 5y - 29\\ y &= \frac{2x+29}{5}\end{aligned}$$

So, $f^{-1}(x) = \dfrac{2x+29}{5}$

15

f is $y = 5x - 2$

$\therefore\ f^{-1}$ is $x = 5y - 2$

$\therefore\ y = \dfrac{x+2}{5}$

$\therefore\ f^{-1}(x) = \dfrac{x+2}{5}$

h is $y = \dfrac{3x}{4}$

$\therefore\ h^{-1}$ is $x = \dfrac{3y}{4}$

$\therefore\ y = \dfrac{4x}{3}$

$\therefore\ h^{-1}(x) = \dfrac{4x}{3}$

$$\begin{aligned}(h \circ f)(x) &= h(f(x))\\ &= h(5x-2)\\ &= \frac{3(5x-2)}{4}\\ &= \frac{15x-6}{4}\end{aligned}$$

$\therefore\ (h \circ f)(x) = \dfrac{15x-6}{4}$ .... $(*)$

$h \circ f$ is $y = \dfrac{15x-6}{4}$ {using $(*)$}

$\therefore\ (h \circ f)^{-1}$ is $x = \dfrac{15y-6}{4}$

$\therefore\ 4x = 15y - 6$

$\therefore\ y = \dfrac{4x+6}{15}$

$\therefore\ (h \circ f)^{-1}(x) = \dfrac{4x+6}{15}$

Now
$$\begin{aligned}(f^{-1} \circ h^{-1})(x) &= f^{-1}(h^{-1}(x))\\ &= f^{-1}\left(\frac{4x}{3}\right)\\ &= \frac{\frac{4x}{3}+2}{5}\\ &= \frac{4x+6}{15}\end{aligned}$$

So, $(f^{-1} \circ h^{-1})(x) = (h \circ f)^{-1}(x)$ as required.

16 $f(x) = 3x+1$ and $g(x) = \dfrac{2}{x}$

a
$$\begin{aligned}(g \circ f)(x) &= g(f(x))\\ &= g(3x+1)\\ &= \frac{2}{3x+1}\end{aligned}$$

b $(g \circ f)(x) = -4$

$\therefore\ \dfrac{2}{3x+1} = -4$

$\therefore\ -4(3x+1) = 2$

$\therefore\ -12x - 4 = 2$

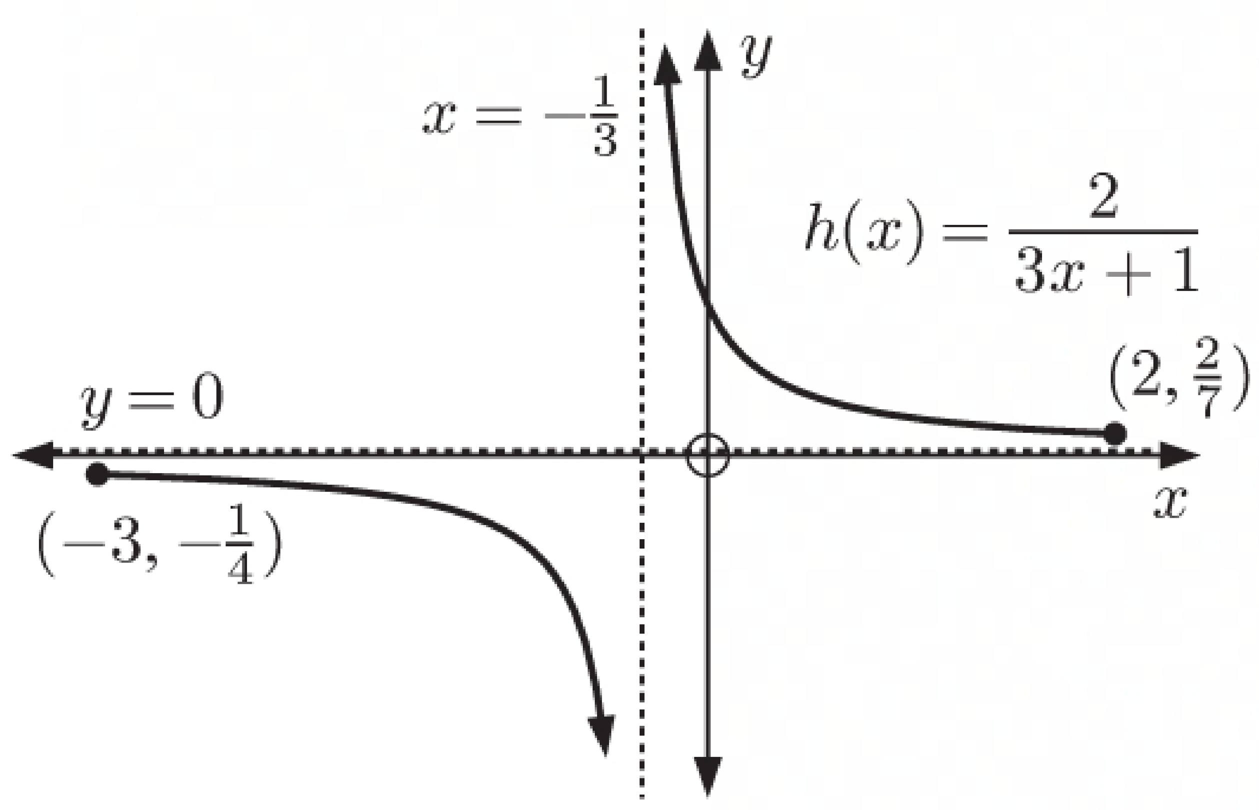
$\therefore\ -12x = 6$

$\therefore\ x = -\frac{1}{2}$

c $h(x) = \dfrac{2}{3x+1}$, $x \neq -\frac{1}{3}$

i The vertical asymptote is $x = -\frac{1}{3}$.
The horizontal asymptote is $y = 0$.

ii



iii The range of h is $\{y \mid y \leqslant -\frac{1}{4}$ or $y \geqslant \frac{2}{7}\}$.

17 $f(x) = 2x+11$ and $g(x) = x^2$

f is $y = 2x + 11$

$\therefore\ f^{-1}$ is $x = 2y + 11$

$\therefore\ 2y = x - 11$

$\therefore\ y = \dfrac{x-11}{2}$

So, $f^{-1}(x) = \dfrac{x-11}{2}$ .... $(*)$

$$\begin{aligned}(g \circ f^{-1})(3) &= g(f^{-1}(3))\\ &= g\left(\frac{3-11}{2}\right) \quad \{\text{using } (*)\}\\ &= g(-4)\\ &= (-4)^2\\ &= 16\end{aligned}$$

18 $f(x) = \dfrac{ax+3}{x-b}$

a $f(x)$ has vertical asymptote $x=-1$, so the function is undefined when $x=-1$.

$\therefore\ -1-b=0$

$\therefore\ b=-1$

$$\begin{aligned} f(x) &= \frac{ax+3}{x-1} \\ &= \frac{a(x-1)+a+3}{x-1} \\ &= a + \frac{a+3}{x-1} \quad \text{which has horizontal asymptote } y=2 \end{aligned}$$

$\therefore\ a=2$

So, $a=2$, $b=-1$.

b $f(x)$ has domain $\{x \mid x \neq -1\}$ and range $\{y \mid y \neq 2\}$.

$\therefore\ f^{-1}(x)$ has domain $\{x \mid x \neq 2\}$ and range $\{y \mid y \neq -1\}$.

19 $f : x \mapsto 2x+1, \quad g : x \mapsto \dfrac{x+1}{x-2}$

a
$$\begin{aligned} (f \circ g)(x) &= f(g(x)) \\ &= f\left(\frac{x+1}{x-2}\right) \\ &= 2\left(\frac{x+1}{x-2}\right)+1 \\ &= \frac{2x+2}{x-2} + \frac{x-2}{x-2} \\ &= \frac{3x}{x-2} \end{aligned}$$

b
$$\begin{aligned} g \text{ is } y &= \frac{x+1}{x-2} \\ \therefore\ g^{-1} \text{ is } x &= \frac{y+1}{y-2} \\ \therefore\ x(y-2) &= y+1 \\ \therefore\ xy-2x &= y+1 \\ \therefore\ xy-y &= 2x+1 \\ \therefore\ y(x-1) &= 2x+1 \\ \therefore\ y &= \frac{2x+1}{x-1} \end{aligned}$$

So, $g^{-1}(x) = \dfrac{2x+1}{x-1}$

20 **a**

f is $y = x^4 - 8x^2 + 3, \quad 0 \leqslant x \leqslant 2$

$\therefore\ f^{-1}$ is $x = y^4 - 8y^2 + 3, \quad 0 \leqslant y \leqslant 2$

$\therefore\ x = y^4 - 8y^2 + (-4)^2 + 3 - (-4)^2$

$\therefore\ x = (y^2-4)^2 - 13$

$\therefore\ (y^2-4)^2 = x+13$

$\therefore\ y^2 - 4 = \pm\sqrt{x+13}, \quad 0 \leqslant y \leqslant 2$

$\therefore\ y^2 - 4 = -\sqrt{x+13}$

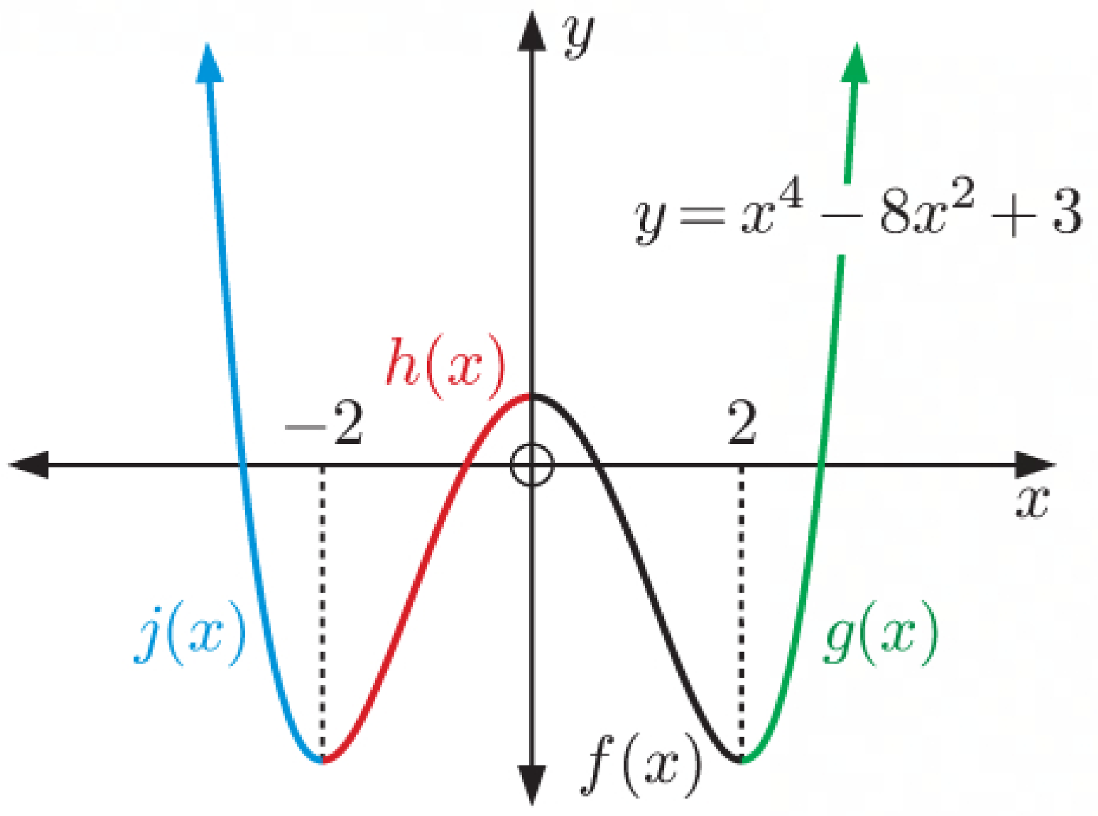
$\{\text{as } y^2-4 \leqslant 0 \text{ and } -\sqrt{x+13} \leqslant 0\}$

$\therefore\ y^2 = 4 - \sqrt{x+13}, \quad -13 \leqslant x \leqslant 3$

$\therefore\ y = \pm\sqrt{4-\sqrt{x+13}}, \quad 0 \leqslant y \leqslant 2$

$\therefore\ y = \sqrt{4-\sqrt{x+13}} \quad \{\text{as } y \geqslant 0 \text{ and } \sqrt{4-\sqrt{x+13}} \geqslant 0\}$

So, $f^{-1}(x) = \sqrt{4-\sqrt{x+13}}$



The domain is $\{x \mid -13 \leqslant x \leqslant 3\}$. The range is $\{y \mid 0 \leqslant y \leqslant 2\}$.

b g is $y = x^4 - 8x^2 + 3, \quad x \geqslant 2$

$\therefore\ g^{-1}$ is $x = y^4 - 8y^2 + 3, \quad y \geqslant 2$

$\therefore\ (y^2 - 4)^2 = x + 13$ {similar to **a**}

$\therefore\ y^2 - 4 = \pm\sqrt{x+13}, \quad y \geqslant 2$

$\therefore\ y^2 - 4 = \sqrt{x+13}$ {as $y^2 - 4 \geqslant 0$ and $\sqrt{x+13} \geqslant 0$}

$\therefore\ y^2 = 4 + \sqrt{x+13}, \quad x \geqslant -13$

$\therefore\ y = \pm\sqrt{4 + \sqrt{x+13}}, \quad y \geqslant 2$

$\therefore\ y = \sqrt{4 + \sqrt{x+13}}$ {as $y > 0$ and $\sqrt{4+\sqrt{x+13}} \geqslant 0$}

So, $g^{-1}(x) = \sqrt{4 + \sqrt{x+13}}$

The domain is $\{x \mid x \geqslant -13\}$. The range is $\{y \mid y \geqslant 2\}$.

c h is $y = x^4 - 8x^2 + 3, \quad -2 \leqslant x \leqslant 0$

$\therefore\ h^{-1}$ is $x = y^4 - 8y^2 + 3, \quad -2 \leqslant y \leqslant 0$

$\therefore\ (y^2 - 4)^2 = x + 13$ {similar to **a**}

$\therefore\ y^2 - 4 = \pm\sqrt{x+13}, \quad -2 \leqslant y \leqslant 0$

$\therefore\ y^2 - 4 = -\sqrt{x+13}$ {as $y^2 - 4 \leqslant 0$ and $-\sqrt{x+13} \leqslant 0$}

$\therefore\ y^2 = 4 - \sqrt{x+13}, \quad -13 \leqslant x \leqslant 3$

$\therefore\ y = \pm\sqrt{4 - \sqrt{x+13}}, \quad -2 \leqslant y \leqslant 0$

$\therefore\ y = -\sqrt{4 - \sqrt{x+13}}$ {as $y \leqslant 0$ and $-\sqrt{4-\sqrt{x+13}} \leqslant 0$}

So, $h^{-1}(x) = -\sqrt{4 - \sqrt{x+13}}$

The domain is $\{x \mid -13 \leqslant x \leqslant 3\}$. The range is $\{y \mid -2 \leqslant y \leqslant 0\}$.

d j is $y = x^4 - 8x^2 + 3, \quad x \leqslant -2$

$\therefore\ j^{-1}$ is $x = y^4 - 8y^2 + 3, \quad y \leqslant -2$

$\therefore\ (y^2 - 4)^2 = x + 13$ {similar to **a**}

$\therefore\ y^2 - 4 = \pm\sqrt{x+13}, \quad y \leqslant -2$

$\therefore\ y^2 - 4 = \sqrt{x+13}$ {as $y^2 - 4 \geqslant 0$ and $\sqrt{x+13} \geqslant 0$}

$\therefore\ y^2 = 4 + \sqrt{x+13}, \quad x \geqslant -13$

$\therefore\ y = \pm\sqrt{4 + \sqrt{x+13}}, \quad y \leqslant -2$

$\therefore\ y = -\sqrt{4 + \sqrt{x+13}}$ {as $y < 0$ and $-\sqrt{4+\sqrt{x+13}} \leqslant 0$}

So, $j^{-1}(x) = -\sqrt{4 + \sqrt{x+13}}$

The domain is $\{x \mid x \geqslant -13\}$. The range is $\{y \mid y \leqslant -2\}$.

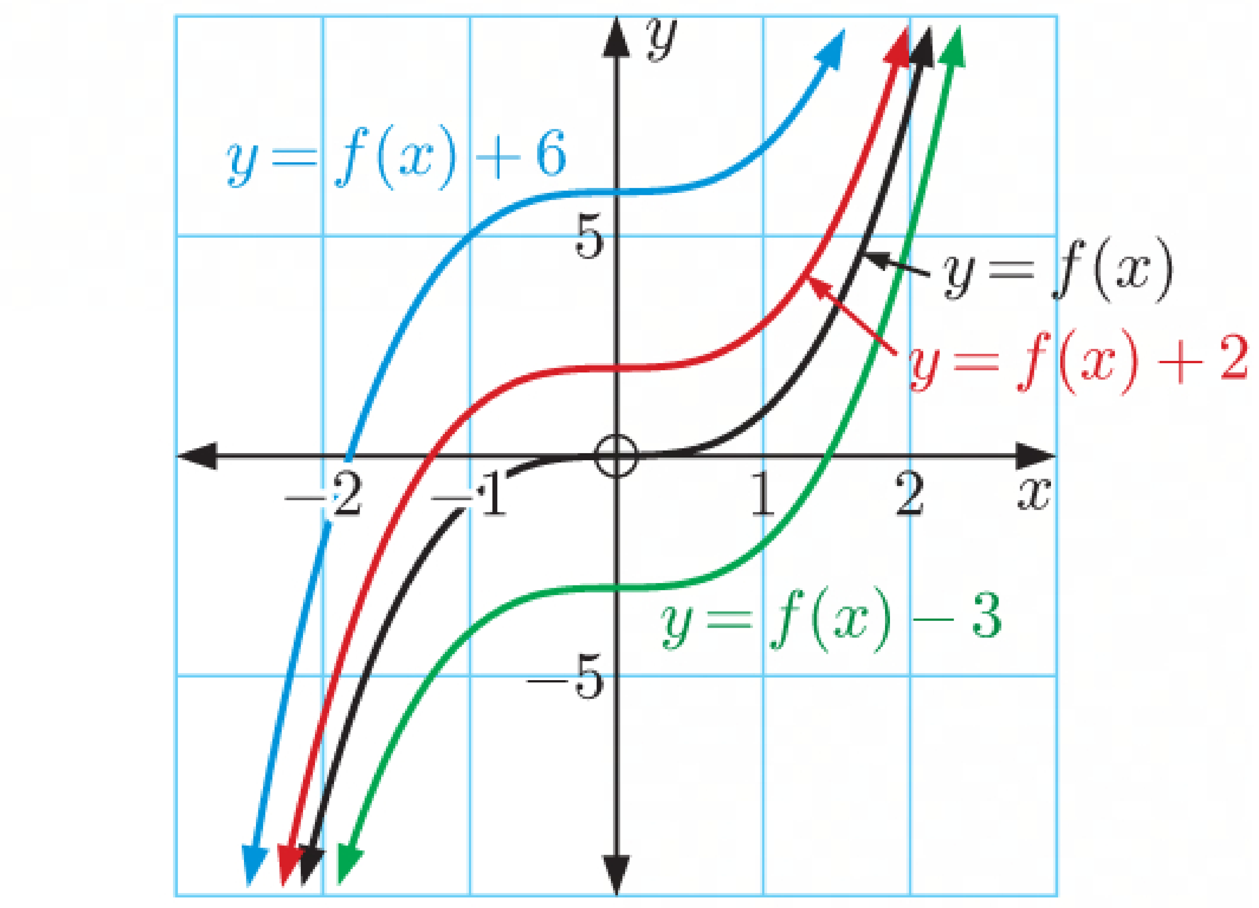
Chapter 16

TRANSFORMATIONS OF FUNCTIONS

INVESTIGATION 1 — TRANSLATIONS

1 $f(x) = x^3$

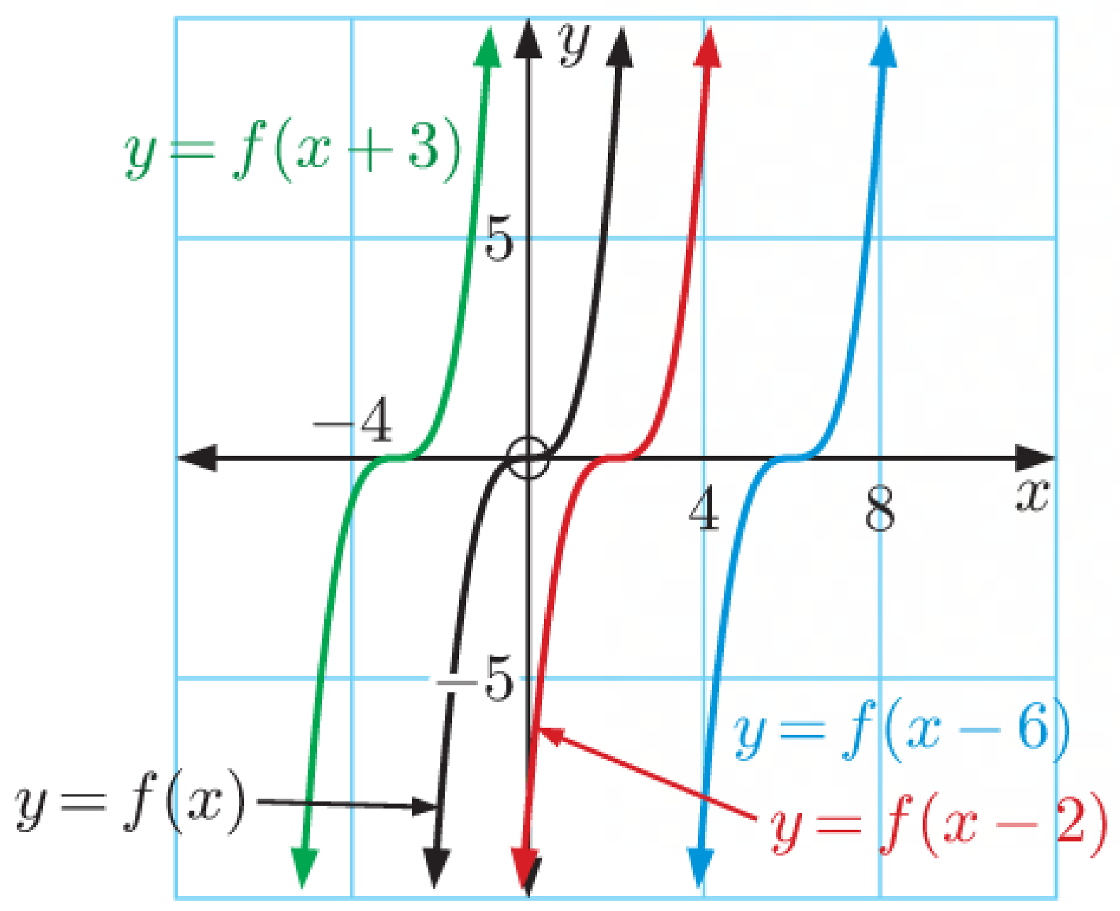
a **i** $f(x) + 2 = x^3 + 2$ **ii** $f(x) - 3 = x^3 - 3$ **iii** $f(x) + 6 = x^3 + 6$



For $y = f(x) + b$, the effect of b is to **translate** the graph **vertically** through b units.

- If $b > 0$ it moves **upwards**.
- If $b < 0$ it moves **downwards**.

b **i** $f(x-2) = (x-2)^3$ **ii** $f(x+3) = (x+3)^3$ **iii** $f(x-6) = (x-6)^3$

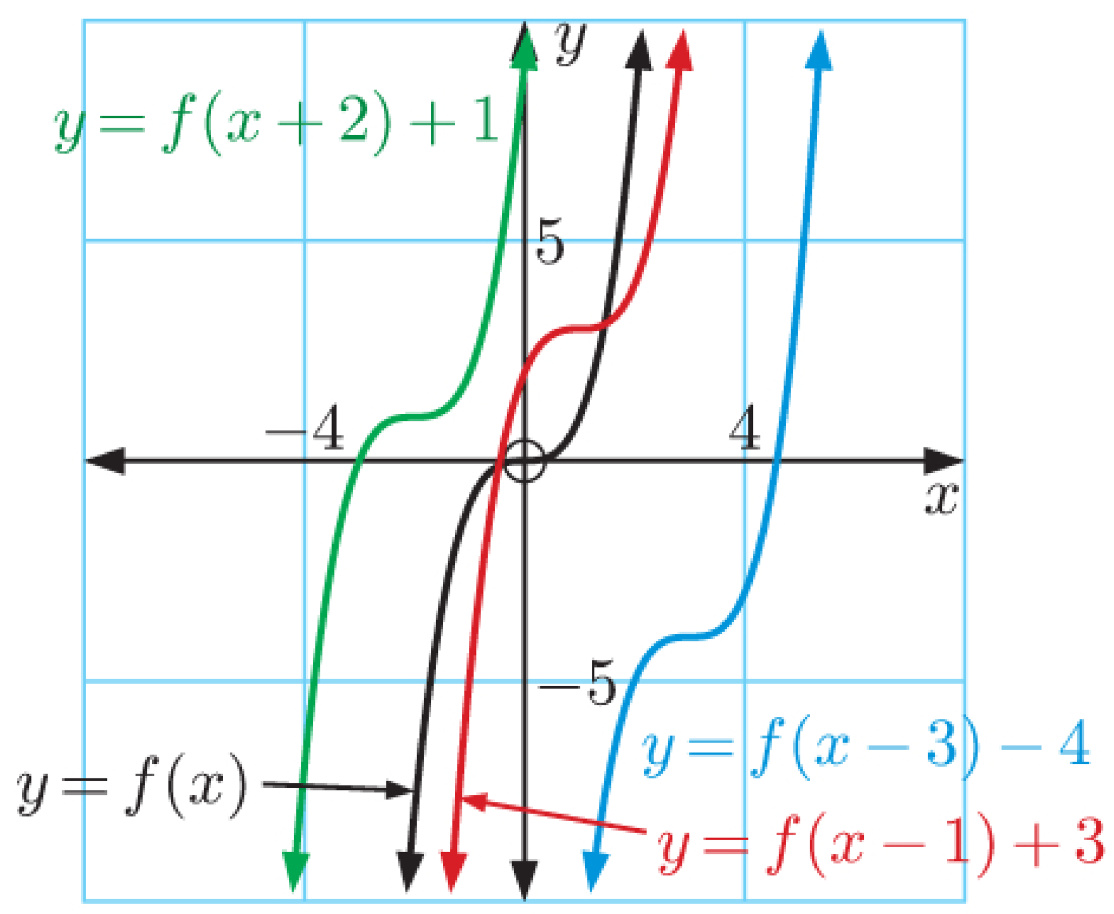


For $y = f(x-a)$, the effect of a is to **translate** the graph **horizontally** through a units.

- If $a > 0$ it moves to the **right**.
- If $a < 0$ it moves to the **left**.

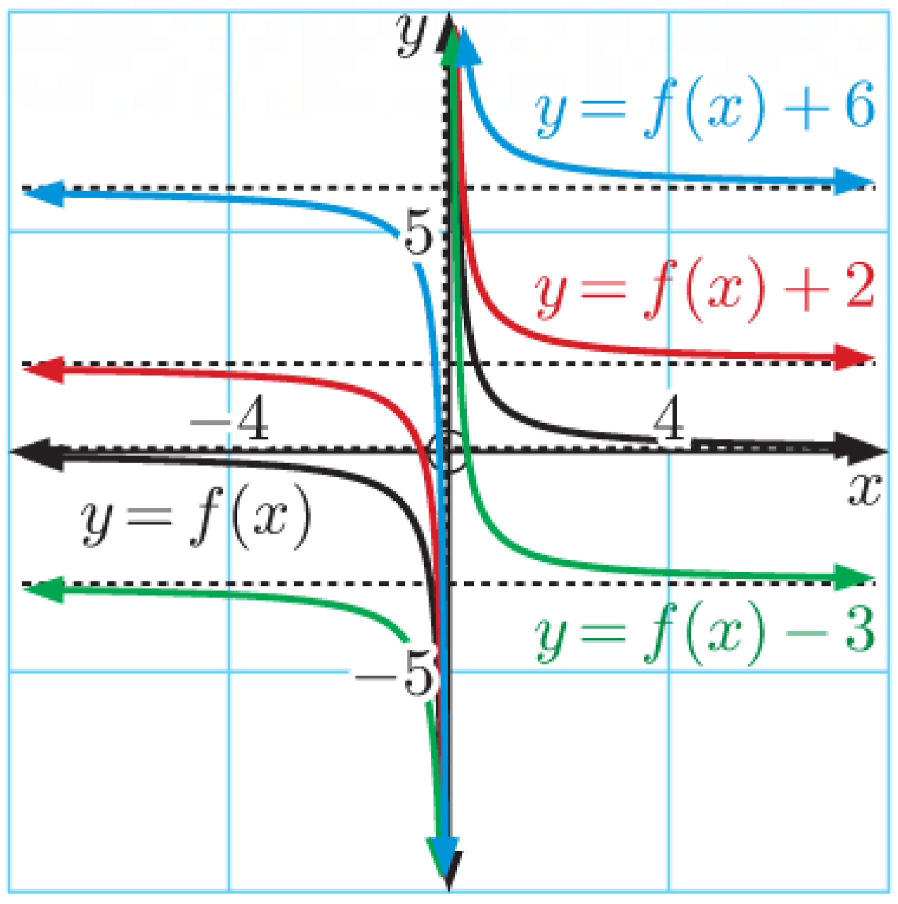
c **i** $f(x-1) + 3 = (x-1)^3 + 3$ **ii** $f(x+2) + 1 = (x+2)^3 + 1$

iii $f(x-3) - 4 = (x-3)^3 - 4$



2 $f(x) = \frac{1}{x}$

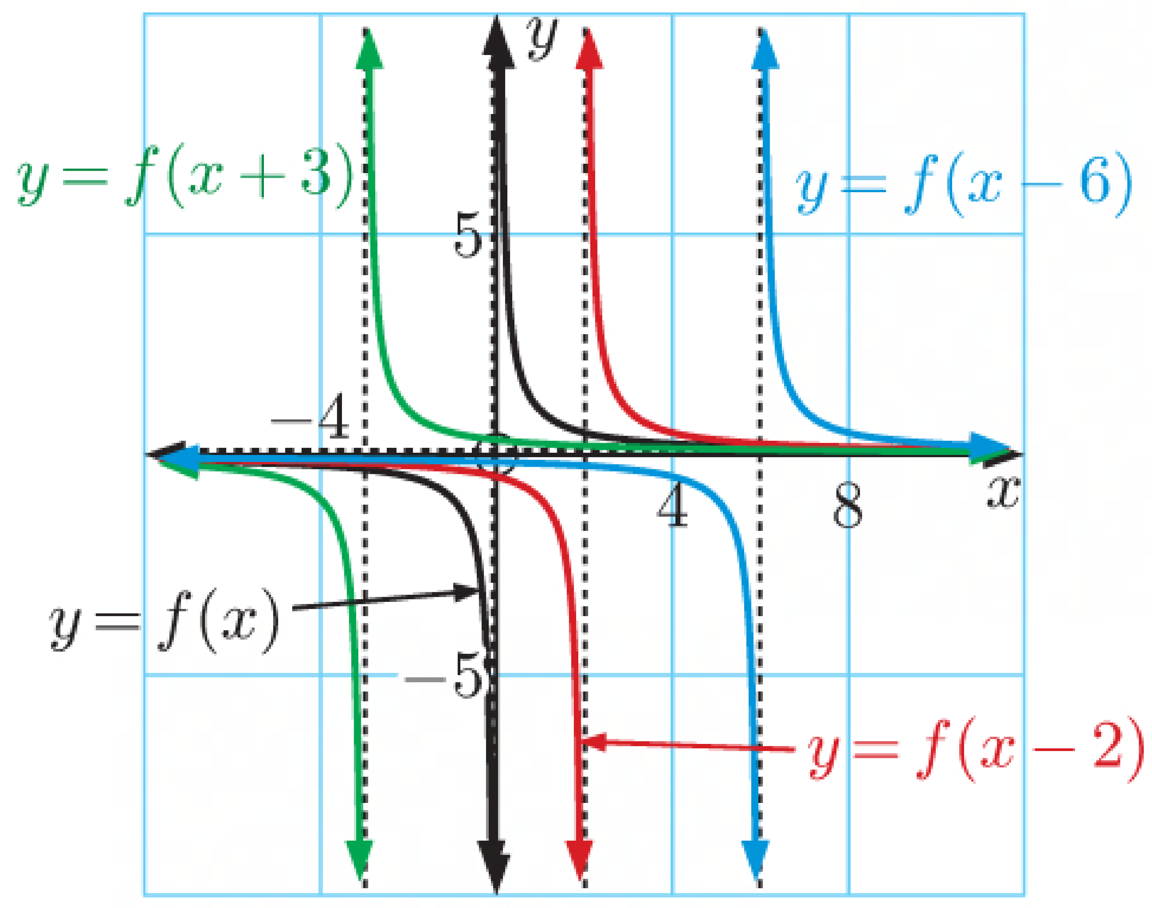
a **i** $f(x)+2 = \frac{1}{x}+2$ **ii** $f(x)-3 = \frac{1}{x}-3$ **iii** $f(x)+6 = \frac{1}{x}+6$



For $y = f(x) + b$, the effect of b is to **translate** the graph **vertically** through b units.

- If $b > 0$ it moves **upwards**.
- If $b < 0$ it moves **downwards**.

b **i** $f(x-2) = \frac{1}{x-2}$ **ii** $f(x+3) = \frac{1}{x+3}$ **iii** $f(x-6) = \frac{1}{x-6}$

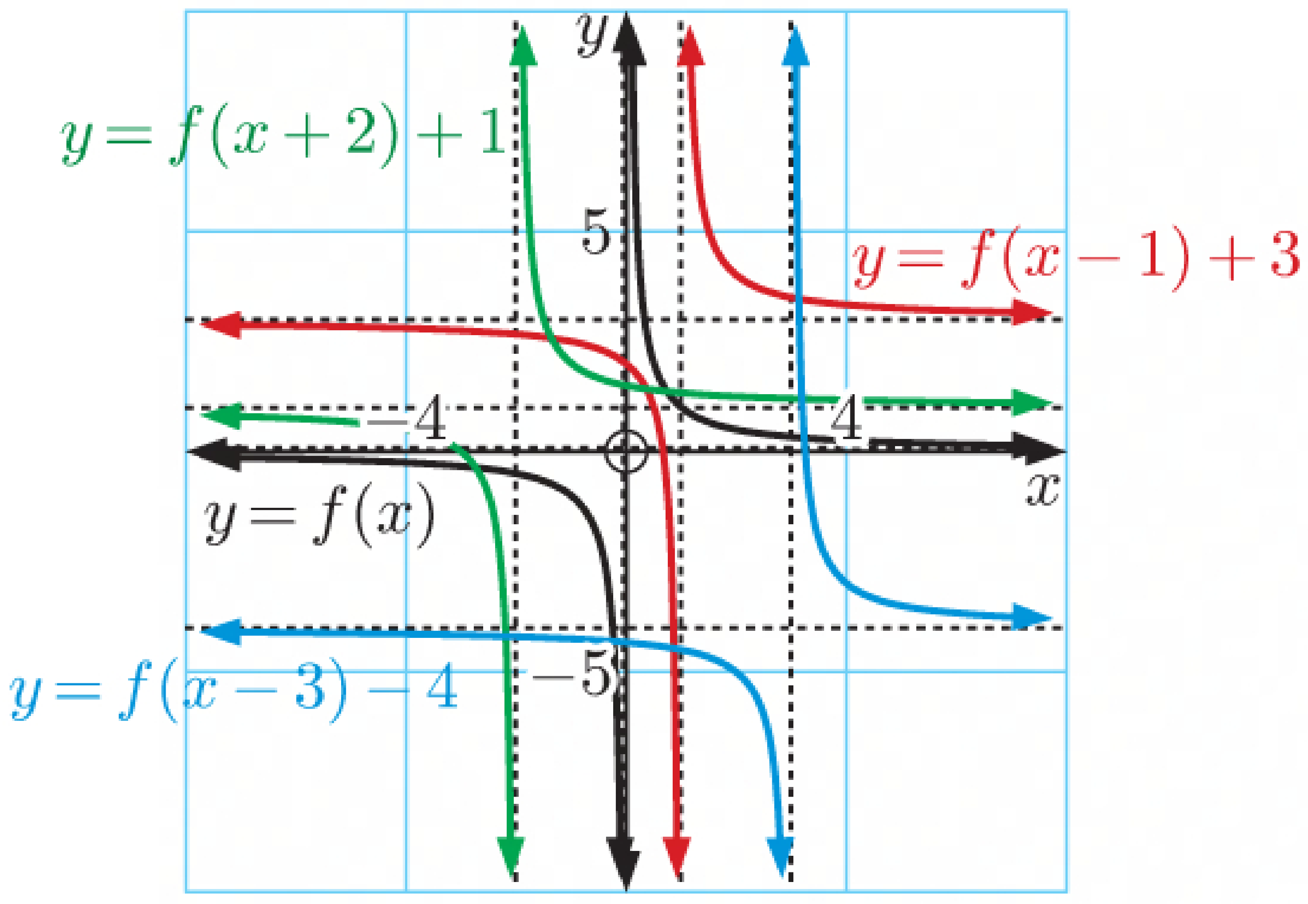


For $y = f(x-a)$, the effect of a is to **translate** the graph **horizontally** through a units.

- If $a > 0$ it moves to the **right**.
- If $a < 0$ it moves to the **left**.

c **i** $f(x-1)+3 = \frac{1}{x-1}+3$ **ii** $f(x+2)+1 = \frac{1}{x+2}+1$

iii $f(x-3)-4 = \frac{1}{x-3}-4$



3 **a** A translation b units vertically will map $y = f(x)$ onto $y = f(x) + b$.

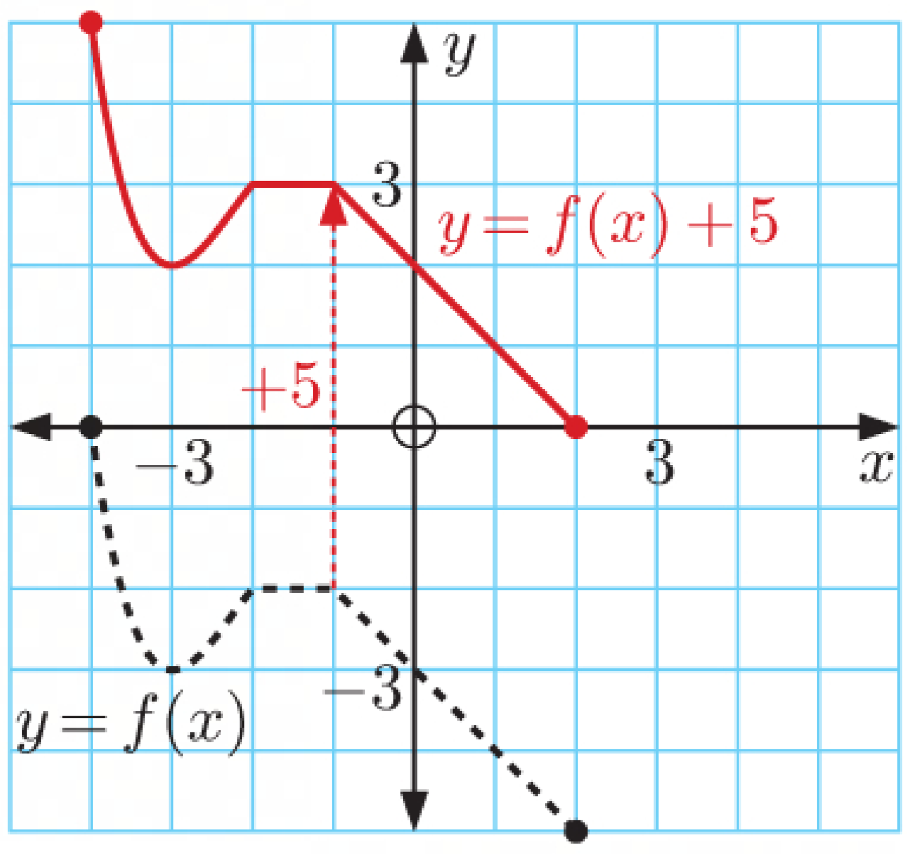
b A translation a units horizontally will map $y = f(x)$ onto $y = f(x-a)$.

c A translation a units horizontally and a translation b units vertically will map $y = f(x)$ onto $y = f(x-a)+b$.

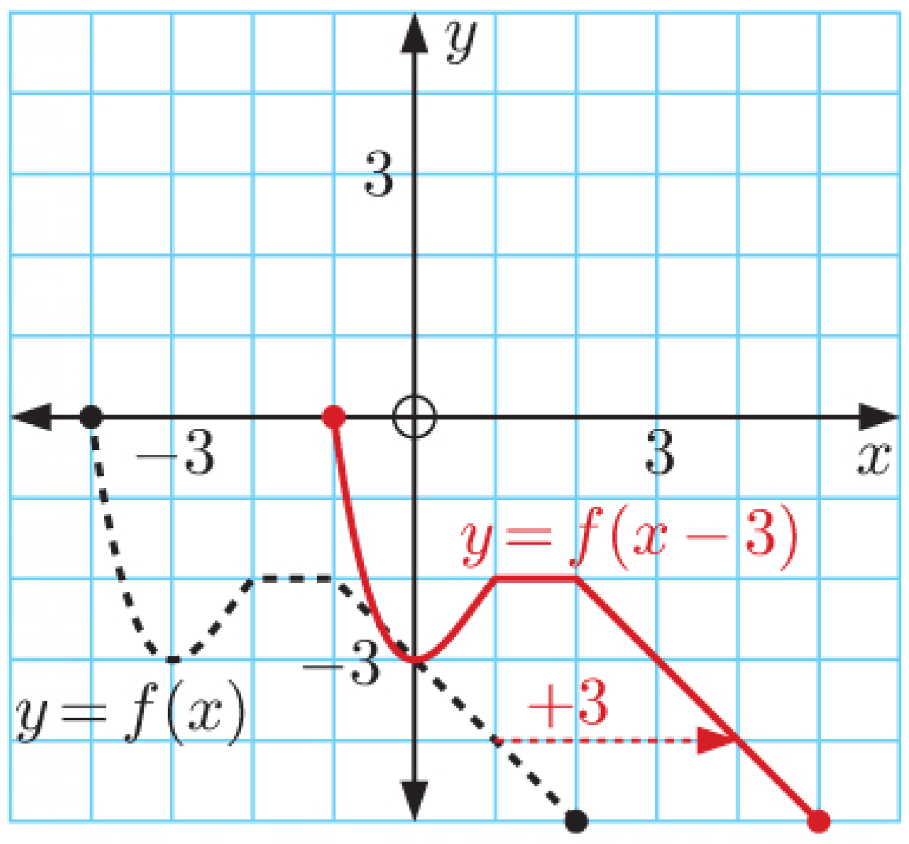
4 No, none of these transformations change the *shape* of the graph.

EXERCISE 16A

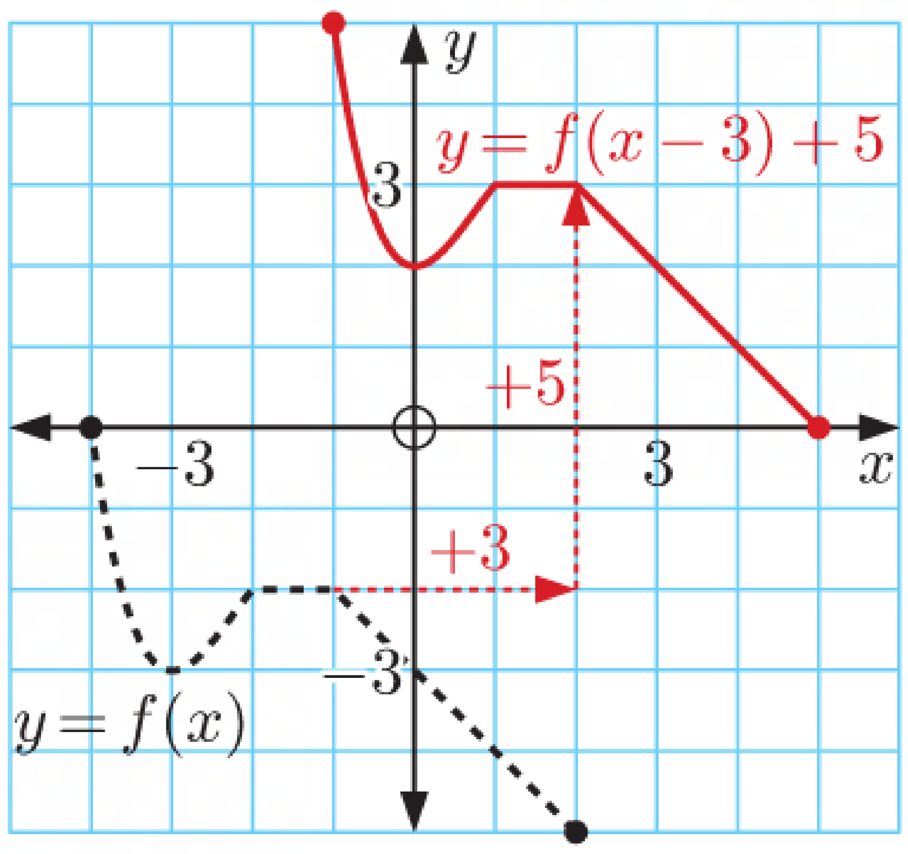
1 **a** The graph of $y = f(x) + 5$ is found by translating $y = f(x)$ 5 units upwards.



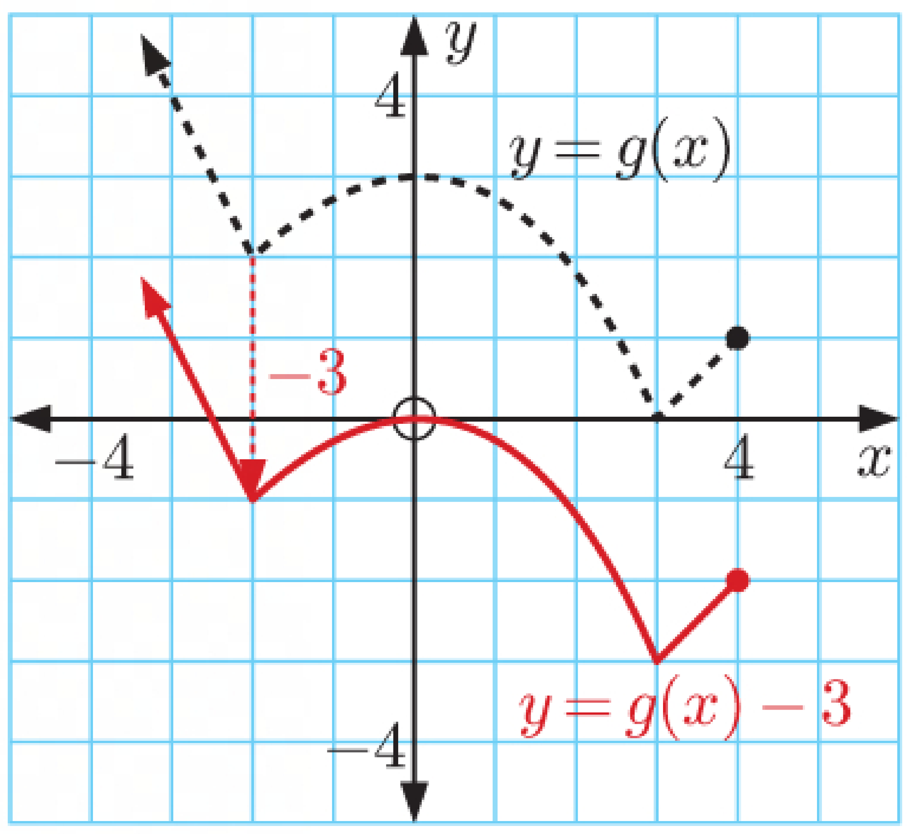
b The graph of $y = f(x - 3)$ is found by translating $y = f(x)$ 3 units to the right.



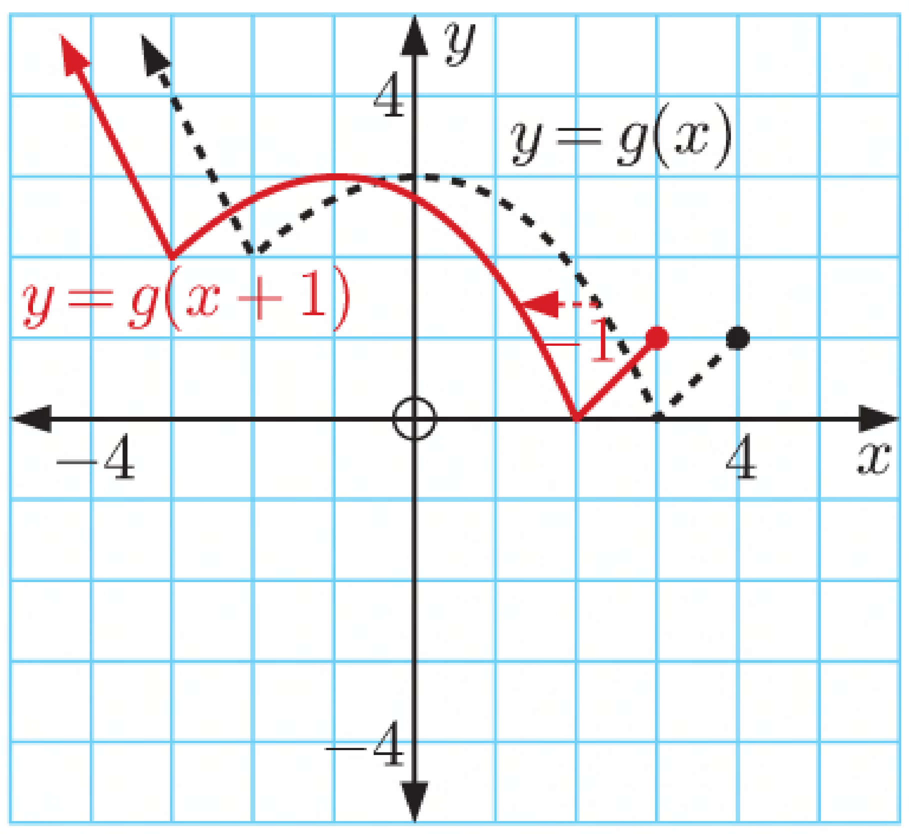
c The graph of $y = f(x - 3) + 5$ is found by translating $y = f(x)$ 3 units to the right and 5 units upwards.



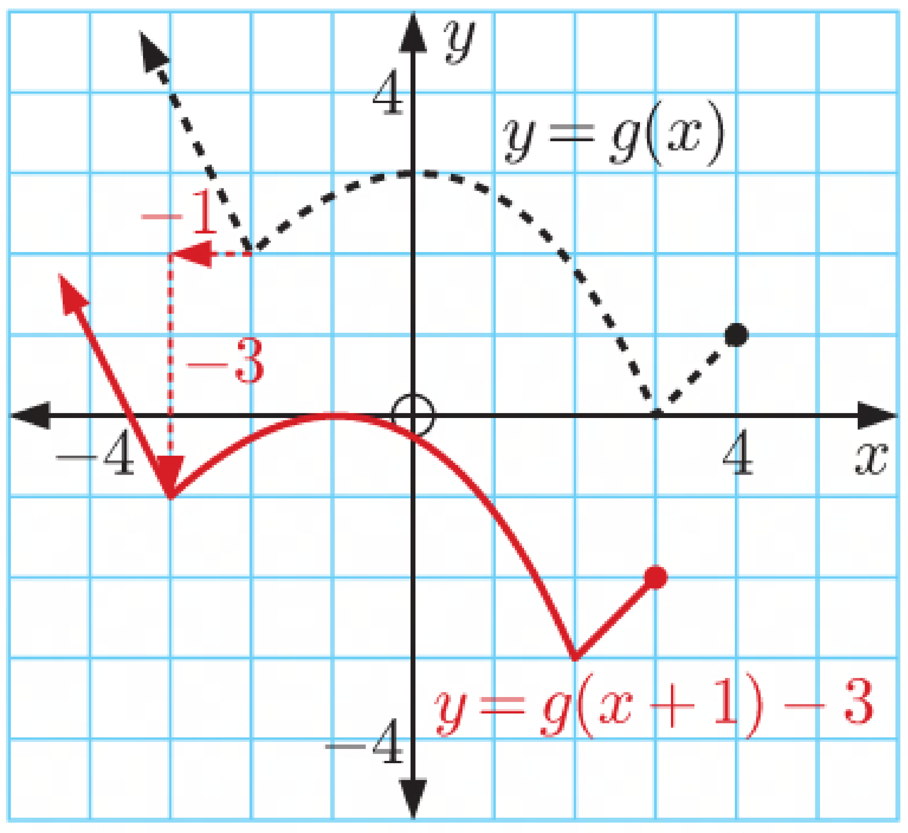
2 **a** The graph of $y = g(x) - 3$ is found by translating $y = g(x)$ 3 units downwards.



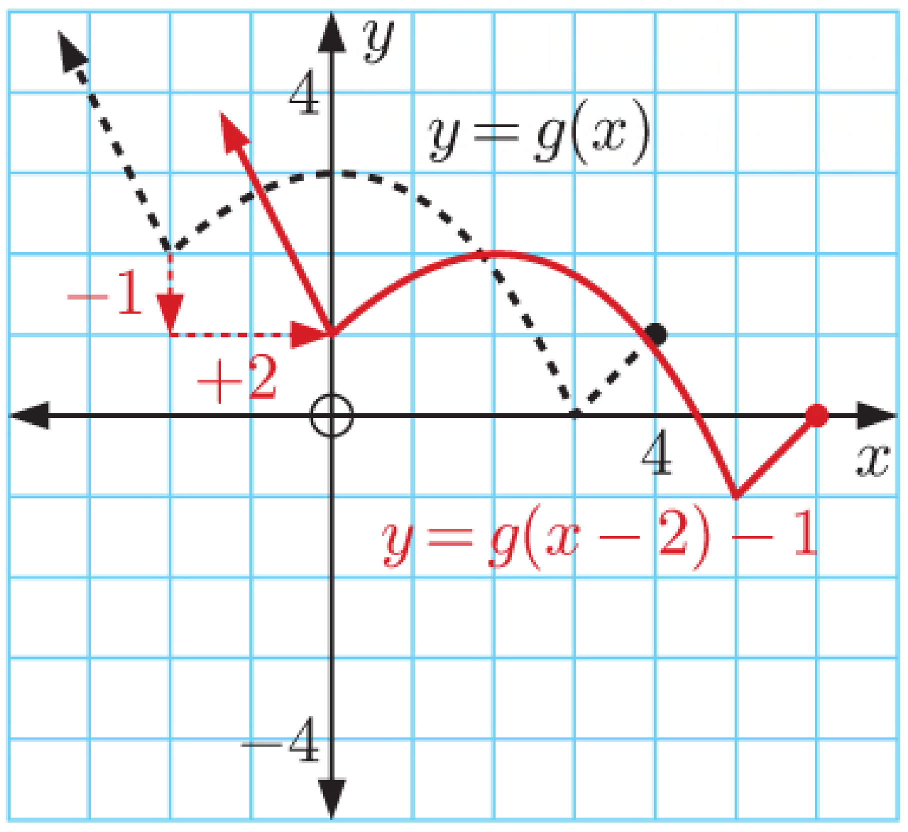
b The graph of $y = g(x + 1)$ is found by translating $y = g(x)$ 1 unit to the left.



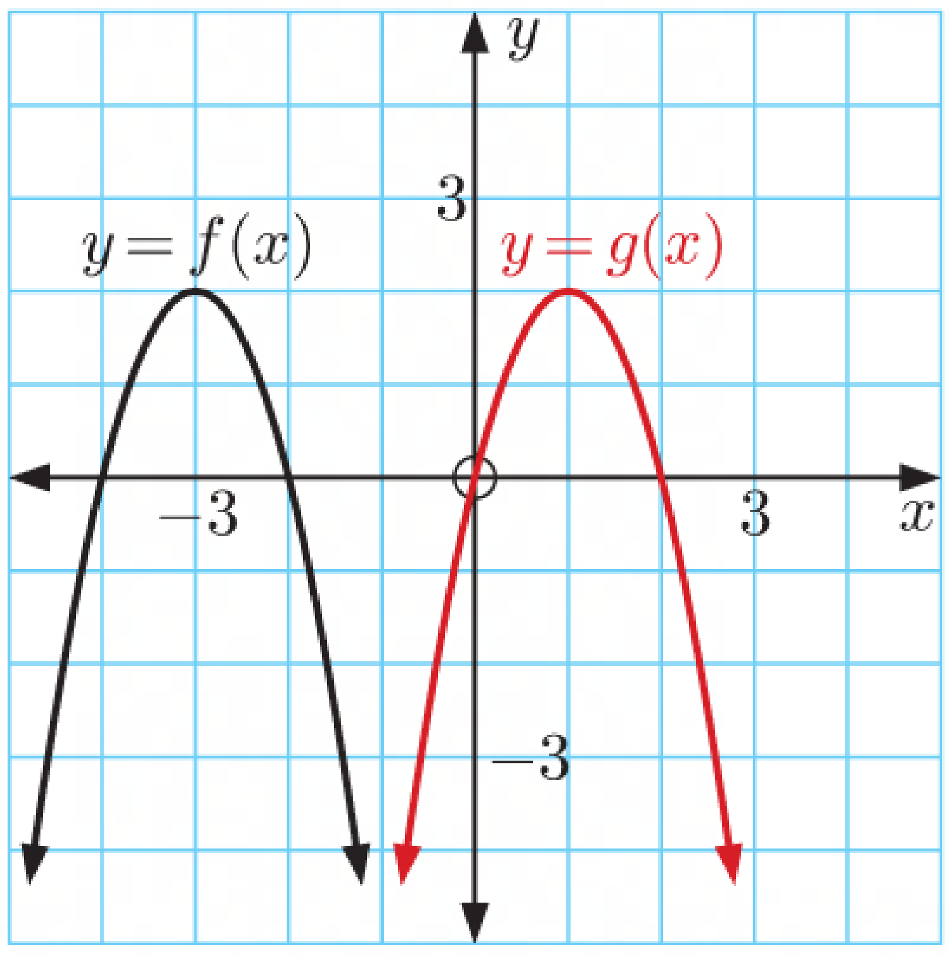
c The graph of $y = g(x+1) - 3$ is found by translating $y = g(x)$ 1 unit to the left and 3 units downwards.



d The graph of $y = g(x-2) - 1$ is found by translating $y = g(x)$ 2 units to the right and 1 unit downwards.



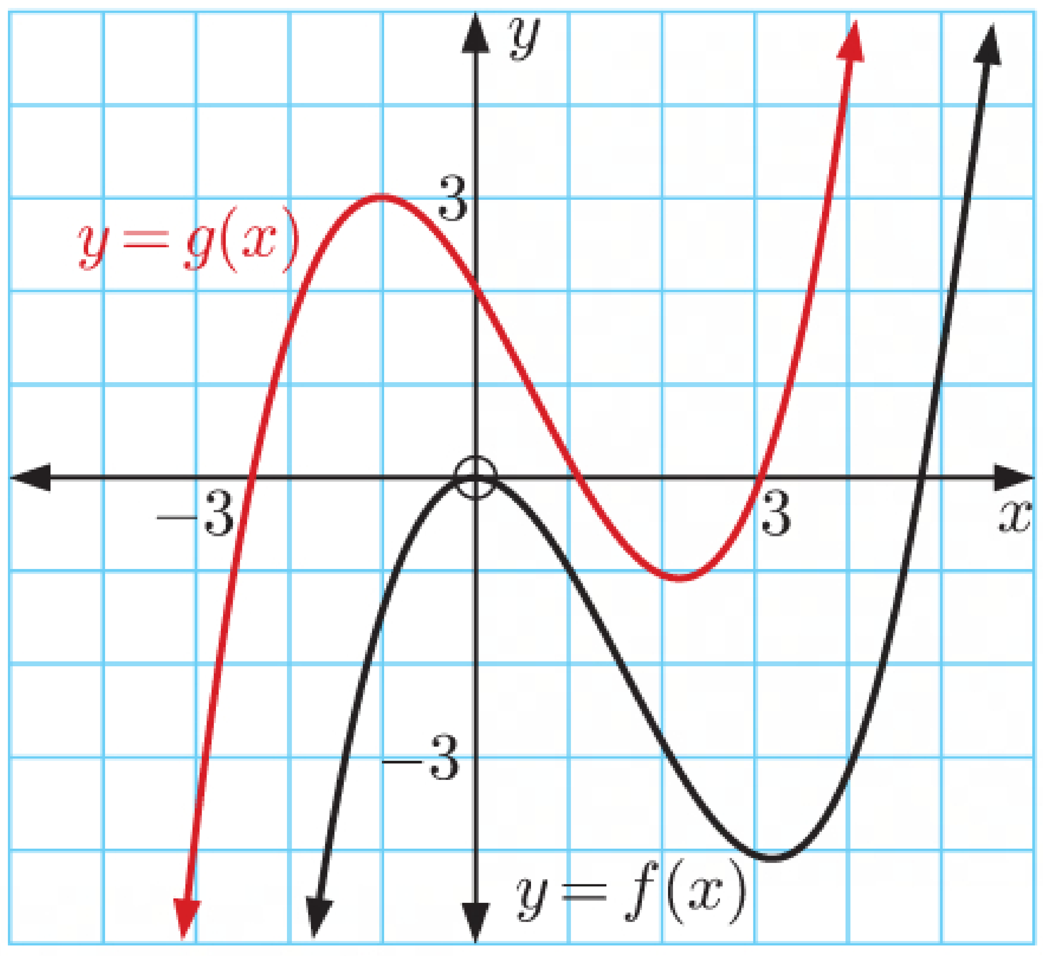
3 **a**



The graph of $y = f(x)$ has been translated 4 units to the right to result in $y = g(x)$.

So, $g(x) = f(x-4)$.

b



The graph of $y = f(x)$ has been translated 1 unit to the left, and 3 units upwards, to result in $y = g(x)$.

So, $g(x) = f(x+1) + 3$.

4 **a** The graph of $y = g(x)$ is found by translating $y = f(x)$ 4 units downwards.

$\therefore\ g(x) = f(x) - 4$

$\therefore\ g(x) = (2x+3) - 4$ {since $f(x) = 2x + 3$}

$\therefore\ g(x) = 2x - 1$

b The graph of $y = g(x)$ is found by translating $y = f(x)$ 2 units to the left.

$\therefore\ g(x) = f(x+2)$

$\therefore\ g(x) = 3(x+2) - 4$ {since $f(x) = 3x - 4$}

$\therefore\ g(x) = 3x + 2$

c The graph of $y = g(x)$ is found by translating $y = f(x)$ 3 units upwards.

$\therefore\ g(x) = f(x) + 3$

$\therefore\ g(x) = (-x^2 + 5x - 7) + 3$ {since $f(x) = -x^2 + 5x - 7$}

$\therefore\ g(x) = -x^2 + 5x - 4$

d The graph of $y = g(x)$ is found by translating $y = f(x)$ 5 units to the right.

$\therefore \; g(x) = f(x-5)$

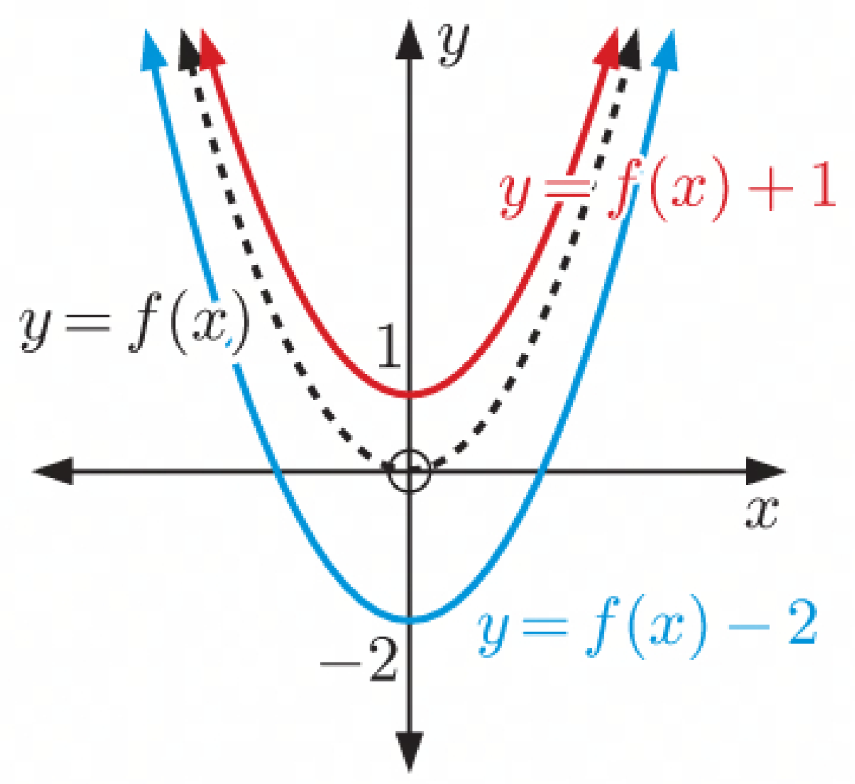
$\therefore \; g(x) = (x-5)^2 + 4(x-5) - 1 \qquad \{\text{since } f(x) = x^2 + 4x - 1\}$

$\therefore \; g(x) = x^2 - 10x + 25 + 4x - 20 - 1$

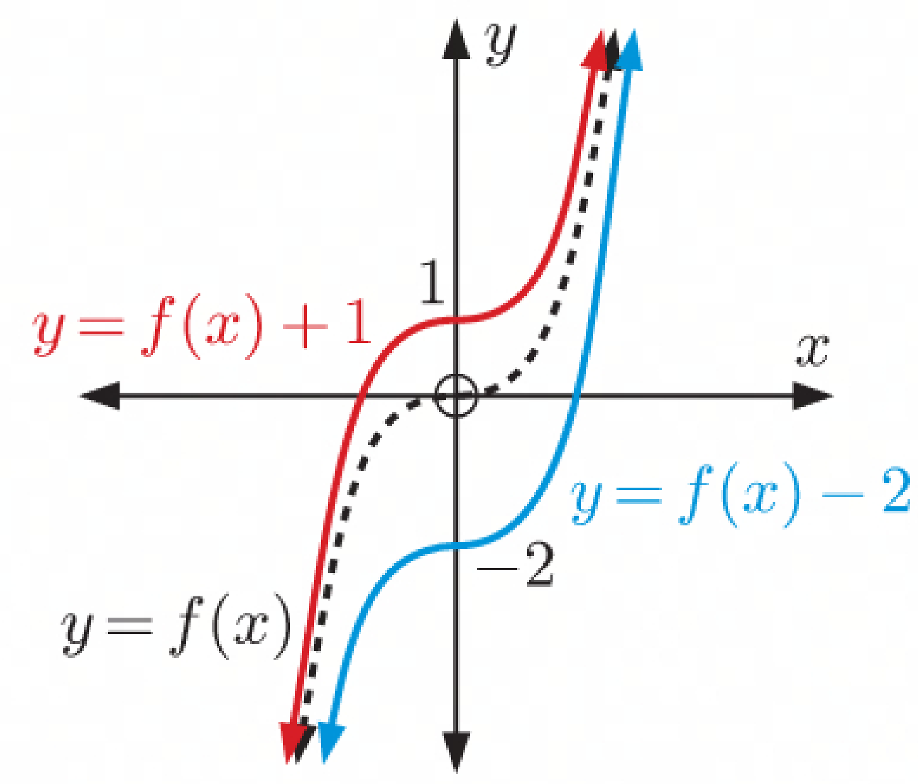
$\therefore \; g(x) = x^2 - 6x + 4$

5 $y = f(x) + 1$ is found by translating $y = f(x)$ 1 unit upwards, $y = f(x) - 2$ is found by translating $y = f(x)$ 2 units downwards.

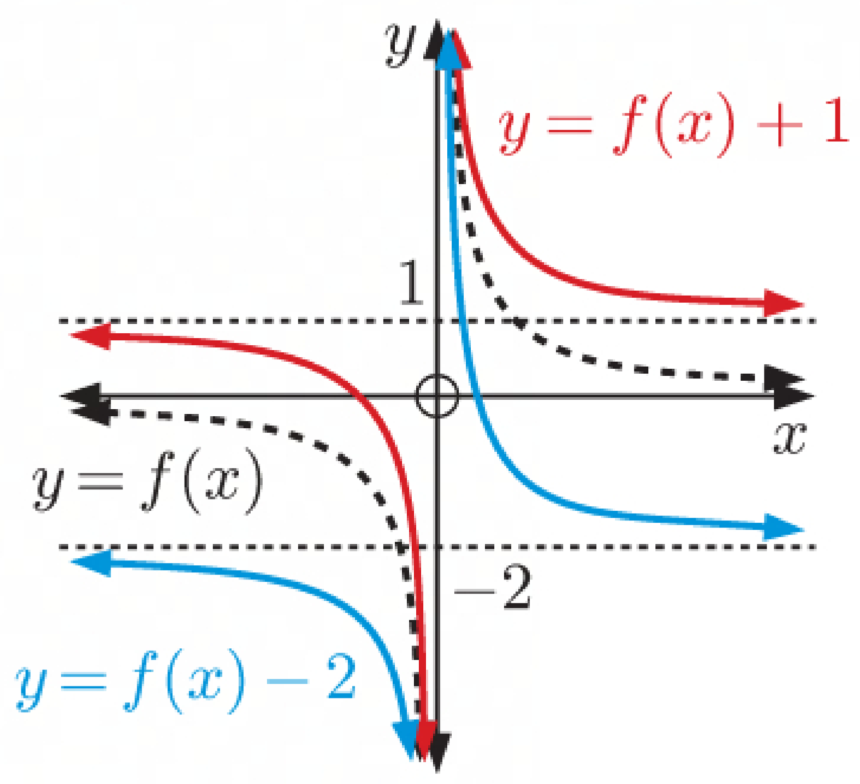
a $f(x) = x^2$



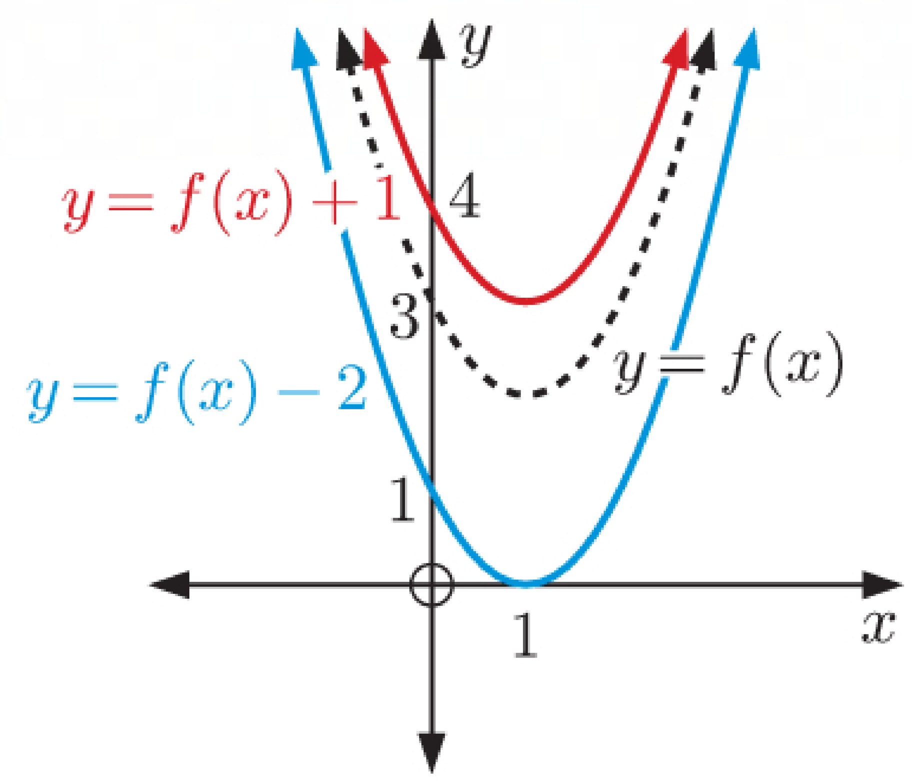
b $f(x) = x^3$



c $f(x) = \dfrac{1}{x}$

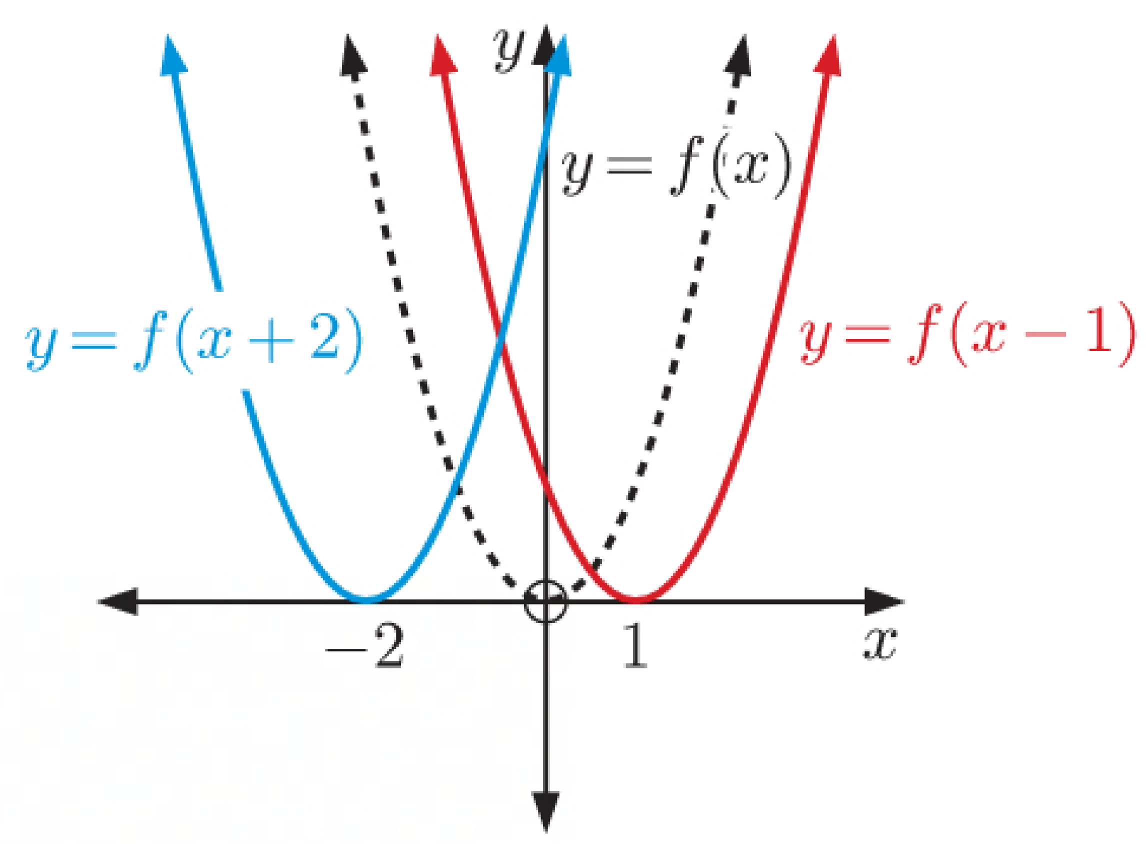


d $f(x) = (x-1)^2 + 2$

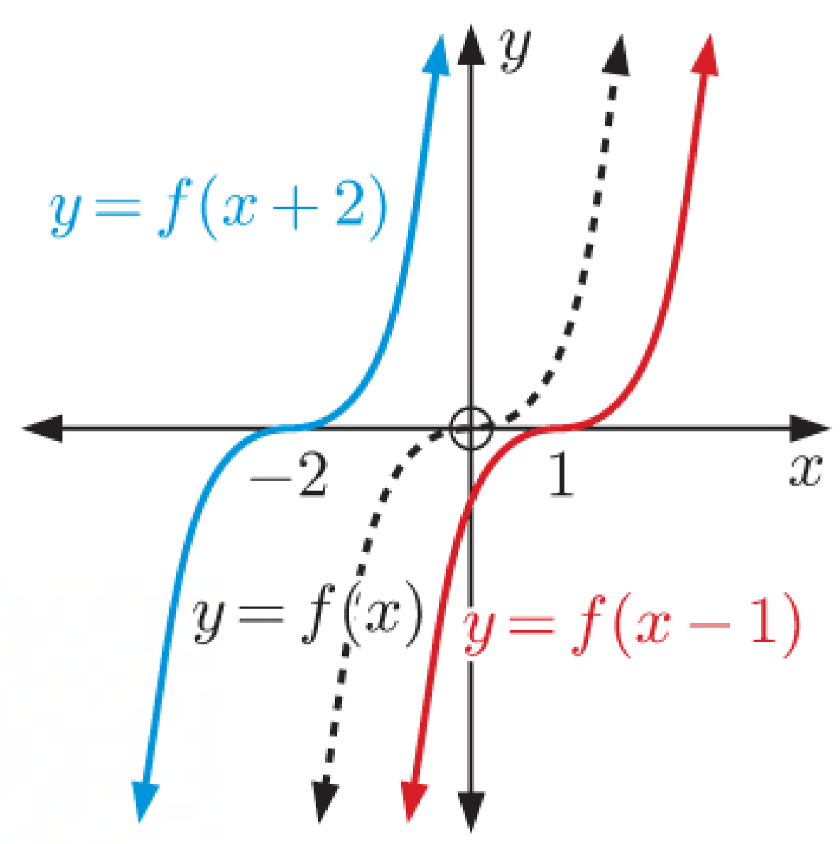


6 $y = f(x-1)$ is found by translating $y = f(x)$ 1 unit to the right, $y = f(x+2)$ is found by translating $y = f(x)$ 2 units to the left.

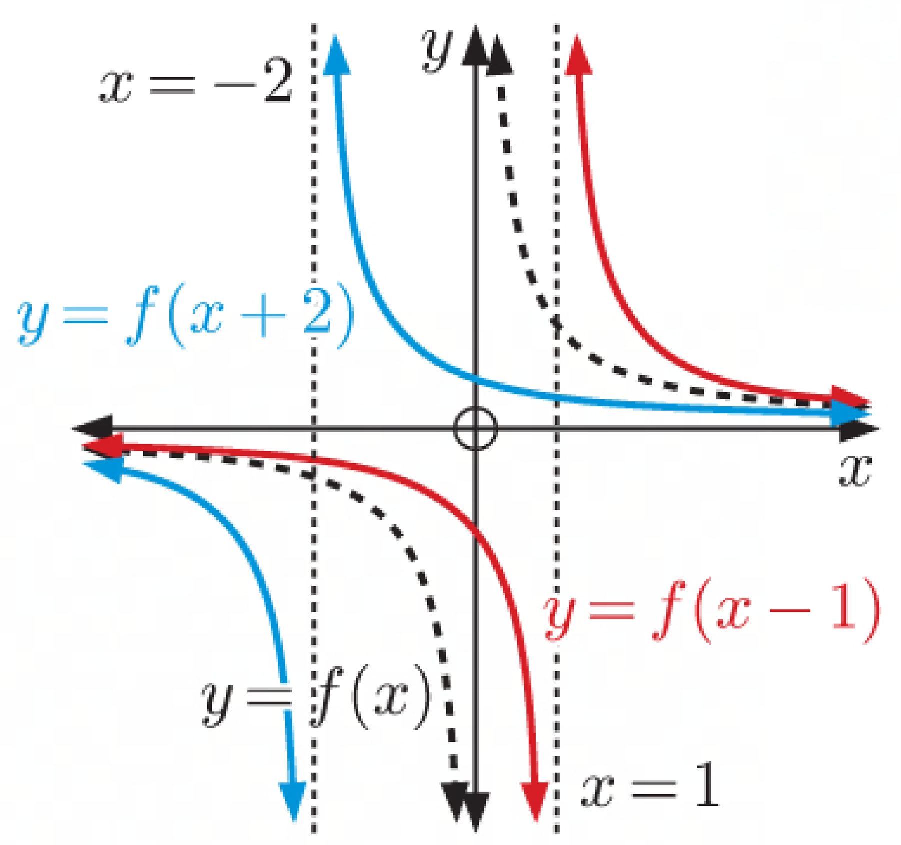
a $f(x) = x^2$



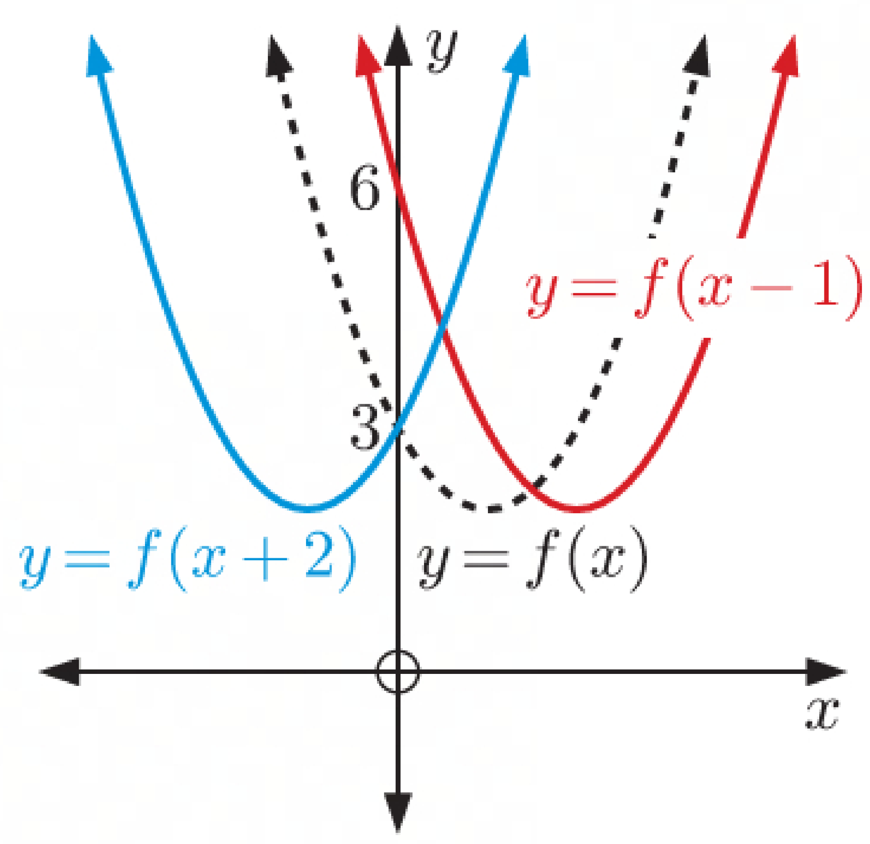
b $f(x) = x^3$



c $f(x) = \frac{1}{x}$

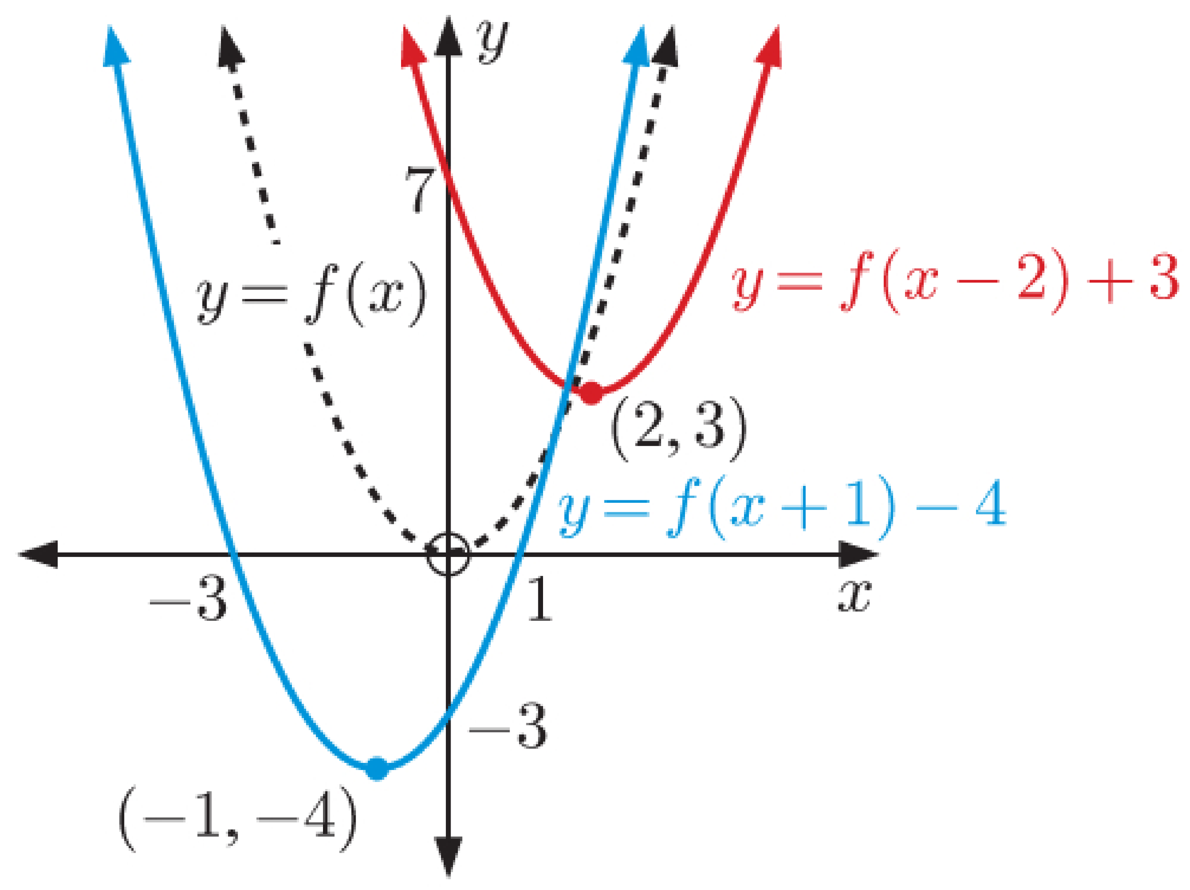


d $f(x) = (x-1)^2 + 2$

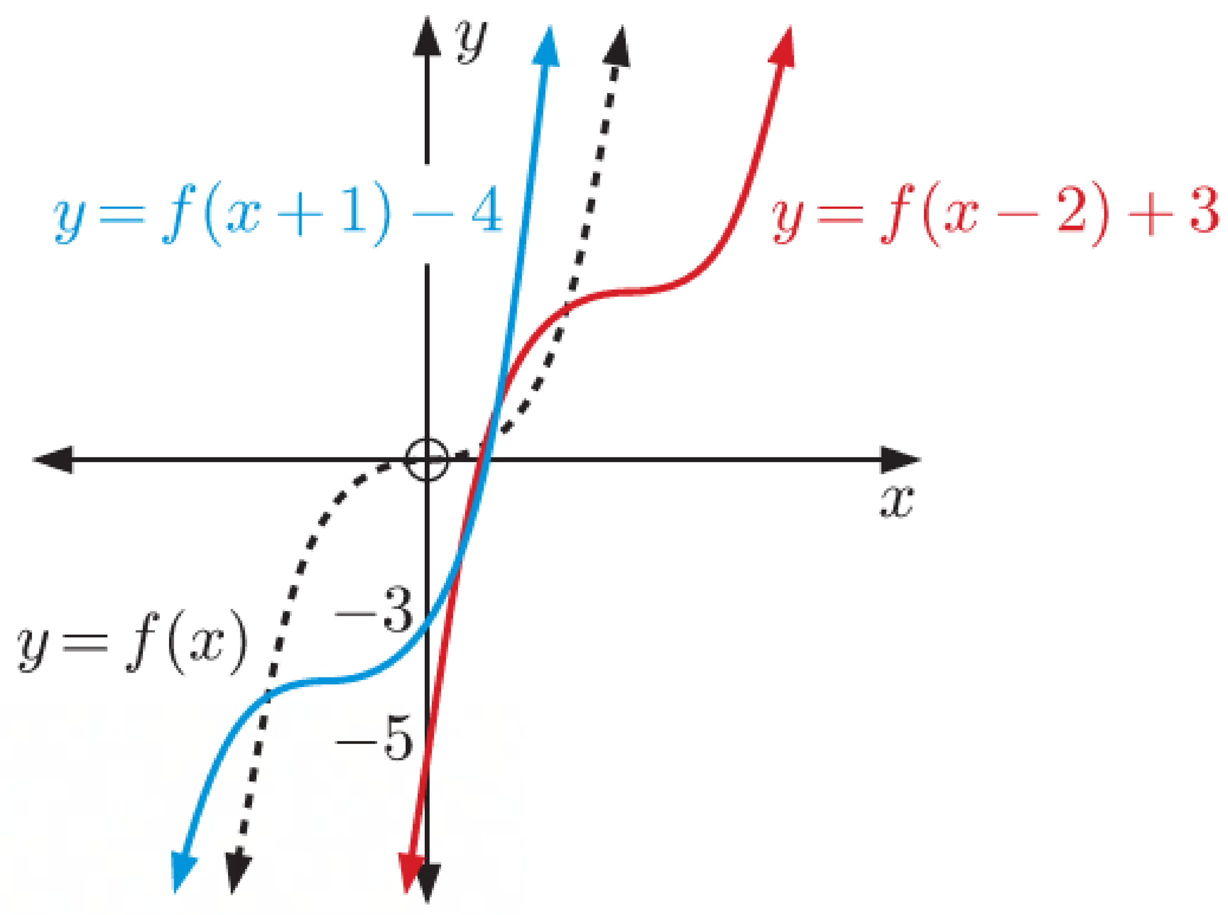


7 $y = f(x-2) + 3$ is found by translating $y = f(x)$ 2 units to the right and 3 units upwards, $y = f(x+1) - 4$ is found by translating $y = f(x)$ 1 unit to the left and 4 units downwards.

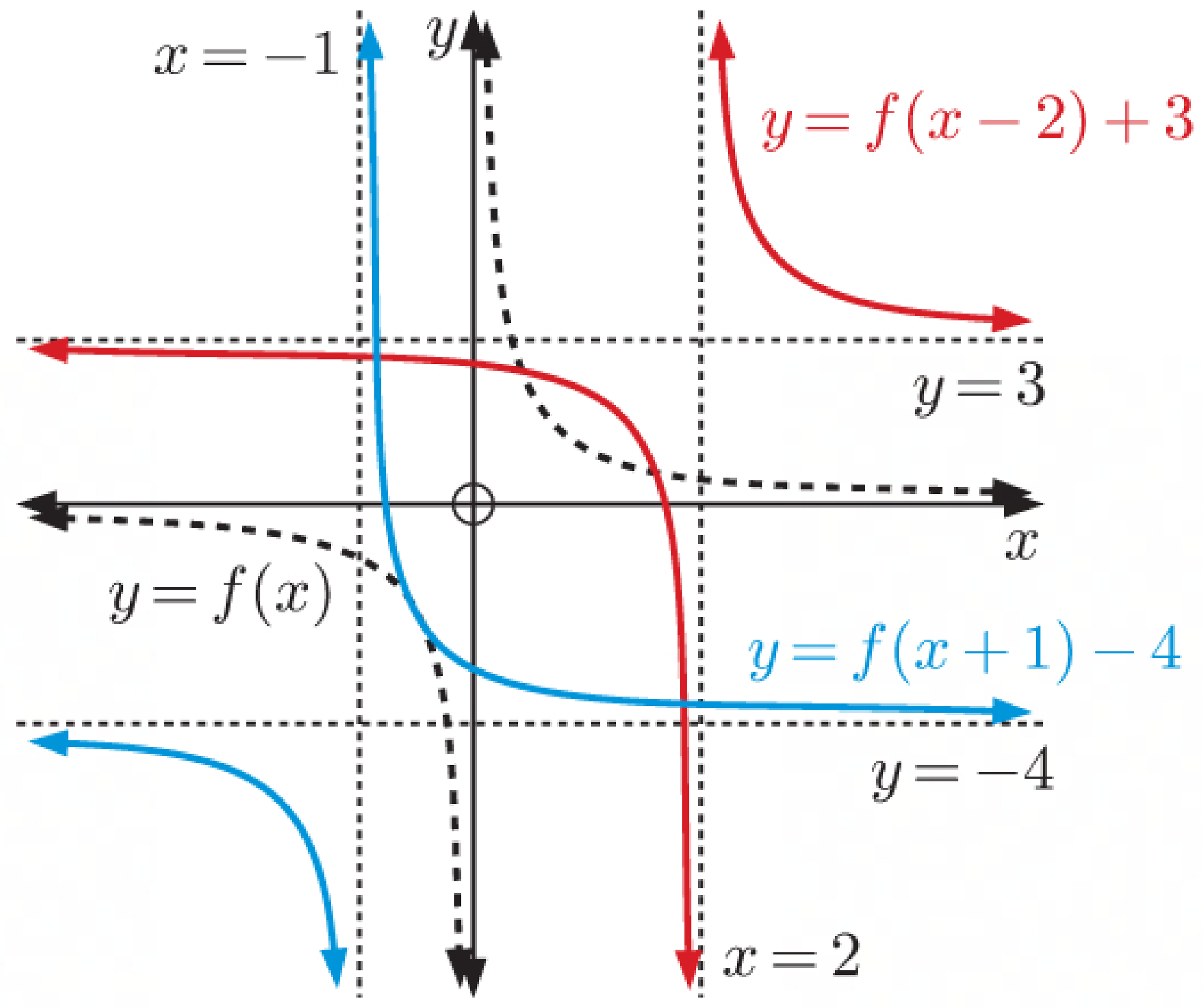
a $f(x) = x^2$



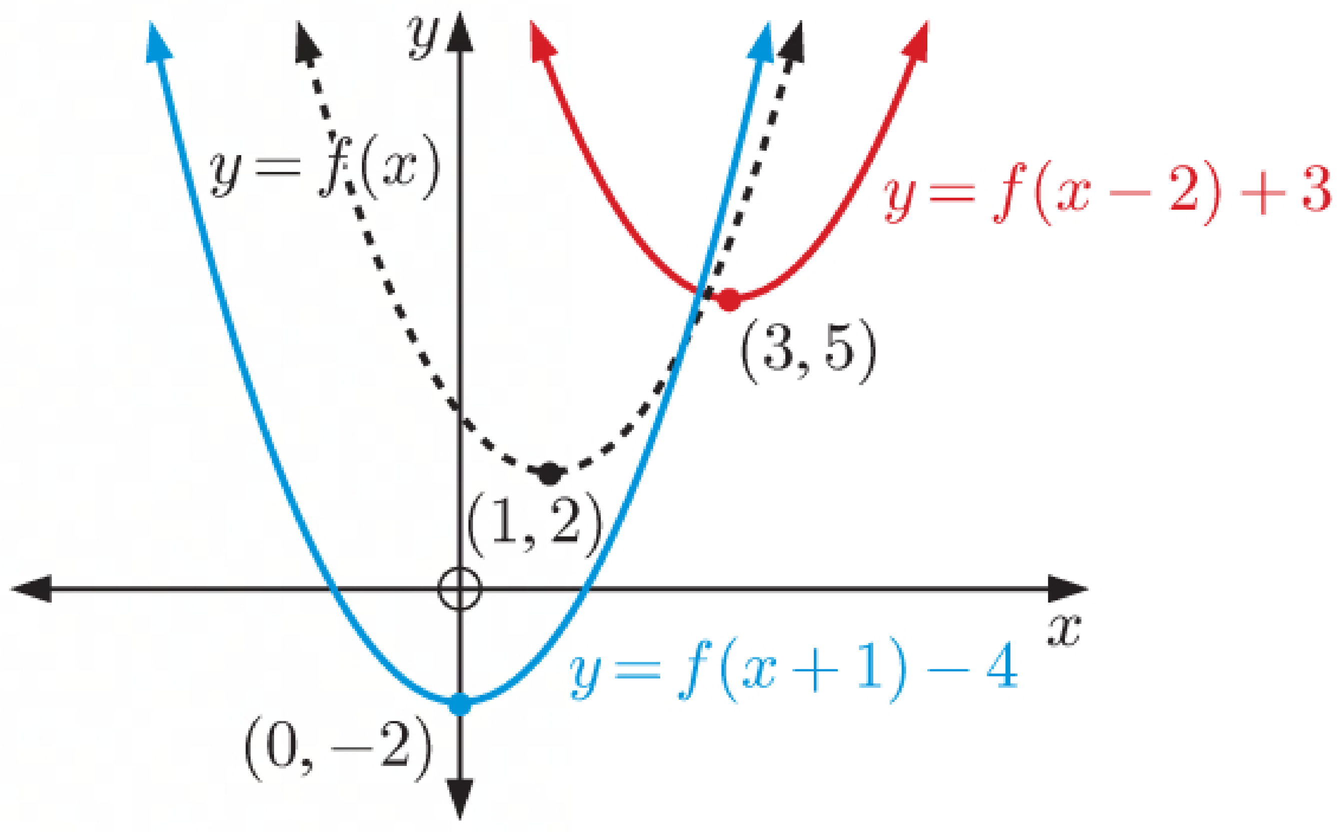
b $f(x) = x^3$



c $f(x) = \frac{1}{x}$



d $f(x) = (x-1)^2 + 2$



8 The graph of $y = g(x) = f(x-3) - 4$ is a translation of $y = f(x)$ 3 units to the right and 4 units downwards.

So, $y = f(x)$ has been translated by the vector $\begin{pmatrix} 3 \\ -4 \end{pmatrix}$.

The point $(-2, -5)$ on the graph of $y = f(x)$ will therefore be translated by the vector $\begin{pmatrix} 3 \\ -4 \end{pmatrix}$, to give the point $(-2+3, \ -5-4)$, or $(1, -9)$, on the graph of $y = g(x)$.

9 **a** The graph of $y = g(x) = f(x) - 3$ is a translation of $y = f(x)$ 3 units downwards.
So, the graph of $y = g(x)$ has y-intercept $2 - 3 = -1$.
There is not enough information to determine the x-intercepts.

b The graph of $y = h(x) = f(x-1)$ is a translation of $y = f(x)$ 1 unit to the right.
So, the graph of $y = h(x)$ has x-intercepts $-3 + 1 = -2$ and $4 + 1 = 5$.
There is not enough information to determine the y-intercept.

c The graph of $y = j(x) = f(x+2) - 4$ is a translation of $y = f(x)$ 2 units to the left and 4 units downwards.
There is not enough information to determine the x or the y-intercepts.

10 $$\begin{aligned} g(x) = f(x-3) - 5 &= (x-3)^2 - 2(x-3) + 2 - 5 \qquad \{\text{since } f(x) = x^2 - 2x + 2\} \\ &= x^2 - 6x + 9 - 2x + 6 + 2 - 5 \\ &= x^2 - 8x + 12 \end{aligned}$$

11 $$\begin{aligned} g(x) = f(x+2) + 7 &= \frac{1}{x+2} + 7 \qquad \{\text{since } f(x) = \frac{1}{x}\} \\ &= \frac{1}{x+2} + \frac{7(x+2)}{x+2} \\ &= \frac{1 + 7x + 14}{x+2} \\ &= \frac{7x+15}{x+2} \end{aligned}$$

12 $g(x) = (x-3)^2 + 2$ is found by translating $f(x) = x^2$ 3 units to the right and 2 units upwards.

a The points on $y = f(x)$ are translated by $\begin{pmatrix} 3 \\ 2 \end{pmatrix}$ to find the image points on $y = g(x)$.

i $(0, 0)$ is translated to $(3, 2)$

ii $(-3, 9)$ is translated to $(-3+3, 9+2)$, or $(0, 11)$

iii $(2, 4)$ is translated to $(2+3, 4+2)$, or $(5, 6)$

b The points on $y = g(x)$ are translated by $\begin{pmatrix} -3 \\ -2 \end{pmatrix}$ to find the corresponding points on $y = f(x)$.

i $(1, 6)$ corresponds to $(1-3, 6-2)$, or $(-2, 4)$

ii $(-2, 27)$ corresponds to $(-2-3, 27-2)$, or $(-5, 25)$

iii $(1\frac{1}{2}, 4\frac{1}{4})$ corresponds to $(1\frac{1}{2} - 3, 4\frac{1}{4} - 2)$, or $(-1\frac{1}{2}, 2\frac{1}{4})$

INVESTIGATION 2 STRETCHES

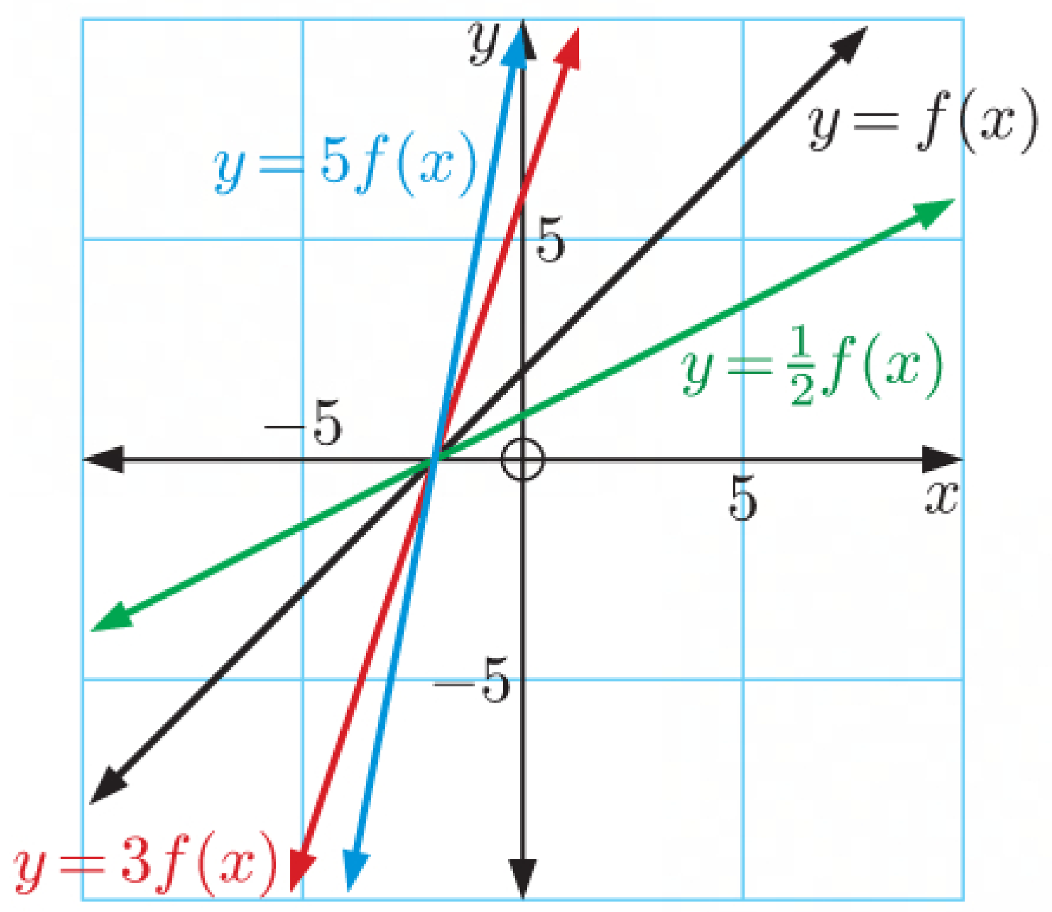
1 $f(x) = x + 2$

a **i** $3f(x) = 3(x+2) = 3x + 6$

ii $\frac{1}{2}f(x) = \frac{1}{2}(x+2) = \frac{1}{2}x + 1$

iii $5f(x) = 5(x+2) = 5x + 10$

b



c All transformations of the form $p\,f(x)$, $p > 0$ do not move the point $(-2, 0)$.

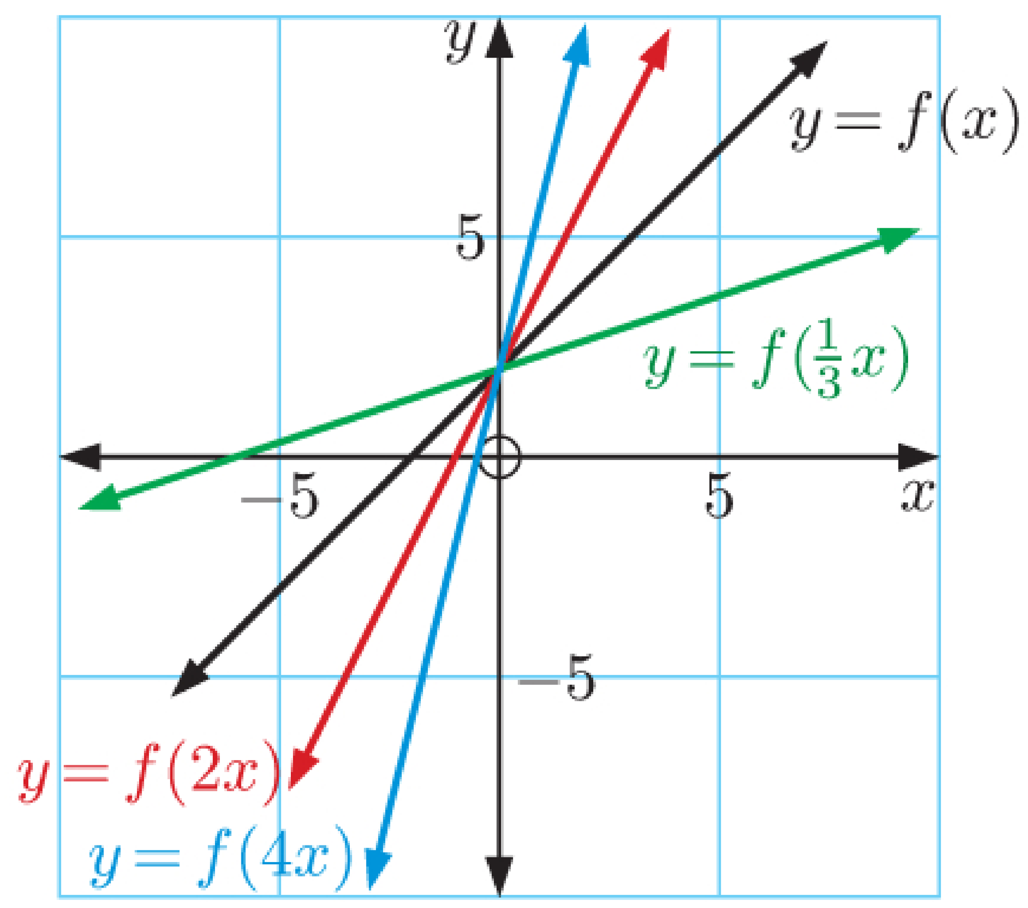
$\therefore$ $(-2, 0)$ is *invariant.*

d For the transformation $y = p\,f(x)$, each point becomes p times its previous distance from the x-axis.

2 $f(x) = x + 2$

a **i** $f(2x) = 2x + 2$ **ii** $f(\frac{1}{3}x) = \frac{1}{3}x + 2$ **iii** $f(4x) = 4x + 2$

b



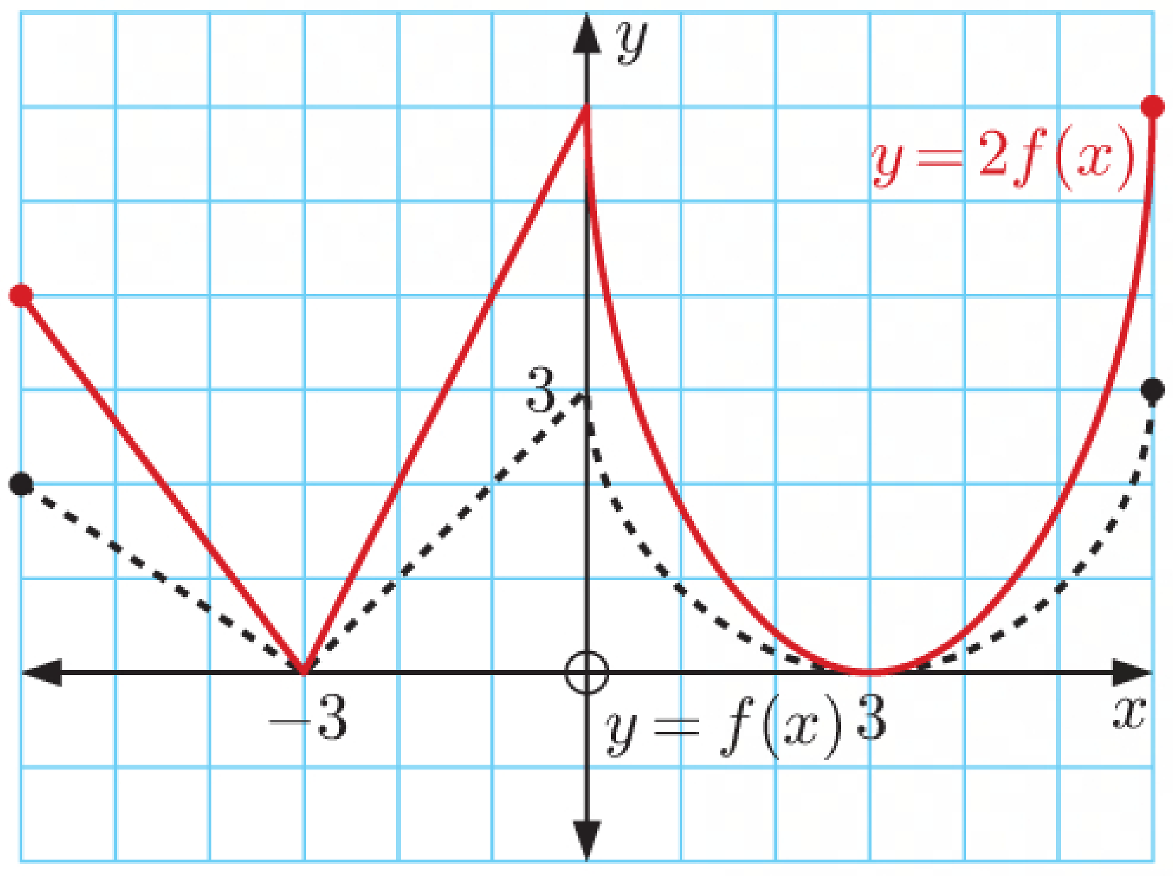
c All transformations of the form $f(qx)$, $q > 0$ do not move the point $(0, 2)$.

$\therefore$ $(0, 2)$ is *invariant.*

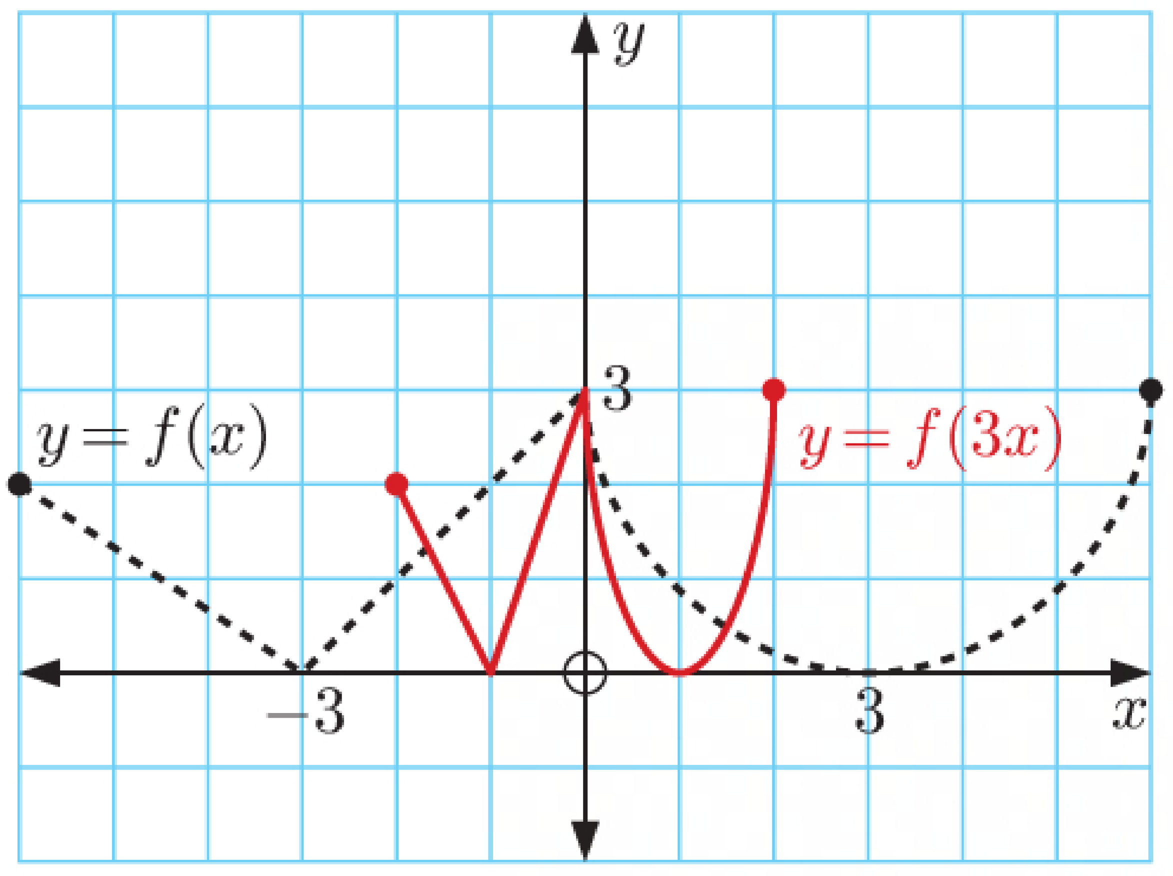
d For the transformation $y = f(qx)$, each point becomes $\dfrac{1}{q}$ times its previous distance from the y-axis.

EXERCISE 16B

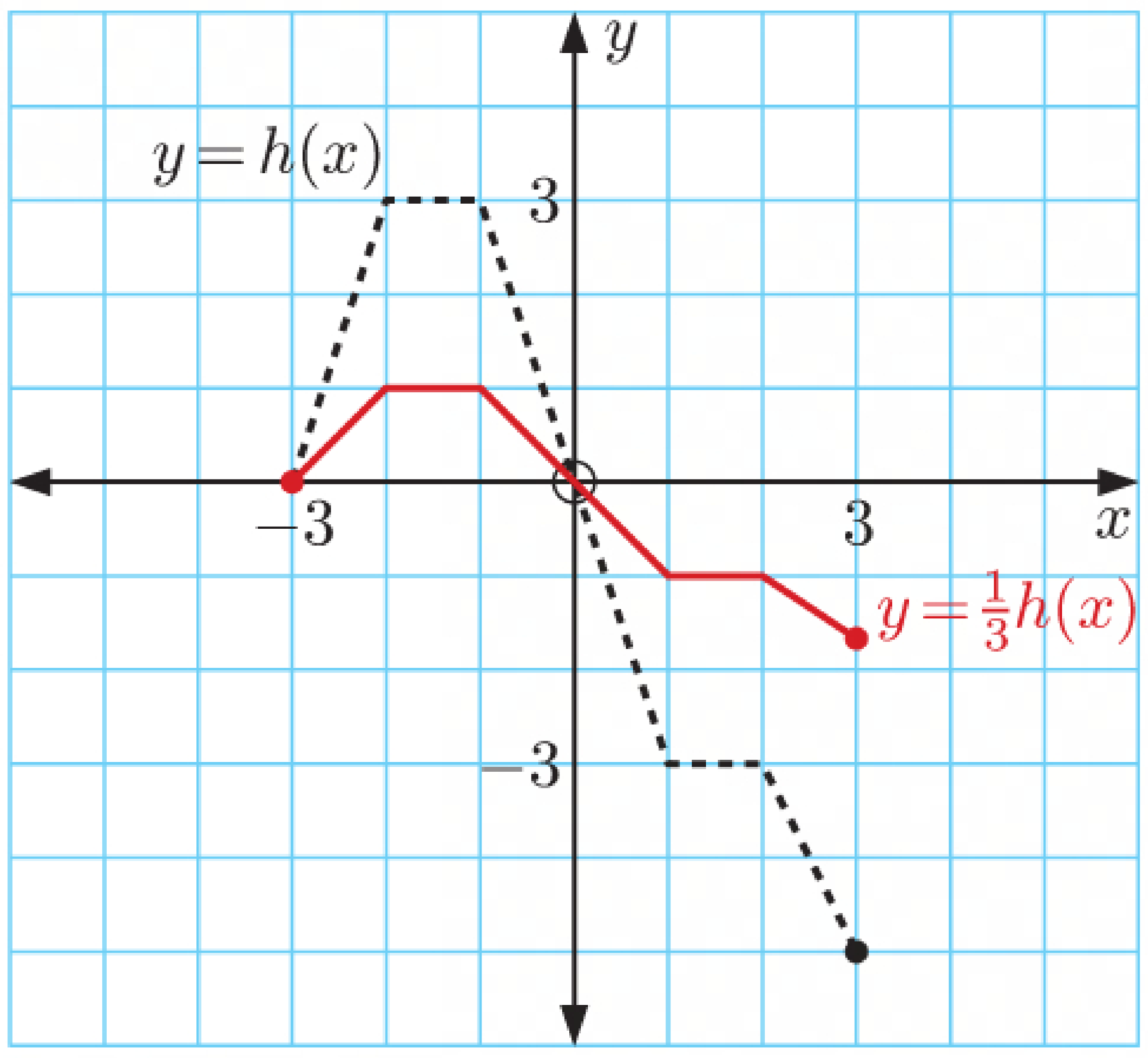
1 **a** The graph of $y = 2\,f(x)$ is a vertical stretch of $y = f(x)$ with scale factor 2.



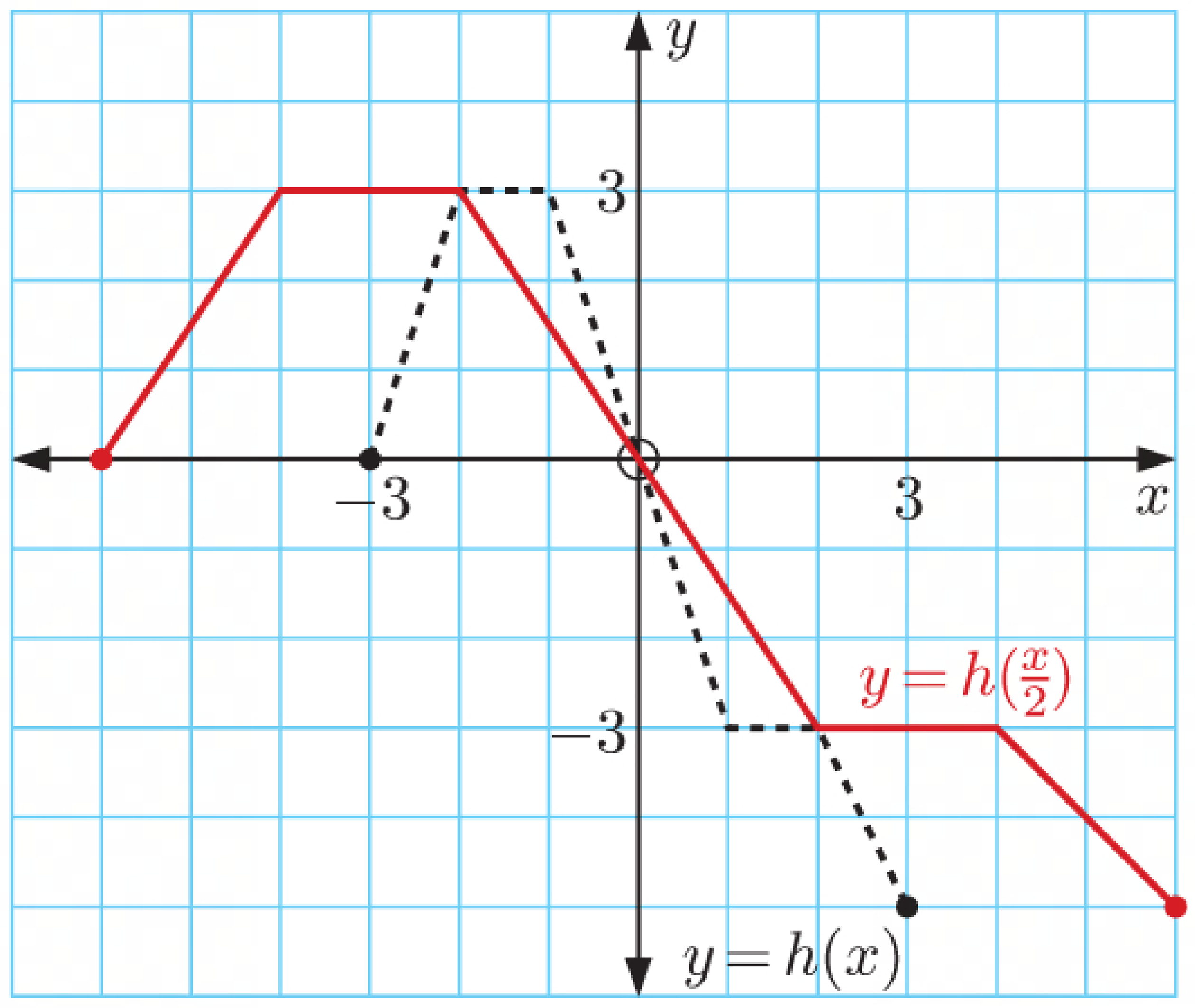
b The graph of $y = f(3x)$ is a horizontal stretch of $y = f(x)$ with scale factor $\frac{1}{3}$.



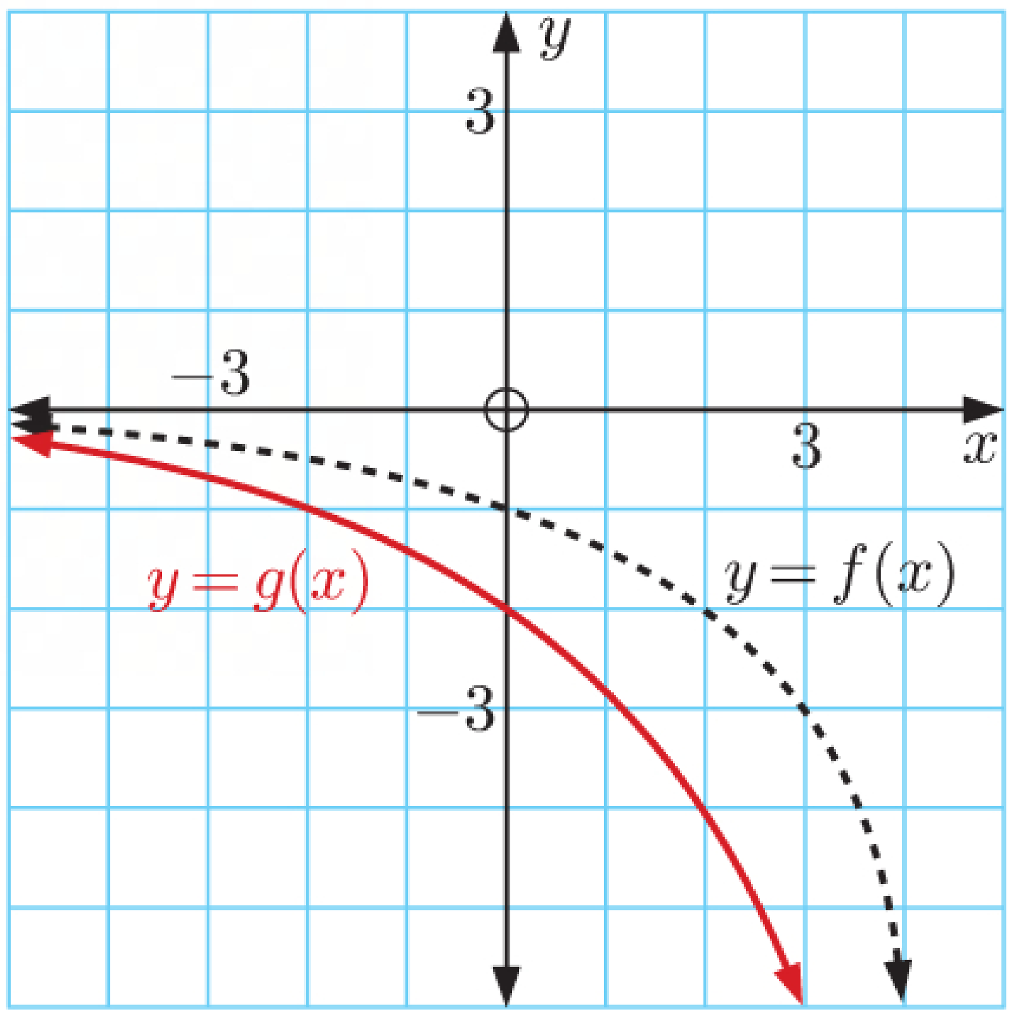
2 **a** The graph of $y = \frac{1}{3}h(x)$ is a vertical stretch of $y = h(x)$ with scale factor $\frac{1}{3}$.



b The graph of $y = h\left(\frac{x}{2}\right)$ is a horizontal stretch of $y = h(x)$ with scale factor 2.



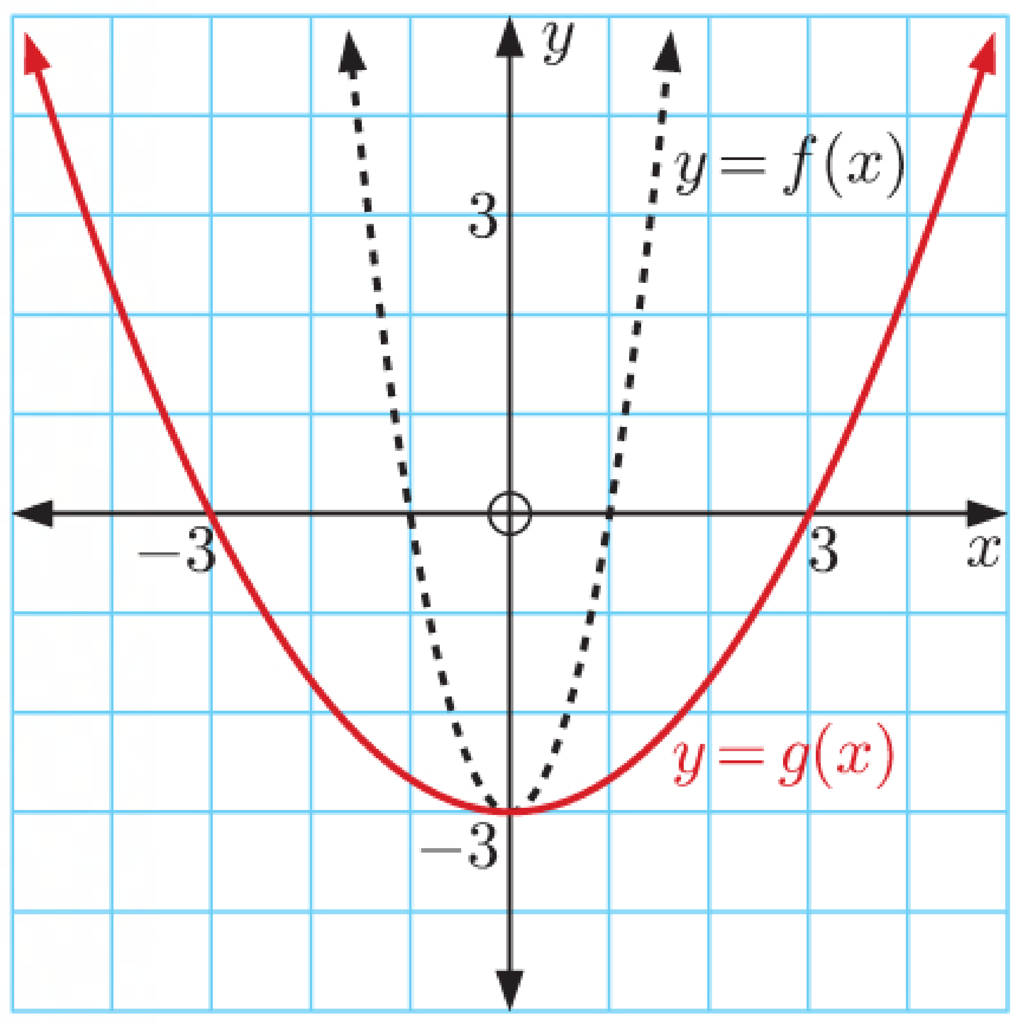
3 **a**



The graph of $y = f(x)$ has been vertically stretched with scale factor 2 to give $y = g(x)$.

So, $g(x) = 2f(x)$.

b



The graph of $y = f(x)$ has been horizontally stretched with scale factor 3 to give $y = g(x)$.

So, $g(x) = f\left(\frac{x}{3}\right)$.

4 Let the original linear function be $y = f(x) = mx + a$.

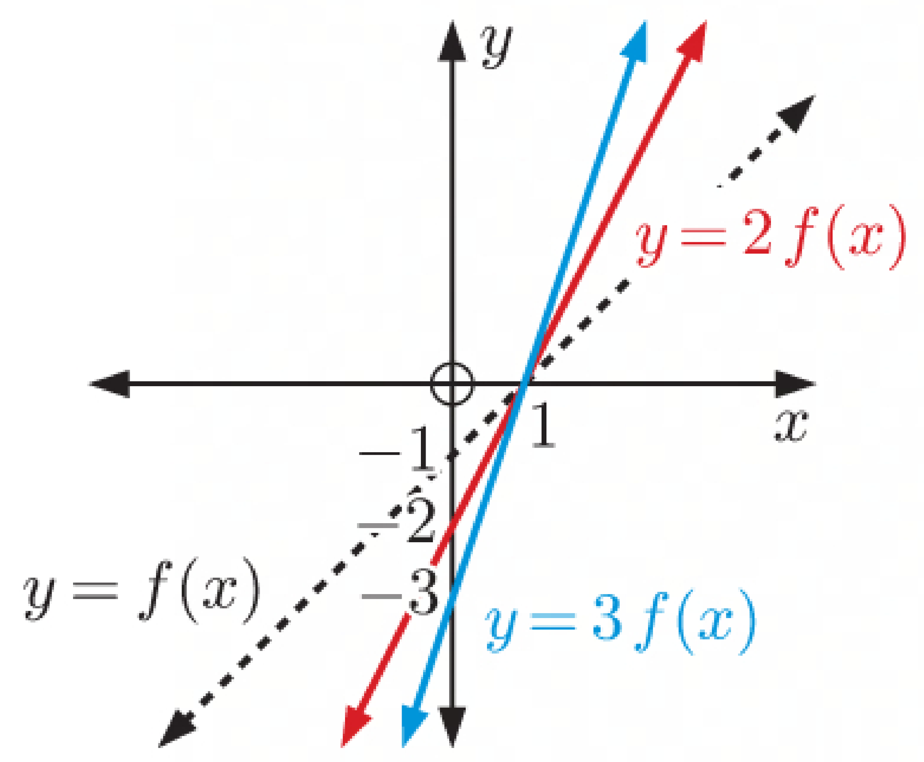
When $y = f(x)$ is vertically stretched with scale factor c, the resulting function is

$$y = c(f(x))$$
$$\therefore \quad y = c(mx + a)$$
$$\therefore \quad y = cmx + ac$$

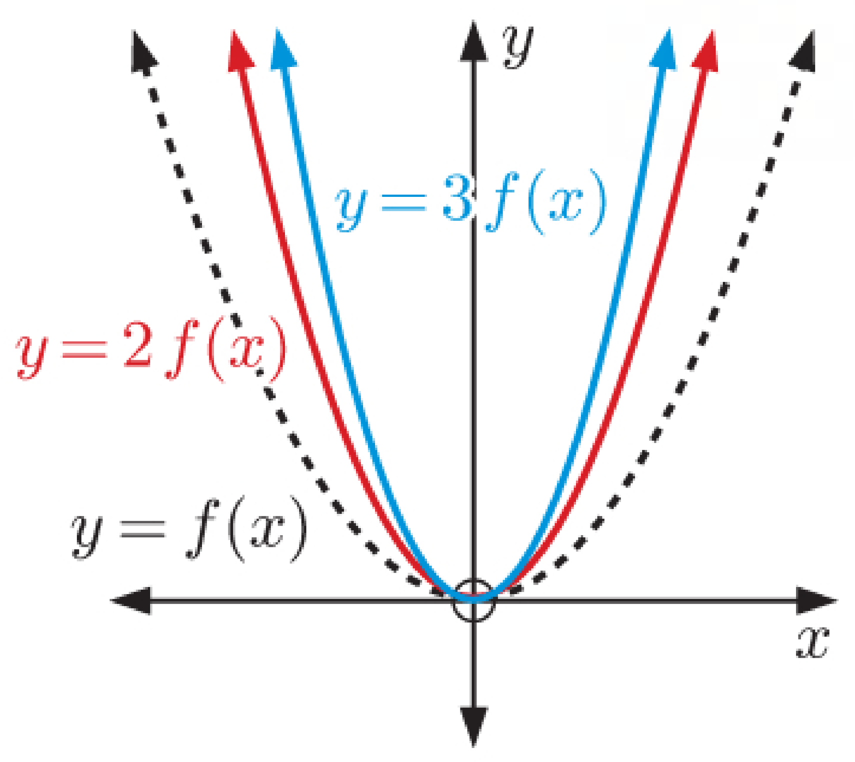
So, the resulting line has gradient cm.

5 The graphs of $y = 2\,f(x)$ and $y = 3\,f(x)$ are vertical stretches of $y = f(x)$ with scale factors 2 and 3, respectively.

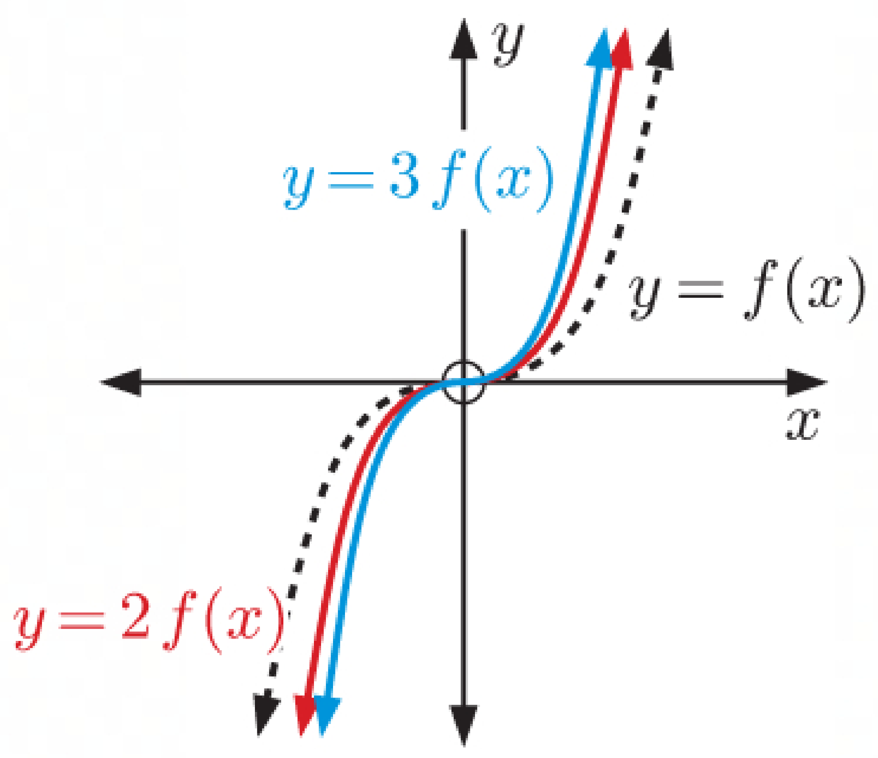
a $f(x) = x - 1$



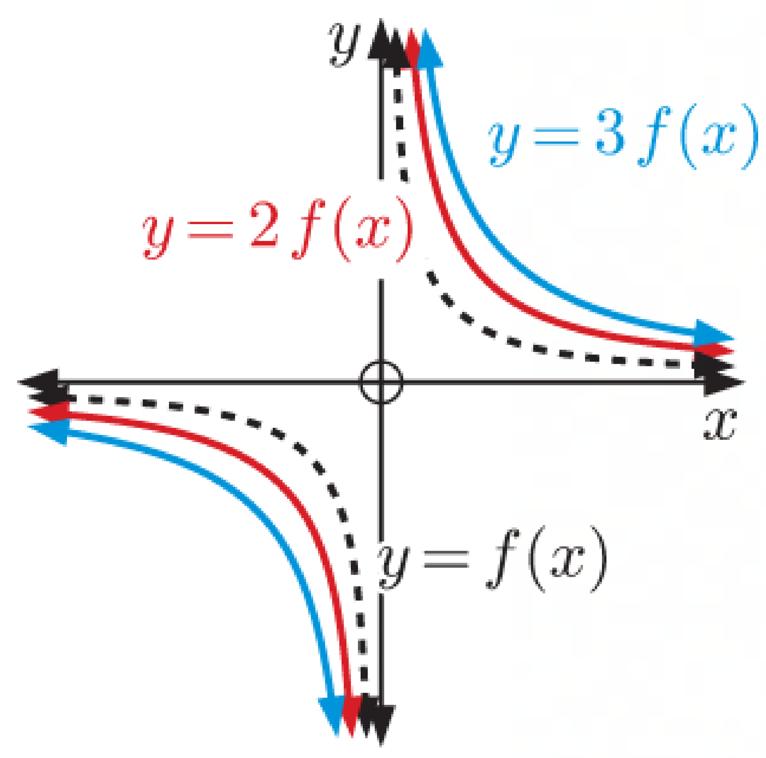
b $f(x) = x^2$



c $f(x) = x^3$

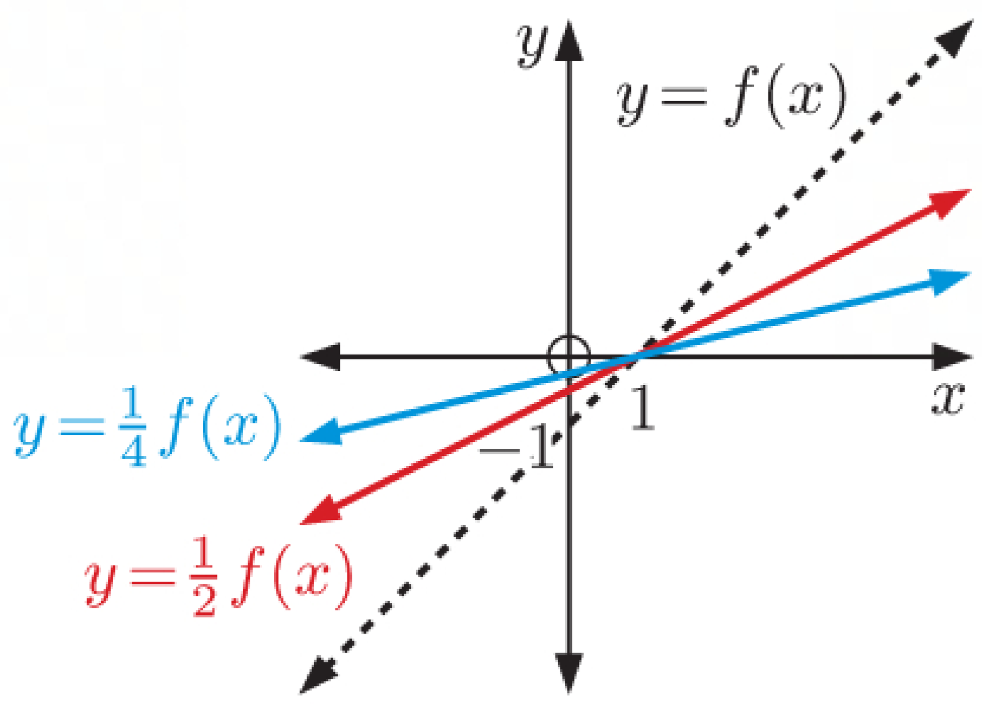


d $f(x) = \dfrac{1}{x}$

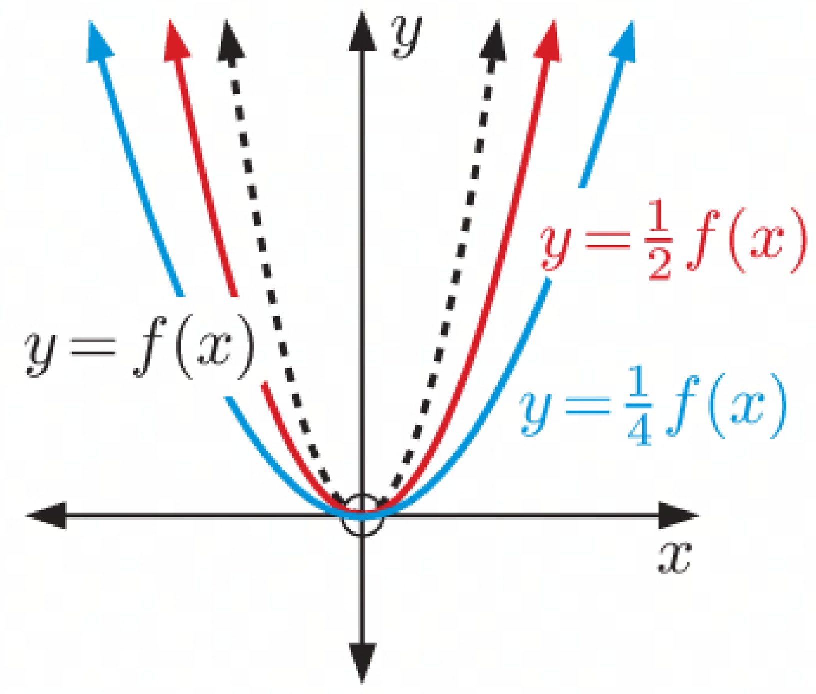


6 The graphs of $y = \frac{1}{2}\,f(x)$ and $y = \frac{1}{4}\,f(x)$ are vertical stretches of $y = f(x)$ with scale factors $\frac{1}{2}$ and $\frac{1}{4}$, respectively.

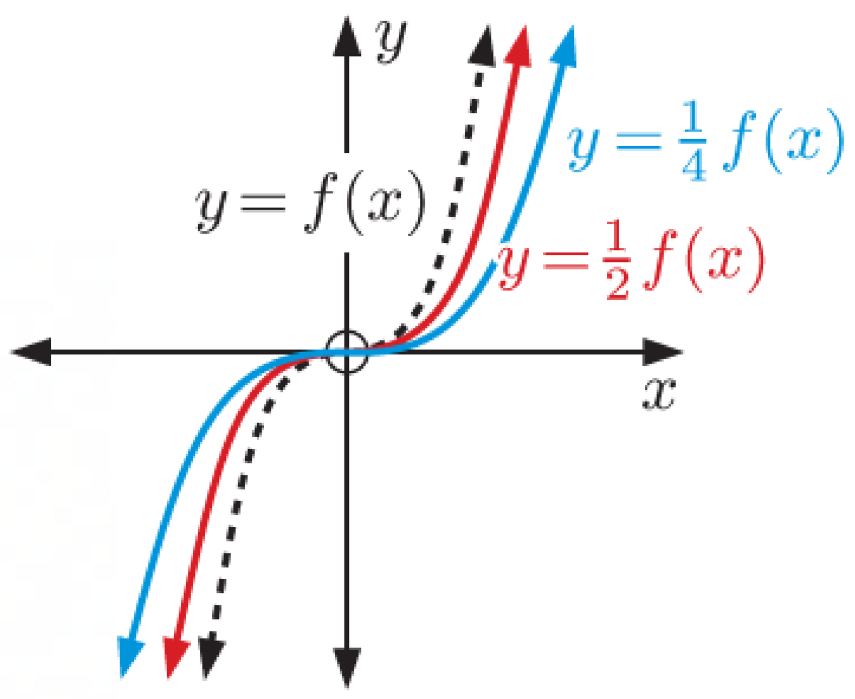
a $f(x) = x - 1$



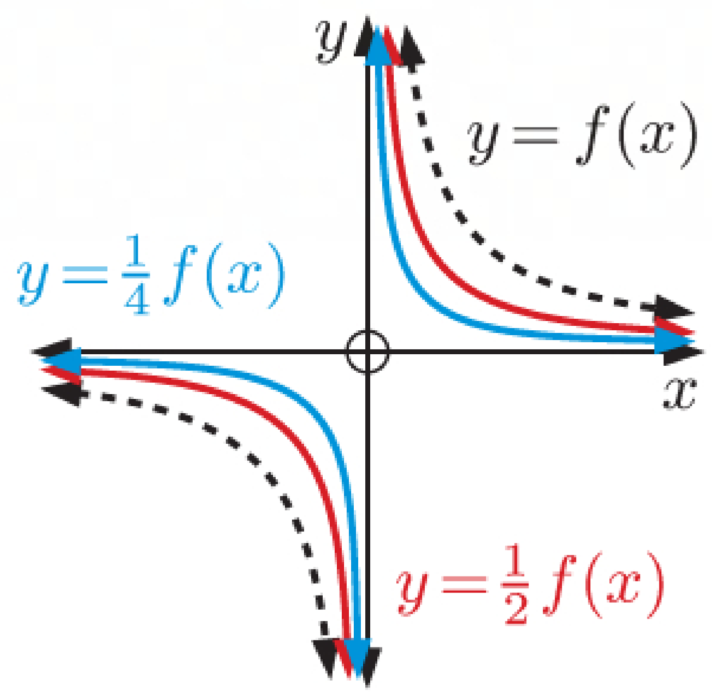
b $f(x) = x^2$



c $f(x) = x^3$

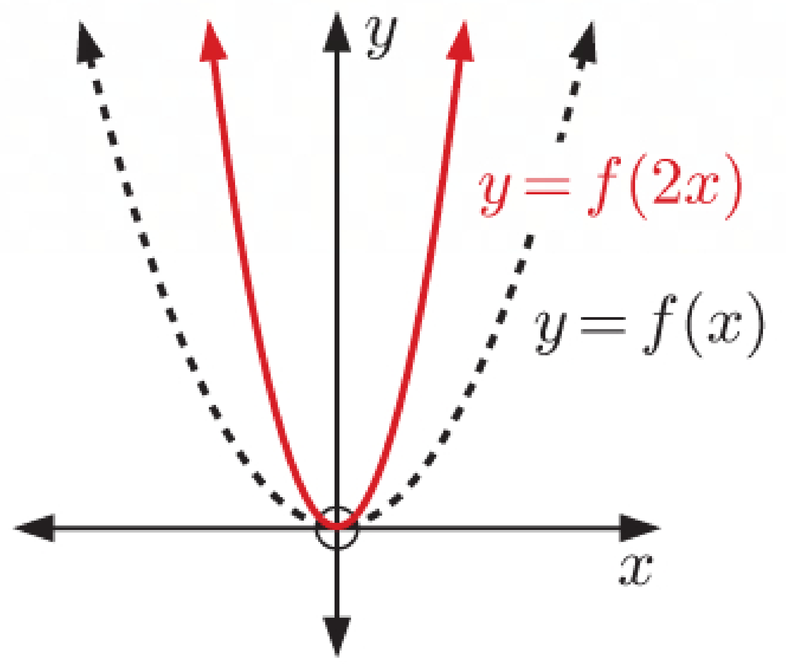


d $f(x) = \dfrac{1}{x}$

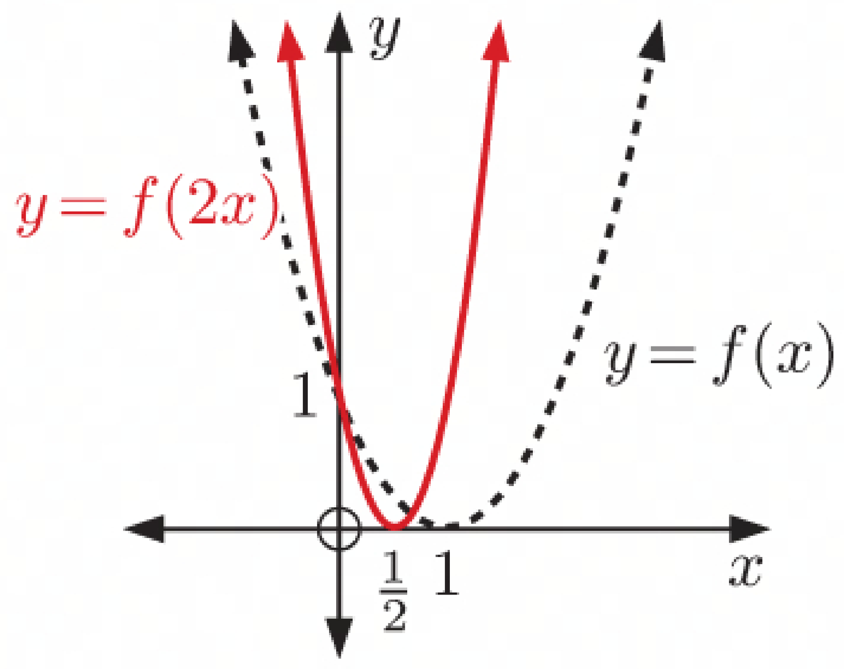


7 The graph of $y = f(2x)$ is a horizontal stretch of $y = f(x)$ with scale factor $\frac{1}{2}$.

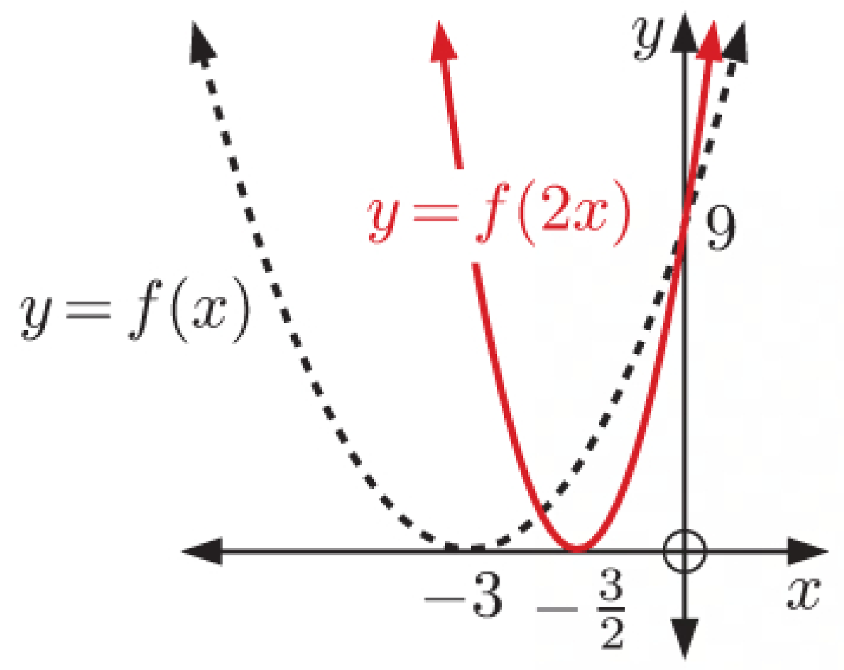
a $y = x^2$



b $y = (x-1)^2$

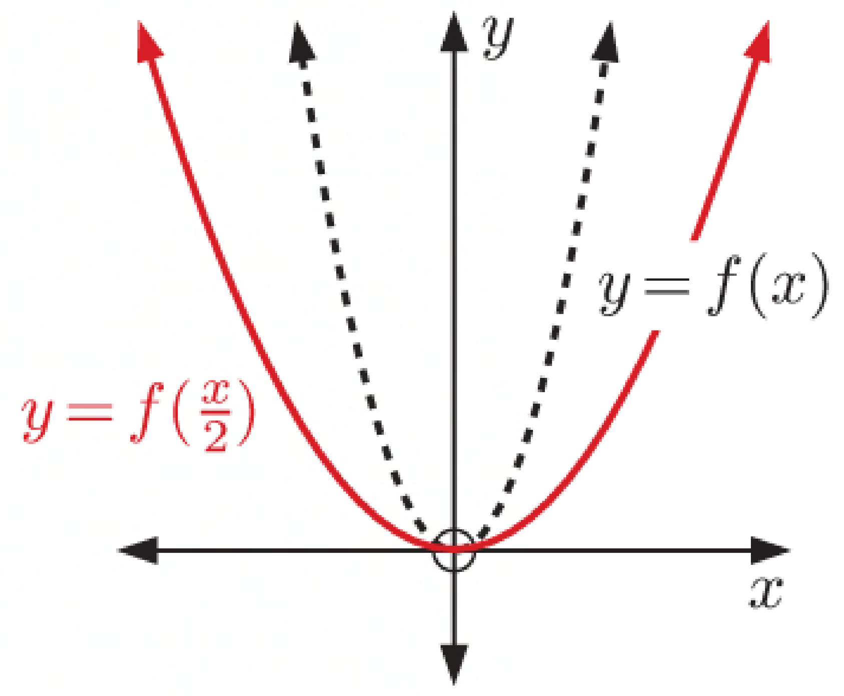


c $y = (x+3)^2$

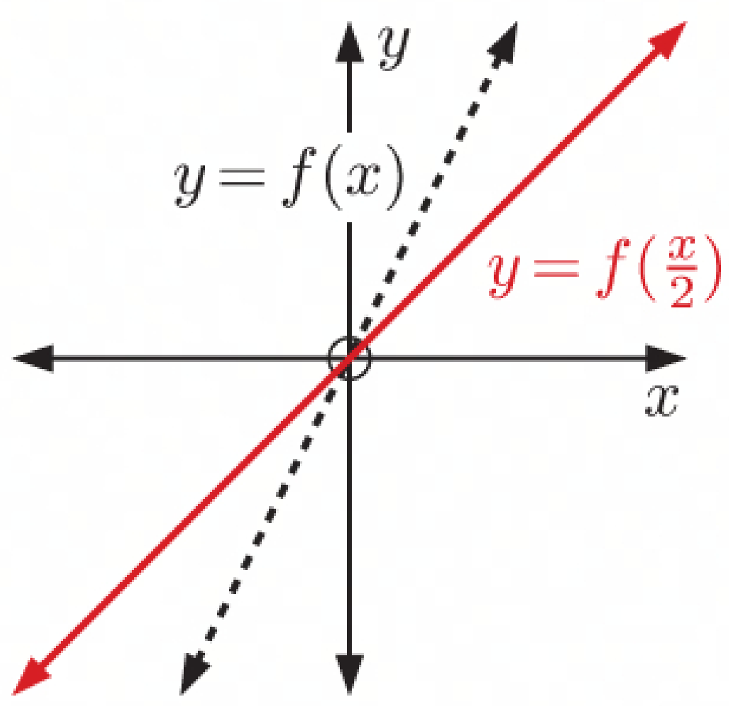


8 The graph of $y = f\left(\frac{x}{2}\right)$ is a horizontal stretch of $y = f(x)$ with scale factor 2.

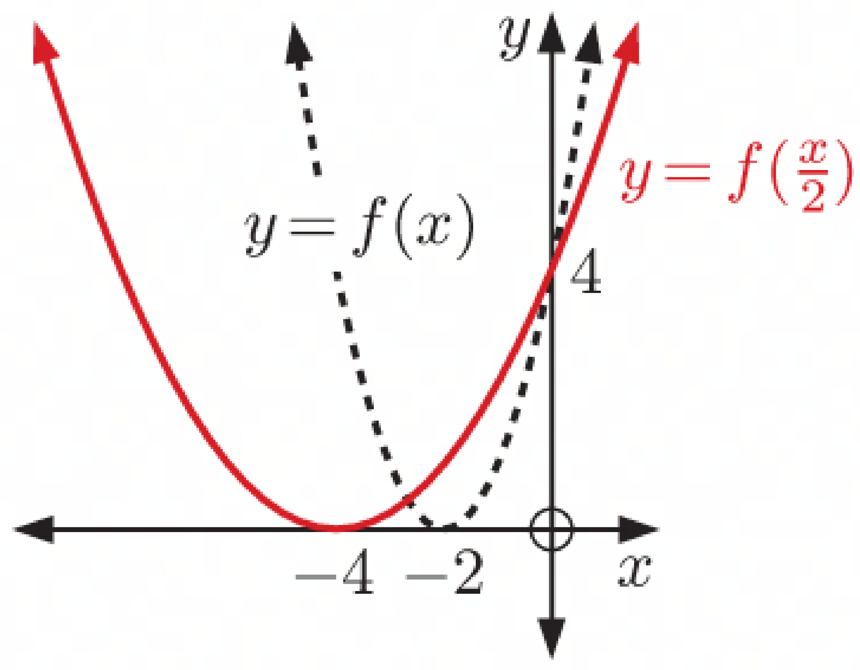
a $y = x^2$



b $y = 2x$



c $y = (x+2)^2$



9 $g(x) = f(5x)$

The graph of $y = g(x)$ is a horizontal stretch of $y = f(x)$ with scale factor $\frac{1}{5}$.

a Each point on $y = g(x)$ is $\frac{1}{5}$ times the distance that $y = f(x)$ is from the y-axis. The point $(10, 25)$ on $y = f(x)$ is 10 units from the y-axis. The corresponding point on $y = g(x)$, which is $\frac{1}{5} \times 10 = 2$ units from the y-axis, is $(2, 25)$.

b Each point on $y = f(x)$ is 5 times the distance that $y = g(x)$ is from the y-axis.
The point $(-5, -15)$ on $y = g(x)$ is 5 units from the y-axis. The corresponding point on $y = f(x)$, which is $5 \times 5 = 25$ units from the y-axis, is $(-25, -15)$.

10 **a** The graph of $y = g(x)$ is a vertical stretch of $y = f(x)$ with scale factor 2.

$\therefore\ g(x) = 2\,f(x)$

$\therefore\ g(x) = 2(x^2 + 2)$ $\quad$ {since $f(x) = x^2 + 2$}

$\therefore\ g(x) = 2x^2 + 4$

b The graph of $y = g(x)$ is a horizontal stretch of $y = f(x)$ with scale factor 3.

$\therefore\ g(x) = f\left(\frac{x}{3}\right)$

$\therefore\ g(x) = 5 - 3\left(\frac{x}{3}\right)$ $\quad$ {since $f(x) = 5 - 3x$}

$\therefore\ g(x) = 5 - x$

c The graph of $y = g(x)$ is a vertical dilation of $y = f(x)$ with scale factor $\frac{1}{4}$.

$\therefore\ g(x) = \frac{1}{4}\,f(x)$

$\therefore\ g(x) = \frac{1}{4}(x^3 + 8x^2 - 2)$ $\quad$ {since $f(x) = x^3 + 8x^2 - 2$}

$\therefore\ g(x) = \frac{1}{4}x^3 + 2x^2 - \frac{1}{2}$

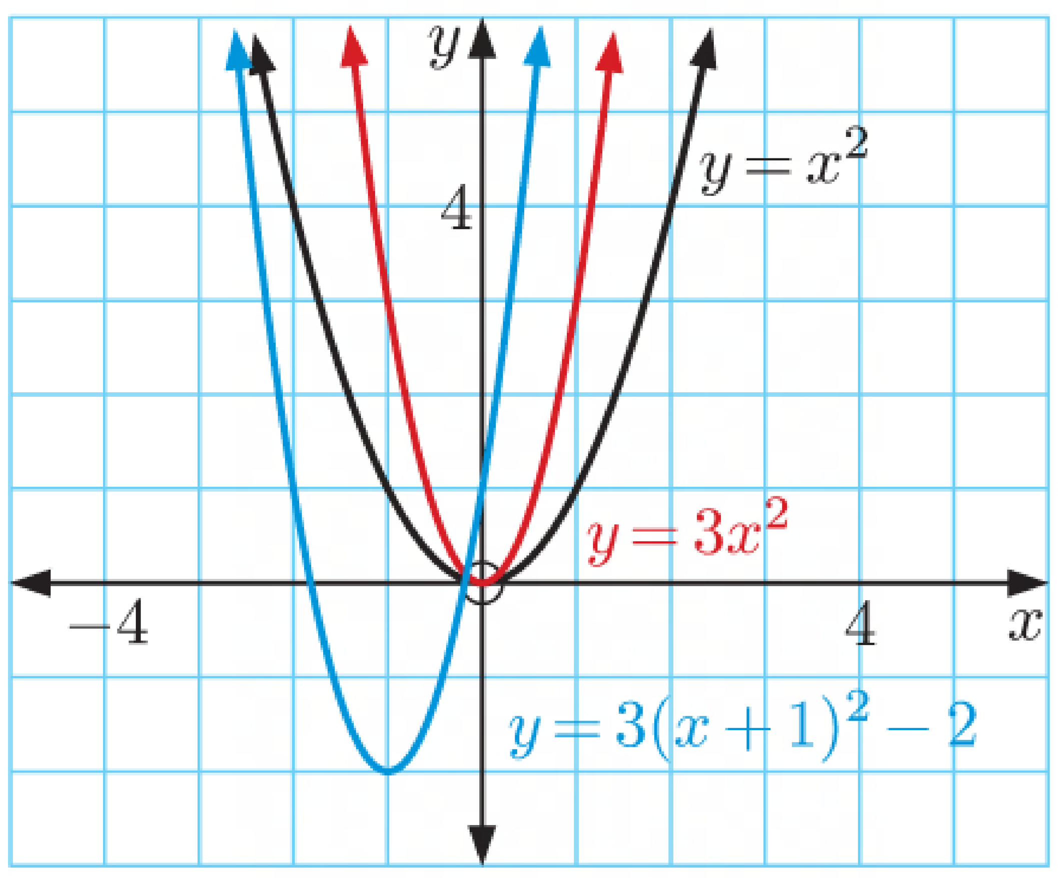
d The graph of $y = g(x)$ is a horizontal dilation of $y = f(x)$ with scale factor $\frac{1}{2}$.

$\therefore\ g(x) = f(2x)$

$\therefore\ g(x) = 2(2x)^2 + (2x) - 3$ $\quad$ {since $f(x) = 2x^2 + x - 3$}

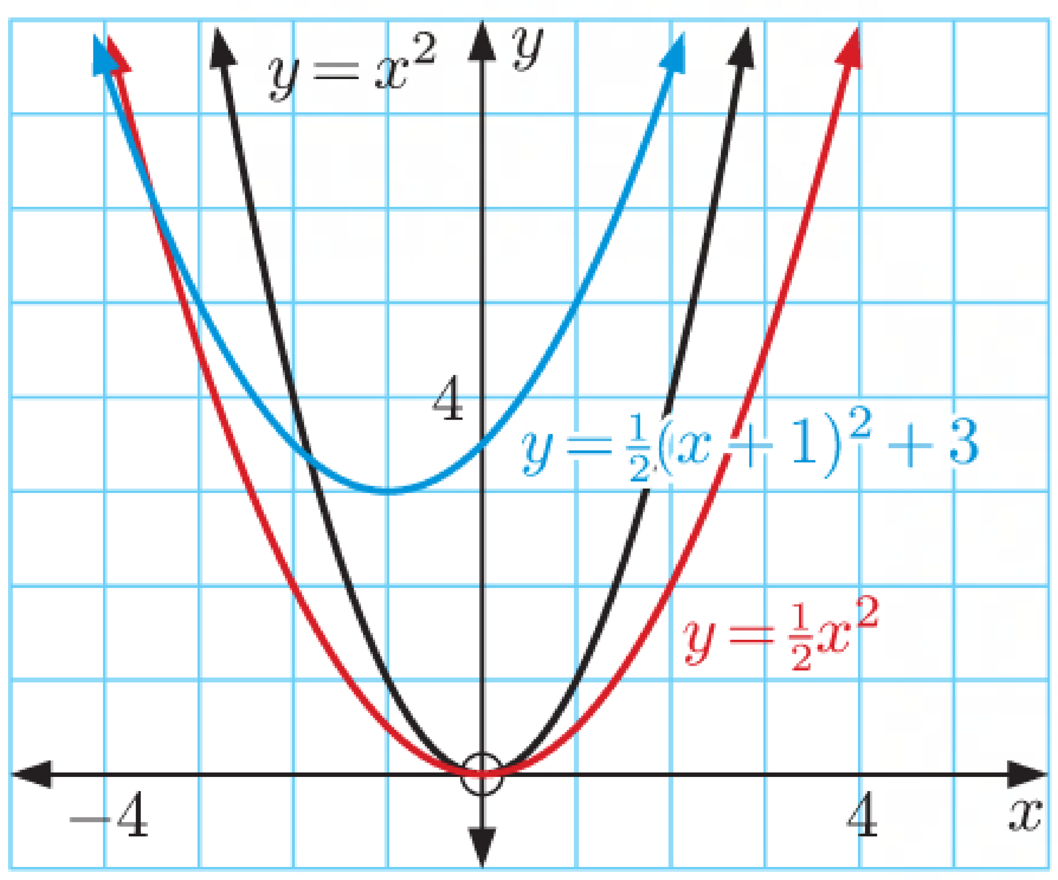
$\therefore\ g(x) = 8x^2 + 2x - 3$

11



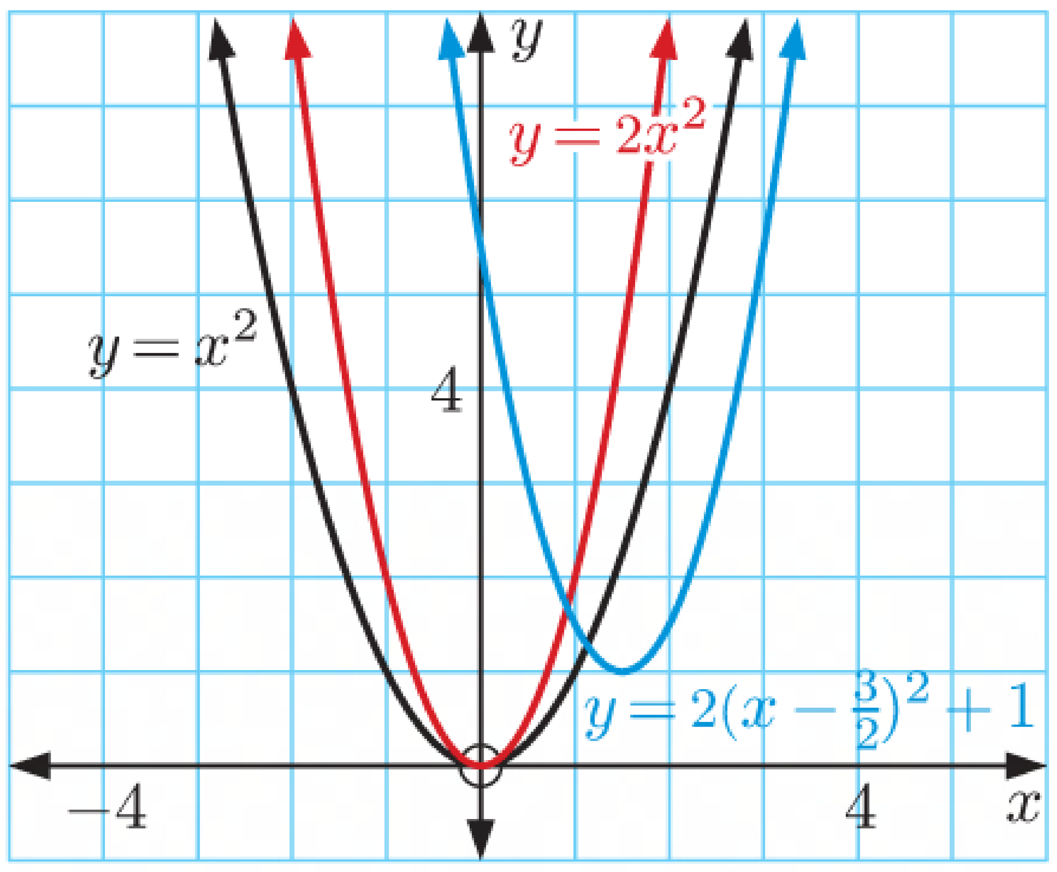
$y = x^2$ is transformed to $y = 3(x+1)^2 - 2$ by vertically stretching with scale factor 3 and then translating through $\begin{pmatrix} -1 \\ -2 \end{pmatrix}$.

12



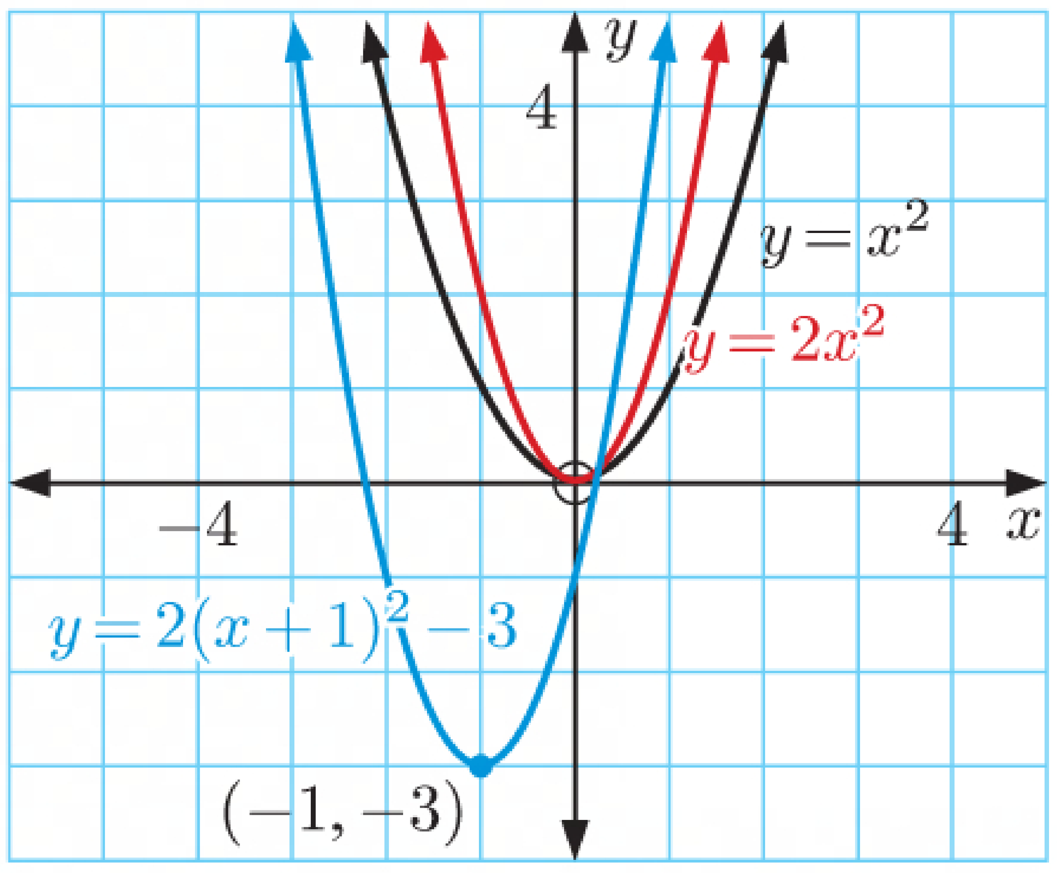
$y = x^2$ is transformed to $y = \frac{1}{2}(x+1)^2 + 3$ by vertically stretching with scale factor $\frac{1}{2}$ and then translating through $\begin{pmatrix} -1 \\ 3 \end{pmatrix}$.

13



$y = x^2$ is transformed to $y = 2(x - \frac{3}{2})^2 + 1$ by vertically stretching with scale factor 2 and then translating through $\begin{pmatrix} \frac{3}{2} \\ 1 \end{pmatrix}$.

14 We vertically stretch $y = x^2$ with scale factor 2 to give $y = 2x^2$. We then translate $y = 2x^2$ through $\begin{pmatrix} -1 \\ -3 \end{pmatrix}$ to give $y = 2(x+1)^2 - 3$.



15 **a** $y = f(2x)$ is a horizontal stretch of $y = f(x)$ with scale factor $\frac{1}{2}$.
$y = 3f(2x)$ is a vertical stretch of $y = f(2x)$ with scale factor 3.
To map $y = f(x)$ onto $y = g(x)$ we apply a horizontal stretch with scale factor $\frac{1}{2}$, then a vertical stretch with scale factor 3.

b The points on $y = f(x)$ become $\frac{1}{2}$ times their distance from the y-axis, and 3 times their distance from the x-axis.

i The image of $(3, -5)$ on $f(x)$ is $(3 \times \frac{1}{2}, -5 \times 3)$, or $(\frac{3}{2}, -15)$.

ii The image of $(1, 2)$ on $f(x)$ is $(1 \times \frac{1}{2}, 2 \times 3)$, or $(\frac{1}{2}, 6)$.

iii The image of $(-2, 1)$ on $f(x)$ is $(-2 \times \frac{1}{2}, 1 \times 3)$, or $(-1, 3)$.

c We multiply the distance from the x-axis by $\frac{1}{3}$ and the distance from the y-axis by 2 to find the corresponding point on $y = f(x)$.

i $(2 \times 2, 1 \times \frac{1}{3})$ or $(4, \frac{1}{3})$ is the point on $y = f(x)$ which maps onto $(2, 1)$.

ii $(-3 \times 2, 2 \times \frac{1}{3})$ or $(-6, \frac{2}{3})$ is the point on $y = f(x)$ which maps onto $(-3, 2)$.

iii $(-7 \times 2, 3 \times \frac{1}{3})$ or $(-14, 1)$ is the point on $y = f(x)$ which maps onto $(-7, 3)$.

16 We apply a translation of a units vertically to $y = x^2$ to give $y = x^2 + a$.

We then apply a horizontal stretch with scale factor b to give $y = \left(\frac{1}{b}x\right)^2 + a$

$$\therefore \quad y = \frac{1}{b^2}x^2 + a$$

Now, $y = 0.1x^2 + 5$, so we equate coefficients: $a = 5$, $\frac{1}{b^2} = 0.1$

$$\therefore \quad b^2 = 10$$

$$\therefore \quad b = \sqrt{10} \qquad \{b > 0\}$$

$\therefore \quad a = 5, \ b = \sqrt{10}$

17 **a** Under a vertical dilation with scale factor $\frac{1}{2}$, $f(x)$ becomes $\frac{1}{2}f(x)$.

$\therefore \quad y = \frac{1}{x}$ becomes $y = \frac{1}{2}\left(\frac{1}{x}\right) = \frac{1}{2x}$.

b Under a horizontal dilation with scale factor 3, $f(x)$ becomes $f(\frac{1}{3}x)$.

$\therefore \quad y = \frac{1}{x}$ becomes $y = \frac{1}{\frac{1}{3}x} = \frac{3}{x}$.

c Under a horizontal translation of -3, $f(x)$ becomes $f(x+3)$.

$\therefore \quad y = \frac{1}{x}$ becomes $y = \frac{1}{x+3}$.

d Under a vertical translation of 4, $f(x)$ becomes $f(x) + 4$.

$\therefore \quad y = \frac{1}{x}$ becomes $y = \frac{1}{x} + 4$

$$\therefore \quad y = \frac{1}{x} + \frac{4x}{x}$$

$$\therefore \quad y = \frac{4x+1}{x}$$

18 **a** Under a vertical stretch with scale factor 3, $f(x)$ becomes $3f(x)$.

$\therefore \quad y = \frac{1}{x}$ becomes $y = 3\left(\frac{1}{x}\right) = \frac{3}{x}$.

Under a translation of $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$, $f(x)$ becomes $f(x-1) - 1$.

$\therefore \quad y = \frac{3}{x}$ becomes $y = \frac{3}{x-1} - 1$.

So, $y = \frac{1}{x}$ becomes
$$g(x) = \frac{3}{x-1} - 1$$
$$= \frac{3-(x-1)}{x-1}$$
$$= \frac{3-x+1}{x-1}$$
$$= \frac{-x+4}{x-1}$$

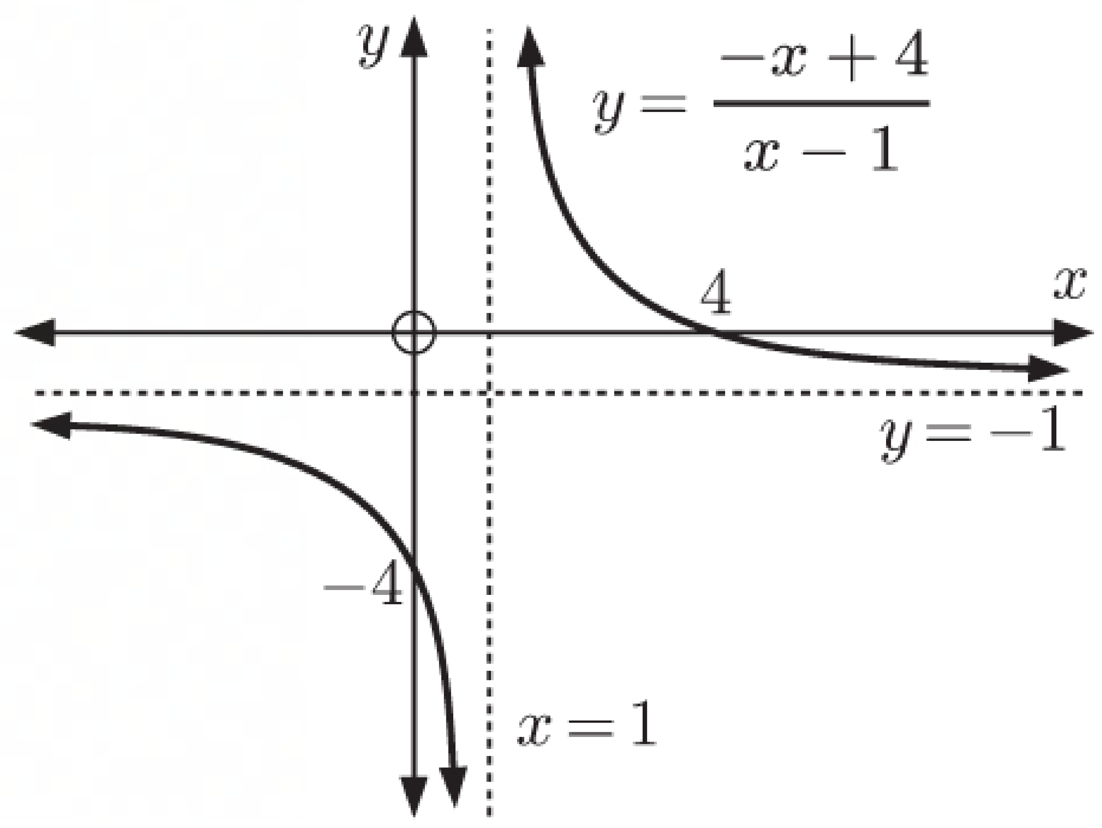
b The asymptotes of $y = \frac{1}{x}$ are $x = 0$ and $y = 0$.

These are unchanged by the vertical stretch, and shifted $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ by the translation.

$\therefore$ the vertical asymptote is $x = 1$ and the horizontal asymptote is $y = -1$.

c The domain is $\{x \mid x \neq 1\}$.
The range is $\{y \mid y \neq -1\}$.

d



19 a Under a translation of $\begin{pmatrix} -1 \\ 3 \end{pmatrix}$, $f(x)$ becomes $f(x+1)+3$.

$\therefore$ $y = \frac{1}{x}$ becomes $y = \frac{1}{x+1} + 3$.

Under a horizontal stretch with scale factor $\frac{1}{2}$, $f(x)$ becomes $f(2x)$.

$\therefore$ $y = \frac{1}{x+1} + 3$ becomes $y = \frac{1}{2x+1} + 3$.

So, $y = \frac{1}{x}$ becomes
$$g(x) = \frac{1}{2x+1} + 3 = \frac{1+3(2x+1)}{2x+1} = \frac{1+6x+3}{2x+1} = \frac{6x+4}{2x+1}$$

b The asymptotes of $y = \frac{1}{x}$ are $x = 0$ and $y = 0$.

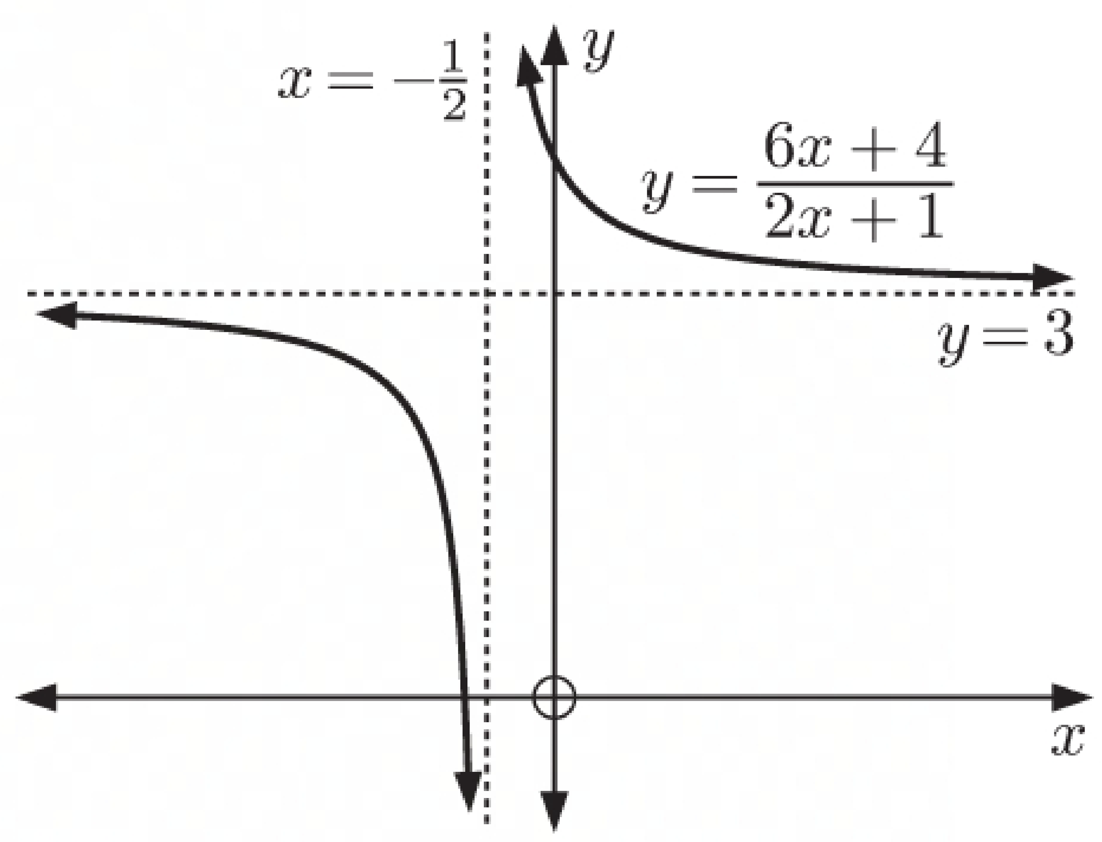
These are shifted $\begin{pmatrix} -1 \\ 3 \end{pmatrix}$ by the translation to $x = -1$ and $y = 3$.

The distance of the vertical asymptote then becomes $\frac{1}{2}$ times its distance from the y-axis.

$\therefore$ the vertical asymptote is $x = -\frac{1}{2}$ and the horizontal asymptote is $y = 3$.

c The domain is $\{x \mid x \neq -\frac{1}{2}\}$.
The range is $\{y \mid y \neq 3\}$.

d



20 $2x^2+8x-1 \xrightarrow[\text{scale factor } 4]{\text{vertical stretch}} 8x^2+32x-4 \xrightarrow{\text{translation } \binom{a}{b}} 8(x-a)^2+32(x-a)-4+b$

Now, $g(x) = 8x^2-16x+5 = 8(x-a)^2+32(x-a)-4+b$

$\therefore\ 8x^2-16x+5 = 8(x^2-2ax+a^2)+32x-32a-4+b$

$\therefore\ 8x^2-16x+5 = 8x^2-16ax+8a^2+32x-32a-4+b$

$\therefore\ (16a-48)x-8a^2+32a-b+9=0$

$\therefore\ 16a-48=0$ and $-8a^2+32a-b+9=0$

$\therefore\ 16a=48$ $\qquad \therefore\ -8(1)^2+32(1)-b+9=0$ {using $(*)$}

$\therefore\ a=3$ .... $(*)$ $\qquad \therefore\ -8+32-b+9=0$

$\therefore\ b=33$

A vertical stretch with scale factor 4 followed by a translation through $\begin{pmatrix}3\\33\end{pmatrix}$ maps $f(x)=2x^2+8x-1$ onto $g(x)=8x^2-16x+5$.

We obtain another combination of transformations by applying a translation first, followed by a vertical stretch.

The horizontal translation is the same, but we require a vertical translation $\frac{1}{4}$ times as large, since the vertical stretch has scale factor 4.

$\therefore$ a translation through $\begin{pmatrix}3\\33\times\frac{1}{4}\end{pmatrix} = \begin{pmatrix}3\\8\frac{1}{4}\end{pmatrix}$ followed by a vertical stretch with scale factor 4 maps $f(x)=2x^2+8x-1$ onto $g(x)=8x^2-16x+5$.

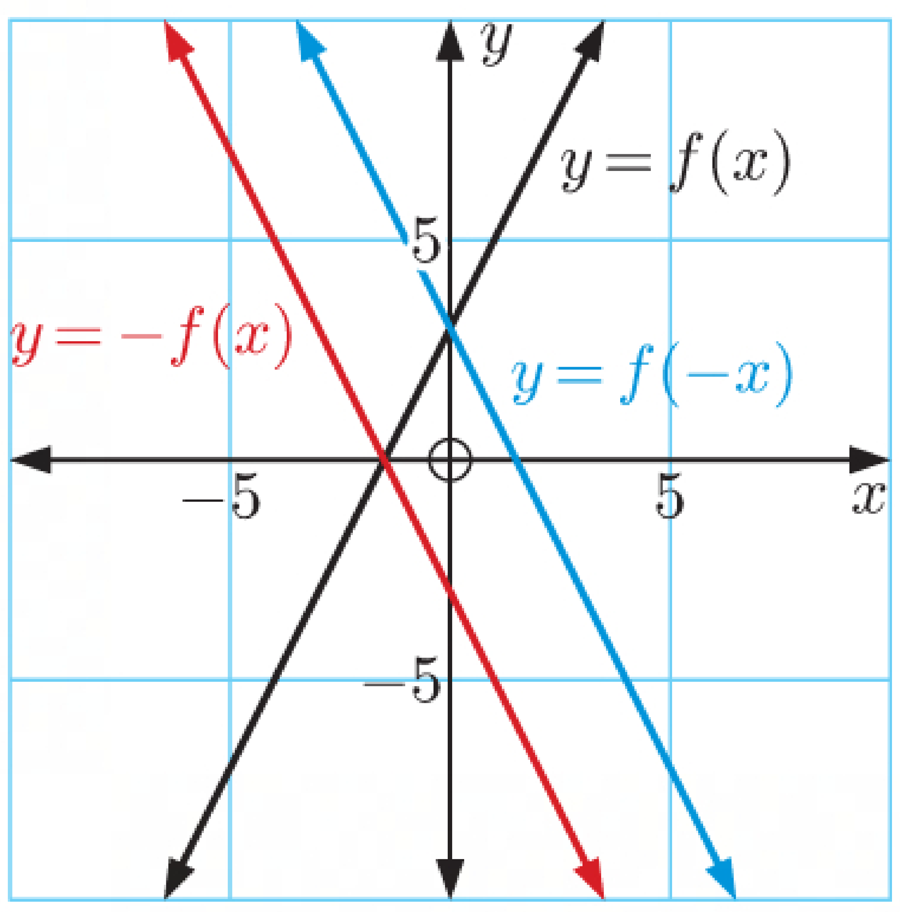
INVESTIGATION 3 REFLECTIONS

1 $f(x)=2x+3$

a **i** $-f(x) = -(2x+3)$
$= -2x-3$

ii $f(-x) = 2(-x)+3$
$= -2x+3$

b

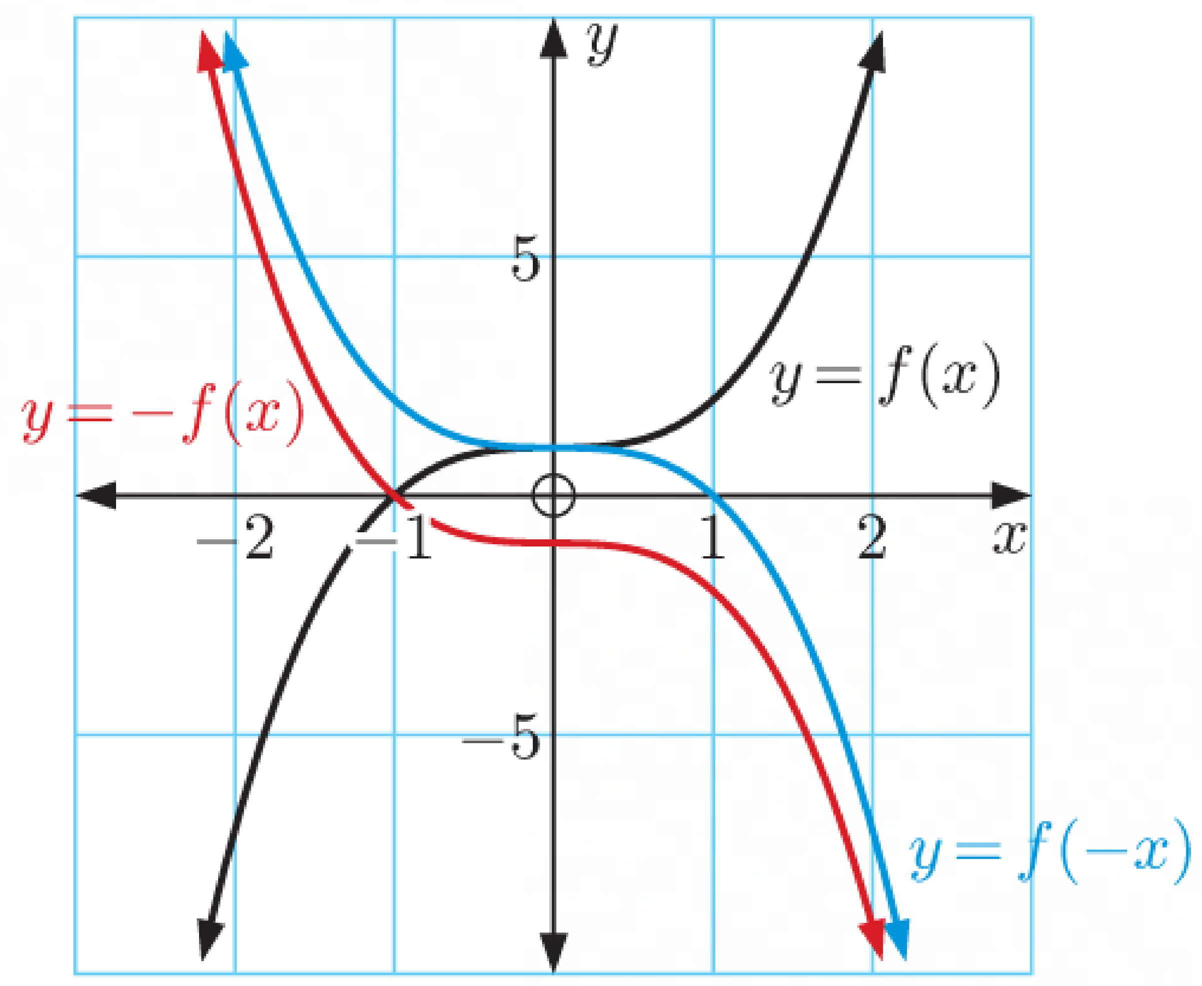


2 $f(x)=x^3+1$

a **i** $-f(x) = -(x^3+1)$
$= -x^3-1$

ii $f(-x) = (-x)^3+1$
$= -x^3+1$

b

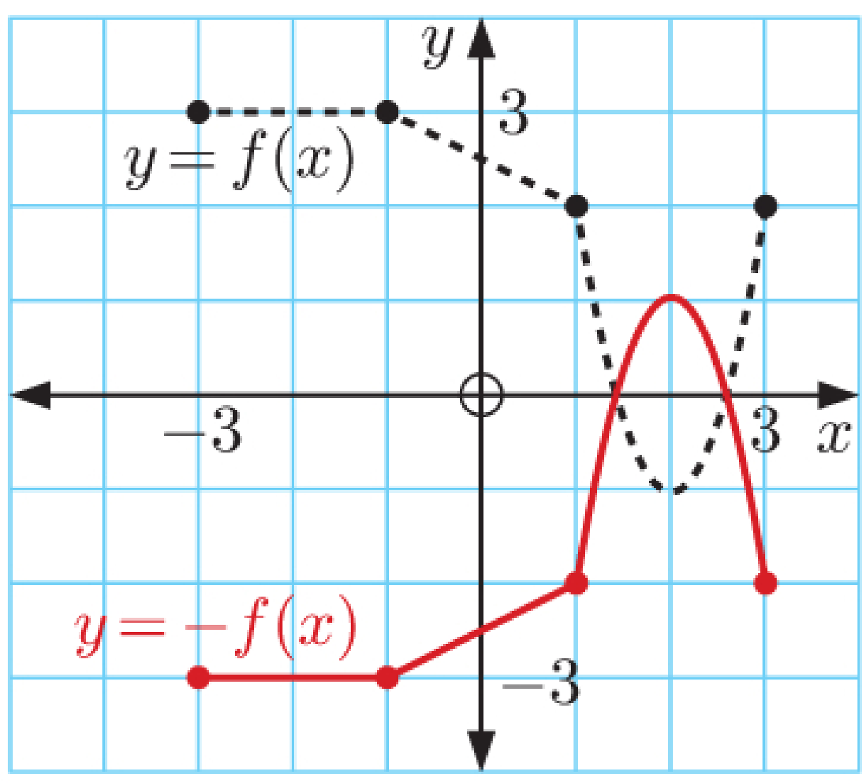


3 **a** A reflection of the graph in the **x-axis** moves $y = f(x)$ to $y = -f(x)$.

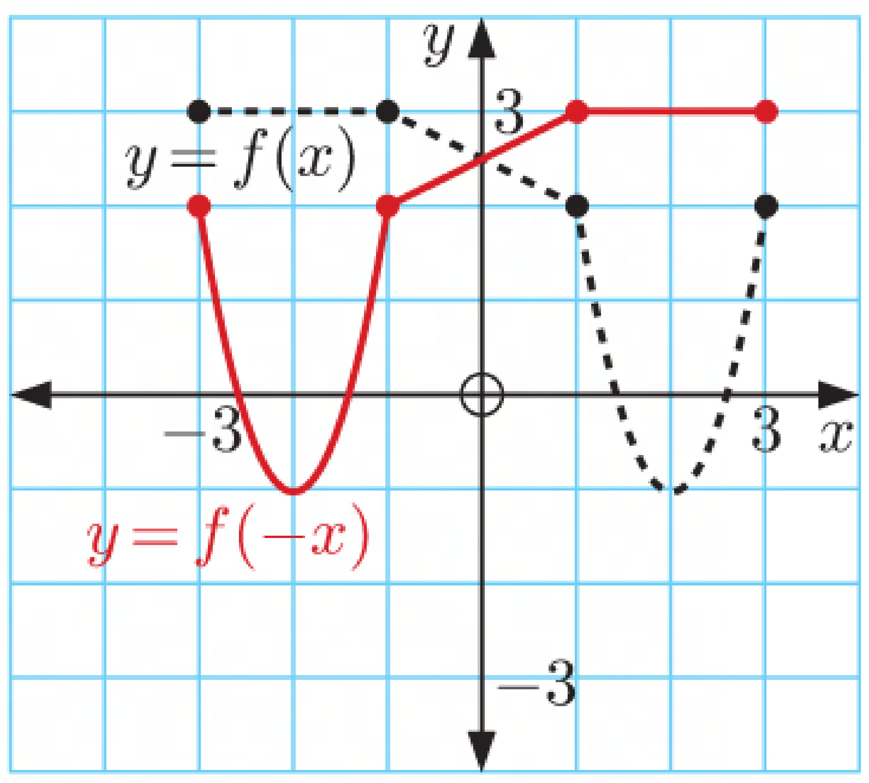
b A reflection of the graph in the **y-axis** moves $y = f(x)$ to $y = f(-x)$.

EXERCISE 16C

1 **a** The graph of $y = -f(x)$ is found by reflecting $y = f(x)$ in the x-axis.

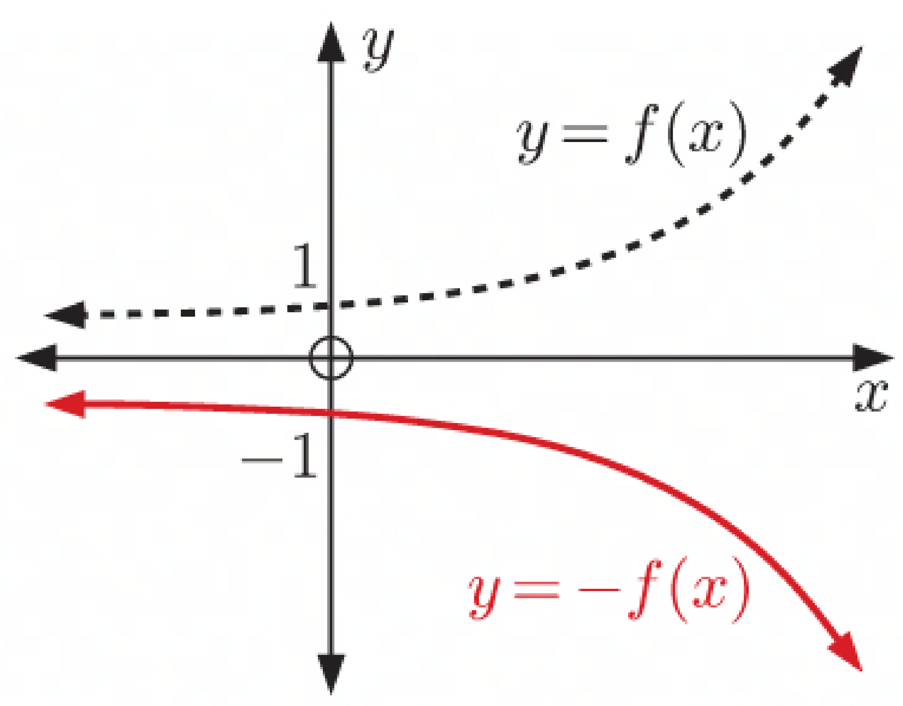


b The graph of $y = f(-x)$ is found by reflecting $y = f(x)$ in the y-axis.

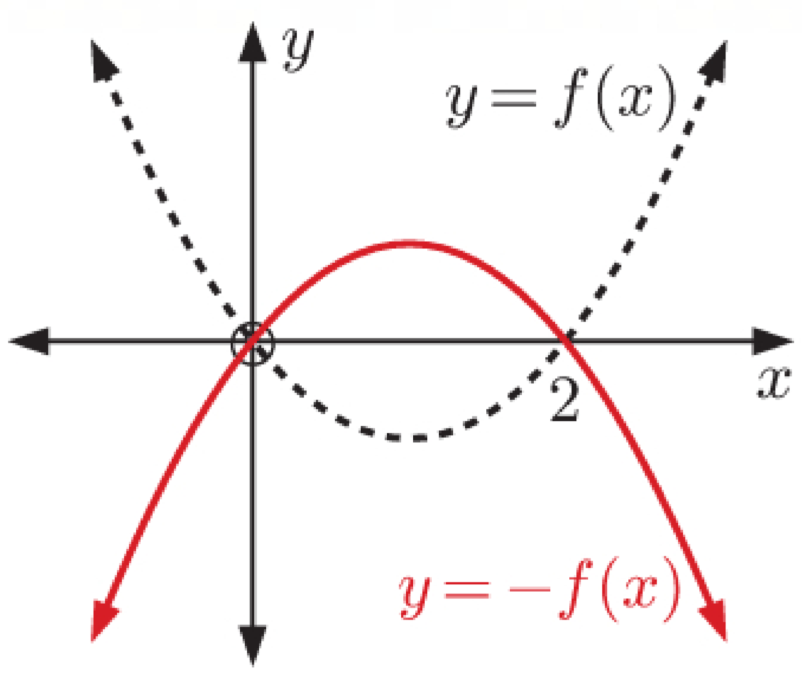


2 The graph of $y = -f(x)$ is found by reflecting $y = f(x)$ in the x-axis.

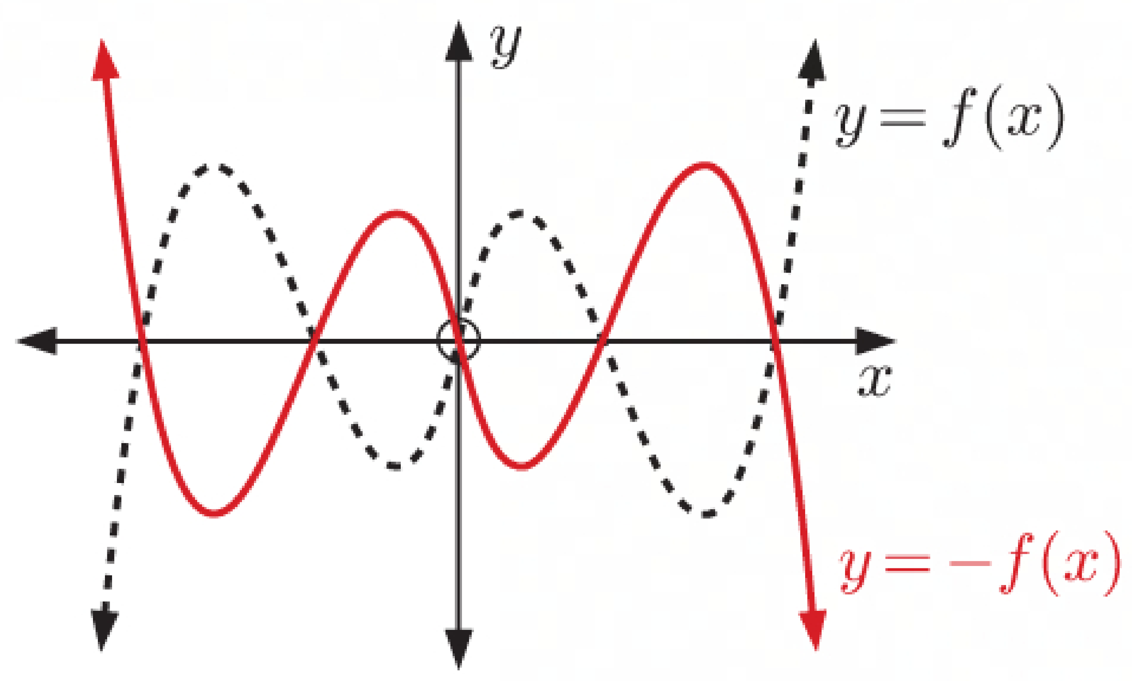
a



b

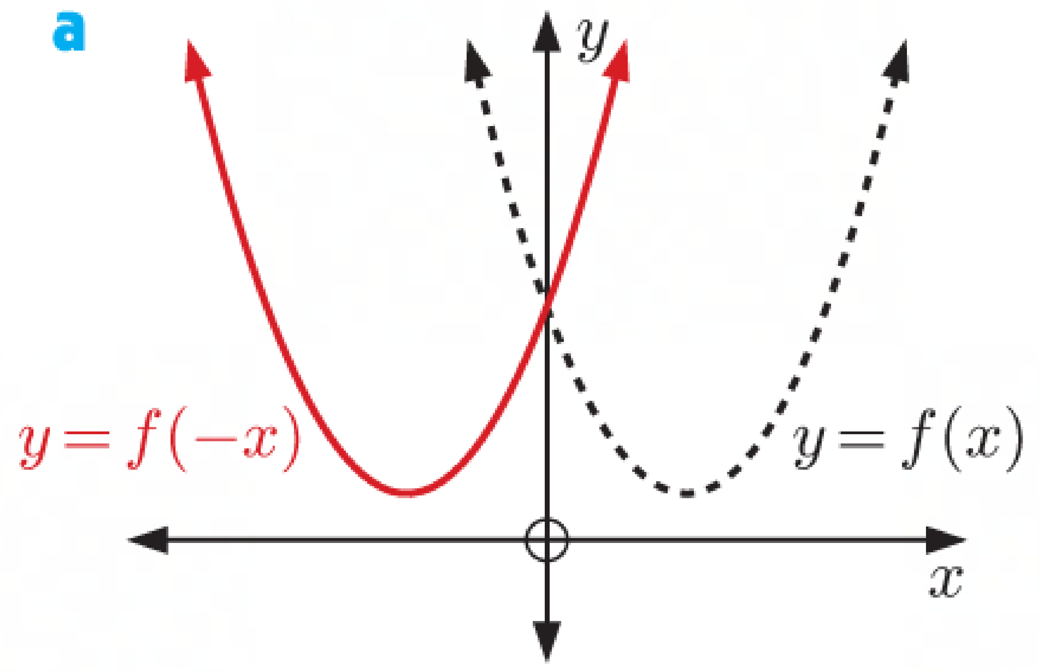


c

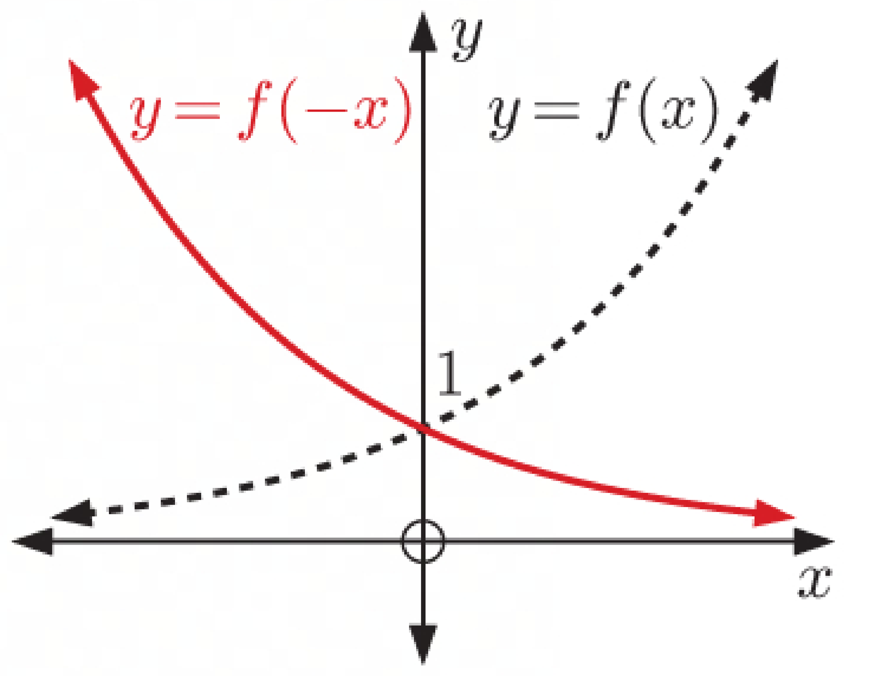


3 The graph of $y=f(-x)$ is found by reflecting $y=f(x)$ in the y-axis.

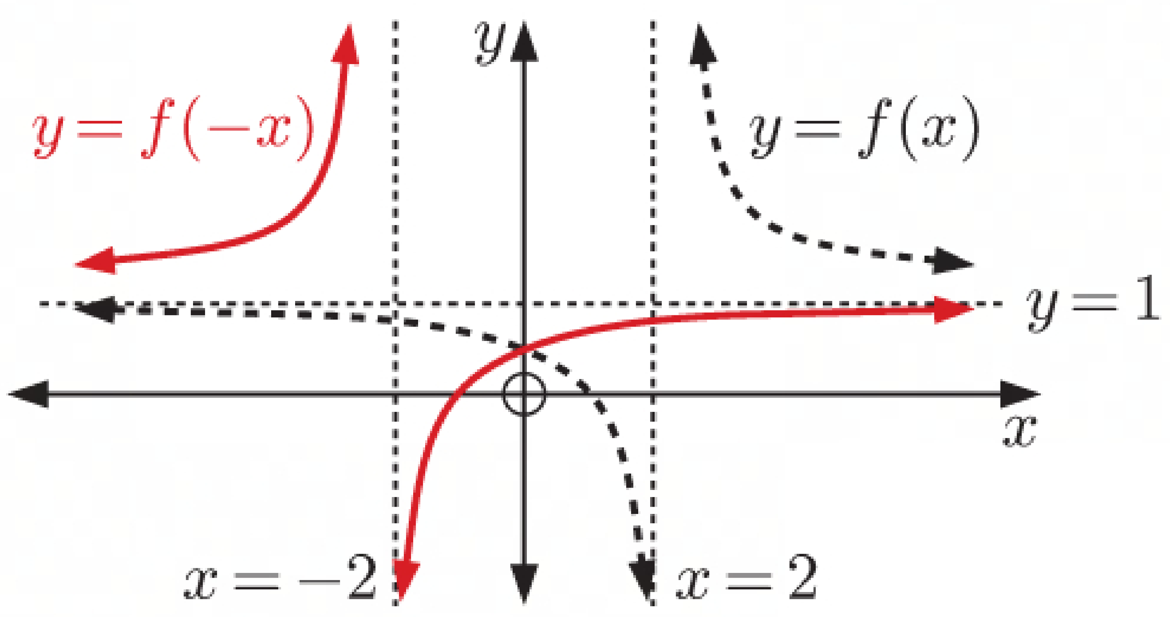
a



b

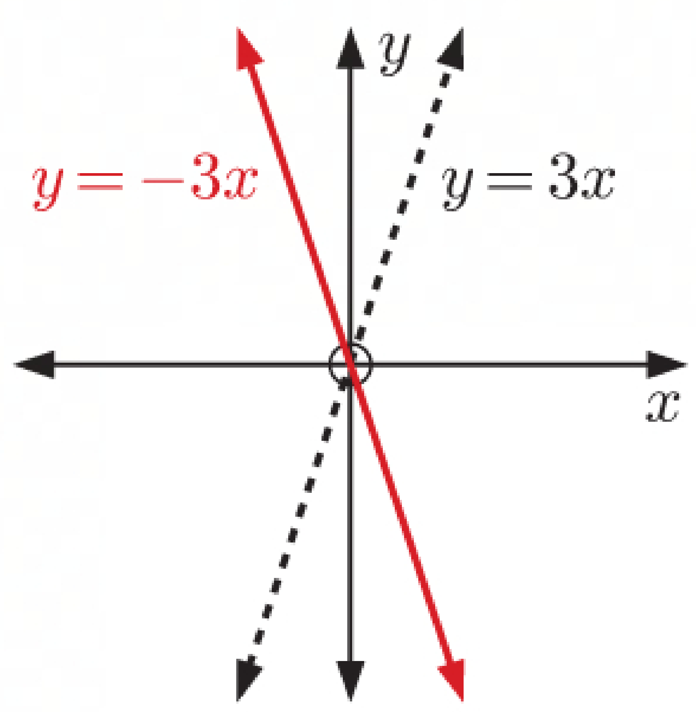


c

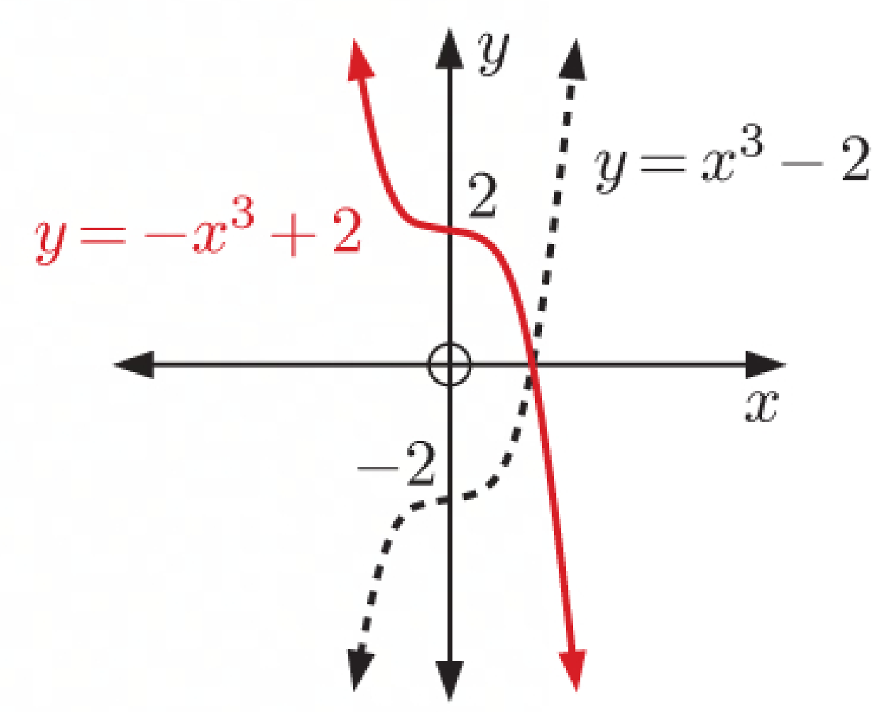


4 The graph of $y=-f(x)$ is found by reflecting $y=f(x)$ in the x-axis.

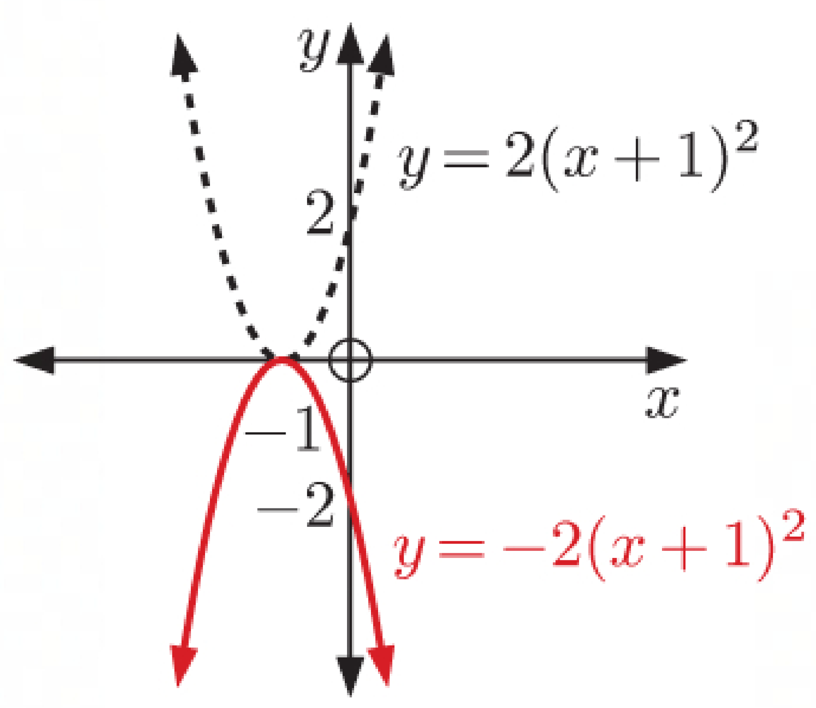
a $f(x)=3x$



b $f(x)=x^3-2$

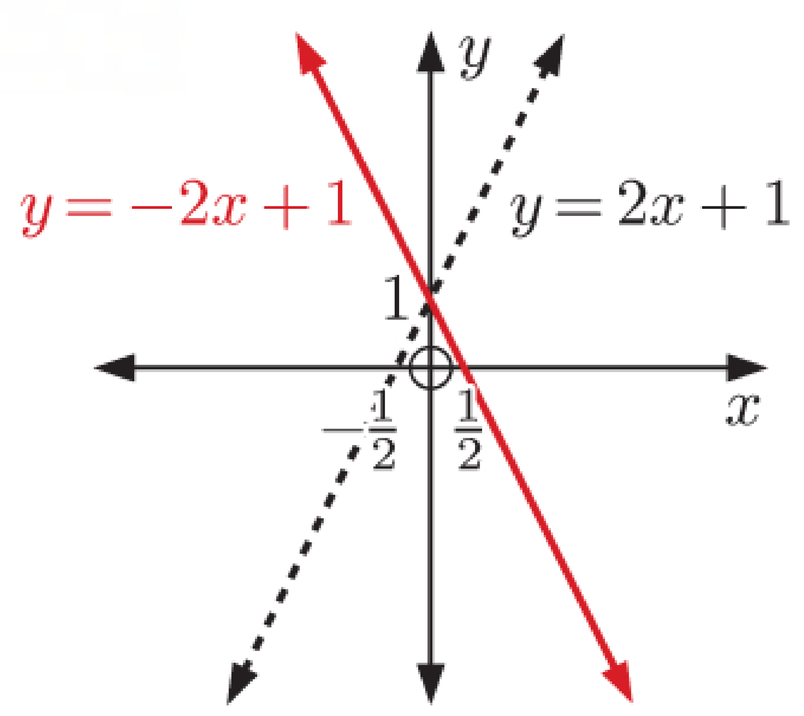


c $f(x)=2(x+1)^2$

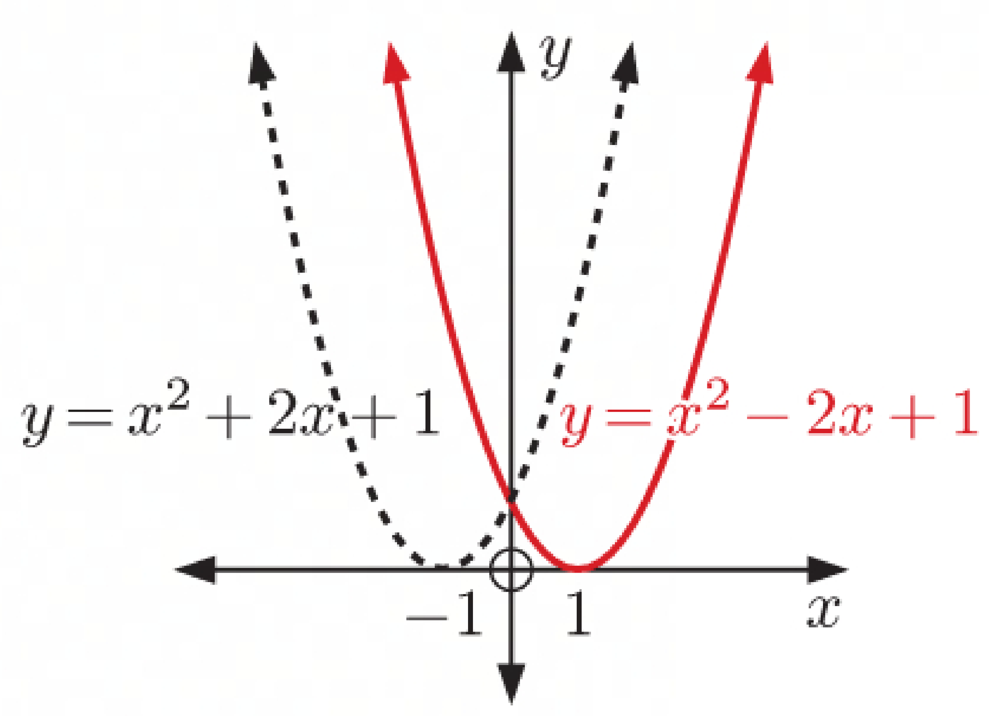


5 The graph of $y=f(-x)$ is found by reflecting $y=f(x)$ in the y-axis.

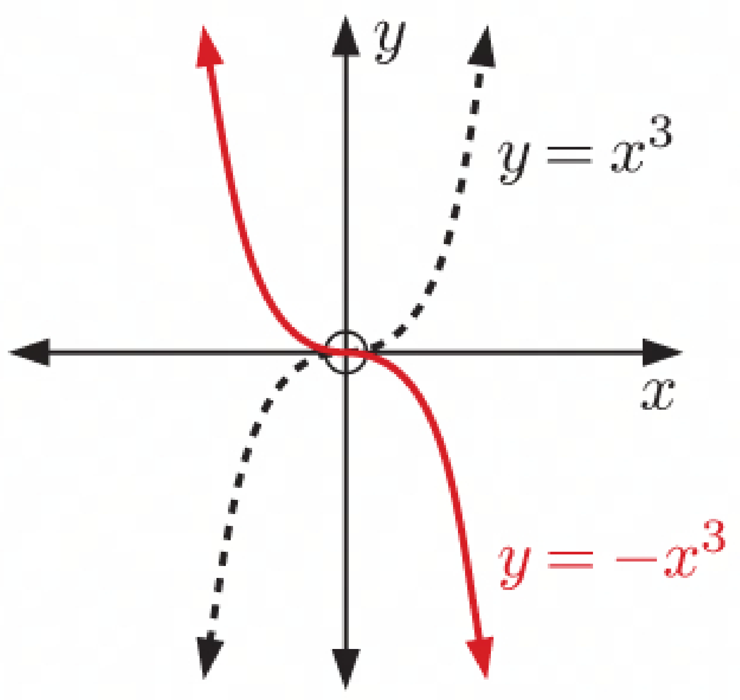
a $f(x)=2x+1$



b $f(x)=x^2+2x+1$



c $f(x) = x^3$



6 **a** $f(x) = 5x + 7$

$\therefore\ g(x) = -f(x)$ {reflected in the x-axis}

$= -(5x + 7)$

$= -5x - 7$

b $f(x) = 2^x$

$\therefore\ g(x) = f(-x)$ {reflected in the y-axis}

$= 2^{-x}$

c $f(x) = 2x^2 + 1$

$\therefore\ g(x) = -f(x)$ {reflected in the x-axis}

$= -2x^2 - 1$

d $f(x) = x^4 - 2x^3 - 3x^2 + 5x - 7$

$\therefore\ g(x) = f(-x)$ {reflected in the y-axis}

$= (-x)^4 - 2(-x)^3 - 3(-x)^2 + 5(-x) - 7$

$= x^4 + 2x^3 - 3x^2 - 5x - 7$

7 To transform $y = f(x)$ to $g(x) = -f(x)$, we reflect $y = f(x)$ in the x-axis.
By doing this, the x-coordinate stays the same, and we take the negative of the y-coordinate.

a **i** The point $(3, 0)$ on $y = f(x)$ will be transformed to $(3, 0)$ on $y = g(x)$.

ii The point $(2, -1)$ on $y = f(x)$ will be transformed to $(2, 1)$ on $y = g(x)$.

iii The point $(-3, 2)$ on $y = f(x)$ will be transformed to $(-3, -2)$ on $y = g(x)$.

b **i** The point on $y = f(x)$ which has been transformed to $(7, -1)$ on $y = g(x)$ is $(7, 1)$.

ii The point on $y = f(x)$ which has been transformed to $(-5, 0)$ on $y = g(x)$ is $(-5, 0)$.

iii The point on $y = f(x)$ which has been transformed to $(-3, -2)$ on $y = g(x)$ is $(-3, 2)$.

8 To transform $y = f(x)$ to $h(x) = f(-x)$, we reflect $y = f(x)$ in the y-axis.
By doing this, the y-coordinate stays the same, and we take the negative of the x-coordinate.

a **i** The point $(2, -1)$ on $y = f(x)$ will be transformed to $(-2, -1)$ on $y = h(x)$.

ii The point $(0, 3)$ on $y = f(x)$ will be transformed to $(0, 3)$ on $y = h(x)$.

iii The point $(-1, 2)$ on $y = f(x)$ will be transformed to $(1, 2)$ on $y = h(x)$.

b **i** The point on $y = f(x)$ which has been transformed to $(5, -4)$ on $y = h(x)$ is $(-5, -4)$.

ii The point on $y = f(x)$ which has been transformed to $(0, 3)$ on $y = h(x)$ is $(0, 3)$.

iii The point on $y = f(x)$ which has been transformed to $(2, 3)$ on $y = h(x)$ is $(-2, 3)$.

9 **a** To transform $y = f(x)$ to $g(x) = -f(-x)$, we first reflect $y = f(x)$ in the y-axis, and then reflect it in the x axis.

b By reflecting in the x-axis and y-axis, we take the negative of the x-coordinate and the negative of the y-coordinate.
The point $(3, -7)$ on $y = f(x)$ will be transformed to $(-3, 7)$.

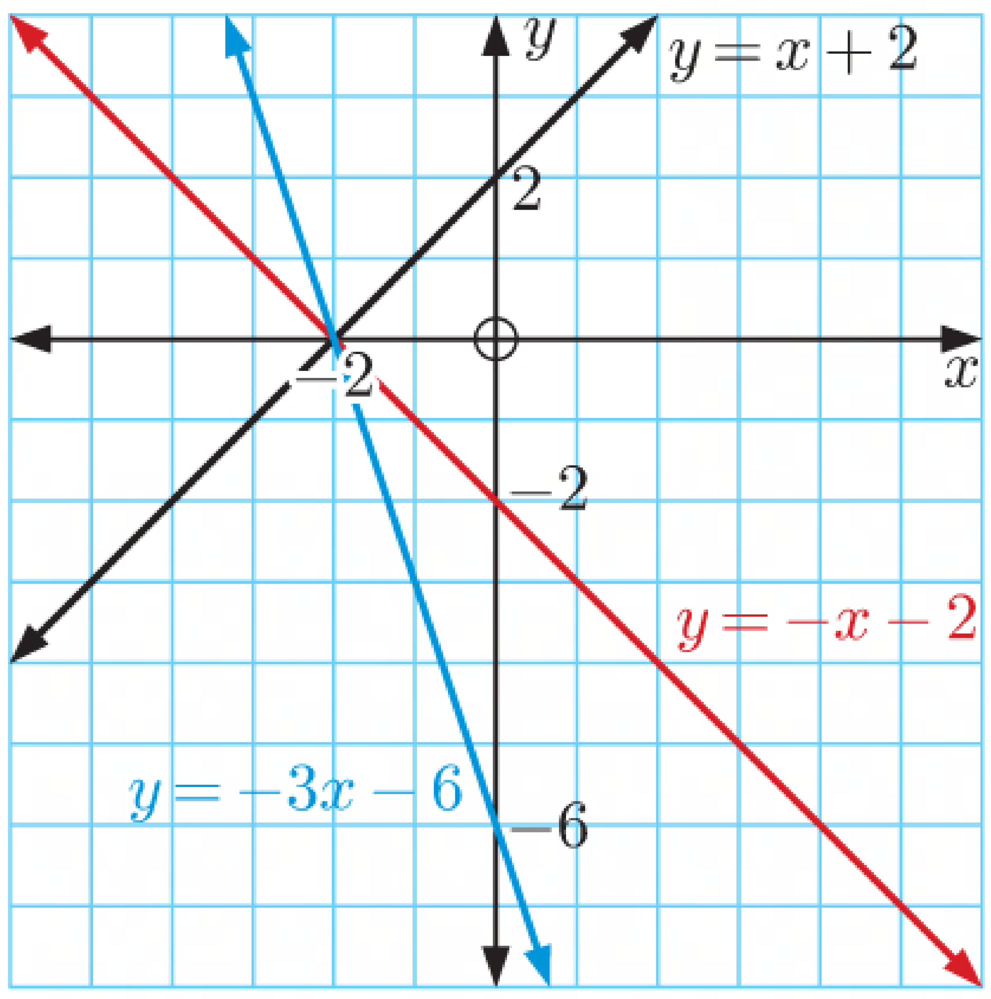
c The point on $y = f(x)$ which has been transformed to $(-5, -1)$ on $y = g(x)$ is $(5, 1)$.

10 $f(x) = x + 2$

a To transform $y = f(x)$ to $y = -f(x)$, we reflect $y = f(x)$ in the x-axis.

b To transform $y = -f(x)$ to $y = -3f(x)$, we stretch $y = -f(x)$ vertically with scale factor 3.

c
$$f(x) = x + 2$$
$$\therefore \quad -f(x) = -(x + 2) = -x - 2$$
$$\text{and} \quad -3f(x) = 3(-x - 2) = -3x - 6$$

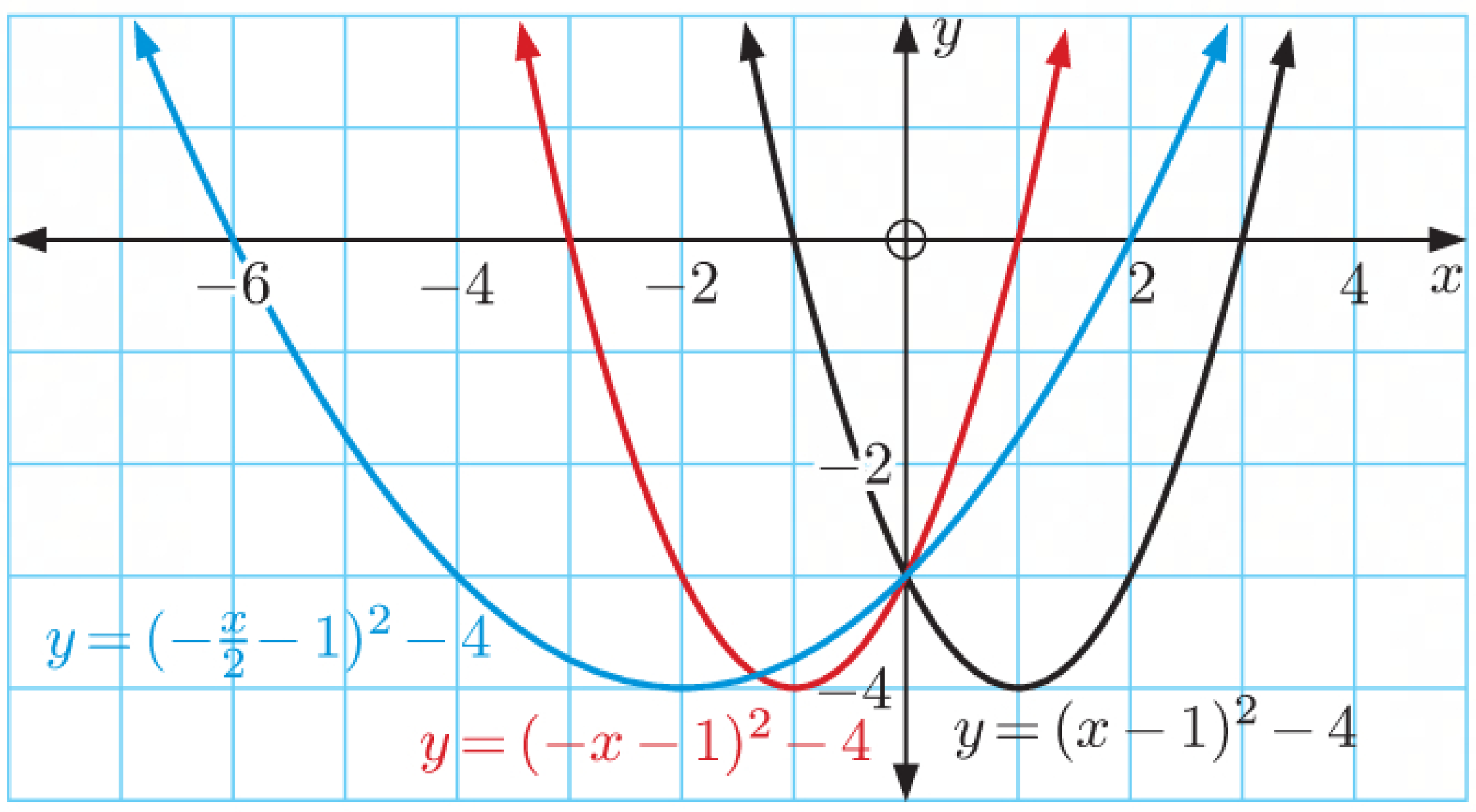


11 $f(x) = (x - 1)^2 - 4$

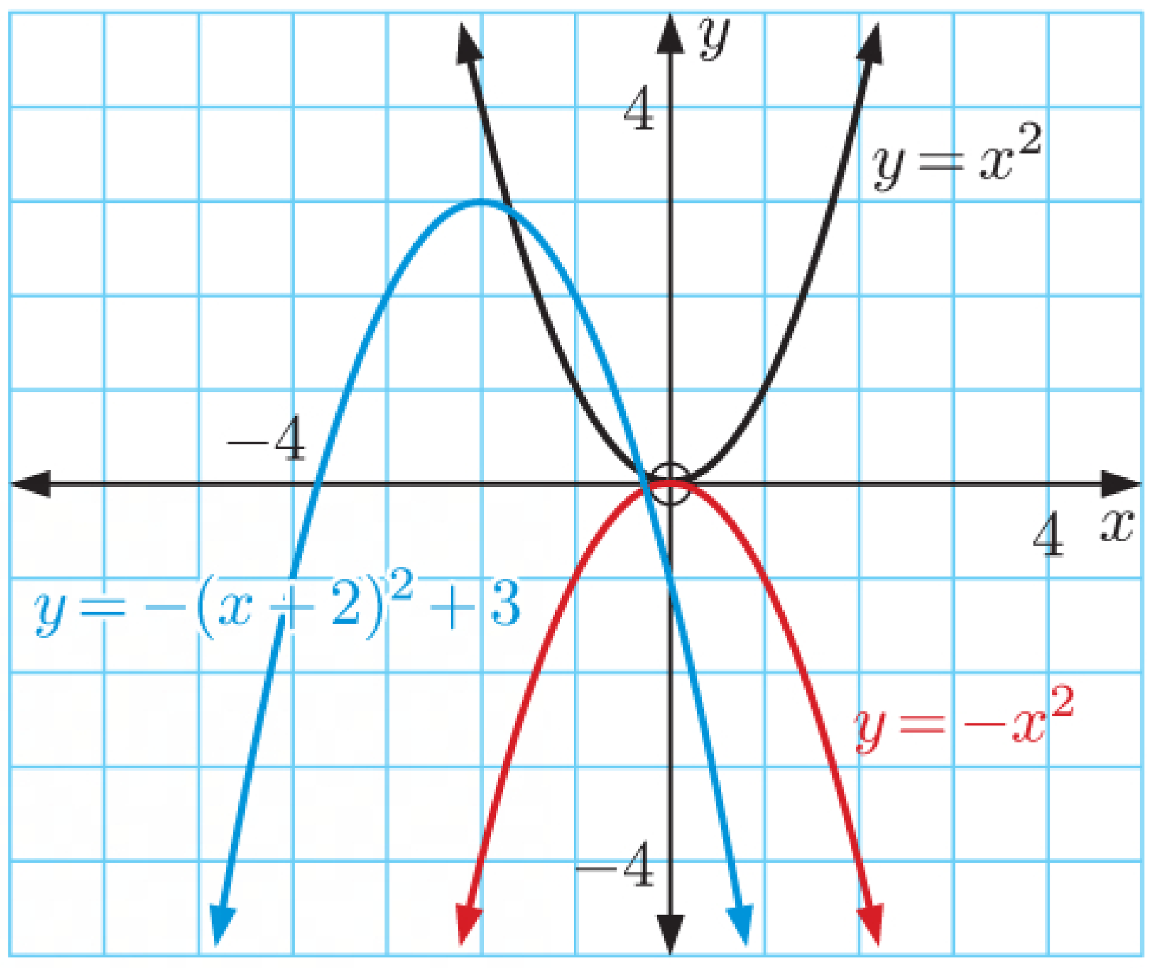
a To transform $y = f(x)$ to $y = f(-x)$, we reflect $y = f(x)$ in the y-axis.

b To transform $y = f(-x)$ to $y = f(-\frac{1}{2}x)$, we stretch $y = f(-x)$ horizontally with scale factor 2.

c
$$f(x) = (x - 1)^2 - 4$$
$$\therefore \quad f(-x) = (-x - 1)^2 - 4$$
$$\text{and} \quad f(-\tfrac{1}{2}x) = \left(-\frac{x}{2} - 1\right)^2 - 4$$



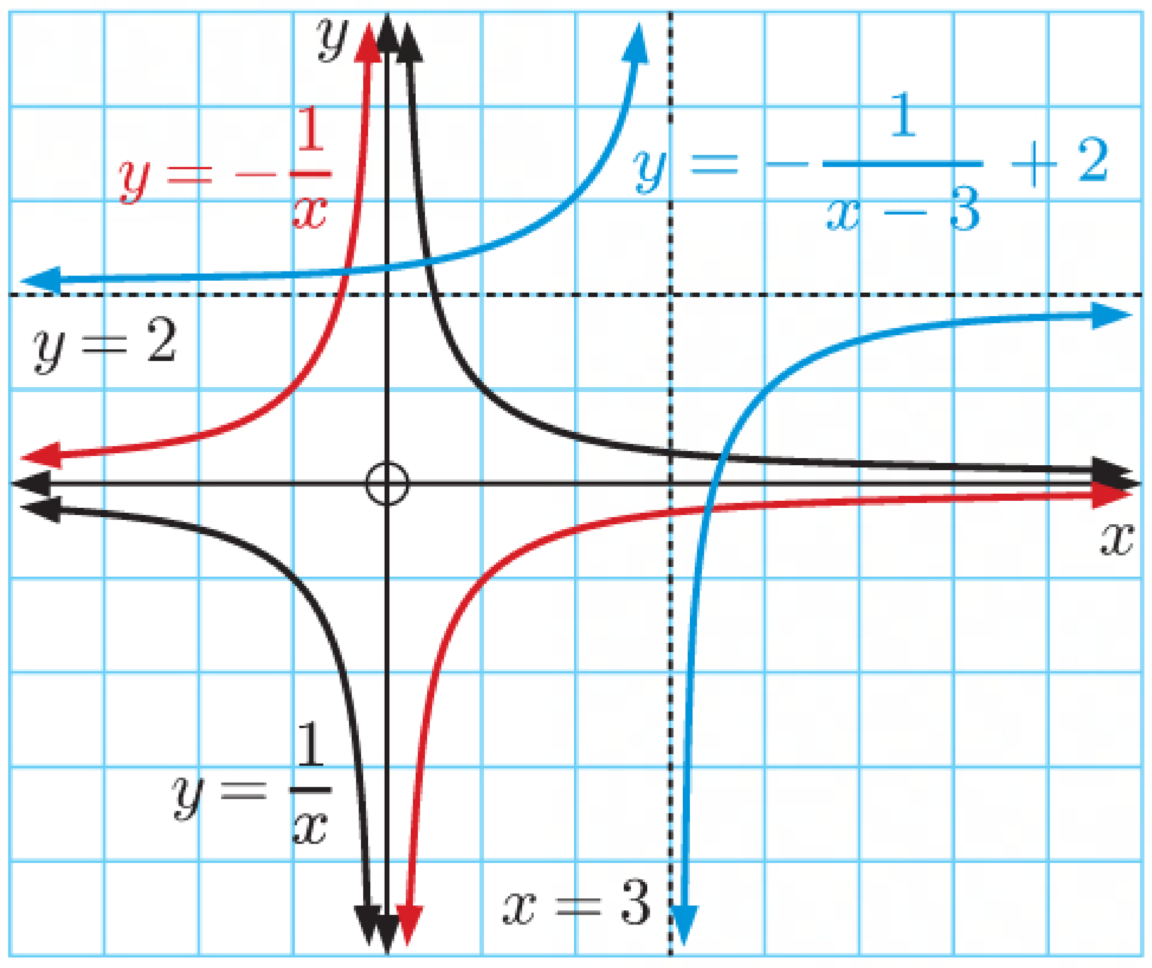
12



To transform $y = x^2$ to $y = -x^2$, we reflect $y = x^2$ in the x-axis.

To transform $y = -x^2$ to $y = -(x+2)^2 + 3$, we translate $y = -x^2$ through $\begin{pmatrix} -2 \\ 3 \end{pmatrix}$.

13



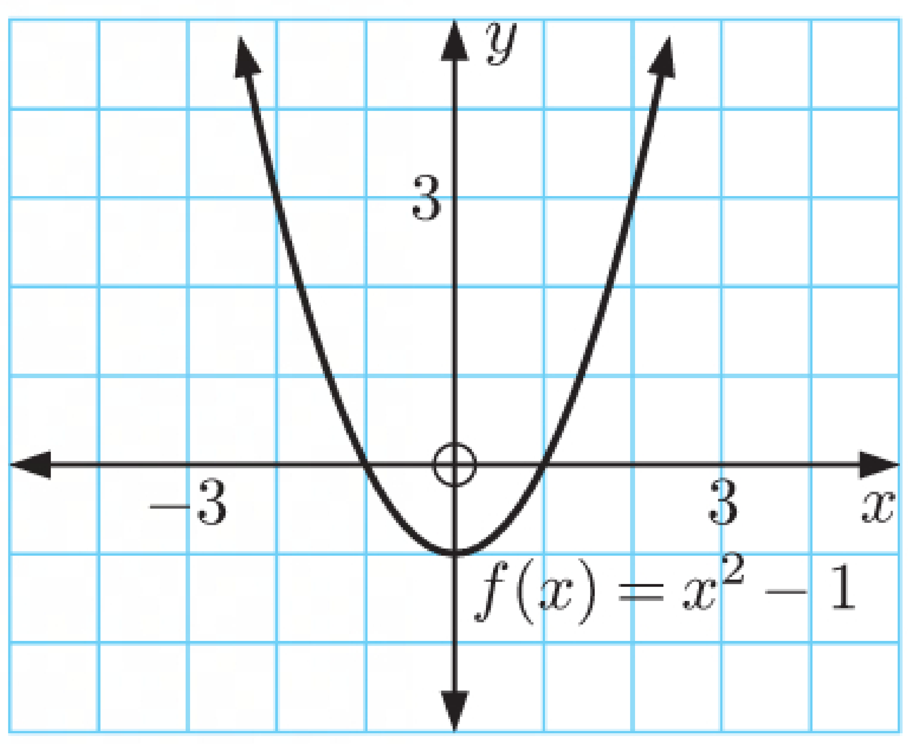
To transform $y = \frac{1}{x}$ to $y = -\frac{1}{x}$, we reflect $y = \frac{1}{x}$ in the x-axis.

To transform $y = -\frac{1}{x}$ to $y = -\frac{1}{x-3} + 2$, we translate $y = -\frac{1}{x}$ through $\begin{pmatrix} 3 \\ 2 \end{pmatrix}$.

EXERCISE 16D

1 $f(x) = x^2 - 1$

a



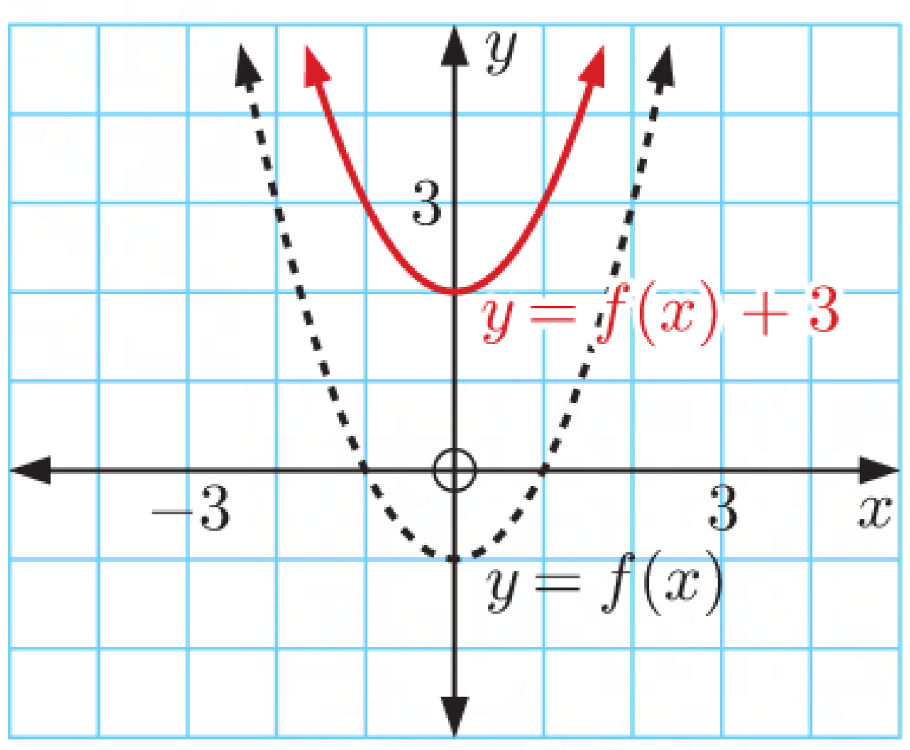
$$f(0) = (0)^2 - 1$$
$$= -1$$

$\therefore$ the y-intercept is -1.

When $f(x) = 0$, $x^2 - 1 = 0$
$$x^2 = 1$$
$$x = \pm 1$$

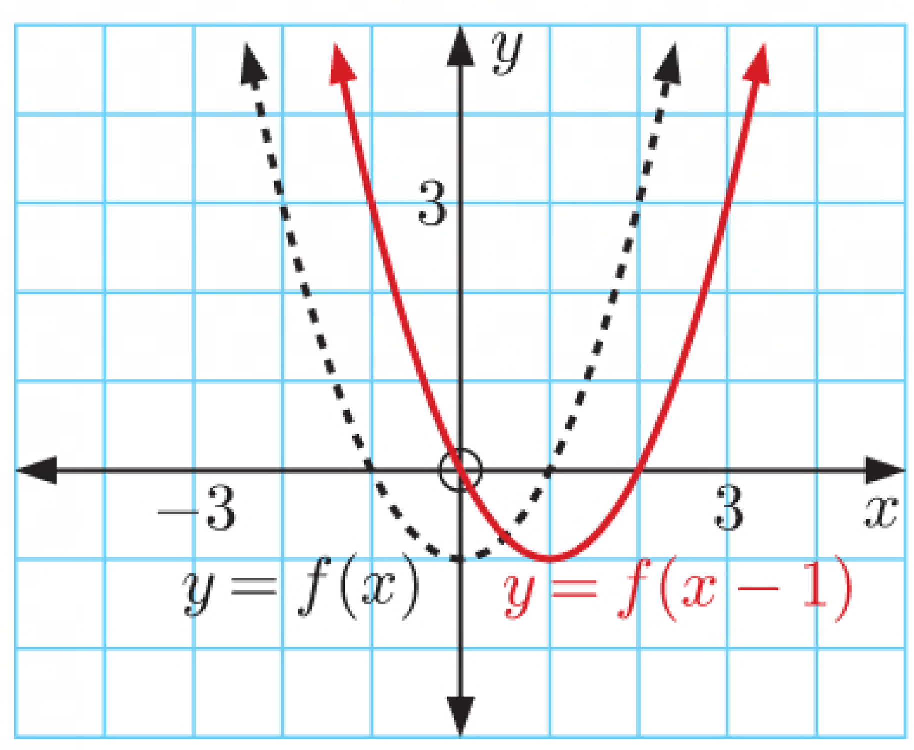
$\therefore$ the x-intercepts are 1 and -1.

b **i**



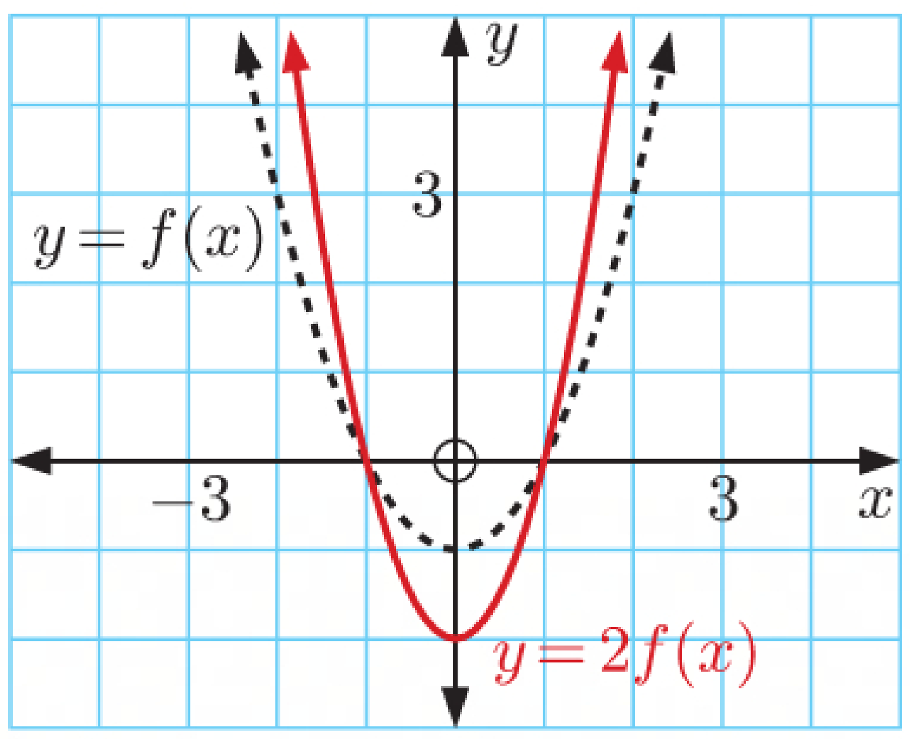
$y = f(x)$ has been translated 3 units upwards.

ii



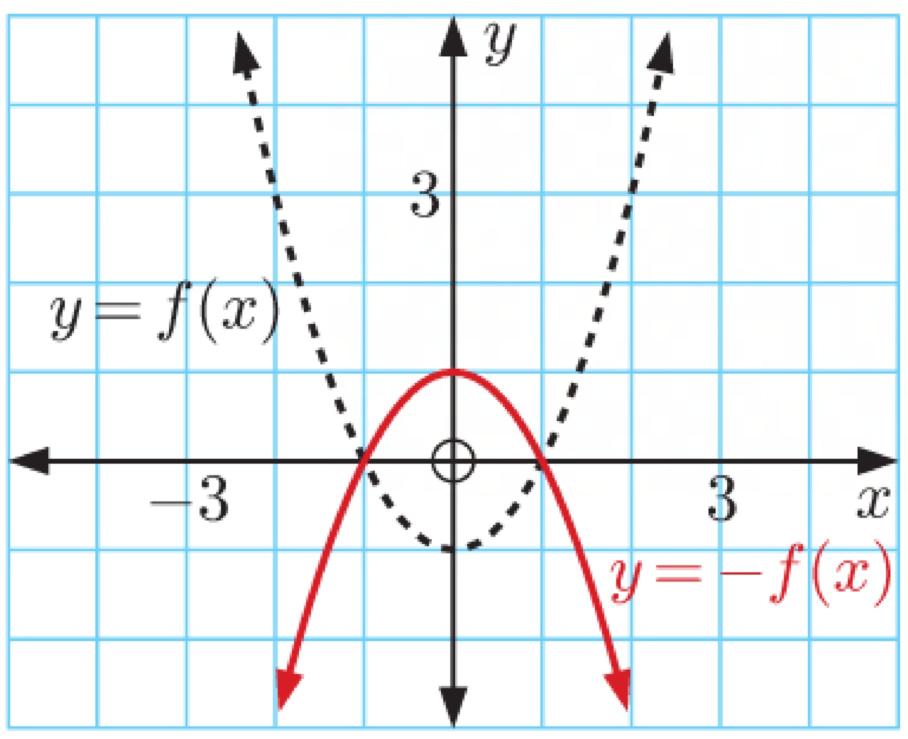
$y = f(x)$ has been translated 1 unit to the right.

iii



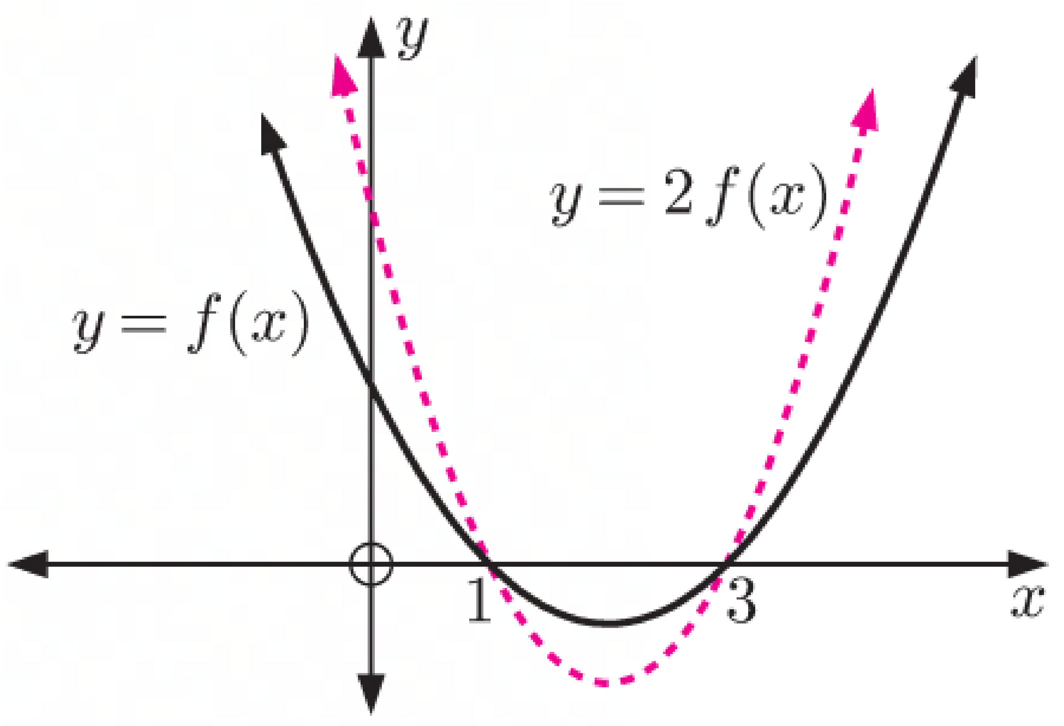
$y=f(x)$ has been vertically stretched with scale factor 2.

iv

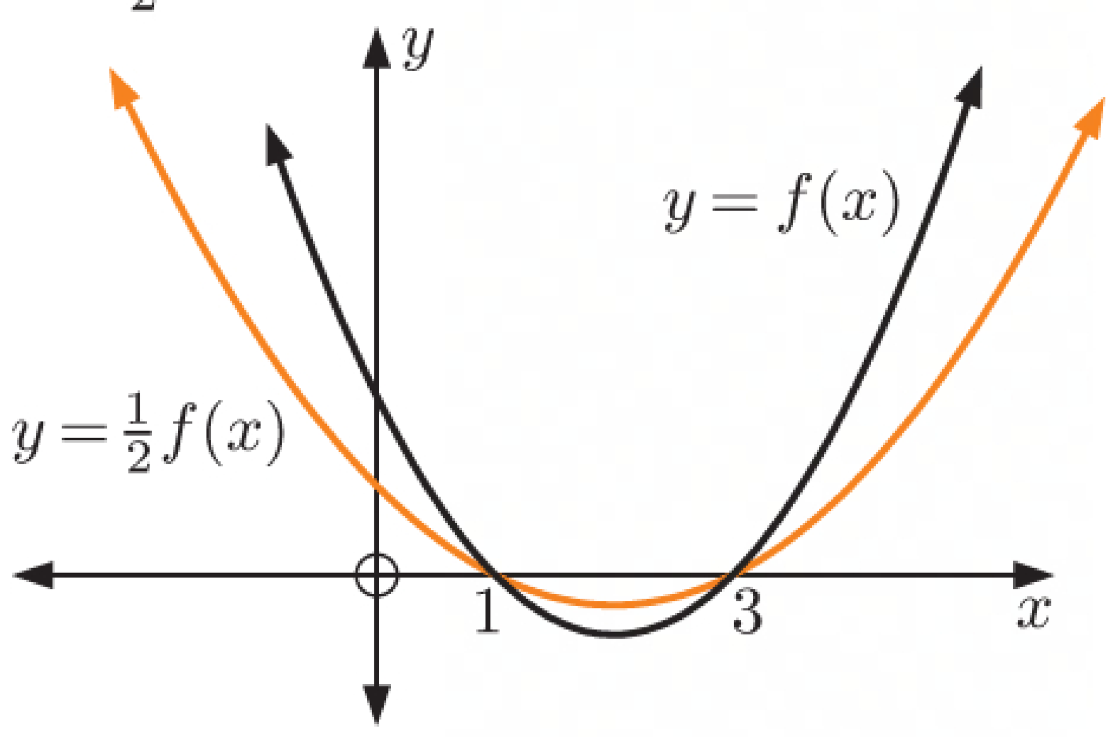


$y=f(x)$ has been reflected in the x-axis.

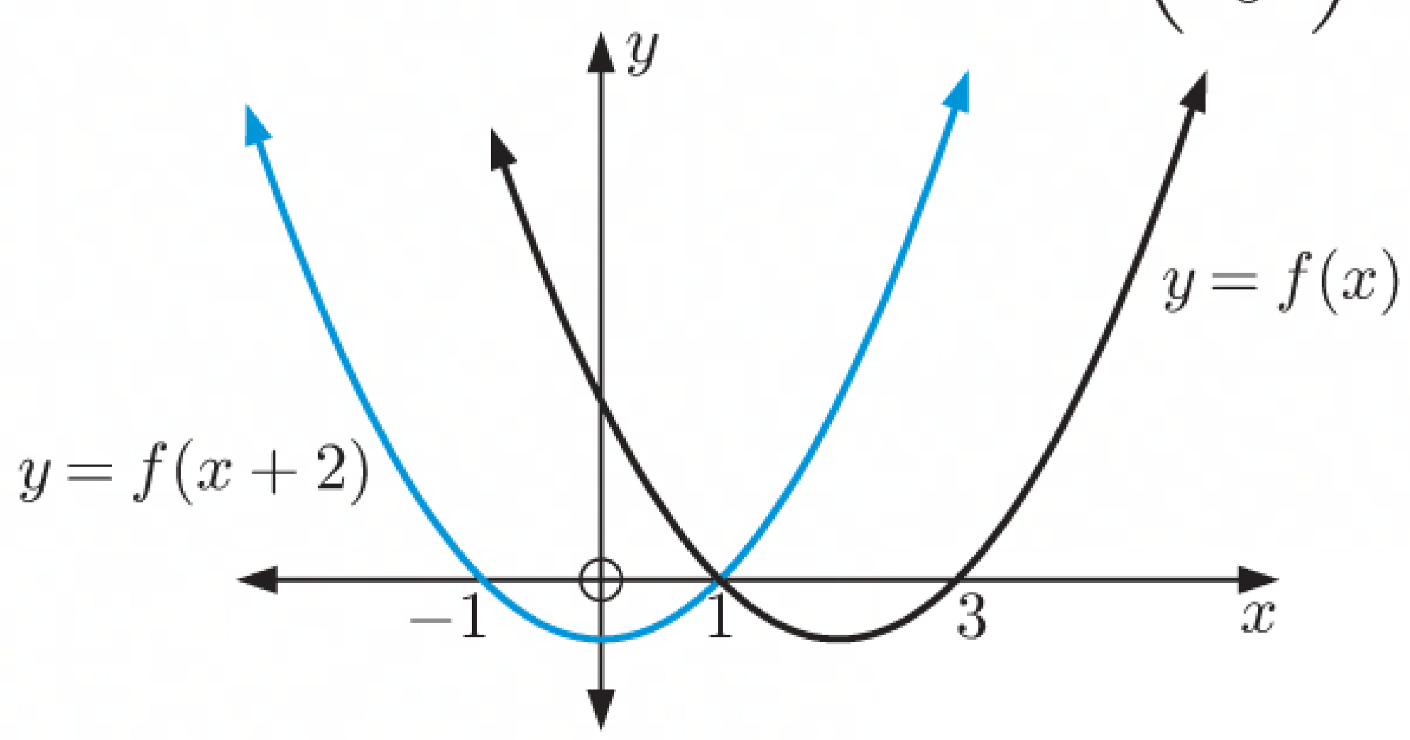
2 **a** To transform $y=f(x)$ to $y=2f(x)$, we vertically stretch $y=f(x)$ with scale factor 2.



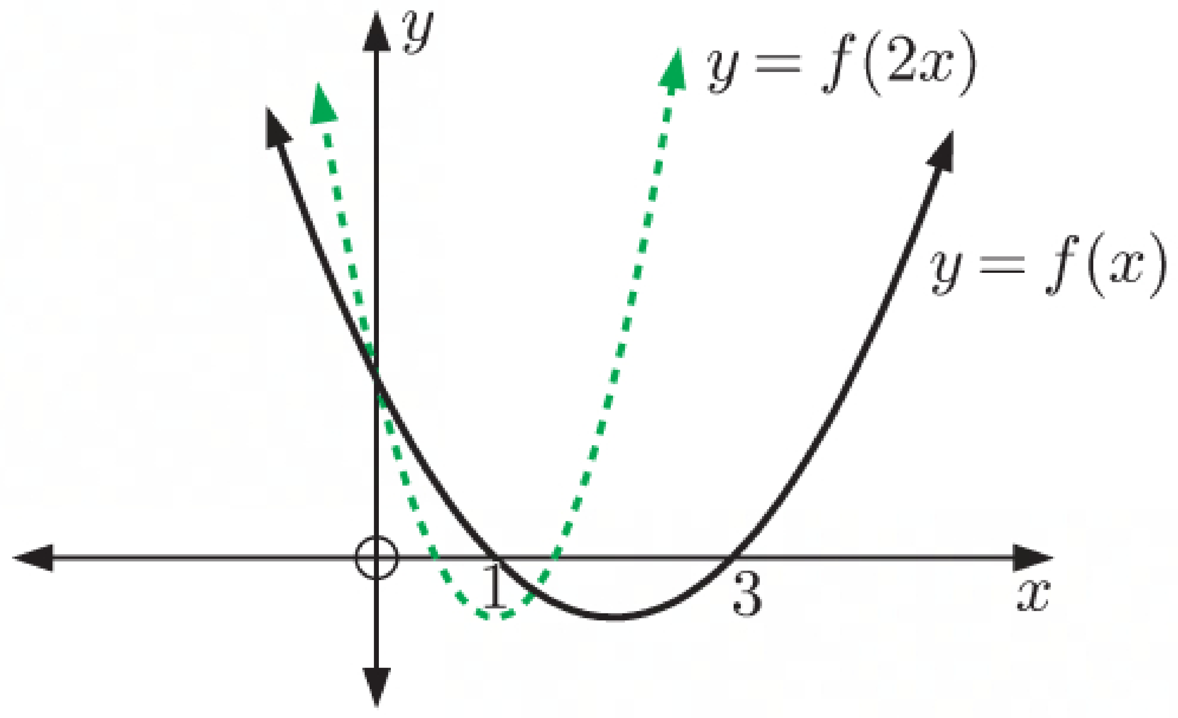
b To transform $y=f(x)$ to $y=\frac{1}{2}f(x)$, we vertically stretch $y=f(x)$ with scale factor $\frac{1}{2}$.



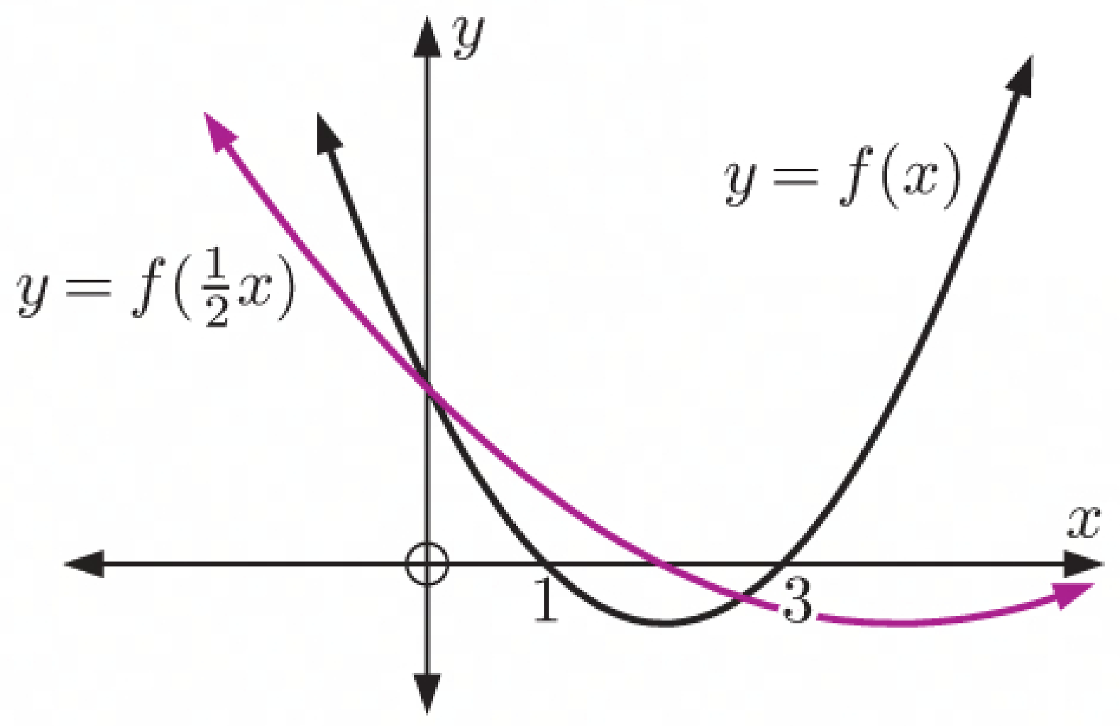
c To transform $y=f(x)$ to $y=f(x+2)$, we translate $y=f(x)$ through $\begin{pmatrix}-2\\0\end{pmatrix}$.

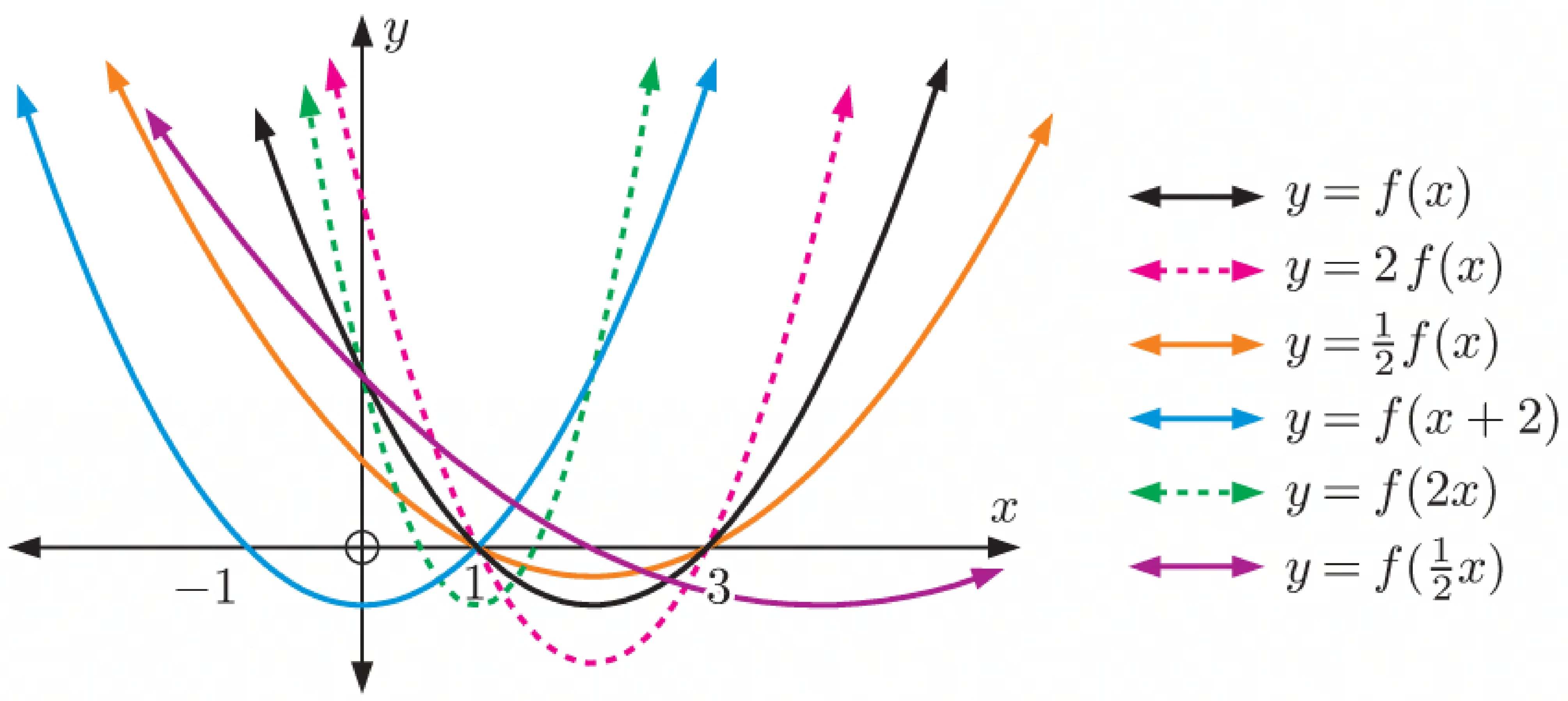


d To transform $y=f(x)$ to $y=f(2x)$, we horizontally stretch $y=f(x)$ with scale factor $\frac{1}{2}$.

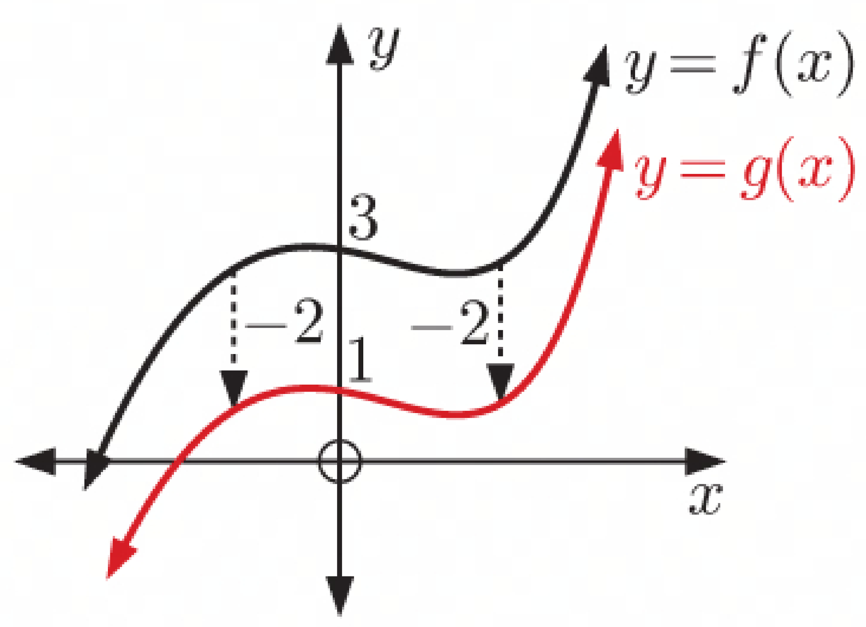


e To transform $y=f(x)$ to $y=f(\frac{1}{2}x)$, we horizontally stretch $y=f(x)$ with scale factor 2.





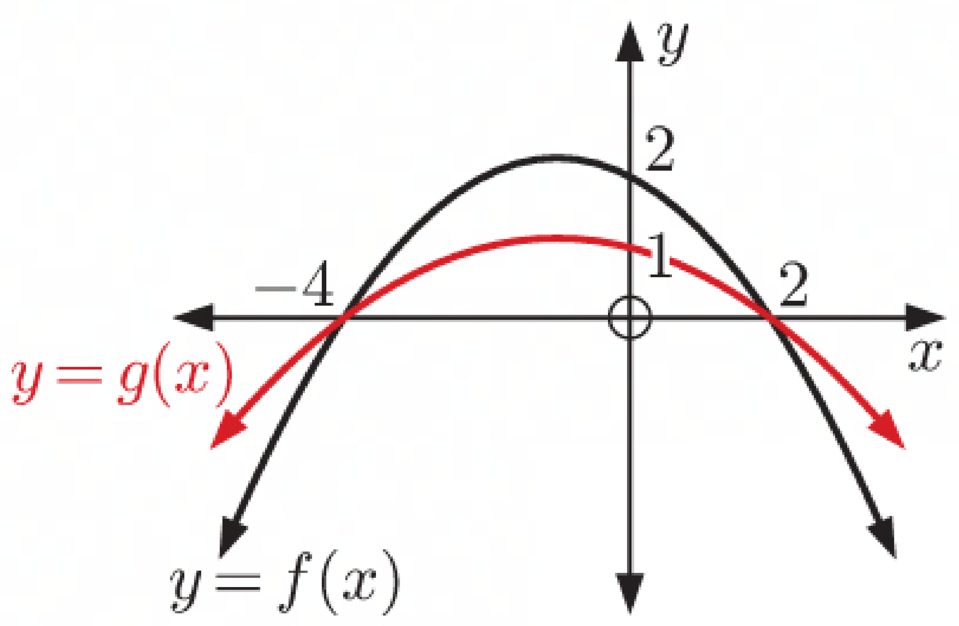
3 **a**



i $y = f(x)$ has been translated through $\begin{pmatrix} 0 \\ -2 \end{pmatrix}$.

ii $g(x) = f(x) - 2$

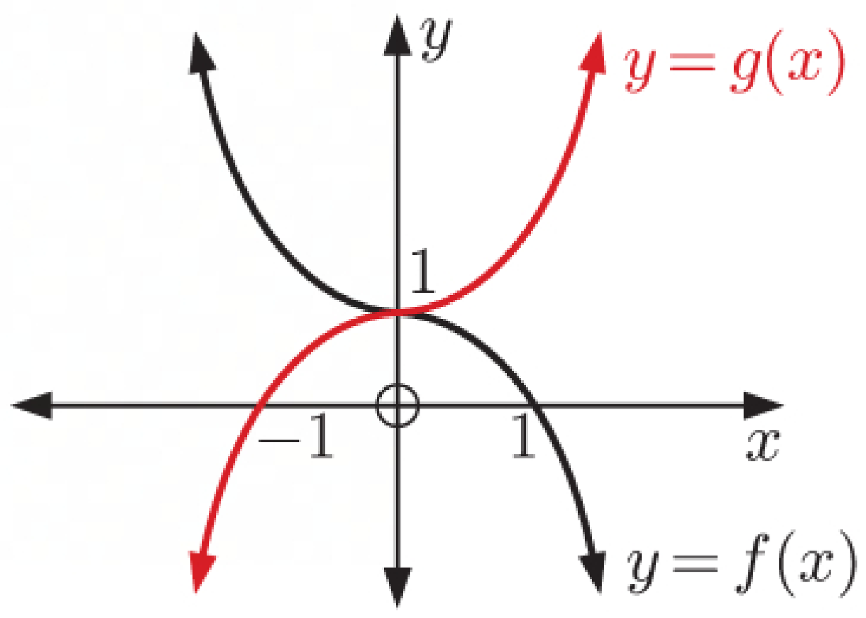
b



i $y = f(x)$ has been stretched vertically with scale factor $\frac{1}{2}$.

ii $g(x) = \frac{1}{2}f(x)$

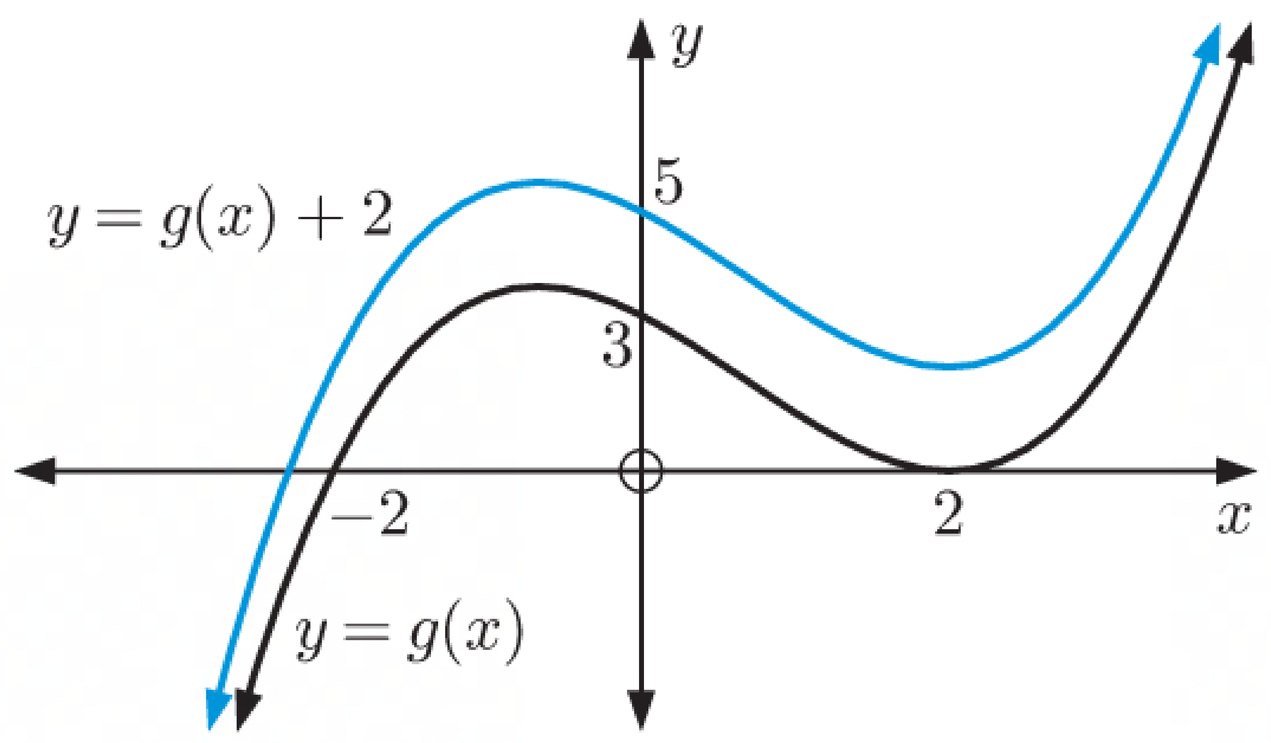
c



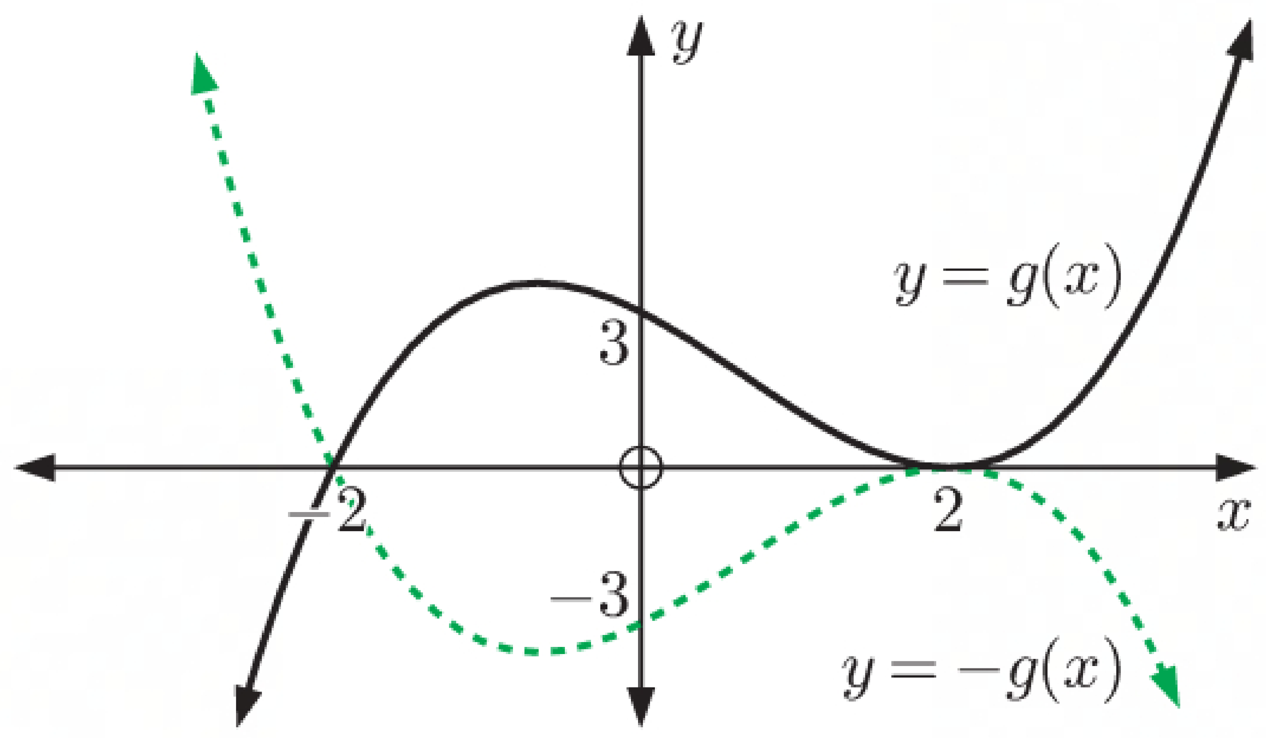
i $y = f(x)$ has been reflected in the y-axis.

ii $g(x) = f(-x)$

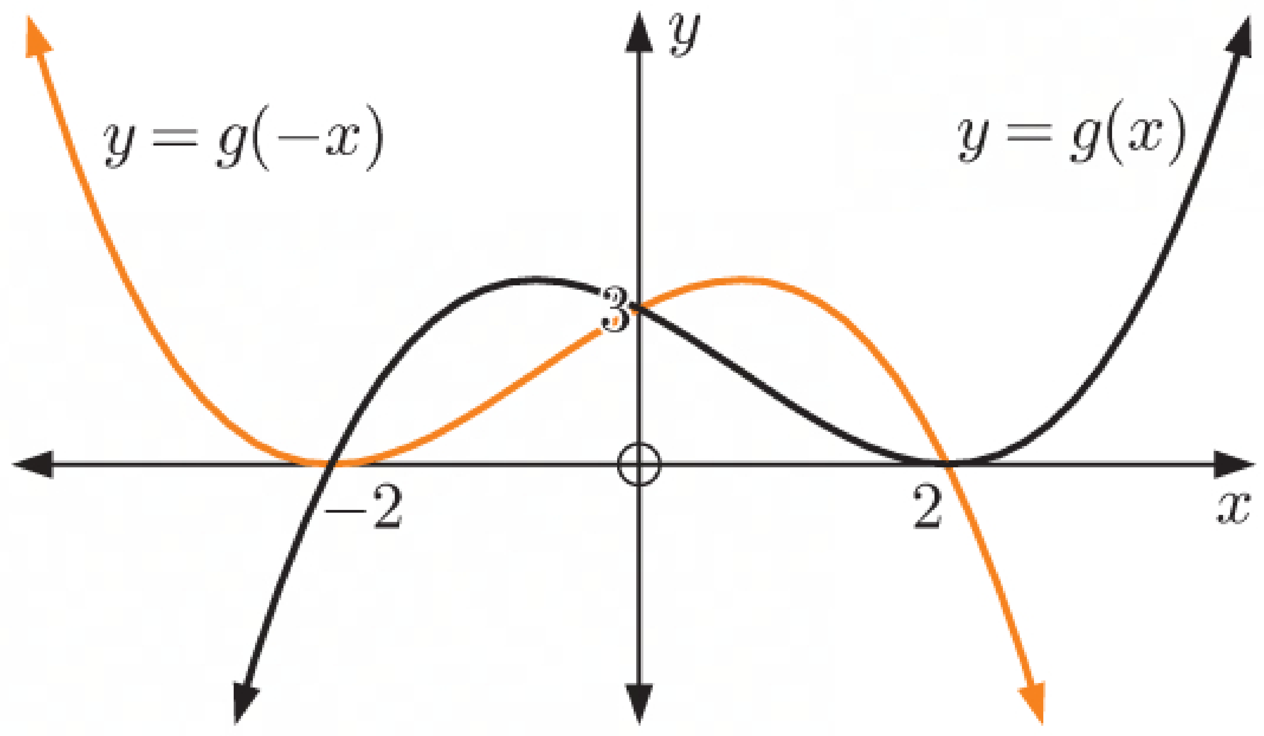
4 **a** To transform $y = g(x)$ to $y = g(x) + 2$, we translate $y = g(x)$ through $\begin{pmatrix} 0 \\ 2 \end{pmatrix}$.



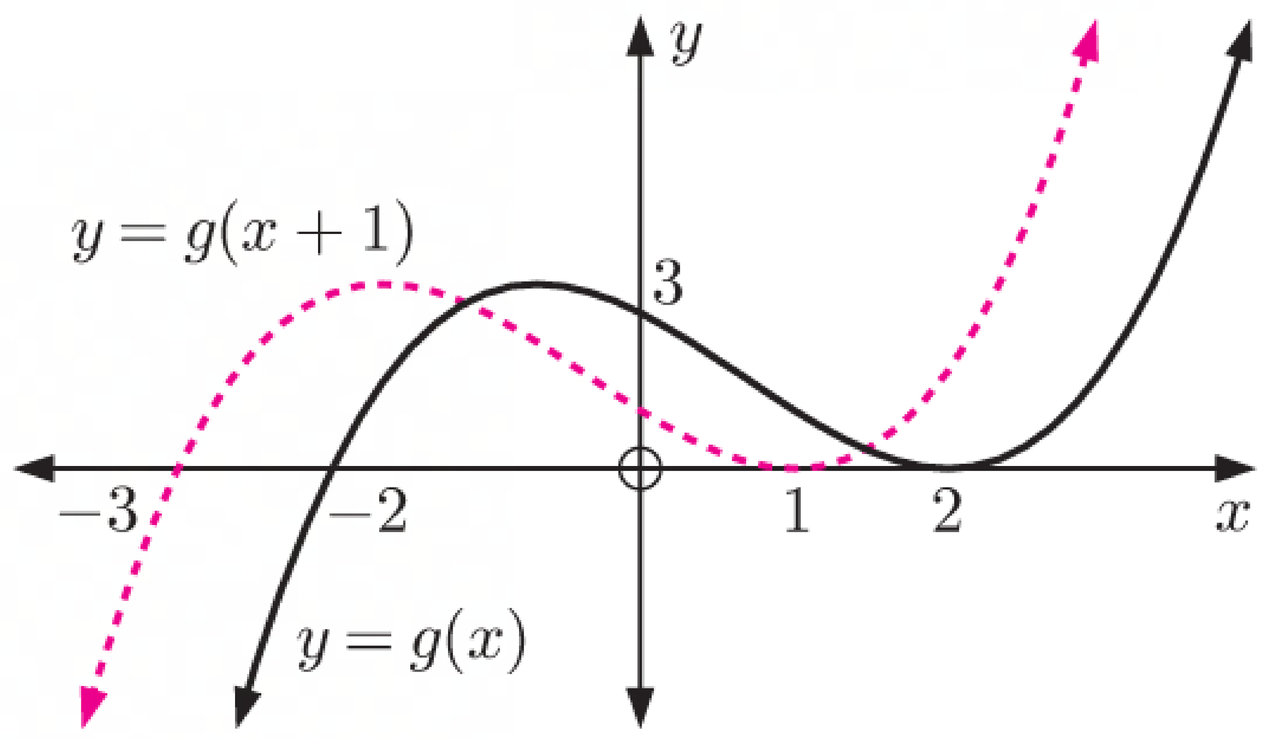
b To transform $y = g(x)$ to $y = -g(x)$, we reflect $y = g(x)$ in the x-axis.

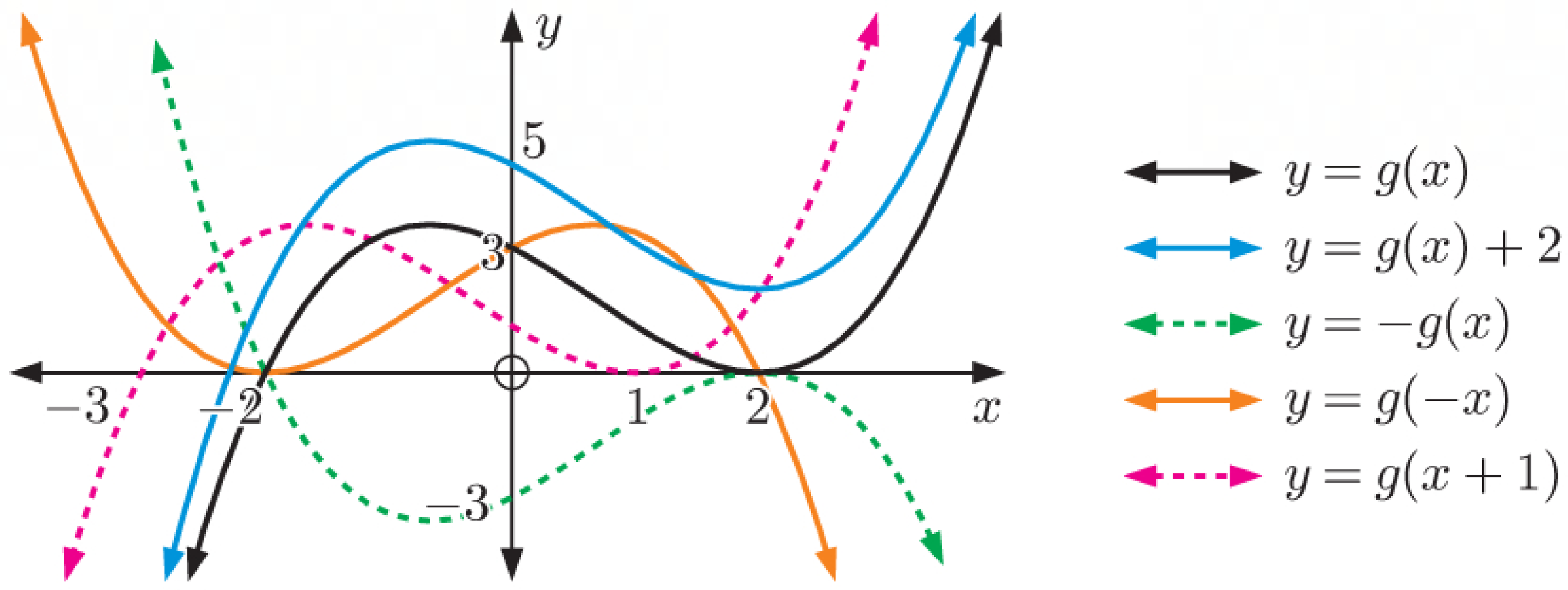


c To transform $y = g(x)$ to $y = g(-x)$, we reflect $y = g(x)$ in the y-axis.

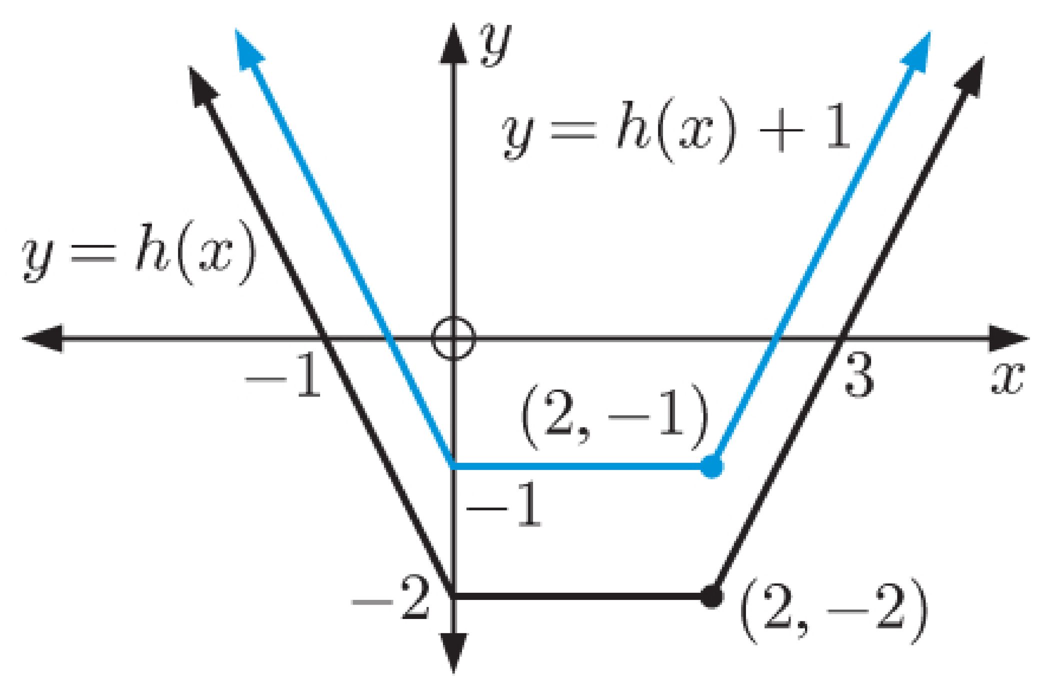


d To transform $y = g(x)$ to $y = g(x+1)$, we translate $y = g(x)$ through $\begin{pmatrix} -1 \\ 0 \end{pmatrix}$.

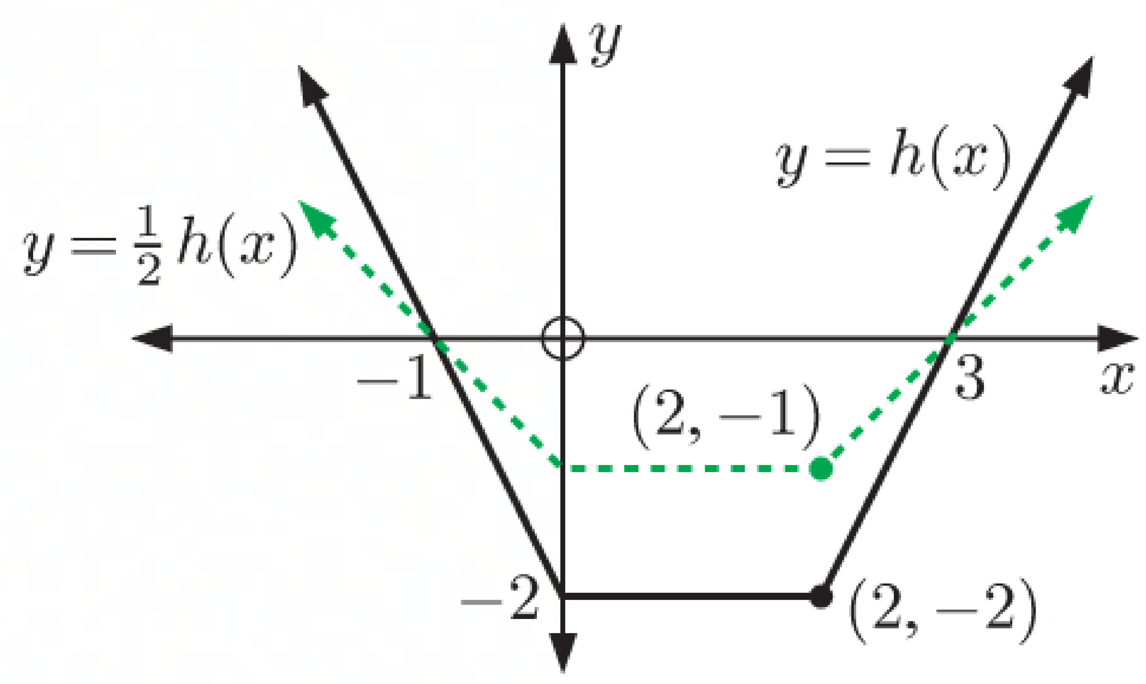




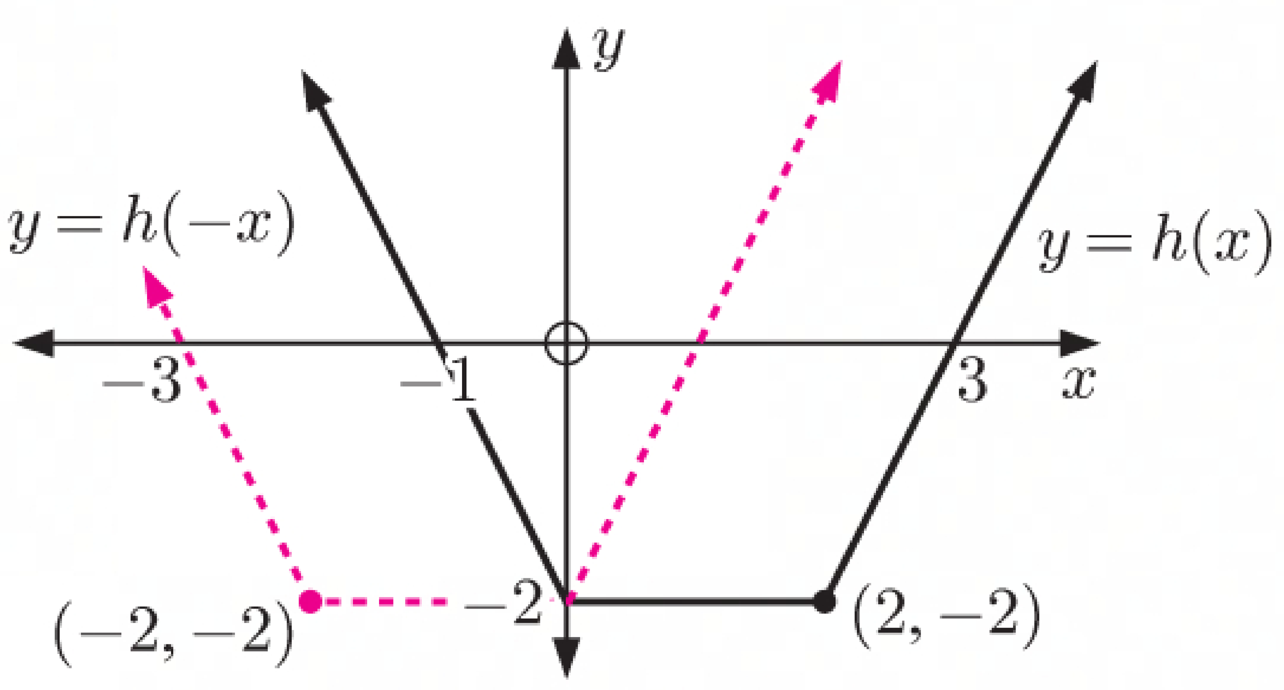
5 a To transform $y = h(x)$ to $y = h(x) + 1$, we translate $y = h(x)$ through $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$.



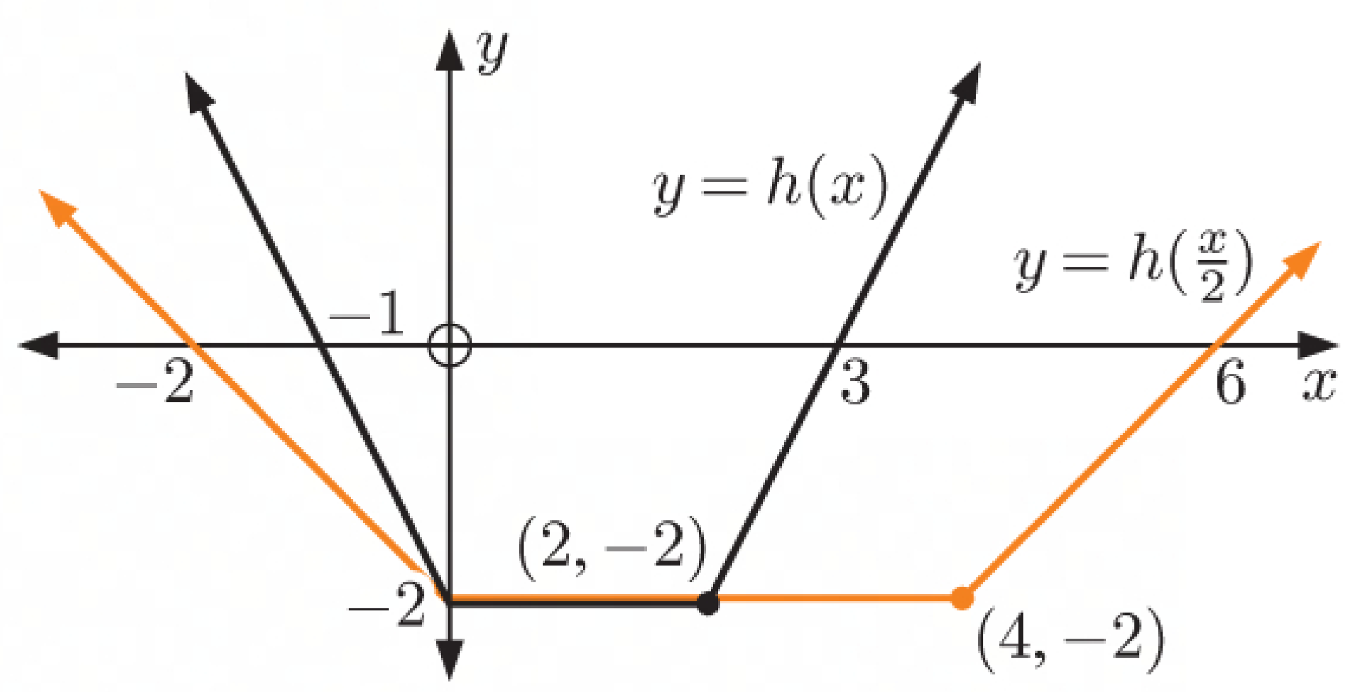
b To transform $y = h(x)$ to $y = \frac{1}{2}h(x)$, we vertically stretch $y = h(x)$ with scale factor $\frac{1}{2}$.

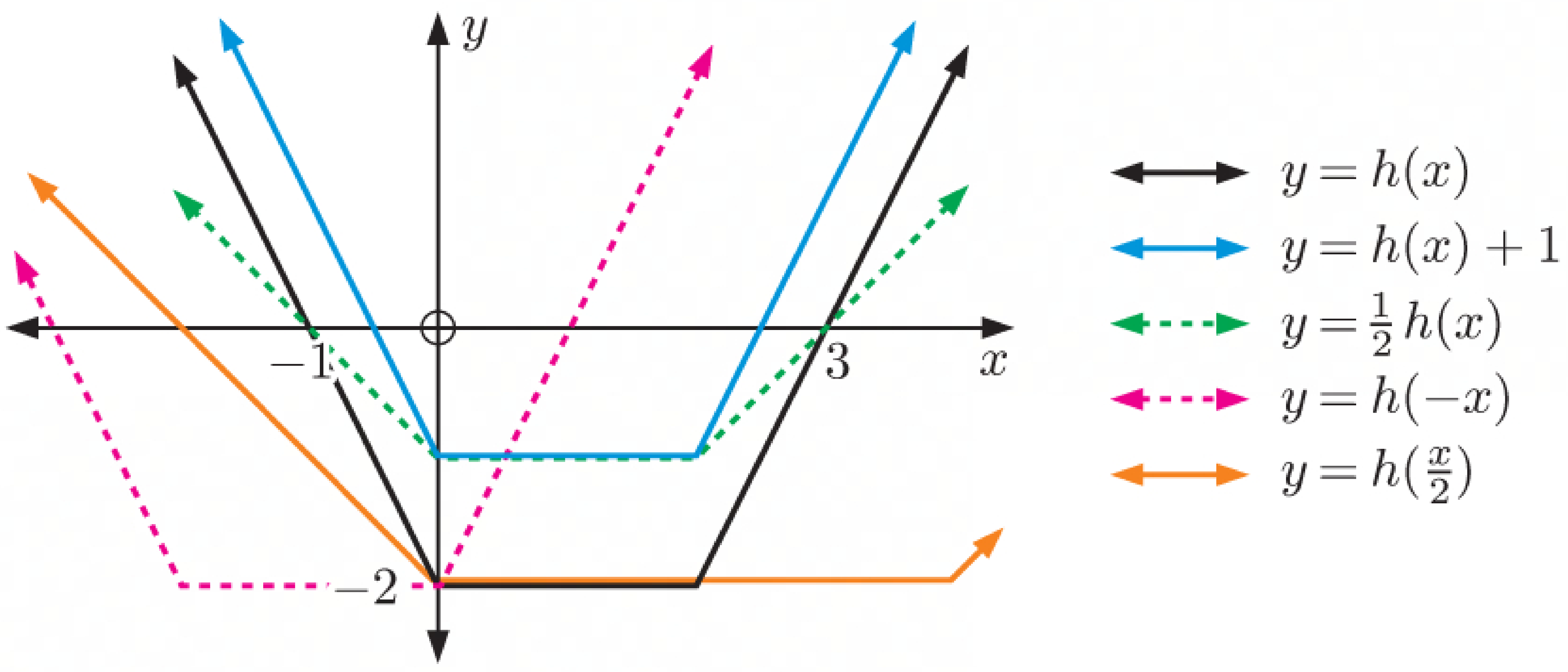


c To transform $y = h(x)$ to $y = h(-x)$, we reflect $y = h(x)$ in the y-axis.



d To transform $y = h(x)$ to $y = h\left(\frac{x}{2}\right)$, we horizontally stretch $y = h(x)$ with scale factor 2.





6 **a** $f(x) = (x+1)(x-\beta)$

$= 0$ when $x = -1$ or $x = \beta$

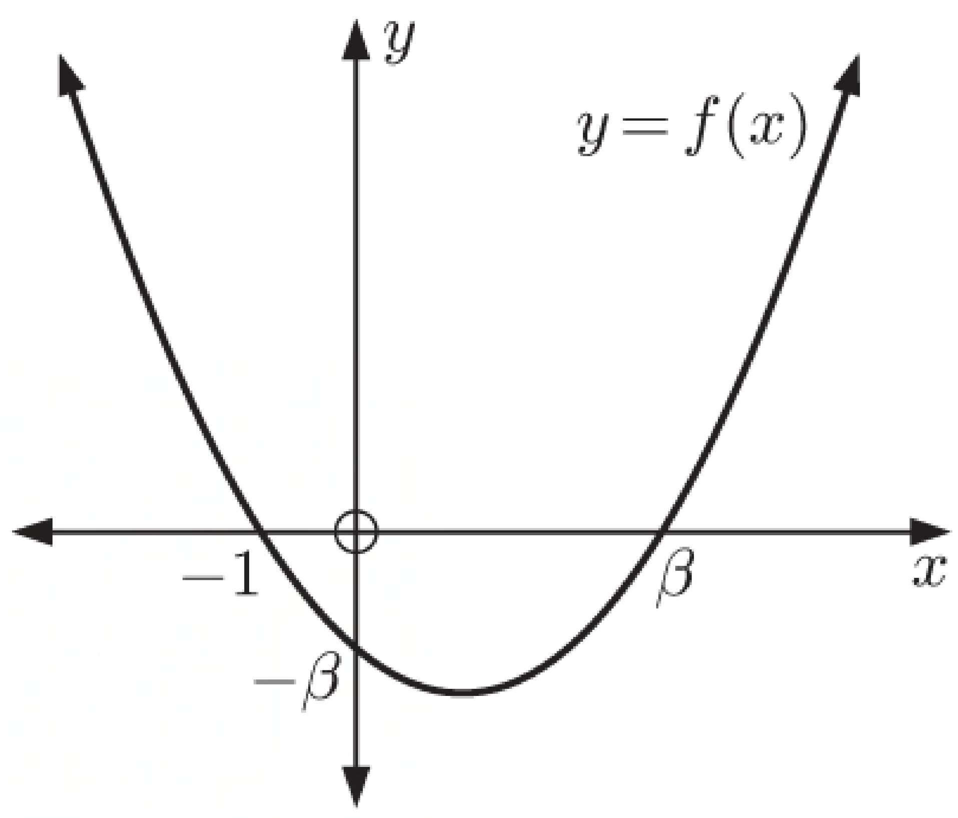
$\therefore$ the x-intercepts are -1 and β.

$f(0) = (0+1)(0-\beta)$

$= (1)(-\beta)$

$= -\beta$

$\therefore$ the y-intercept is $-\beta$.



b To transform $f(x)$ to $g(x) = -f(x-1)$, we reflect $y = f(x)$ in the x-axis, then translate it through $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$.

c The x-intercepts of $y = g(x)$ are the x-intercepts of $y = f(x)$ translated 1 unit to the right.

$\therefore$ the x-intercepts are 0 and $\beta + 1$.

Now $g(0) = -f(0-1)$

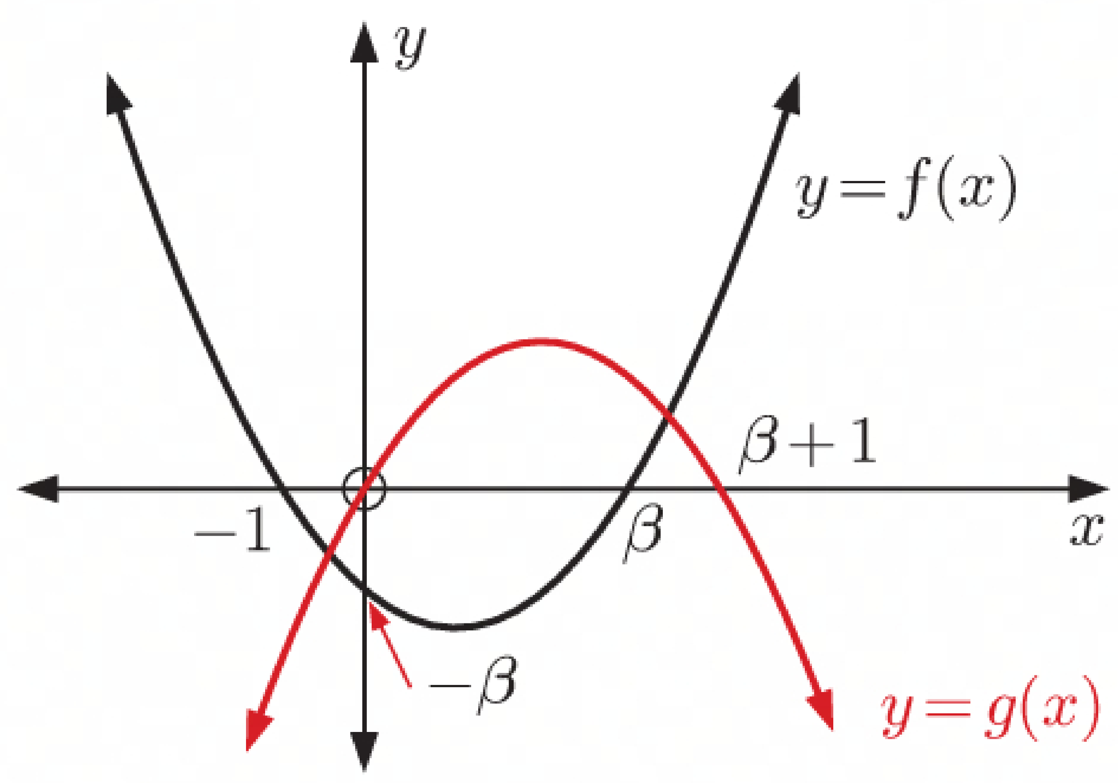
$= -f(-1)$

$= -(-1+1)(-1-\beta)$

$= -(0)(-1-\beta)$

$= 0$

$\therefore$ the y-intercept is 0.



7 **a** $f(x) \xrightarrow{\text{translation } \begin{pmatrix} 4 \\ -1 \end{pmatrix}} f(x-4) - 1 \xrightarrow{\text{reflection in } y\text{-axis}} f(-x-4) - 1$

The resulting function is $f(-x-4) - 1$.

b $f(x) \xrightarrow{\text{reflection in } y\text{-axis}} f(-x) \xrightarrow{\text{translation } \begin{pmatrix} 4 \\ -1 \end{pmatrix}} f(-(x-4)) - 1$

The resulting function is $f(-x+4) - 1$.

c $f(x) \xrightarrow{\text{translation } \begin{pmatrix} -2 \\ 1 \end{pmatrix}} f(x+2)+1 \xrightarrow{\substack{\text{vertical stretch} \\ \text{scale factor } \frac{1}{2}}} \frac{1}{2}(f(x+2)+1)$

The resulting function is $\frac{1}{2}f(x+2)+\frac{1}{2}$.

d $f(x) \xrightarrow{\substack{\text{vertical stretch} \\ \text{scale factor } \frac{1}{2}}} \frac{1}{2}f(x) \xrightarrow{\text{translation } \begin{pmatrix} -2 \\ 1 \end{pmatrix}} \frac{1}{2}f(x+2)+1$

The resulting function is $\frac{1}{2}f(x+2)+1$.

e $f(x) \xrightarrow{\text{translation } \begin{pmatrix} 3 \\ -5 \end{pmatrix}} f(x-3)-5 \xrightarrow{\substack{\text{horizontal stretch} \\ \text{scale factor } 4}} f(\frac{1}{4}x-3)-5$

The resulting function is $f(\frac{1}{4}x-3)-5$.

f $f(x) \xrightarrow{\substack{\text{horizontal stretch} \\ \text{scale factor } 4}} f(\frac{1}{4}x) \xrightarrow{\text{translation } \begin{pmatrix} 3 \\ -5 \end{pmatrix}} f(\frac{1}{4}(x-3))-5$

The resulting function is $f\left(\dfrac{x-3}{4}\right)-5$.

8 a $f(x) \xrightarrow{\substack{\text{reflection} \\ \text{in } x\text{-axis}}} -f(x) \xrightarrow{\text{translation } \begin{pmatrix} -1 \\ 3 \end{pmatrix}} -f(x+1)+3$

A reflection in the x-axis, then a translation through $\begin{pmatrix} -1 \\ 3 \end{pmatrix}$ maps $y=f(x)$ onto $y=-f(x+1)+3$.

b $f(x) \xrightarrow{\substack{\text{horizontal stretch} \\ \text{scale factor } 2}} f(\frac{1}{2}x) \xrightarrow{\text{translation } \begin{pmatrix} 0 \\ -7 \end{pmatrix}} f(\frac{1}{2}x)-7$

A horizontal stretch with scale factor 2, then a translation through $\begin{pmatrix} 0 \\ -7 \end{pmatrix}$ maps $y=f(x)$ onto $y=f(\frac{1}{2}x)-7$.

c $f(x) \xrightarrow{\text{translation } \begin{pmatrix} 1 \\ 0 \end{pmatrix}} f(x-1) \xrightarrow{\substack{\text{horizontal stretch} \\ \text{scale factor } \frac{1}{3}}} f(3x-1)$

A translation through $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$, then a horizontal stretch with scale factor $\frac{1}{3}$ maps $y=f(x)$ onto $y=f(3x-1)$.

d $f(x) \xrightarrow[\text{scale factor } 2]{\text{vertical stretch}} 2f(x) \xrightarrow{\text{translation } \binom{1}{-1}} -1+2f(x-1) \xrightarrow[\text{scale factor } 4]{\text{horizontal stretch}} -1+2f(\frac{1}{4}x-1)$

A vertical stretch with scale factor 2, a translation through $\binom{1}{-1}$, then a horizontal stretch with scale factor 4 maps $y = f(x)$ onto $y = -1 + 2f(\frac{1}{4}x - 1)$.

e $f(x) \xrightarrow[\text{scale factor } 2]{\text{vertical stretch}} 2\,f(x) \xrightarrow[\text{scale factor } \frac{1}{3}]{\text{horizontal stretch}} 2\,f(3x) \xrightarrow{\text{translation } \binom{1}{5}} 5 + 2\,f(3(x-1))$

A vertical stretch with scale factor 2, a horizontal stretch with scale factor $\frac{1}{3}$, then a translation through $\binom{1}{5}$ maps $y = f(x)$ onto $y = 5 + 2\,f(3(x-1))$.

f

$f(x) \xrightarrow[\text{in } x\text{-axis}]{\text{reflection}} -f(x) \xrightarrow[\text{scale factor } 4]{\text{vertical stretch}} -4\,f(x) \xrightarrow[\text{scale factor } 2]{\text{horizontal stretch}} -4\,f(\frac{1}{2}x) \xrightarrow{\text{translation } \binom{-3}{-1}} -4\,f(\frac{1}{2}(x+3))-1$

A reflection in the x-axis, a vertical stretch with scale factor 4, a horizontal stretch with scale factor 2, then a translation through $\binom{-3}{-1}$ maps $y = f(x)$ onto $y = -4\,f(\frac{1}{2}(x+3)) - 1$.

9 $f(x)$ has domain $\{x \mid x \geqslant 1\}$ and range $\{y \mid -2 \leqslant y < 5\}$.

a $g(x) = f(x+4) - 1$ translates every point on $y = f(x)$ 4 units to the left and 1 unit downwards.

$\therefore$ $g(x)$ has domain $\{x \mid x \geqslant -3\}$ and range $\{y \mid -3 \leqslant y < 4\}$.

b $f(x) \xrightarrow[\text{in } x\text{-axis}]{\text{reflection}} -f(x) \xrightarrow[\text{scale factor } 2]{\text{vertical stretch}} -2\,f(x) \xrightarrow[\text{scale factor } \frac{1}{3}]{\text{horizontal stretch}} -2\,f(3x)$

$\therefore$ $-f(x)$ has domain $\{x \mid x \geqslant 1\}$ and range $\{y \mid -5 < y \leqslant 2\}$.

$\therefore$ $-2\,f(x)$ has domain $\{x \mid x \geqslant 1\}$ and range $\{y \mid -10 < y \leqslant 4\}$.

$\therefore$ $g(x) = -2\,f(3x)$ has domain $\{x \mid x \geqslant \frac{1}{3}\}$ and range $\{y \mid -10 < y \leqslant 4\}$.

c $f(x) \xrightarrow[\text{scale factor } \frac{1}{3}]{\text{vertical stretch}} \frac{1}{3}\,f(x) \xrightarrow{\text{translation } \binom{5}{4}} \frac{1}{3}\,f(x-5)+4 \xrightarrow[\text{scale factor } \frac{1}{2}]{\text{horizontal stretch}} \frac{1}{3}\,f(2x-5)+4$

$\therefore$ $\frac{1}{3}\,f(x)$ has domain $\{x \mid x \geqslant 1\}$ and range $\{y \mid -\frac{2}{3} \leqslant y < \frac{5}{3}\}$.

$\therefore$ $\frac{1}{3}\,f(x-5)+4$ has domain $\{x \mid x \geqslant 6\}$ and range $\{y \mid \frac{10}{3} \leqslant y < \frac{17}{3}\}$.

$\therefore$ $g(x) = \frac{1}{3}\,f(2x-5)+4$ has domain $\{x \mid x \geqslant 3\}$ and range $\{y \mid \frac{10}{3} \leqslant y < \frac{17}{3}\}$.

10 T_A: a translation through $\begin{pmatrix} -2 \\ 3 \end{pmatrix}$

T_B: a reflection in the y-axis

T_C: a vertical stretch with scale factor 5

$f(x) = \sqrt{x}$ has domain $\{x \mid x \geqslant 0\}$ and range $\{y \mid y \geqslant 0\}$.

a $\sqrt{x} \xrightarrow{T_A} \sqrt{x+2}+3 \xrightarrow{T_B} \sqrt{2-x}+3 \xrightarrow{T_C} 5\sqrt{2-x}+15$

The resulting function is $5\sqrt{2-x}+15$.

$\sqrt{x+2}+3$ has domain $\{x \mid x \geqslant -2\}$ and range $\{y \mid y \geqslant 3\}$.

$\therefore$ $\sqrt{2-x}+3$ has domain $\{x \mid x \leqslant 2\}$ and range $\{y \mid y \geqslant 3\}$.

$\therefore$ $5\sqrt{2-x}+15$ has domain $\{x \mid x \leqslant 2\}$ and range $\{y \mid y \geqslant 15\}$.

b $\sqrt{x} \xrightarrow{T_C} 5\sqrt{x} \xrightarrow{T_A} 5\sqrt{x+2}+3 \xrightarrow{T_B} 5\sqrt{2-x}+3$

The resulting function is $5\sqrt{2-x}+3$.

$5\sqrt{x}$ has domain $\{x \mid x \geqslant 0\}$ and range $\{y \mid y \geqslant 0\}$.

$\therefore$ $5\sqrt{x+2}+3$ has domain $\{x \mid x \geqslant -2\}$ and range $\{y \mid y \geqslant 3\}$.

$\therefore$ $5\sqrt{2-x}+3$ has domain $\{x \mid x \leqslant 2\}$ and range $\{y \mid y \geqslant 3\}$.

c $\sqrt{x} \xrightarrow{T_C} 5\sqrt{x} \xrightarrow{T_B} 5\sqrt{-x} \xrightarrow{T_A} 5\sqrt{-(x+2)}+3$

The resulting function is $5\sqrt{-x-2}+3$.

$5\sqrt{x}$ has domain $\{x \mid x \geqslant 0\}$ and range $\{y \mid y \geqslant 0\}$.

$\therefore$ $5\sqrt{-x}$ has domain $\{x \mid x \leqslant 0\}$ and range $\{y \mid y \geqslant 0\}$.

$\therefore$ $5\sqrt{-x-2}+3$ has domain $\{x \mid x \leqslant -2\}$ and range $\{y \mid y \geqslant 3\}$.

11 **a** The graph is stretched vertically with scale factor $|\,a\,|$, and reflected in the x-axis. It is then translated h units horizontally and k units vertically.

b The function has shape  after it is reflected in the x-axis.

The function has vertex $(h,\, k)$, and y-intercept $ah^2 + k$.

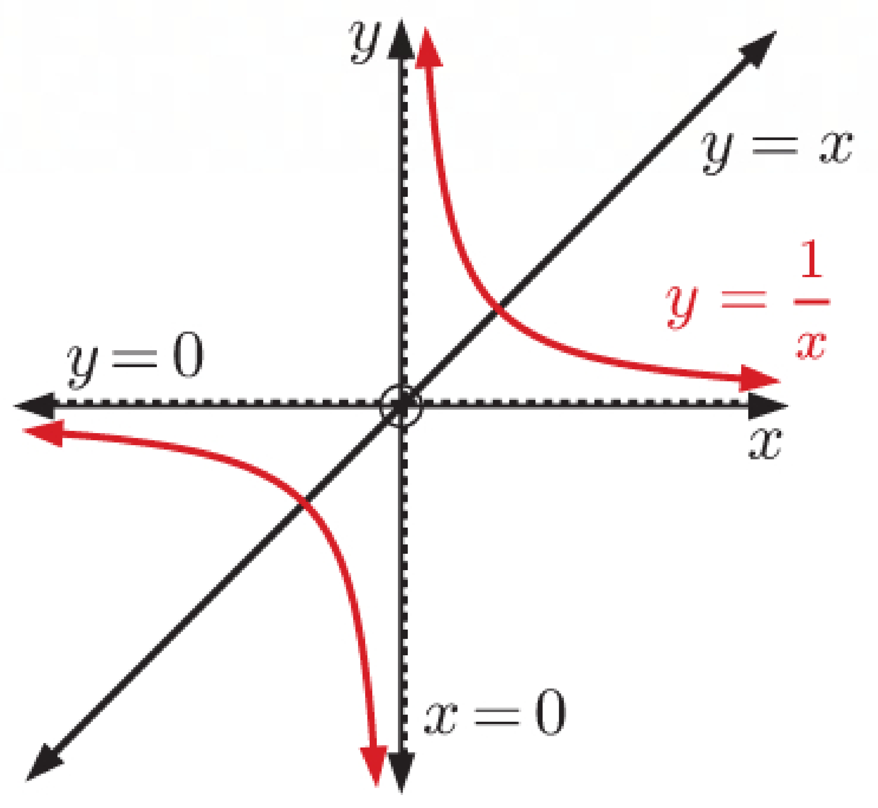
12 **a** $$\frac{10x+11}{2x+3} = \frac{5(2x+3)-4}{2x+3}$$
$$= 5 + \frac{-4}{2x+3}$$

b $$\frac{1}{x} \xrightarrow[\text{in } x\text{-axis}]{\text{reflection}} -\frac{1}{x} \xrightarrow[\text{scale factor } 4]{\text{vertical stretch}} \frac{-4}{x} \xrightarrow{\text{translation } \begin{pmatrix} -3 \\ 5 \end{pmatrix}} 5 + \frac{-4}{x+3} \xrightarrow[\text{scale factor } \frac{1}{2}]{\text{horizontal stretch}} 5 + \frac{-4}{2x+3}$$

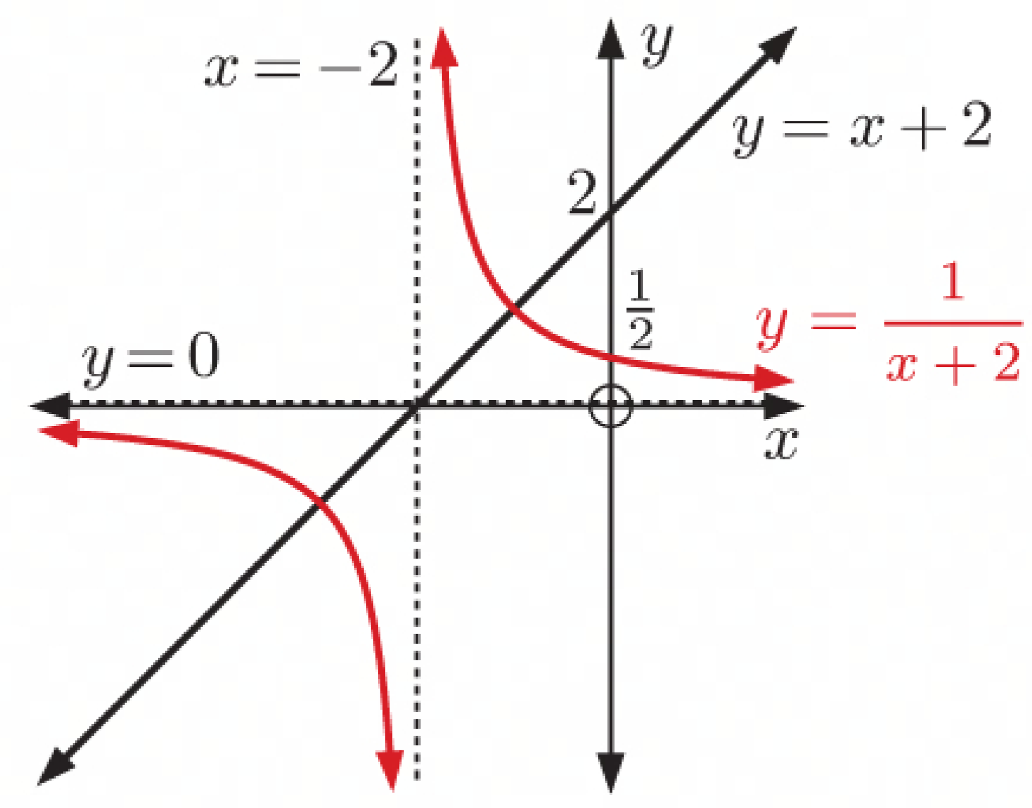
A reflection in the x-axis, a vertical stretch with scale factor 4, a translation through $\begin{pmatrix} -3 \\ 5 \end{pmatrix}$, then a horizontal stretch with scale factor $\frac{1}{2}$ maps $y = \dfrac{1}{x}$ onto $y = \dfrac{10x+11}{2x+3}$.

INVESTIGATION 4 THE GRAPH OF $y = \frac{1}{f(x)}$

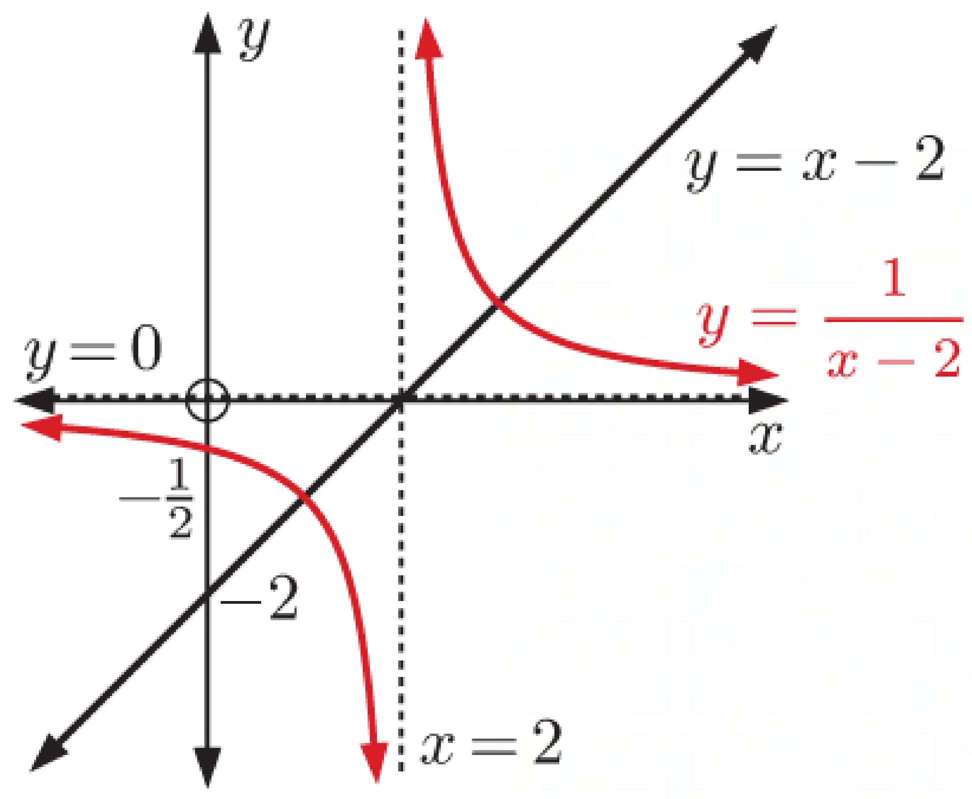
1 **a**



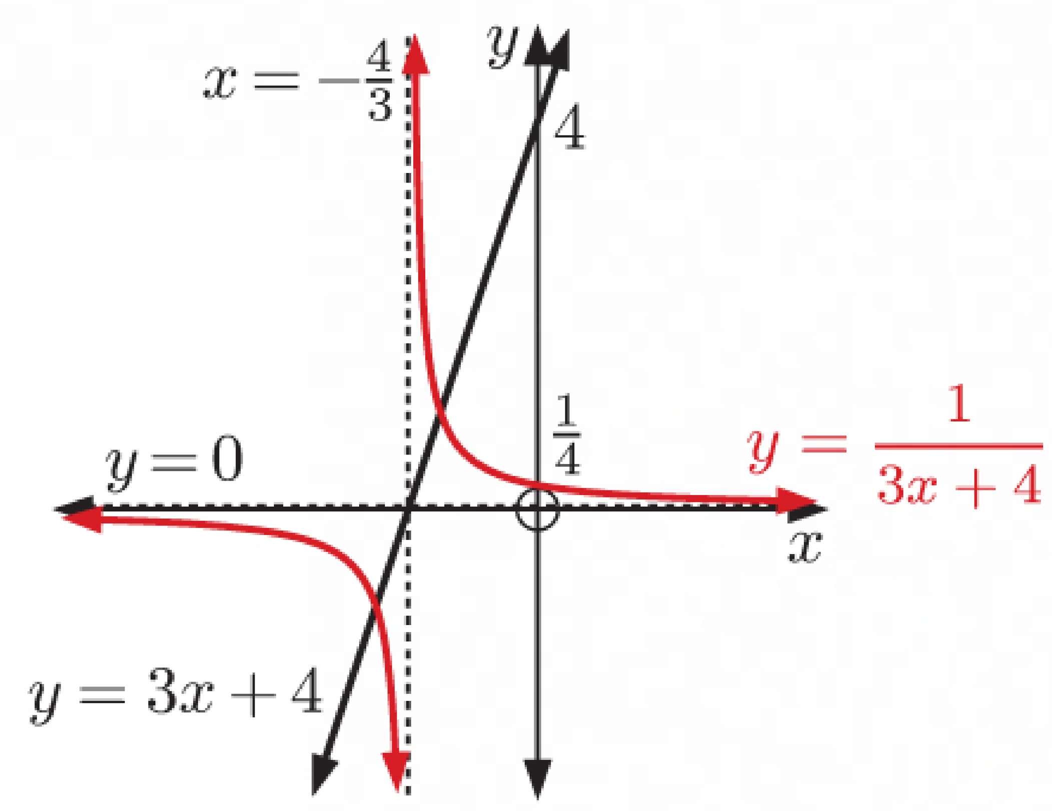
b



c

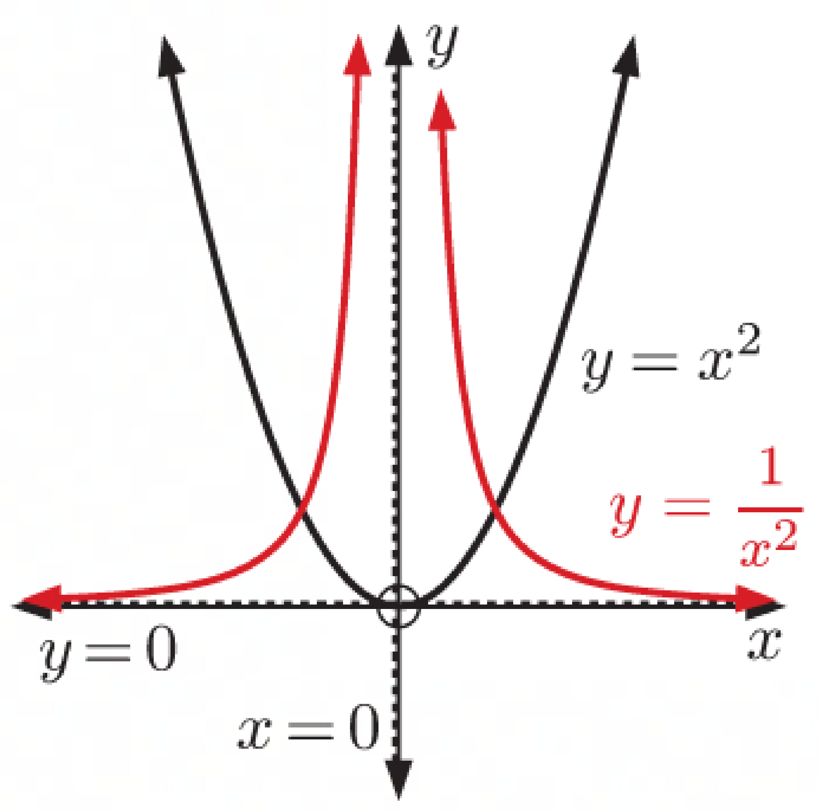


d

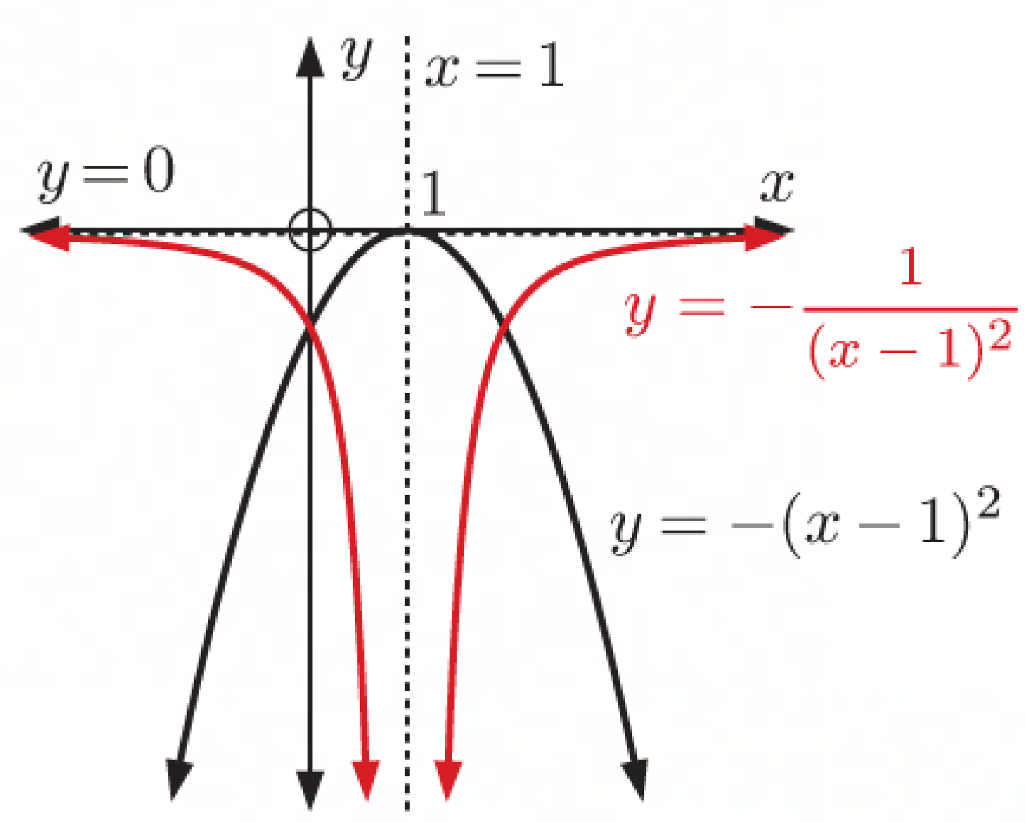


If $f(x)$ is a linear function with x-intercept a, then $\frac{1}{f(x)}$ has a vertical asymptote $x = a$ and a horizontal asymptote $y = 0$.

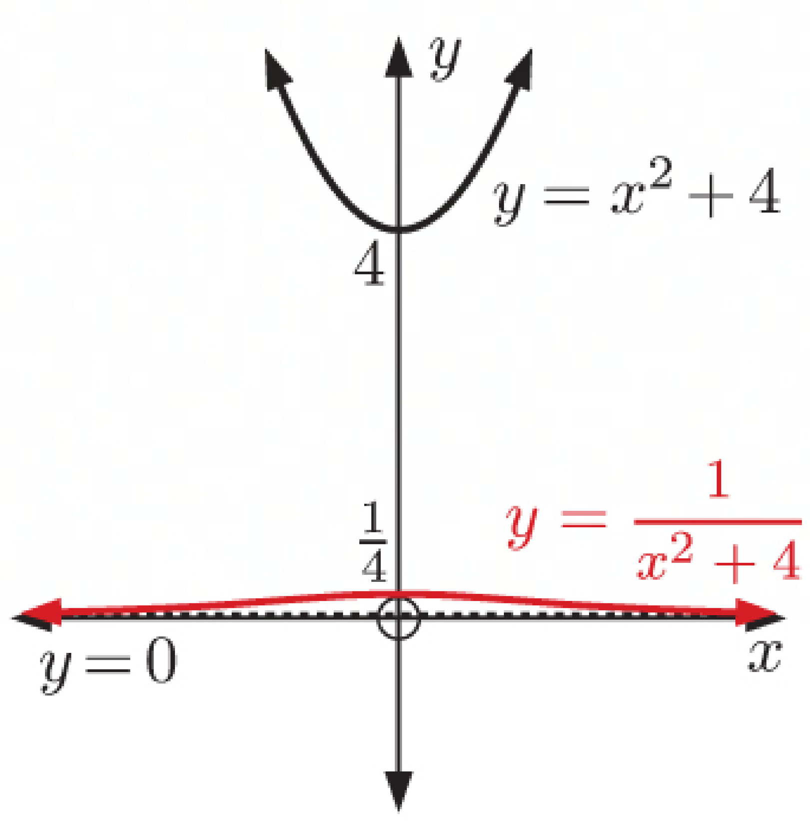
2 **a** **i**



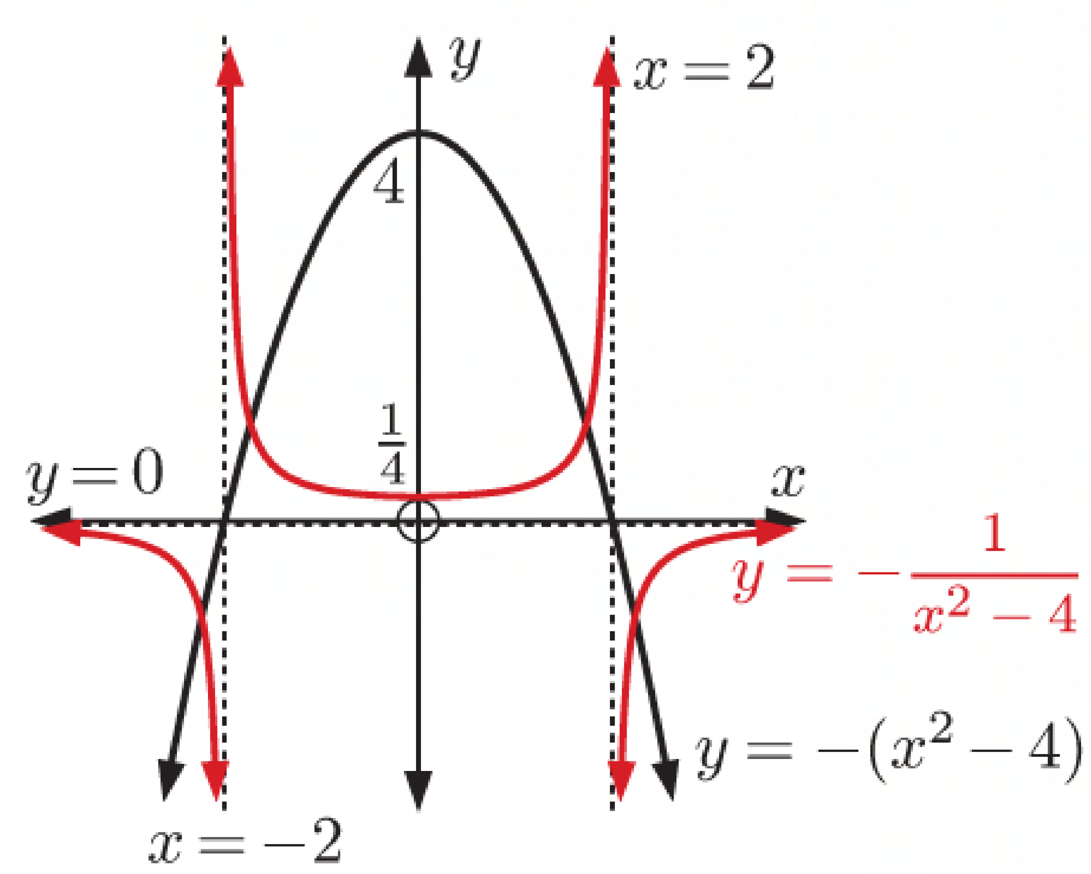
ii



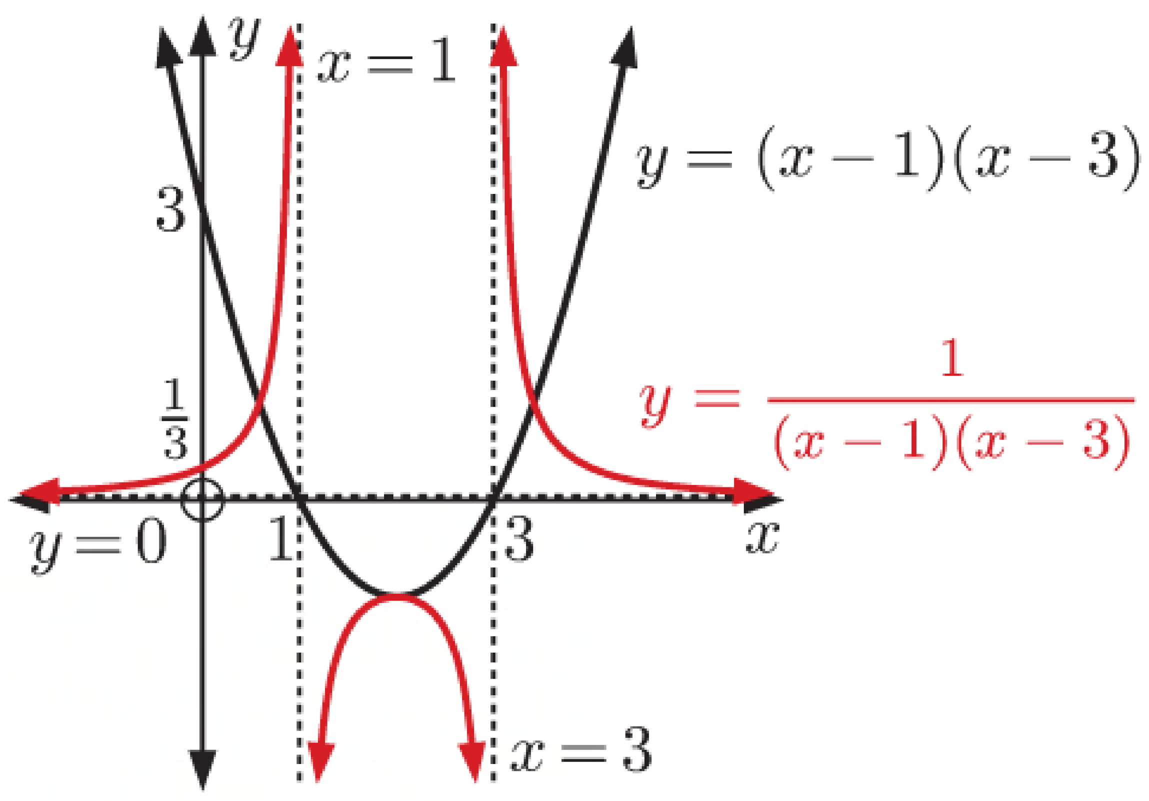
iii



iv



v



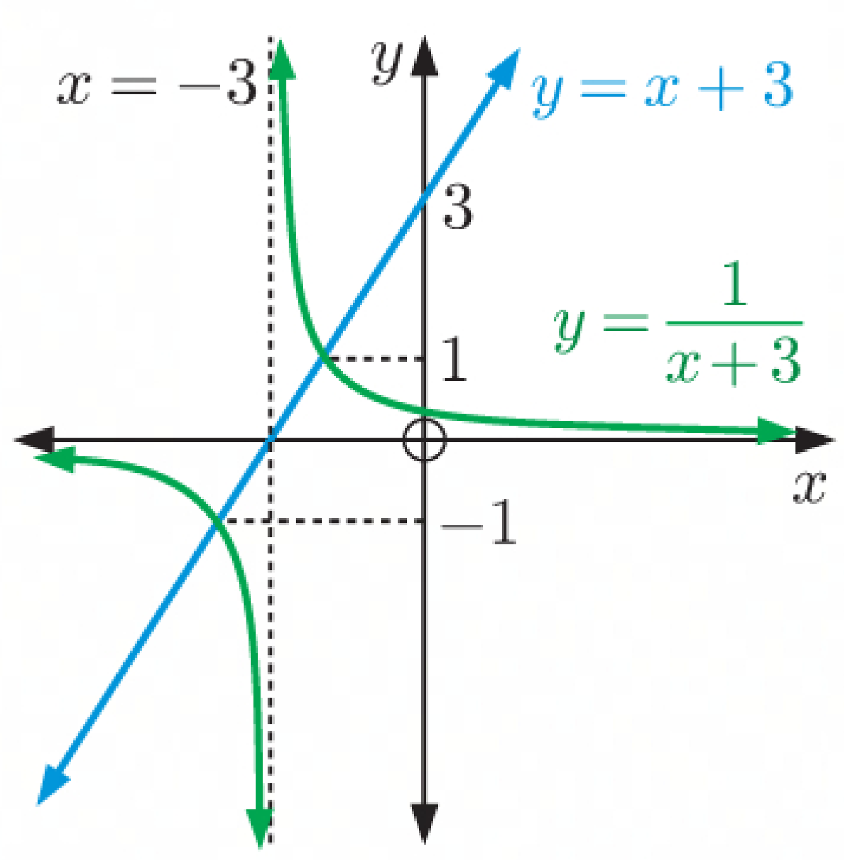
b The zeros of $y = f(x)$ become vertical asymptotes of $y = \dfrac{1}{f(x)}$.

c We observe that:

- vertical asymptotes of $y = f(x)$ become zeros of $y = \dfrac{1}{f(x)}$
- local maxima of $y = f(x)$ which are not zeros correspond to local minima of $y = \dfrac{1}{f(x)}$
- local minima of $y = f(x)$ which are not zeros correspond to local maxima of $y = \dfrac{1}{f(x)}$
- when $f(x) > 0$, $\dfrac{1}{f(x)} > 0$ and when $f(x) < 0$, $\dfrac{1}{f(x)} < 0$
- when $f(x) \to 0$, $\dfrac{1}{f(x)} \to \pm\infty$ and when $f(x) \to \pm\infty$, $\dfrac{1}{f(x)} \to 0$.

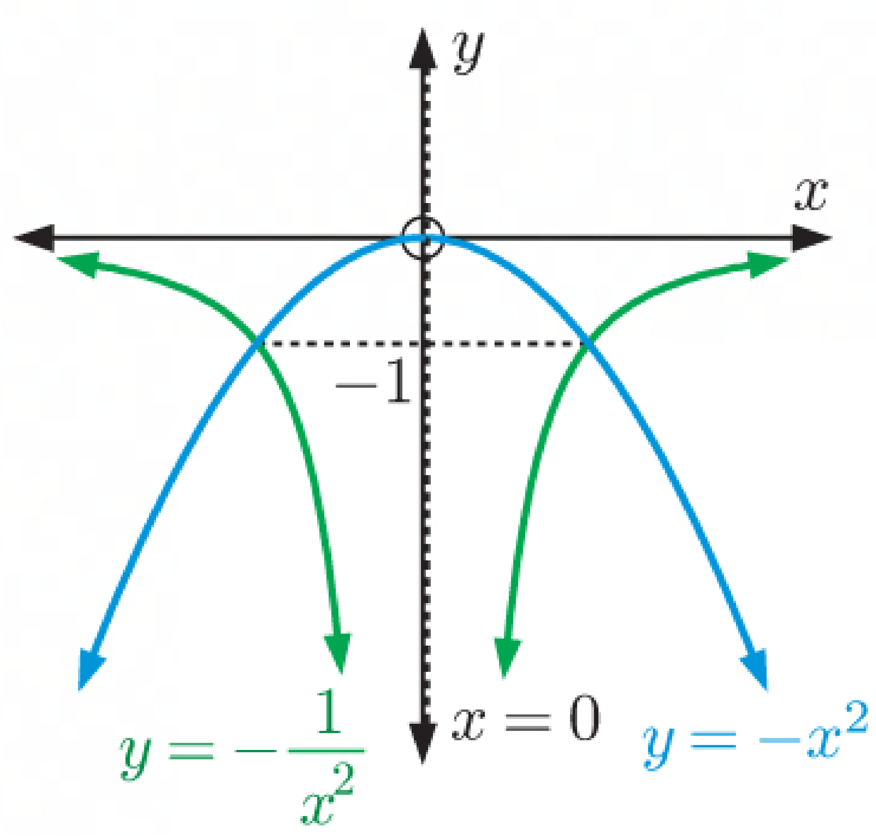
EXERCISE 16E

1 **a** $y = x + 3$ has x-intercept -3, so $y = \dfrac{1}{x+3}$ has vertical asymptote $x = -3$.

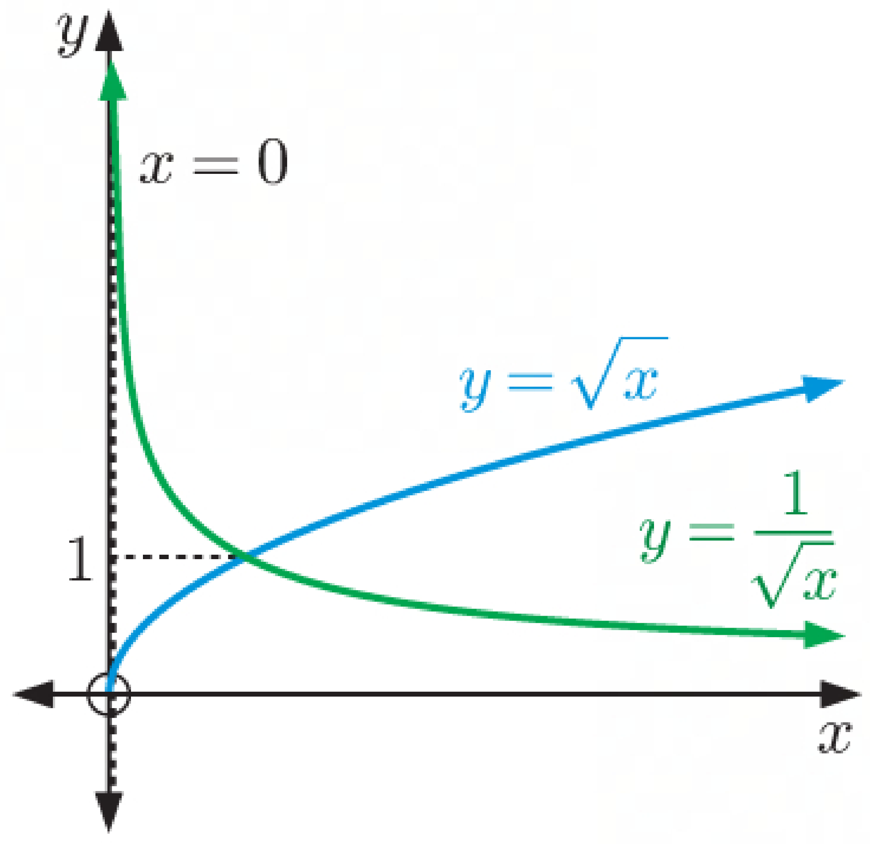


b $y = -x^2$ has x-intercept 0, so $y = -\dfrac{1}{x^2}$ has vertical asymptote $x = 0$.

$y = -x^2$ has a local maximum at $(0, 0)$, but $y = -\dfrac{1}{x^2}$ is undefined at $x = 0$.

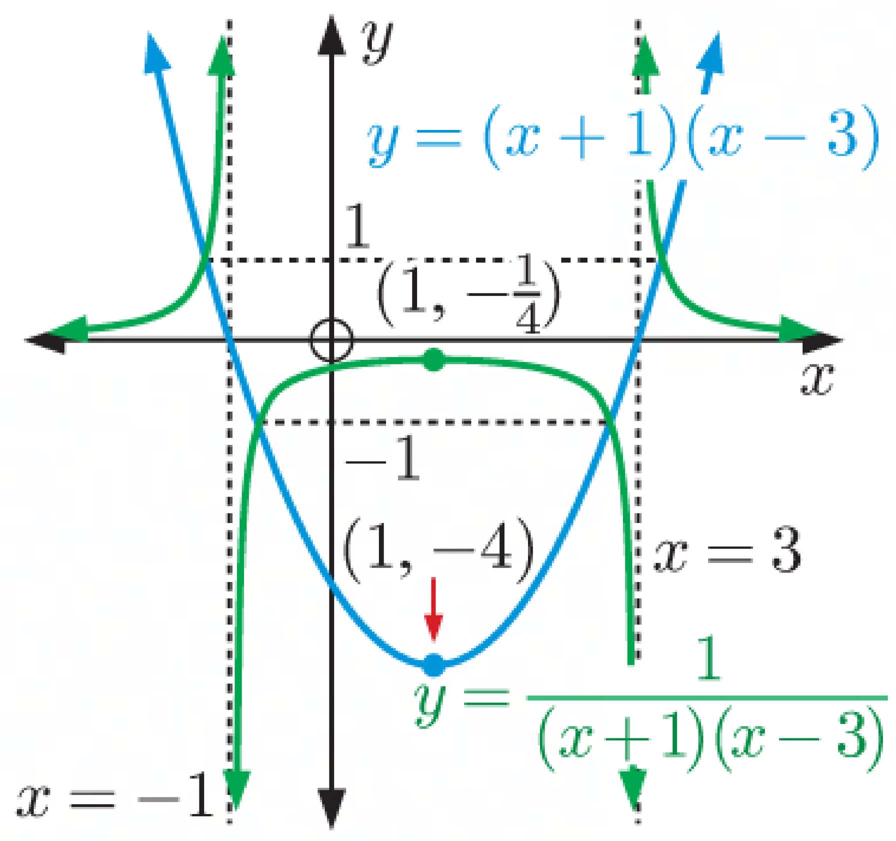


c $y = \sqrt{x}$ has x-intercept 0, so $y = \dfrac{1}{\sqrt{x}}$ has vertical asymptote $x = 0$.



d $y = (x+1)(x-3)$ has x-intercepts -1 and 3, so $y = \dfrac{1}{(x+1)(x-3)}$ has vertical asymptotes $x = -1$ and $x = 3$.

$y = (x+1)(x-3)$ has a local minimum at $(1, -4)$, so $y = \dfrac{1}{(x+1)(x-3)}$ has a local maximum at $(1, -\frac{1}{4})$.



2 If $f(x) = \dfrac{1}{f(x)}$ then $y = \dfrac{1}{y}$

$$\therefore\ y^2 = 1$$
$$\therefore\ y = \pm 1$$

For **1 a**: When $y = 1$, $x + 3 = 1$ $\therefore\ x = -2$ When $y = -1$, $x + 3 = -1$ $\therefore\ x = -4$

So, the invariant points are $(-2, 1)$ and $(-4, -1)$. ✓

For **1 b**: When $y = 1$, $-x^2 = 1$ which has no real solutions. When $y = -1$, $-x^2 = -1$ $\therefore\ x^2 = 1$ $\therefore\ x = \pm 1$

So, the invariant points are $(1, -1)$ and $(-1, -1)$. ✓

For **1 c**: When $y = 1$, $\sqrt{x} = 1$ $\therefore\ x = 1$ When $y = -1$, $\sqrt{x} = -1$ which has no real solutions.

So, the invariant point is $(1, 1)$. ✓

For **1 d**:

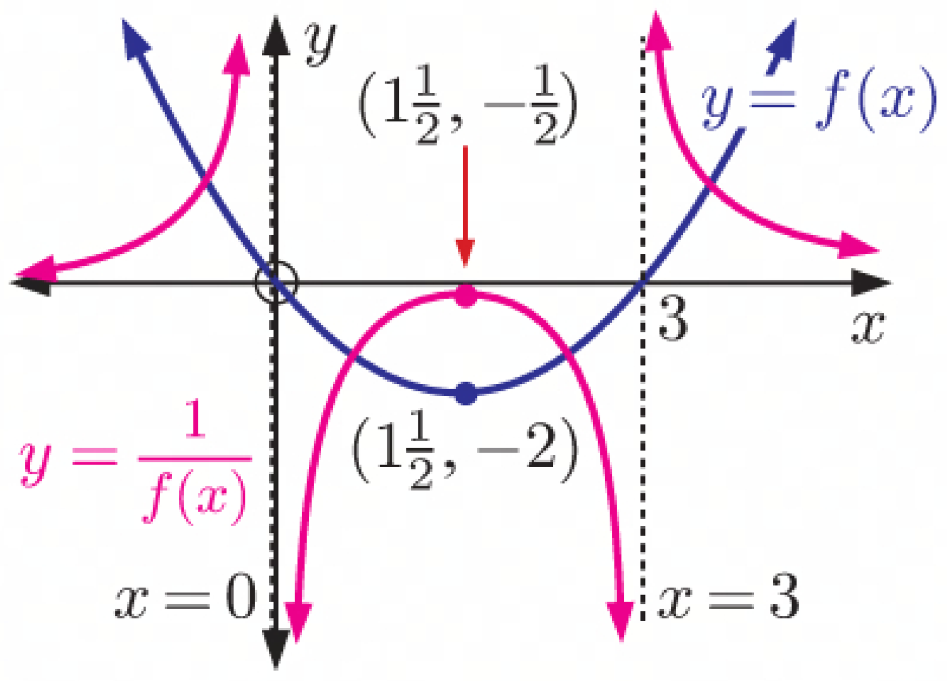
When $y = 1$,
$$(x+1)(x-3) = 1$$
$$\therefore\ x^2 - 2x - 3 = 1$$
$$\therefore\ x^2 - 2x - 4 = 0$$
$$\therefore\ x = \frac{2 \pm \sqrt{20}}{2} = 1 \pm \sqrt{5}$$
$$\approx 3.24 \text{ or } -1.24$$

When $y = -1$,
$$(x+1)(x-3) = -1$$
$$\therefore\ x^2 - 2x - 3 = -1$$
$$\therefore\ x^2 - 2x - 2 = 0$$
$$\therefore\ x = \frac{2 \pm \sqrt{12}}{2} = 1 \pm \sqrt{3}$$
$$\approx 2.73 \text{ or } -0.732$$

So, the invariant points are $(-1.24, 1)$, $(-0.732, -1)$, $(2.73, -1)$, $(3.24, 1)$. ✓

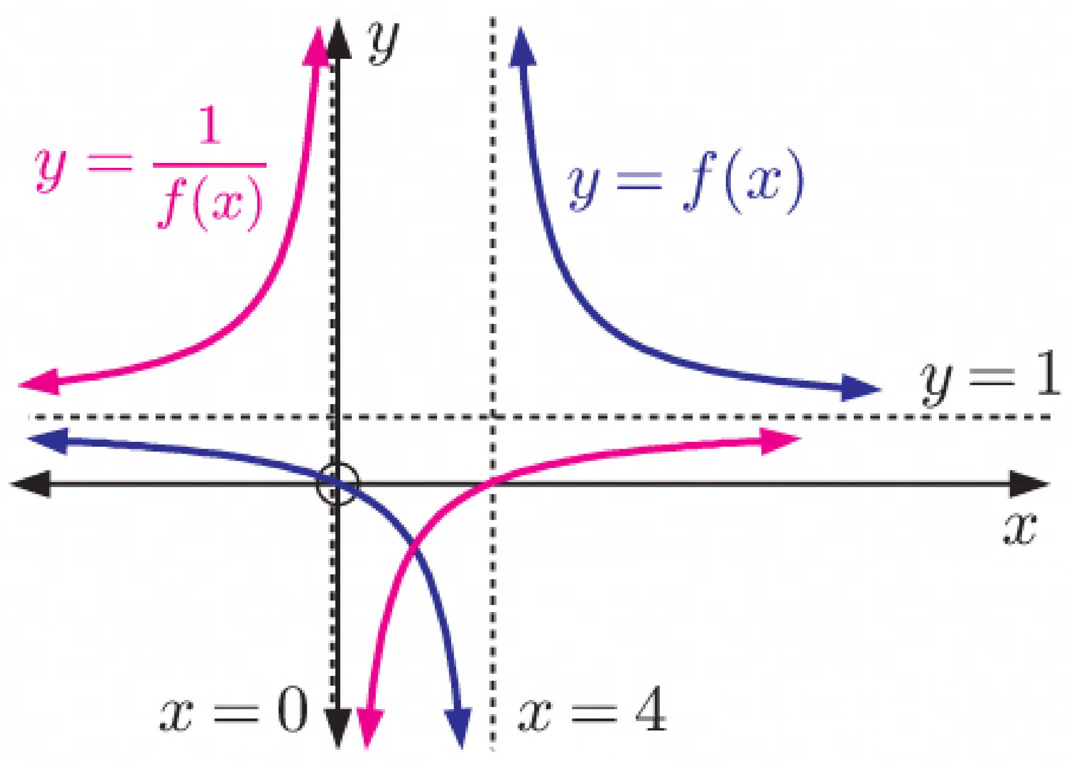
3 **a** $y = f(x)$ has x-intercepts 0 and 3, so $y = \dfrac{1}{f(x)}$ has vertical asymptotes $x = 0$ and $x = 3$.

$y = f(x)$ has a local minimum at $(1\frac{1}{2}, -2)$, so $y = \dfrac{1}{f(x)}$ has a local maximum at $(1\frac{1}{2}, -\frac{1}{2})$.



b $y = f(x)$ has x-intercept 0, so $y = \dfrac{1}{f(x)}$ has vertical asymptote $x = 0$.

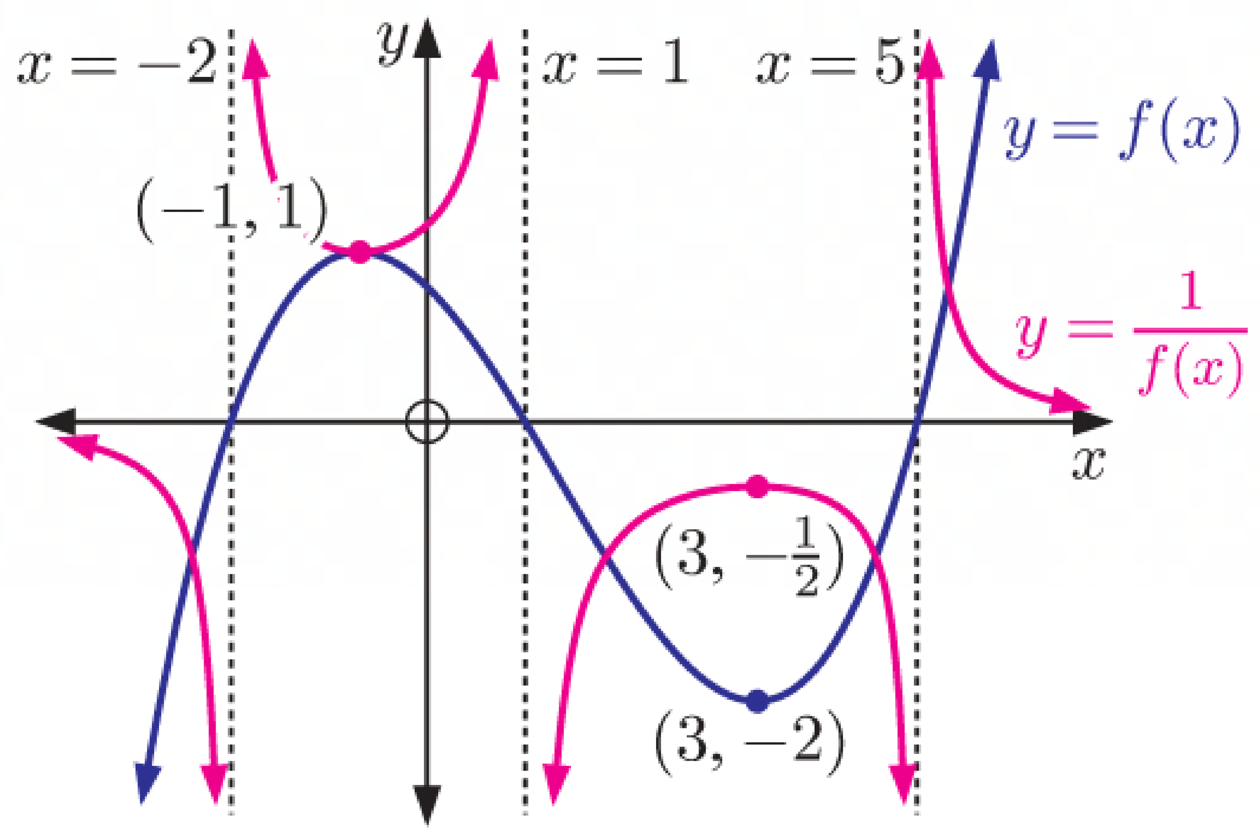
$y = f(x)$ has vertical asymptote $x = 4$, so $y = \dfrac{1}{f(x)}$ has x-intercept 4.



c $y = f(x)$ has x-intercepts -2, 1, and 5, so $y = \dfrac{1}{f(x)}$ has vertical asymptotes $x = -2$, $x = 1$, and $x = 5$.

$y = f(x)$ has a local maximum $(-1, 1)$, so $y = \dfrac{1}{f(x)}$ has a local minimum $(-1, \frac{1}{2})$.

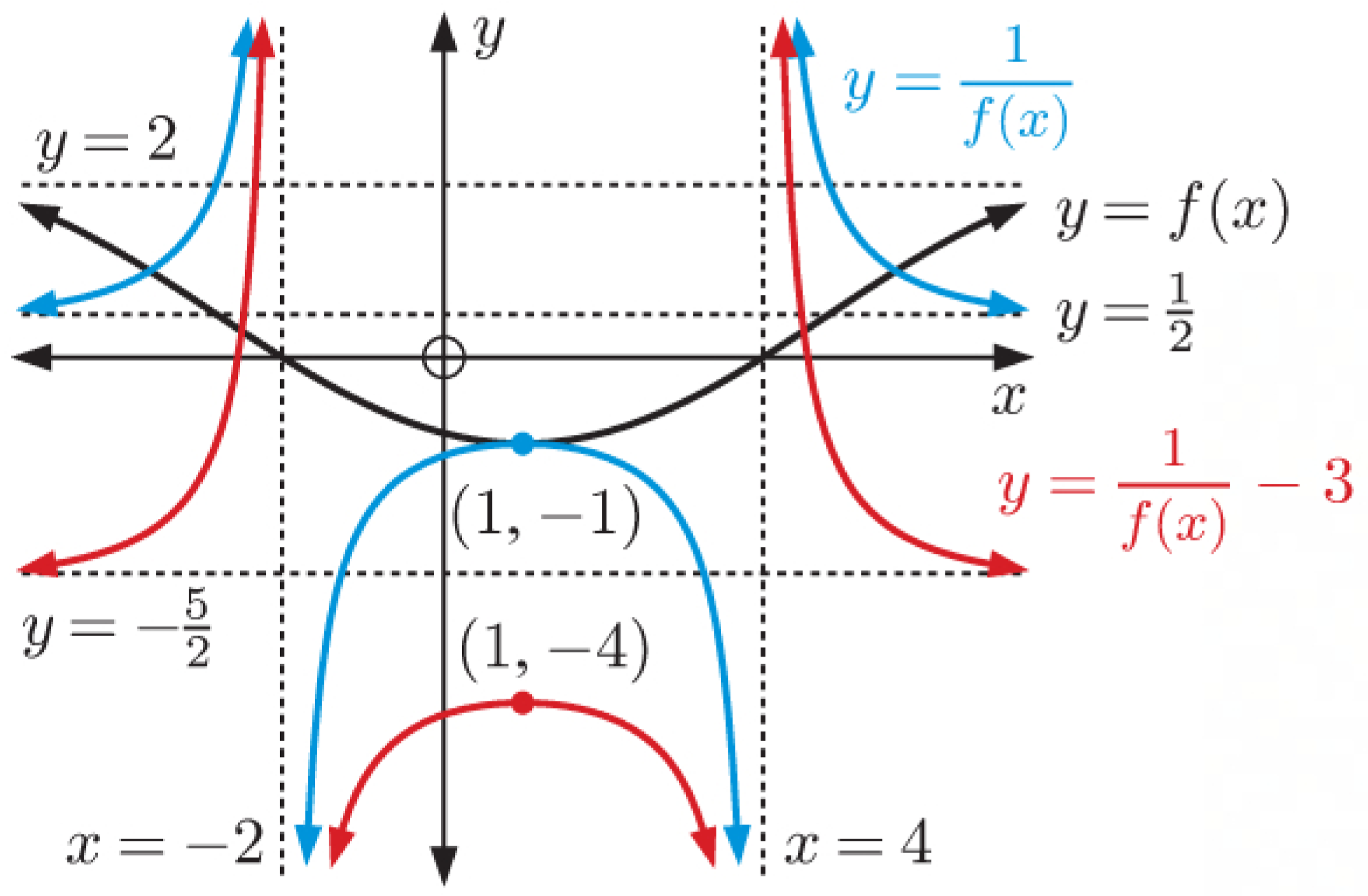
$y = f(x)$ has a local minimum $(3, -2)$, so $y = \dfrac{1}{f(x)}$ has a local maximum $(3, -\frac{1}{2})$.



4 $y = f(x)$ has x-intercepts -2 and 4, so $y = \dfrac{1}{f(x)}$ and $y = \dfrac{1}{f(x)} - 3$ have vertical asymptotes $x = -2$ and $x = 4$.

$y = f(x)$ has horizontal asymptote $y = 2$, so $y = \dfrac{1}{f(x)}$ has horizontal asymptote $y = \frac{1}{2}$, and $y = \dfrac{1}{f(x)} - 3$ has horizontal asymptote $y = -\frac{5}{2}$.

$y = f(x)$ has a local minimum $(1, -1)$, so $y = \dfrac{1}{f(x)}$ has a local maximum $(1, -1)$, and $y = \dfrac{1}{f(x)} - 3$ has a local maximum $(1, -4)$.



5 **a** $f(x) = x^2 + 4x + 3$

$= (x+3)(x+1)$ which has shape 

$f(x) = 0$ when $x = -3$ or -1

$\therefore$ the x-intercepts are -3 and -1.

Now, $f(0) = (0+3)(0+1) = 3$

$\therefore$ the y-intercept is 3.

The vertex lies on the axis of symmetry, half way between $x = -3$ and $x = -1$.

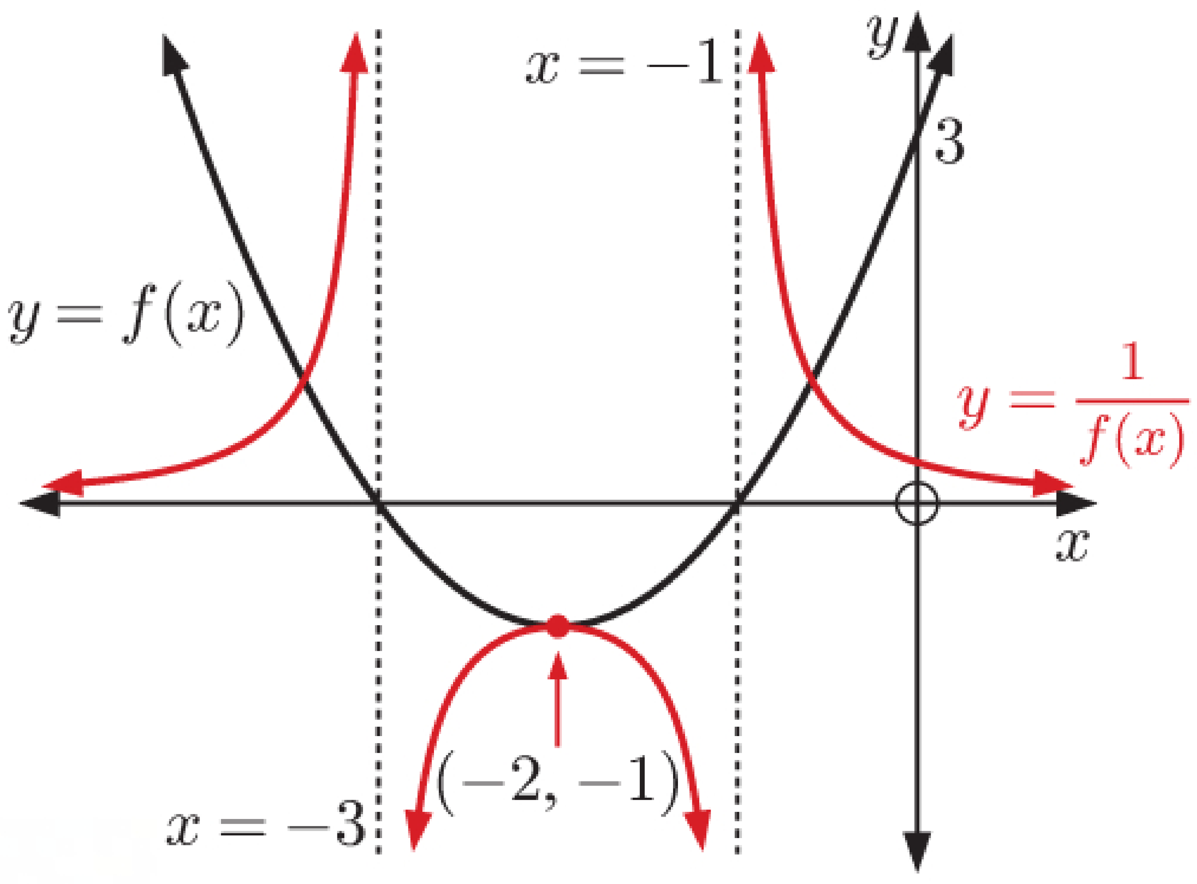
$\therefore$ the vertex has x-coordinate -2.

Now $f(-2) = (-2+3)(-2+1)$
$= (1)(-1)$
$= -1$

So, the vertex (and local minimum) is $(-2, -1)$.

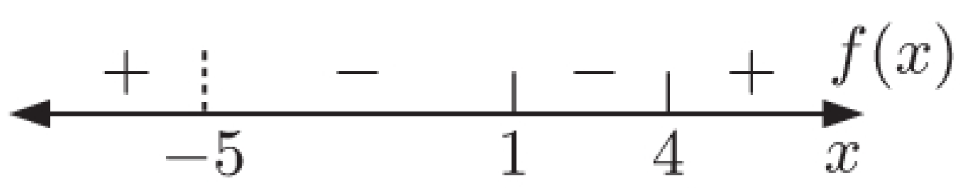
b $y = f(x)$ has x-intercepts -3 and -1, so $y = \dfrac{1}{f(x)}$ has vertical asymptotes $x = -3$ and $x = -1$.

$y = f(x)$ has a local minimum $(-2, -1)$, so $y = \dfrac{1}{f(x)}$ has a local maximum $(-2, -1)$.



c
$$\frac{1}{f(x)} = \frac{4}{21}$$
$$\therefore\ f(x) = \tfrac{21}{4}$$
$$\therefore\ x^2 + 4x + 3 = \tfrac{21}{4}$$
$$\therefore\ x^2 + 4x + 2^2 - 2^2 + 3 = \tfrac{21}{4}$$
$$\therefore\ (x+2)^2 - 4 + 3 = \tfrac{21}{4}$$
$$\therefore\ (x+2)^2 = \tfrac{25}{4}$$
$$\therefore\ x + 2 = \pm\tfrac{5}{2}$$
$$\therefore\ x = -2 \pm \tfrac{5}{2}$$
$$\therefore\ x = -\tfrac{9}{2} \text{ or } \tfrac{1}{2}$$

6 $y = f(x)$ has x-intercepts 1 and 4, so $y = \dfrac{1}{f(x)}$ has vertical asymptotes $x = 1$ and $x = 4$.



$y = f(x)$ has vertical asymptote $x = -5$, so $y = \dfrac{1}{f(x)}$ has x-intercept -5.

The signs of $y = f(x)$ and $y = \dfrac{1}{f(x)}$ are the same on each interval.

$\therefore$ $\dfrac{1}{f(x)}$ has sign diagram 

7 **a** $f(x)$ is always positive, so $\dfrac{1}{f(x)}$ is always a defined value and hence has the same domain as $f(x)$.

$\therefore$ the domain of $\dfrac{1}{f(x)}$ is $\{x \mid -1 \leqslant x \leqslant 6\}$.

Now $2 \leqslant y < 5$, so $y \geqslant 2$ and $y < 5$

$\therefore \dfrac{1}{y} \leqslant \dfrac{1}{2}$ $\{y > 0\}$ $\quad\therefore \dfrac{1}{y} > \dfrac{1}{5}$ $\{y > 0\}$

$\therefore \frac{1}{5} < \dfrac{1}{y} \leqslant \frac{1}{2}$

$\therefore$ the range of $\dfrac{1}{f(x)}$ is $\{y \mid \frac{1}{5} < y \leqslant \frac{1}{2}\}$.

b We do not know where or if $f(x) = 0$. We are only told that the range is $-3 \leqslant y \leqslant 3$.

$\therefore$ we cannot comment about the domain of $\dfrac{1}{f(x)}$.

Now $-3 \leqslant y \leqslant 3$, which we divide into three parts:

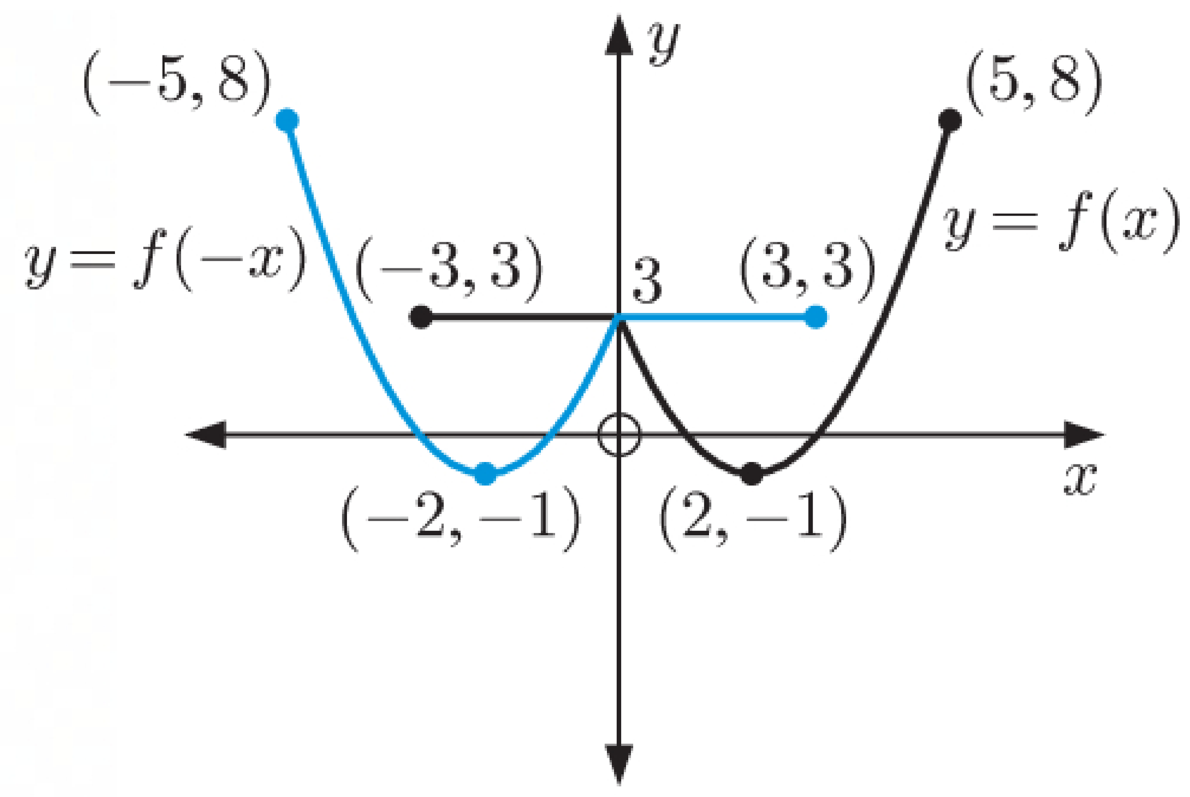
$-3 \leqslant y < 0$ *or* $0 < y \leqslant 3$ *or* $y = 0$

$\therefore \dfrac{1}{y} \leqslant -\dfrac{1}{3}$ $\{y < 0\}$ $\quad\therefore \dfrac{1}{y} \geqslant \dfrac{1}{3}$ $\{y > 0\}$ $\quad$ But $\dfrac{1}{f(x)} \neq 0$ for all x, so 0 is not in the range of $\dfrac{1}{f(x)}$.

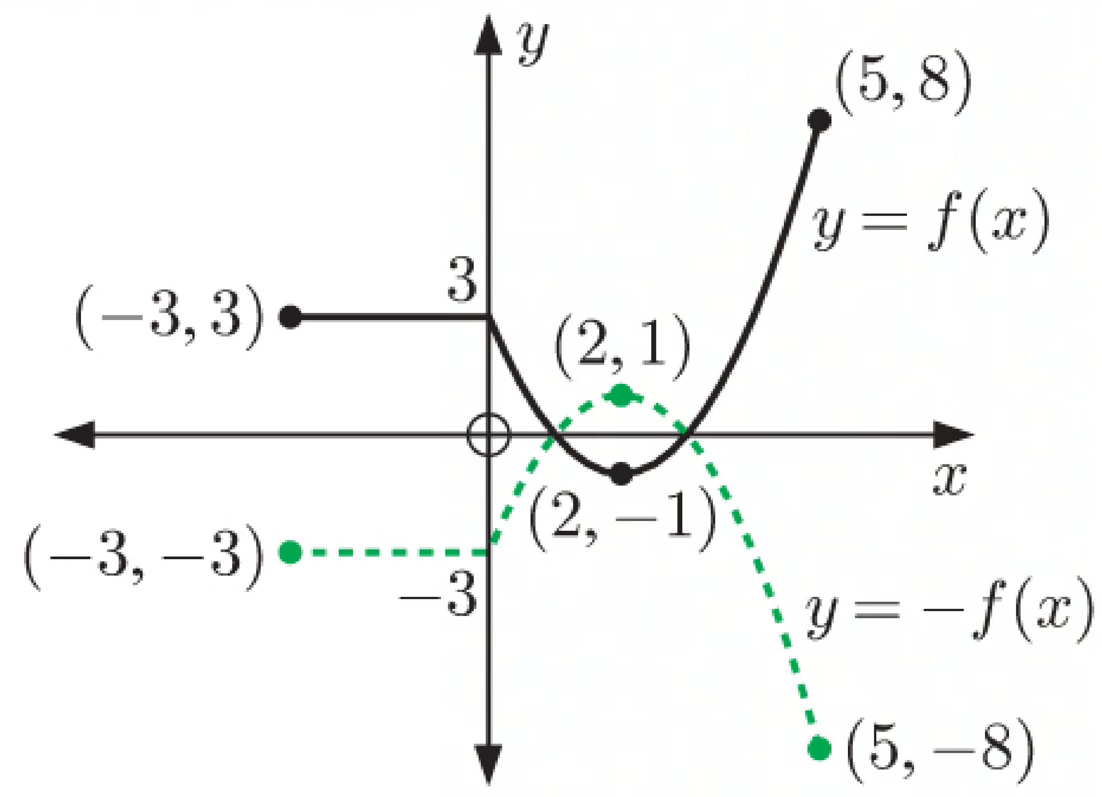
$\therefore$ the range of $\dfrac{1}{f(x)}$ is $\{y \mid y \leqslant -\frac{1}{3}$ or $y \geqslant \frac{1}{3}\}$.

REVIEW SET 16A

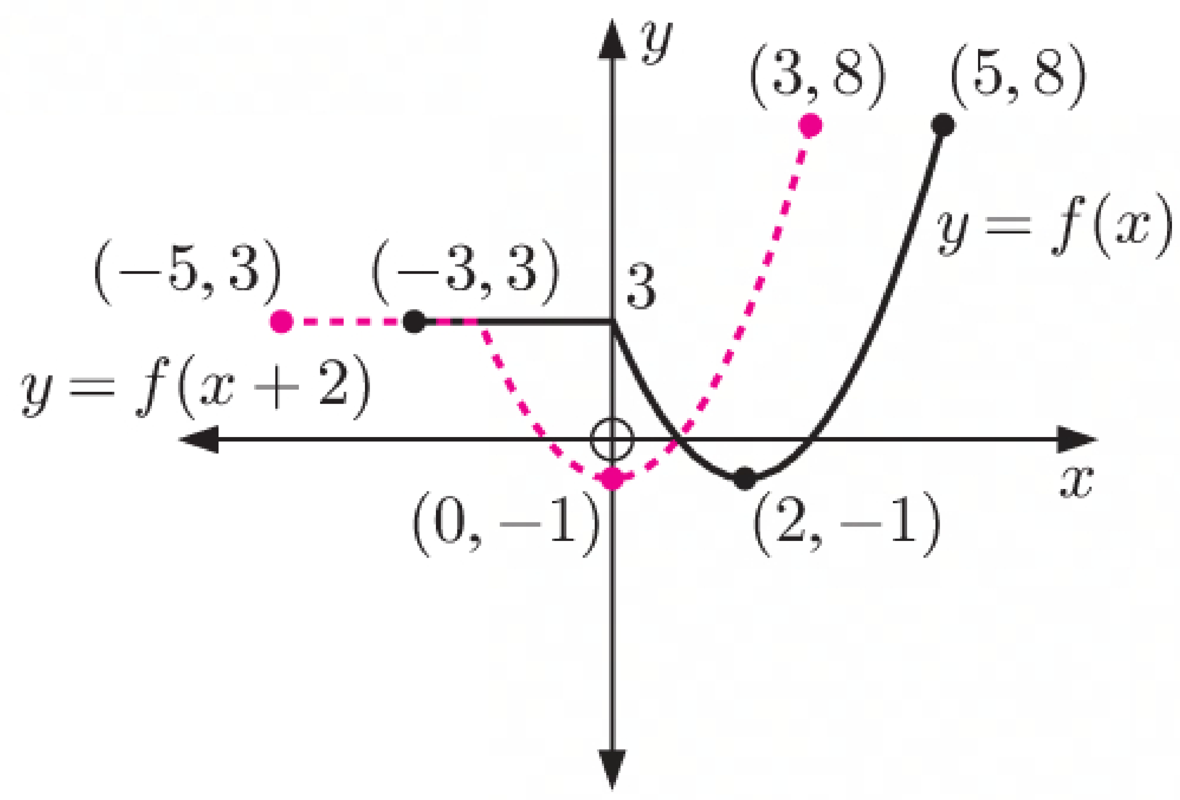
1 **a** To transform $y = f(x)$ to $y = f(-x)$, we reflect $y = f(x)$ in the y-axis.



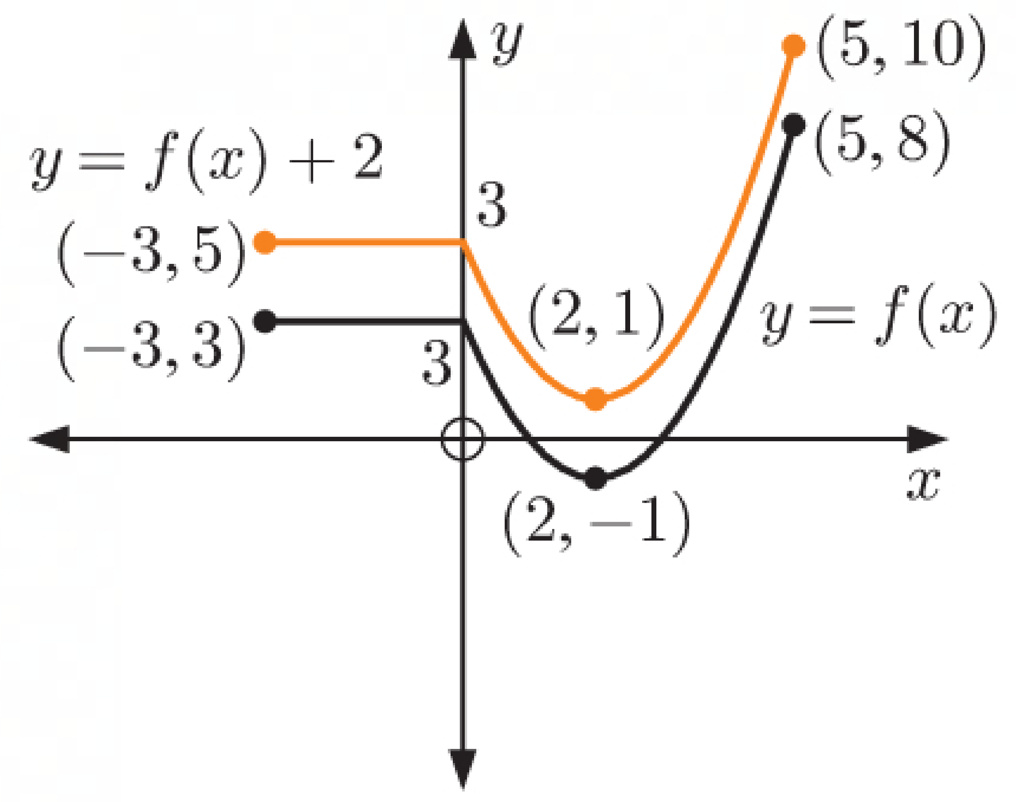
b To transform $y = f(x)$ to $y = -f(x)$, we reflect $y = f(x)$ in the x-axis.

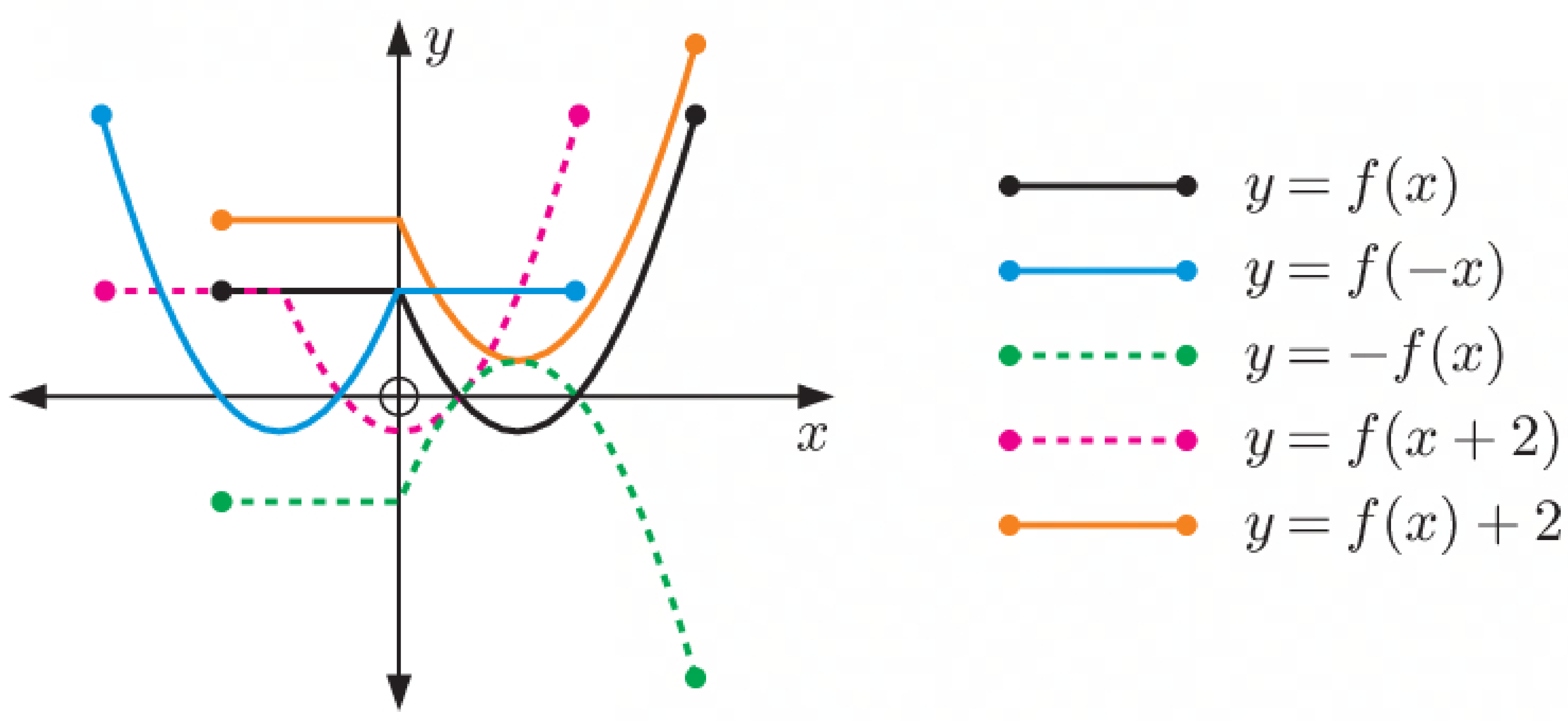


c To transform $y = f(x)$ to $y = f(x+2)$, we translate $y = f(x)$ through $\begin{pmatrix} -2 \\ 0 \end{pmatrix}$.



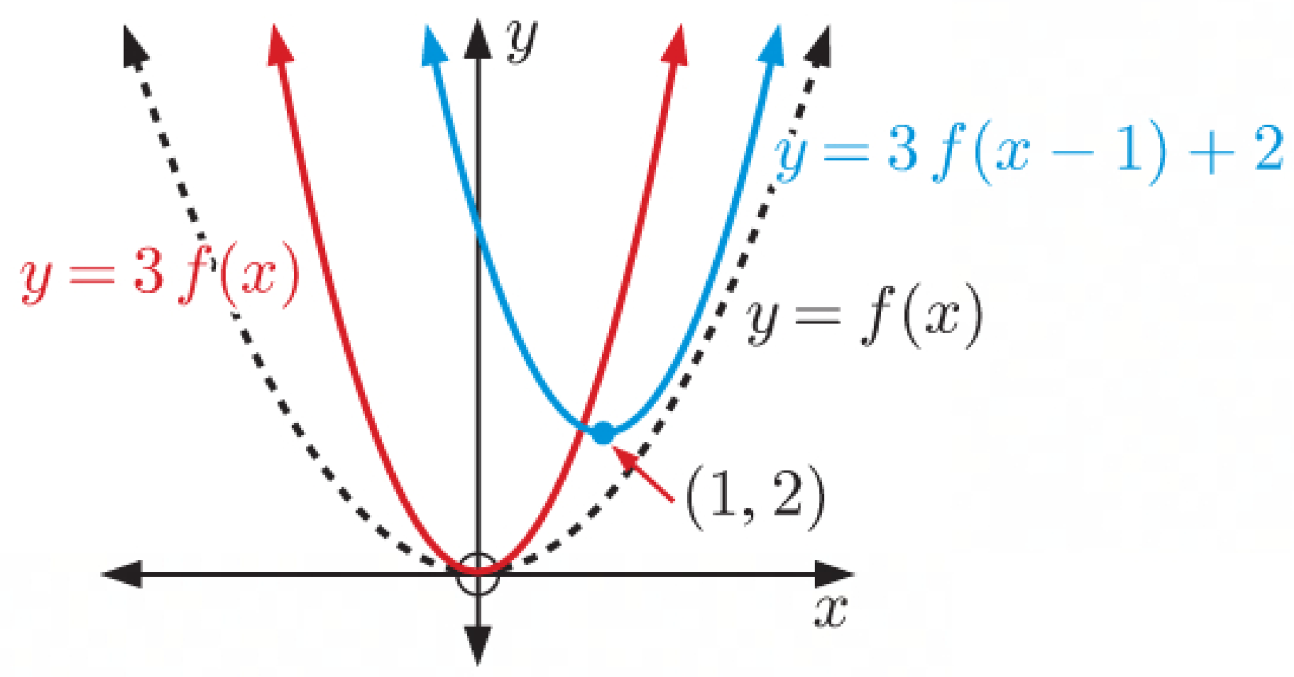
d To transform $y = f(x)$ to $y = f(x)+2$, we translate $y = f(x)$ through $\begin{pmatrix} 0 \\ 2 \end{pmatrix}$.





2 $y = 3f(x)$ is a vertical stretch of $y = f(x)$ with scale factor 3.

$y = 3f(x-1)+2$ is a translation of $y = 3f(x)$ through $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$.



3 **a** $g(x) = f(x) - 3$
$= 4x - 7 - 3$
$= 4x - 10$

b $g(x) = 5f(x)$
$= 5(x^2 + 6)$
$= 5x^2 + 30$

c $g(x) = f(x+4)$
$= 7 - 3(x+4)$
$= 7 - 3x - 12$
$= -3x - 5$

d $g(x) = f(\frac{1}{3}x)$
$= 2(\frac{1}{3}x)^2 - (\frac{1}{3}x) + 4$
$= \frac{2}{9}x^2 - \frac{1}{3}x + 4$

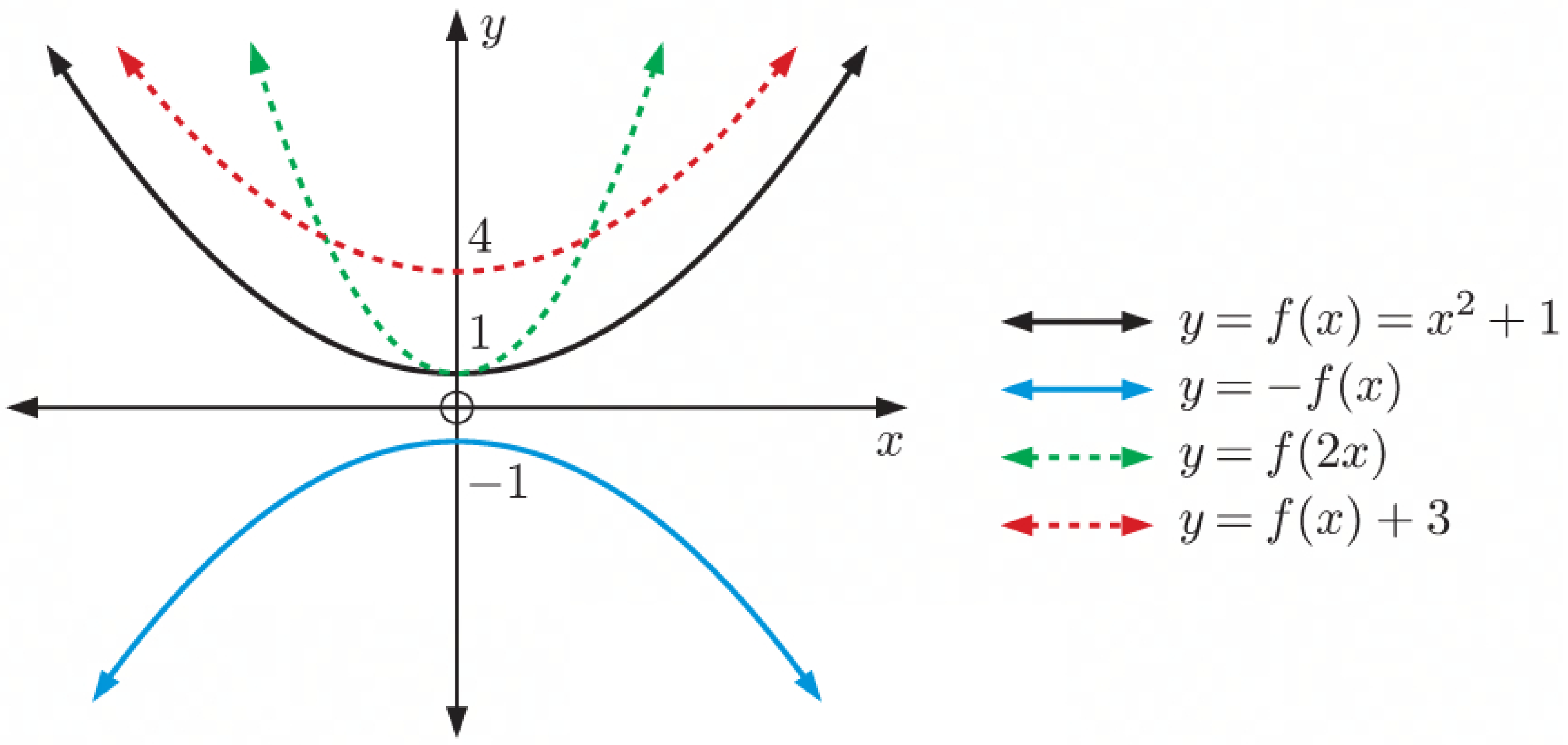
e $g(x) = f(-x)$
$= (-x)^3$
$= -x^3$

4 $f(x) = x^2 + 1$

a To transform $y = f(x)$ to $y = -f(x)$, we reflect $y = f(x)$ in the x-axis.

b To transform $y = f(x)$ to $y = f(2x)$, we horizontally stretch $y = f(x)$ with scale factor $\frac{1}{2}$.

c To transform $y = f(x)$ to $y = f(x) + 3$, we translate $y = f(x)$ through $\begin{pmatrix} 0 \\ 3 \end{pmatrix}$.



5 $f(x)$ has domain $\{x \mid -2 \leqslant x \leqslant 3\}$ and range $\{y \mid -1 \leqslant y \leqslant 7\}$.
$g(x) = f(x+3) - 4$ translates every point on $y = f(x)$ 3 units to the left and 4 units downwards.
$\therefore$ $g(x)$ has domain $\{x \mid -5 \leqslant x \leqslant 0\}$ and range $\{y \mid -5 \leqslant y \leqslant 3\}$.

6 **a**
$$\begin{aligned} g(x) &= f(x-2) + 4 \\ &= [(x-2)+1]^2 + 4 + 4 \\ &= (x-1)^2 + 8 \end{aligned}$$

b **i** $f(x) = (x+1)^2 + 4$ has vertex $(-1, 4)$ and shape  $(a > 0)$.
$\therefore$ the minimum value is 4.
$\therefore$ the range of $f(x)$ is $\{y \mid y \geqslant 4\}$.

ii $g(x) = (x-1)^2 + 8$ has vertex $(1, 8)$ and shape  $(a > 0)$.
$\therefore$ the minimum value is 8.
$\therefore$ the range is $\{y \mid y \geqslant 8\}$.

7 Let the quadratic be $f(x) = ax^2 + bx + c$, $\Delta = b^2 - 4ac$.

a The reflection of $f(x)$ in the x-axis is
$$\begin{aligned} -f(x) &= -(ax^2 + bx + c) \\ &= -ax^2 - bx - c \end{aligned}$$
which has
$$\begin{aligned} \Delta &= (-b)^2 - 4(-a)(-c) \\ &= b^2 - 4ac \end{aligned}$$
$\therefore$ the discriminant is unchanged.

b The reflection of $f(x)$ in the y-axis is
$$\begin{aligned} f(-x) &= a(-x)^2 + b(-x) + c \\ &= ax^2 - bx + c \end{aligned}$$
which has
$$\begin{aligned} \Delta &= (-b)^2 - 4(a)(c) \\ &= b^2 - 4ac \end{aligned}$$
$\therefore$ the discriminant is unchanged.

c The translation of $f(x)$ h units to the right is

$$\begin{aligned} f(x-h) &= a(x-h)^2 + b(x-h) + c \\ &= a(x^2 - 2hx + h^2) + bx - bh + c \\ &= ax^2 - 2ahx + ah^2 + bx - bh + c \\ &= ax^2 + (b-2ah)x + ah^2 - bh + c \end{aligned}$$

which has $\Delta = (b-2ah)^2 - 4(a)(ah^2 - bh + c)$

$$\begin{aligned} &= b^2 - 4abh + 4a^2h^2 - 4a^2h^2 + 4abh - 4ac \\ &= b^2 - 4ac \end{aligned}$$

$\therefore$ the discriminant is unchanged.

8 $f(x) = 3x^2 - x + 4$ is transformed to $g(x)$ by translating through $\begin{pmatrix} -1 \\ 3 \end{pmatrix}$.

$$\begin{aligned} \therefore\ g(x) &= f(x+1) + 3 \\ &= 3(x+1)^2 - (x+1) + 4 + 3 \\ &= 3(x^2 + 2x + 1) - x - 1 + 4 + 3 \\ &= 3x^2 + 6x + 3 - x + 6 \\ &= 3x^2 + 5x + 9 \end{aligned}$$

9 **a** $f(x) \xrightarrow[\text{in } x\text{-axis}]{\text{reflection}} -f(x) \xrightarrow{\text{translation } \begin{pmatrix} -2 \\ 3 \end{pmatrix}} -f(x+2) + 3$

The resulting function is $-f(x+2) + 3$.

b $f(x) \xrightarrow{\text{translation } \begin{pmatrix} 4 \\ -1 \end{pmatrix}} f(x-4) - 1 \xrightarrow[\text{scale factor } 2]{\text{vertical stretch}} 2[f(x-4) - 1]$

The resulting function is $2f(x-4) - 2$.

10 A$(-2,\ 3)$ lies on the graph of $y = f(x)$.

a $y = f(x-2) + 1$ is found by translating $y = f(x)$ through $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$.

$\therefore$ the image of A on $y = f(x-2) + 1$ is $(-2+2,\ 3+1)$, or $(0,\ 4)$.

b $f(x) \xrightarrow[\text{scale factor } 2]{\text{vertical stretch}} 2f(x) \xrightarrow{\text{translation } \begin{pmatrix} 2 \\ 0 \end{pmatrix}} 2f(x-2)$

$\therefore$ the image of A on $y = 2f(x)$ is $(-2,\ 2 \times 3)$, or $(-2,\ 6)$.

$\therefore$ the image of A on $y = 2f(x-2)$ is $(-2+2,\ 6)$, or $(0,\ 6)$.

c $f(x) \xrightarrow{\text{translation } \begin{pmatrix} 3 \\ 0 \end{pmatrix}} f(x-3) \xrightarrow[\text{scale factor } \frac{1}{2}]{\text{horizontal stretch}} f(2x-3)$

$\therefore$ the image of A on $y = f(x-3)$ is $(-2+3,\ 3)$, or $(1,\ 3)$.

$\therefore$ the image of A on $y = f(2x-3)$ is $(\frac{1}{2},\ 3)$.

11 **a** The graph of $y = f(x+4)$ is a translation of $y = f(x)$ 4 units to the left.
So, the graph of $y = f(x+4)$ will have x-intercepts $-5-4=-9$ and $1-4=-3$.
There is not enough information to determine the y-intercept.

b The graph of $y = 3f(x)$ is a vertical stretch of $y = f(x)$ with scale factor 3.
Each point on the graph of $y = f(x)$ becomes 3 times its previous distance from the x-axis.
So, the graph of $y = 3f(x)$ will have x-intercepts -5 and 1 (unchanged), and y-intercept $3 \times -3 = -9$.

c The graph of $y = f\left(\frac{x}{2}\right)$ is a horizontal stretch of $y = f(x)$ with scale factor 2.
Each point on the graph of $y = f(x)$ becomes 2 times its previous distance from the y-axis.
So, the graph of $y = f\left(\frac{x}{2}\right)$ will have x-intercepts $-5 \times 2 = -10$ and $1 \times 2 = 2$, and y-intercept -3 (unchanged).

d The graph of $y = -f(x)$ is a reflection of the graph of $y = f(x)$ in the x-axis.
The y-coordinate of each point on $y = f(x)$ becomes negative.
So, the graph of $y = -f(x)$ has x-intercepts -5 and 1 (unchanged), and y-intercept $-3 \times -1 = 3$.

12 $$\frac{1}{x} \xrightarrow{\text{translation } \begin{pmatrix} -1 \\ 2 \end{pmatrix}} \frac{1}{x+1} + 2 \xrightarrow[\text{in } y\text{-axis}]{\text{reflection}} \frac{1}{1-x} + 2$$

The resulting function is $g(x) = \dfrac{1}{1-x} + 2$.

a $$\begin{aligned} g(x) &= \frac{1}{1-x} + 2 \\ &= \frac{1}{1-x} + \frac{2(1-x)}{(1-x)} \\ &= \frac{1+2(1-x)}{1-x} \\ &= \frac{1+2-2x}{1-x} \\ &= \frac{3-2x}{1-x} \\ &= \frac{2x-3}{x-1} \end{aligned}$$

b The asymptotes of $y = \dfrac{1}{x}$ are $x = 0$ and $y = 0$.

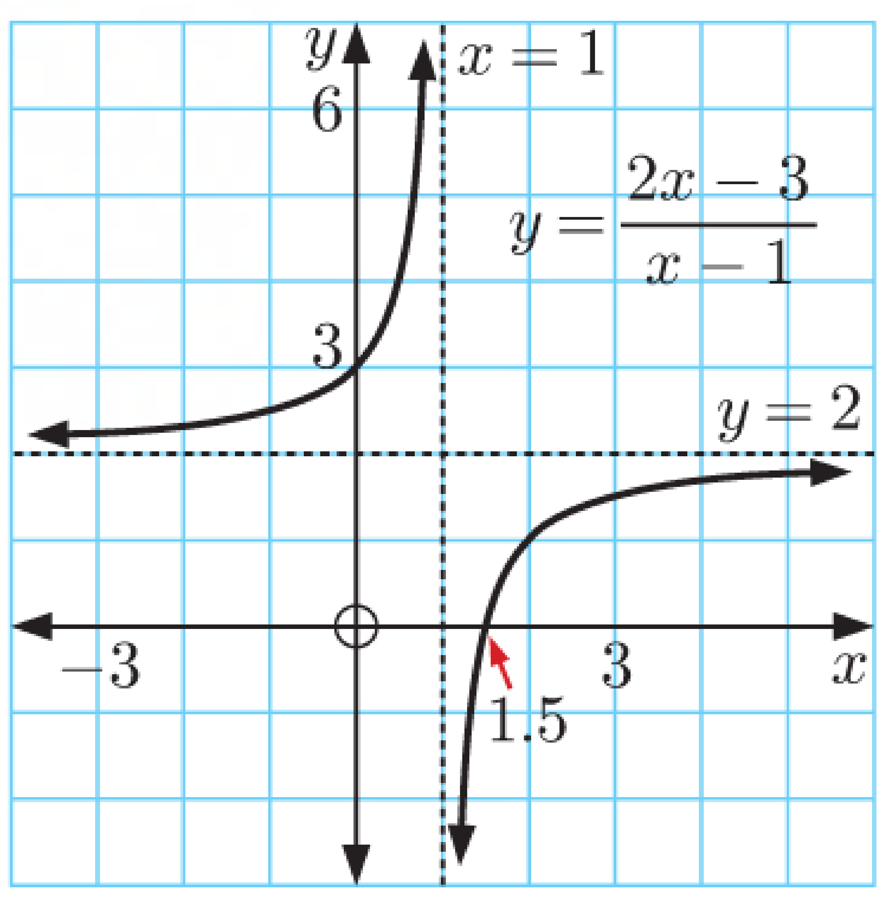
These are translated $\begin{pmatrix} -1 \\ 2 \end{pmatrix}$ and reflected in the y-axis.

$\therefore$ the vertical asymptote is $x = 1$ and the horizontal asymptote is $y = 2$.

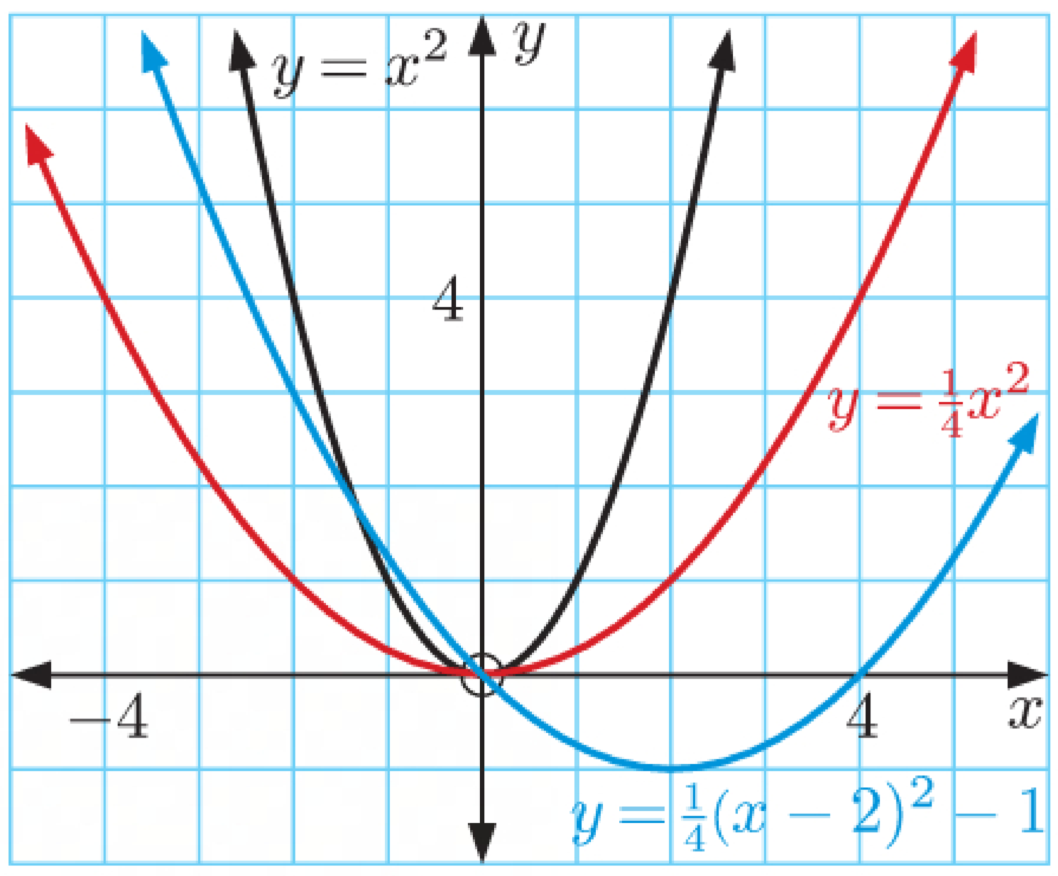
c $g(x) = \dfrac{1}{1-x} + 2$

The domain of $g(x)$ is $\{x \mid x \neq 1\}$.
The range of $g(x)$ is $\{y \mid y \neq 2\}$.

d



13



To transform $y = x^2$ to $y = \frac{1}{4}x^2$, we vertically stretch $y = x^2$ with scale factor $\frac{1}{4}$.

To transform $y = \frac{1}{4}x^2$ to $y = \frac{1}{4}(x-2)^2 - 1$, we translate $y = \frac{1}{4}x^2$ through $\begin{pmatrix} 2 \\ -1 \end{pmatrix}$.

14 $y = (x-2)(x+3)$ has x-intercepts 2 and -3.

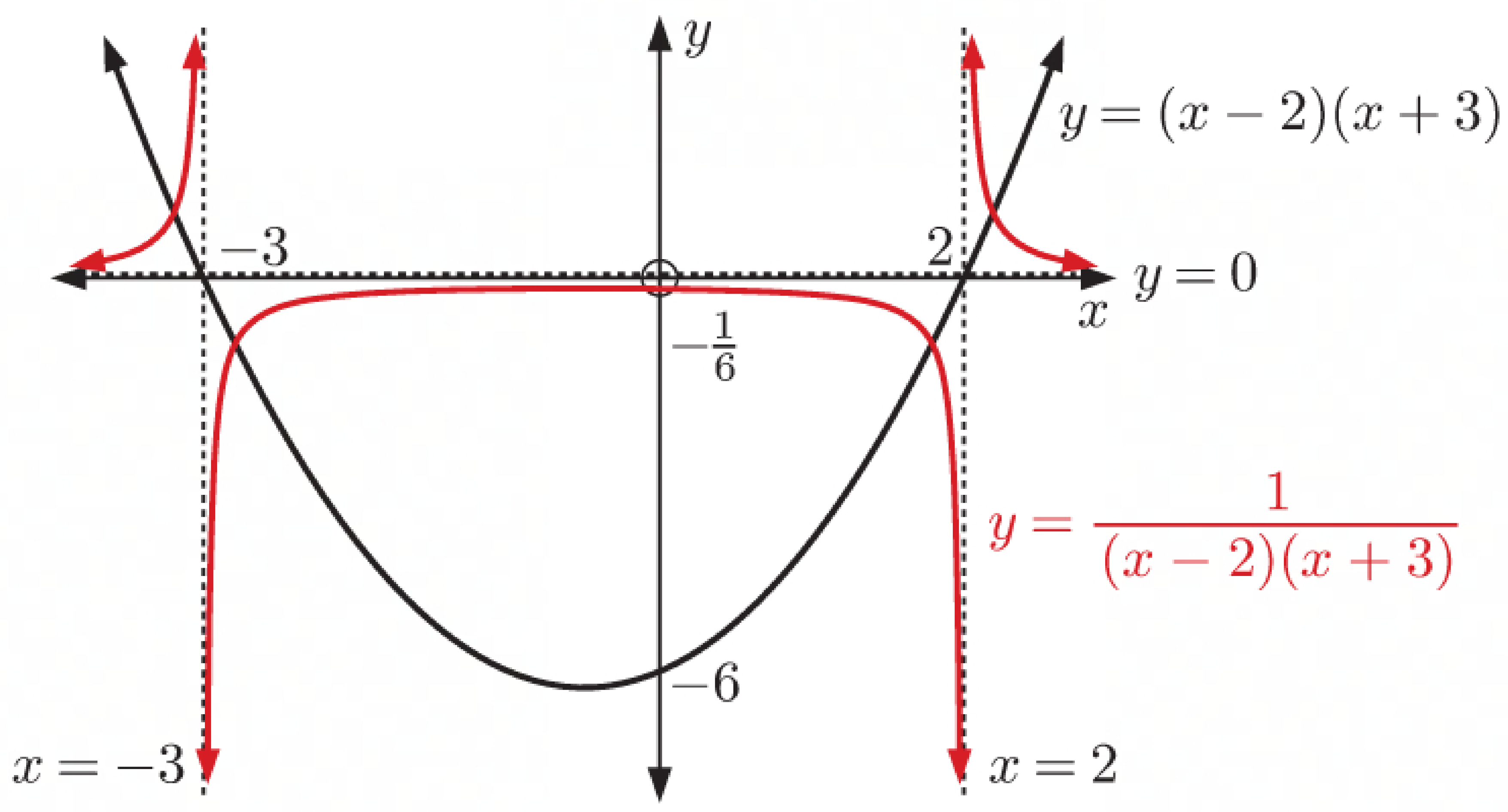
$\therefore \quad y = \dfrac{1}{(x-2)(x+3)}$ has vertical asymptotes $x = 2$ and $x = -3$.

$y = \dfrac{1}{(x-2)(x+3)} \neq 0$ as $1 \neq 0$

$\therefore \quad y = \dfrac{1}{(x-2)(x+3)}$ has a horizontal asymptote $y = 0$

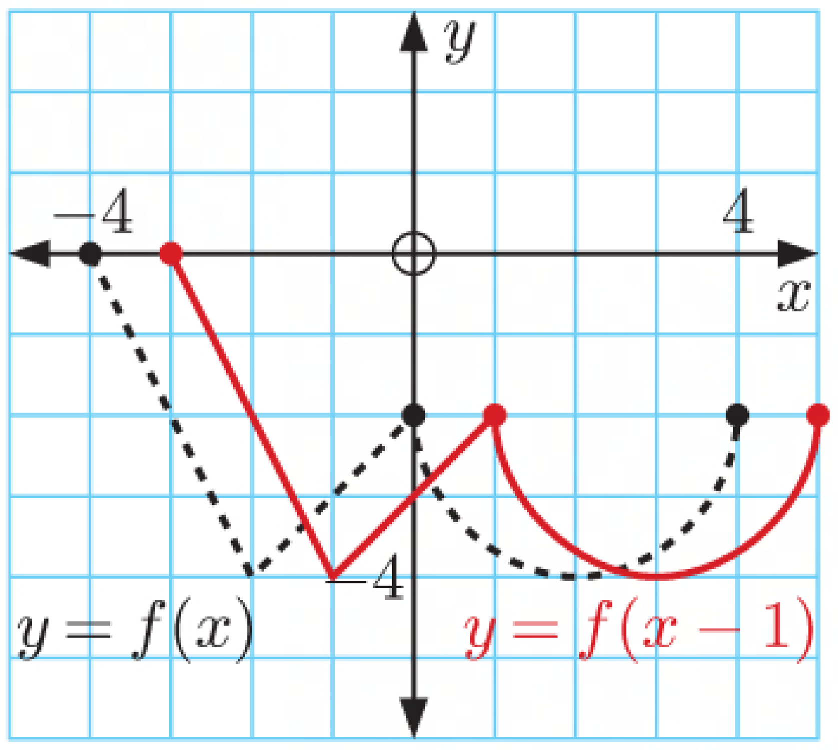
$y = (x-2)(x+3)$ has y-intercept $(0-2)(0+3) = -6$.

$\therefore \quad y = \dfrac{1}{(x-2)(x+3)}$ has y-intercept $-\frac{1}{6}$.

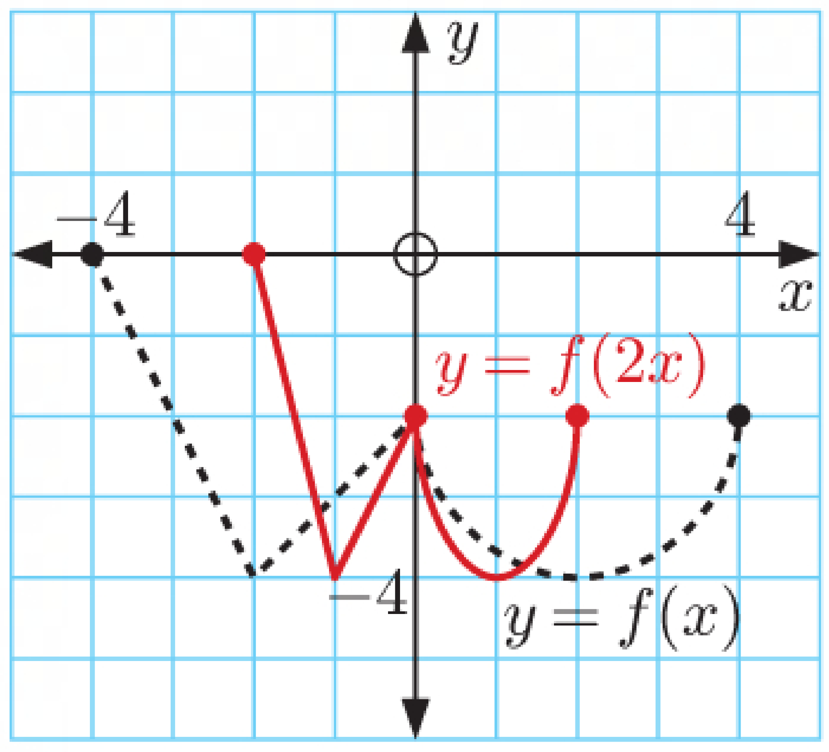


REVIEW SET 16B

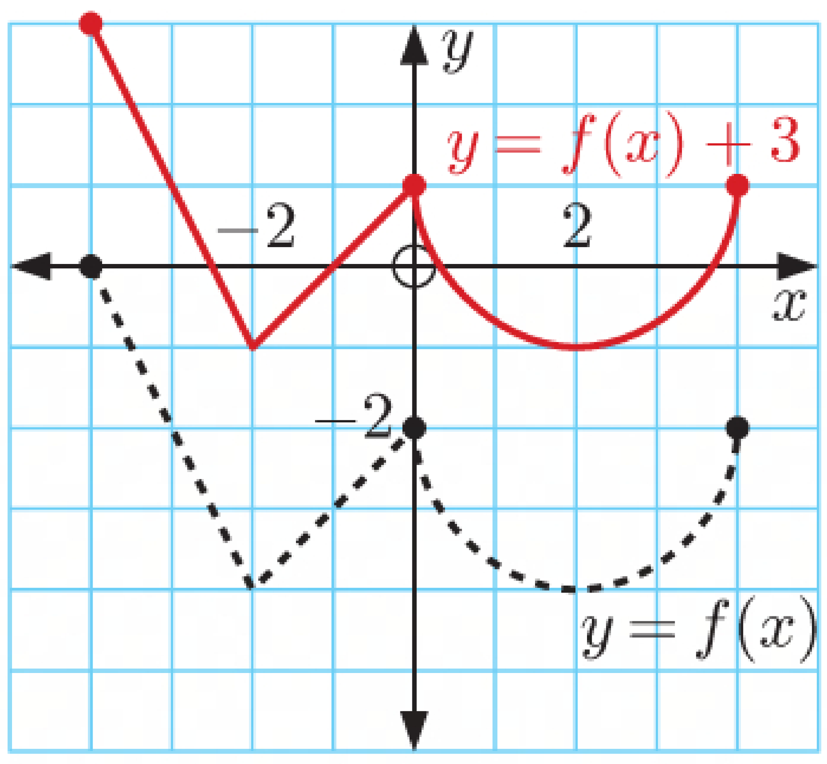
1 **a** The graph of $y = f(x-1)$ is found by translating $y = f(x)$ 1 unit to the right.



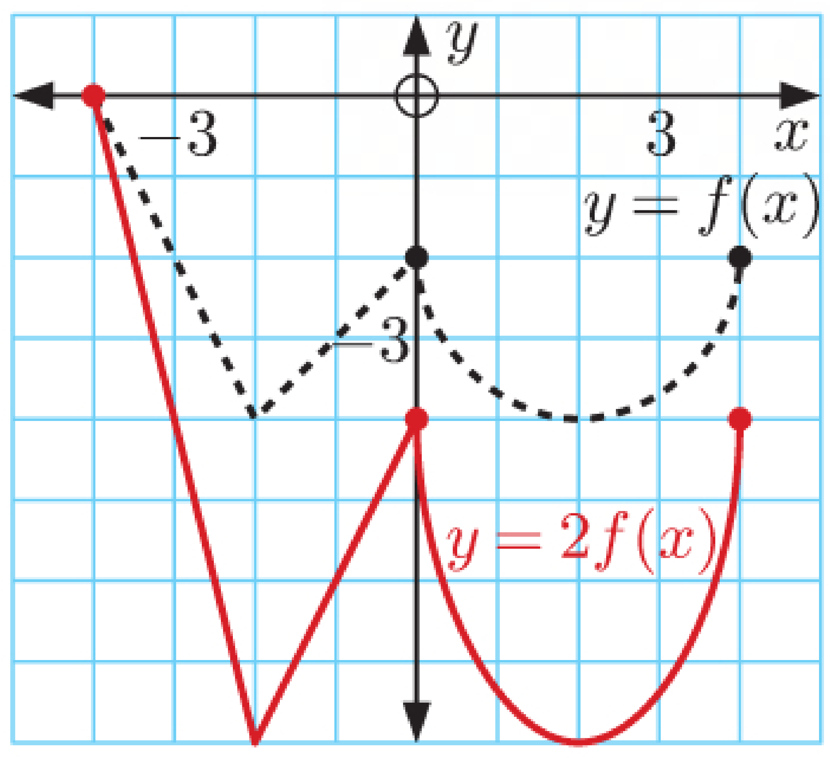
b The graph of $y = f(2x)$ is found by horizontally stretching $y = f(x)$ with scale factor $\frac{1}{2}$.



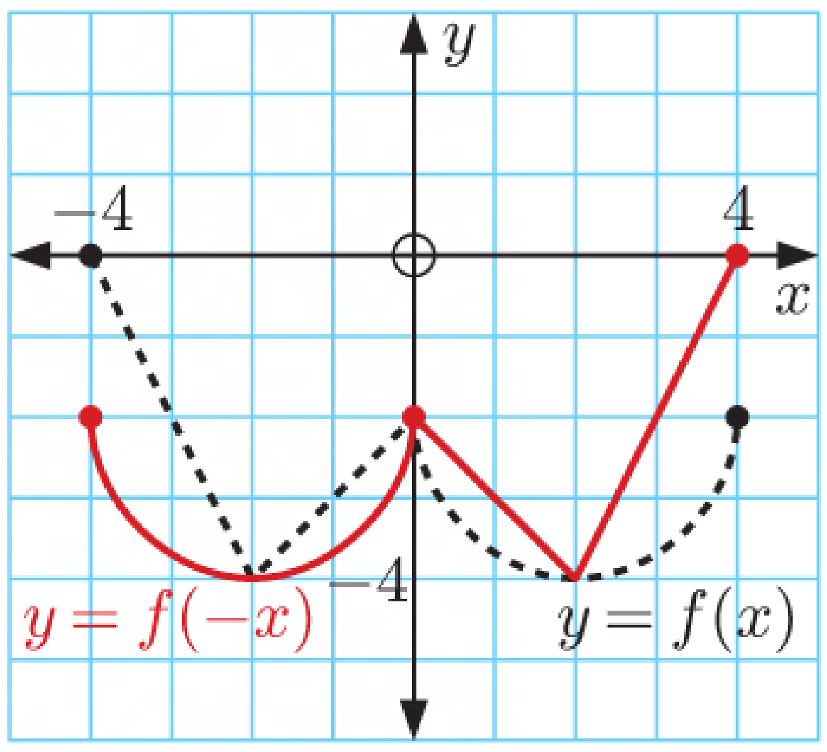
c The graph of $y = f(x) + 3$ is found by translating $y = f(x)$ 3 units upwards.



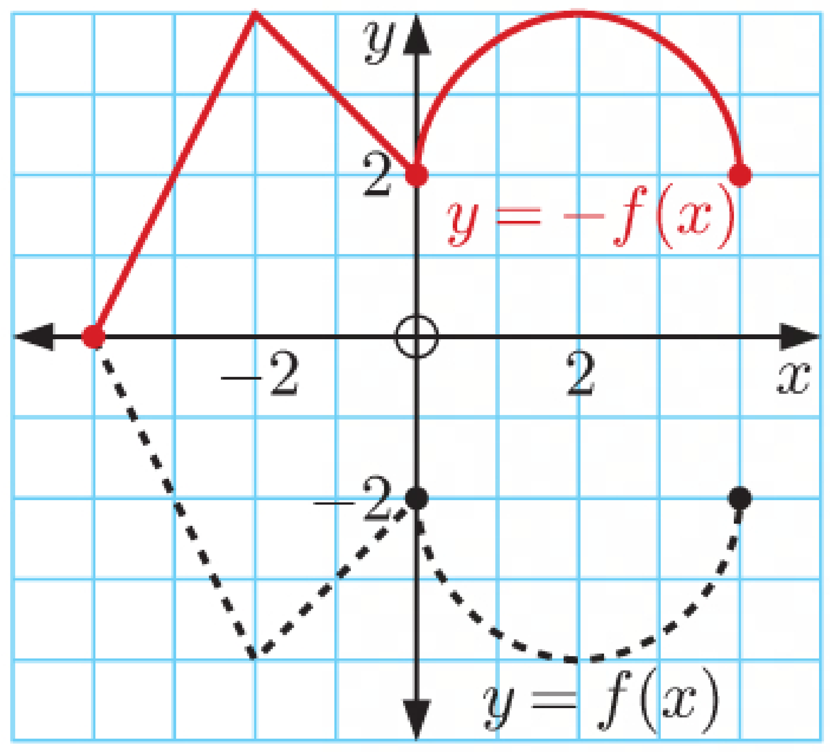
d The graph of $y = 2f(x)$ is found by vertically stretching $y = f(x)$ with scale factor 2.



e The graph of $y = f(-x)$ is found by reflecting $y = f(x)$ in the y-axis.



f The graph of $y = -f(x)$ is found by reflecting $y = f(x)$ in the x-axis.



2 **a** $g(x) = -f(x)$
$= -(x^2 - 3x)$
$= 3x - x^2$

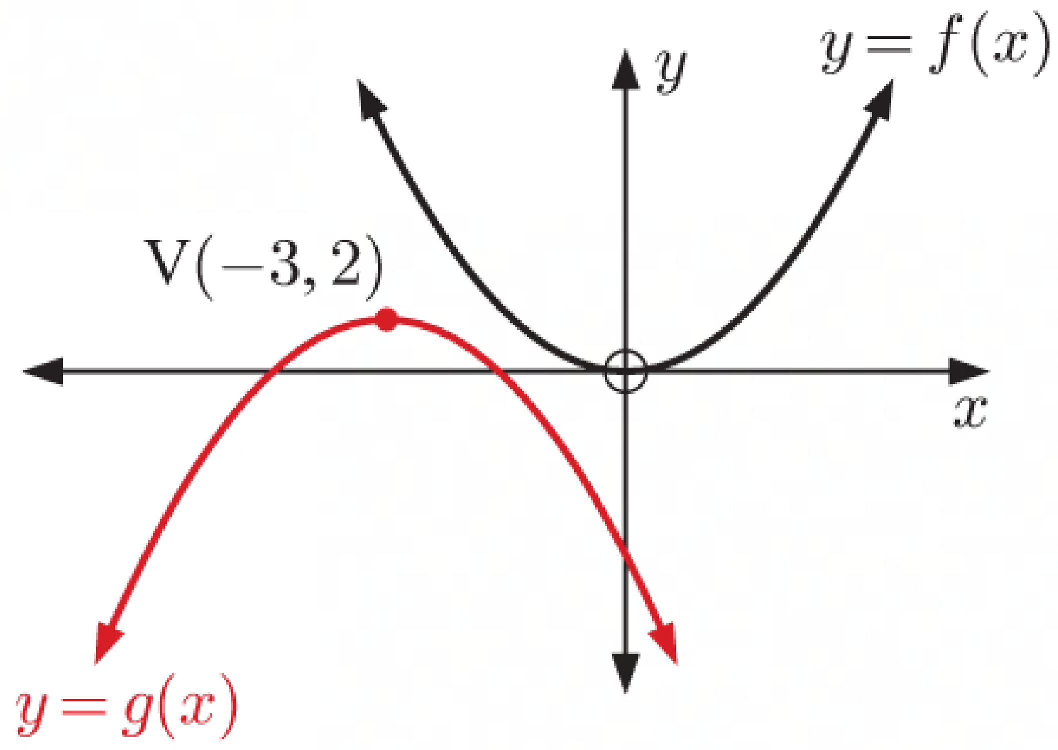
b $g(x) = f(x) + 2$
$= 14 - x + 2$
$= 16 - x$

c $g(x) = f(\frac{1}{4}x)$
$= \frac{1}{3}(\frac{1}{4}x) + 2$
$= \frac{1}{12}x + 2$

3 $y = x^2$ is transformed to $y = -x^2$ by reflecting $y = x^2$ in the x-axis. The vertex is $(0, 0)$.

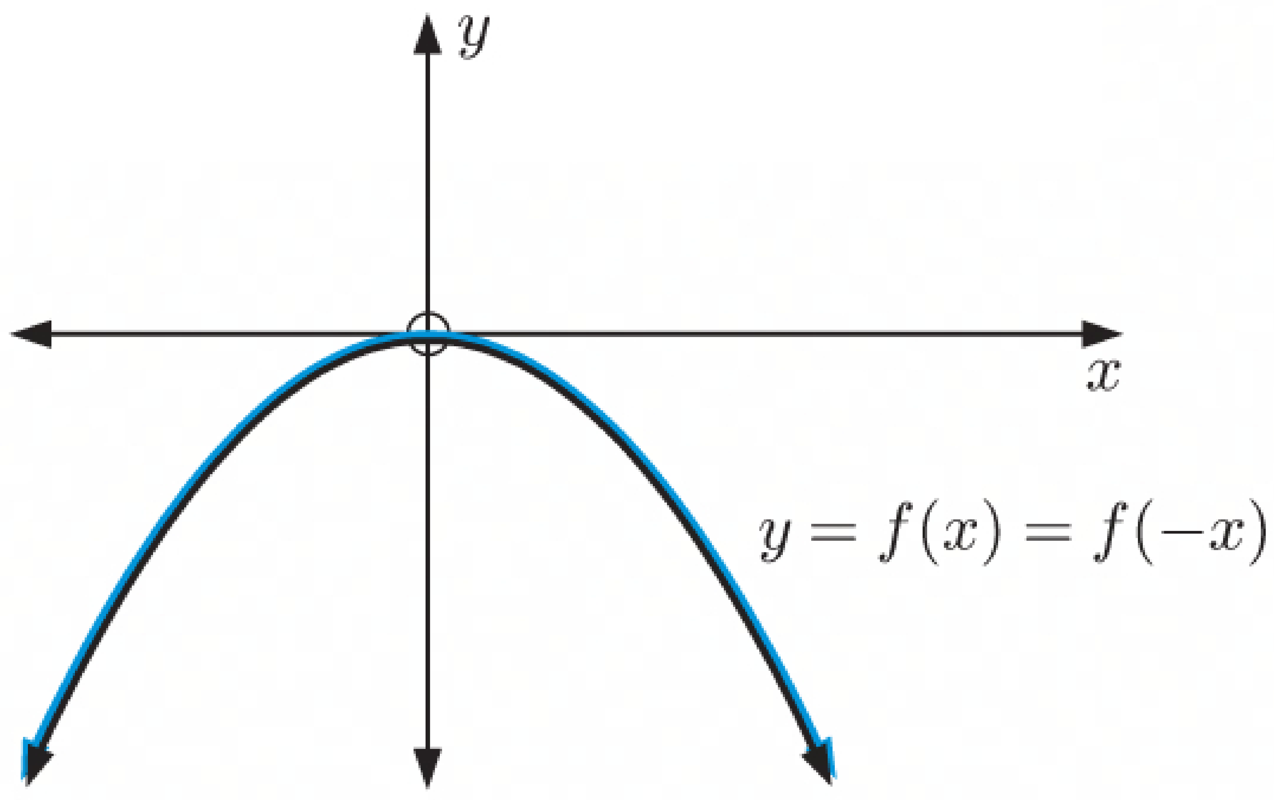
$y = -x^2$ is transformed to $y = g(x)$ by translating $y = -x^2$ through $\begin{pmatrix} -3 \\ 2 \end{pmatrix}$.

$\therefore$ $g(x) = -(x+3)^2 + 2$
$= -(x^2 + 6x + 9) + 2$
$= -x^2 - 6x - 9 + 2$
$= -x^2 - 6x - 7$

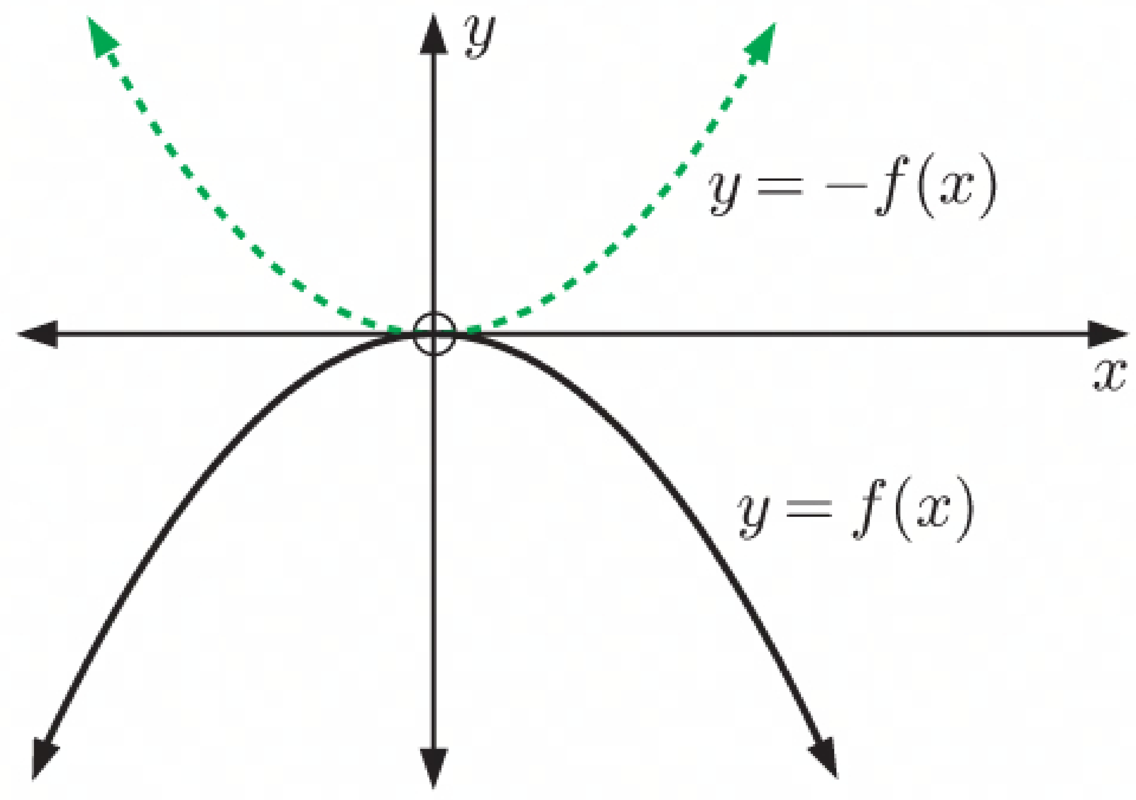


4 $f(x) = -x^2$

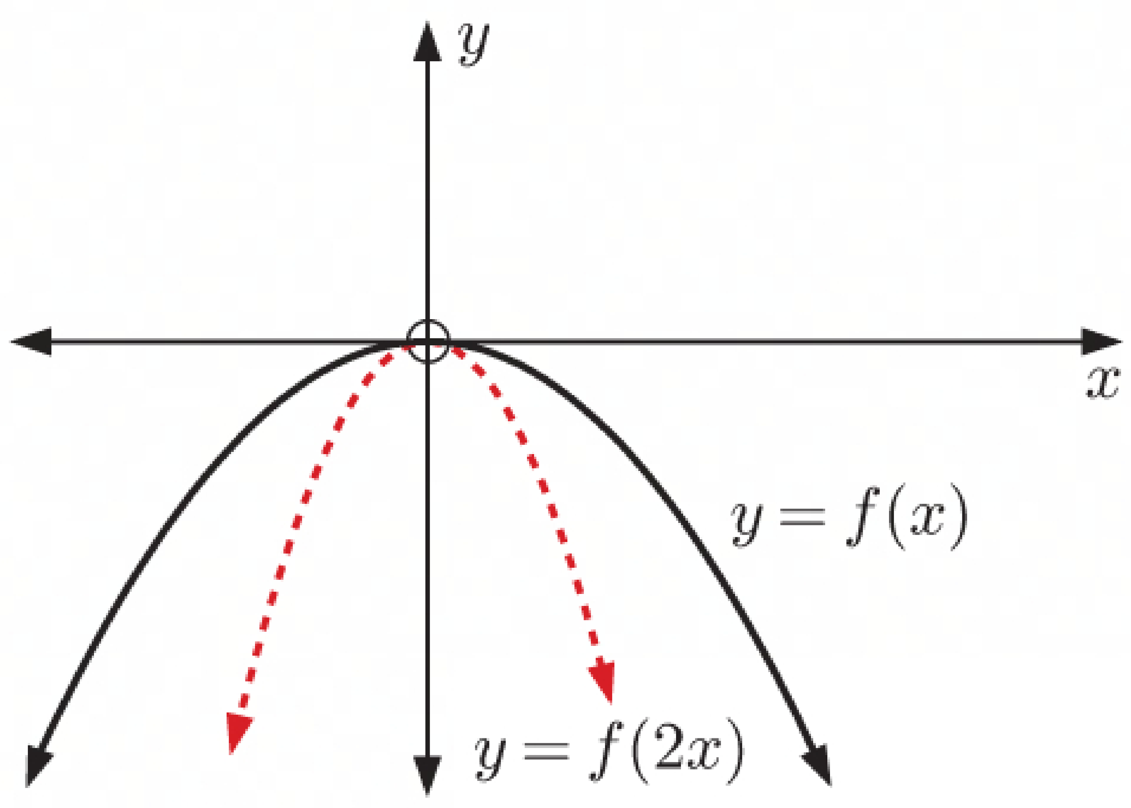
a To transform $y = f(x)$ to $y = f(-x)$, we reflect $y = f(x)$ in the y-axis.



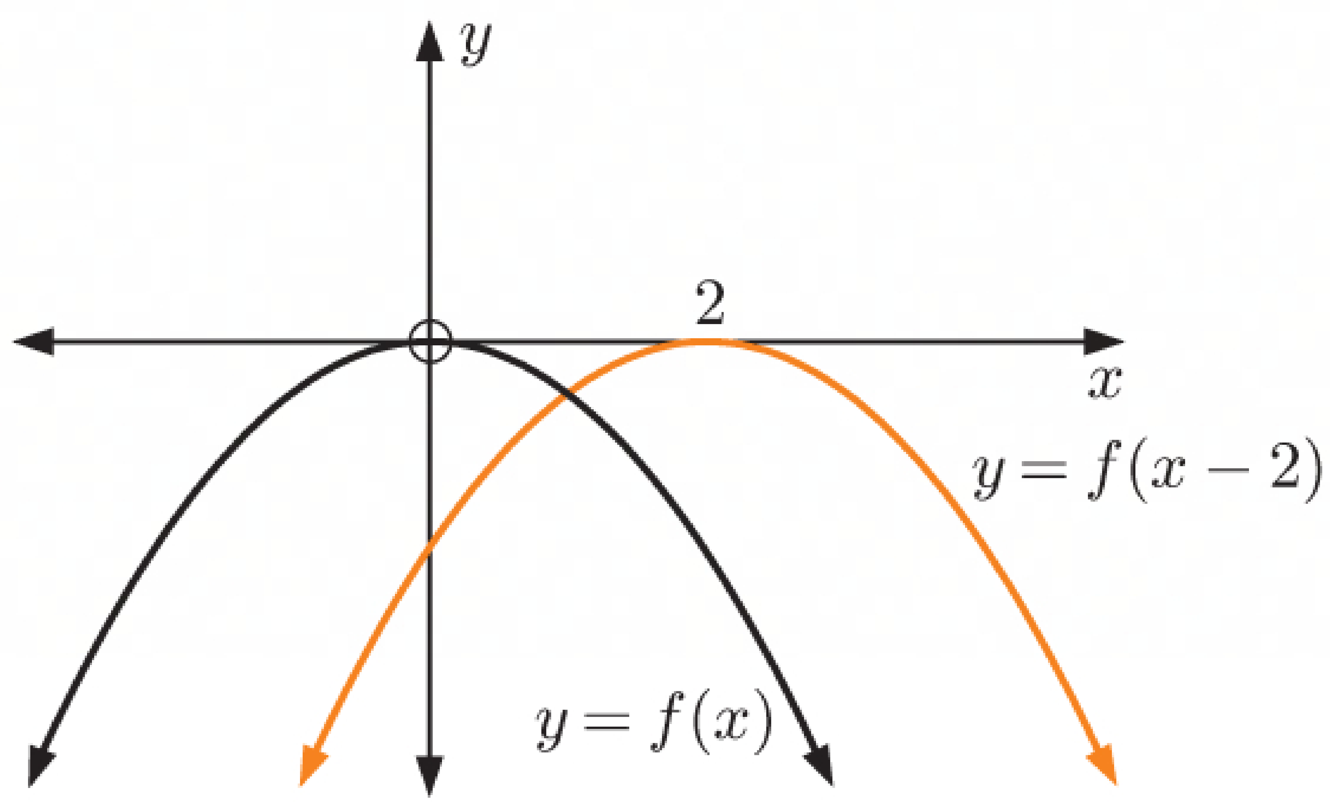
b To transform $y = f(x)$ to $y = -f(x)$, we reflect $y = -f(x)$ in the x-axis.

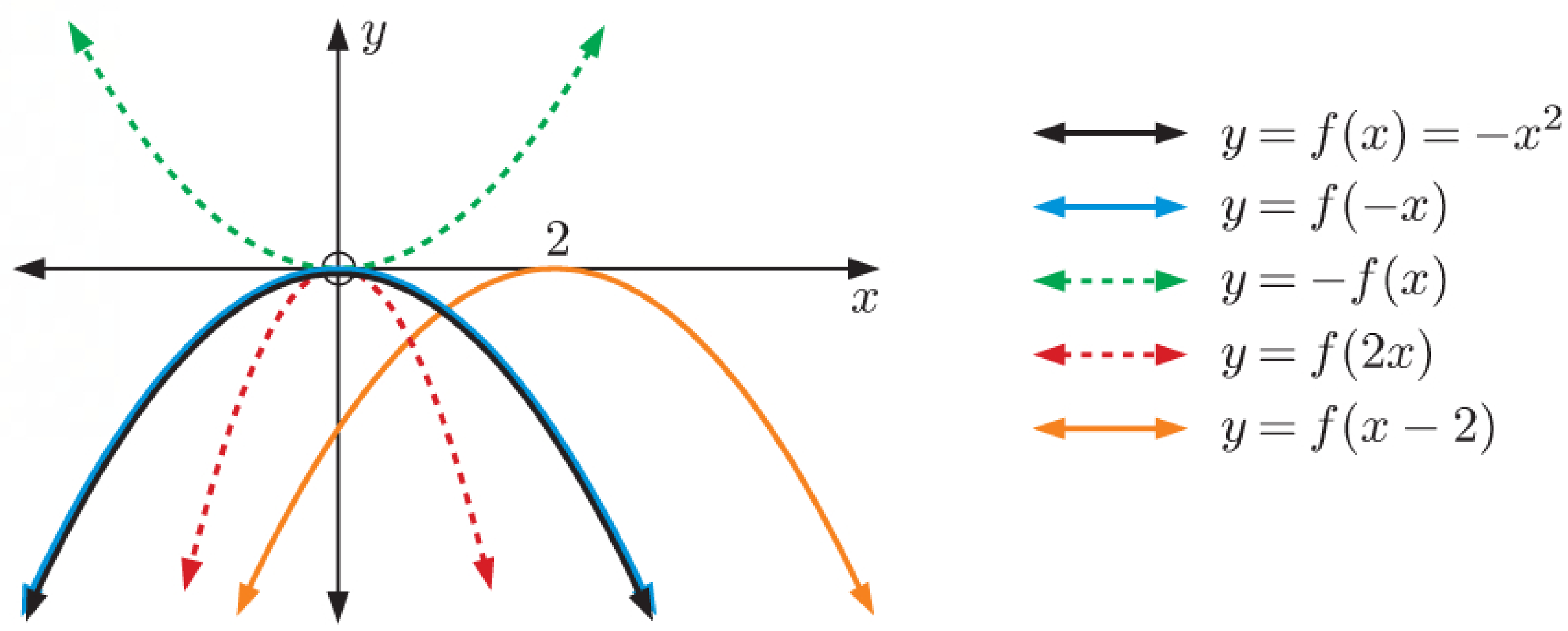


c To transform $y = f(x)$ to $y = f(2x)$, we horizontally stretch $y = f(x)$ with scale factor $\frac{1}{2}$.



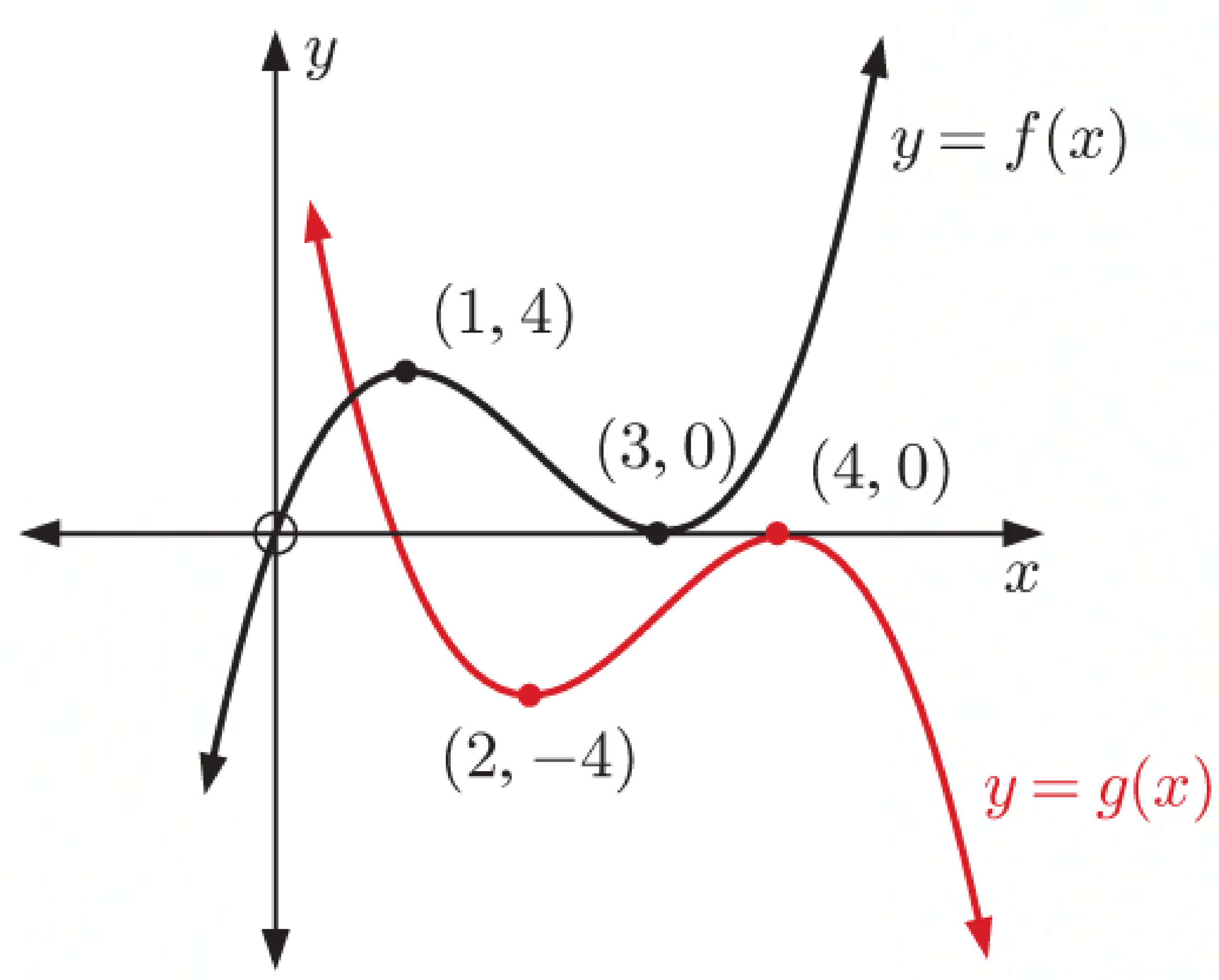
d To transform $y = f(x)$ to $y = f(x-2)$, we translate $y = f(x)$ through $\begin{pmatrix} 2 \\ 0 \end{pmatrix}$.





5 **a** $y = f(x)$ is transformed to $y = -f(x)$ by reflecting $y = f(x)$ in the x-axis.

$y = -f(x)$ is transformed to $g(x) = -f(x-1)$ by translating $y = -f(x)$ through $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$.



b $(1, 4)$ on $y = f(x)$ is reflected in the x-axis to $(1, -4)$, then translated through $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ to $(2, -4)$ on $y = g(x)$.

$(3, 0)$ on $y = f(x)$ is unchanged by a reflection in the x-axis, then translated through $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ to $(4, 0)$.

$\therefore$ the turning points of $y = g(x)$ are $(2, -4)$ and $(4, 0)$.

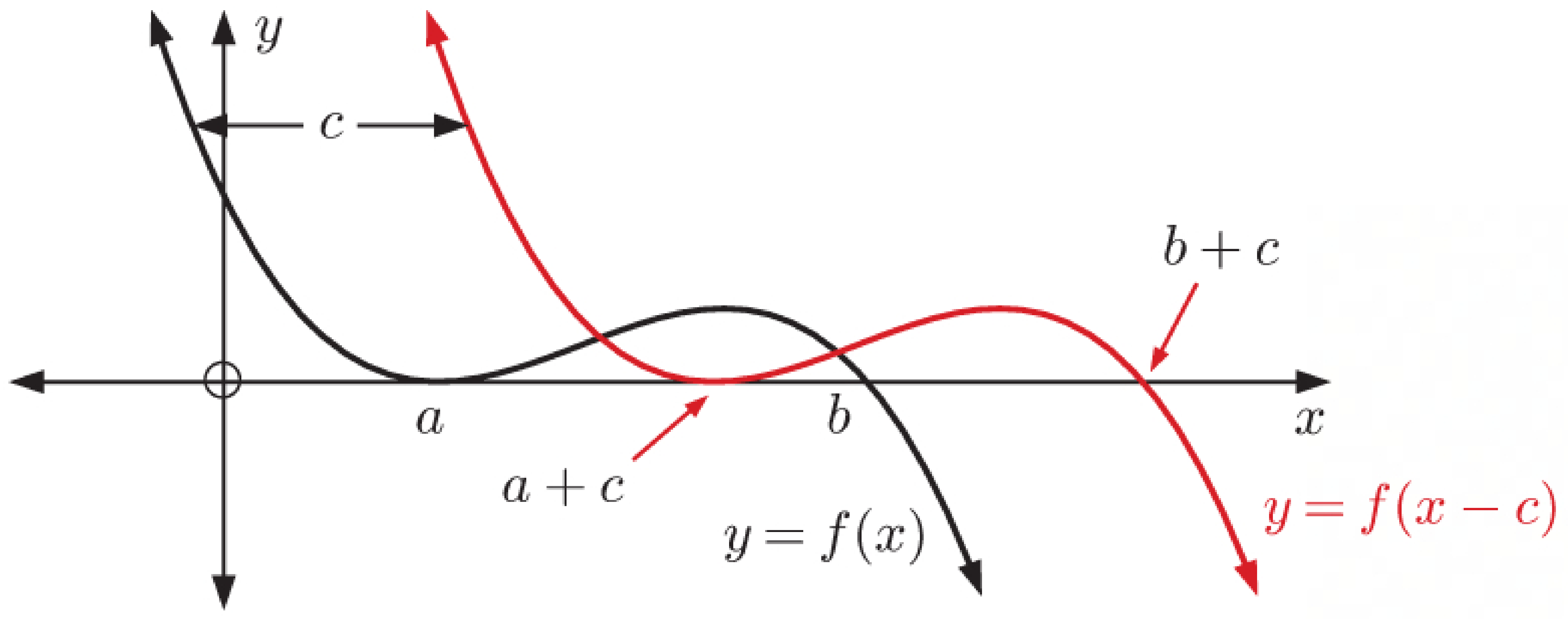
6 $f(x) = -2x^2 + x + 2$ is translated through $\begin{pmatrix} 1 \\ -2 \end{pmatrix}$.

$$\begin{aligned} f(x-1) - 2 &= -2(x-1)^2 + (x-1) + 2 - 2 \\ &= -2(x^2 - 2x + 1) + x - 1 \\ &= -2x^2 + 4x - 2 + x - 1 \\ &= -2x^2 + 5x - 3 \end{aligned}$$

$\therefore$ the image is $y = -2x^2 + 5x - 3$.

7 $y = f(x)$ is transformed to $y = f(x - c)$ by translating $y = f(x)$ through $\begin{pmatrix} c \\ 0 \end{pmatrix}$, $0 < c < b - a$.

The x-intercepts on $y = f(x - c)$ are $a + c$ and $b + c$.



8 $2x^2 + 8x - 3 \xrightarrow[\text{in } x\text{-axis}]{\text{reflection}} -2x^2 - 8x + 3 \xrightarrow{\text{translation} \begin{pmatrix} a \\ b \end{pmatrix}} -2(x-a)^2 - 8(x-a) + 3 + b$

Now, $g(x) = -2x^2 + 2x + 7 = -2(x-a)^2 - 8(x-a) + 3 + b$

$\therefore\ -2x^2 + 2x + 7 = -2(x^2 - 2ax + a^2) - 8x + 8a + 3 + b$

$\therefore\ -2x^2 + 2x + 7 = -2x^2 + 4ax - 2a^2 - 8x + 8a + 3 + b$

$\therefore\ (10 - 4a)x + 2a^2 - 8a - b + 4 = 0$

$\therefore\ 10 - 4a = 0$ and $2a^2 - 8a - b + 4 = 0$

$\therefore\ 4a = 10$ $\qquad \therefore\ 2(\frac{5}{2})^2 - 8(\frac{5}{2}) - b + 4 = 0$ {using $(*)$}

$\therefore\ a = \frac{5}{2}$ .... $(*)$ $\qquad \therefore\ \frac{25}{2} - 20 - b + 4 = 0$

$\therefore\ b = -\frac{7}{2}$

A reflection in the x-axis, then a translation through $\begin{pmatrix} \frac{5}{2} \\ -\frac{7}{2} \end{pmatrix}$ maps $f(x) = 2x^2 + 8x - 3$ onto $g(x) = -2x^2 + 2x + 7$.

9 $f(x) \xrightarrow[\text{scale factor } \frac{1}{2}]{\text{vertical stretch}} \frac{1}{2}f(x) \xrightarrow{\text{translation } \binom{2}{3}} \frac{1}{2}f(x-2)+3$

$\therefore$ $(-1, 6)$ on $y = f(x)$ is transformed to $(-1, 3)$ on $y = \frac{1}{2}f(x)$, then transformed to $(1, 6)$ on $y = \frac{1}{2}f(x-2)+3$.

10 **a** $f(x) \xrightarrow[\text{scale factor } 2]{\text{vertical stretch}} 2f(x) \xrightarrow{\text{translation } \binom{-1}{3}} 2f(x+1)+3$

A vertical stretch with scale factor 2, then a translation through $\begin{pmatrix} -1 \\ 3 \end{pmatrix}$ maps $y = f(x)$ onto $y = 2f(x+1)+3$.

b $f(x) \xrightarrow[\text{in } x\text{-axis}]{\text{reflection}} -f(x) \xrightarrow[\text{scale factor } \frac{3}{2}]{\text{horizontal stretch}} -f(\frac{2}{3}x) \xrightarrow{\text{translation } \binom{0}{-6}} -f(\frac{2}{3}x)-6$

A reflection in the x-axis, a horizontal stretch with scale factor $\frac{3}{2}$, then a translation through $\begin{pmatrix} 0 \\ -6 \end{pmatrix}$ maps $y = f(x)$ onto $y = -f(\frac{2}{3}x)-6$.

c $f(x) \xrightarrow[\text{scale factor } \frac{1}{3}]{\text{vertical stretch}} \frac{1}{3}f(x) \xrightarrow[\text{in } y\text{-axis}]{\text{reflection}} \frac{1}{3}f(-x) \xrightarrow{\text{translation } \binom{2}{0}} \frac{1}{3}f(-(x-2))$

A vertical stretch with scale factor $\frac{1}{3}$, a reflection in the y-axis, then a translation through $\begin{pmatrix} 2 \\ 0 \end{pmatrix}$ maps $y = f(x)$ onto $y = \frac{1}{3}f(-x+2)$.

11 $x^2 + bx + c \xrightarrow[\text{in } y\text{-axis}]{\text{reflection}} x^2 - bx + c \xrightarrow[\text{scale factor } \frac{3}{2}]{\text{horizontal stretch}} \left(\frac{2x}{3}\right)^2 - b\left(\frac{2x}{3}\right) + c = \frac{4}{9}x^2 - \frac{2b}{3}x + c$

$$\xrightarrow{\text{translation } \binom{-10}{20}} \frac{4}{9}(x+10)^2 - \frac{2b}{3}(x+10) + c + 20$$

$$= \frac{4}{9}(x^2 + 20x + 100) - \frac{2b}{3}x - \frac{20b}{3} + c + 20$$

$$= \frac{4}{9}x^2 + \left(\frac{80}{9} - \frac{2b}{3}\right)x + \left(\frac{400}{9} - \frac{20b}{3} + c + 20\right)$$

Now for this quadratic to have the same x-intercepts as $f(x)$, the sums and products of their x-intercepts must be equal.

The sums of the x-intercepts are equal, so $-\dfrac{\frac{80}{9} - \frac{2b}{3}}{\frac{4}{9}} = -\dfrac{b}{1}$

$$\therefore \frac{80}{9} - \frac{2b}{3} = \frac{4b}{9}$$

$$\therefore \frac{80}{9} = \frac{10b}{9}$$

$$\therefore b = 8$$

The products of the x-intercepts are equal, so $\dfrac{\frac{400}{9}-\frac{20b}{3}+c+20}{\frac{4}{9}}=\dfrac{c}{1}$

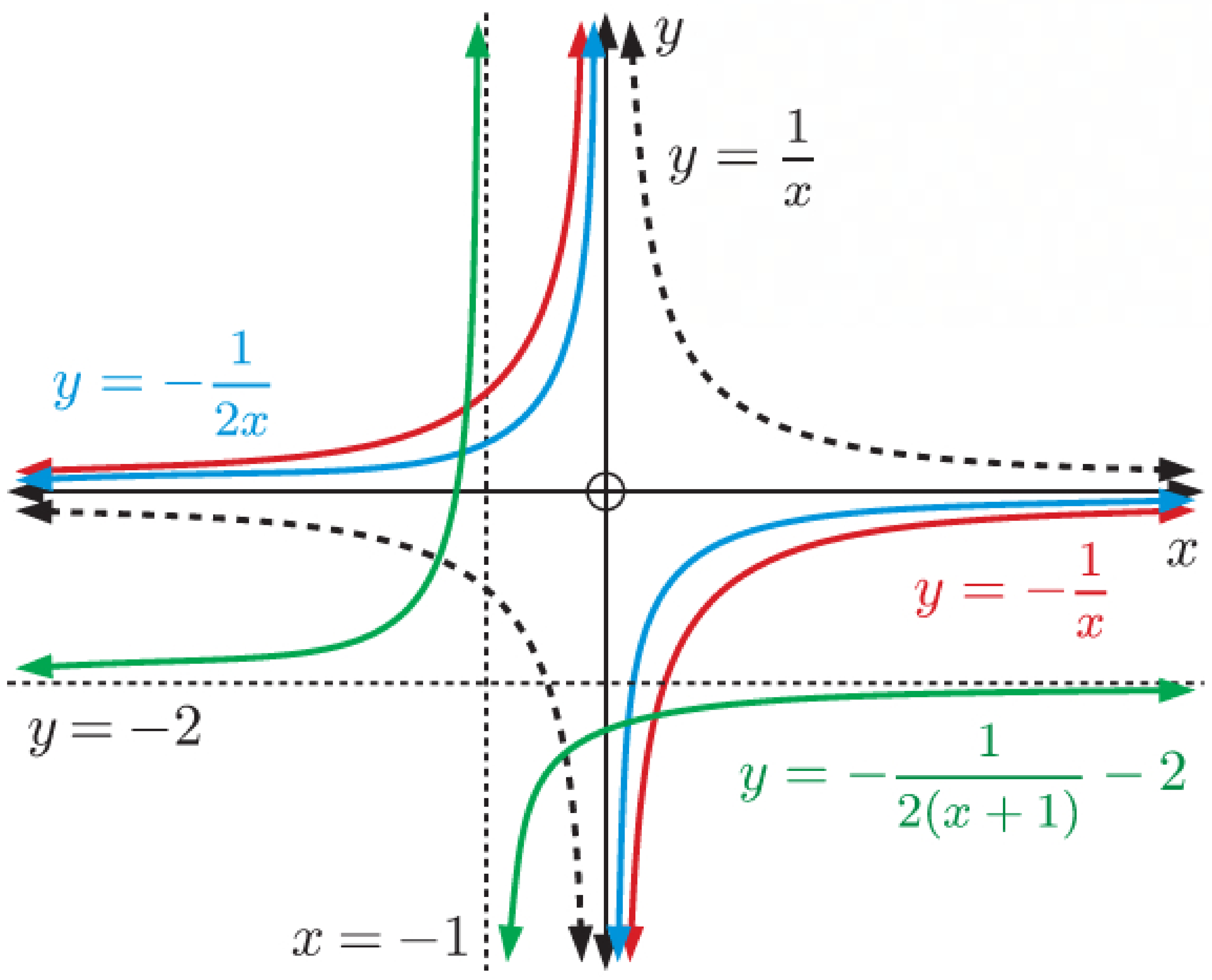
$$\therefore\ \frac{400}{9}-\frac{160}{3}+c+20=\frac{4c}{9}$$

$$\therefore\ \frac{100}{9}=-\frac{5c}{9}$$

$$\therefore\ c=-20$$

So, $b=8$ and $c=-20$.

12 **a**



b $$\frac{1}{x}\xrightarrow{\text{reflection in } x\text{-axis}}-\frac{1}{x}\xrightarrow{\text{vertical stretch scale factor } \frac{1}{2}}-\frac{1}{2x}\xrightarrow{\text{translation } \binom{-1}{-2}}-\frac{1}{2(x+1)}-2$$

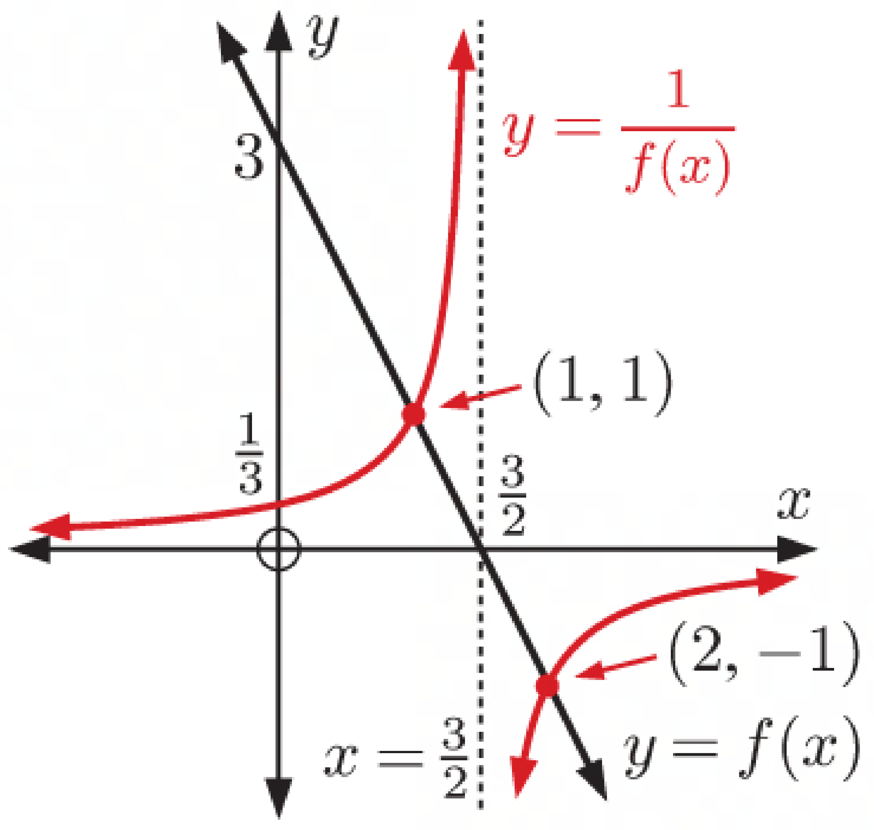
A reflection in the x-axis, a vertical stretch with scale factor $\frac{1}{2}$, then a translation through $\begin{pmatrix}-1\\-2\end{pmatrix}$ transforms $y=\dfrac{1}{x}$ into $y=-\dfrac{1}{2(x+1)}-2$.

c
$$y=-\frac{1}{2(x+1)}-2$$
$$\therefore\ y=-\frac{1}{2x+2}-\frac{2(2x+2)}{2x+2}$$
$$\therefore\ y=-\frac{1}{2x+2}-\frac{4x+4}{2x+2}$$
$$\therefore\ y=\frac{-1-4x-4}{2x+2}$$
$$\therefore\ y=\frac{-4x-5}{2x+2}$$

y is undefined when $x=-1$, and as $|x|\to\infty$, $y\to-2$.

$\therefore$ the domain is $\{x\mid x\neq-1\}$, and the range is $\{y\mid y\neq-2\}$.

13 **a, d**



b The invariant points of the graph

$y = \dfrac{1}{f(x)} = \dfrac{1}{-2x+3}$ correspond to the values of x such that $\dfrac{1}{-2x+3} = \pm 1$

$\therefore\ -2x + 3 = \pm 1$

$\therefore\ -2x = -2$ or -4

$\therefore\ x = 1$ or 2

$\therefore$ the invariant points are $(1, 1)$ and $(2, -1)$.

c $\dfrac{1}{-2x+3}$ is undefined when $-2x + 3 = 0$

$\therefore\ 2x = 3$

$\therefore\ x = \frac{3}{2}$

$\therefore$ the vertical asymptote is $x = \frac{3}{2}$.

When $x = 0$, $\dfrac{1}{-2x+3} = \dfrac{1}{-2(0)+3}$

$= \frac{1}{3}$

$\therefore$ the y-intercept is $\frac{1}{3}$.

14 $f(x) = \dfrac{c}{x+c}$

$f(0) = \dfrac{c}{0+c} = 1$

$\therefore$ the y-intercept of $y = f(x)$ is 1.

$\therefore$ the y-intercept of $y = \dfrac{1}{f(x)}$ is 1.

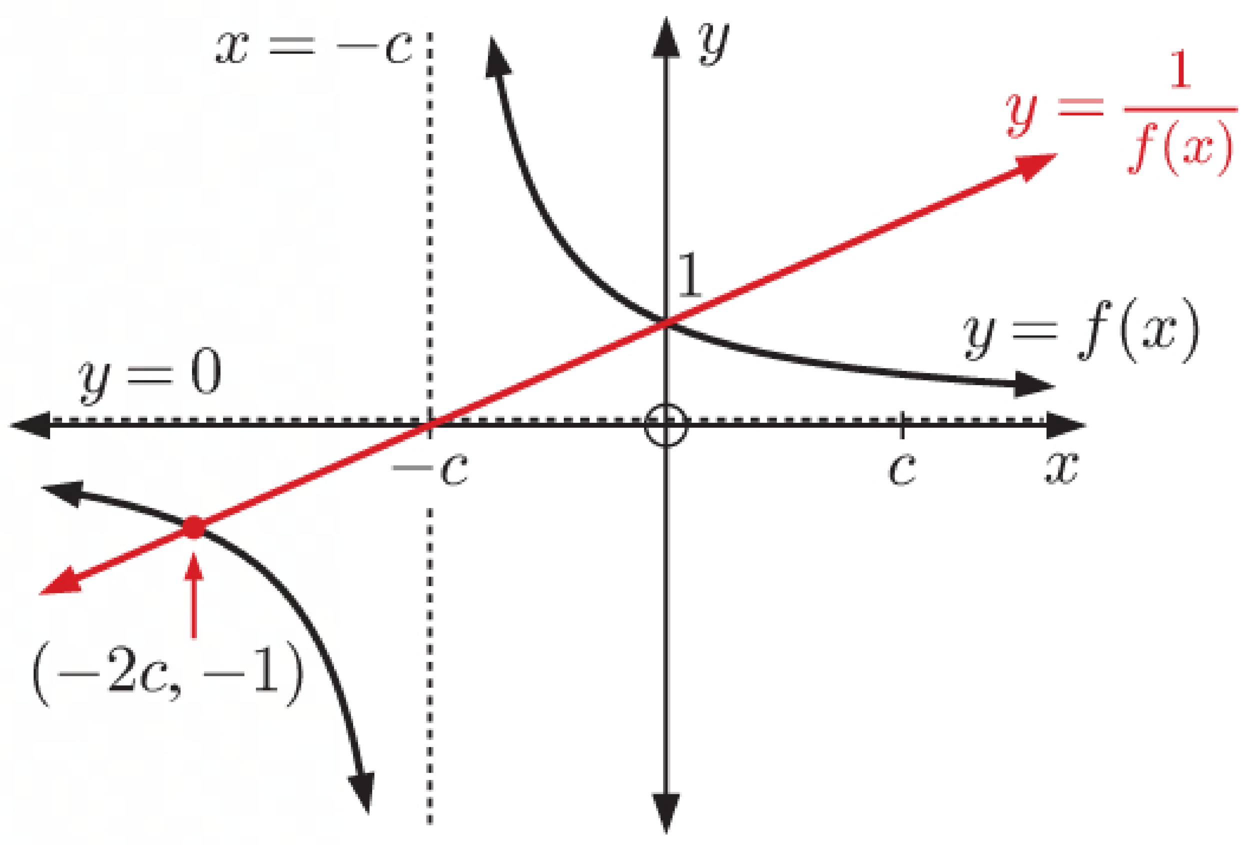
$f(x)$ is undefined when $x = -c$

$\therefore$ $y = f(x)$ has vertical asymptote $x = -c$

$\therefore$ $y = \dfrac{1}{f(x)}$ has x-intercept $-c$

$f(x) = \dfrac{c}{x+c} \neq 0$ as $c > 0$

$\therefore$ $y = f(x)$ has horizontal asymptote $y = 0$.



$y = f(x)$ and $y = \dfrac{1}{f(x)}$ meet when $f(x) = \dfrac{1}{f(x)}$

$\therefore\ \dfrac{c}{x+c} = \dfrac{x+c}{c}$

$\therefore\ c^2 = (x+c)^2$

$\therefore\ c^2 = x^2 + 2cx + c^2$

$\therefore\ x(x + 2c) = 0$

$\therefore\ x = 0$ or $x = -2c$

$f(-2c) = \dfrac{c}{-2c+c}$, $f(0) = 1$

$= \dfrac{c}{-c}$

$= -1$

$\therefore$ the graphs meet at $(-2c, -1)$ and $(0, 1)$.

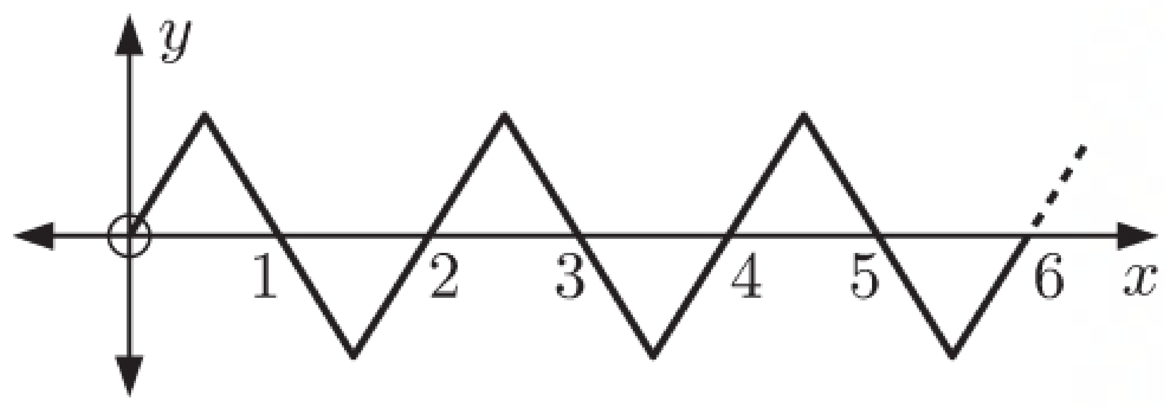
Chapter 17

TRIGONOMETRIC FUNCTIONS

EXERCISE 17A

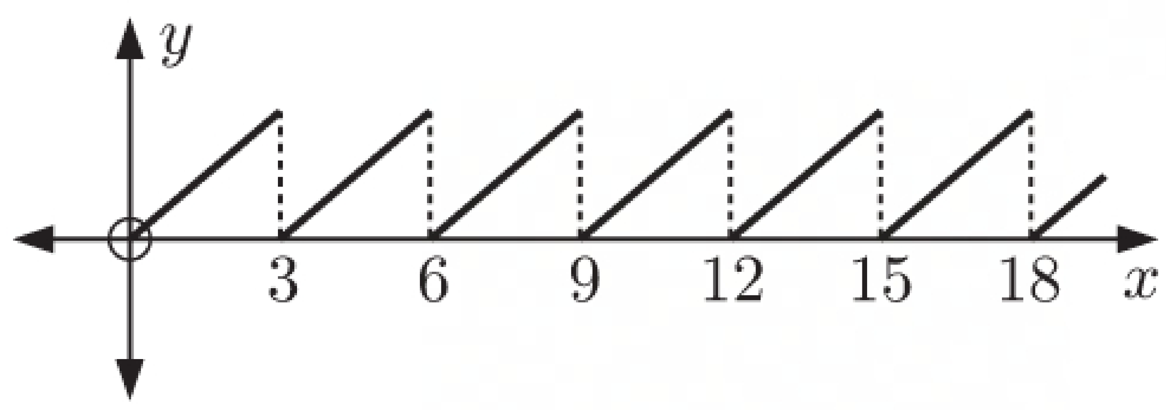
1 **a** This graph repeats itself over and over in a horizontal direction, in intervals of the same length.

$\therefore$ this graph shows periodic behaviour.



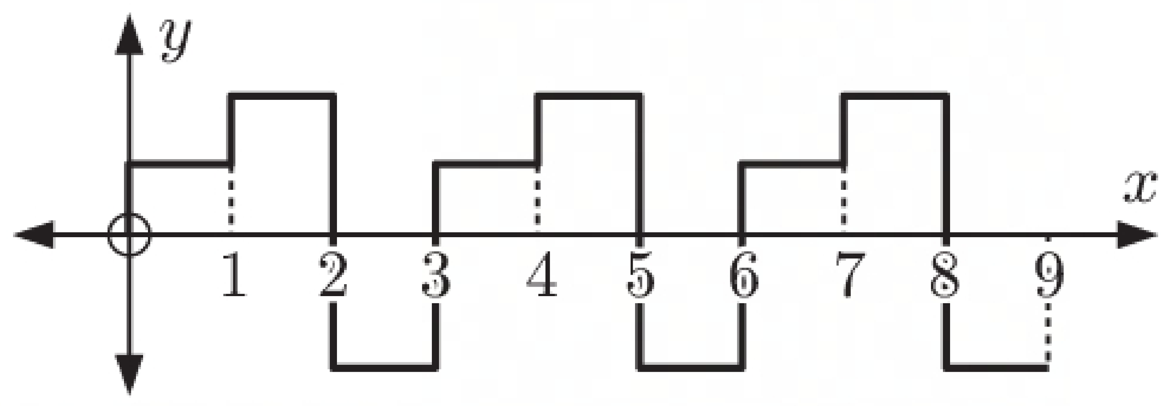
b This graph repeats itself over and over in a horizontal direction, in intervals of the same length.

$\therefore$ this graph shows periodic behaviour.



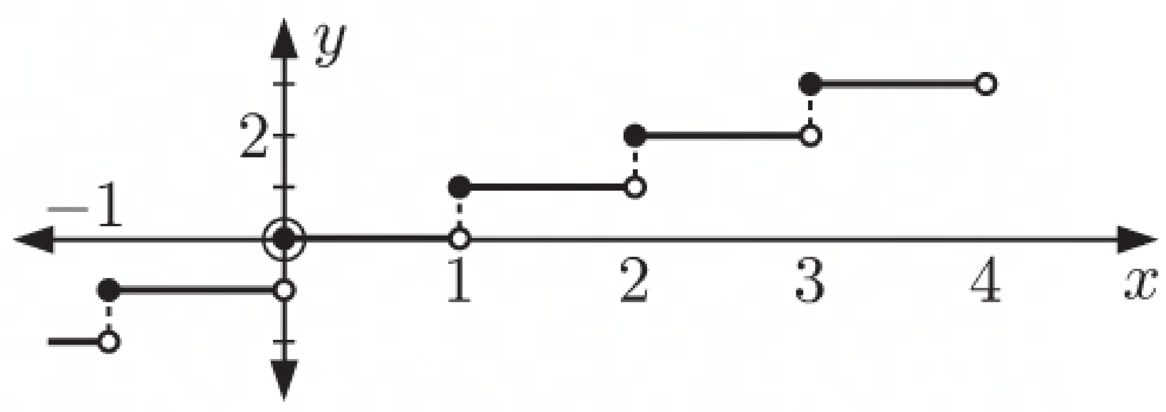
c This graph repeats itself over and over in a horizontal direction, in intervals of the same length.

$\therefore$ this graph shows periodic behaviour.



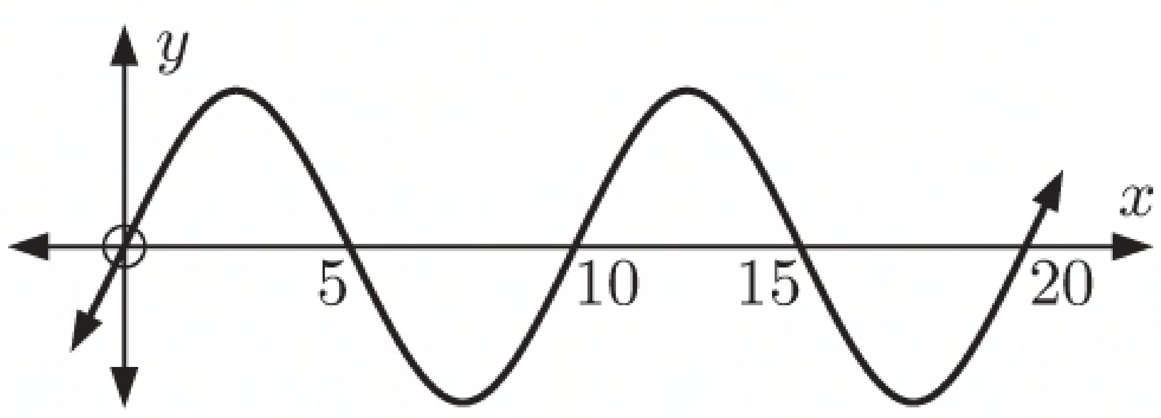
d This graph repeats itself over and over in intervals of the same length, but not in a horizontal direction.

$\therefore$ this graph does not show periodic behaviour.



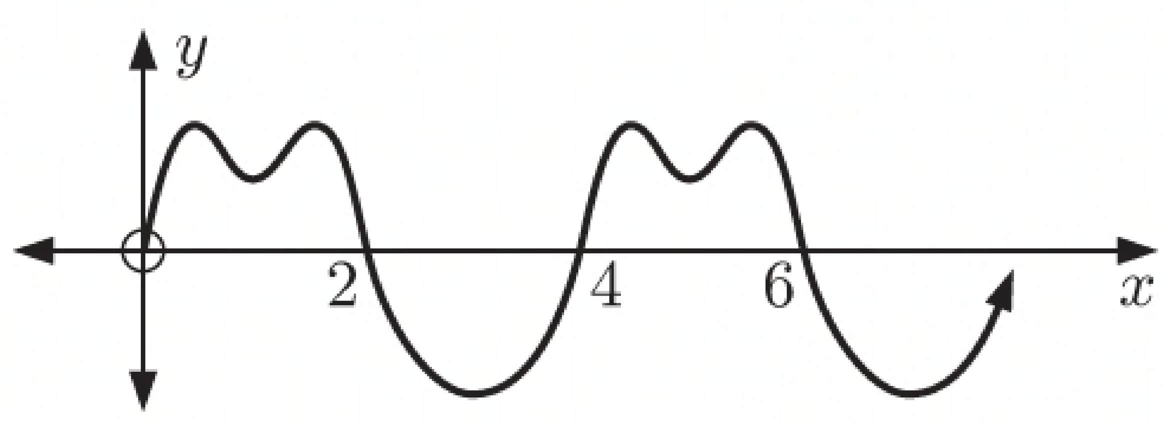
e This graph repeats itself over and over in a horizontal direction, in intervals of the same length.

$\therefore$ this graph shows periodic behaviour.



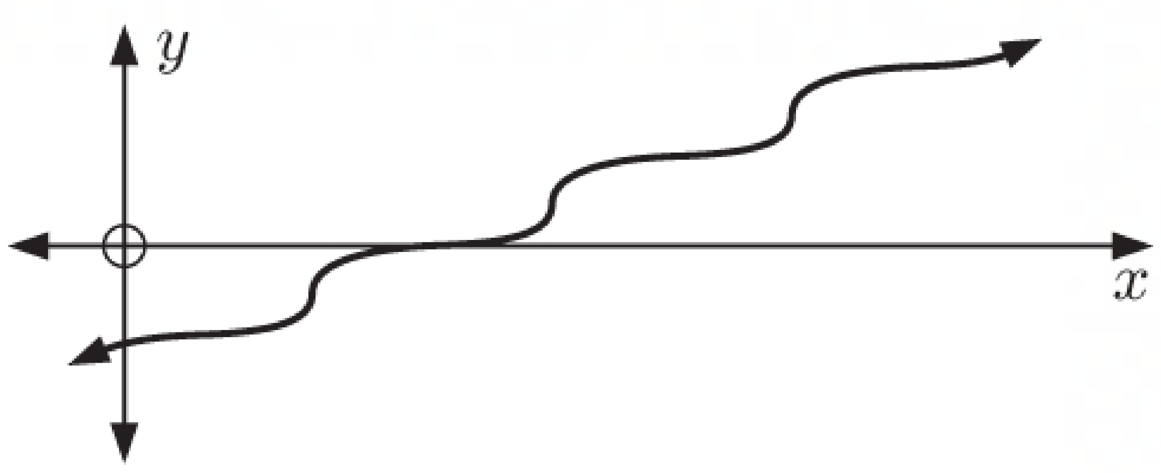
f This graph repeats itself over and over in a horizontal direction, in intervals of the same length.

$\therefore$ this graph shows periodic behaviour.



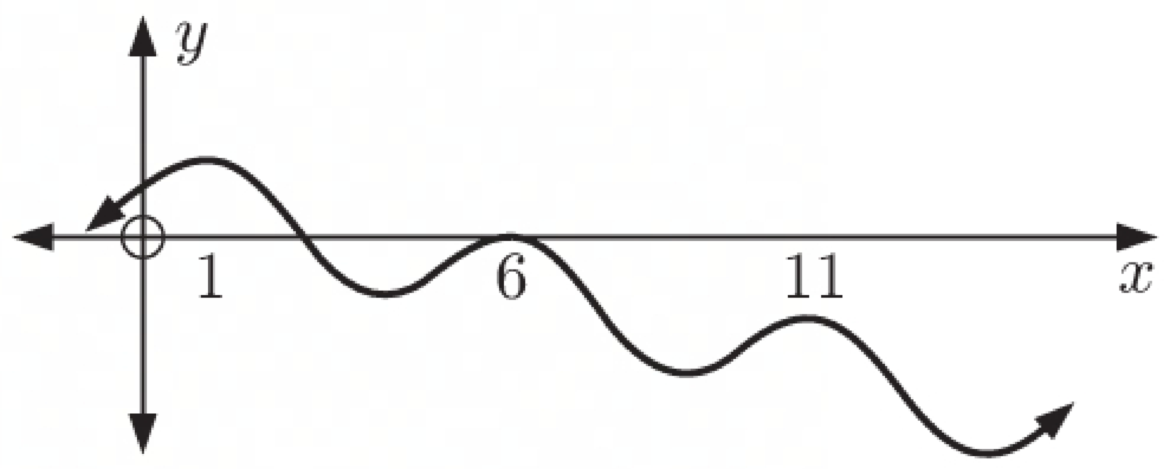
g This graph repeats itself over and over in intervals of the same length, but not in a horizontal direction.

$\therefore$ this graph does not show periodic behaviour.



h This graph repeats itself over and over in intervals of the same length, but not in a horizontal direction.

$\therefore$ this graph does not show periodic behaviour.

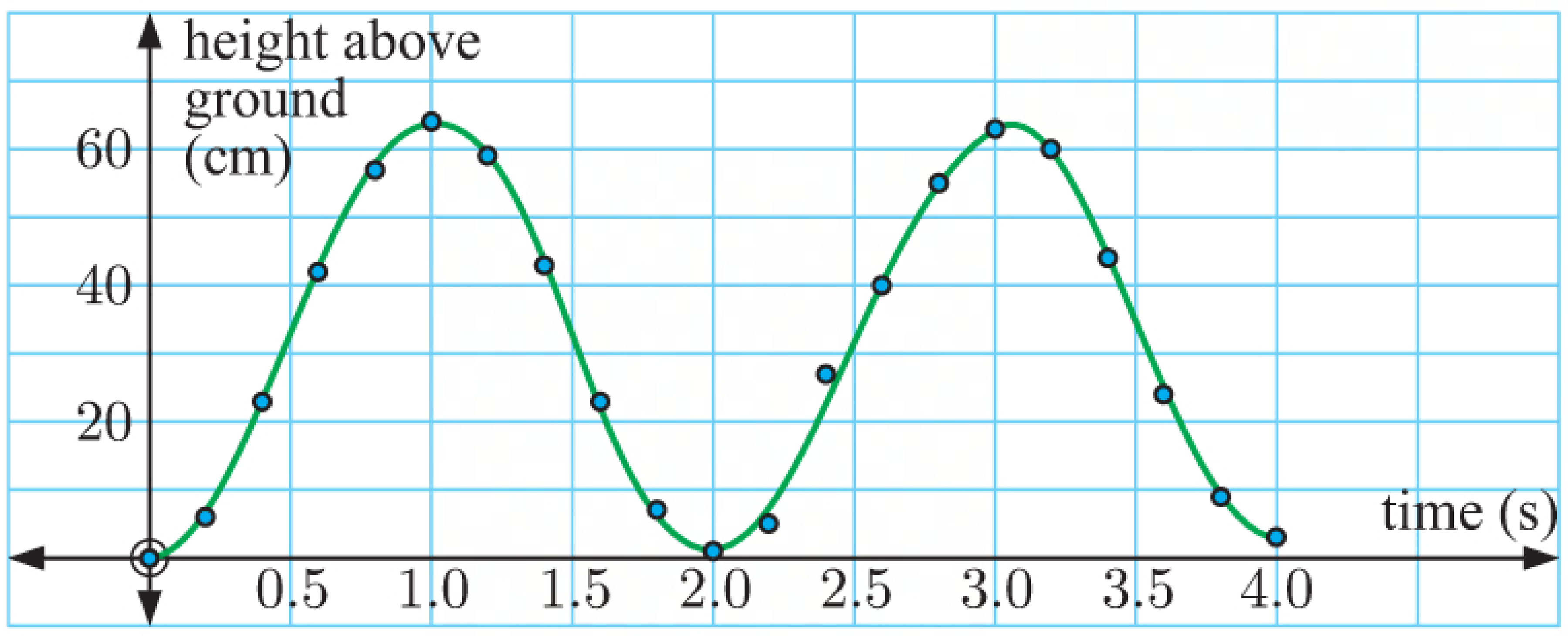


2

Time (seconds)	0	0.2	0.4	0.6	0.8	1	1.2	1.4	1.6	1.8	2
Height above ground (cm)	0	6	23	42	57	64	59	43	23	7	1

Time (seconds)	2.2	2.4	2.6	2.8	3	3.2	3.4	3.6	3.8	4
Height above ground (cm)	5	27	40	55	63	60	44	24	9	3

a



b A curve can be fitted to the data, as time is continuous.

c The graph repeats itself in a horizontal direction, in intervals of the same length.

$\therefore$ this data shows periodic behaviour.

i The minimum data value is 0 cm and the maximum data value is 64 cm.

The principal axis is $y = \dfrac{\text{max} + \text{min}}{2}$

$\approx \dfrac{64+0}{2}$

$\therefore\ y \approx 32$

ii The maximum value is ≈ 64 cm.

iii The period $\approx 3.0 - 1.0$

≈ 2 seconds

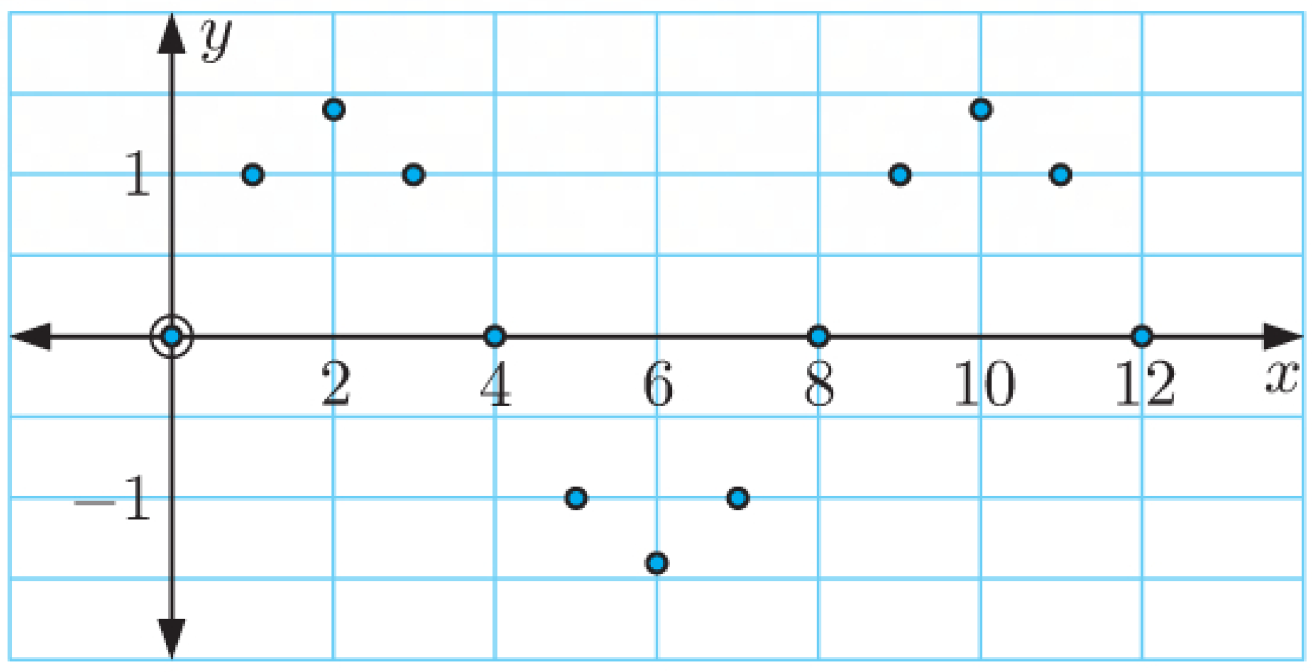
iv The amplitude $= \dfrac{\text{max} - \text{min}}{2}$

$\approx \dfrac{64-0}{2}$

≈ 32 cm

3 **a**

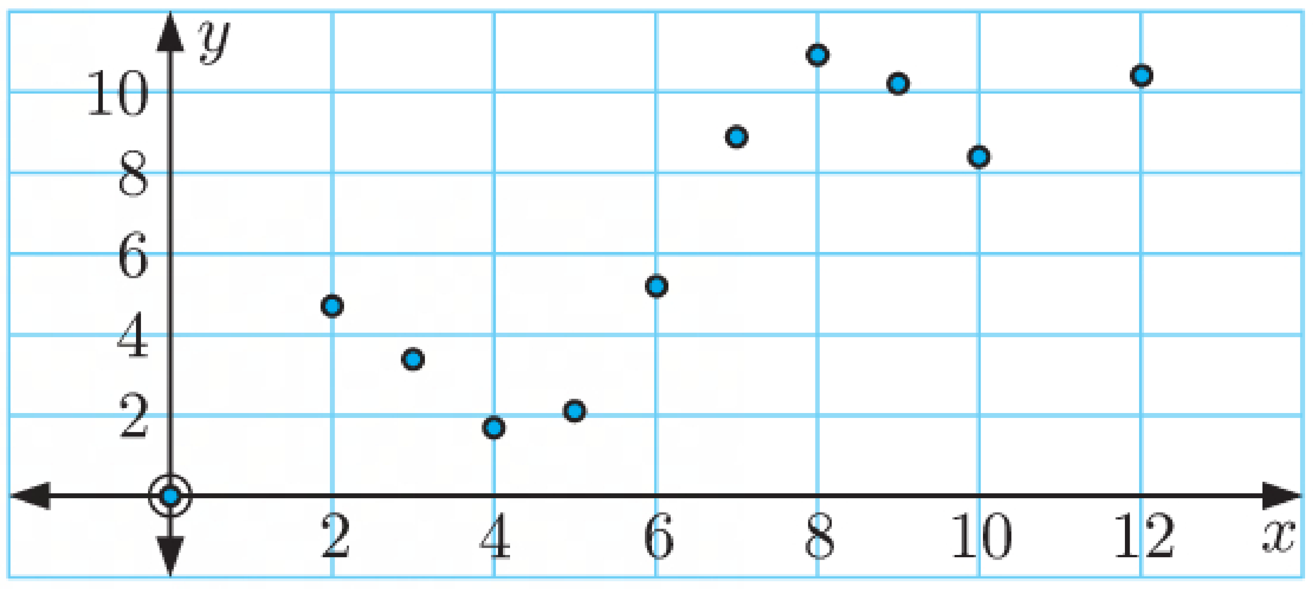
x	0	1	2	3	4	5	6	7	8	9	10	11	12
y	0	1	1.4	1	0	-1	-1.4	-1	0	1	1.4	1	0



This data exhibits periodic behaviour, as the graph repeats itself in intervals of the same length in a horizontal direction.

b

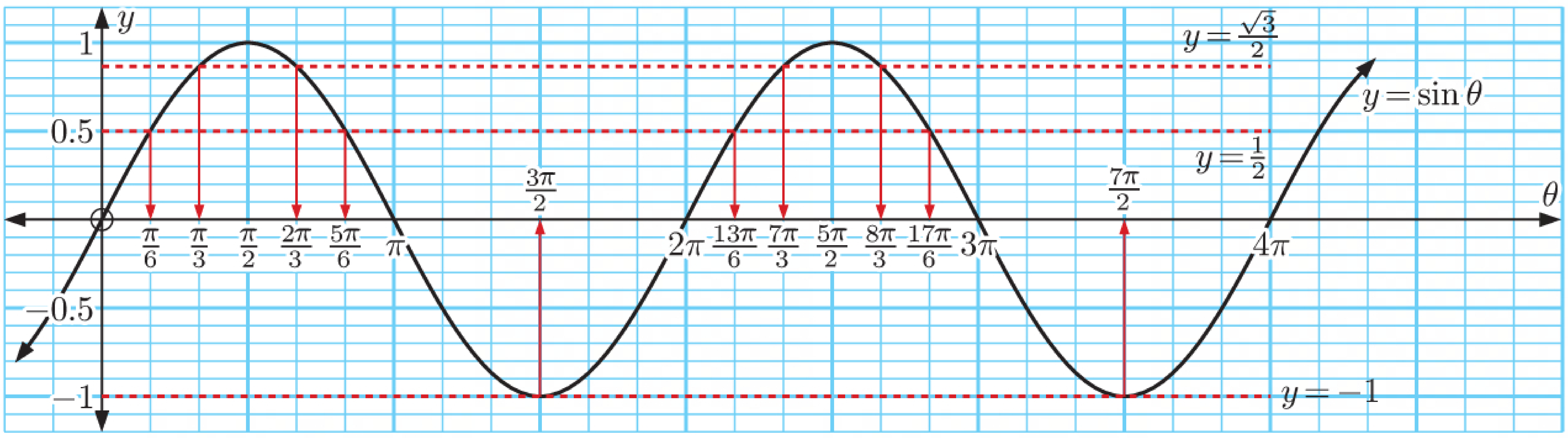
x	0	2	3	4	5	6	7	8	9	10	12
y	0	4.7	3.4	1.7	2.1	5.2	8.9	10.9	10.2	8.4	10.4



There is not enough information to say this data is periodic.

EXERCISE 17B

1



a The y-intercept is 0.

b **i** When $\sin\theta=0$, $0\leqslant\theta\leqslant 4\pi$, $\theta=0, \pi, 2\pi, 3\pi$, or 4π.

ii When $\sin\theta=-1$, $0\leqslant\theta\leqslant 4\pi$, $\theta=\frac{3\pi}{2}$ or $\frac{7\pi}{2}$.

iii When $\sin\theta=\frac{1}{2}$, $0\leqslant\theta\leqslant 4\pi$, $\theta=\frac{\pi}{6}, \frac{5\pi}{6}, \frac{13\pi}{6}$, or $\frac{17\pi}{6}$.

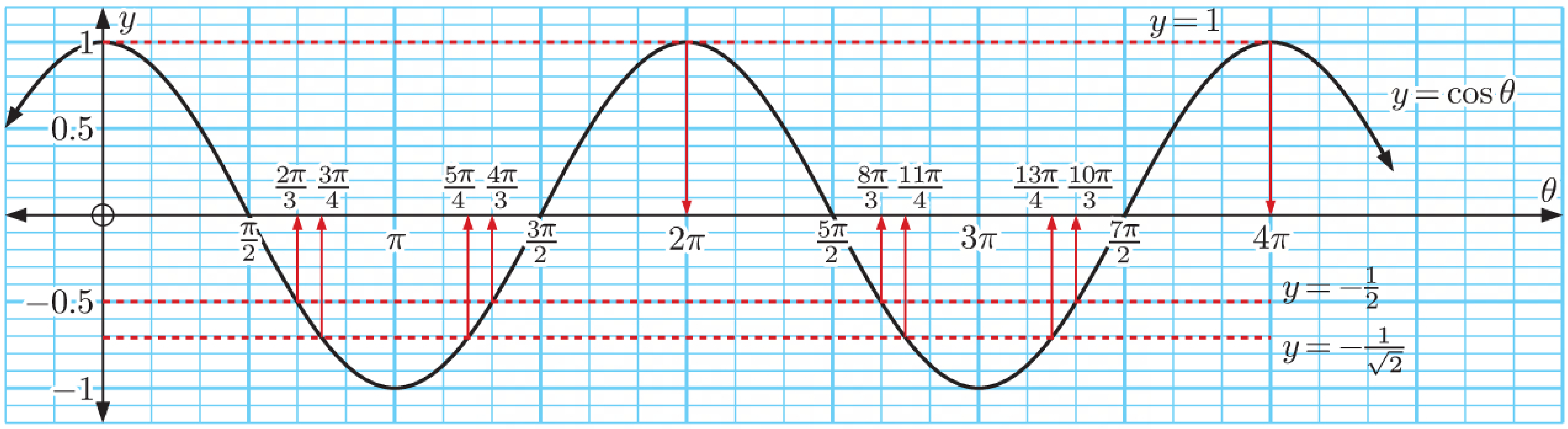
iv When $\sin\theta=\frac{\sqrt{3}}{2}$, $0\leqslant\theta\leqslant 4\pi$, $\theta=\frac{\pi}{3}, \frac{2\pi}{3}, \frac{7\pi}{3}$, or $\frac{8\pi}{3}$.

c **i** On $0\leqslant\theta\leqslant 4\pi$, $\sin\theta$ is positive for $0<\theta<\pi$, $2\pi<\theta<3\pi$.

ii On $0\leqslant\theta\leqslant 4\pi$, $\sin\theta$ is negative for $\pi<\theta<2\pi$, $3\pi<\theta<4\pi$.

d The range is $\{y \mid -1\leqslant y\leqslant 1\}$.

2



a The y-intercept is 1.

b **i** When $\cos\theta=0$, $0\leqslant\theta\leqslant 4\pi$, $\theta=\frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}$, or $\frac{7\pi}{2}$.

ii When $\cos\theta=1$, $0\leqslant\theta\leqslant 4\pi$, $\theta=0, 2\pi$, or 4π.

iii When $\cos\theta = -\frac{1}{2}$, $0 \leqslant \theta \leqslant 4\pi$, $\theta = \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{8\pi}{3}$, or $\frac{10\pi}{3}$.

iv When $\cos\theta = -\frac{1}{\sqrt{2}}$, $0 \leqslant \theta \leqslant 4\pi$, $\theta = \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{11\pi}{4}$, or $\frac{13\pi}{4}$.

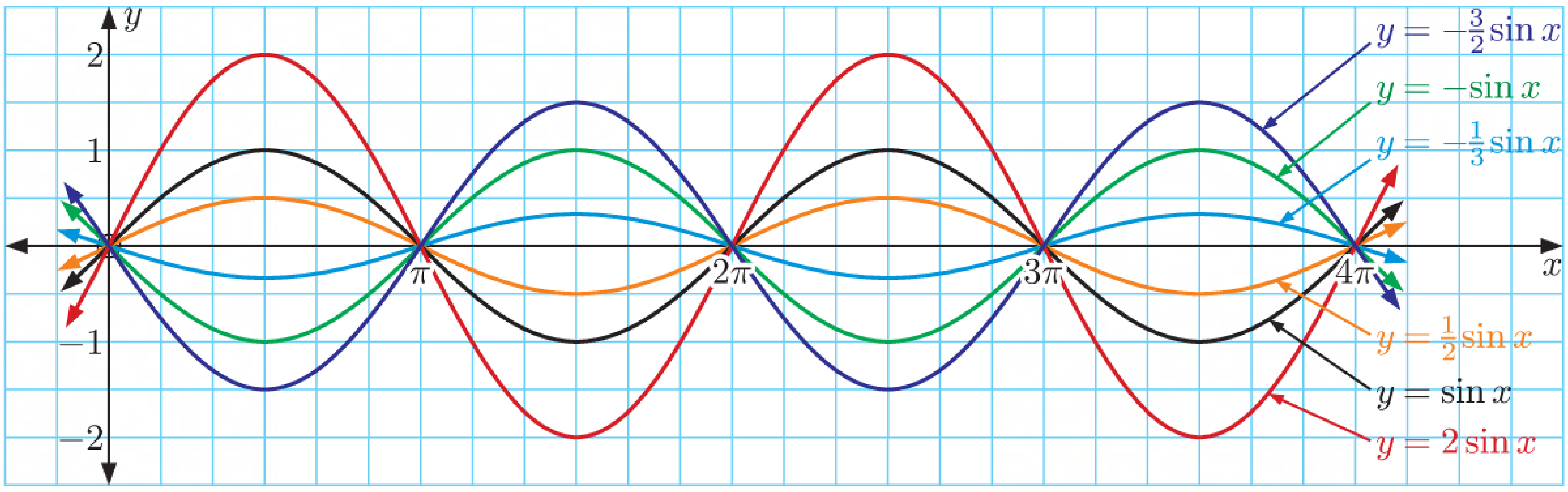
c **i** On $0 \leqslant \theta \leqslant 4\pi$, $\cos\theta$ is positive for $0 \leqslant \theta < \frac{\pi}{2}$, $\frac{3\pi}{2} < \theta < \frac{5\pi}{2}$, $\frac{7\pi}{2} < \theta \leqslant 4\pi$.

ii On $0 \leqslant \theta \leqslant 4\pi$, $\cos\theta$ is negative for $\frac{\pi}{2} < \theta < \frac{3\pi}{2}$, $\frac{5\pi}{2} < \theta < \frac{7\pi}{2}$.

d The range is $\{y \mid -1 \leqslant y \leqslant 1\}$.

INVESTIGATION FAMILIES OF TRIGONOMETRIC FUNCTIONS

1 **a**

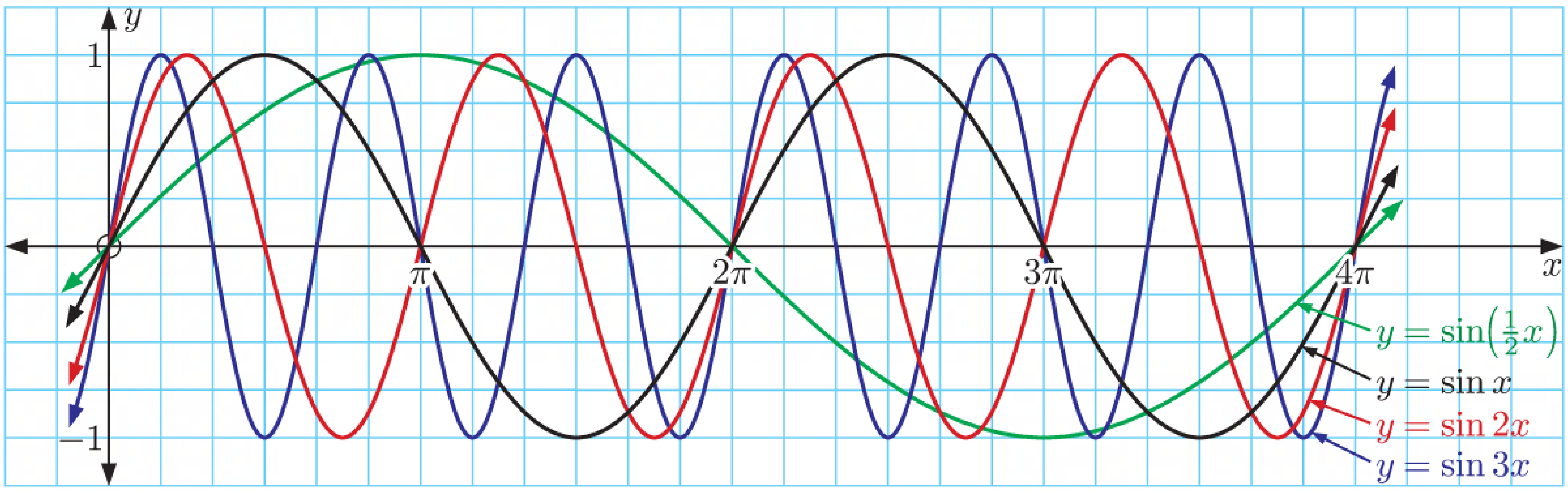


b For graphs of the form $y = a\sin x$:

i the sign of a affects where the graph is positive or negative

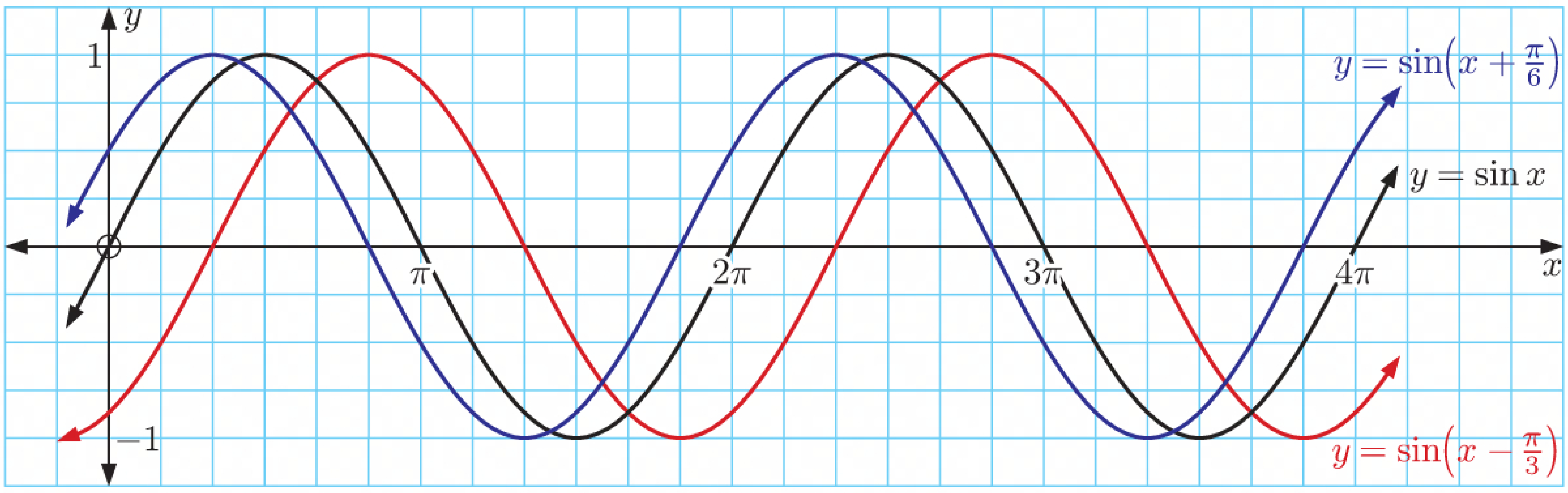
ii the size of a affects the **amplitude** of the graph.

2 **a**



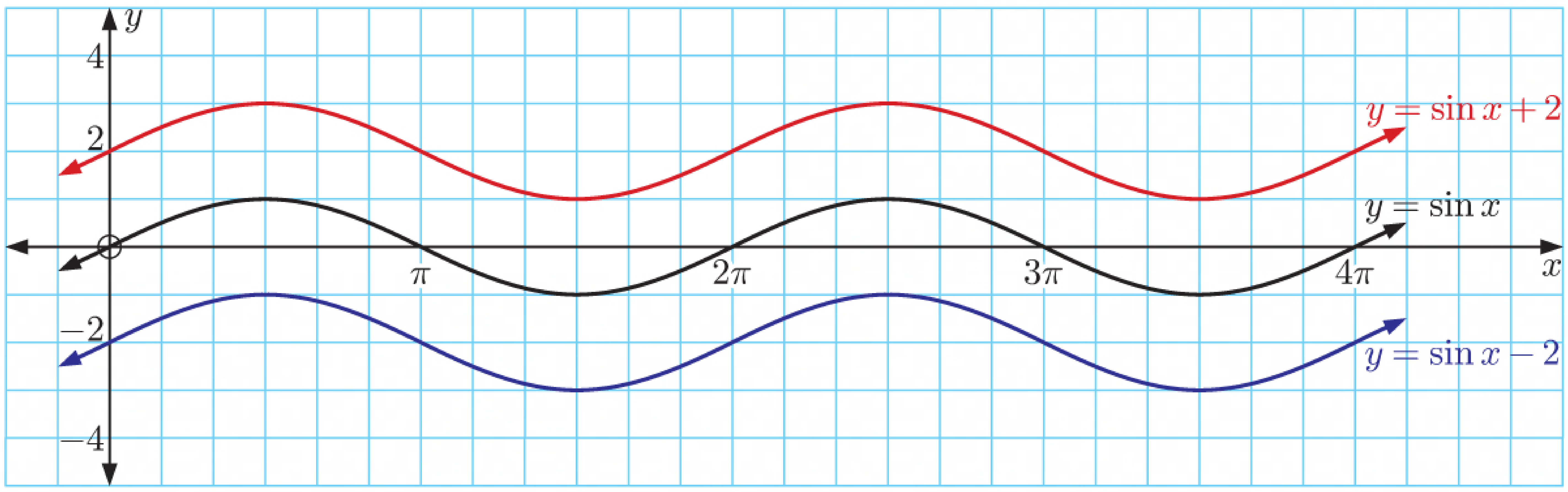
b For graphs of the form $y = \sin bx$, $b > 0$, the period is $\frac{2\pi}{b}$.

3 **a**



b A horizontal translation of c units moves $y = \sin x$ to $y = \sin(x - c)$.

4 **a**



b A vertical translation of d units moves $y = \sin x$ to $y = \sin x + d$.

c The principal axis of $y = \sin x + d$ is $y = d$.

5 $y = a\sin(b(x-c)) + d$ is obtained from $y = \sin x$ by a vertical stretch with scale factor a and a horizontal stretch with scale factor $\frac{1}{b}$, followed by a horizontal translation of c units and a vertical translation of d units.

EXERCISE 17C

1 **a** $y = \sin x - 1$ is a vertical translation of $y = \sin x$ downwards by 1 unit.
So, a vertical translation 1 unit downwards will map $y = \sin x$ onto $y = \sin x - 1$.

b $y = \sin\left(x - \frac{\pi}{4}\right)$ is a horizontal translation of $y = \sin x$ to the right by $\frac{\pi}{4}$ units.
So, a horizontal translation of $\frac{\pi}{4}$ units to the right will map $y = \sin x$ onto $y = \sin\left(x - \frac{\pi}{4}\right)$.

c $y = 2\sin x$ is a vertical stretch of $y = \sin x$ with scale factor 2.
So, a vertical stretch with scale factor 2 will map $y = \sin x$ onto $y = 2\sin x$.

d $y = \sin 4x$ is a horizontal stretch of $y = \sin x$ with scale factor $\frac{1}{4}$.
So, a horizontal stretch with scale factor $\frac{1}{4}$ will map $y = \sin x$ onto $y = \sin 4x$.

e $y = \sin\frac{x}{4}$ is a horizontal stretch of $y = \sin x$ with scale factor $\dfrac{1}{\frac{1}{4}} = 4$.
So, a horizontal stretch with scale factor 4 will map $y = \sin x$ onto $y = \sin\frac{x}{4}$.

f $y = \sin\left(x - \frac{\pi}{3}\right) + 2$ is a horizontal translation of $y = \sin x$ to the right by $\frac{\pi}{3}$ units followed by a vertical translation upwards by 2 units.
So, a translation of $\begin{pmatrix} \frac{\pi}{3} \\ 2 \end{pmatrix}$ will map $y = \sin x$ onto $y = \sin\left(x - \frac{\pi}{3}\right) + 2$.

2 **a** $y = \frac{1}{2}\cos x$ is a vertical stretch of $y = \cos x$ with scale factor $\frac{1}{2}$.
So, a vertical stretch with scale factor $\frac{1}{2}$ will map $y = \cos x$ on $y = \frac{1}{2}\cos x$.

b $y = -\cos x$ is a reflection of $y = \cos x$ in the x-axis.
So, a reflection of $y = \cos x$ in the x-axis will map $y = \cos x$ onto $y = -\cos x$.

c $y = \cos\left(x + \frac{\pi}{6}\right) - 2$ is a horizontal translation of $y = \cos x$ to the left $\frac{\pi}{6}$ units followed by a vertical translation downwards by 2 units.

3 **a** $y = \sin 5x$ has period $\frac{2\pi}{b} = \frac{2\pi}{5}$

b $y = \sin(0.6x)$ has period $\frac{2\pi}{b} = \frac{2\pi}{0.6}$
$= \frac{2\pi}{\frac{3}{5}}$
$= \frac{10\pi}{3}$

c $y = \sin \pi x$ has period $\frac{2\pi}{b} = \frac{2\pi}{\pi}$
$= 2$

d $y = \cos 3x$ has period $\frac{2\pi}{b} = \frac{2\pi}{3}$

e $y = \cos \frac{x}{3}$ has period $\frac{2\pi}{b} = \frac{2\pi}{\frac{1}{3}}$
$= 6\pi$

f $y = \cos \frac{\pi x}{50}$ has period $\frac{2\pi}{b} = \frac{2\pi}{\frac{\pi}{50}}$
$= 100$

4 $y = \sin bx, \quad b > 0$

a period $= \frac{2\pi}{b}$
$\therefore \quad 5\pi = \frac{2\pi}{b}$
$\therefore \quad b = \frac{2}{5}$

b period $= \frac{2\pi}{b}$
$\therefore \quad \frac{2\pi}{3} = \frac{2\pi}{b}$
$\therefore \quad b = 3$

c period $= \frac{2\pi}{b}$
$\therefore \quad 12\pi = \frac{2\pi}{b}$
$\therefore \quad b = \frac{1}{6}$

d period $= \frac{2\pi}{b}$
$\therefore \quad 4 = \frac{2\pi}{b}$
$\therefore \quad 4b = 2\pi$
$\therefore \quad b = \frac{\pi}{2}$

e period $= \frac{2\pi}{b}$
$\therefore \quad 100 = \frac{2\pi}{b}$
$\therefore \quad 100b = 2\pi$
$\therefore \quad b = \frac{\pi}{50}$

5 **a** $y = 4\cos 2x$ has maximum value $4(1) = 4$ $\{$when $\cos 2x = 1\}$
and minimum value $4(-1) = -4$ $\{$when $\cos 2x = -1\}$

b $y = 3\cos x + 5$ has maximum value $3(1) + 5 = 8$ $\{$when $\cos x = 1\}$
and minimum value $3(-1) + 5 = 2$ $\{$when $\cos x = -1\}$

c $y = -2\cos(x-3) - 4$ has maximum value $-2(-1) - 4 = -2$
$\{$when $\cos(x-3) = -1\}$
and minimum value $-2(1) - 4 = -6$
$\{$when $\cos(x-3) = 1\}$

6 **a** The amplitude is $a = 4$.

b The period is $\frac{2\pi}{b} = \frac{2\pi}{3}$.

c $y = 4\sin 3x + 2$ has maximum value $4(1) + 2 = 6$ $\{$when $\sin 3x = 1\}$
and minimum value $4(-1) + 2 = -2$ $\{$when $\sin 3x = -1\}$

$\therefore$ the range is $\{y \mid -2 \leqslant y \leqslant 6\}$.

7 $y = a\cos(b(x-c)) + d$

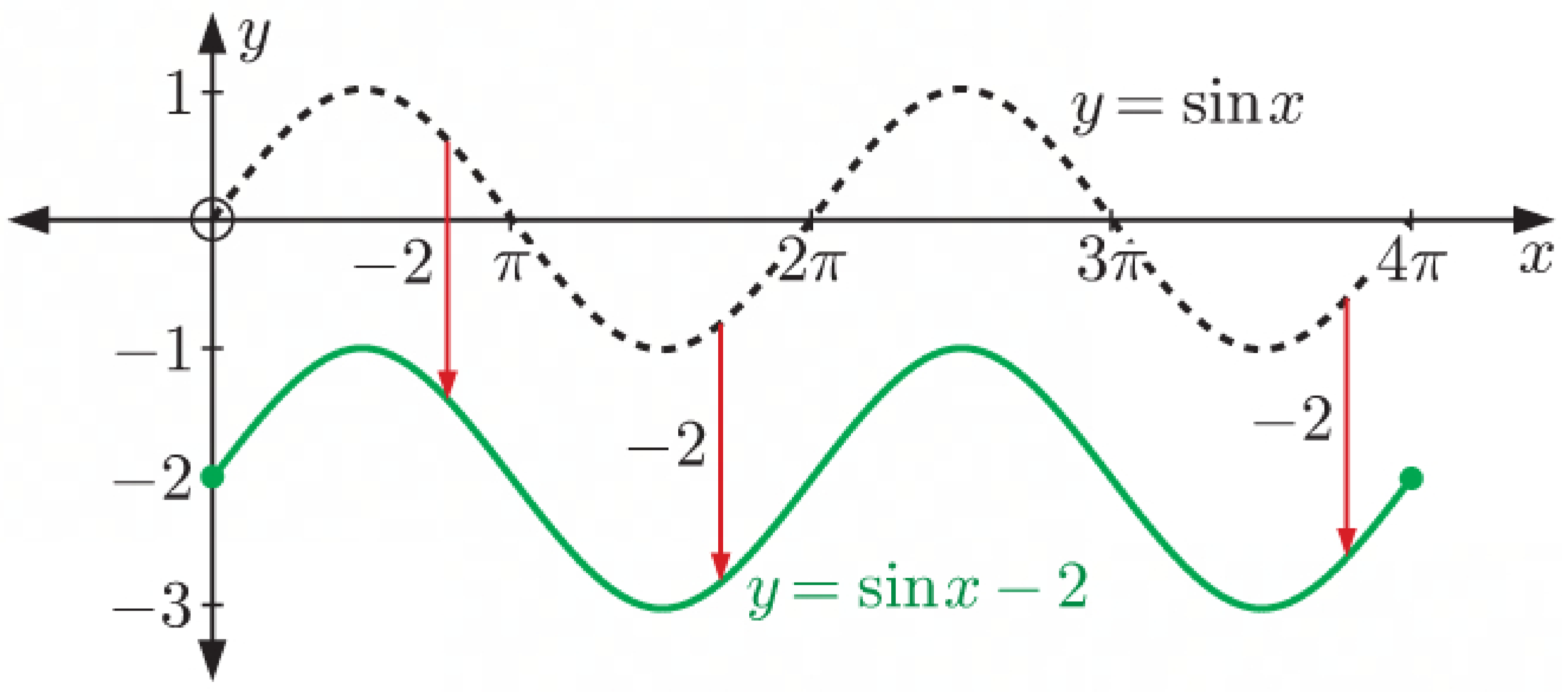
a controls the amplitude $\{\text{amplitude} = |\,a\,|\}$.

b controls the period $\{\text{period} = \frac{2\pi}{b}\}$.

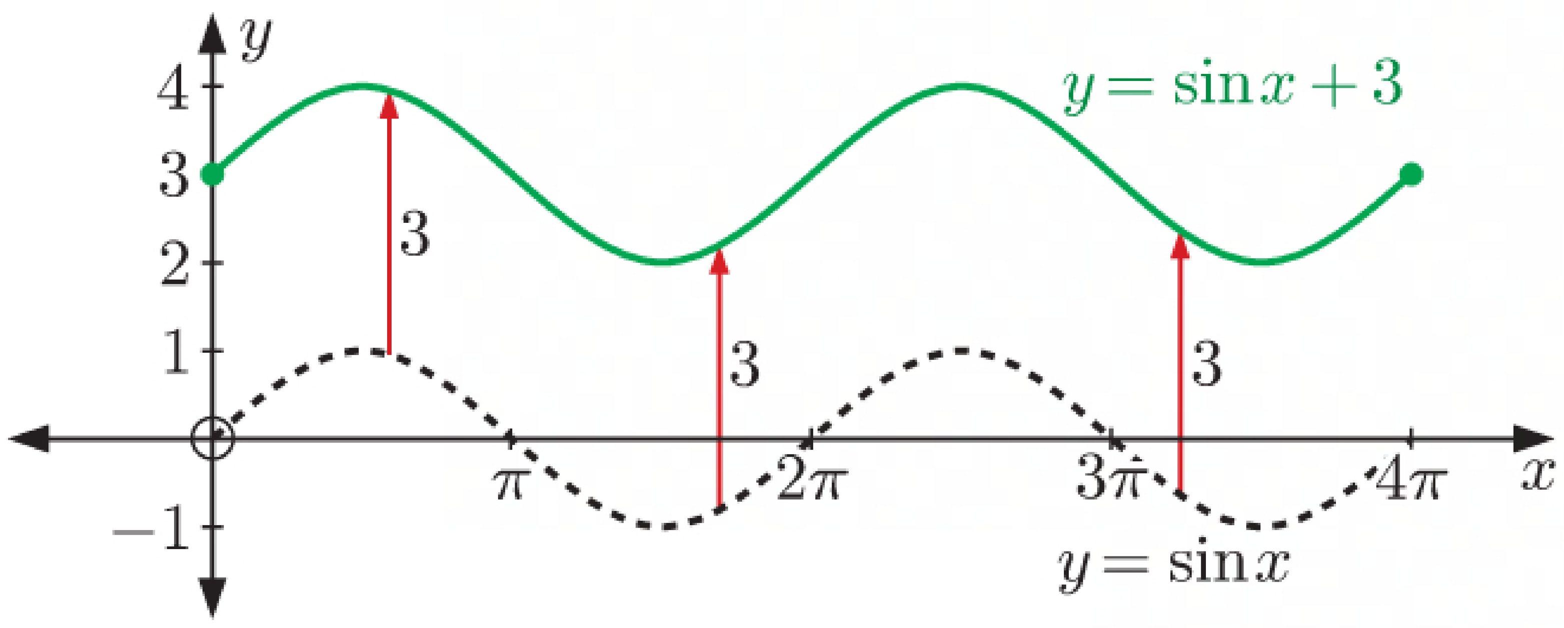
c controls the horizontal translation.

d controls the vertical translation.

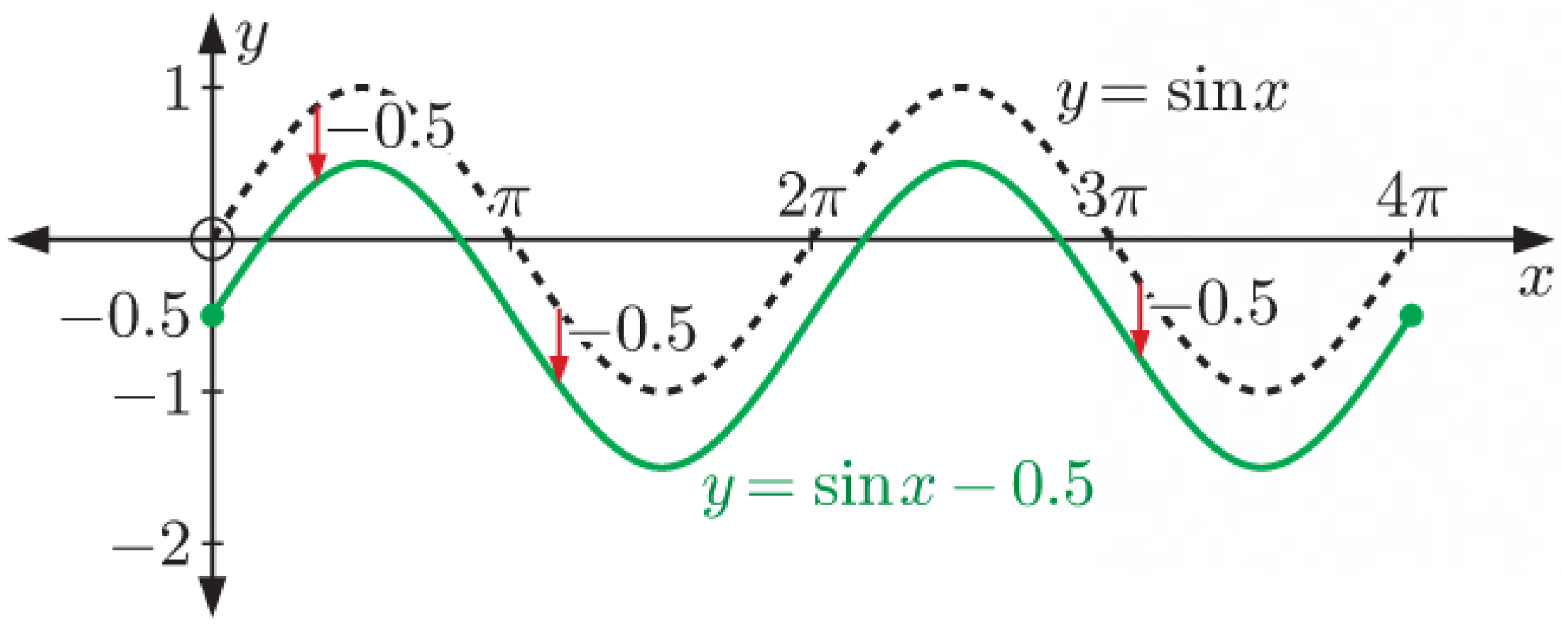
8 **a** $y = \sin x - 2$ is a vertical translation of $y = \sin x$ downwards by 2 units.



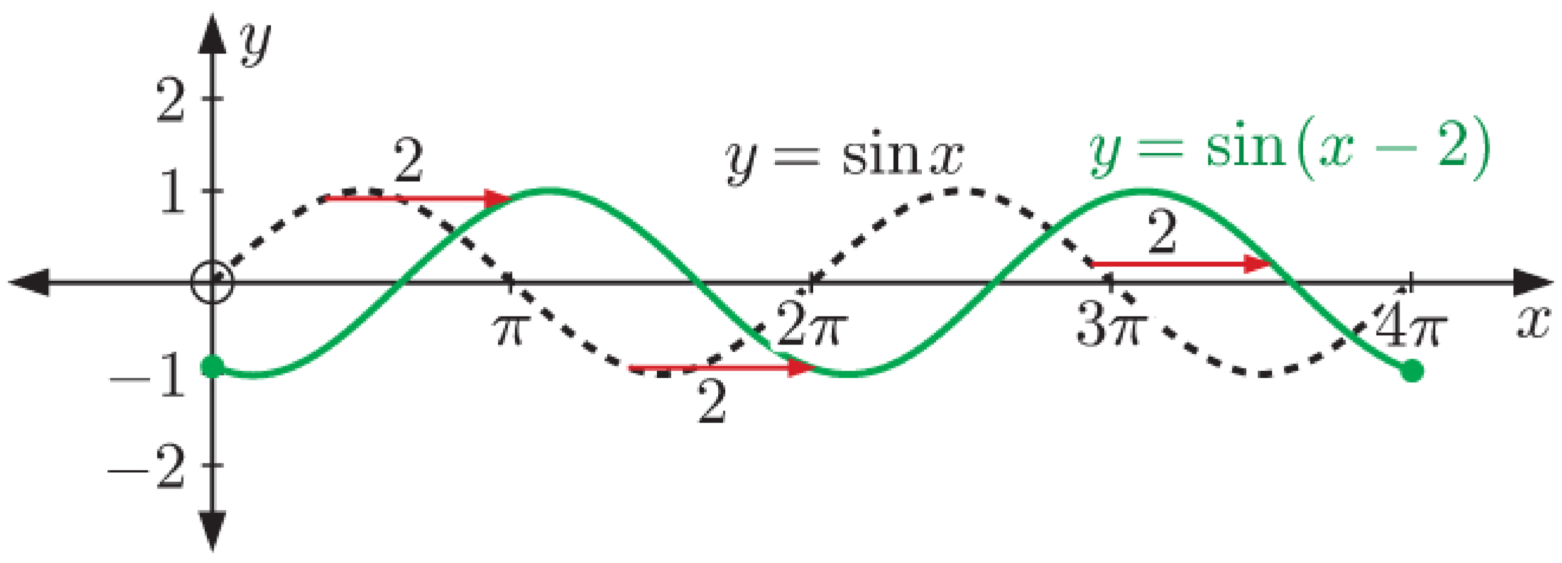
b $y = \sin x + 3$ is a vertical translation of $y = \sin x$ upwards by 3 units.



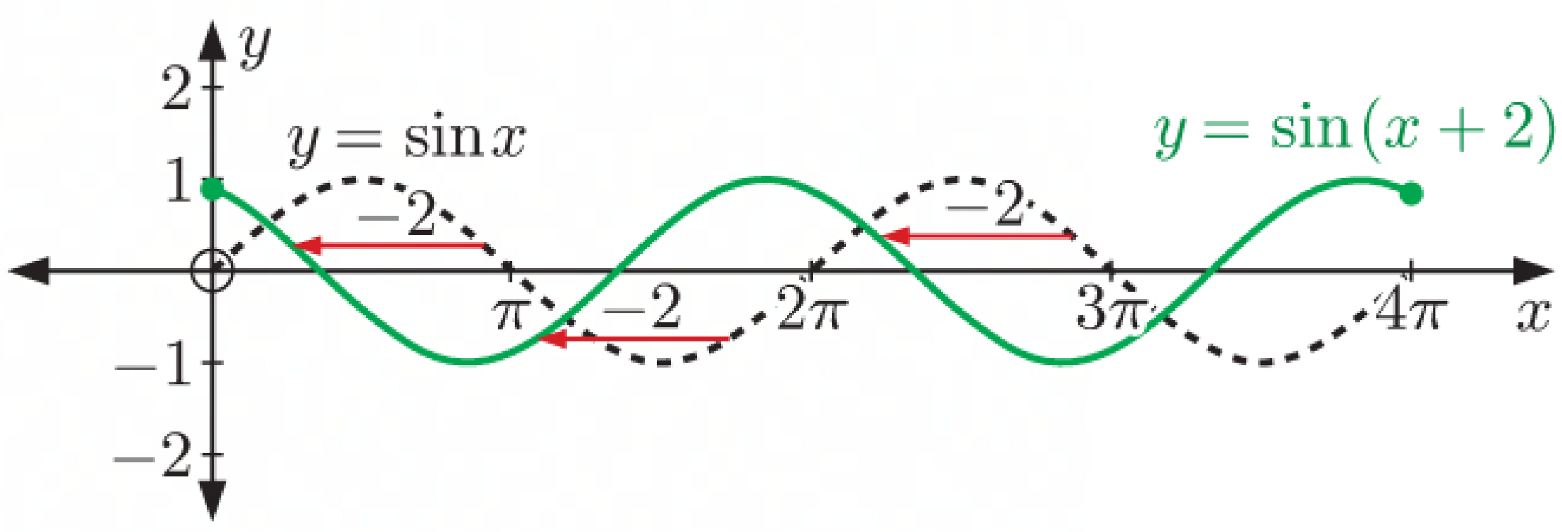
c $y = \sin x - 0.5$ is a vertical translation of $y = \sin x$ downwards by 0.5 units.



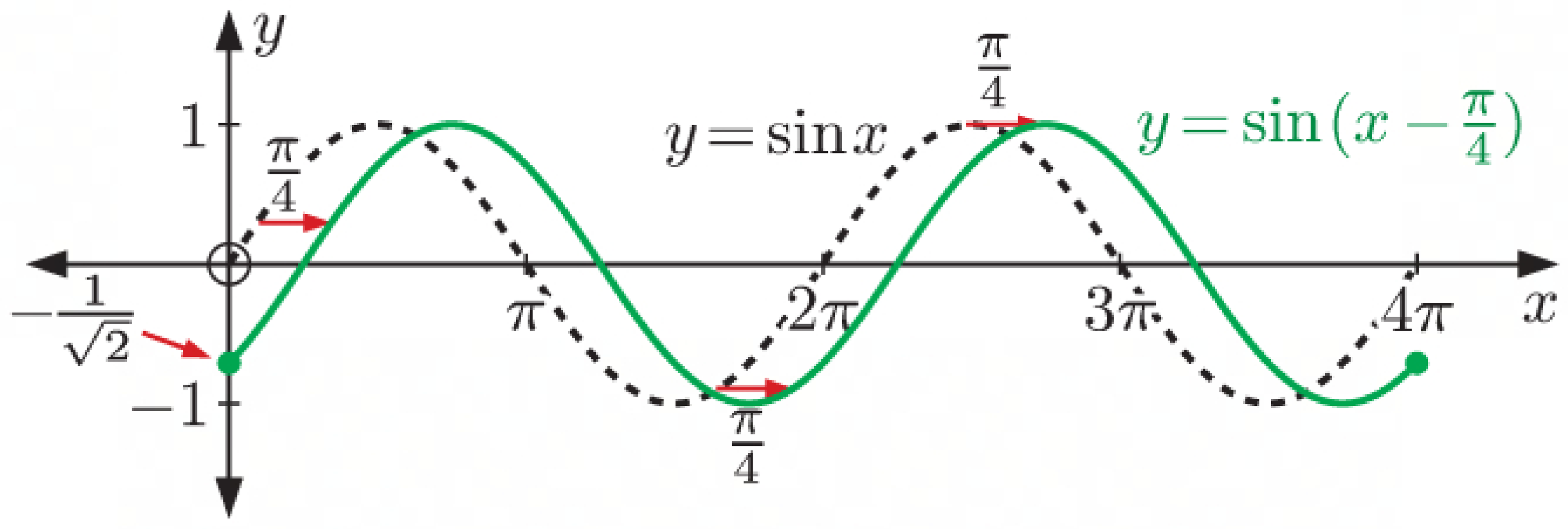
d $y = \sin(x-2)$ is a horizontal translation of $y = \sin x$ to the right by 2 units.



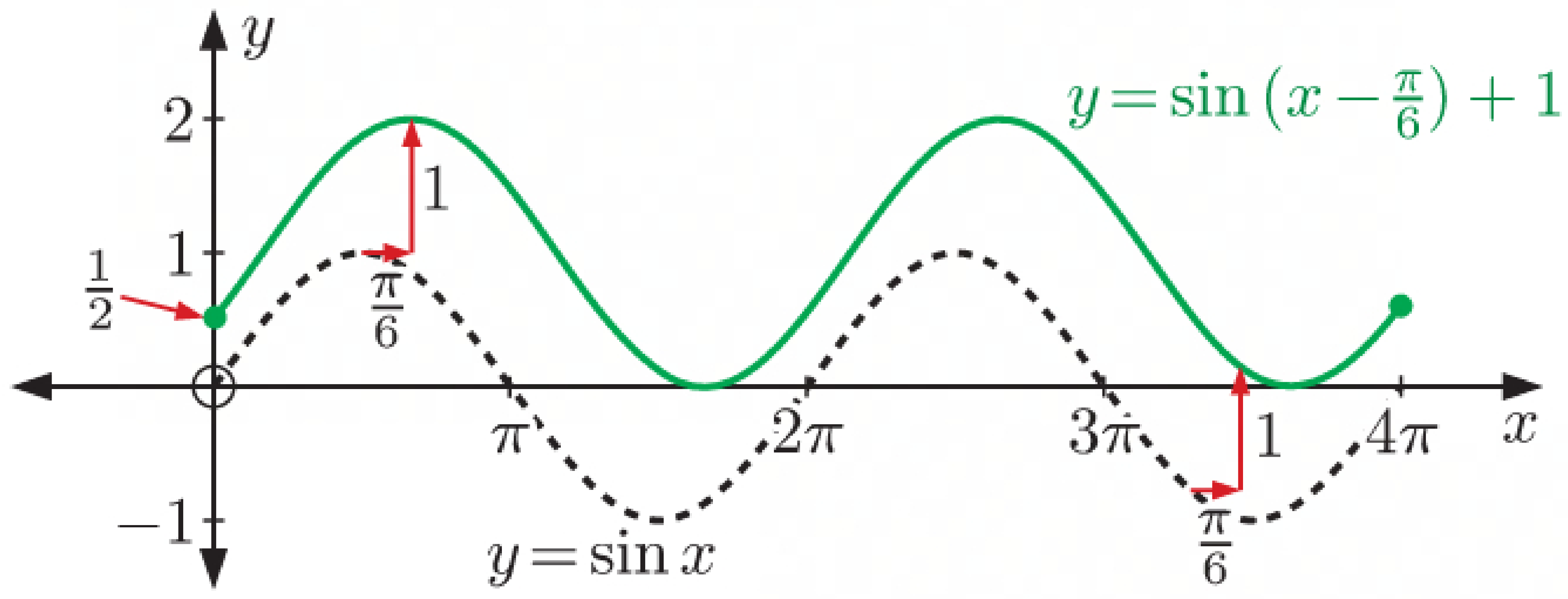
e $y = \sin(x+2)$ is a horizontal translation of $y = \sin x$ to the left by 2 units.



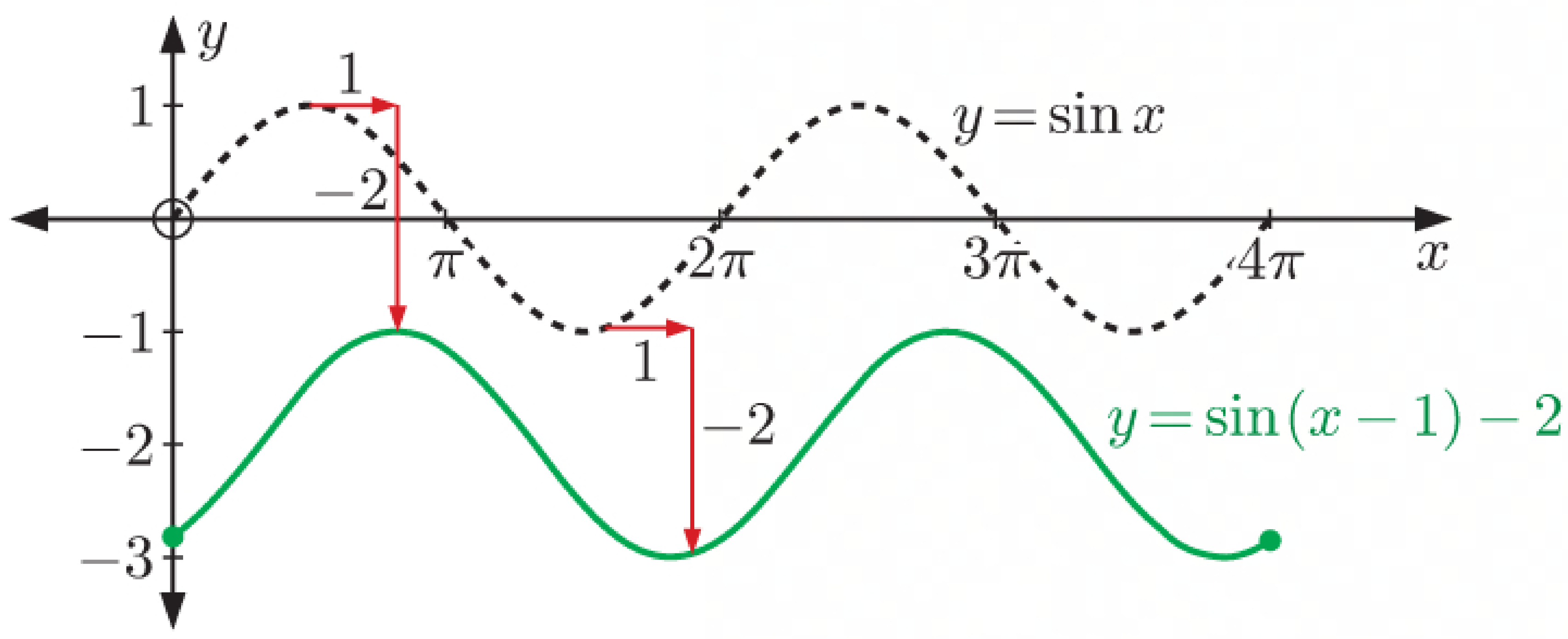
f $y = \sin\left(x - \frac{\pi}{4}\right)$ is a horizontal translation of $y = \sin x$ to the right by $\frac{\pi}{4}$ units.



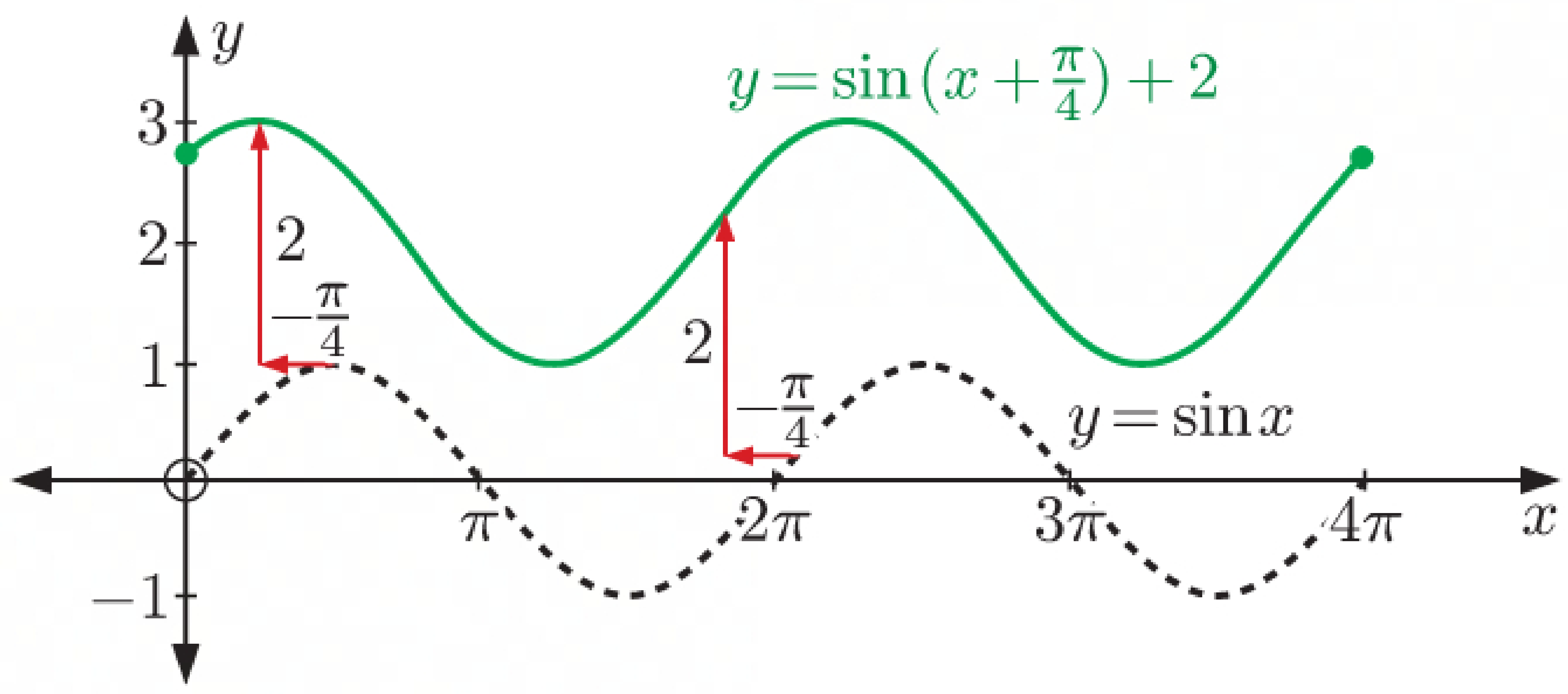
g $y = \sin\left(x - \frac{\pi}{6}\right) + 1$ is a translation of $y = \sin x$ to the right by $\frac{\pi}{6}$ units and upwards by 1 unit.



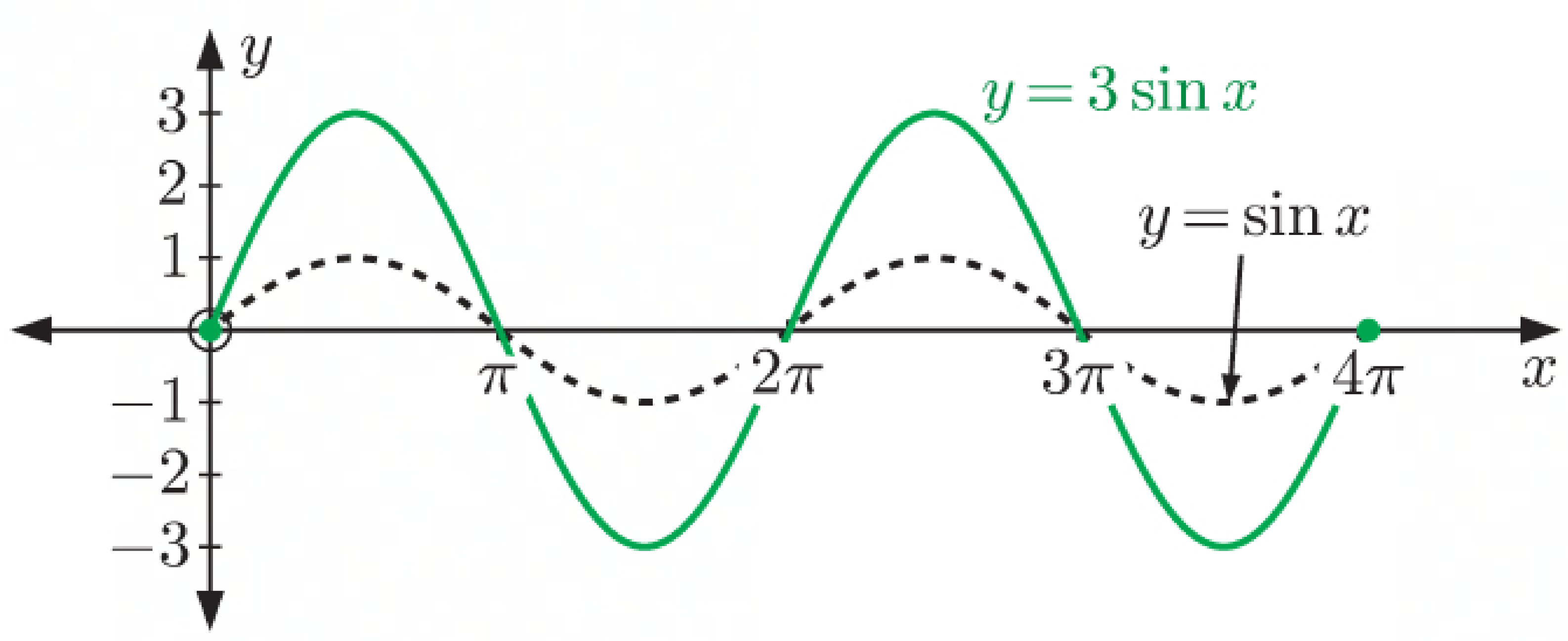
h $y = \sin(x - 1) - 2$ is a translation of $y = \sin x$ to the right by 1 unit and downwards by 2 units.



i $y = \sin\left(x + \frac{\pi}{4}\right) + 2$ is a translation of $y = \sin x$ to the left by $\frac{\pi}{4}$ units and upwards by 2 units.

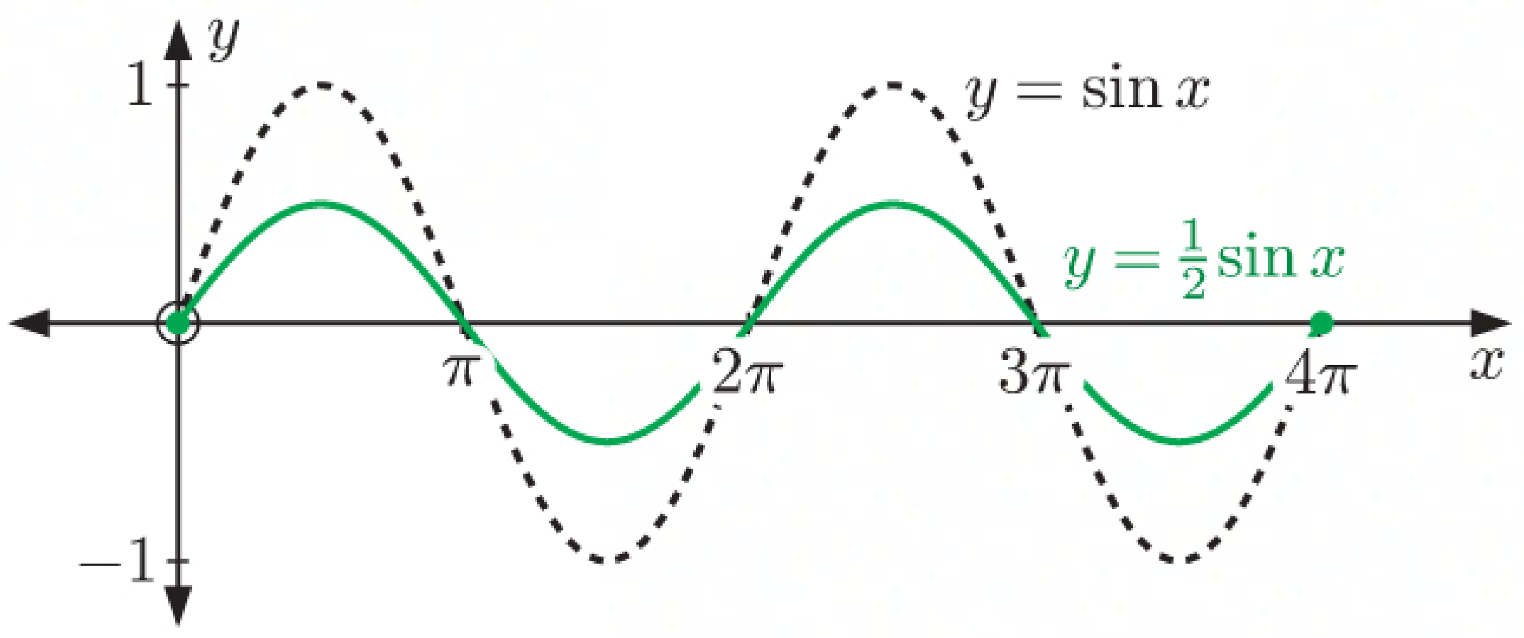


j $y = 3\sin x$ is a vertical stretch of $y = \sin x$ with scale factor 3.
The amplitude is 3 and the period is 2π.



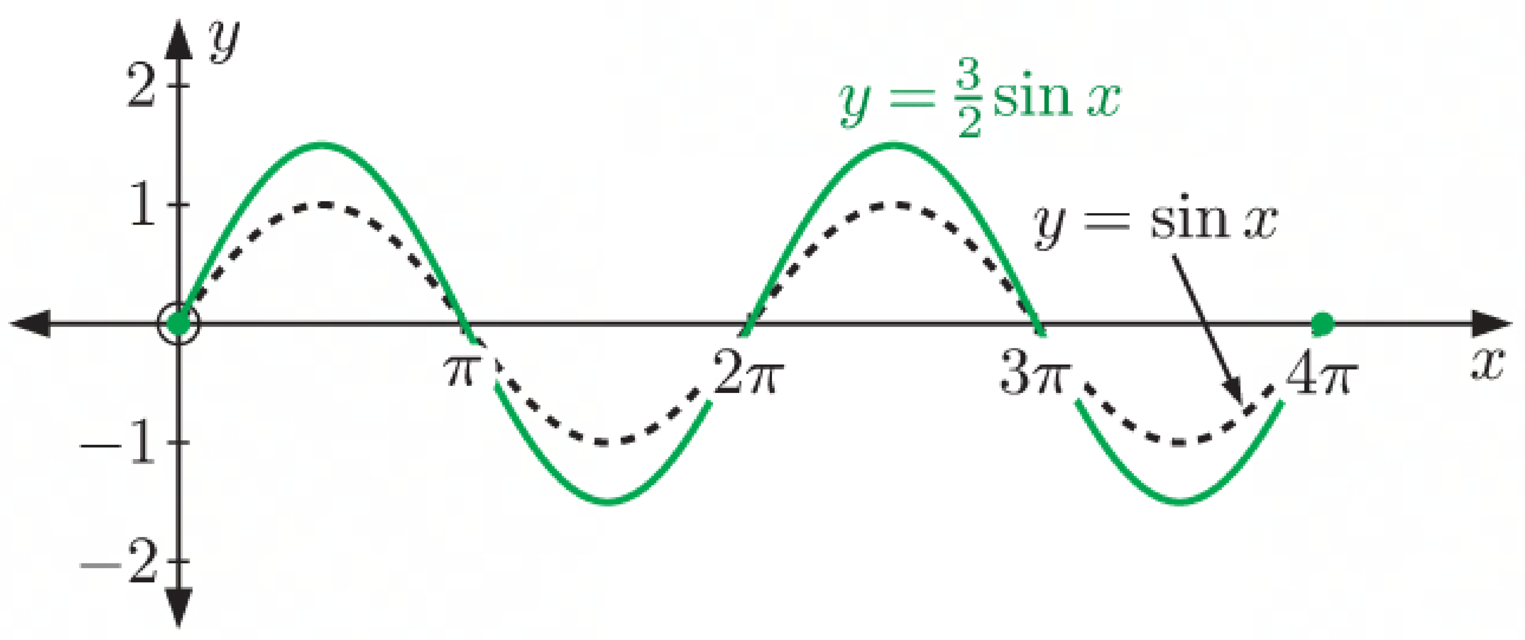
k $y = \frac{1}{2}\sin x$ is a vertical stretch of $y = \sin x$ with scale factor $\frac{1}{2}$.

The amplitude is $\frac{1}{2}$ and the period is 2π.



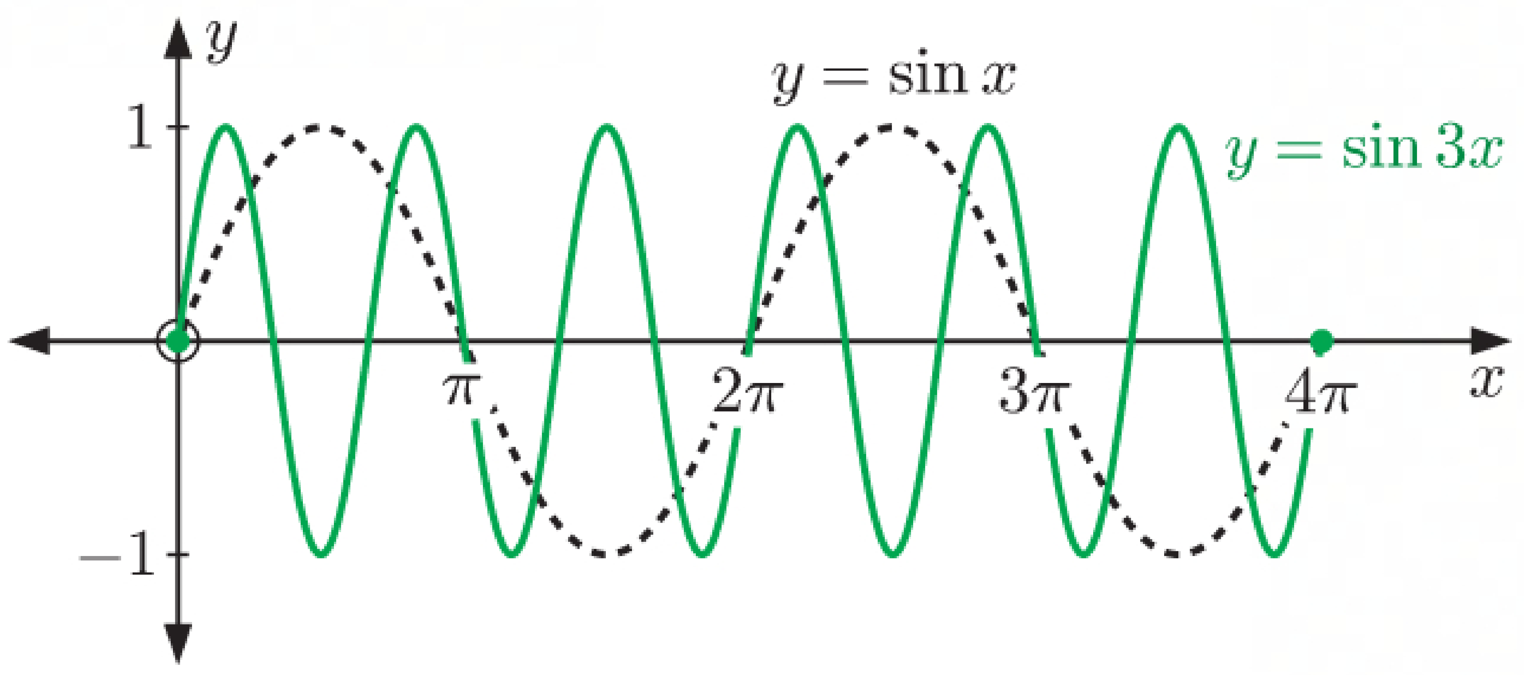
l $y = \frac{3}{2}\sin x$ is a vertical stretch of $y = \sin x$ with scale factor $\frac{3}{2}$.

The amplitude is $\frac{3}{2}$ and the period is 2π.



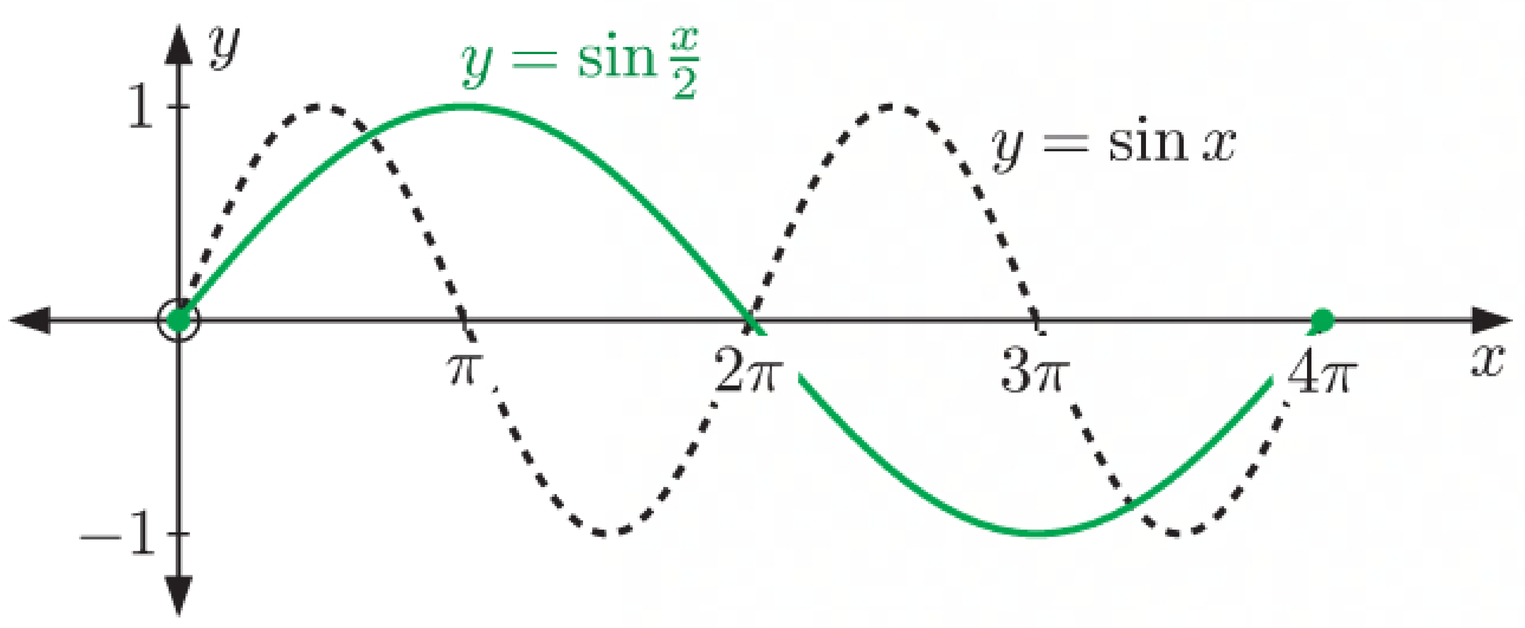
m $y = \sin 3x$ is a horizontal stretch of $y = \sin x$ with scale factor $\frac{1}{3}$.

The period is $\frac{2\pi}{3}$. $\therefore$ the maximum values are $\frac{2\pi}{3}$ units apart.

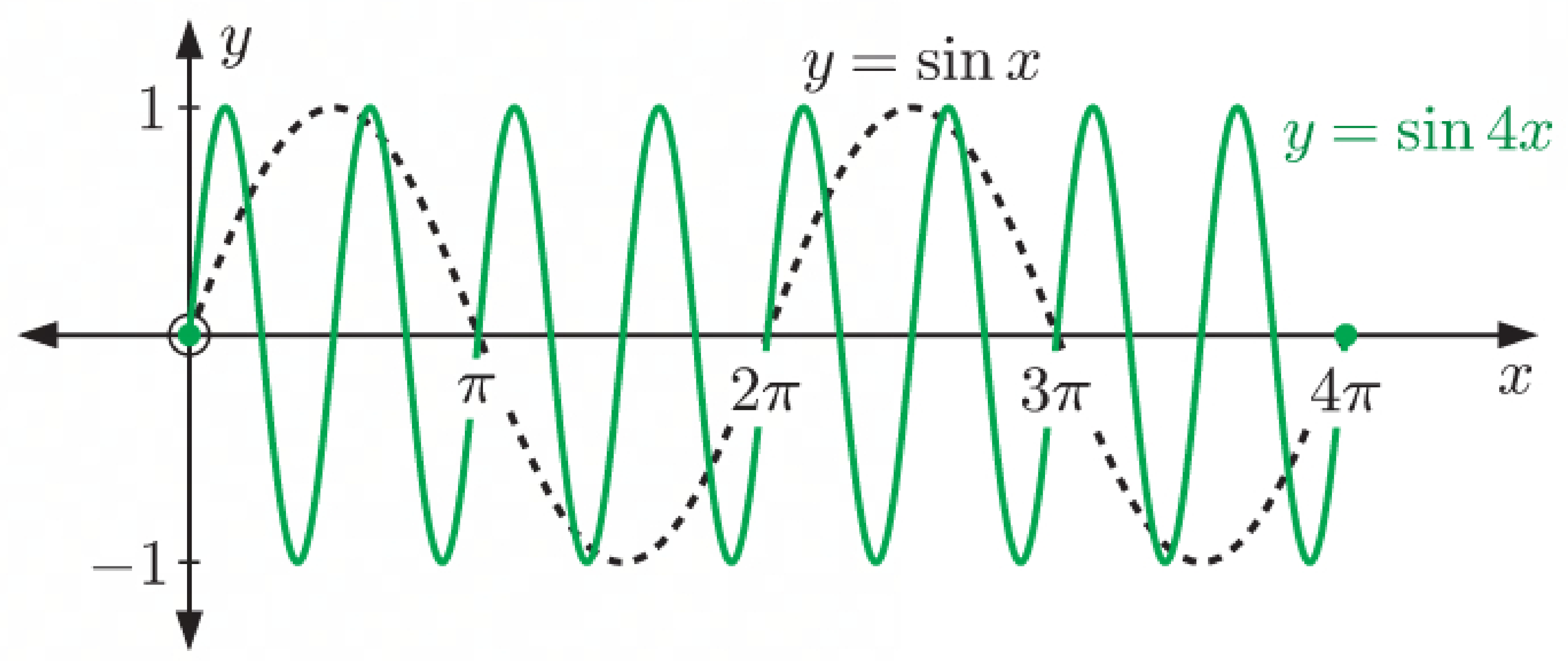


n $y = \sin \frac{x}{2}$ is a horizontal stretch of $y = \sin x$ with scale factor $\dfrac{1}{\frac{1}{2}} = 2$.

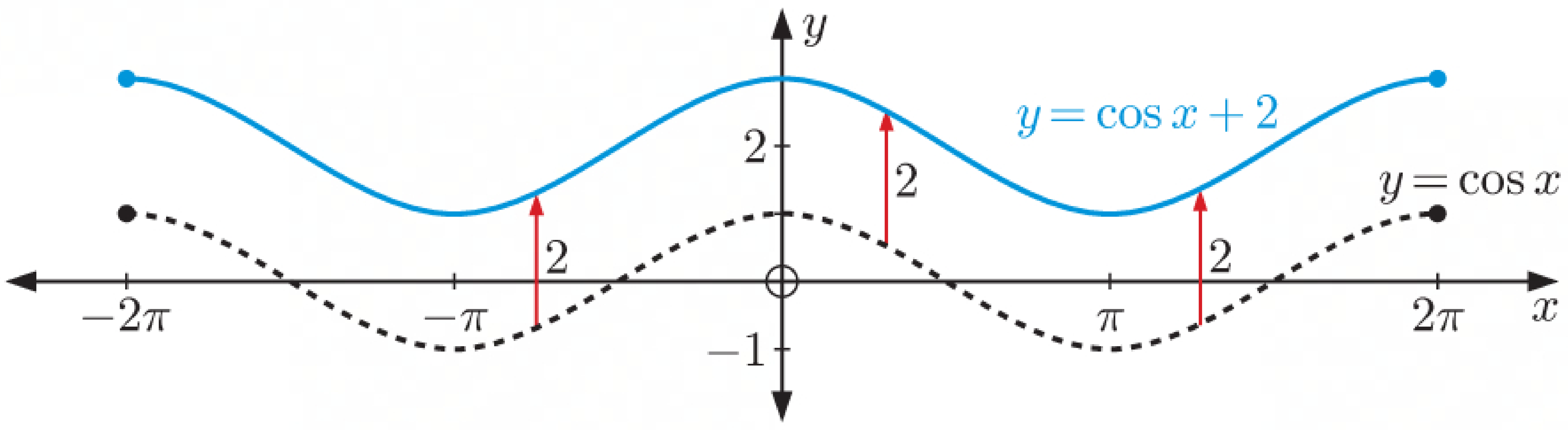
The period is $\dfrac{2\pi}{\frac{1}{2}} = 4\pi$. $\therefore$ the maximum values are 4π units apart.



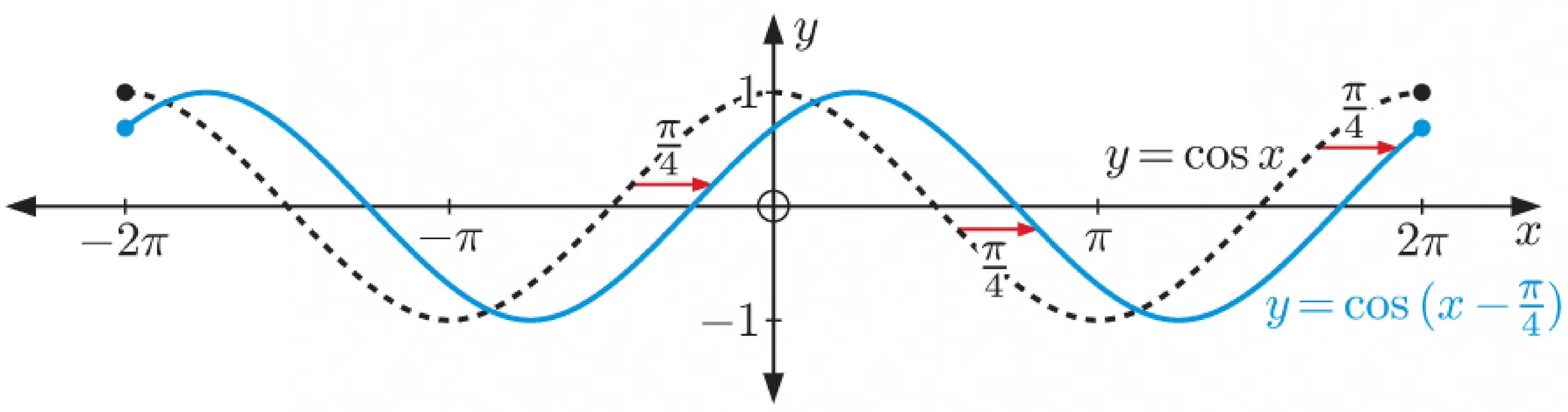
o $y = \sin 4x$ is a horizontal stretch of $y = \sin x$ with scale factor $\frac{1}{4}$.
The period is $\frac{2\pi}{4} = \frac{\pi}{2}$. $\therefore$ the maximum values are $\frac{\pi}{2}$ units apart.



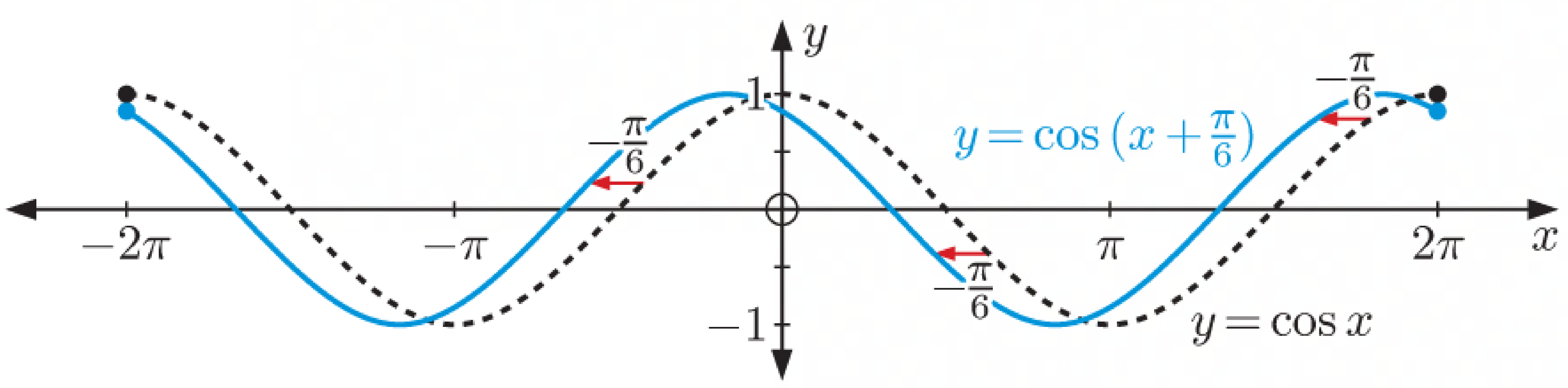
9 **a** $y = \cos x + 2$ is a vertical translation of $y = \cos x$ upwards by 2 units.



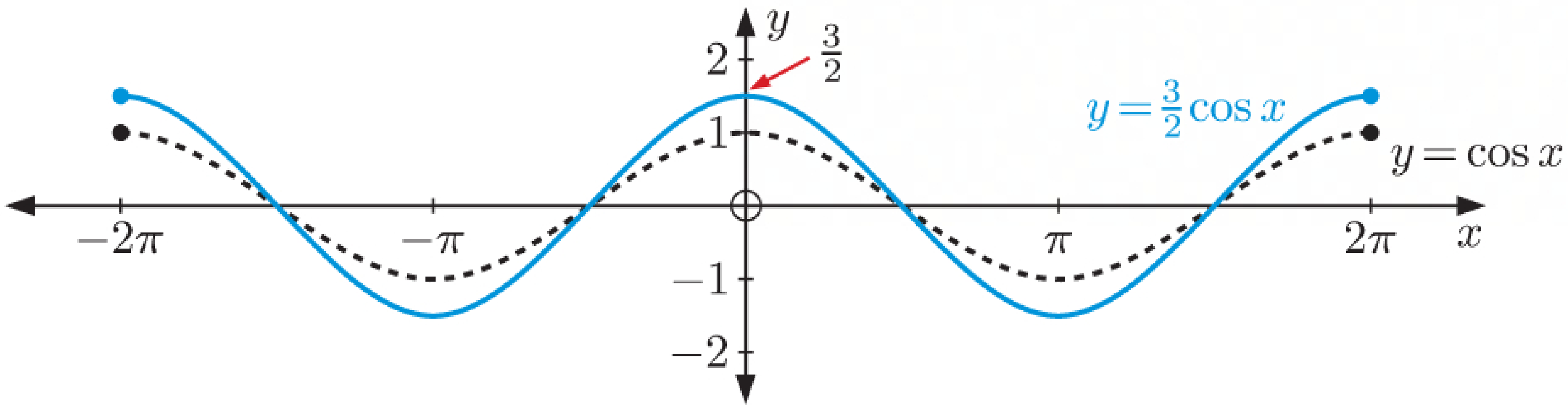
b $y = \cos\left(x - \frac{\pi}{4}\right)$ is a horizontal translation of $y = \cos x$ to the right by $\frac{\pi}{4}$ units.



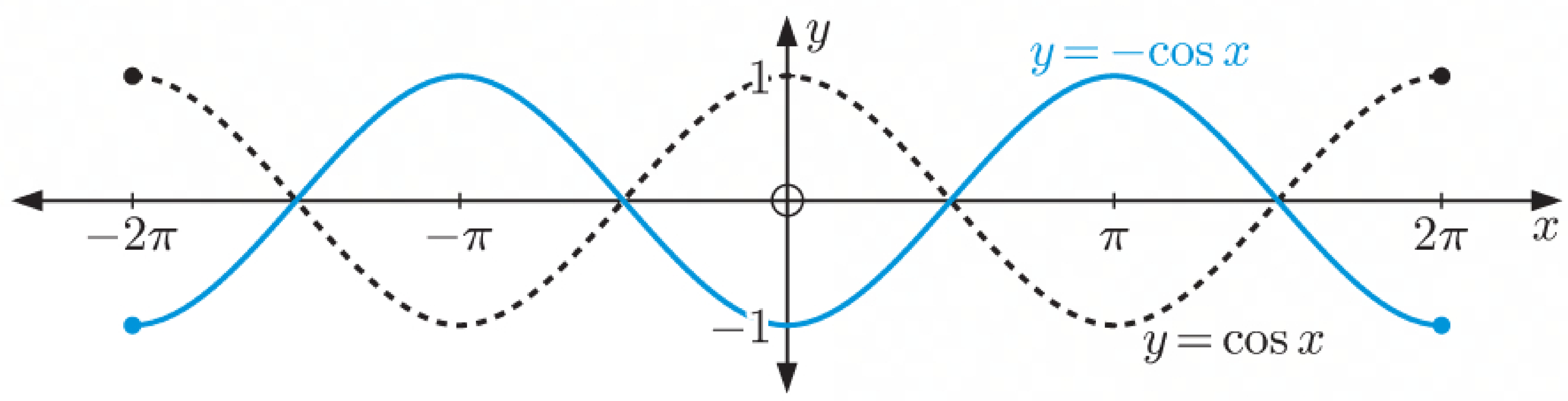
c $y = \cos\left(x + \frac{\pi}{6}\right)$ is a horizontal translation of $y = \cos x$ to the left by $\frac{\pi}{6}$ units.



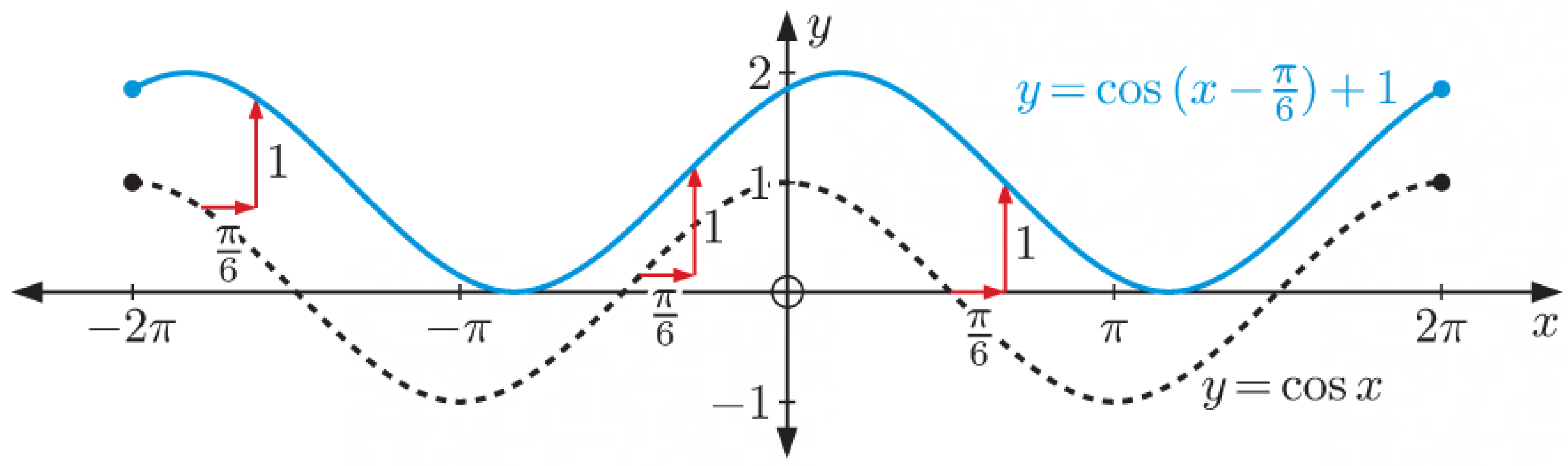
d $y = \frac{3}{2}\cos x$ is a vertical stretch of $y = \cos x$ with scale factor $\frac{3}{2}$.
The amplitude is $\frac{3}{2}$ and the period is 2π.



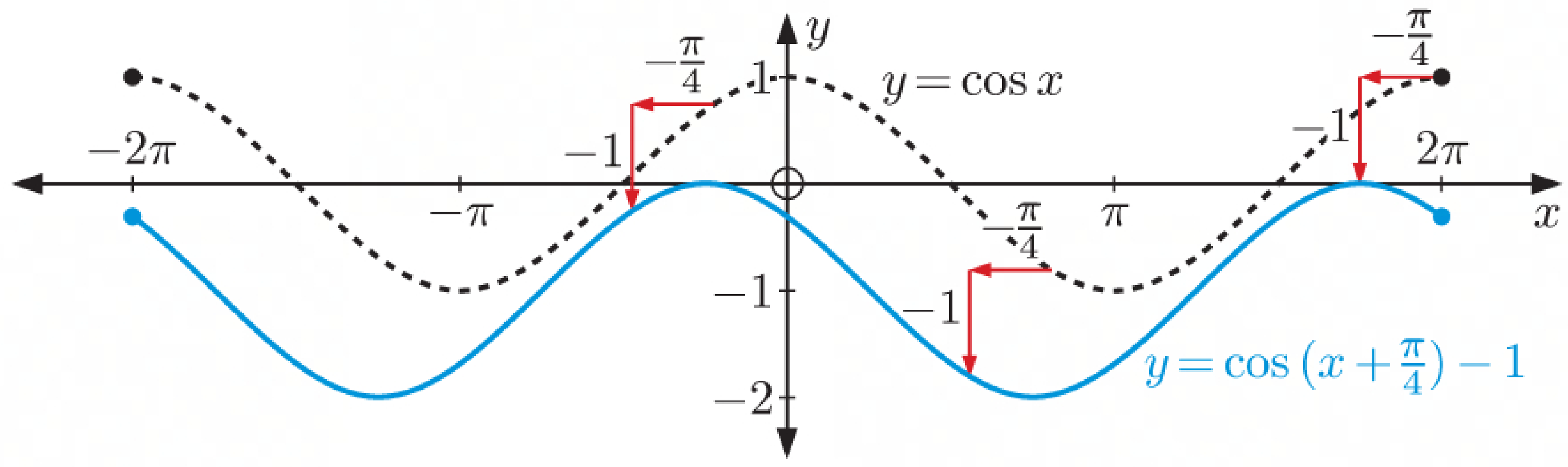
e $y = -\cos x$ is the reflection of $y = \cos x$ in the x-axis.
The amplitude is 1 and the period is 2π.



f $y = \cos\left(x - \frac{\pi}{6}\right) + 1$ is a translation of $y = \cos x$ to the right by $\frac{\pi}{6}$ units and upwards by 1 unit.

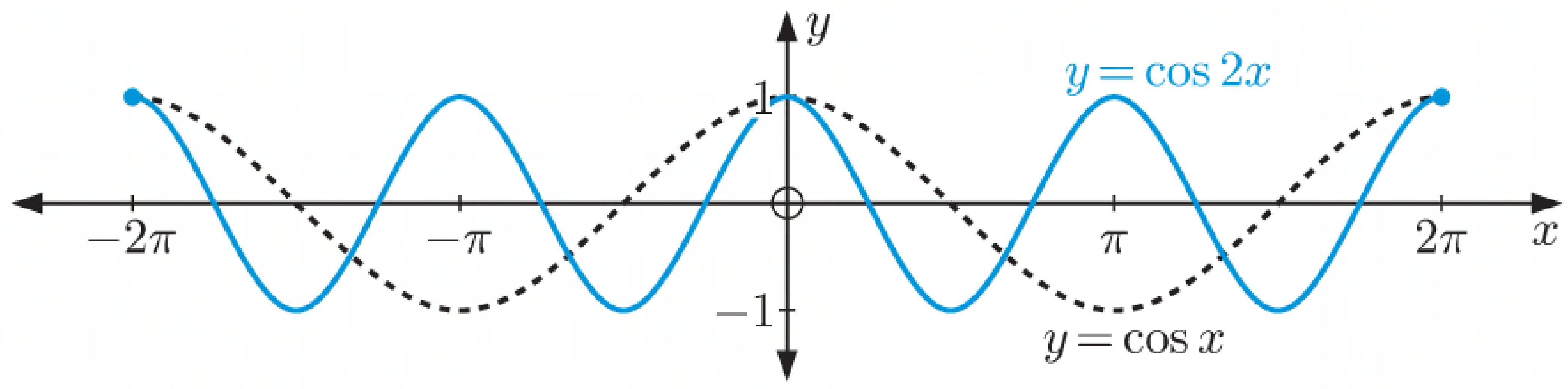


g $y = \cos\left(x + \frac{\pi}{4}\right) - 1$ is a translation of $y = \cos x$ to the left by $\frac{\pi}{4}$ units and downwards by 1 unit.



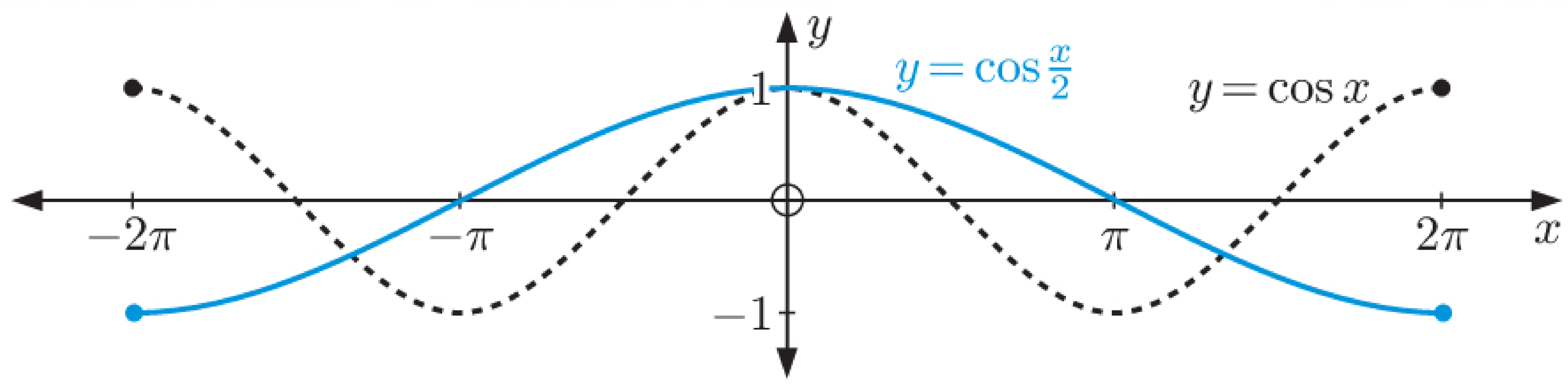
h $y = \cos 2x$ is a horizontal stretch of $y = \cos x$ with scale factor $\frac{1}{2}$.

The period is $\dfrac{2\pi}{2} = \pi$. $\therefore$ the maximum values are π units apart.

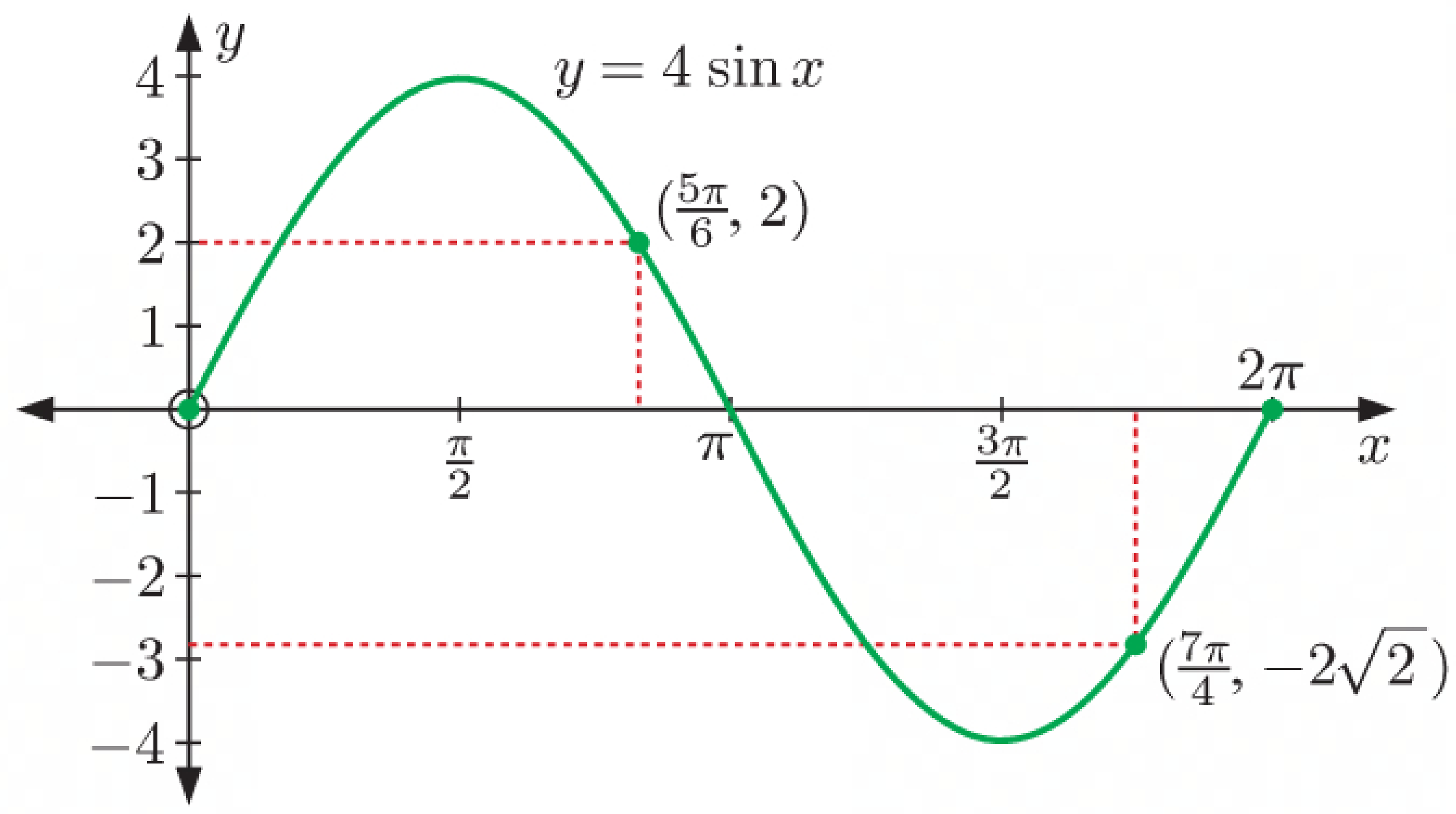


i $y = \cos \frac{x}{2}$ is a horizontal stretch of $y = \cos x$ with scale factor $\dfrac{1}{\frac{1}{2}} = 2$.

The period is $\dfrac{2\pi}{\frac{1}{2}} = 4\pi$. $\therefore$ the maximum values are 4π units apart.



10 **a** $y = 4\sin x$ is a vertical stretch of $y = \sin x$ with scale factor 4.
The amplitude is 4 and the period is 2π.



b **i** When $x = \frac{5\pi}{6}$, $y = 4\sin\frac{5\pi}{6}$

$$= 4 \times \tfrac{1}{2}$$

$$= 2$$

ii When $x = \frac{7\pi}{4}$, $y = 4\sin\frac{7\pi}{4}$

$$= 4 \times \left(-\tfrac{1}{\sqrt{2}}\right)$$

$$= -2\sqrt{2}$$

$$\approx -2.83$$

11 $y = 3\cos x$ has minimum value -3 and maximum value 3.
Now, $y = 3\cos x + d$ is a vertical translation of $y = 3\cos x$ by d units.

a $y = 3\cos x + d$ will lie entirely above the x-axis when $y = 3\cos x$ has been translated more than 3 units upwards.
$\therefore\ d > 3$

b $y = 3\cos x + d$ will lie entirely below the x-axis when $y = 3\cos x$ has been translated more than 3 units downwards.
$\therefore\ d < -3$

c $y = 3\cos x + d$ will lie partially above and partially below the x-axis when $y = 3\cos x$ has been translated between 0 and 3 units upwards or downwards.
$\therefore\ -3 < d < 3$

12 **a** $\sin x \xrightarrow[\text{scale factor } \frac{1}{3}]{\text{horizontal stretch}} \sin 3x \xrightarrow[\text{scale factor } 2]{\text{vertical stretch}} 2\sin 3x$

A horizontal stretch with scale factor $\frac{1}{3}$, then a vertical stretch with scale factor 2 maps $y = \sin x$ onto $y = 2\sin 3x$.

b $\cos x \xrightarrow[\text{scale factor } 2]{\text{vertical stretch}} 2\cos x \xrightarrow[\text{in } x\text{-axis}]{\text{reflection}} -2\cos x$

A vertical stretch with scale factor 2, then a reflection in the x-axis maps $y = \cos x$ onto $y = -2\cos x$.

c $\sin x \xrightarrow[\text{scale factor } 3]{\text{vertical stretch}} 3\sin x \xrightarrow{\text{translation } \begin{pmatrix} 0 \\ -5 \end{pmatrix}} 3\sin x - 5$

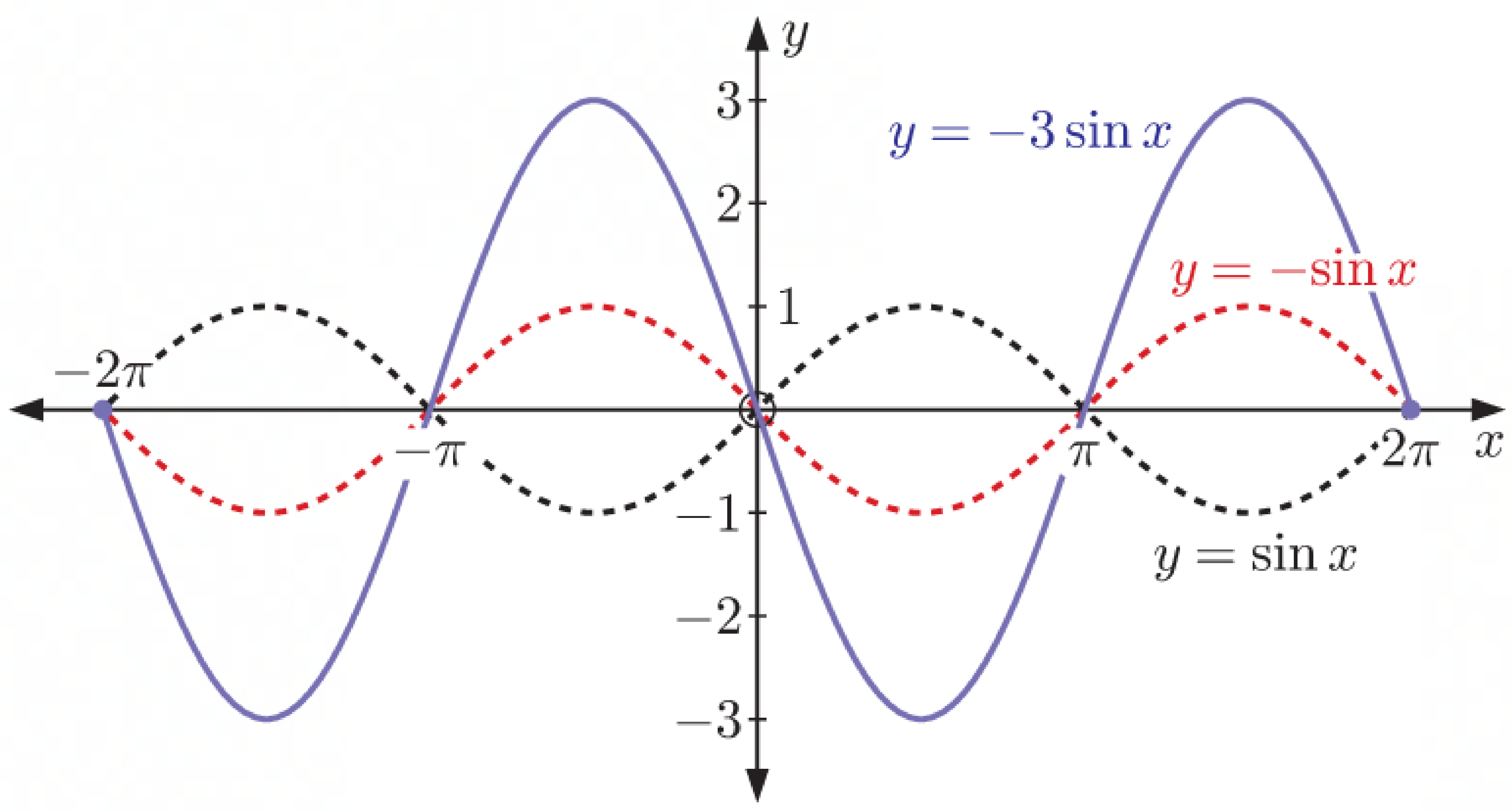
A vertical stretch with scale factor 3, then a translation 5 units downwards maps $y = \sin x$ onto $y = 3\sin x - 5$.

d $\cos x \xrightarrow[\text{scale factor } \frac{1}{2}]{\text{horizontal stretch}} \cos 2x \xrightarrow{\text{translation } \begin{pmatrix} -\frac{\pi}{6} \\ 0 \end{pmatrix}} \cos\left(2\left(x+\frac{\pi}{6}\right)\right)$

A horizontal stretch with scale factor $\frac{1}{2}$, then a translation $\frac{\pi}{6}$ units left maps $y = \cos x$ onto $y = \cos\left(2\left(x+\frac{\pi}{6}\right)\right)$.

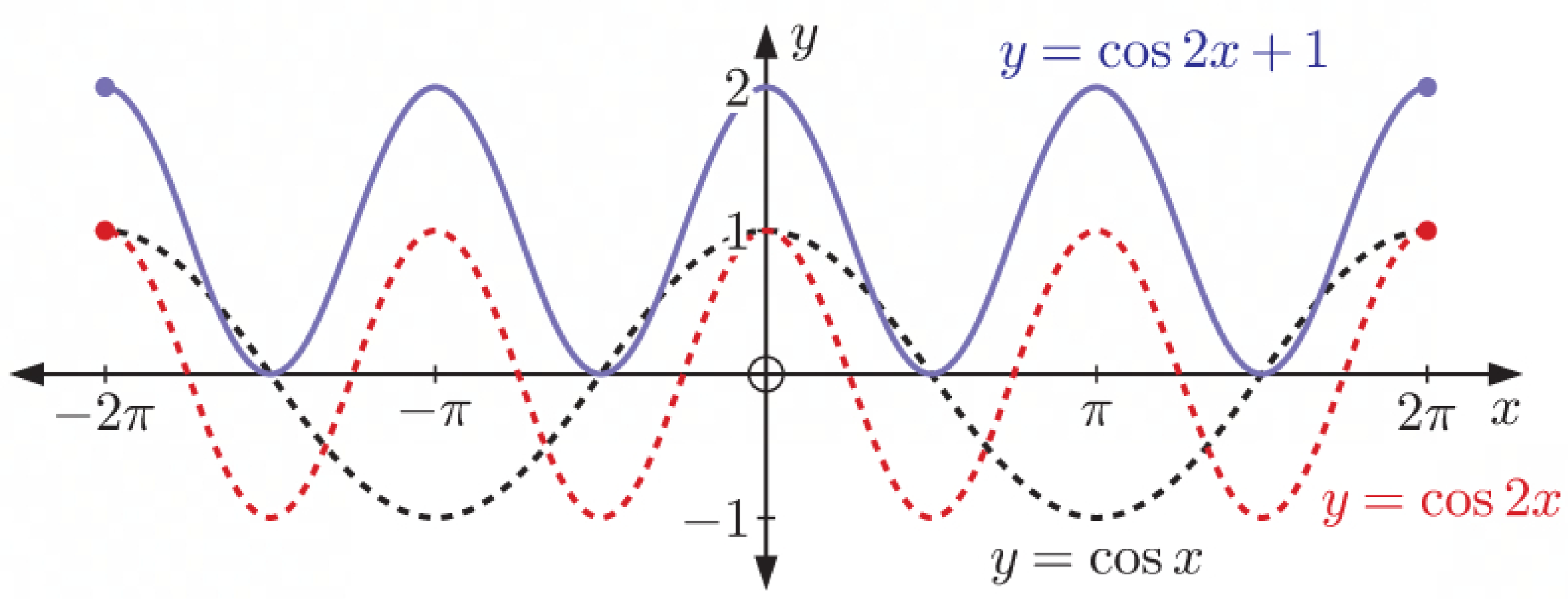
13 **a** $a = -3$, so the amplitude is $|-3| = 3$.

We reflect $y = \sin x$ in the x-axis to give $y = -\sin x$, then stretch $y = -\sin x$ vertically with scale factor 3.



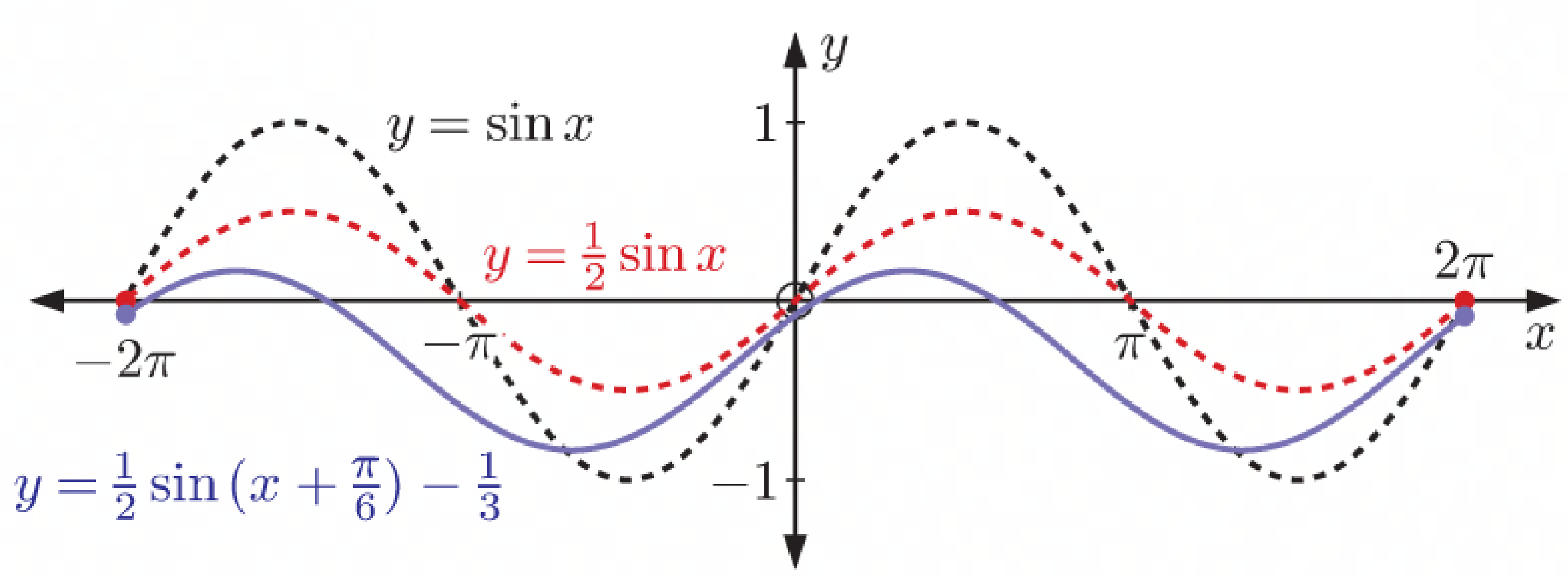
b $b = 2$, so the period is $\dfrac{2\pi}{b} = \dfrac{2\pi}{2} = \pi$.

We stretch $y = \cos x$ horizontally with scale factor $\frac{1}{2}$ to give $y = \cos 2x$, then translate $y = \cos 2x$ 1 unit upwards to give $y = \cos 2x + 1$.



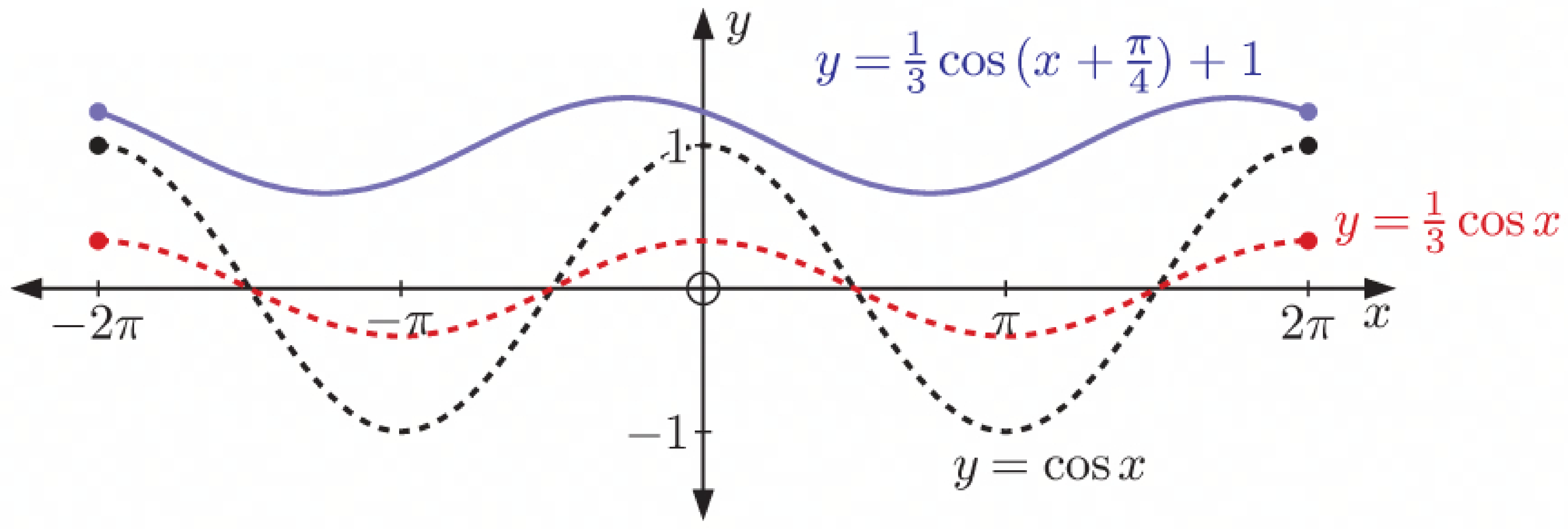
c $a = \frac{1}{2}$, so the amplitude is $\left|\frac{1}{2}\right| = \frac{1}{2}$.

We stretch $y = \sin x$ vertically with scale factor $\frac{1}{2}$ to give $y = \frac{1}{2}\sin x$, then translate $y = \frac{1}{2}\sin x$ $\frac{\pi}{6}$ units to the left and $\frac{1}{3}$ units downwards to give $y = \frac{1}{2}\sin\left(x+\frac{\pi}{6}\right) - \frac{1}{3}$.



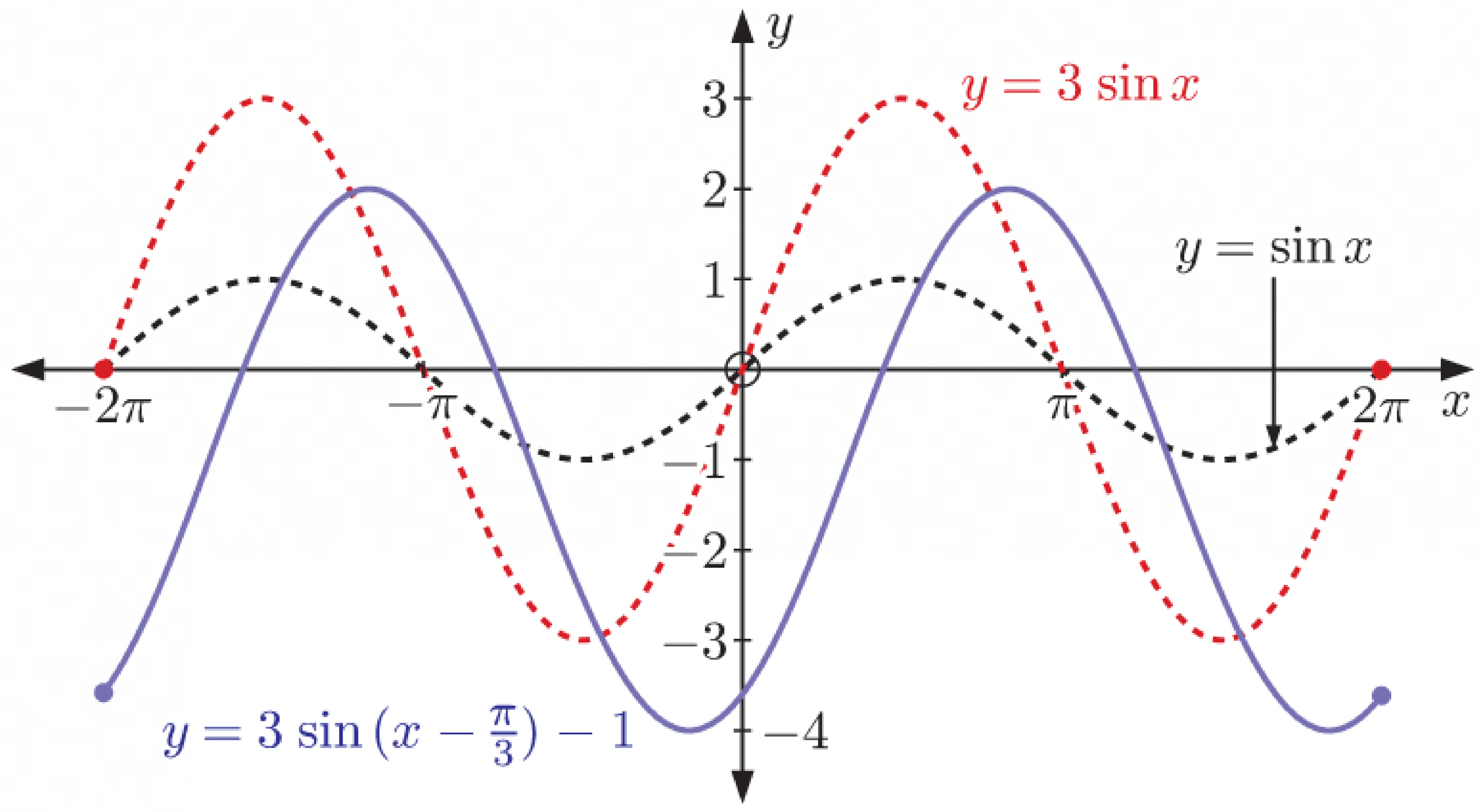
d $a = \frac{1}{3}$, so the amplitude is $\left|\frac{1}{3}\right| = \frac{1}{3}$.

We stretch $y = \cos x$ vertically with scale factor $\frac{1}{3}$ to give $y = \frac{1}{3}\cos x$, then translate $y = \frac{1}{3}\cos x$ $\frac{\pi}{4}$ units to the left and 1 unit upwards to give $y = \frac{1}{3}\cos\left(x + \frac{\pi}{4}\right) + 1$.



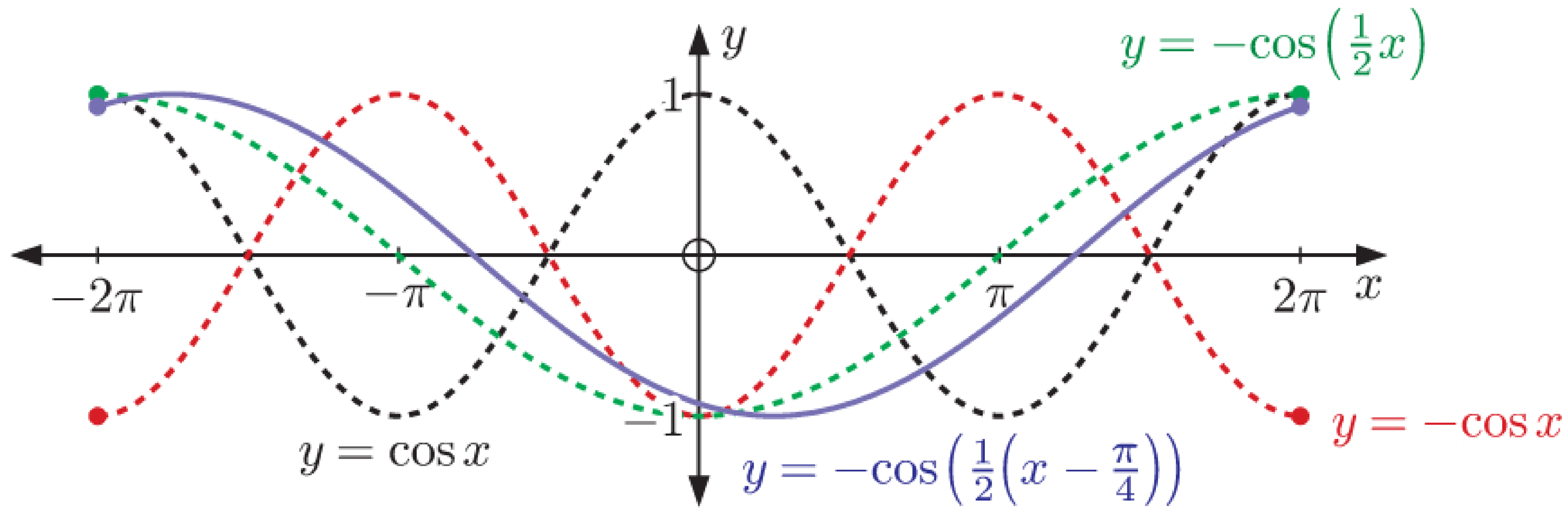
e $a = 3$, so the amplitude is $|3| = 3$.

We stretch $y = \sin x$ vertically with scale factor 3 to give $y = 3\sin x$, then translate $y = 3\sin x$ $\frac{\pi}{3}$ units to the right and 1 unit downwards to give $y = 3\sin\left(x - \frac{\pi}{3}\right) - 1$.



f $a = -1$, so the amplitude is $|-1| = 1$. $b = \frac{1}{2}$, so the period is $\dfrac{2\pi}{b} = \dfrac{2\pi}{\frac{1}{2}} = 4\pi$.

We reflect $y = \cos x$ in the x-axis to give $y = -\cos x$, then stretch $y = -\cos x$ horizontally with scale factor 2 to give $y = -\cos\left(\frac{1}{2}x\right)$, then translate $y = -\cos\left(\frac{1}{2}x\right)$ $\frac{\pi}{4}$ units to the right to give $y = -\cos\left(\frac{1}{2}\left(x - \frac{\pi}{4}\right)\right)$.



14 $y = a\sin(b(x - c)) + d$

a b affects the period of the function. So, a change in b will produce a change in the x-intercepts of the function.

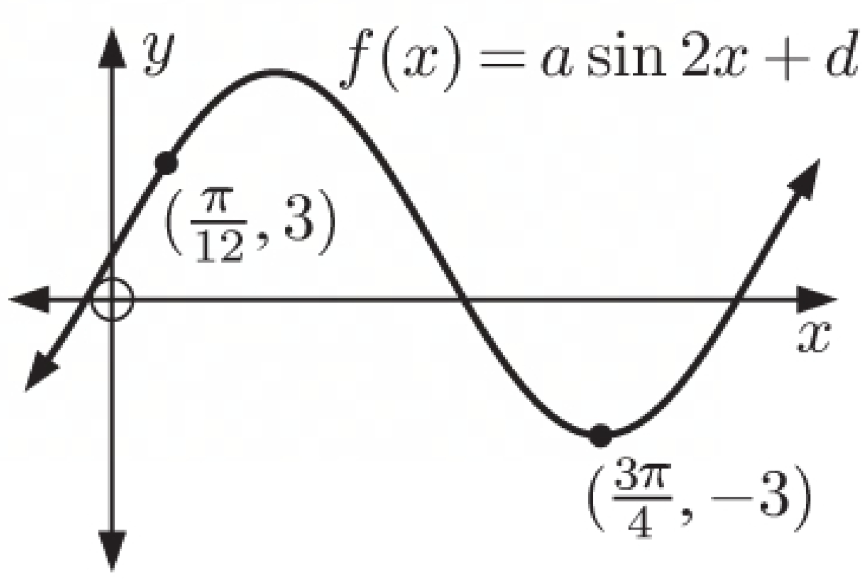
c affects horizontal translation. So, a change in c will produce a change in the x-intercepts of the function (provided the change in c is not a multiple of π).

d affects vertical translation. So, a change in d will produce a change in the x-intercepts of the function.

b c affects horizontal translation. So, a change in c will produce a change in the y-intercept of the function (provided the change in c is not a multiple of π).
d affects vertical translation. So, a change in d will produce a change in the y-intercept of the function.

c a affects the amplitude of the function. So, a change in a will produce a change in the range of the function (provided a is not changed to -1).
d affects vertical translation. So, a change in d will produce a change in the range of the function.

15 **a**



$f(x) = a\sin 2x + d$

$$f\left(\tfrac{\pi}{12}\right) = 3 \quad\text{and}\quad f\left(\tfrac{3\pi}{4}\right) = -3$$

$$\therefore\ a\sin\left(2\times\tfrac{\pi}{12}\right) + d = 3 \qquad \therefore\ a\sin\left(2\times\tfrac{3\pi}{4}\right) + d = -3$$

$$\therefore\ a\sin\tfrac{\pi}{6} + d = 3 \qquad \therefore\ a\sin\tfrac{3\pi}{2} + 3 - \tfrac{1}{2}a = -3 \quad \{\text{using } (*)\}$$

$$\therefore\ \tfrac{1}{2}a + d = 3 \qquad \therefore\ -a + 3 - \tfrac{1}{2}a = -3$$

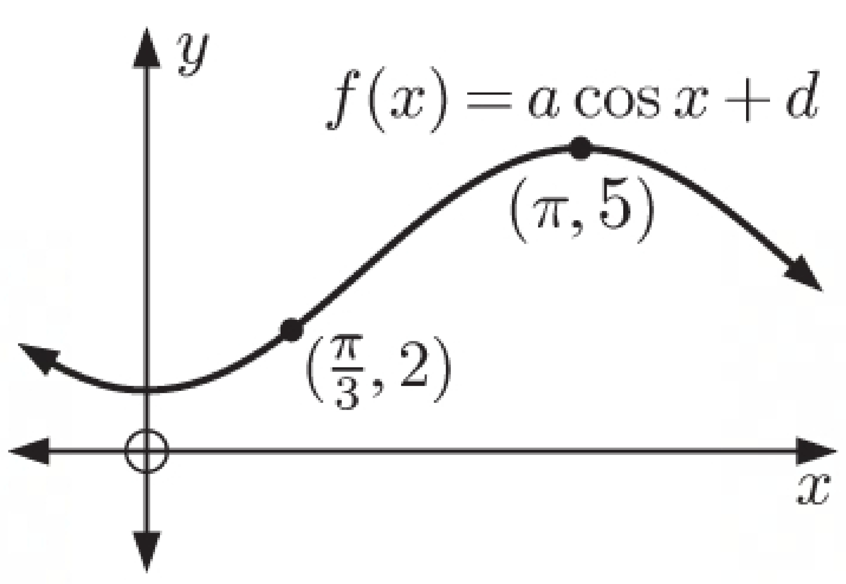
$$\therefore\ d = 3 - \tfrac{1}{2}a \quad .... (*) \qquad \therefore\ -\tfrac{3}{2}a = -6$$

$$\therefore\ a = 4$$

Subsituting $a = 4$ into $(*)$ gives $d = 3 - \frac{1}{2}(4)$
$= 3 - 2$
$= 1$

$\therefore\ a = 4,\ d = 1$

b



$f(x) = a\cos x + d$

$$f\left(\tfrac{\pi}{3}\right) = 2 \quad\text{and}\quad f(\pi) = 5$$

$$\therefore\ a\cos\tfrac{\pi}{3} + d = 2 \qquad \therefore\ a\cos\pi + d = 5$$

$$\therefore\ \tfrac{1}{2}a + d = 2 \qquad \therefore\ -a + 2 - \tfrac{1}{2}a = 5 \quad \{\text{using } (*)\}$$

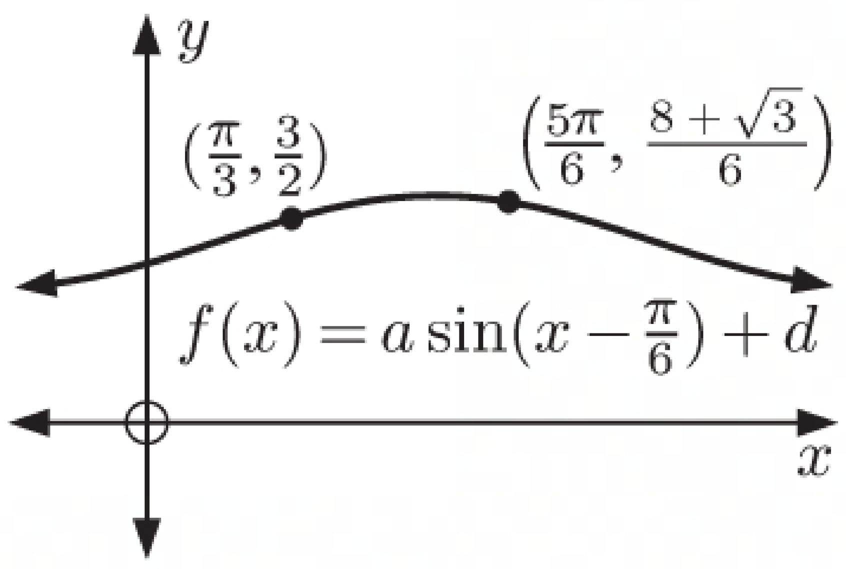
$$\therefore\ d = 2 - \tfrac{1}{2}a \quad .... (*) \qquad \therefore\ -\tfrac{3}{2}a = 3$$

$$\therefore\ a = -2$$

Substituting $a = -2$ into $(*)$ gives $d = 2 - \frac{1}{2}(-2)$
$= 2 + 1$
$= 3$

$\therefore\ a = -2,\ d = 3$

c



$f(x) = a\sin\left(x - \frac{\pi}{6}\right) + d$

$$\begin{aligned} f\left(\tfrac{\pi}{3}\right) &= \tfrac{3}{2} \\ \therefore\ a\sin\left(\tfrac{\pi}{3} - \tfrac{\pi}{6}\right) + d &= \tfrac{3}{2} \\ \therefore\ a\sin\tfrac{\pi}{6} + d &= \tfrac{3}{2} \\ \therefore\ \tfrac{1}{2}a + d &= \tfrac{3}{2} \\ \therefore\ d &= \tfrac{3}{2} - \tfrac{1}{2}a \quad .... \ (*) \end{aligned}$$

and

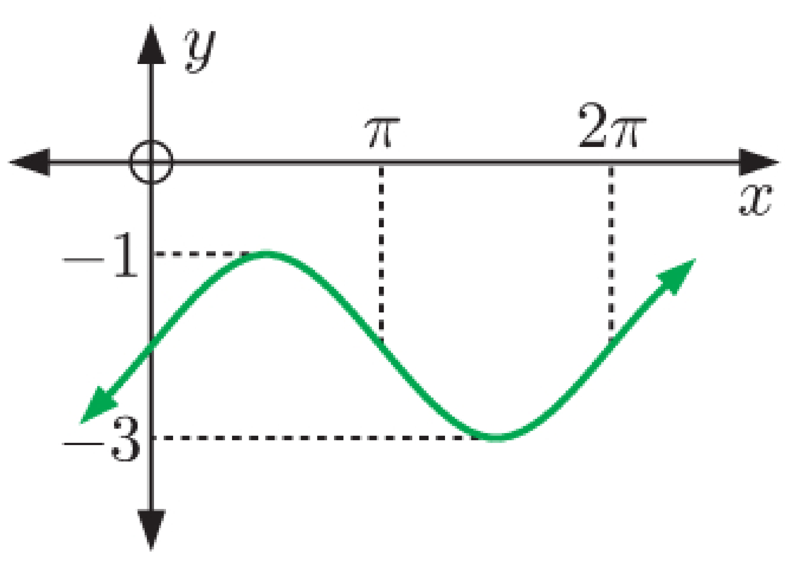
$$\begin{aligned} f\left(\tfrac{5\pi}{6}\right) &= \frac{8+\sqrt{3}}{6} \\ \therefore\ a\sin\left(\tfrac{5\pi}{6} - \tfrac{\pi}{6}\right) + d &= \frac{8+\sqrt{3}}{6} \\ \therefore\ a\sin\tfrac{2\pi}{3} + d &= \frac{8+\sqrt{3}}{6} \\ \therefore\ \tfrac{\sqrt{3}}{2}a + \tfrac{3}{2} - \tfrac{1}{2}a &= \frac{8+\sqrt{3}}{6} \\ \therefore\ \left(\frac{-1+\sqrt{3}}{2}\right)a &= \frac{-1+\sqrt{3}}{6} \\ \therefore\ a &= \tfrac{1}{3} \end{aligned}$$

Substituting $a = \frac{1}{3}$ into $(*)$ gives

$$\begin{aligned} d &= \tfrac{3}{2} - \tfrac{1}{2}\left(\tfrac{1}{3}\right) \\ &= \tfrac{3}{2} - \tfrac{1}{6} \\ &= \tfrac{4}{3} \end{aligned}$$

$\therefore\ a = \frac{1}{3},\ d = \frac{4}{3}$

16 **a**



The amplitude is 1, so $a = 1$.

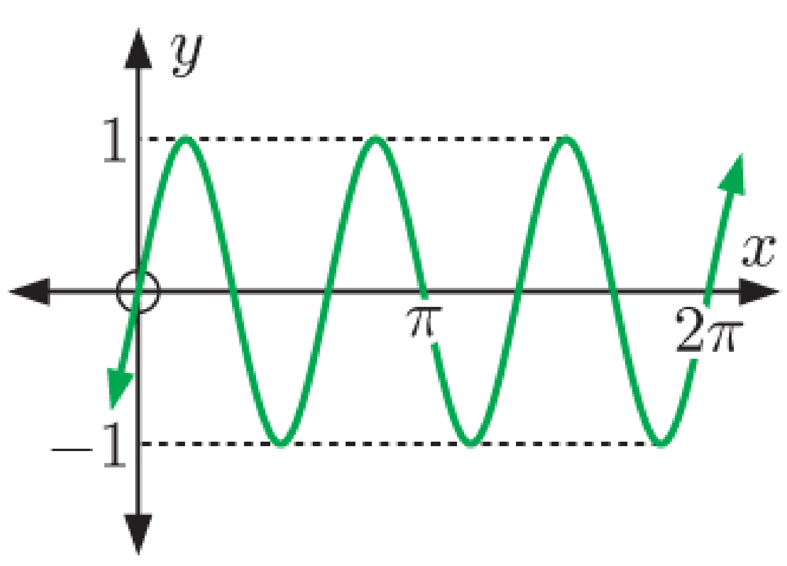
The period is 2π, so $\dfrac{2\pi}{b} = 2\pi$ and $\therefore\ b = 1$.

There is no horizontal translation, so $c = 0$.

The principal axis is $y = -2$, so $d = -2$.

$\therefore$ the equation of the function is $y = \sin x - 2$.

b



The amplitude is 1, so $a = 1$.

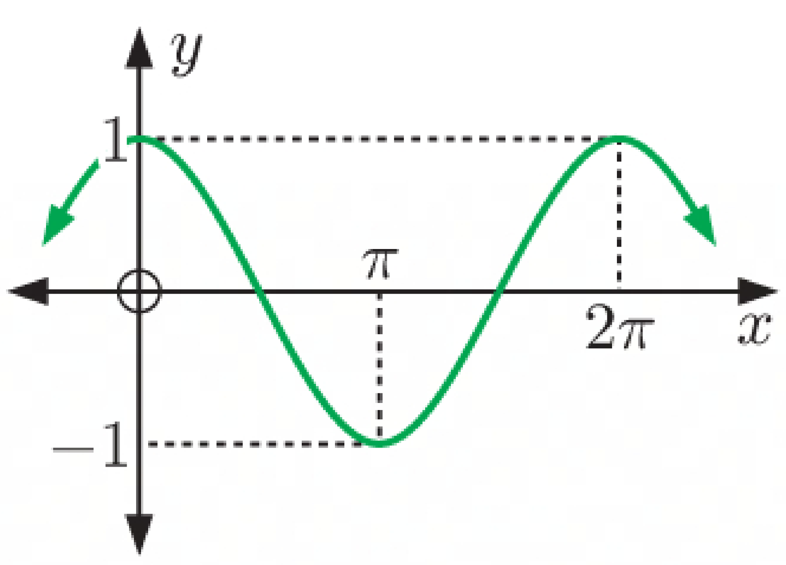
The period is $\frac{2\pi}{3}$, so $\dfrac{2\pi}{b} = \dfrac{2\pi}{3}$ and $\therefore\ b = 3$.

There is no horizontal translation, so $c = 0$.

The principal axis is $y = 0$, so $d = 0$.

$\therefore$ the equation of the function is $y = \sin 3x$.

c



The amplitude is 1, so $a = 1$.

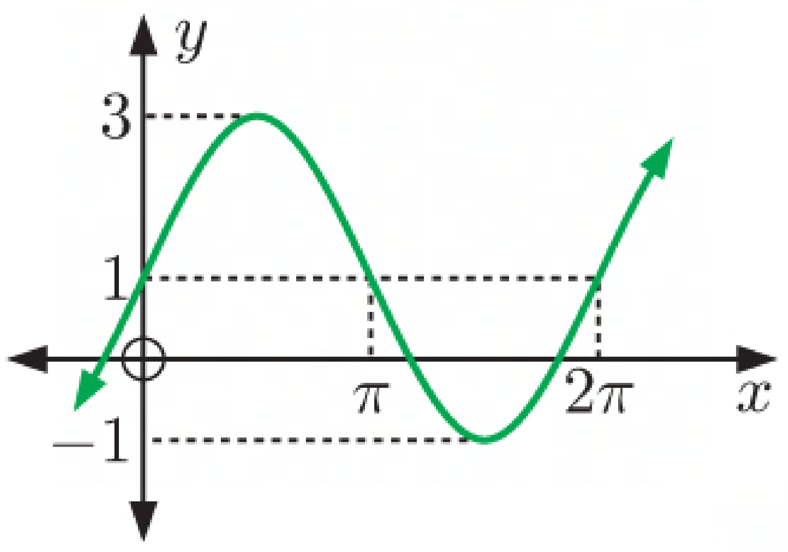
The period is 2π, so $\dfrac{2\pi}{b} = 2\pi$ and $\therefore\ b = 1$.

There is a horizontal translation of $\frac{\pi}{2}$ units to the left, so $c = -\frac{\pi}{2}$.

The principal axis is $y = 0$, so $d = 0$.

$\therefore$ the equation of the function is $y = \sin\left(x + \frac{\pi}{2}\right)$.

d



The amplitude is 2, so $a = 2$.

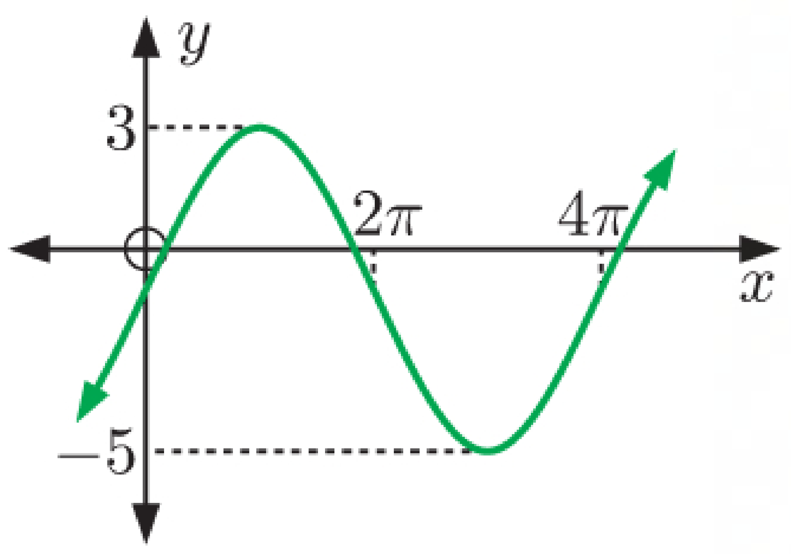
The period is 2π, so $\frac{2\pi}{b} = 2\pi$ and $\therefore$ $b = 1$.

There is no horizontal translation, so $c = 0$.

The principal axis is $y = 1$, so $d = 1$.

$\therefore$ the equation of the function is $y = 2\sin x + 1$.

e



The amplitude is 4, so $a = 4$.

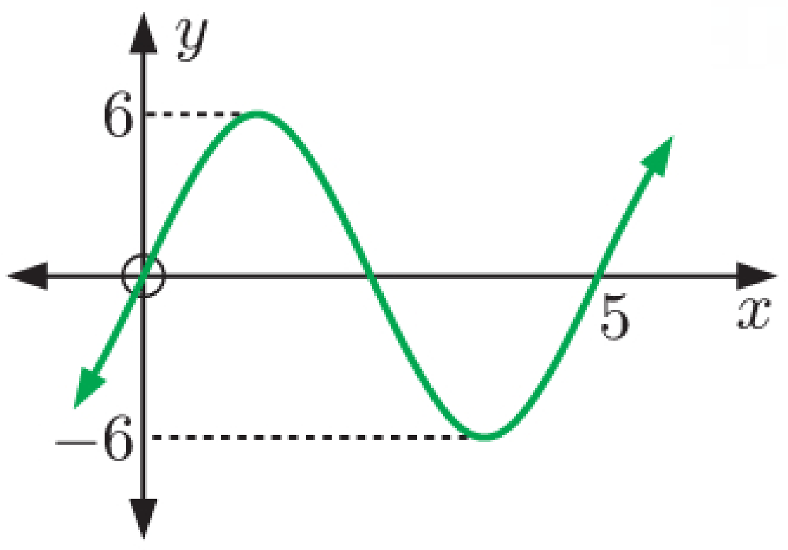
The period is 4π, so $\frac{2\pi}{b} = 4\pi$ and $\therefore$ $b = \frac{1}{2}$.

There is no horizontal translation, so $c = 0$.

The principal axis is $y = -1$, so $d = -1$.

$\therefore$ the equation of the function is $y = 4\sin\frac{x}{2} - 1$.

f



The amplitude is 6, so $a = 6$.

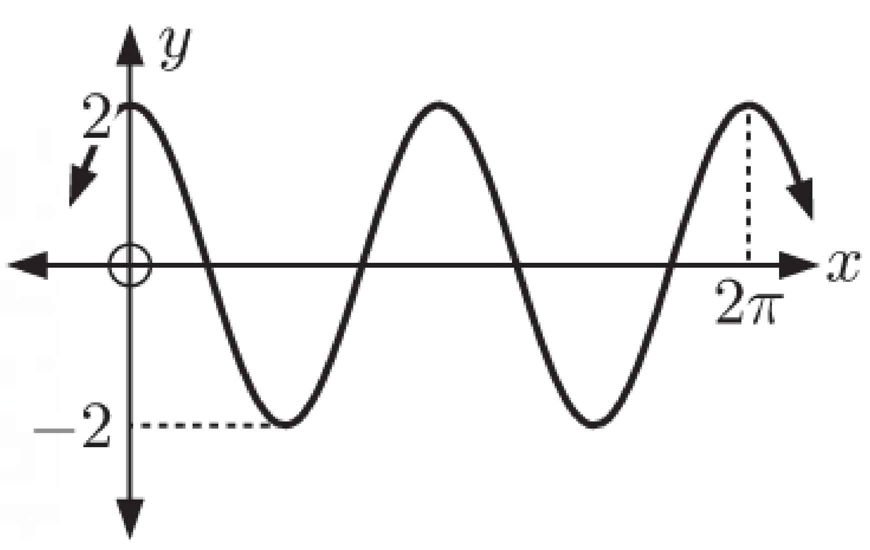
The period is 5, so $\frac{2\pi}{b} = 5$ and $\therefore$ $b = \frac{2\pi}{5}$.

There is no horizontal translation, so $c = 0$.

The principal axis is $y = 0$, so $d = 0$.

$\therefore$ the equation of the function is $y = 6\sin\frac{2\pi x}{5}$.

17 **a**



The amplitude is 2, so $a = 2$.

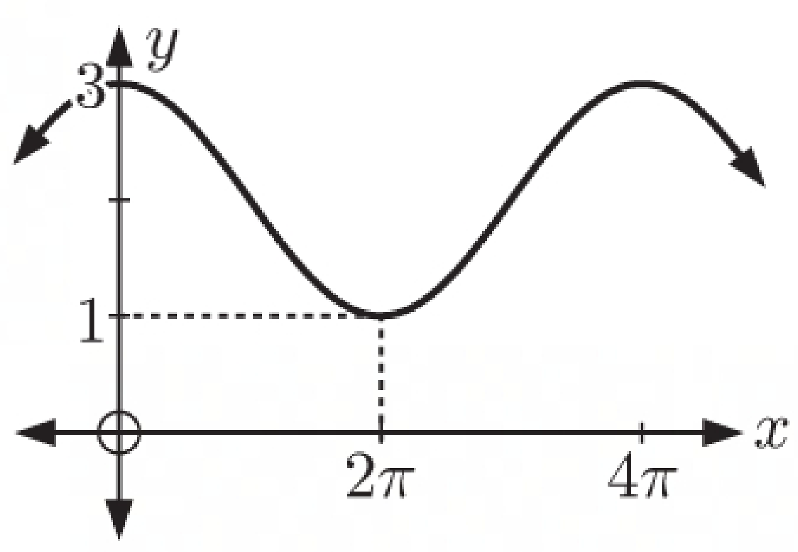
The period is π, so $\frac{2\pi}{b} = \pi$ and $\therefore$ $b = 2$.

There is no horizontal translation, so $c = 0$.

The principal axis is $y = 0$, so $d = 0$.

$\therefore$ the equation of the function is $y = 2\cos 2x$.

b



The amplitude is 1, so $a = 1$.

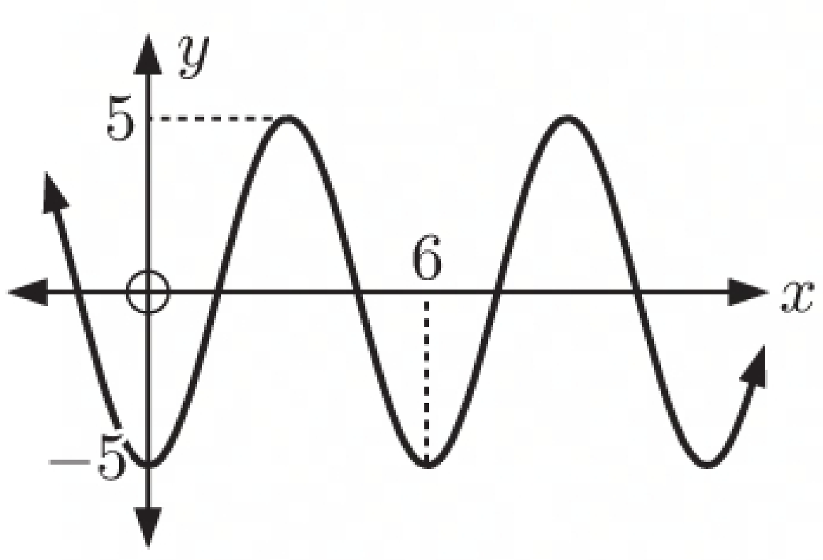
The period is 4π, so $\frac{2\pi}{b} = 4\pi$ and $\therefore$ $b = \frac{1}{2}$.

There is no horizontal translation, so $c = 0$.

The principal axis is $y = 2$, so $d = 2$.

$\therefore$ the equation of the function is $y = \cos\frac{x}{2} + 2$.

c



The amplitude is 5 and the function has been mirrored in the x-axis, so $a = -5$.

The period is 6, so $\frac{2\pi}{b} = 6$ and $\therefore$ $b = \frac{\pi}{3}$.

There is no horizontal translation, so $c = 0$.

The principal axis is $y = 0$, so $d = 0$.

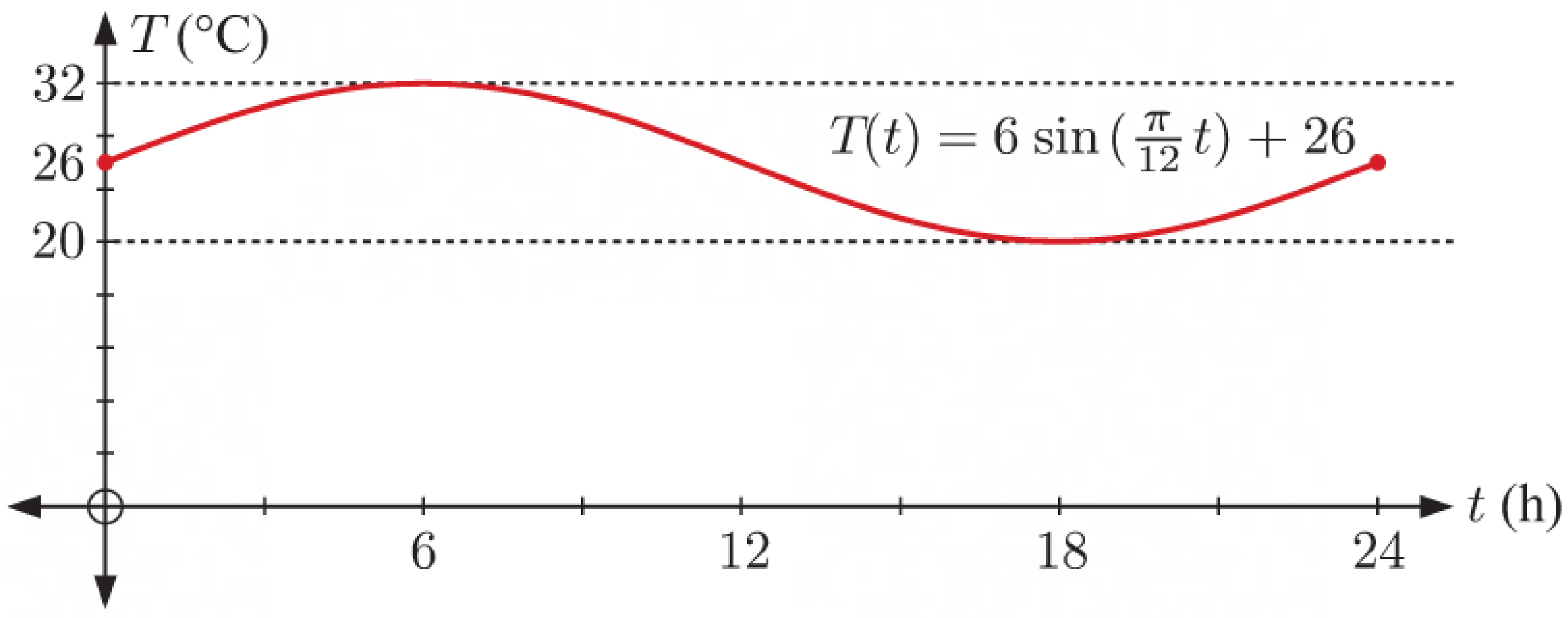
The graph has been reflected in the x-axis.

$\therefore$ the equation of the function is $y = -5\cos\frac{\pi x}{3}$.

EXERCISE 17D

1 **a** For $T(t) = 6\sin\left(\frac{\pi}{12}t\right) + 26$:

- the amplitude is 6
- the period is $\dfrac{2\pi}{\left(\frac{\pi}{12}\right)} = 24$ hours
- the principal axis is $T = 26$.



b **i** Midnight is 12 hours after midday.

When $t = 12$,

$$\begin{aligned} T &= 6\sin\left(\tfrac{\pi}{12} \times 12\right) + 26 \\ &= 6\sin\pi + 26 \\ &= 6 \times 0 + 26 \\ &= 26 \end{aligned}$$

$\therefore$ at midnight the temperature inside Vanessa's house is 26°C.

ii 2 pm is 2 hours after midday.

When $t = 2$,

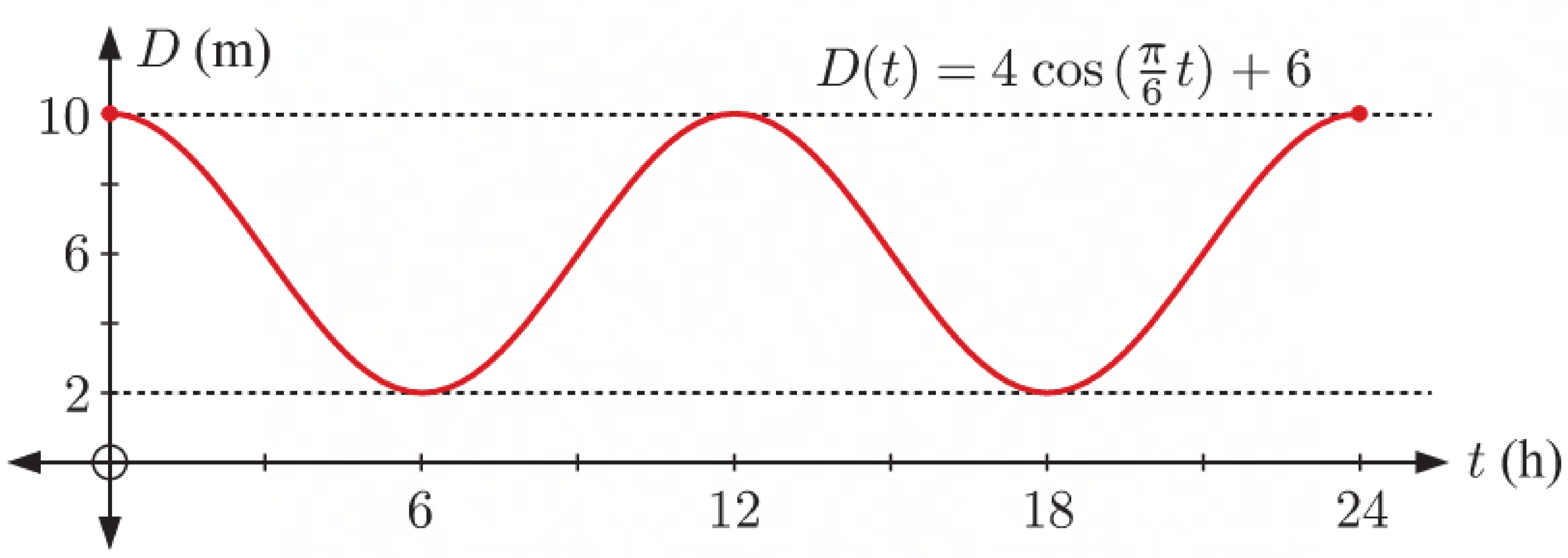
$$\begin{aligned} T &= 6\sin(\tfrac{\pi}{12} \times 2) + 26 \\ &= 6\sin\tfrac{\pi}{6} + 26 \\ &= 6 \times \tfrac{1}{2} + 26 \\ &= 29 \end{aligned}$$

$\therefore$ at 2 pm the temperature inside Vanessa's house is 29°C.

c The maximum temperature inside Vanessa's house is $26 + 6 = 32$°C, which occurs when $t = 6$.
So, the maximum temperature inside Vanessa's house occurs at 6 pm.

2 **a** For $D(t) = 4\cos\left(\frac{\pi}{6}t\right) + 6$:

- the amplitude is 4
- the period is $\dfrac{2\pi}{\left(\frac{\pi}{6}\right)} = 12$ hours
- the principal axis is $D = 6$.



b The highest water depth is $6 + 4 = 10$ metres, which occurs when $t = 0$, 12, or 24.
So, the highest water depth occurs at midnight, midday, and midnight the next day.
The lowest water depth is $6 - 4 = 2$ metres, which occurs when $t = 6$ or 18.
So, the lowest water depth occurs at 6 am or 6 pm.

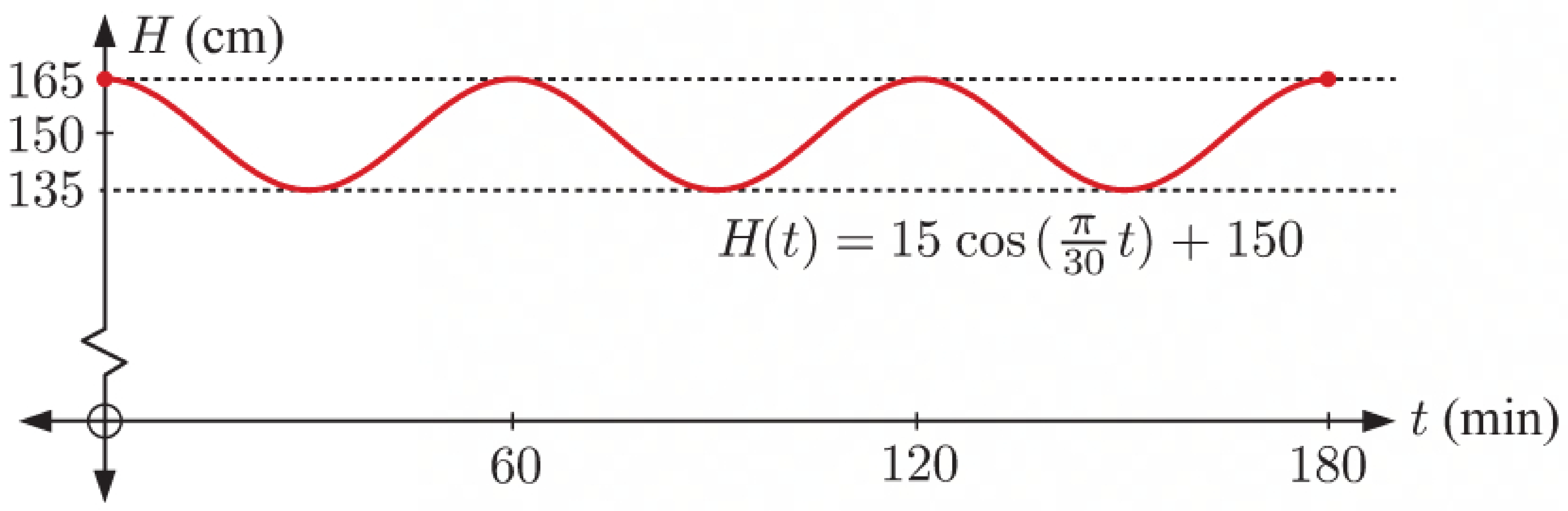
c 8 pm is 20 hours after midnight.

When $t = 20$,
$$\begin{aligned} D &= 4\cos\left(\tfrac{\pi}{6} \times 20\right) + 6 \\ &= 4\cos\left(\tfrac{10\pi}{3}\right) + 6 \\ &= 4 \times \left(-\tfrac{1}{2}\right) + 6 \\ &= 4 \end{aligned}$$

$\therefore$ at 8 pm the water depth is 4 metres, but the boat requires 5 metres, so it cannot enter the harbour at that time.

3 **a** For $H(t) = 15\cos\left(\frac{\pi}{30}t\right) + 150$:

- the amplitude is 15
- the period is $\dfrac{2\pi}{\left(\frac{\pi}{30}\right)} = 60$ minutes
- the principal axis is $H = 150$.



b The minute hand's length is represented by the amplitude in the function.
$\therefore$ the minute hand's length is 15 cm.

c **i** 5:08 pm is 8 minutes after 5 pm.
When $t = 8$,
$$\begin{aligned} H &= 15\cos\left(\tfrac{\pi}{30} \times 8\right) + 150 \\ &\approx 160.037 \end{aligned}$$
$\therefore$ at 5:08 pm, the minute hand's tip is approximately 160.0 cm above ground level.

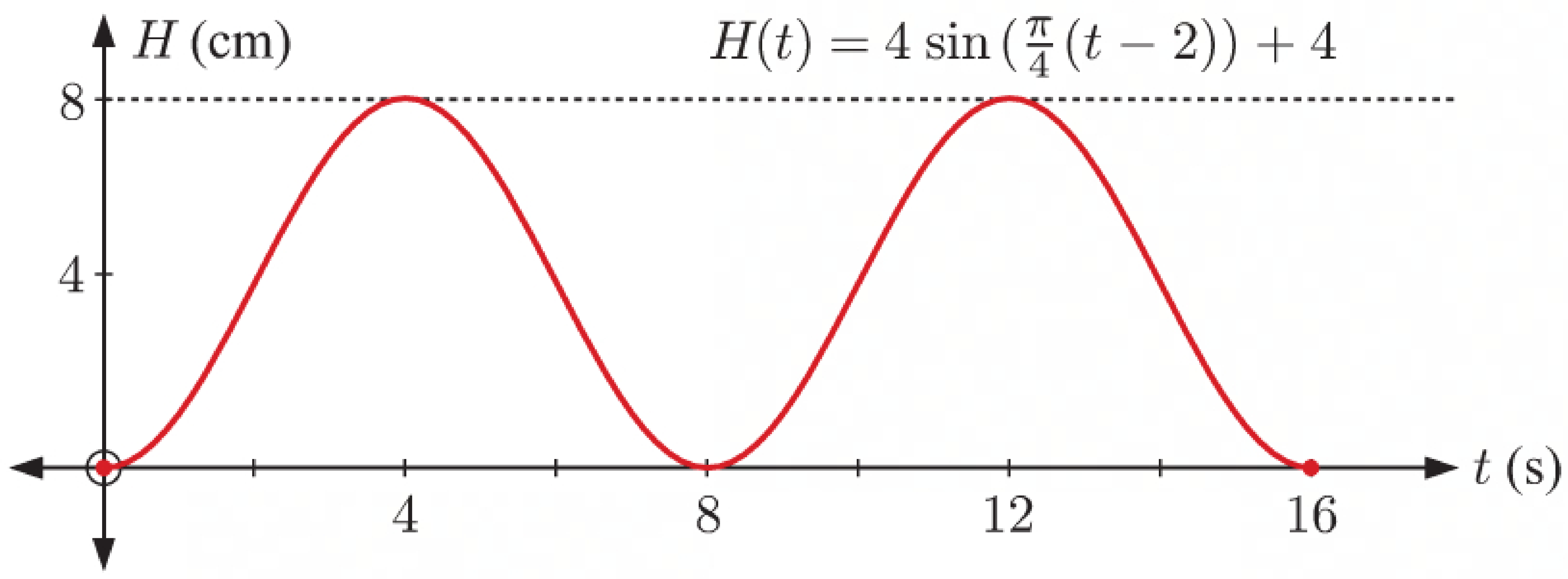
ii 5:37 pm is 37 minutes after 5 pm.
When $t = 37$,
$$\begin{aligned} H &= 15\cos\left(\tfrac{\pi}{30} \times 37\right) + 150 \\ &\approx 138.853 \end{aligned}$$
$\therefore$ at 5:37 pm, the minute hand's tip is approximately 138.9 cm above ground level.

iii 5:51 pm is 51 minutes after 5 pm.
When $t = 51$,
$$\begin{aligned} H &= 15\cos\left(\tfrac{\pi}{30} \times 51\right) + 150 \\ &\approx 158.817 \end{aligned}$$
$\therefore$ at 5:51 pm, the minute hand's tip is approximately 158.8 cm above ground level.

iv 6:23 pm is 83 minutes after 5 pm.
When $t = 83$,
$$\begin{aligned} H &= 15\cos\left(\tfrac{\pi}{30} \times 83\right) + 150 \\ &\approx 138.853 \end{aligned}$$
$\therefore$ at 6:23 pm, the minute hand's tip is approximately 138.9 cm above ground level.

4 **a** For $H(t) = 4\sin\left(\frac{\pi}{4}(t-2)\right) + 4$:

- the amplitude is 4
- the period is $\dfrac{2\pi}{\left(\frac{\pi}{4}\right)} = 8$ seconds
- the horizontal translation is 2 seconds to the right
- the principal axis is $H = 4$.



b When $t = 2$,

$$\begin{aligned} H &= 4\sin\left(\tfrac{\pi}{4}(2-2)\right) + 4 \\ &= 4\sin\left(\tfrac{\pi}{4} \times 0\right) + 4 \\ &= 4\sin 0 + 4 \\ &= 4 \times 0 + 4 \\ &= 4 \end{aligned}$$

$\therefore$ 2 seconds after the gate touches the ground it is 4 cm above ground level.

c When $t = 5.3 + 1 = 6.3$,

$$\begin{aligned} H &= 4\sin\left(\tfrac{\pi}{4}(6.3-2)\right) + 4 \\ &= 4\sin\left(\tfrac{\pi}{4} \times 4.3\right) + 4 \\ &\approx 3.066 \end{aligned}$$

$\therefore$ the ball will not pass through the entrance as its diameter is $2 \times 2.14 = 4.28$ cm but the gate height is only approximately 3.07 cm above ground level.

5 The mean temperature $= \dfrac{15.8 + 5.4}{2} = 10.6°\text{C}$, so $d = 10.6$.

The amplitude $= \dfrac{15.8 - 5.4}{2} = 5.2°\text{C}$

$\therefore\ a = 5.2$

The period is 24 hours, so $b = \frac{2\pi}{24} = \frac{\pi}{12}$.

The maximum occurs at 2 pm, so we assume the temperature passed its mean value 6 hours earlier, at 8 am.

The day begins at mignight, so the function is shifted 8 hours to the right, thus $c = 8$.

If t is the number of hours after midnight, the temperature T is modelled by

$$T(t) = 5.2\sin\left(\tfrac{\pi}{12}(t-8)\right) + 10.6\ °\text{C}.$$

6 The mean height $= \dfrac{1.36 + 0.16}{2} = 0.76$ m, so $d = 0.76$.

The amplitude $= \dfrac{1.36 - 0.16}{2} = 0.6$ m

$\therefore\ a = 0.6$

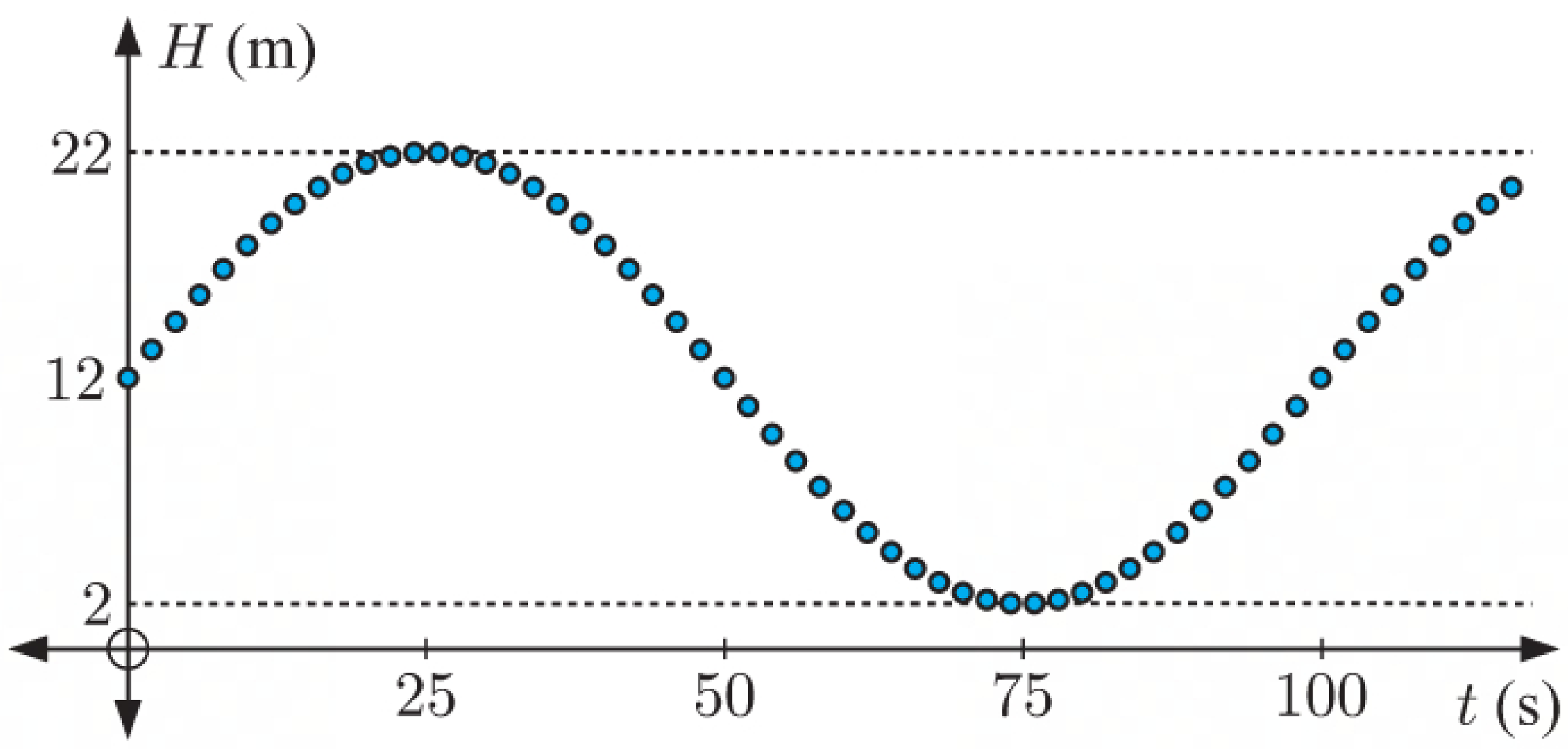
The period is 12.4 hours, so $b = \frac{2\pi}{12.4} = \frac{5\pi}{31}$.

The maximum occurs at 1:30 am, so the function is shifted $1\frac{1}{2}$ hours to the right, thus $c = 1.5$.

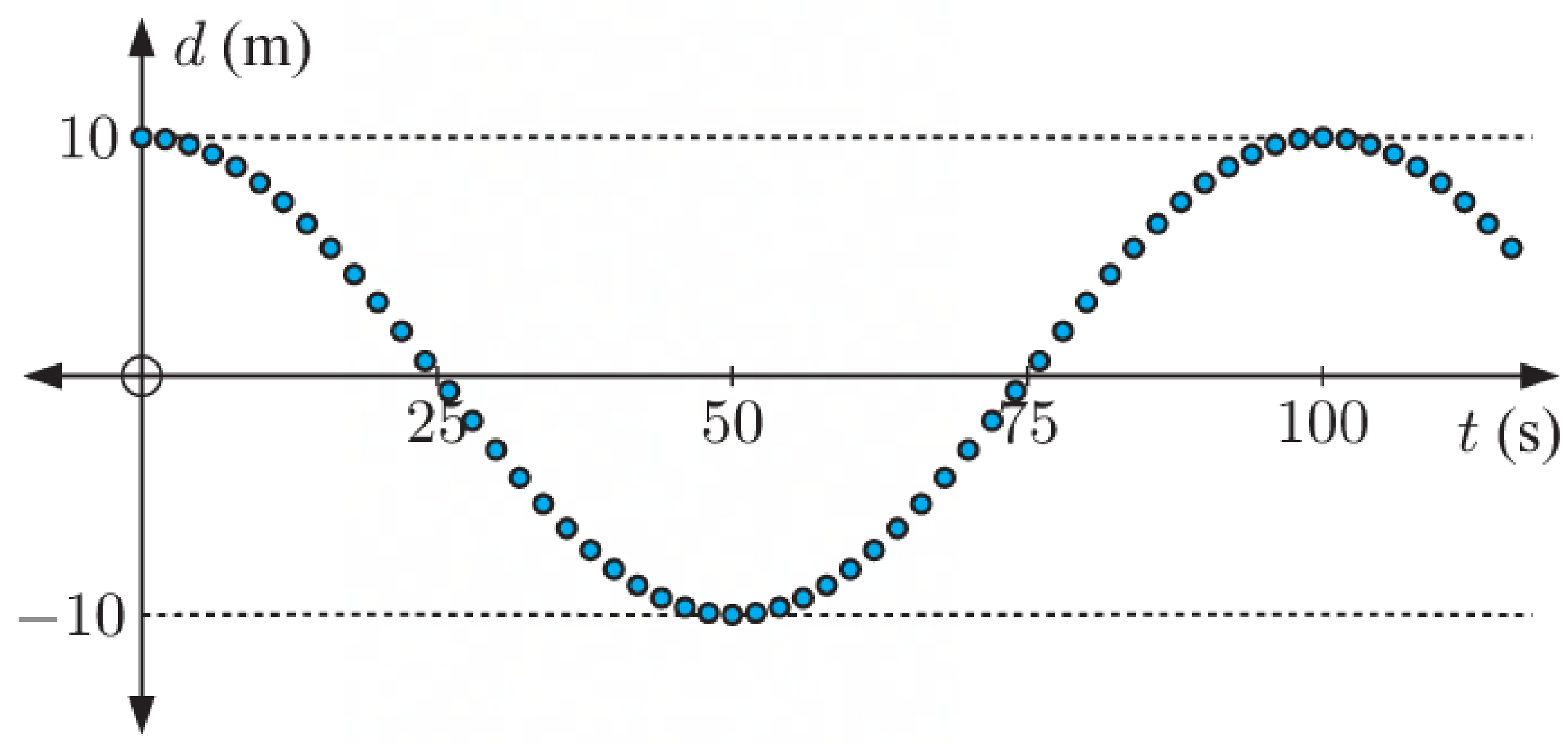
If t is the number of hours after midnight, the height H is modelled by

$$H(t) = 0.6\cos\left(\tfrac{5\pi}{31}(t-1.5)\right) + 0.76 \text{ m}.$$

7 **a** The light is initially $10 + 2 = 12$ m above the ground (its mean height) and oscillates with amplitude 10 m and period 100 seconds.



b Assume that when the light is directly above or below the centre of the Ferris wheel, its horizontal position is 0 m. The horizontal position of the light oscillates with amplitude 10 m and period 100 seconds. The light initially has horizontal position 10 m.



c Both graphs are periodic with an amplitude of 10 m and a period of 100 s. The graphs differ by a horizontal translation of 25 s, and the principal axis is translated by 12 m.

d **i** For the height, $H(t)$:

The amplitude is 10, so $a = 10$.

The period is $\dfrac{2\pi}{b} = 100$,

$$\therefore\ 100b = 2\pi$$
$$\therefore\ b = \tfrac{\pi}{50}$$

The mean height occurs at midnight, so there is no horizontal translation, thus $c = 0$.

The principal axis is $H = 12$, so $d = 12$.

$\therefore\ H(t) = 10\sin\left(\frac{\pi}{50}t\right) + 12$ m

ii The graph of $d(t)$ has the same amplitude and period as the graph of $H(t)$.

The graph of $d(t)$ is obtained by translating the graph of $H(t)$ 25 units to the left and 12 units downwards.

$\therefore\ d(t) = 10\sin\left(\frac{\pi}{50}(t + 25)\right)$ m

8 **a** The mean height $= \dfrac{16.2 + (16.2 - 14)}{2}$

$$= \frac{16.2 + 2.2}{2}$$
$$= 9.2 \text{ m}$$
$$\therefore\ d = 9.2$$

The amplitude $= \frac{14}{2}$

$$= 7 \text{ m}$$
$$\therefore\ a = 7$$

The period is 12.4 hours, so $b = \frac{2\pi}{12.4} = \frac{5\pi}{31}$.

The maximum occurs at 9 am, so we assume the height passed its mean height $\frac{12.4}{4} = 3.1$ hours earlier, or 5.9 hours after midnight.

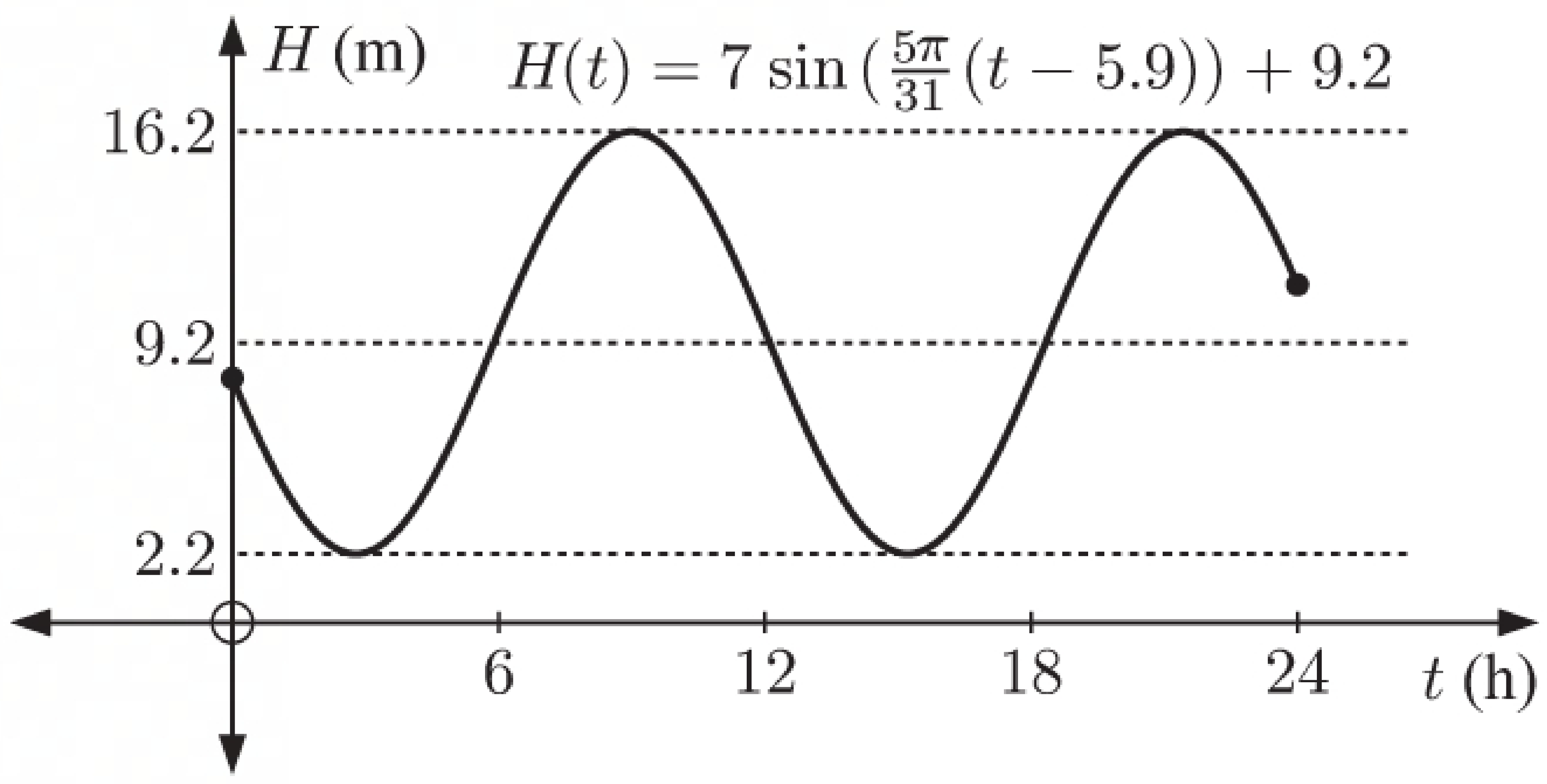
The day begins at midnight, so the function is shifted 5.9 hours to the right, thus $c = 5.9$.

If t is the number of hours after midnight, the height H is modelled by

$$H(t) = 7\sin\left(\tfrac{5\pi}{31}(t - 5.9)\right) + 9.2 \text{ m}.$$

b For $H(t) = 7\sin\left(\frac{5\pi}{31}(t - 5.9)\right) + 9.2$:

- the amplitude is 7
- the period is 12.4 hours
- the horizontal translation is 5.9 hours to the right
- the principal axis is $H = 9.2$.



9 **a** The mean height of the tip of the hour hand $= \dfrac{6 + (-6)}{2}$

$= 0$ cm

$\therefore \ d = 0$

The amplitude of the height of the tip of the hour hand $= 6$ cm

$\therefore \ a = 6$

The period is 12 hours, so $b = \frac{2\pi}{12} = \frac{\pi}{6}$.

The maximum occurs at midnight, so there is no translation.

If t is the number of hours after midnight, the height of the tip of the hour hand relative to the centre of the clock is modelled by $H(t) = 6\cos\left(\frac{\pi}{6}t\right)$ cm.

b The mean horizontal displacement of the tip of the minute hand $= \dfrac{12 + (-12)}{2}$

$= 0$ cm

$\therefore \ d = 0$

The amplitude of the horizontal displacement of the tip of the minute hand $= 12$ cm

$\therefore \ a = 12$

The period is 1 hour, so $b = \frac{2\pi}{1} = 2\pi$.

The mean horizontal displacement occurs at midnight, so there is no translation.

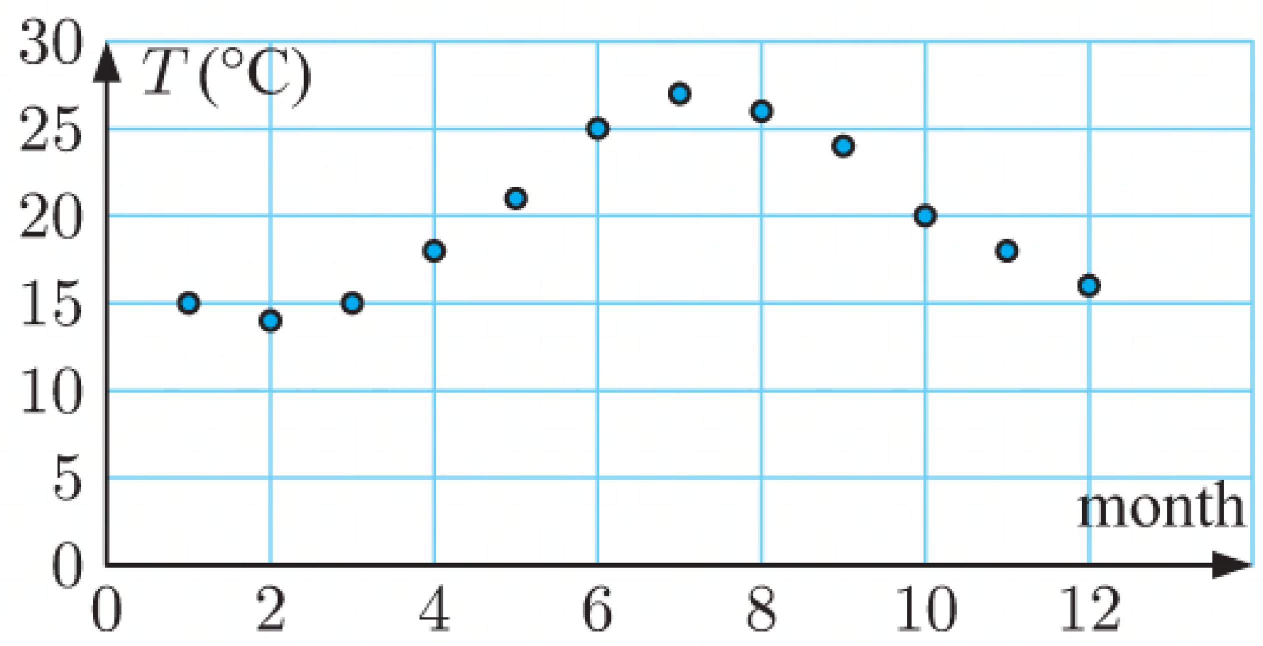
If t is the number of hours after midnight, the horizontal displacement of the tip of the minute hand relative to the centre of the clock is modelled by $d(t) = 12\sin 2\pi t$ cm.

EXERCISE 17E

1

Month	Jan	Feb	Mar	Apr	May	Jun	July	Aug	Sept	Oct	Nov	Dec
Temperature (°C)	15	14	15	18	21	25	27	26	24	20	18	16

a



b The data appears to be periodic, so it is appropriate to fit a trigonometric model.

c **i** The period is 12 months, so $\frac{2\pi}{b} = 12$ and $\therefore$ $b = \frac{\pi}{6}$.

ii The amplitude $= \frac{\text{max} - \text{min}}{2} \approx \frac{27 - 14}{2} \approx 6.5$, so $a \approx 6.5$.

iii The principal axis is midway between the maximum and minimum, so

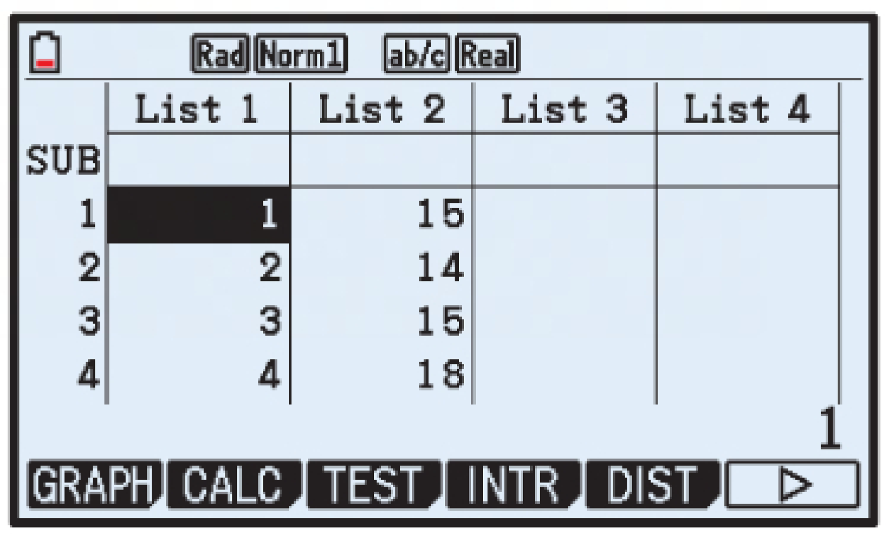
$d \approx \frac{27 + 14}{2} \approx 20.5$.

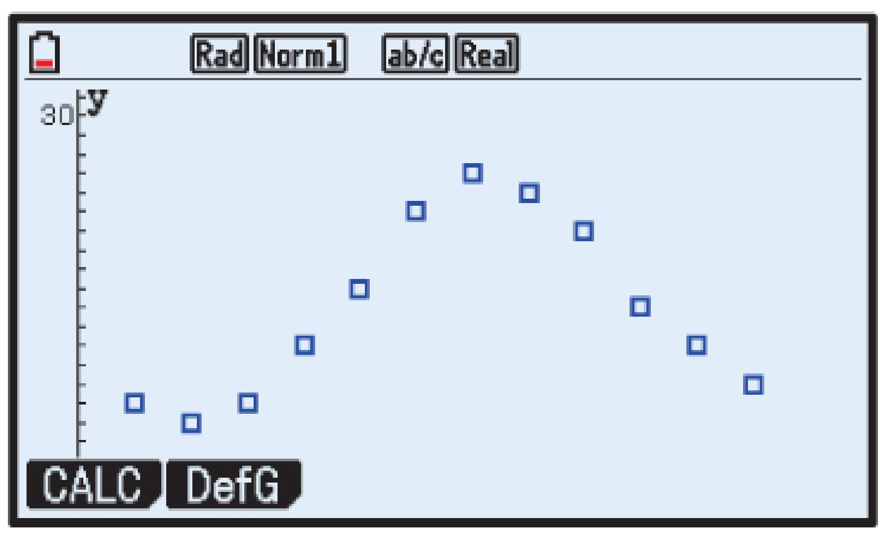
iv The model is $T \approx 6.5 \sin\left(\frac{\pi}{6}(t - c)\right) + 20.5$ for some constant c.

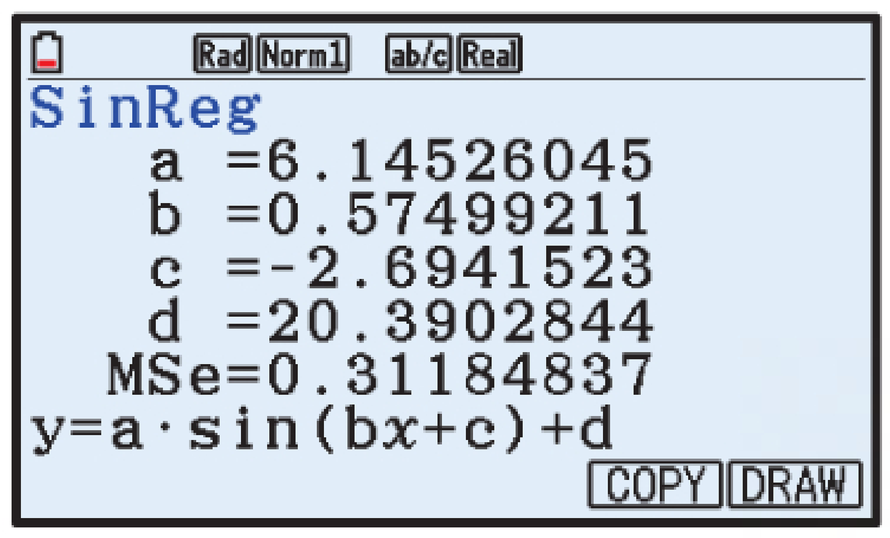
On the original graph, the sine function starts a new period between months 4 and 5.
We estimate that $c \approx 4.5$.

d From **c**, our model is $T \approx 6.5 \sin\left(\frac{\pi}{6}(t - 4.5)\right) + 20.5$

$\approx 6.5 \sin(0.524t - 2.36) + 20.5$







Using technology, $T \approx 6.15 \sin(0.575t - 2.69) + 20.4$.
Our model was a reasonable fit.

2

Month	Jan	Feb	Mar	Apr	May	Jun	July	Aug	Sept	Oct	Nov	Dec
Temperature (°C)	15	16	$14\frac{1}{2}$	12	10	$7\frac{1}{2}$	7	$7\frac{1}{2}$	$8\frac{1}{2}$	$10\frac{1}{2}$	$12\frac{1}{2}$	14

a The period is 12 months, so $\frac{2\pi}{b} = 12$ and $\therefore$ $b = \frac{\pi}{6}$.

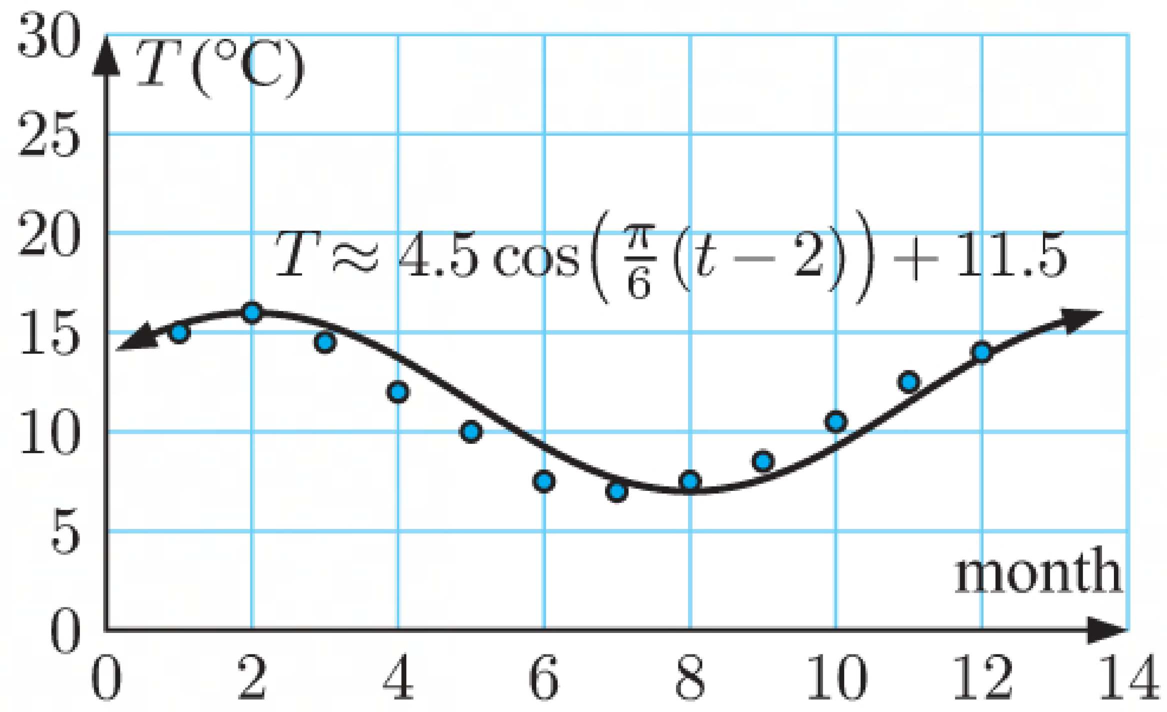
The amplitude $= \frac{\text{max} - \text{min}}{2} \approx \frac{16 - 7}{2} \approx 4.5$, so $a \approx 4.5$.

The principal axis is midway between the maximum and minimum, so $d \approx \frac{16 + 7}{2} \approx 11.5$.

The maximum occurs in February, which is the 2nd month. We estimate that the translation $c \approx 2$.

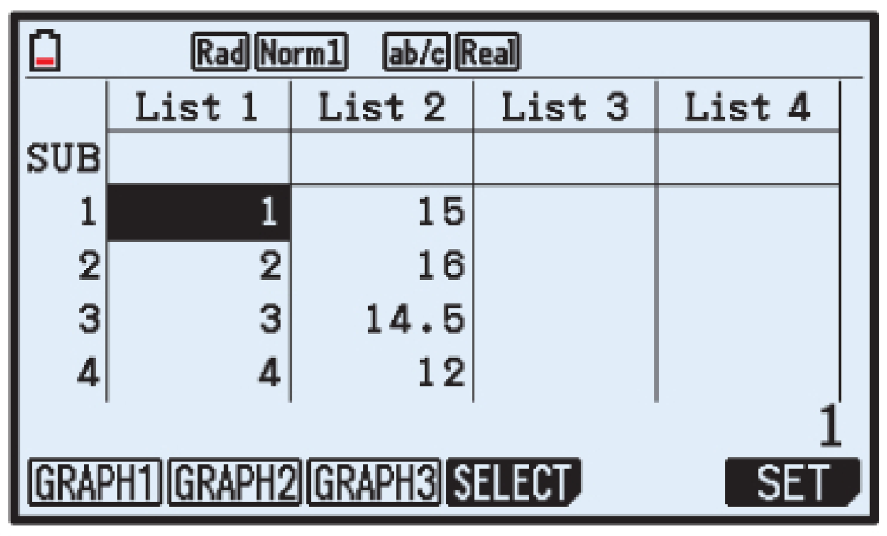
$\therefore \quad T \approx 4.5\cos\left(\frac{\pi}{6}(t-2)\right) + 11.5$

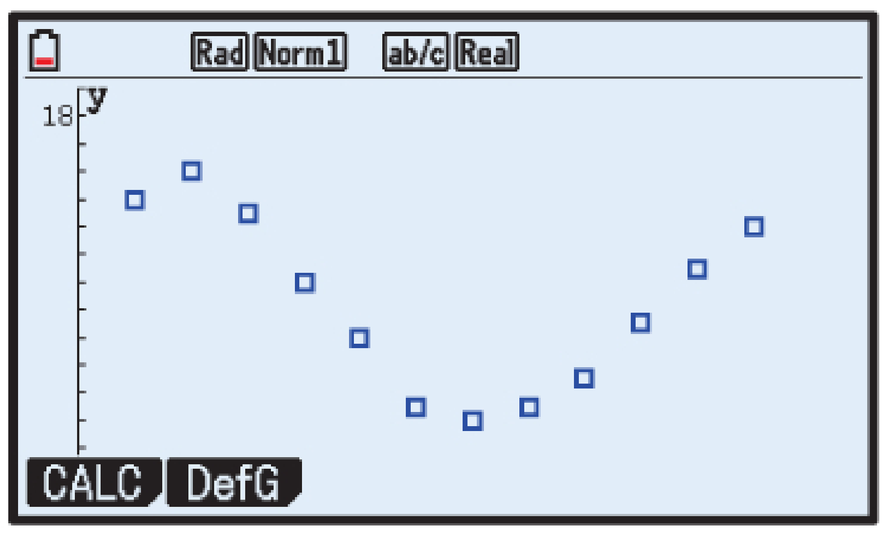
b

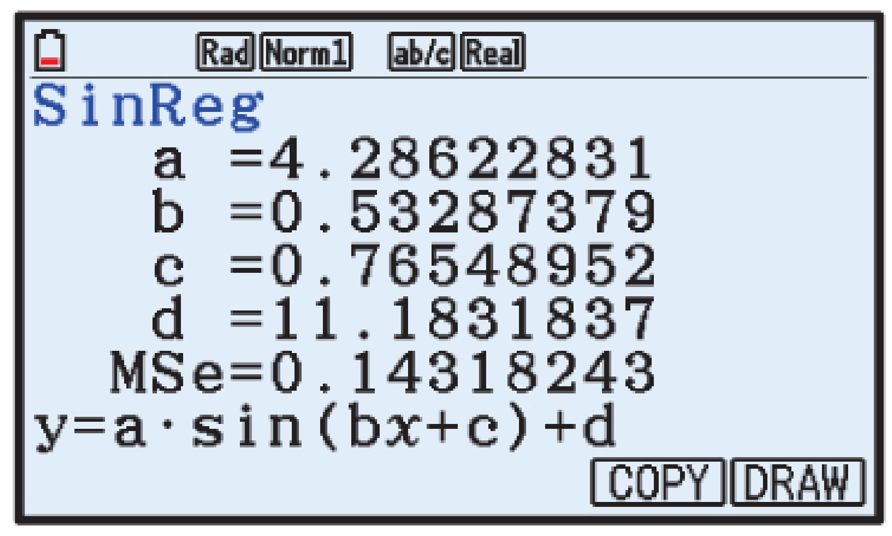


c From **b**, our model is $T \approx 4.5\cos\left(\frac{\pi}{6}(t-2)\right) + 11.5$

$\approx 4.5\cos(0.524t - 1.05) + 11.5$







Using technology, $T \approx 4.29\sin(0.533t + 0.765) + 11.2$

$\approx 4.29\cos(0.533t + 0.76 - \frac{\pi}{2}) + 11.2 \qquad \{\cos x = \sin(x - \frac{\pi}{2})\}$

$\approx 4.29\cos(0.533t - 0.805) + 11.2$

3

Month	Jan	Feb	Mar	Apr	May	Jun	July	Aug	Sept	Oct	Nov	Dec
Temperature (°C)	0	0	-4	-9	-14	-17	-18	-19	-17	-13	-6	-2

a The period is 12 months, so $\dfrac{2\pi}{b} = 12$ and $\therefore \ b = \frac{\pi}{6}$.

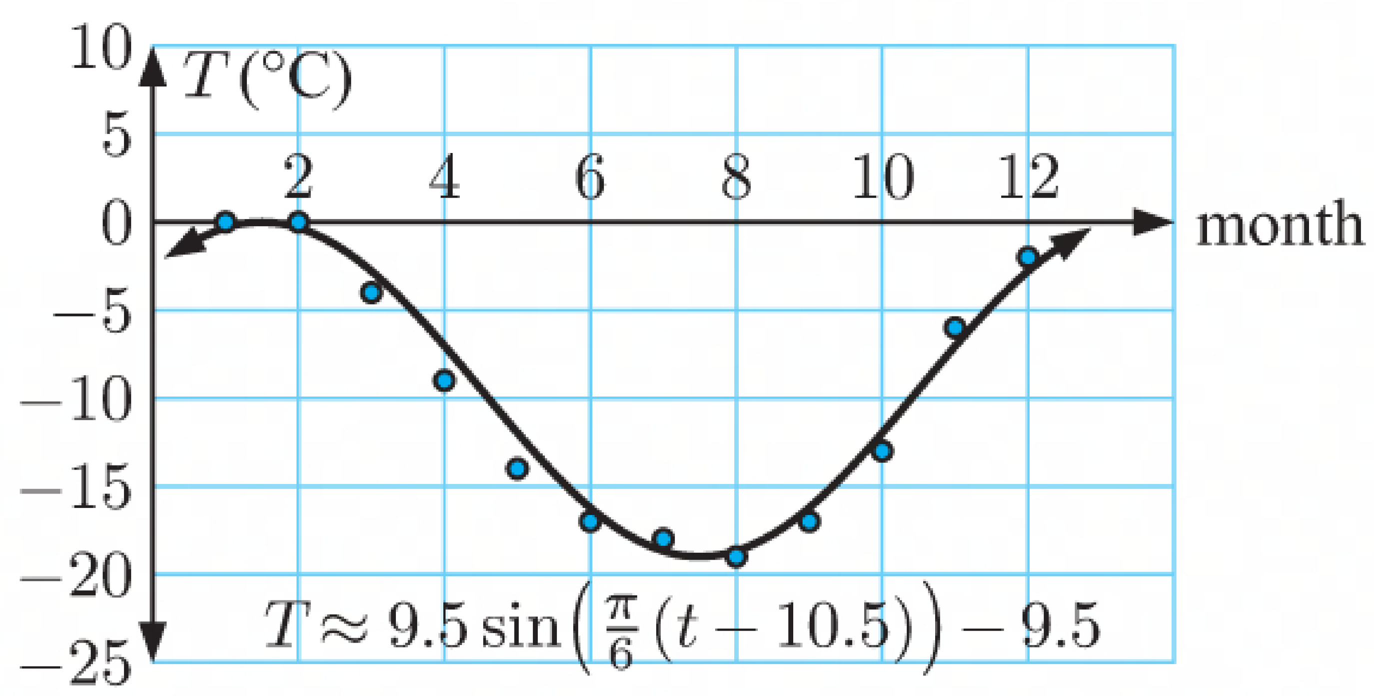
The amplitude $= \dfrac{\max - \min}{2} \approx \dfrac{0-(-19)}{2} \approx 9.5$, so $a \approx 9.5$.

The principal axis is midway between the maximum and minimum, so $d \approx \dfrac{0+(-19)}{2} \approx -9.5$.

The sine function starts a new period midway between August and the following January (months 8 and 13). We estimate that $c \approx \dfrac{8+13}{2} \approx 10.5$.

$\therefore \quad T \approx 9.5\sin\left(\frac{\pi}{6}(t-10.5)\right) - 9.5$

b



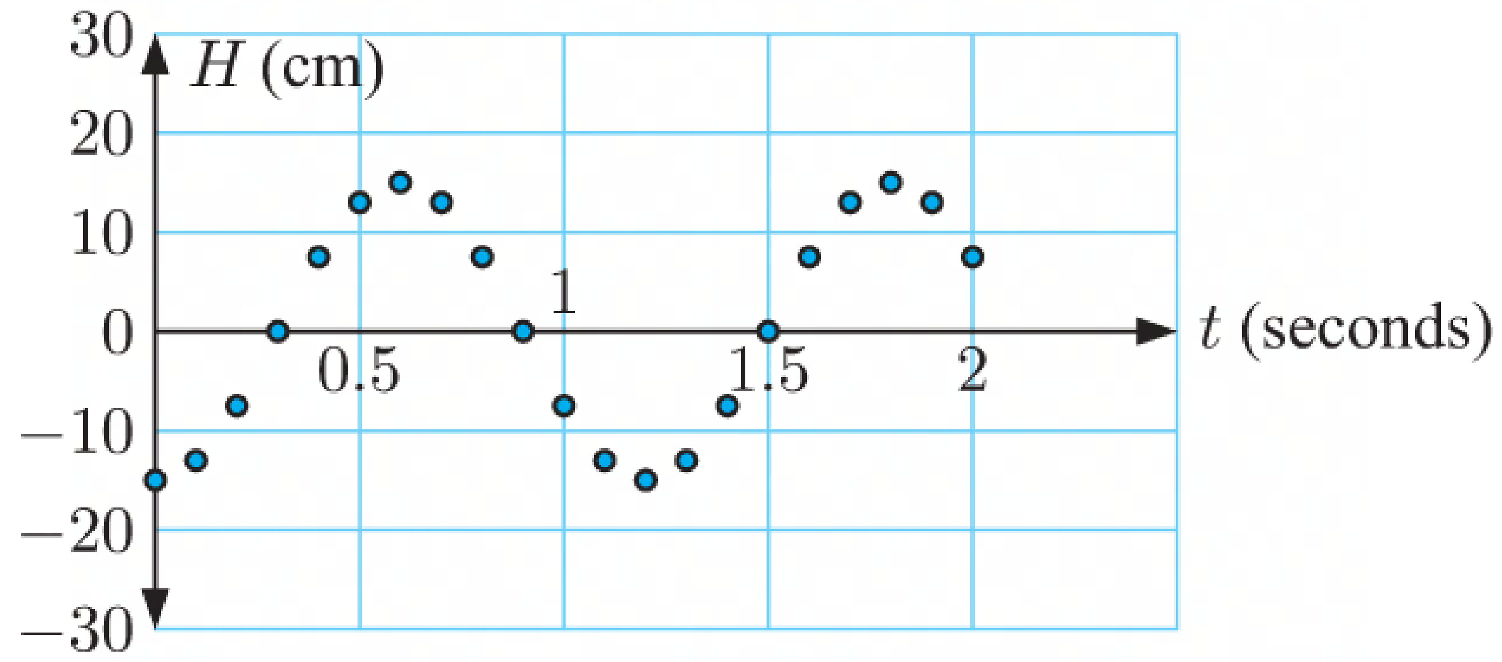
c The model is not perfect, but it is a reasonable fit.

4

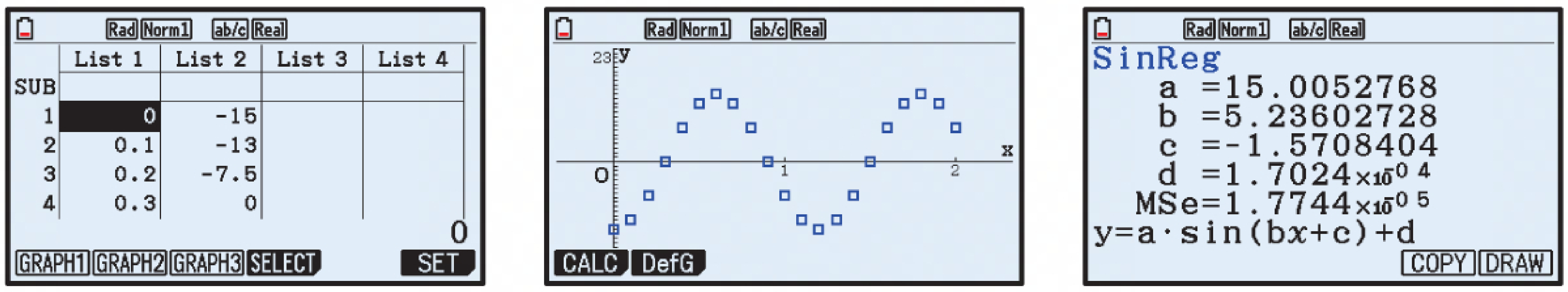
Time (t seconds)	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
Height (H cm)	-15	-13	-7.5	0	7.5	13	15	13	7.5	0

Time (t seconds)	1	1.1	1.2	1.3	1.4	1.5	1.6	1.7	1.8	1.9	2.0
Height (H cm)	-7.5	-13	-15	-13	-7.5	0	7.5	13	15	13	7.5

a



b



Using technology, $H \approx 15.0\sin(5.24t - 1.57) + 0.000\,170$

c When $t = 4.25$, $H \approx 15.0\sin(5.24(4.25) - 1.57) + 0.000\,170$

≈ 14.5

$\therefore$ the height of the object after 4.25 seconds will be about 14.5 cm.

d This model is unrealistic since the spring will not oscillate indefinitely. It will slow down and come to a stop.

EXERCISE 17F

1 **a** $y = \tan\left(x - \frac{\pi}{2}\right)$ is a horizontal translation of $y = \tan x$ to the right by $\frac{\pi}{2}$ units.

So, a horizontal translation of $\frac{\pi}{2}$ units to the right will map $y = \tan x$ onto $y = \tan\left(x - \frac{\pi}{2}\right)$.

b $y = 4\tan x$ is a vertical stretch of $y = \tan x$ with scale factor 4.

So, a vertical stretch with scale factor 4 will map $y = \tan x$ onto $y = 4\tan x$.

c $y = \tan\left(\frac{\pi}{2}x\right)$ is a horizontal stretch of $y = \tan x$ with scale factor $\frac{2}{\pi}$.

So, a horizontal stretch with scale factor $\frac{2}{\pi}$ will map $y = \tan x$ onto $y = \tan\left(\frac{\pi}{2}x\right)$.

d $$\tan x \xrightarrow[\text{scale factor } \frac{1}{2}]{\text{horizontal stretch}} \tan 2x \xrightarrow{\text{translation } \begin{pmatrix} 0 \\ -1 \end{pmatrix}} \tan 2x - 1$$

So, a horizontal stretch with scale factor $\frac{1}{2}$, then a translation 1 unit downwards will map $y = \tan x$ onto $y = \tan 2x - 1$.

e $\tan x \xrightarrow[\text{scale factor } \frac{1}{2}]{\text{vertical stretch}} \frac{1}{2}\tan x \xrightarrow[\text{in } x\text{-axis}]{\text{reflection}} -\frac{1}{2}\tan x$

So, a vertical stretch with scale factor $\frac{1}{2}$, then a reflection in the x-axis will map $y = \tan x$ onto $y = -\frac{1}{2}\tan x$.

f $\tan(x+\pi) = \tan x$ $\{\text{period is } \pi\}$

$\therefore$ $y = \tan(x+\pi) + 2 = \tan x + 2$ is a vertical translation of $y = \tan x$ 2 units upwards.

So, a vertical translation of 2 units upwards will map $y = \tan x$ onto $y = \tan(x+\pi) + 2$.

2 **a** $y = \tan 3x$ has period $\frac{\pi}{b} = \frac{\pi}{3}$

b $y = \tan\frac{x}{4}$ has period $\frac{\pi}{b} = \frac{\pi}{(\frac{1}{4})}$
$= 4\pi$

c $y = \tan \pi x$ has period $\frac{\pi}{b} = \frac{\pi}{\pi}$
$= 1$

d $y = -\tan\left(\frac{\pi}{2}x\right)$ has period $\frac{\pi}{b} = \frac{\pi}{(\frac{\pi}{2})}$
$= 2$

e $y = \tan\left(\frac{2x}{3} - \frac{\pi}{3}\right)$ has period $\frac{\pi}{b} = \frac{\pi}{(\frac{2}{3})}$
$= \frac{3\pi}{2}$

f $y = \tan nx$, $n \neq 0$ has period $\frac{\pi}{b} = \frac{\pi}{n}$

3 **a** **i** $y = \tan 2x = 0$ when

$2x = k\pi$, $k \in \mathbb{Z}$

$\therefore$ $x = \dfrac{k\pi}{2}$, $k \in \mathbb{Z}$

ii $y = \tan 2x$ is undefined when

$2x = \frac{\pi}{2} + k\pi$, $k \in \mathbb{Z}$

$\therefore$ $x = \dfrac{\pi}{4} + \dfrac{k\pi}{2}$, $k \in \mathbb{Z}$

b **i** $y = \tan\left(x + \frac{\pi}{3}\right) = 0$ when

$x + \frac{\pi}{3} = k\pi$, $k \in \mathbb{Z}$

$\therefore$ $x = \frac{2\pi}{3} + k\pi$, $k \in \mathbb{Z}$

ii $y = \tan\left(x + \frac{\pi}{3}\right)$ is undefined when

$x + \frac{\pi}{3} = \frac{\pi}{2} + k\pi$, $k \in \mathbb{Z}$

$\therefore$ $x = \frac{\pi}{6} + k\pi$, $k \in \mathbb{Z}$

c **i** $y = \frac{1}{2}\tan\left(\frac{1}{2}\left(x - \frac{\pi}{6}\right)\right) = 0$ when

$\frac{1}{2}\left(x - \frac{\pi}{6}\right) = k\pi$, $k \in \mathbb{Z}$

$\therefore$ $x - \frac{\pi}{6} = 2k\pi$, $k \in \mathbb{Z}$

$\therefore$ $x = \frac{\pi}{6} + 2k\pi$, $k \in \mathbb{Z}$

ii $y = \frac{1}{2}\tan\left(\frac{1}{2}\left(x - \frac{\pi}{6}\right)\right)$ is undefined

when $\frac{1}{2}\left(x - \frac{\pi}{6}\right) = \frac{\pi}{2} + k\pi$, $k \in \mathbb{Z}$

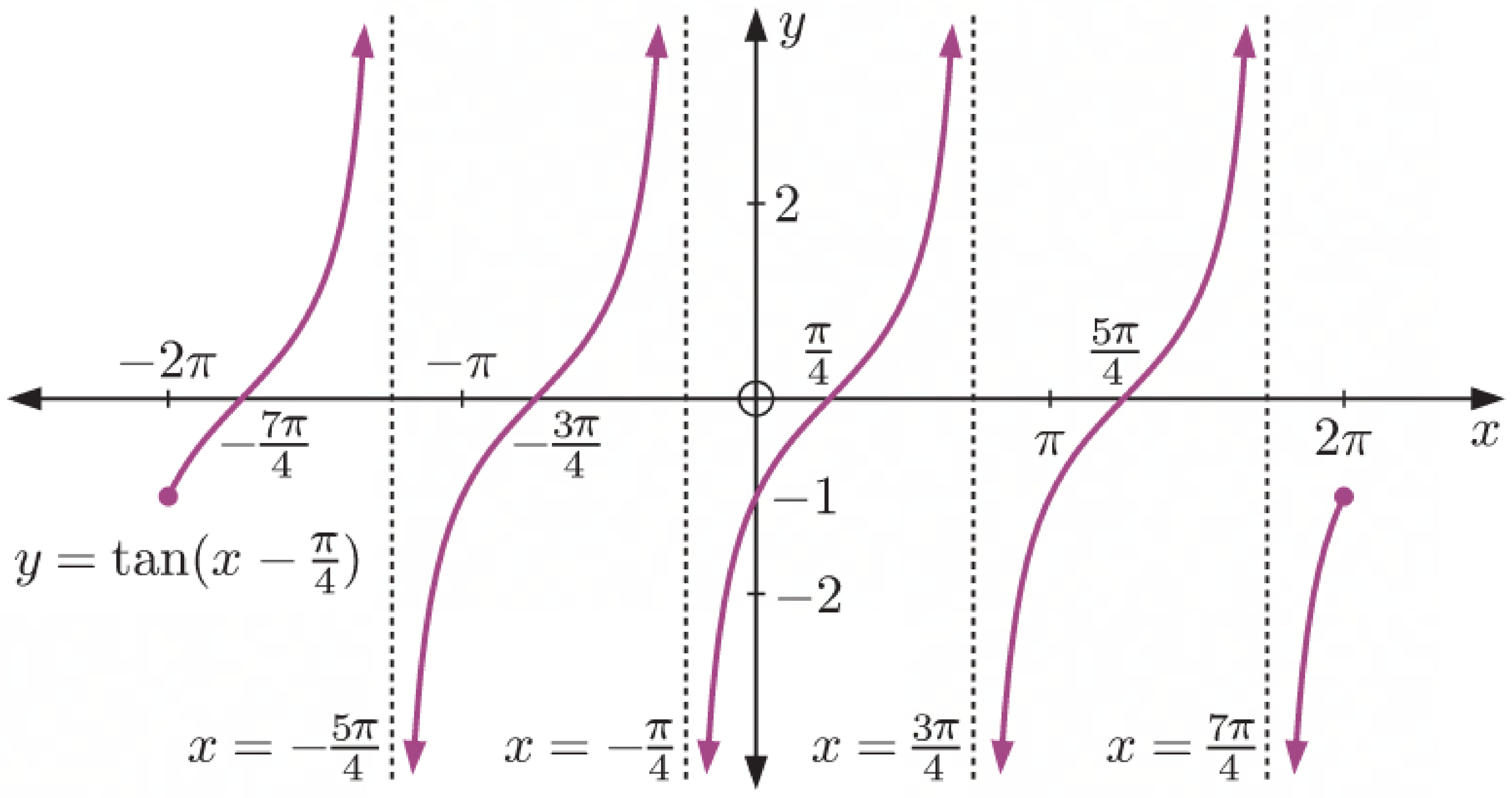
$\therefore$ $x - \frac{\pi}{6} = \pi + 2k\pi$, $k \in \mathbb{Z}$

$\therefore$ $x = \frac{7\pi}{6} + 2k\pi$, $k \in \mathbb{Z}$

4 **a** $y = \tan\left(x - \frac{\pi}{4}\right)$ is a horizontal translation of $y = \tan x$ to the right by $\frac{\pi}{4}$ units.

$y = \tan x$ has vertical asymptotes $x = -\frac{3\pi}{2}$, $x = -\frac{\pi}{2}$, $x = \frac{\pi}{2}$, $x = \frac{3\pi}{2}$, and its x-intercepts are -2π, $-\pi$, 0, π, and 2π.

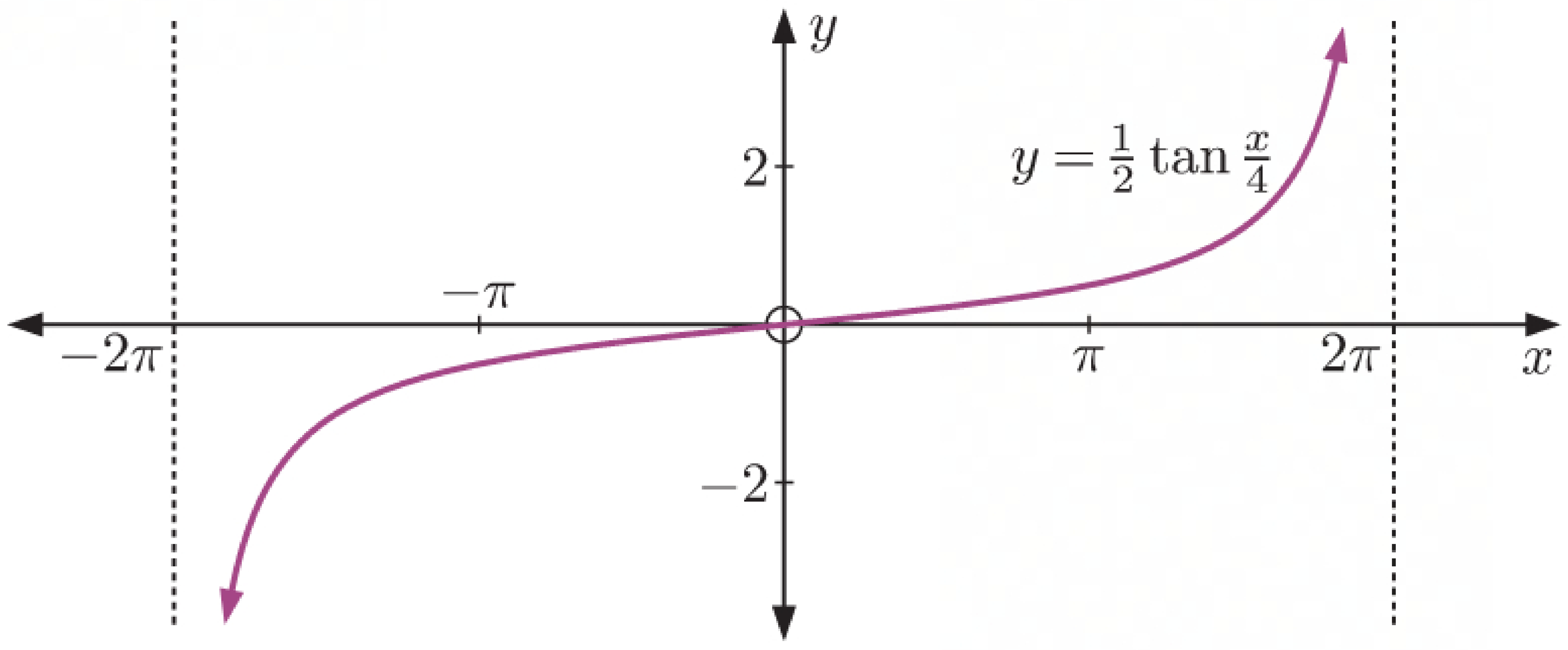
$\therefore$ $y = \tan\left(x - \frac{\pi}{4}\right)$ has vertical asymptotes $x = -\frac{5\pi}{4}$, $x = -\frac{\pi}{4}$, $x = \frac{3\pi}{4}$, $x = \frac{7\pi}{4}$, and x-intercepts $-\frac{7\pi}{4}$, $-\frac{3\pi}{4}$, $\frac{\pi}{4}$, $\frac{5\pi}{4}$.



b $y = \frac{1}{2}\tan\frac{x}{4}$ is a horizontal stretch of $y = \tan x$ with scale factor 4, followed by a vertical stretch with scale factor $\frac{1}{2}$.

Since $b = \frac{1}{4}$, the period is 4π.

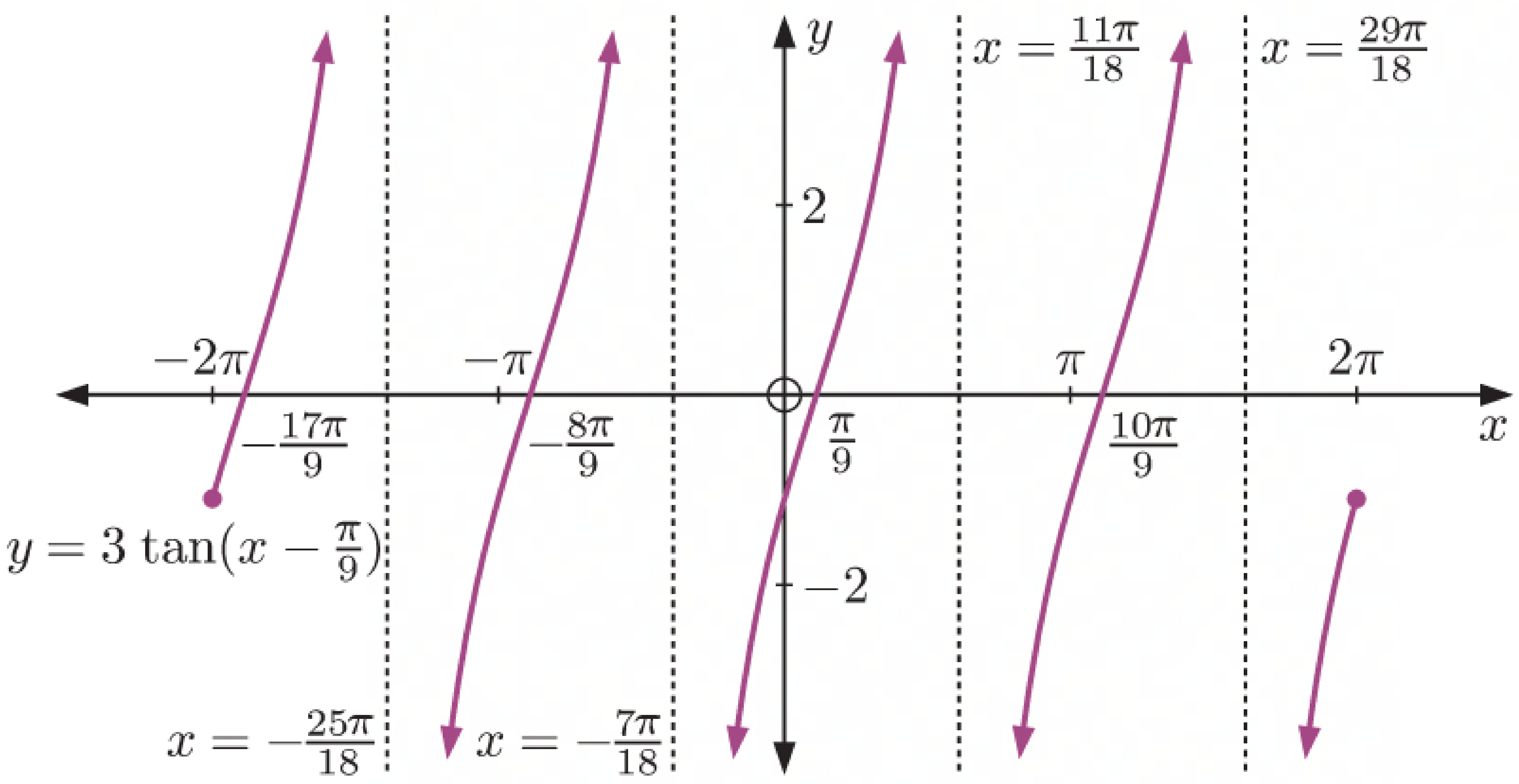
$y = \frac{1}{2}\tan\frac{x}{4}$ has vertical asymptotes $x = \pm 2\pi$, and x-intercept 0.



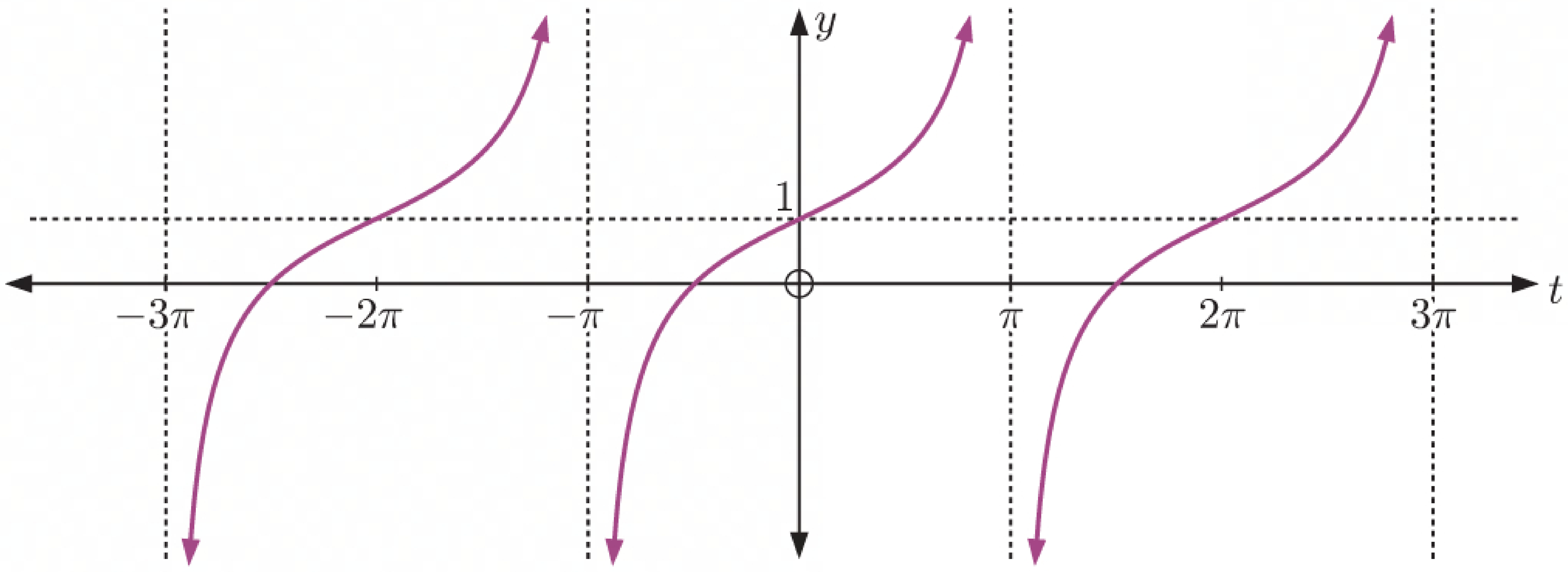
c $y = 3\tan\left(x - \frac{\pi}{9}\right)$ is a horizontal translation of $y = \tan x$ to the right by $\frac{\pi}{9}$ units, followed by a vertical stretch with scale factor 3.

$y = \tan x$ has vertical asymptotes $x = -\frac{3\pi}{2}$, $x = -\frac{\pi}{2}$, $x = \frac{\pi}{2}$, $x = \frac{3\pi}{2}$, and x-intercepts -2π, $-\pi$, 0, π, and 2π.

$\therefore$ $y = 3\tan\left(x - \frac{\pi}{9}\right)$ has vertical asymptotes $x = -\frac{25\pi}{18}$, $x = -\frac{7\pi}{18}$, $x = \frac{11\pi}{18}$, $x = \frac{29\pi}{18}$, and x-intercepts $-\frac{17\pi}{9}$, $-\frac{8\pi}{9}$, $\frac{\pi}{9}$, and $\frac{10\pi}{9}$.



5



The y-intercept is 1, so $y = \tan pt$ has been translated upwards by 1 unit.

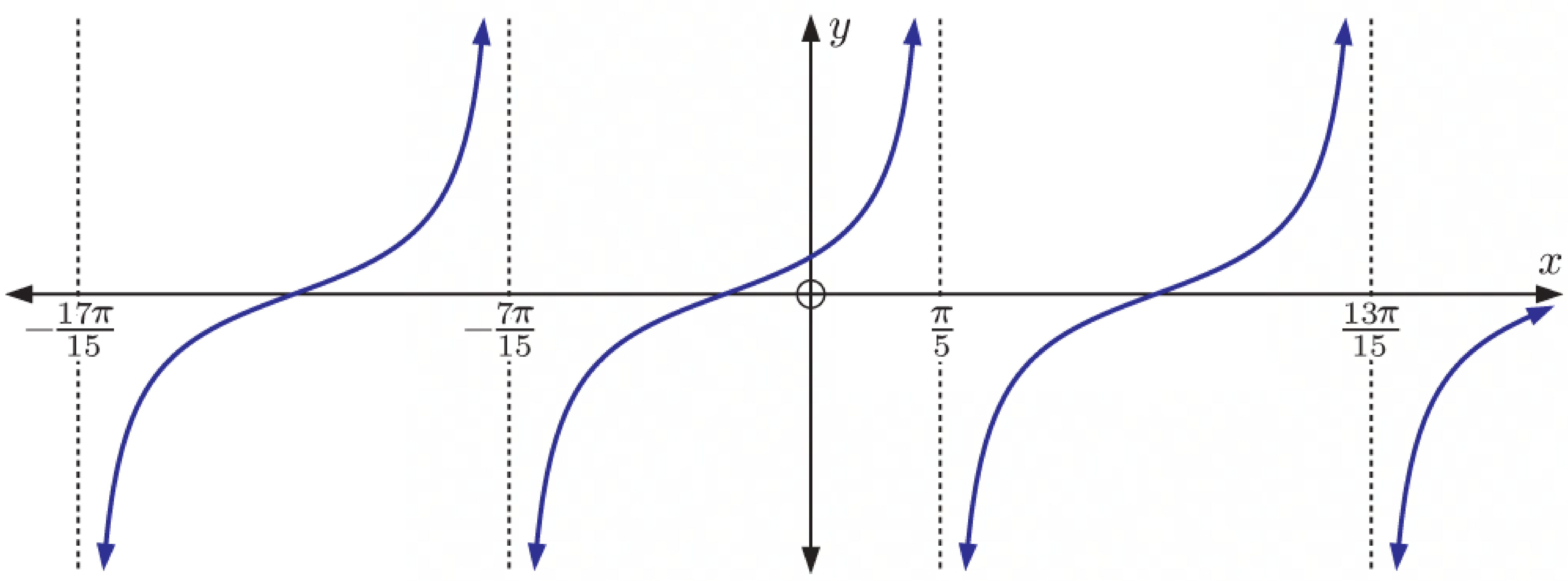
$\therefore \quad q = 1$

$y = \tan pt + 1$ has period $\dfrac{\pi}{p} = 2\pi$

$$\therefore \quad p = \tfrac{1}{2}$$

So, $p = \frac{1}{2}$, $q = 1$.

6



$y = \tan a(x-b)$ has period $\dfrac{\pi}{a} = \dfrac{\pi}{5} - \left(-\dfrac{7\pi}{15}\right) = \frac{2\pi}{3}$

$$\therefore \quad \frac{a}{\pi} = \frac{3}{2\pi}$$

$$\therefore \quad a = \tfrac{3}{2}$$

$y = \tan x$ has vertical asymptotes $x = \frac{\pi}{2} + k\pi$, $k \in \mathbb{Z}$

$\therefore \quad y = \tan \frac{3}{2}x$ has vertical asymptotes $x = \frac{\pi}{3} + \dfrac{2k\pi}{3}$, $k \in \mathbb{Z}$

$\therefore \quad y = \tan \frac{3}{2}(x-b)$ has vertical asymptotes $x = \frac{\pi}{3} + b + \dfrac{2k\pi}{3}$, $k \in \mathbb{Z}$

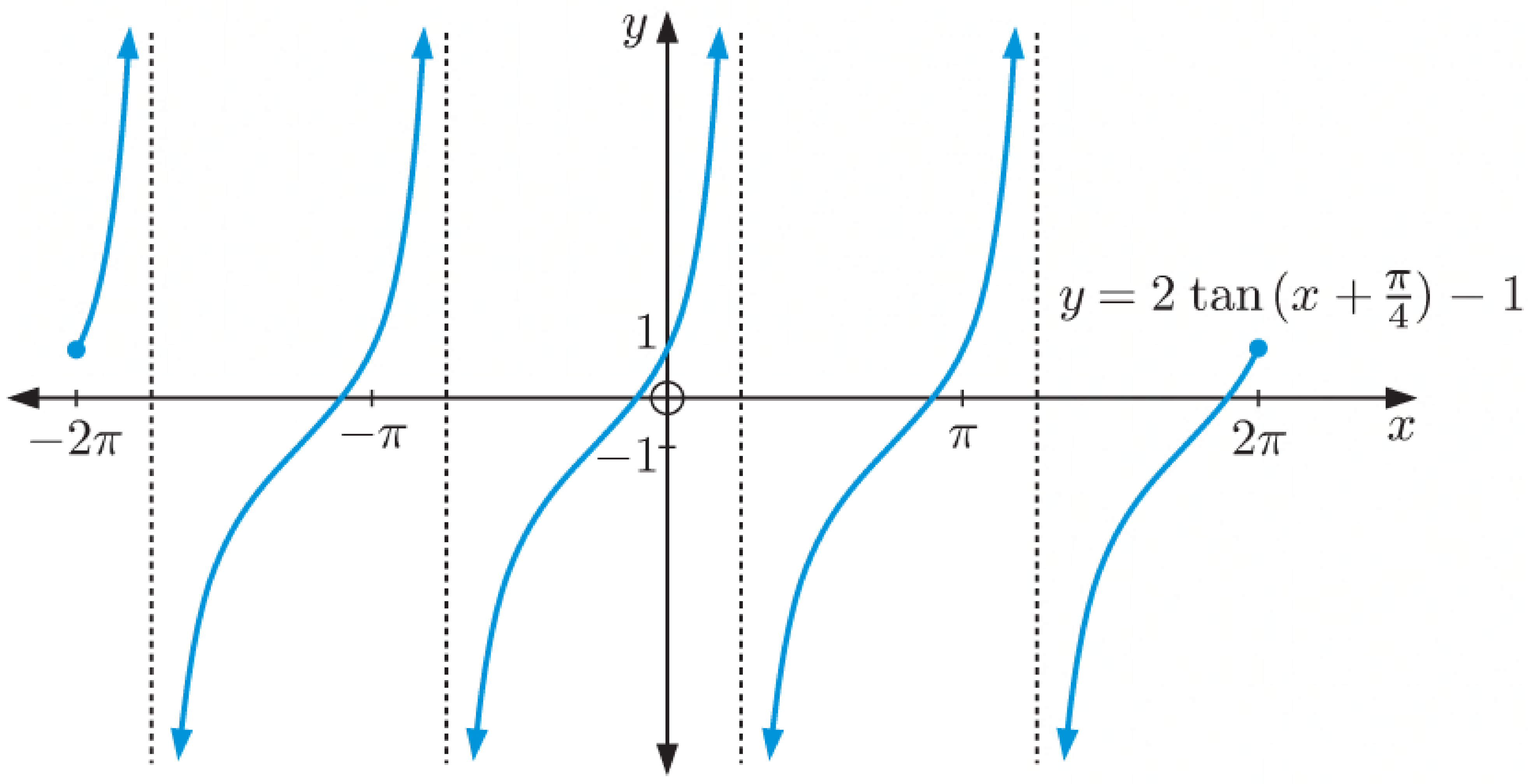
From the graph, $\frac{\pi}{3} + b + \dfrac{2k_1\pi}{3} = \frac{\pi}{5} + \dfrac{2k_2\pi}{3}$, $k_1, k_2 \in \mathbb{Z}$

$$\therefore \quad b = -\tfrac{2\pi}{15} + \frac{2k\pi}{3}, \quad k \in \mathbb{Z} \qquad \{k = k_2 - k_1\}$$

7 **a** $\tan x \xrightarrow[\text{scale factor } 2]{\text{vertical stretch}} 2\tan x \xrightarrow{\text{translation } \begin{pmatrix} -\frac{\pi}{4} \\ -1 \end{pmatrix}} 2\tan\left(x + \frac{\pi}{4}\right) - 1$

So, a vertical stretch with scale factor 2, then a translation $\frac{\pi}{4}$ units left and 1 unit downwards will map $y = \tan x$ onto $y = 2\tan\left(x + \frac{\pi}{4}\right) - 1$.

b



8 $y = a\tan(b(x-c)) + d, \quad a \neq 0, \quad b > 0$ is undefined when

$$b(x-c) = \tfrac{\pi}{2} + k\pi, \quad k \in \mathbb{Z}$$
$$\therefore \quad 2b(x-c) = \pi + 2k\pi$$
$$\therefore \quad 2b(x-c) = \pi(2k+1)$$
$$\therefore \quad x - c = \frac{\pi}{2b}(2k+1)$$
$$\therefore \quad x = \frac{\pi}{2b}(2k+1) + c$$ are the asymptotes of the function for all $k \in \mathbb{Z}$.

9 $f(x) = \tan x, \quad g(x) = 2x - \frac{\pi}{2}$

a **i** $(f \circ g)(x) = f(g(x))$
$= f\left(2x - \frac{\pi}{2}\right)$
$= \tan\left(2x - \frac{\pi}{2}\right)$

ii $(g \circ f)(x) = g(f(x))$
$= g(\tan x)$
$= 2\tan x - \frac{\pi}{2}$

b **i** $(f \circ g)\left(\frac{\pi}{3}\right) = \tan\left(2 \times \frac{\pi}{3} - \frac{\pi}{2}\right)$
$= \tan\left(\frac{2\pi}{3} - \frac{\pi}{2}\right)$
$= \tan\frac{\pi}{6}$
$= \frac{1}{\sqrt{3}}$

ii $(g \circ f)(\pi) = 2\tan\pi - \frac{\pi}{2}$
$= -\frac{\pi}{2}$

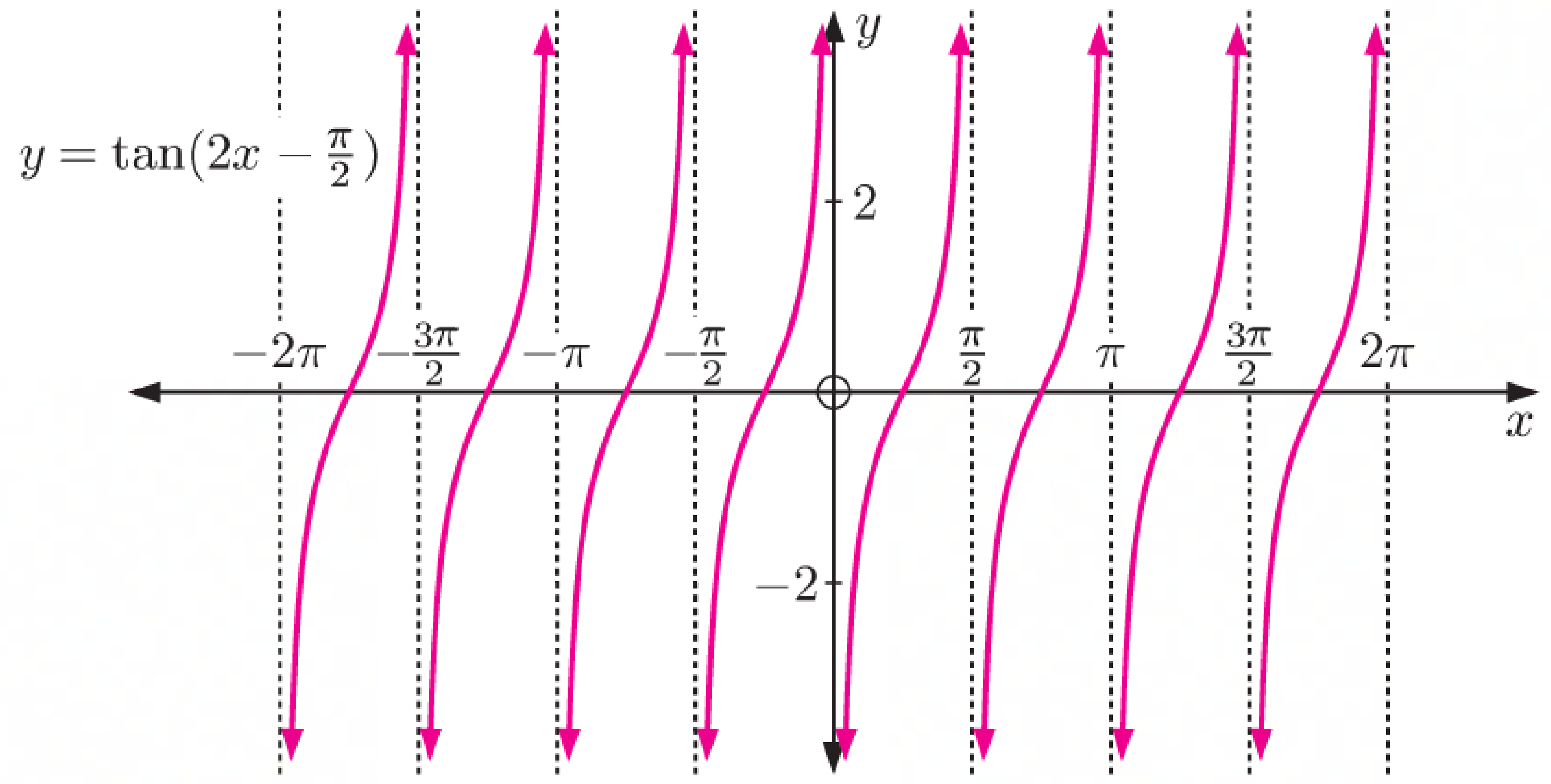
c **i** $(f \circ g)(x) = \tan\left(2x - \frac{\pi}{2}\right)$ has period $\frac{\pi}{2}$.

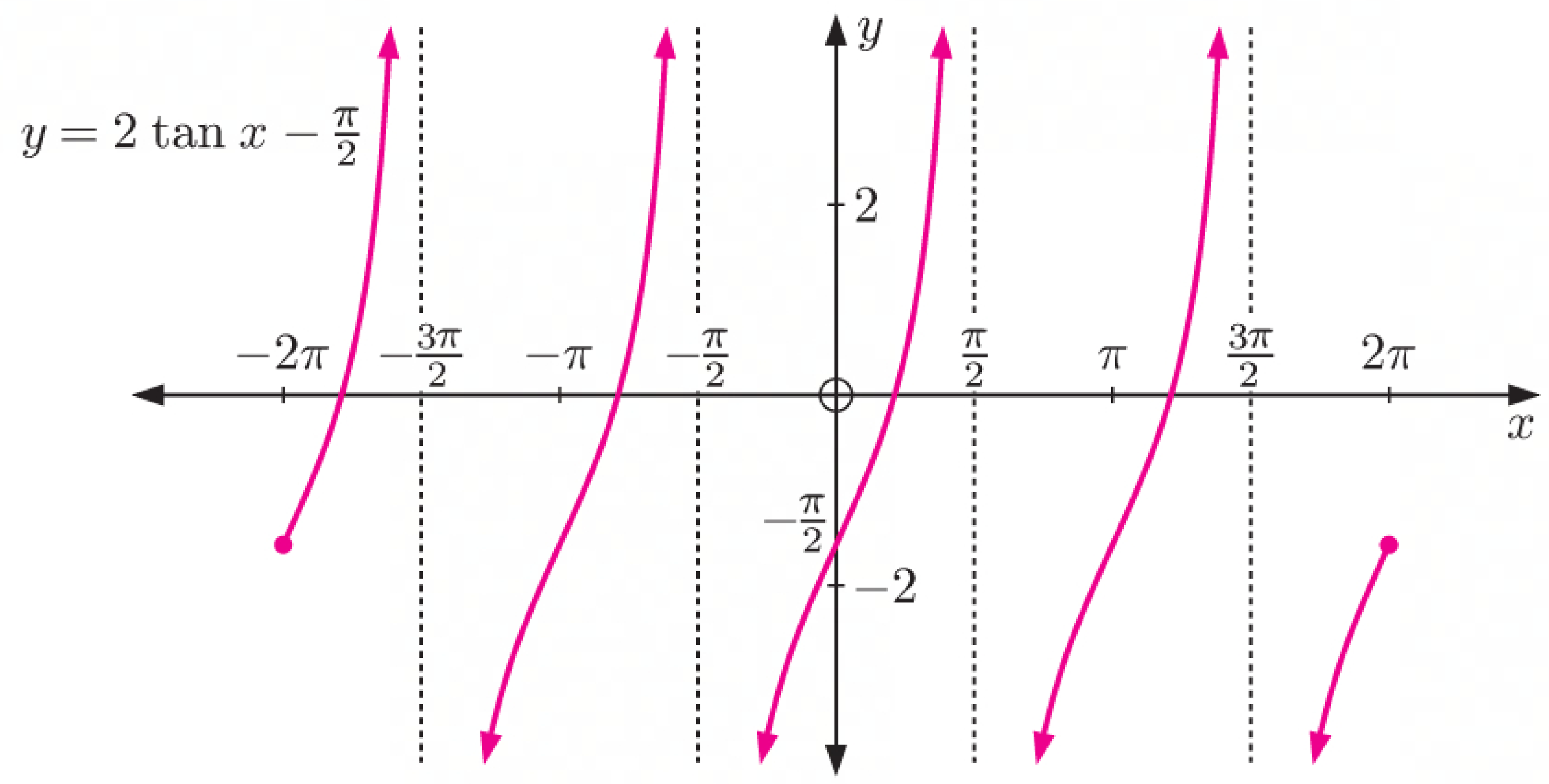
$\tan\left(2x - \frac{\pi}{2}\right)$ is undefined when $2x - \frac{\pi}{2} = \frac{\pi}{2} + k\pi, \quad k \in \mathbb{Z}$
$\therefore \quad 2x = \pi + k\pi, \quad k \in \mathbb{Z}$
$\therefore \quad 2x = k\pi, \quad k \in \mathbb{Z}$
$\therefore \quad x = \frac{k\pi}{2}, \quad k \in \mathbb{Z}$

$\therefore$ the vertical asymptotes of $(f \circ g)(x)$ are $x = \frac{k\pi}{2}, \quad k \in \mathbb{Z}$.

ii $(g \circ f)(x) = 2\tan x - \frac{\pi}{2}$ has period π.
$\tan x$ is undefined when $x = \frac{\pi}{2} + k\pi, \quad k \in \mathbb{Z}$
$\therefore$ the vertical asymptotes of $(g \circ f)(x)$ are $x = \frac{\pi}{2} + k\pi, \quad k \in \mathbb{Z}$.

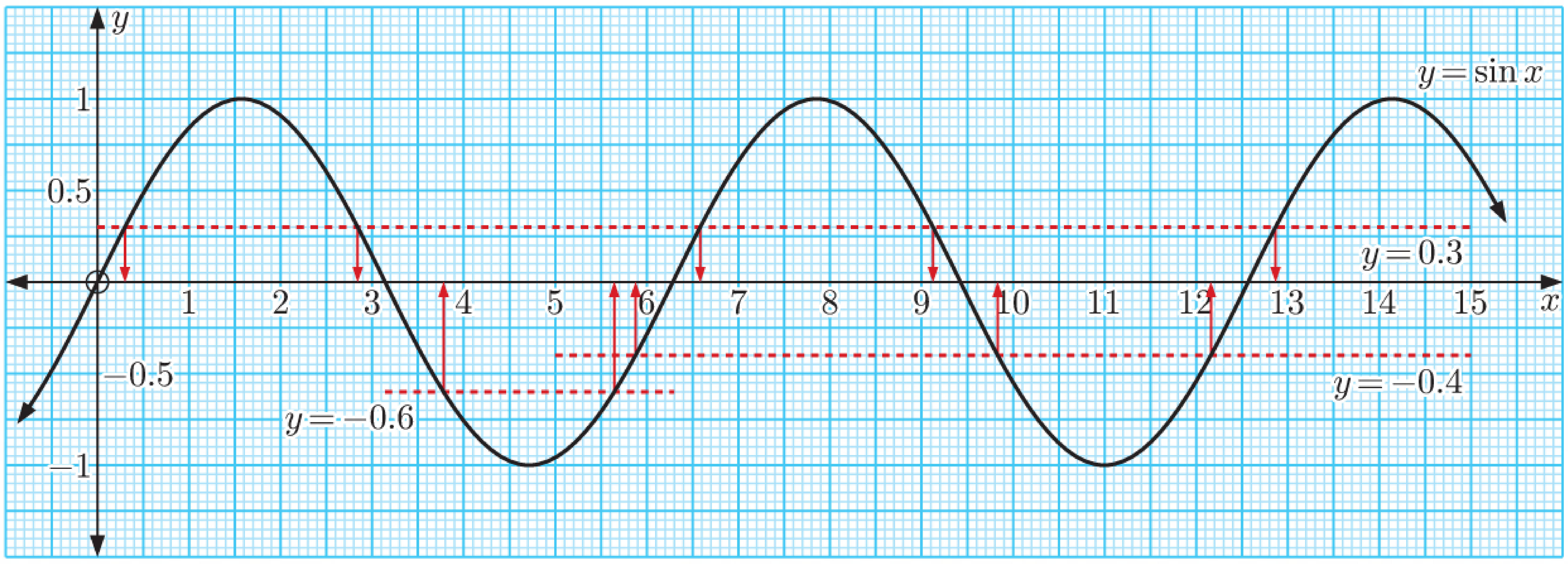
d





EXERCISE 17G.1

1



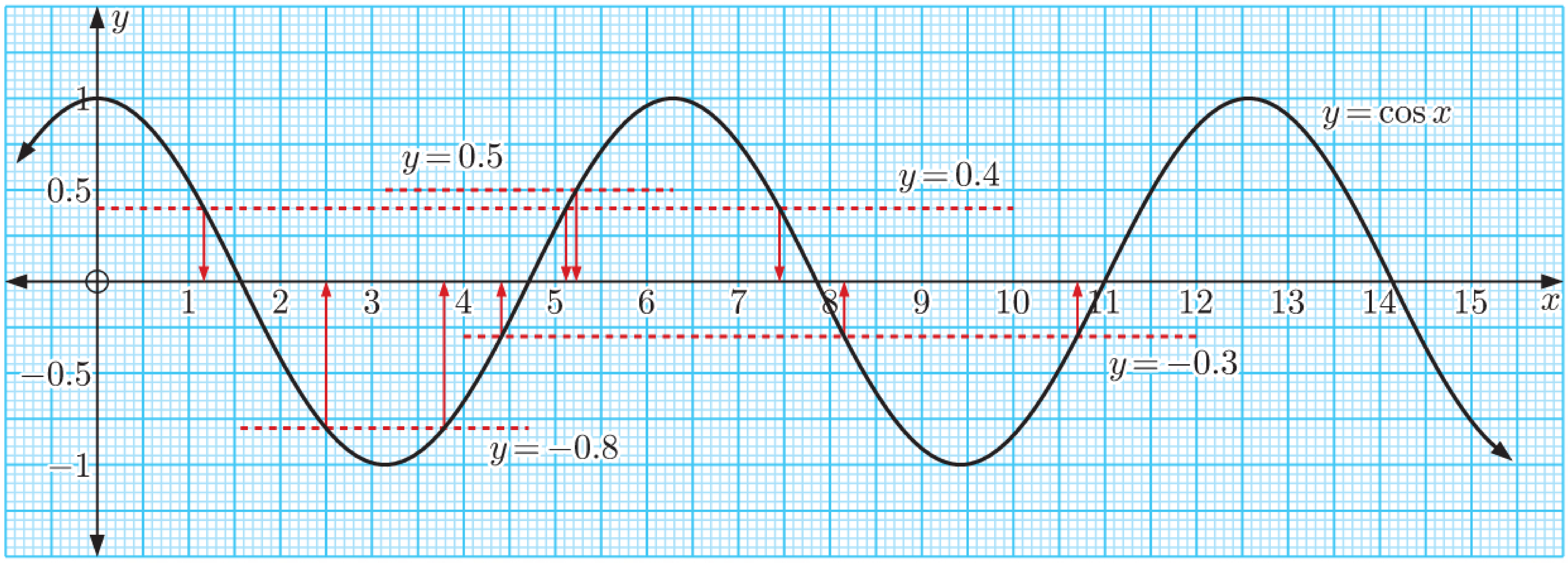
a When $\sin x = 0.3$, $0 \leqslant x \leqslant 15$, $x \approx 0.3, 2.8, 6.6, 9.1, 12.9$

b When $\sin x = -0.4$, $5 \leqslant x \leqslant 15$, $x \approx 5.9, 9.8, 12.2$

c When $\sin x = 0.3$, $0 \leqslant x \leqslant 2\pi$, $x \approx 0.3, 2.8$

d When $\sin x = -0.6$, $\pi \leqslant x \leqslant 2\pi$, $x \approx 3.8, 5.6$

2



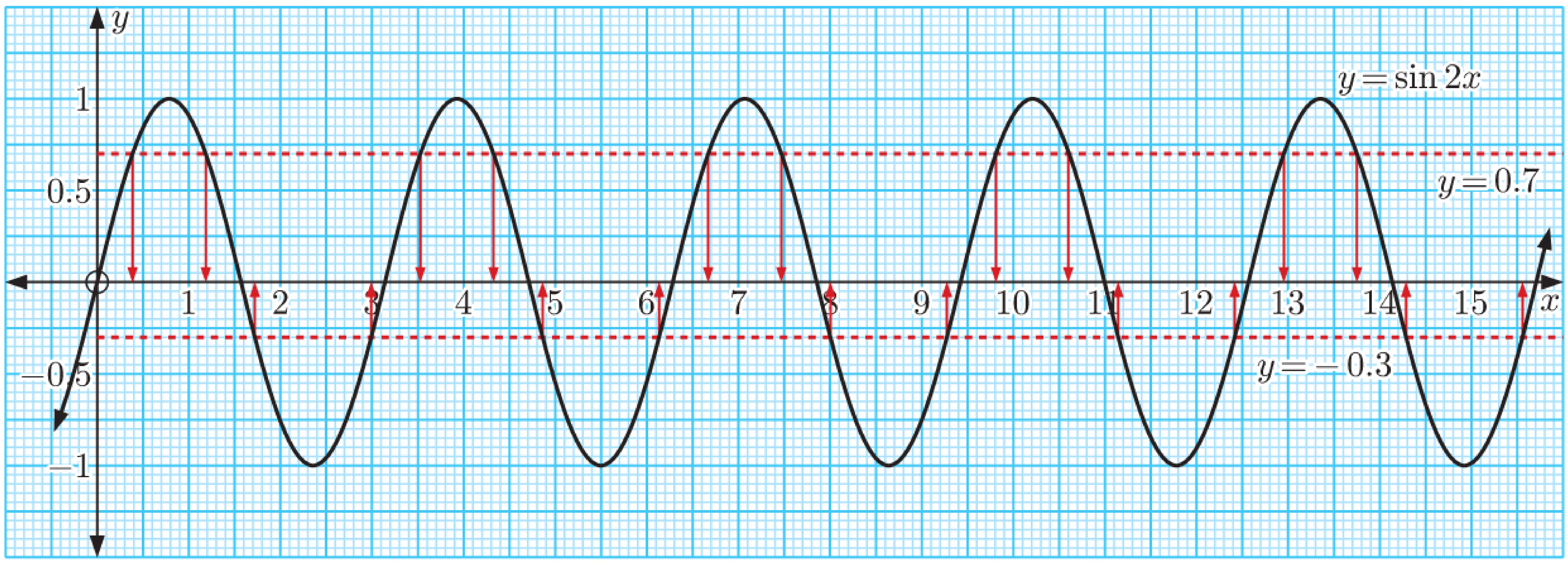
a When $\cos x = 0.4$, $0 \leqslant x \leqslant 10$, $x \approx 1.2, 5.1, 7.4$

b When $\cos x = -0.3$, $4 \leqslant x \leqslant 12$, $x \approx 4.4, 8.2, 10.7$

c When $\cos x = 0.5$, $\pi \leqslant x \leqslant 2\pi$, $x \approx 5.2$

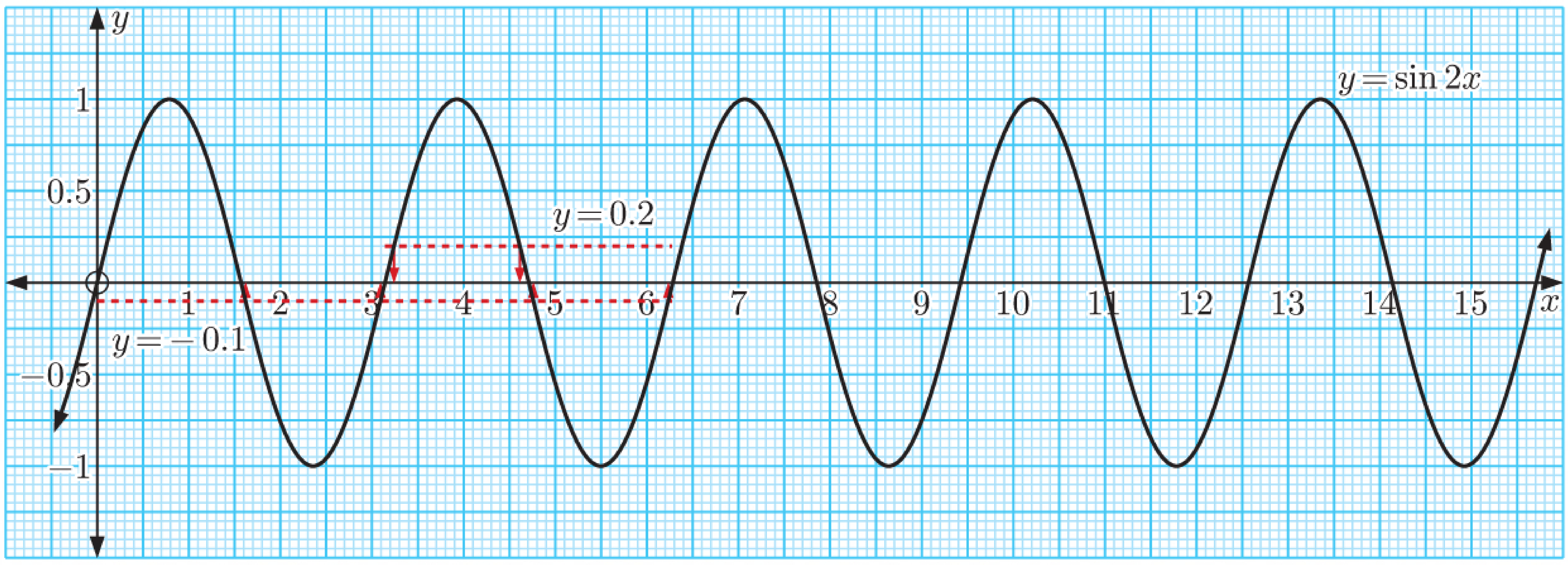
d When $\cos x = -0.8$, $\frac{\pi}{2} \leqslant x \leqslant \frac{3\pi}{2}$, $x \approx 2.5, 3.8$

3



a When $\sin 2x = 0.7$, $0 \leqslant x \leqslant 16$, $x \approx 0.4, 1.2, 3.5, 4.3, 6.7, 7.5, 9.8, 10.6, 13.0, 13.7$

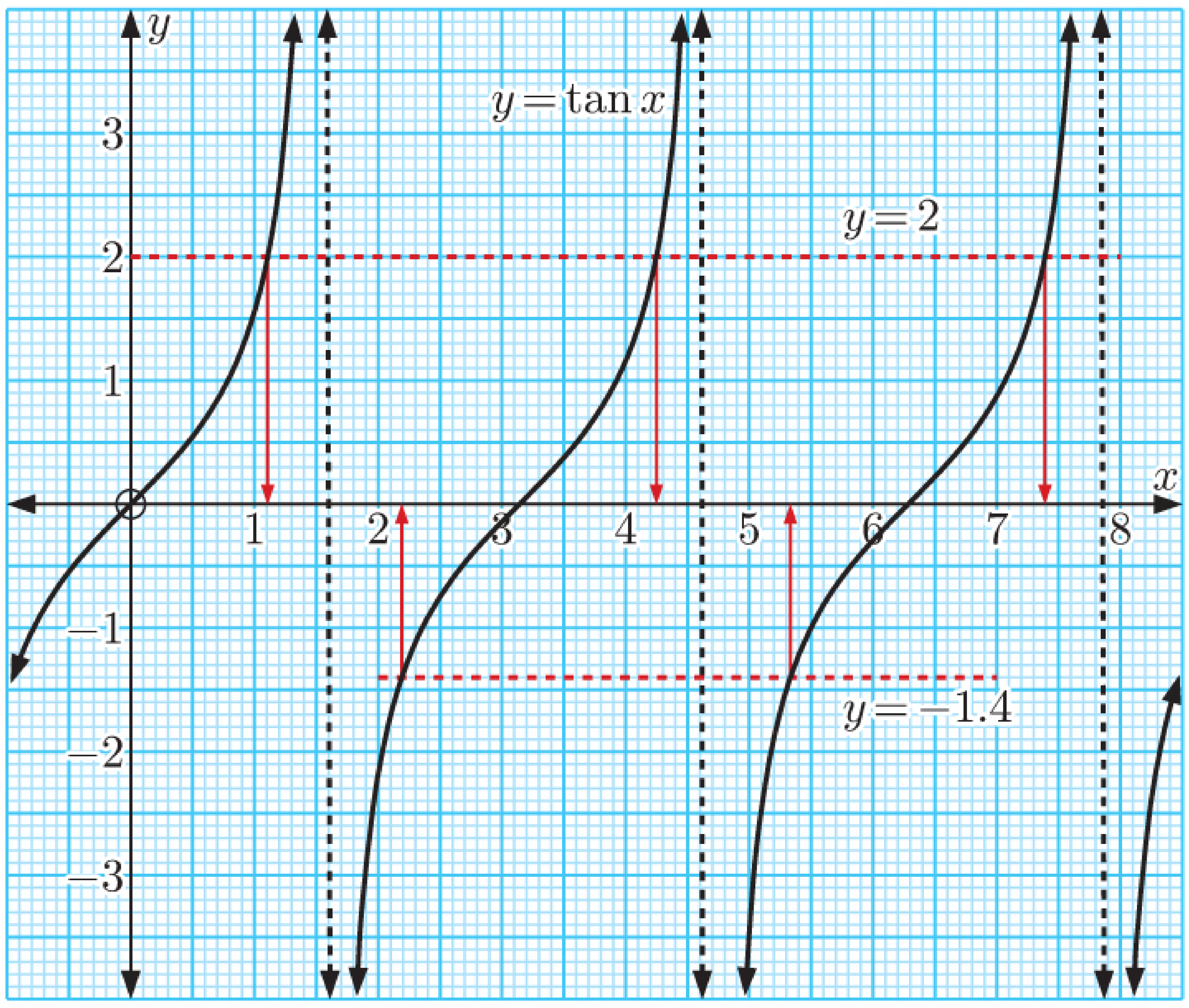
b When $\sin 2x = -0.3$, $0 \leqslant x \leqslant 16$, $x \approx 1.7, 3.0, 4.9, 6.1, 8.0, 9.3, 11.1, 12.4, 14.3, 15.6$



c When $\sin 2x = 0.2$, $\pi \leqslant x \leqslant 2\pi$, $x \approx 3.2, 4.6$

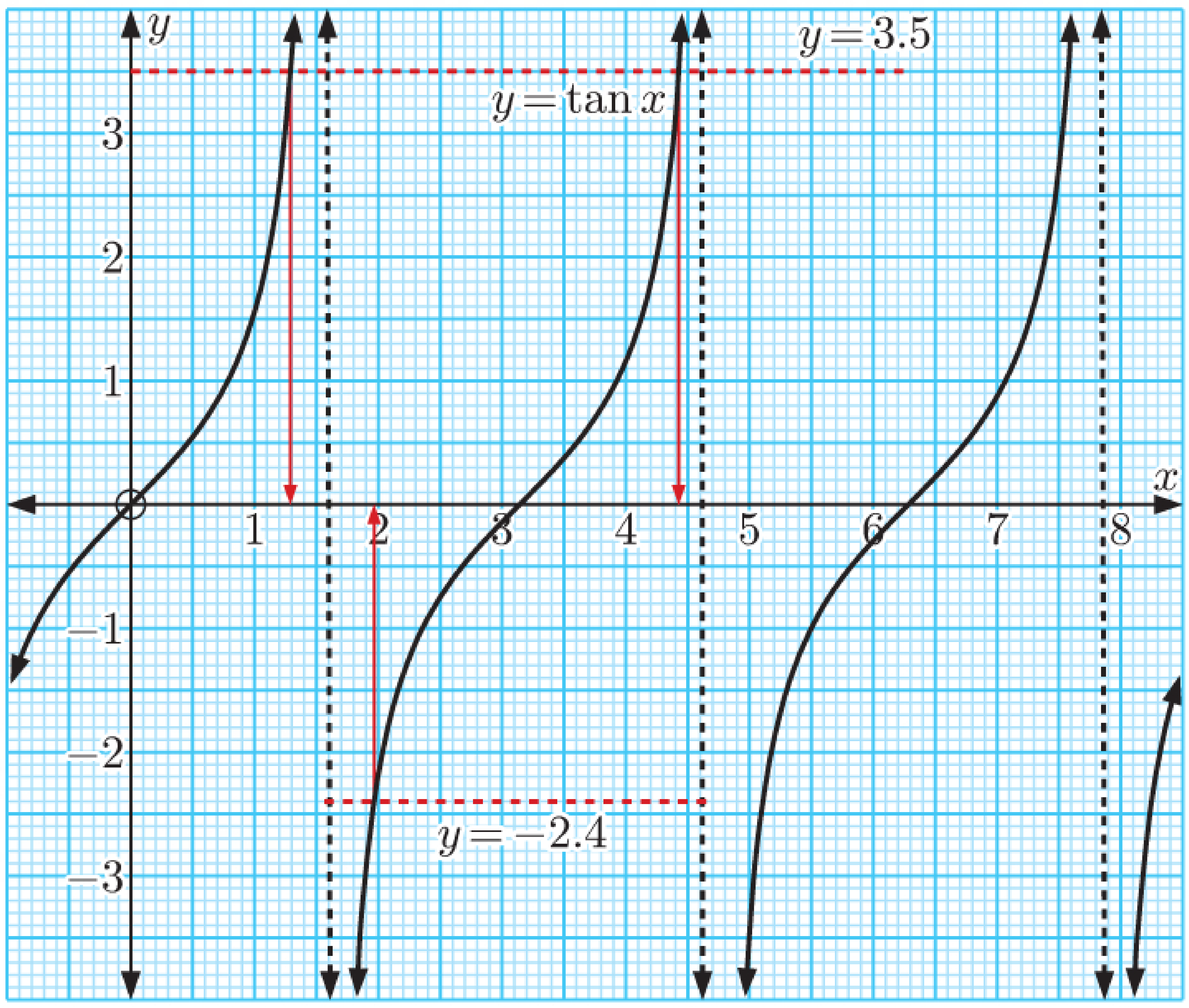
d When $\sin 2x = -0.1$, $0 \leqslant x \leqslant 2\pi$, $x \approx 1.6, 3.1, 4.8, 6.2$

4



a When $\tan x = 2$, $0 \leqslant x \leqslant 8$, $x \approx 1.1, 4.2, 7.4$

b When $\tan x = -1.4$, $2 \leqslant x \leqslant 7$, $x \approx 2.2, 5.3$

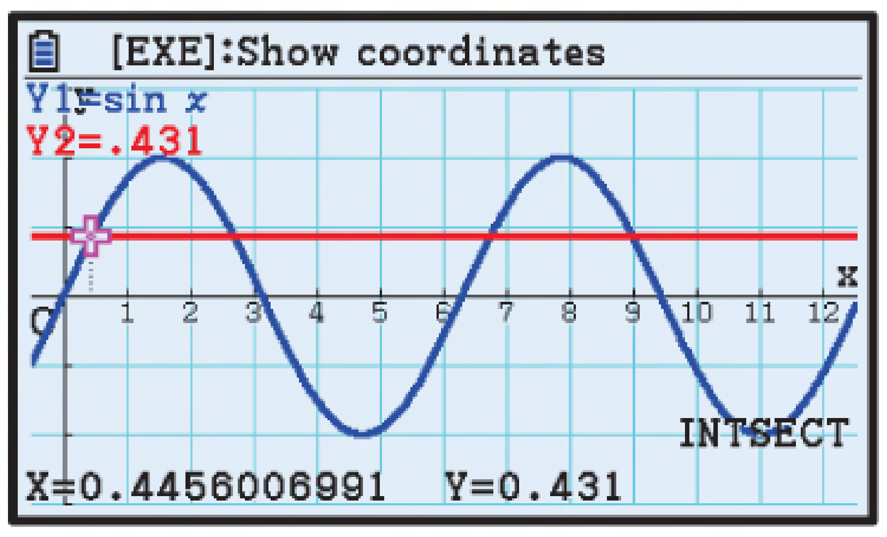


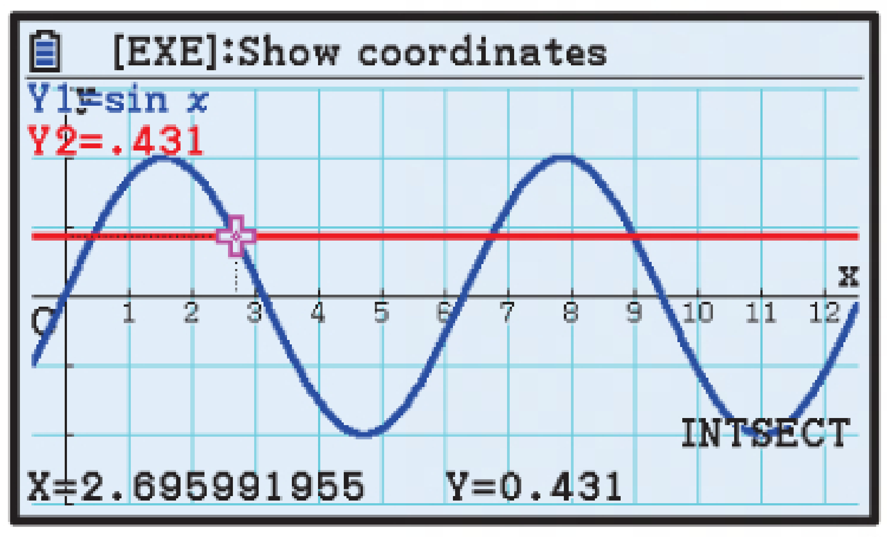
c When $\tan x = 3.5$, $0 \leqslant x \leqslant 2\pi$, $x \approx 1.3, 4.4$

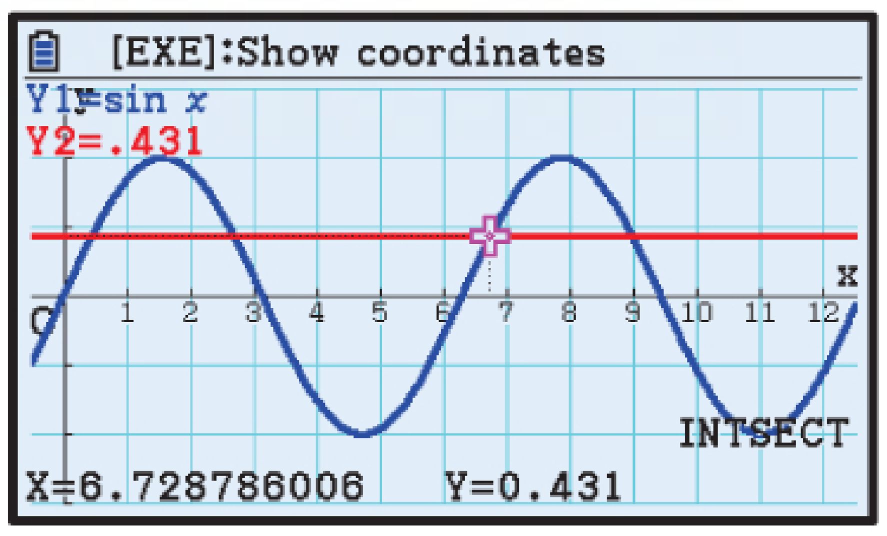
d When $\tan x = -2.4$, $\frac{\pi}{2} \leqslant x \leqslant \frac{3\pi}{2}$, $x \approx 2.0$

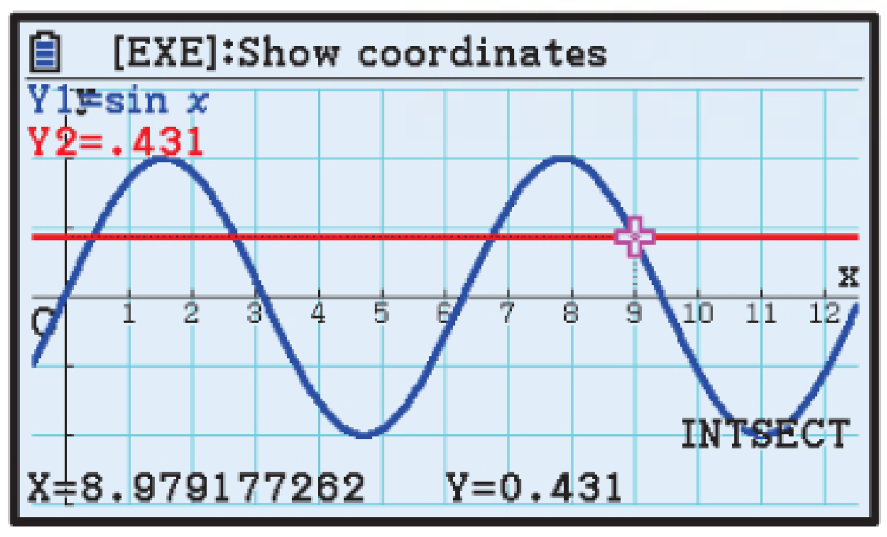
EXERCISE 17G.2

1 **a** We graph the functions $Y_1 = \sin X$ and $Y_2 = 0.431$ on the same set of axes.
We need to use **window** settings just larger than the domain.
In this case, $X\min = -0.5$, $X\max = 12.5$, $X\text{scale} = 1$.



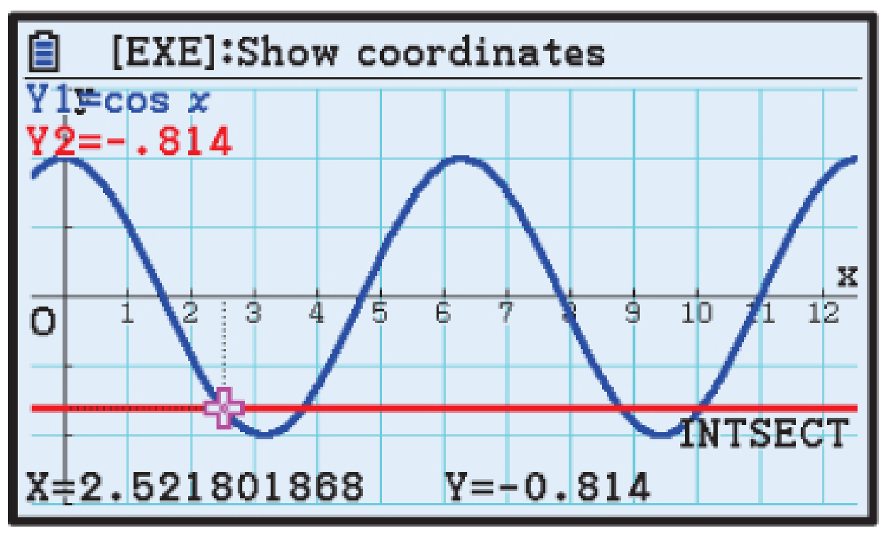


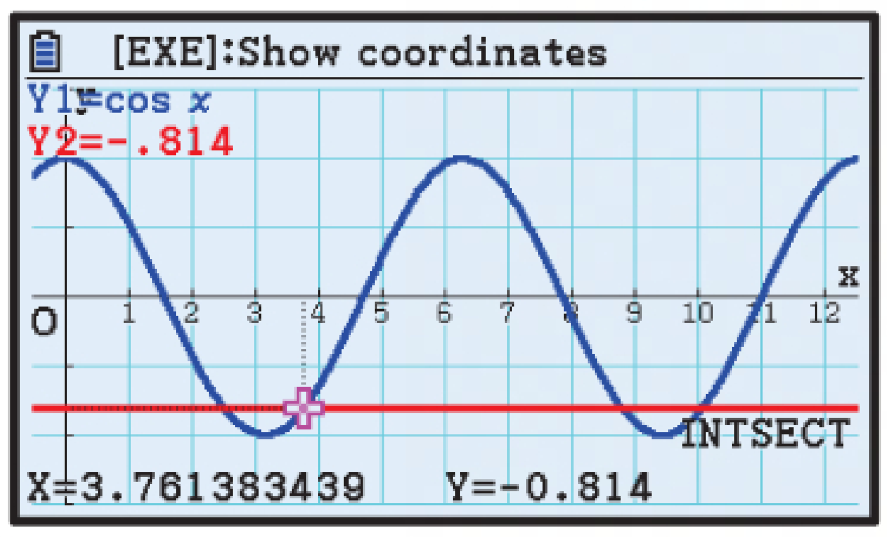


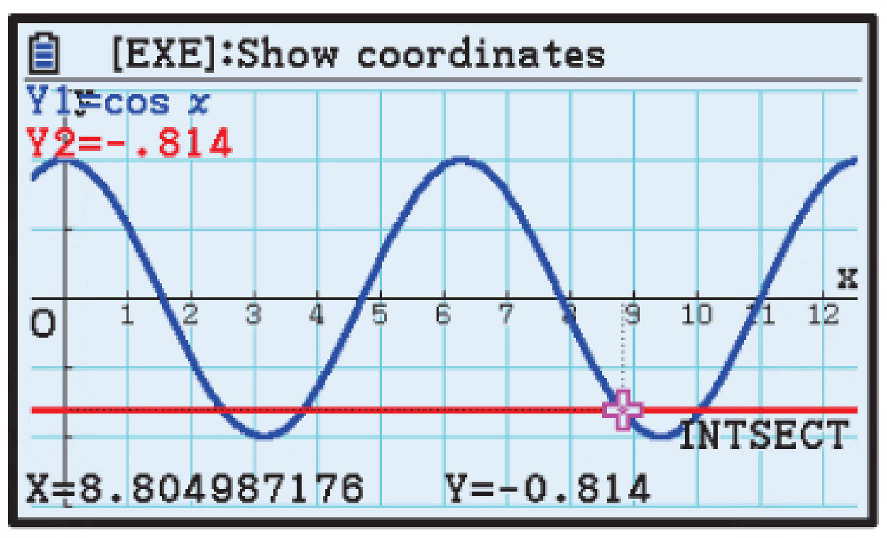


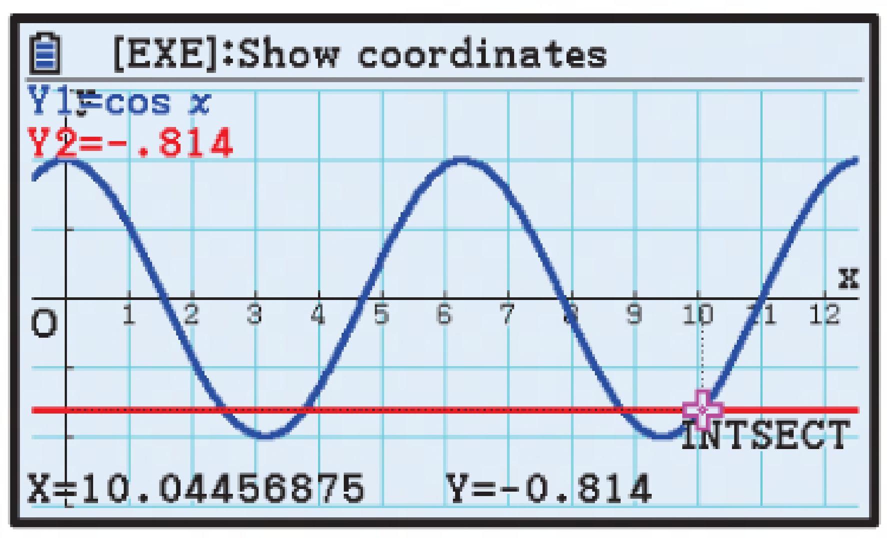
The solutions are $x \approx 0.446,\ 2.70,\ 6.73,\ 8.98$.

b We graph the functions $Y_1 = \cos X$ and $Y_2 = -0.814$ on the same set of axes.
We need to use **window** settings just larger than the domain.
In this case, $X\min = -0.5$, $X\max = 12.5$, $X\text{scale} = 1$.



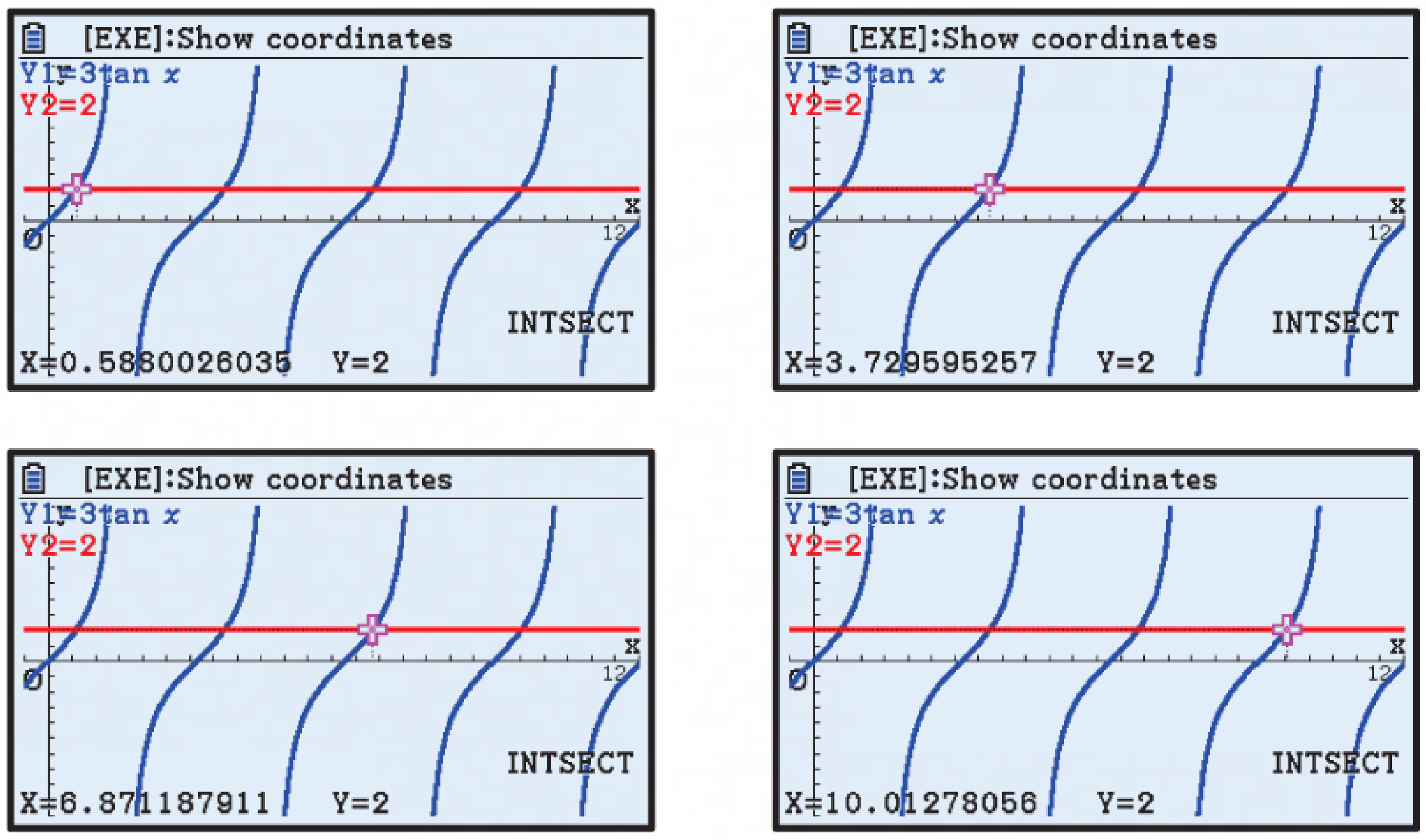






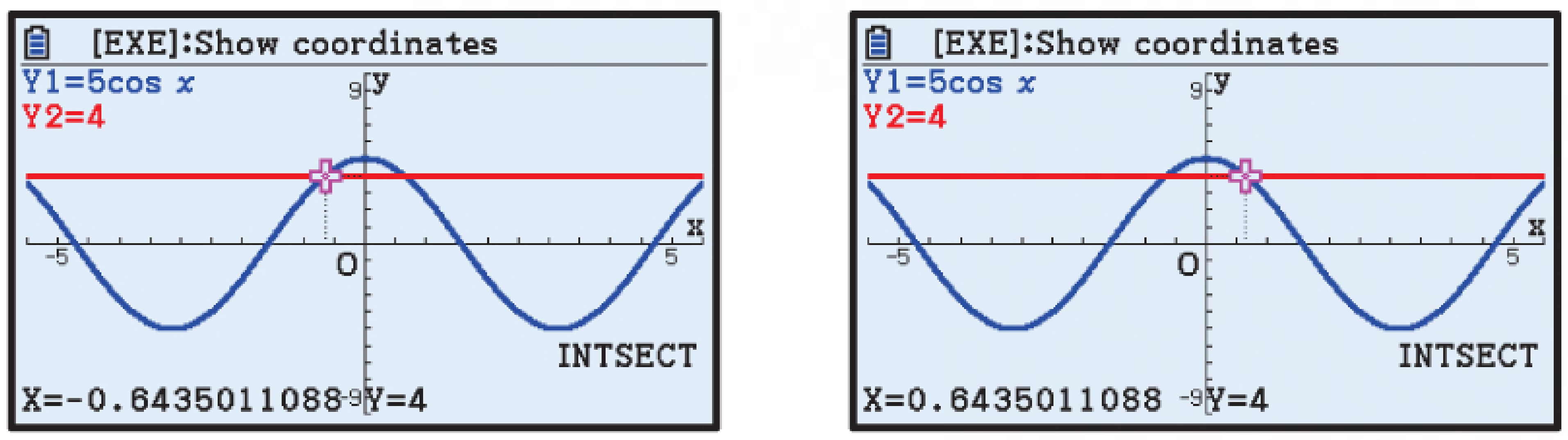
The solutions are $x \approx 2.52,\ 3.76,\ 8.80,\ 10.0$.

c We graph the functions $Y_1 = 3\tan X$ and $Y_2 = 2$ on the same set of axes.
We need to use **window** settings just larger than the domain.
In this case, $X\text{min} = -0.5$, $X\text{max} = 12.5$, $X\text{scale} = 0.5$.



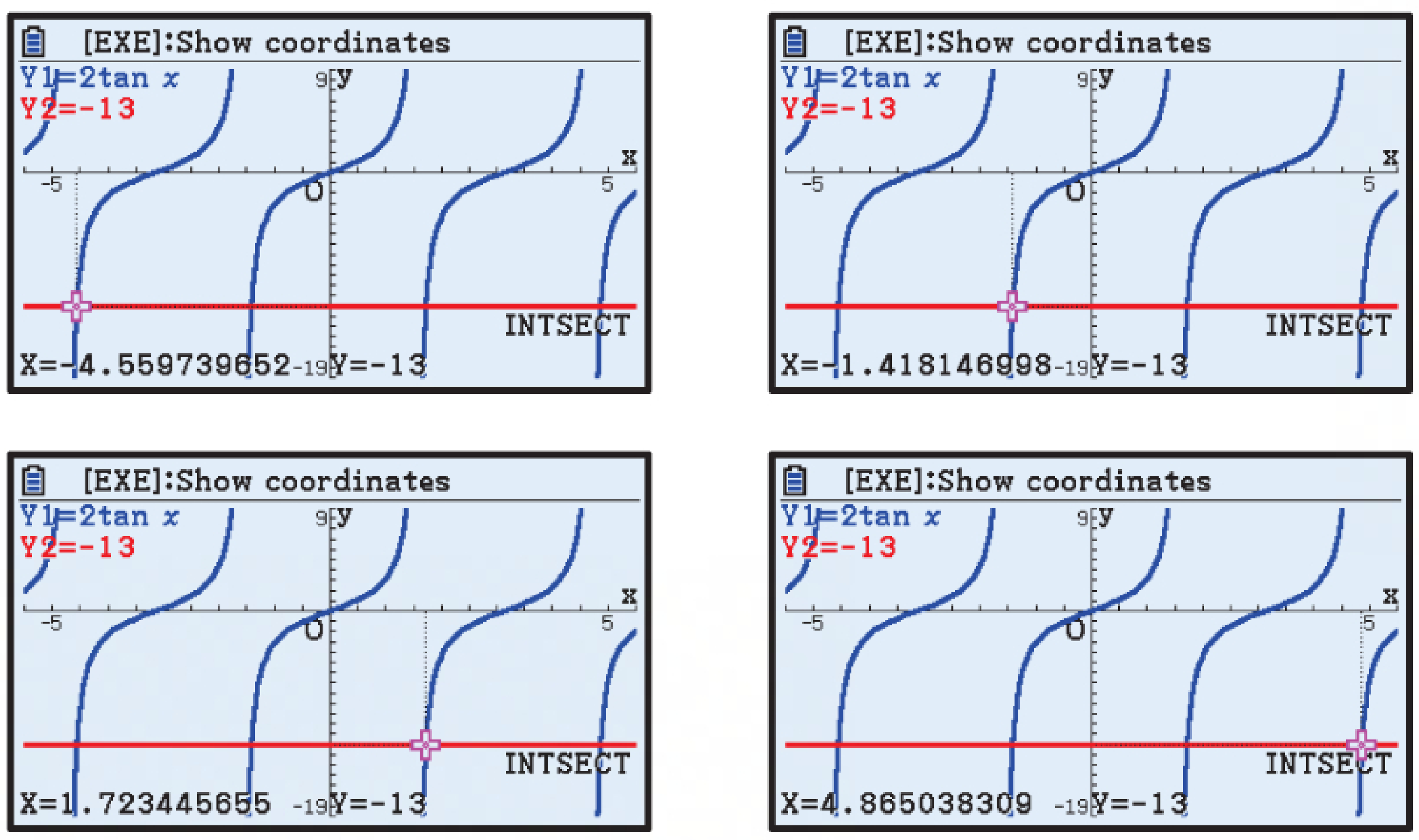
The solutions are $x \approx 0.588,\ 3.73,\ 6.87,\ 10.0$.

2 **a** We graph the functions $Y_1 = 5\cos X$ and $Y_2 = 4$ on the same set of axes.
We need to use **window** settings just larger than the domain.
In this case, $X\text{min} = -5.5$, $X\text{max} = 5.5$, $X\text{scale} = 0.5$.



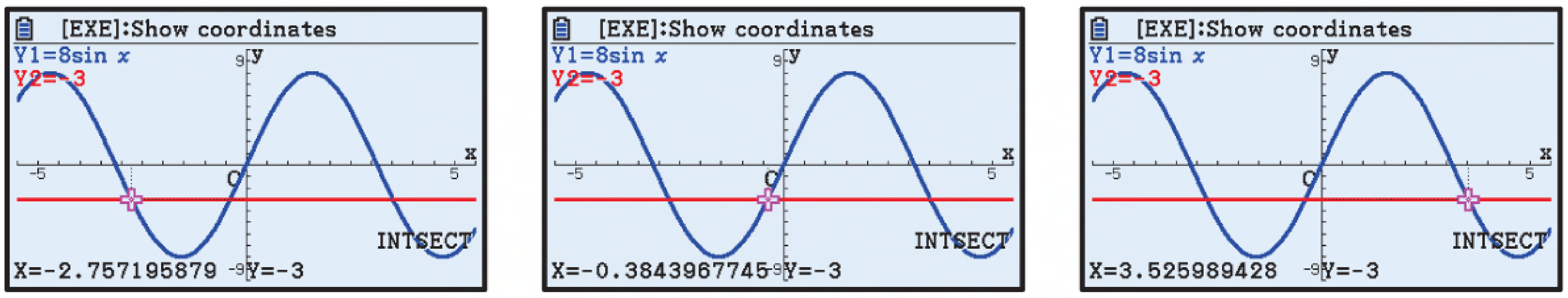
The solutions are $x \approx -0.644,\ 0.644$.

b We graph the functions $Y_1 = 2\tan X$ and $Y_2 = -13$ on the same set of axes.
We need to use **window** settings just larger than the domain.
In this case, $X\text{min} = -5.5$, $X\text{max} = 5.5$, $X\text{scale} = 0.5$.



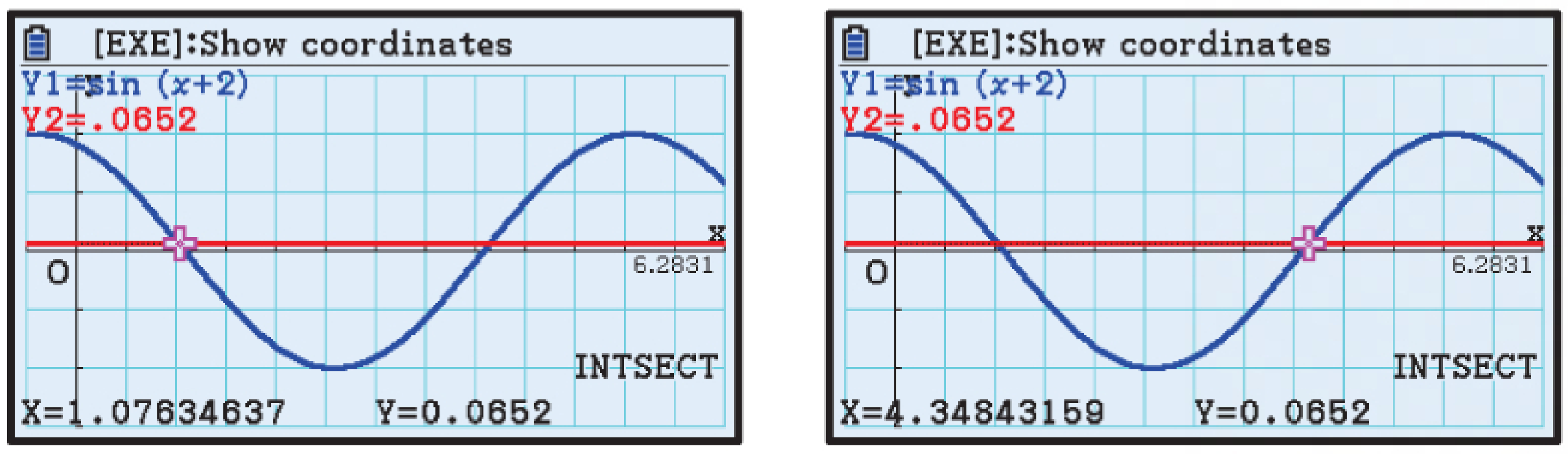
The solutions are $x \approx -4.56,\ -1.42,\ 1.72,\ 4.87$.

c We graph the functions $Y_1 = 8\sin X$ and $Y_2 = -3$ on the same set of axes.
We need to use **window** settings just larger than the domain.
In this case, $X\text{min} = -5.5$, $X\text{max} = 5.5$, $X\text{scale} = 0.5$.



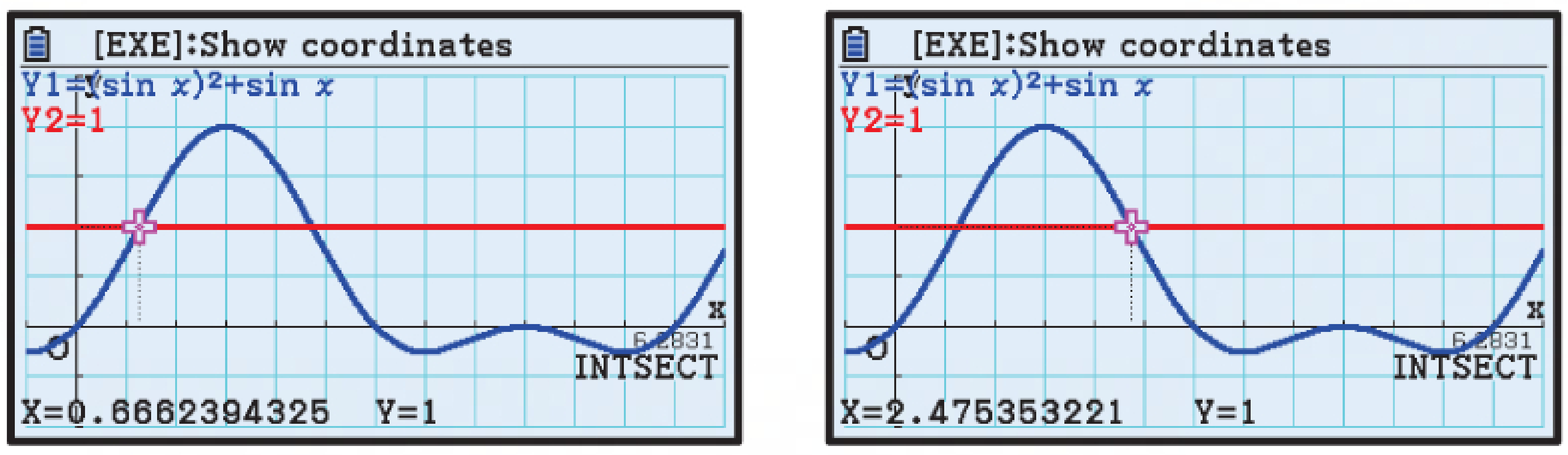
The solutions are $x \approx -2.76$, -0.384, 3.53.

3 **a** We graph the functions $Y_1 = \sin(X + 2)$ and $Y_2 = 0.0652$ on the same set of axes.
We need to use **window** settings just larger than the domain.
In this case, $X\text{min} = -\frac{\pi}{6}$, $X\text{max} = \frac{13\pi}{6}$, $X\text{scale} = \frac{\pi}{6}$.



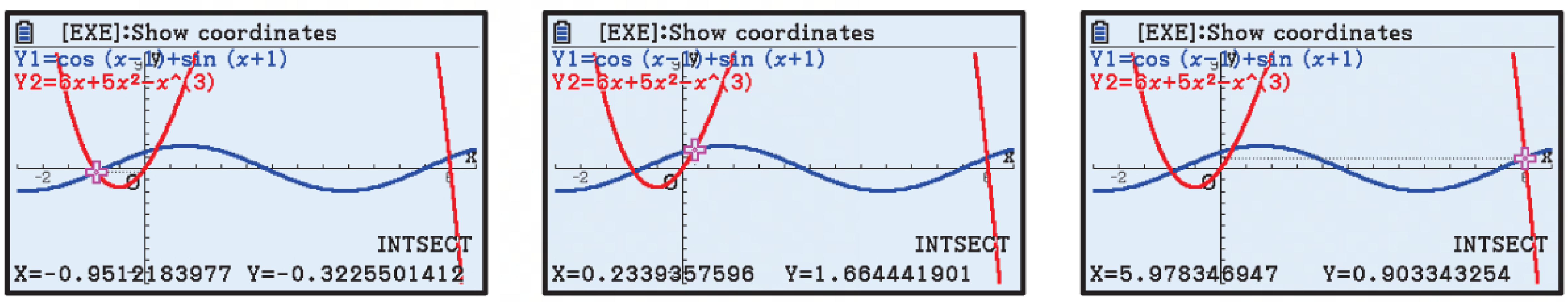
The solutions are $x \approx 1.08$, 4.35.

b We graph the functions $Y_1 = (\sin X)^\wedge 2 + \sin X$ and $Y_2 = 1$ on the same set of axes.
We need to use **window** settings just larger than the domain.
In this case, $X\text{min} = -\frac{\pi}{6}$, $X\text{max} = \frac{13\pi}{6}$, $X\text{scale} = \frac{\pi}{6}$.



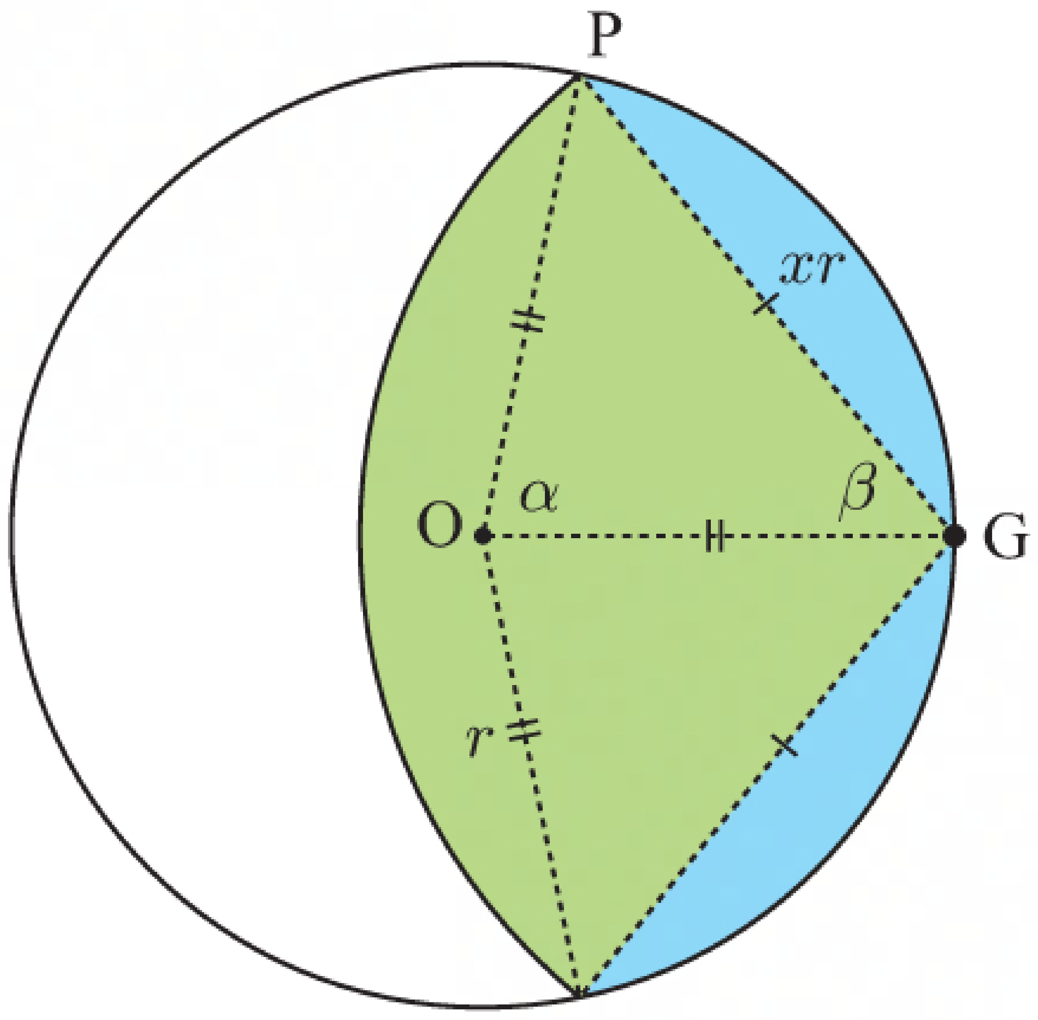
The solutions are $x \approx 0.666$, 2.48.

4 We graph the functions $Y_1 = \cos(X - 1) + \sin(X + 1)$ and $Y_2 = 6X + 5X^\wedge 2 - X^\wedge 3$ on the same set of axes.
We need to use **window** settings just larger than the domain.
In this case, $X\text{min} = -2.5$, $X\text{max} = 6.5$, $X\text{scale} = 0.5$.



The solutions are $x \approx -0.951$, 0.234, 5.98.

5 **a**



Suppose the centre of the field is O, and that the goat is tethered at G.

Let the field have radius r, so the rope has length xr.

Let α and β be the angles shown.

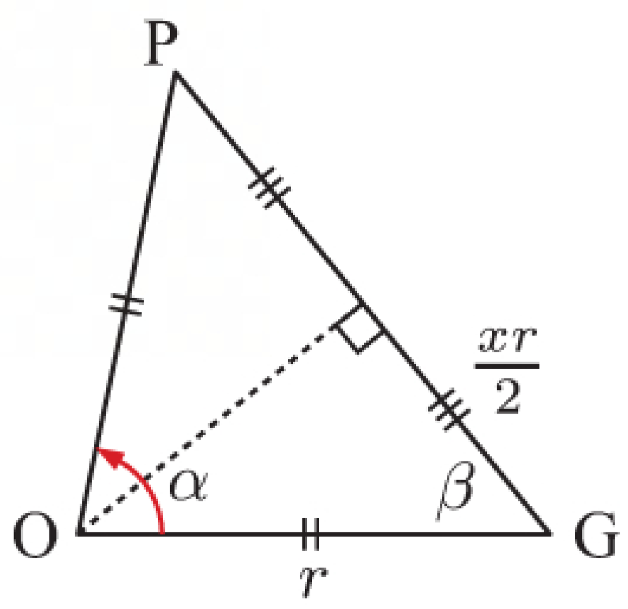
The area of the green sector $= \dfrac{2\beta}{2\pi} \times \pi(xr)^2$

$= \beta x^2 r^2$

The area of each blue segment $= \frac{1}{2}r^2(\alpha - \sin\alpha)$

$\therefore$ the total area the goat can graze $= r^2(\beta x^2 + \alpha - \sin\alpha)$

$\therefore$ the proportion of the field that the goat can graze $= \frac{1}{\pi}(\beta x^2 + \alpha - \sin\alpha)$



Now $\cos\beta = \dfrac{\frac{xr}{2}}{r} = \dfrac{x}{2}$

$\therefore \beta = \arccos\frac{x}{2}$

$\therefore \alpha = \pi - 2\beta$

$= \pi - 2\arccos\frac{x}{2}$

Using two expressions for the area of triangle POG, $\frac{1}{2}r^2\sin\alpha = \frac{1}{2}(xr)\sqrt{r^2 - \left(\frac{xr}{2}\right)^2}$

$= \frac{1}{2}xr^2\sqrt{1 - \left(\frac{x}{2}\right)^2}$

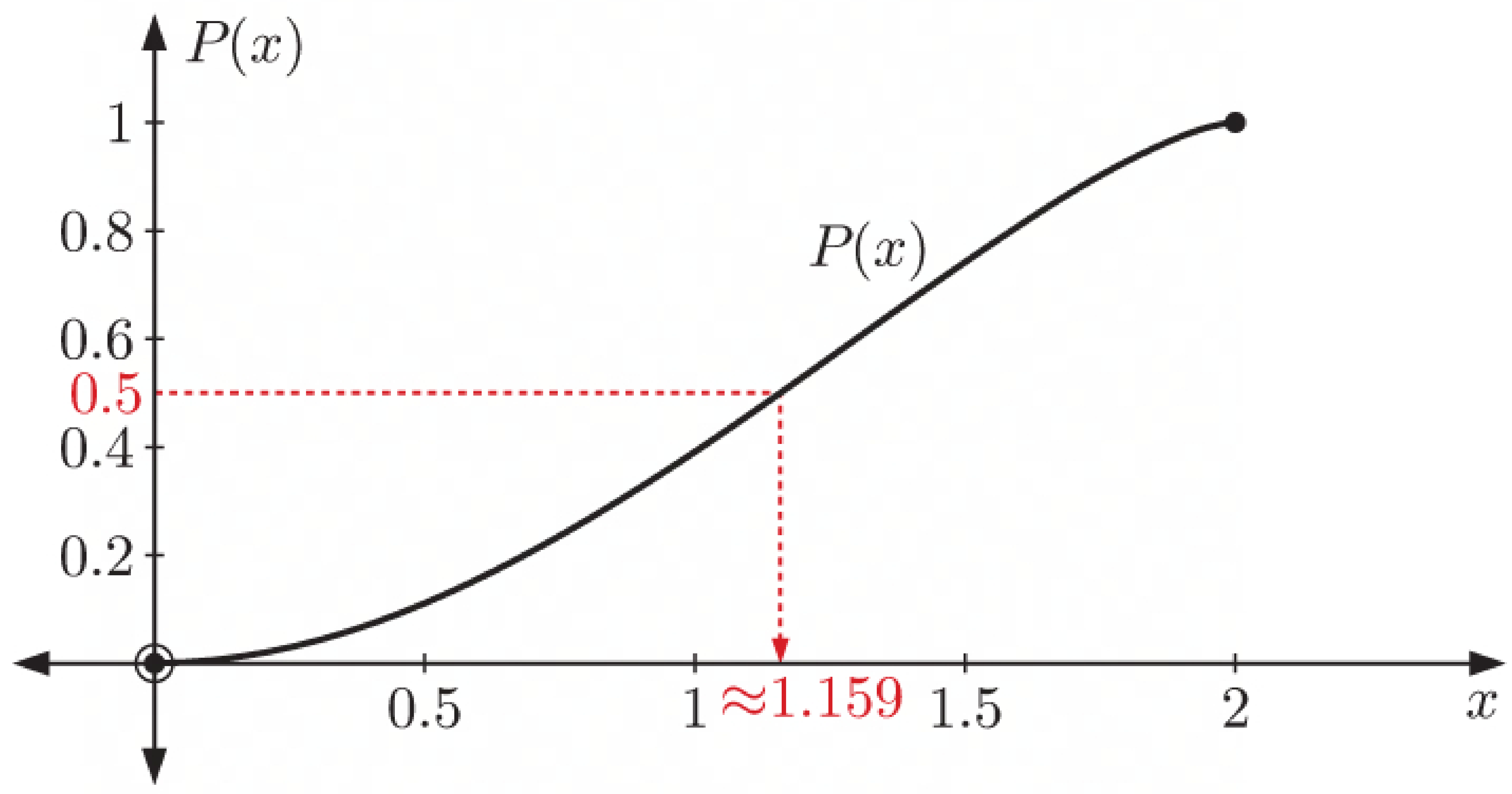
$\therefore \sin\alpha = x\sqrt{1 - \left(\frac{x}{2}\right)^2}$

$$\therefore \; P(x) = \frac{1}{\pi}\left(x^2\arccos\tfrac{x}{2} + \pi - 2\arccos\tfrac{x}{2} - x\sqrt{1 - \left(\tfrac{x}{2}\right)^2}\right)$$

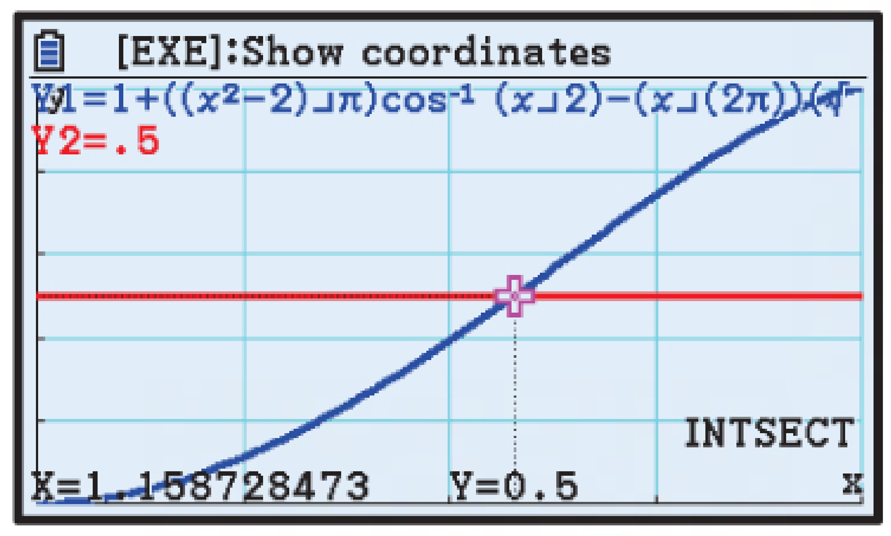
$$= 1 + \frac{x^2 - 2}{\pi}\arccos\tfrac{x}{2} - \frac{x}{2\pi}\sqrt{4 - x^2}$$

Note: The function $P(x)$ can be written in many different ways.

b

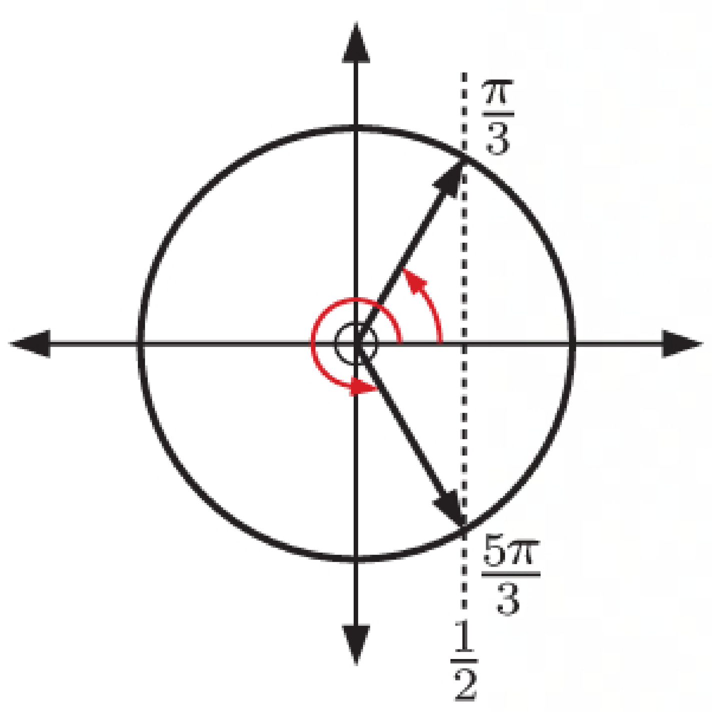


c Using technology, $P(x) = 0.5$ when $x \approx 1.1587$.
The goat can graze exactly half the field when its rope is approximately 1.159 times the field's radius.



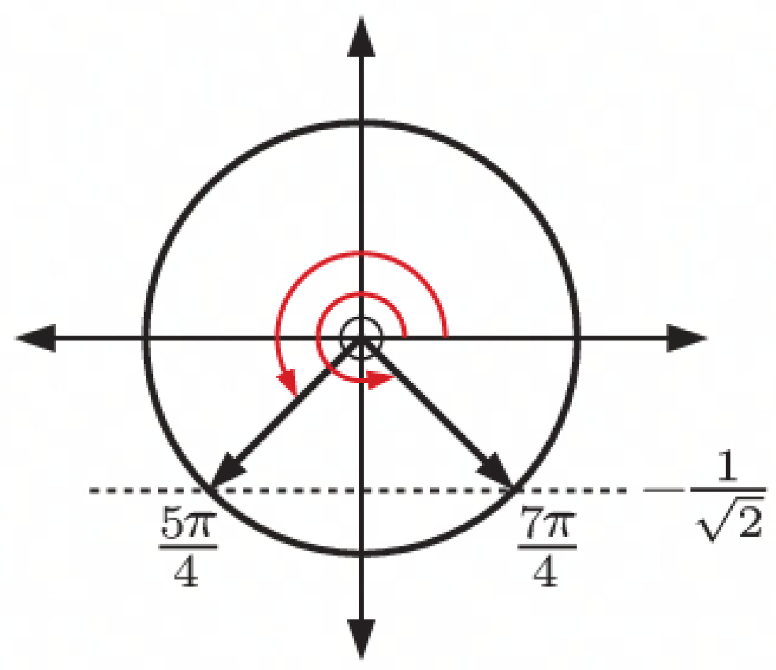
EXERCISE 17G.3

1 **a** On $0 \leqslant x \leqslant 2\pi$, the angles with cosine $\frac{1}{2}$ are $\frac{\pi}{3}$ and $\frac{5\pi}{3}$.



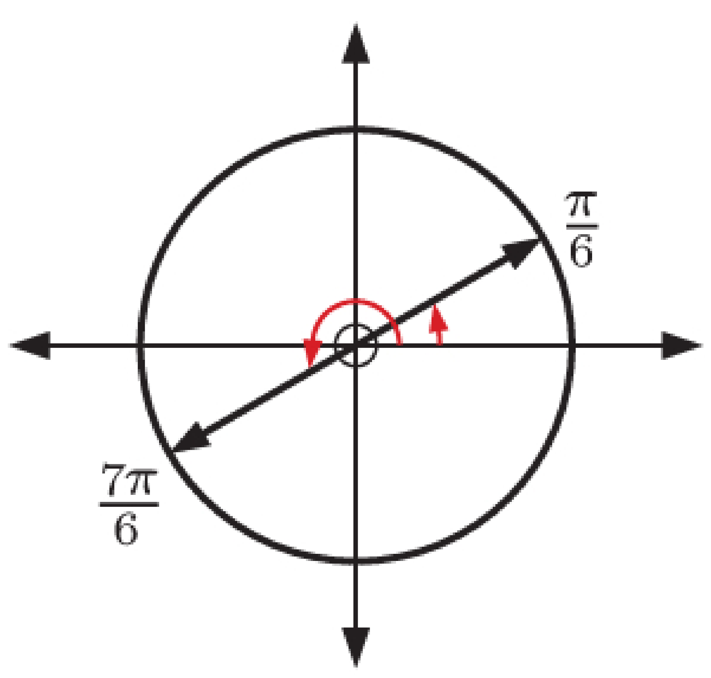
$\therefore$ the solutions are $x = \frac{\pi}{3}$ or $\frac{5\pi}{3}$.

b On $0 \leqslant x \leqslant 2\pi$, the angles with sine $-\frac{1}{\sqrt{2}}$ are $\frac{5\pi}{4}$ and $\frac{7\pi}{4}$.



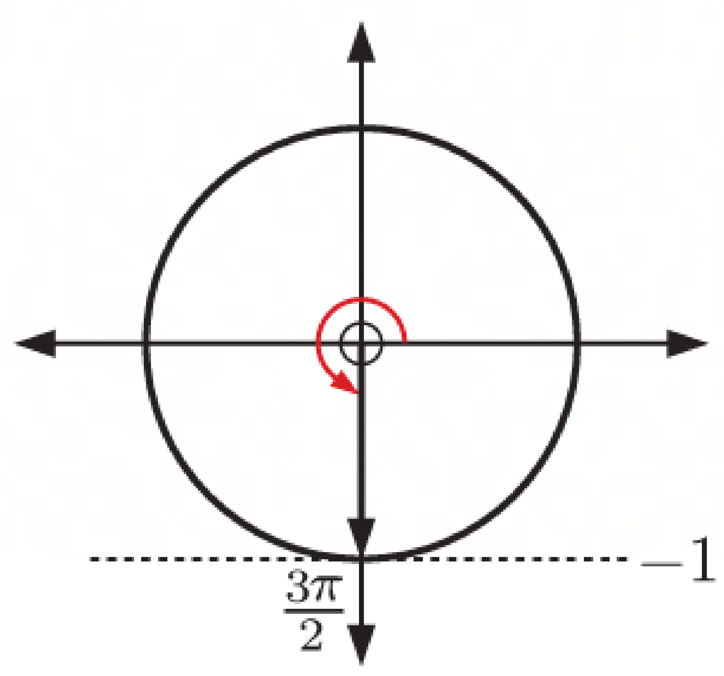
$\therefore$ the solutions are $x = \frac{5\pi}{4}$ or $\frac{7\pi}{4}$.

c On $0 \leqslant x \leqslant 2\pi$, the angles with tangent $\frac{1}{\sqrt{3}}$ are $\frac{\pi}{6}$ and $\frac{7\pi}{6}$.



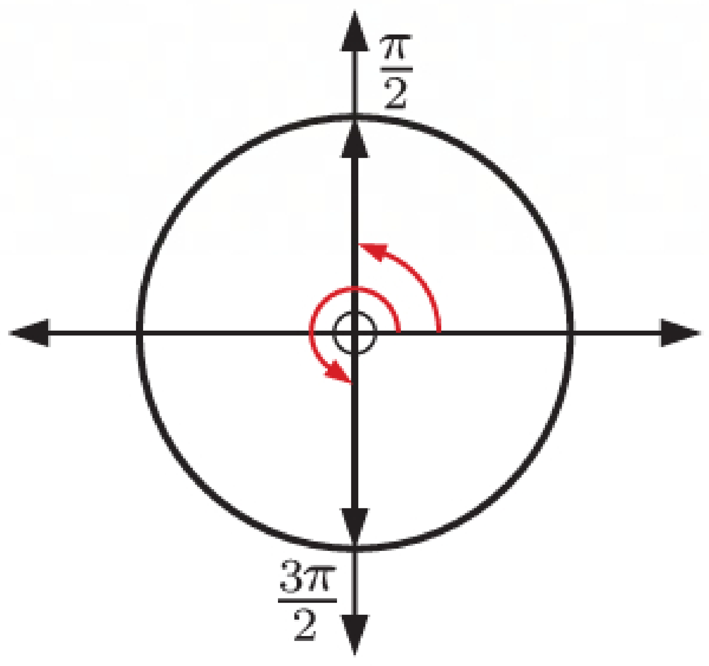
$\therefore$ the solutions are $x = \frac{\pi}{6}$ or $\frac{7\pi}{6}$.

d On $0 \leqslant x \leqslant 2\pi$, the angle with sine -1 is $\frac{3\pi}{2}$.



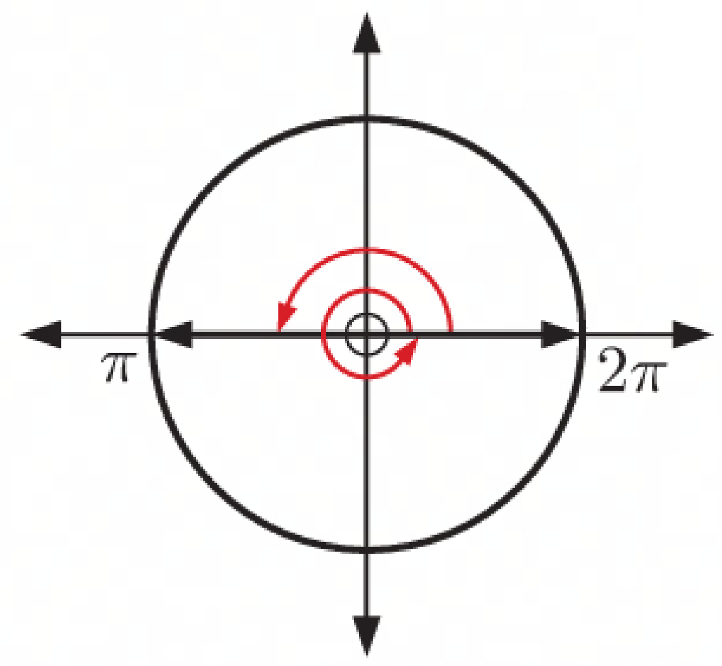
$\therefore$ the solution is $x = \frac{3\pi}{2}$.

e On $0 \leqslant x \leqslant 2\pi$, the angles with cosine 0 are $\frac{\pi}{2}$ and $\frac{3\pi}{2}$.



$\therefore$ the solutions are $x = \frac{\pi}{2}$ or $\frac{3\pi}{2}$.

f On $0 \leqslant x \leqslant 2\pi$, the angles with tangent 0 are 0, π, and 2π.



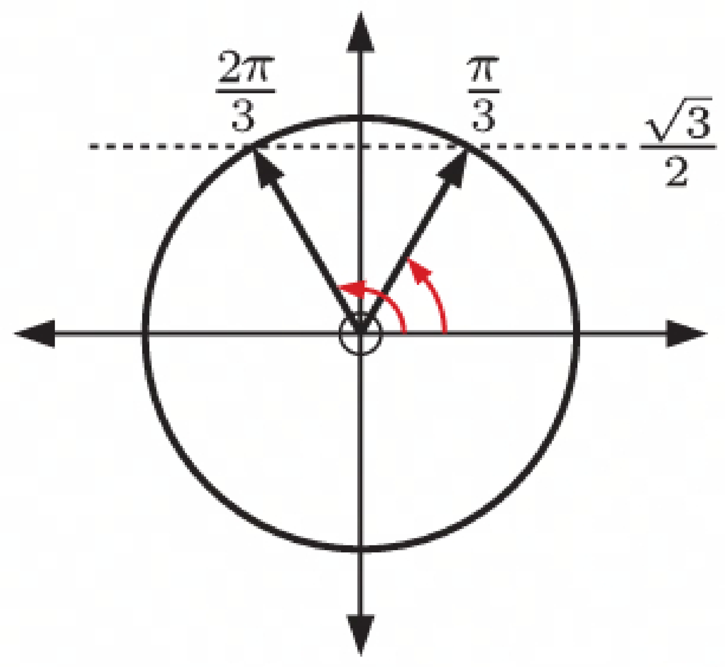
$\therefore$ the solutions are $x = 0$, π, or 2π.

2 **a** $2\sin x = \sqrt{3}$

$\therefore$ $\sin x = \frac{\sqrt{3}}{2}$

On $0 \leqslant x \leqslant 2\pi$, the angles with sine $\frac{\sqrt{3}}{2}$ are $\frac{\pi}{3}$ and $\frac{2\pi}{3}$.

$\therefore$ the solutions are $x = \frac{\pi}{3}$ or $\frac{2\pi}{3}$.

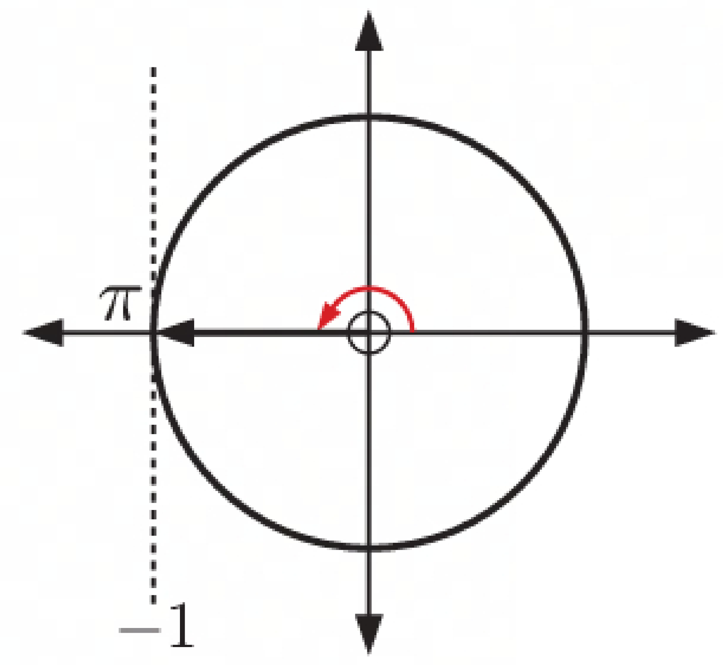


b $3\cos x + 3 = 0$

$\therefore$ $\cos x = -\frac{3}{3} = -1$

On $0 \leqslant x \leqslant 2\pi$, the angle with cosine -1 is π.

$\therefore$ the solution is $x = \pi$.

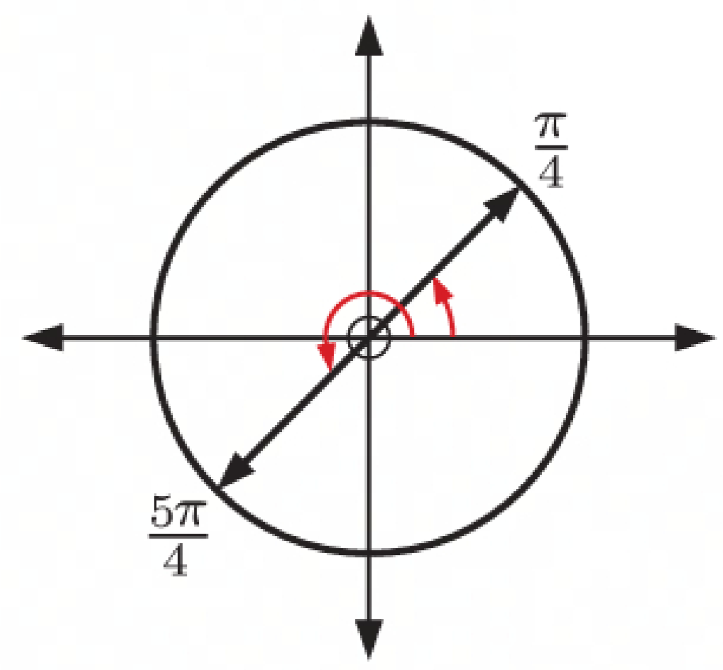


c $2\tan x - 2 = 0$

$\therefore$ $\tan x = \frac{2}{2} = 1$

On $0 \leqslant x \leqslant 2\pi$, the angles with tangent 1 are $\frac{\pi}{4}$ and $\frac{5\pi}{4}$.

$\therefore$ the solutions are $x = \frac{\pi}{4}$ or $\frac{5\pi}{4}$.

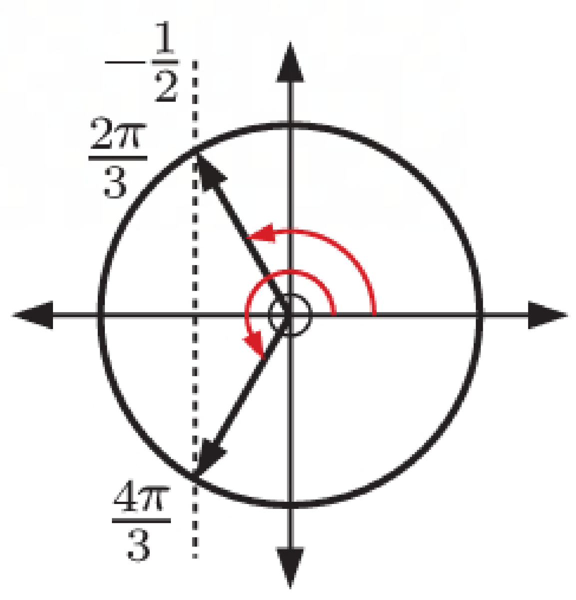


3 **a** $2\cos x + 1 = 0$

$\therefore$ $\cos x = -\frac{1}{2}$

There are two points on the unit circle with cosine $-\frac{1}{2}$.

They correspond to angles $\frac{2\pi}{3}$ and $\frac{4\pi}{3}$.



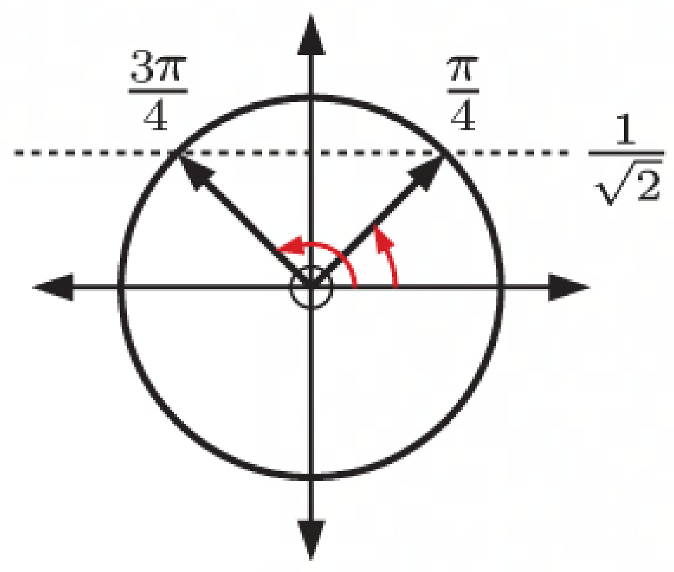
For the domain $0 \leqslant x \leqslant 4\pi$ we have 4 solutions: $x = \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{8\pi}{3}$, or $\frac{10\pi}{3}$.

b $\sqrt{2}\sin x = 1$

$\therefore$ $\sin x = \frac{1}{\sqrt{2}}$

There are two points on the unit circle with sine $\frac{1}{\sqrt{2}}$.

They correspond to angles $\frac{\pi}{4}$ and $\frac{3\pi}{4}$.



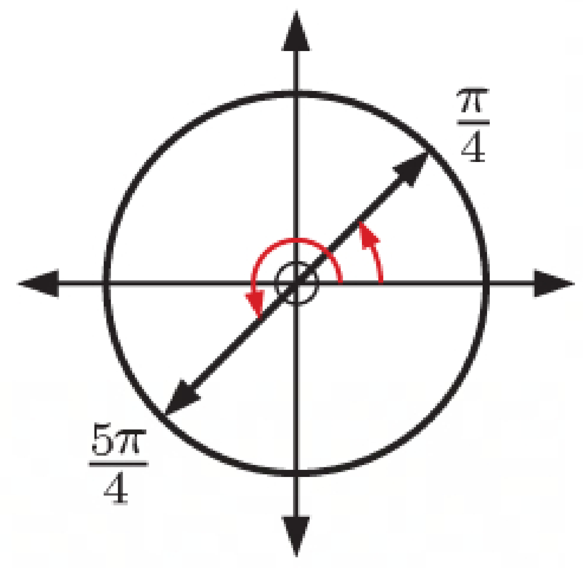
For the domain $0 \leqslant x \leqslant 4\pi$ we have 4 solutions: $x = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{9\pi}{4}$, or $\frac{11\pi}{4}$.

c $\tan x = 1$

There are two points on the unit circle with tangent 1.

They correspond to angles $\frac{\pi}{4}$ and $\frac{5\pi}{4}$.

For the domain $0 \leqslant x \leqslant 4\pi$ we have 4 solutions: $x = \frac{\pi}{4}, \frac{5\pi}{4}, \frac{9\pi}{4}$, or $\frac{13\pi}{4}$.

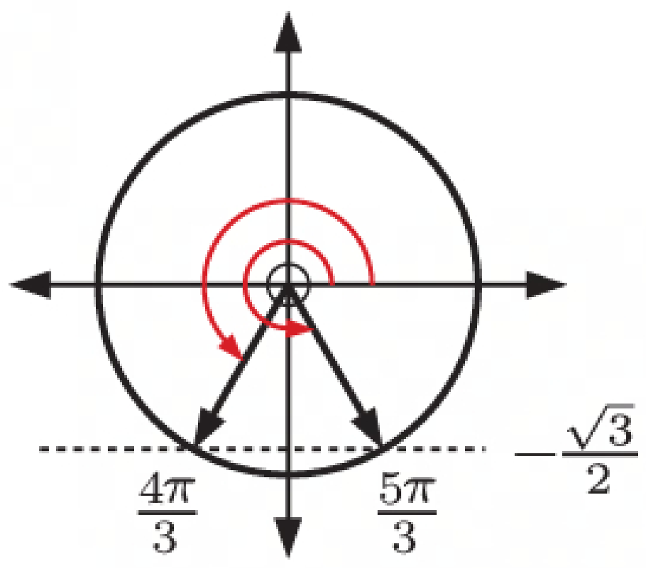


4 **a** $2\sin x + \sqrt{3} = 0$

$\therefore \sin x = -\frac{\sqrt{3}}{2}$

There are two points on the unit circle with sine $-\frac{\sqrt{3}}{2}$.

They correspond to angles $\frac{4\pi}{3}$ and $\frac{5\pi}{3}$.



For the domain $-2\pi \leqslant x \leqslant 2\pi$ we have 4 solutions:

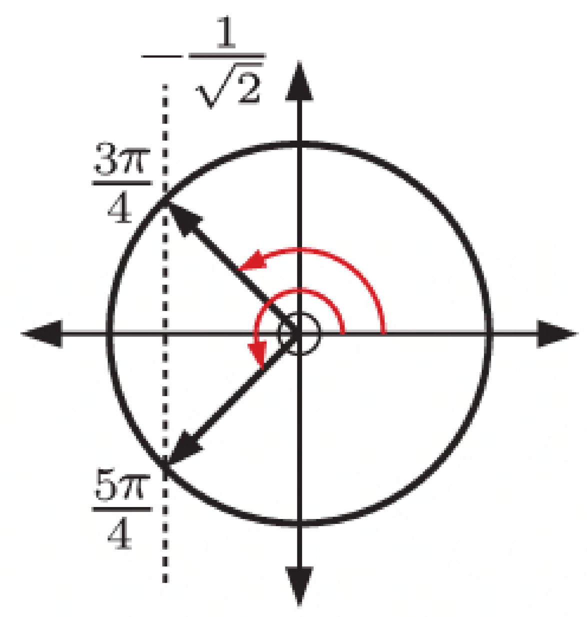
$x = -\frac{2\pi}{3}, -\frac{\pi}{3}, \frac{4\pi}{3}$, or $\frac{5\pi}{3}$.

b $\sqrt{2}\cos x + 1 = 0$

$\therefore \cos x = -\frac{1}{\sqrt{2}}$

There are two points on the unit circle with cosine $-\frac{1}{\sqrt{2}}$.

They correspond to angles $\frac{3\pi}{4}$ and $\frac{5\pi}{4}$.



For the domain $-2\pi \leqslant x \leqslant 2\pi$ we have 4 solutions:

$x = -\frac{5\pi}{4}, -\frac{3\pi}{4}, \frac{3\pi}{4}$, or $\frac{5\pi}{4}$.

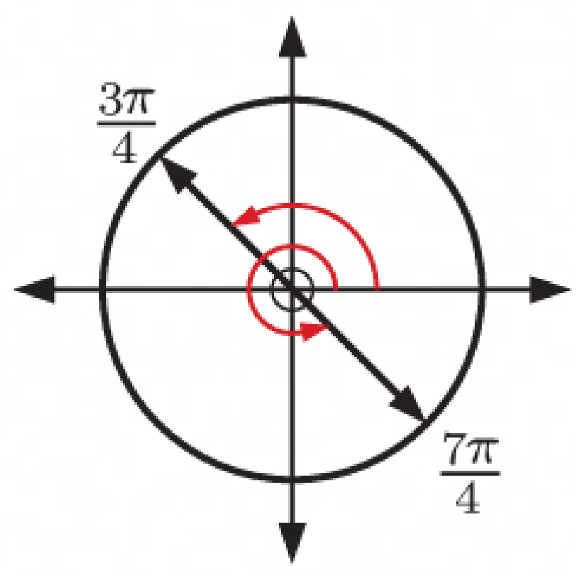
c $\tan x = -1$

There are two points on the unit circle with tangent -1.

They correspond to angles $\frac{3\pi}{4}$ and $\frac{7\pi}{4}$.

For the domain $-2\pi \leqslant x \leqslant 2\pi$ we have 4 solutions:

$x = -\frac{5\pi}{4}, -\frac{\pi}{4}, \frac{3\pi}{4}$, or $\frac{7\pi}{4}$.



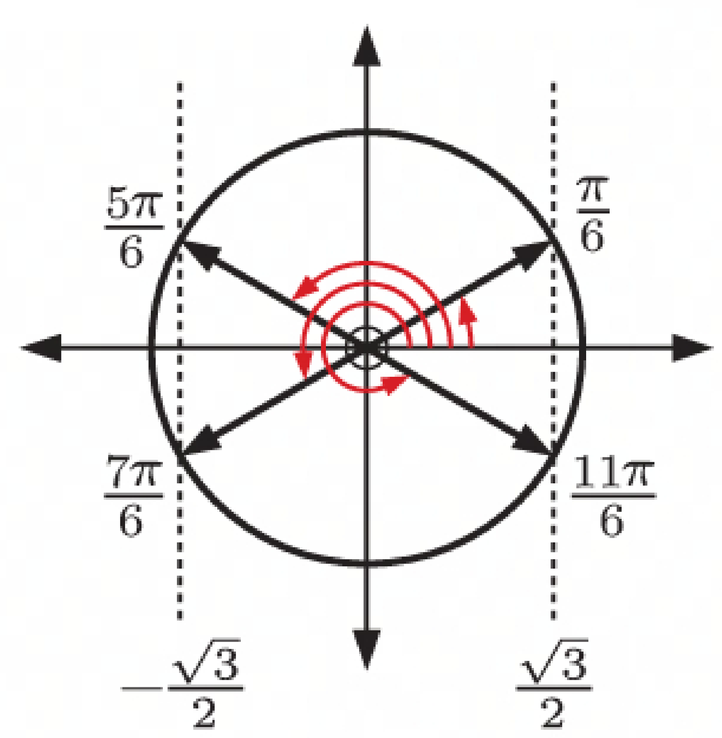
5 **a** $\cos^2 x = \frac{3}{4}$

$\therefore \cos x = \pm\frac{\sqrt{3}}{2}$

On $0 \leqslant x \leqslant 2\pi$, the angles with cosine $\frac{\sqrt{3}}{2}$ are $\frac{\pi}{6}$ and $\frac{11\pi}{6}$,

and the angles with cosine $-\frac{\sqrt{3}}{2}$ are $\frac{5\pi}{6}$ and $\frac{7\pi}{6}$.

$\therefore$ the solutions are $x = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{7\pi}{6}$, or $\frac{11\pi}{6}$.

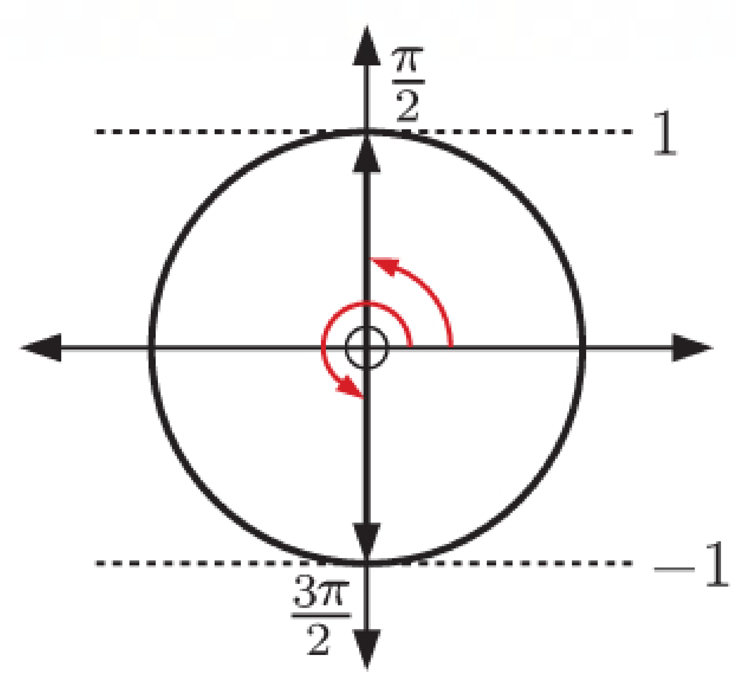


b $\sin^2 x = 1$

$\therefore \sin x = \pm 1$

On $0 \leqslant x \leqslant 2\pi$, the angle with sine 1 is $\frac{\pi}{2}$ and the angle with sine -1 is $\frac{3\pi}{2}$.

$\therefore$ the solutions are $x = \frac{\pi}{2}$ or $\frac{3\pi}{2}$.

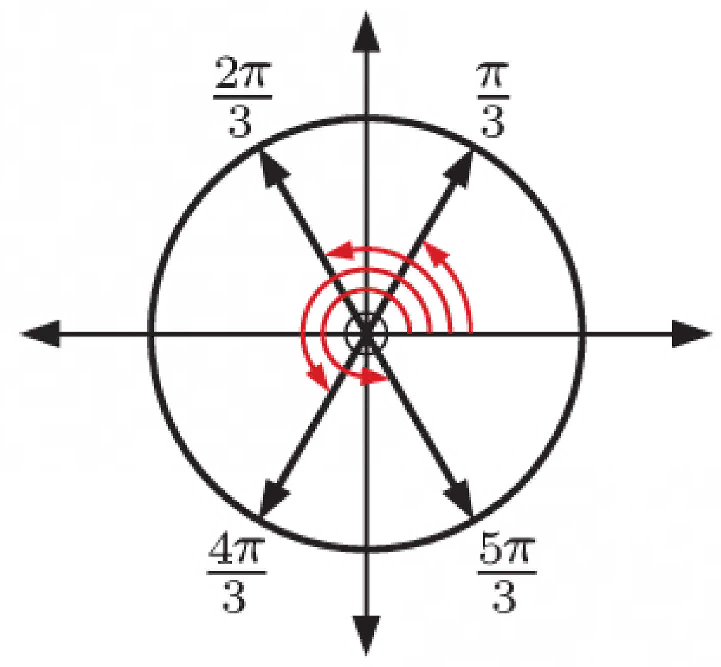


c $\tan^2 x = 3$

$\therefore \tan x = \pm\sqrt{3}$

On $0 \leqslant x \leqslant 2\pi$, the angles with tangent $\sqrt{3}$ are $\frac{\pi}{3}$ and $\frac{4\pi}{3}$, and the angles with tangent $-\sqrt{3}$ are $\frac{2\pi}{3}$ and $\frac{5\pi}{3}$.

$\therefore$ the solutions are $x = \frac{\pi}{3}, \frac{2\pi}{3}, \frac{4\pi}{3}$, or $\frac{5\pi}{3}$.



6 **a** If $0 \leqslant x \leqslant 2\pi$

then $0 \leqslant 2x \leqslant 4\pi$

$\therefore$ the domain is $0 \leqslant 2x \leqslant 4\pi$.

b If $0 \leqslant x \leqslant 2\pi$

then $0 \leqslant \frac{x}{4} \leqslant \frac{2\pi}{4}$

$\therefore 0 \leqslant \frac{x}{4} \leqslant \frac{\pi}{2}$

$\therefore$ the domain is $0 \leqslant \frac{x}{4} \leqslant \frac{\pi}{2}$.

c If $0 \leqslant x \leqslant 2\pi$

then $\frac{\pi}{2} \leqslant x + \frac{\pi}{2} \leqslant \frac{5\pi}{2}$

$\therefore$ the domain is $\frac{\pi}{2} \leqslant x + \frac{\pi}{2} \leqslant \frac{5\pi}{2}$.

d If $0 \leqslant x \leqslant 2\pi$

then $-\frac{\pi}{6} \leqslant x - \frac{\pi}{6} \leqslant \frac{11\pi}{6}$

$\therefore$ the domain is $-\frac{\pi}{6} \leqslant x - \frac{\pi}{6} \leqslant \frac{11\pi}{6}$.

e If $0 \leqslant x \leqslant 2\pi$

then $-\frac{\pi}{4} \leqslant x - \frac{\pi}{4} \leqslant \frac{7\pi}{4}$

and so $-\frac{\pi}{2} \leqslant 2(x - \frac{\pi}{4}) \leqslant \frac{7\pi}{2}$

$\therefore$ the domain is $-\frac{\pi}{2} \leqslant 2(x - \frac{\pi}{4}) \leqslant \frac{7\pi}{2}$.

f If $0 \leqslant x \leqslant 2\pi$

then $-2\pi \leqslant -x \leqslant 0$

$\therefore$ the domain is $-2\pi \leqslant -x \leqslant 0$.

7 **a** If $-\pi \leqslant x \leqslant \pi$

then $-3\pi \leqslant 3x \leqslant 3\pi$

$\therefore$ the domain is $-3\pi \leqslant 3x \leqslant 3\pi$.

b If $-\pi \leqslant x \leqslant \pi$

then $-\frac{\pi}{4} \leqslant \frac{x}{4} \leqslant \frac{\pi}{4}$

$\therefore$ the domain is $-\frac{\pi}{4} \leqslant \frac{x}{4} \leqslant \frac{\pi}{4}$.

c If $-\pi \leqslant x \leqslant \pi$

then $-\frac{3\pi}{2} \leqslant x - \frac{\pi}{2} \leqslant \frac{\pi}{2}$

$\therefore$ the domain is $-\frac{3\pi}{2} \leqslant x - \frac{\pi}{2} \leqslant \frac{\pi}{2}$.

d If $-\pi \leqslant x \leqslant \pi$

then $-2\pi \leqslant 2x \leqslant 2\pi$

and so $-\frac{3\pi}{2} \leqslant 2x + \frac{\pi}{2} \leqslant \frac{5\pi}{2}$

$\therefore$ the domain is $-\frac{3\pi}{2} \leqslant 2x + \frac{\pi}{2} \leqslant \frac{5\pi}{2}$.

e If $-\pi \leqslant x \leqslant \pi$

then $2\pi \geqslant -2x \geqslant -2\pi$

and so $-2\pi \leqslant -2x \leqslant 2\pi$

$\therefore$ the domain is $-2\pi \leqslant -2x \leqslant 2\pi$.

f If $-\pi \leqslant x \leqslant \pi$

then $\pi \geqslant -x \geqslant -\pi$

and so $\pi - \pi \leqslant \pi - x \leqslant \pi + \pi$

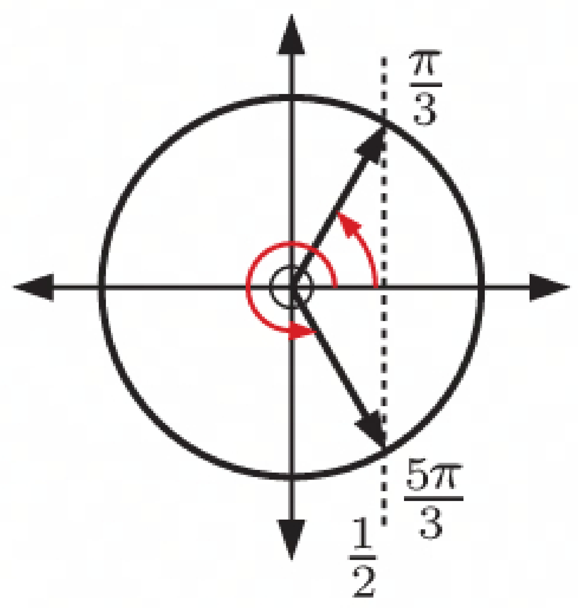
$\therefore 0 \leqslant \pi - x \leqslant 2\pi$

$\therefore$ the domain is $0 \leqslant \pi - x \leqslant 2\pi$.

8 The three equations all have the form $\cos\theta = \frac{1}{2}$.

There are two points on the unit circle with cosine $\frac{1}{2}$.

They correspond to angles $\frac{\pi}{3}$ and $\frac{5\pi}{3}$.



a In this case θ is simply x, so we have the domain $0 \leqslant x \leqslant 3\pi$.

The solutions for this domain are $x = \frac{\pi}{3}, \frac{5\pi}{3}$, or $\frac{7\pi}{3}$.

b In this case θ is $2x$.

If $0 \leqslant x \leqslant 3\pi$ then $0 \leqslant 2x \leqslant 6\pi$.

$\therefore\ 2x = \frac{\pi}{3}, \frac{5\pi}{3}, \frac{7\pi}{3}, \frac{11\pi}{3}, \frac{13\pi}{3}$, or $\frac{17\pi}{3}$

$\therefore\ x = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{11\pi}{6}, \frac{13\pi}{6}$, or $\frac{17\pi}{6}$

c In this case θ is $x + \frac{\pi}{3}$.

If $0 \leqslant x \leqslant 3\pi$ then $\frac{\pi}{3} \leqslant x + \frac{\pi}{3} \leqslant \frac{10\pi}{3}$.

$\therefore\ x + \frac{\pi}{3} = \frac{\pi}{3}, \frac{5\pi}{3}$, or $\frac{7\pi}{3}$

$\therefore\ x = 0, \frac{4\pi}{3}$, or 2π

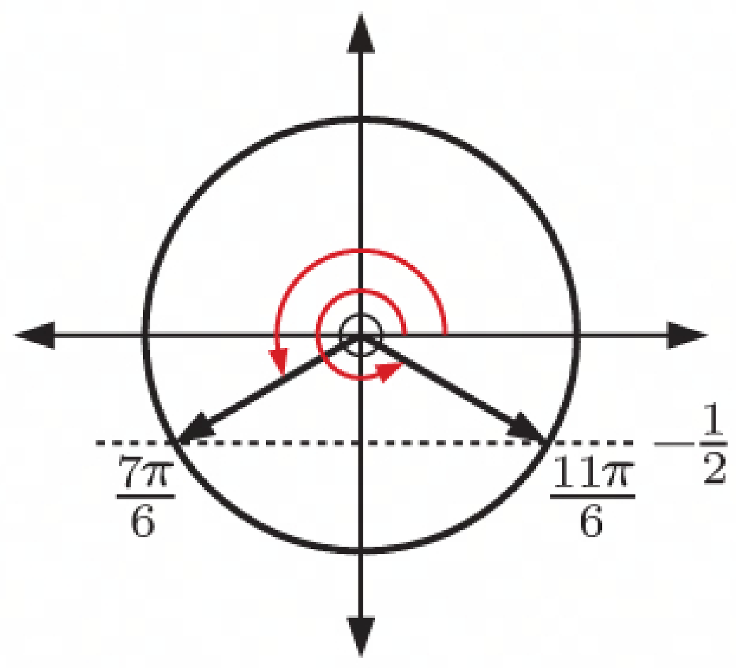
9 **a** If $0 \leqslant x \leqslant 2\pi$, then $0 \leqslant 2x \leqslant 4\pi$.

$\therefore$ the angles between 0 and 4π with sine $-\frac{1}{2}$ are

$\frac{7\pi}{6}, \frac{11\pi}{6}, \frac{19\pi}{6}$, and $\frac{23\pi}{6}$

$\therefore\ 2x = \frac{7\pi}{6}, \frac{11\pi}{6}, \frac{19\pi}{6}$, or $\frac{23\pi}{6}$

$\therefore$ the solutions are $x = \frac{7\pi}{12}, \frac{11\pi}{12}, \frac{19\pi}{12}$, or $\frac{23\pi}{12}$.



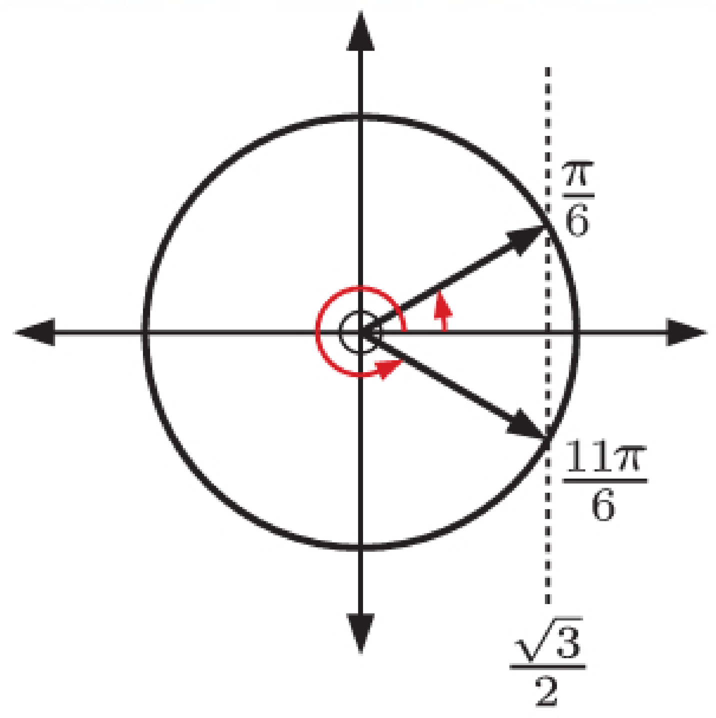
b If $0 \leqslant x \leqslant 2\pi$, then $0 \leqslant 3x \leqslant 6\pi$.

$\therefore$ the angles between 0 and 6π with cosine $\frac{\sqrt{3}}{2}$ are

$\frac{\pi}{6}, \frac{11\pi}{6}, \frac{13\pi}{6}, \frac{23\pi}{6}, \frac{25\pi}{6}$, and $\frac{35\pi}{6}$

$\therefore\ 3x = \frac{\pi}{6}, \frac{11\pi}{6}, \frac{13\pi}{6}, \frac{23\pi}{6}, \frac{25\pi}{6}$, or $\frac{35\pi}{6}$

$\therefore$ the solutions are $x = \frac{\pi}{18}, \frac{11\pi}{18}, \frac{13\pi}{18}, \frac{23\pi}{18}, \frac{25\pi}{18}$, or $\frac{35\pi}{18}$.



c $\tan 2x - \sqrt{3} = 0$

$\therefore\ \tan 2x = \sqrt{3}$

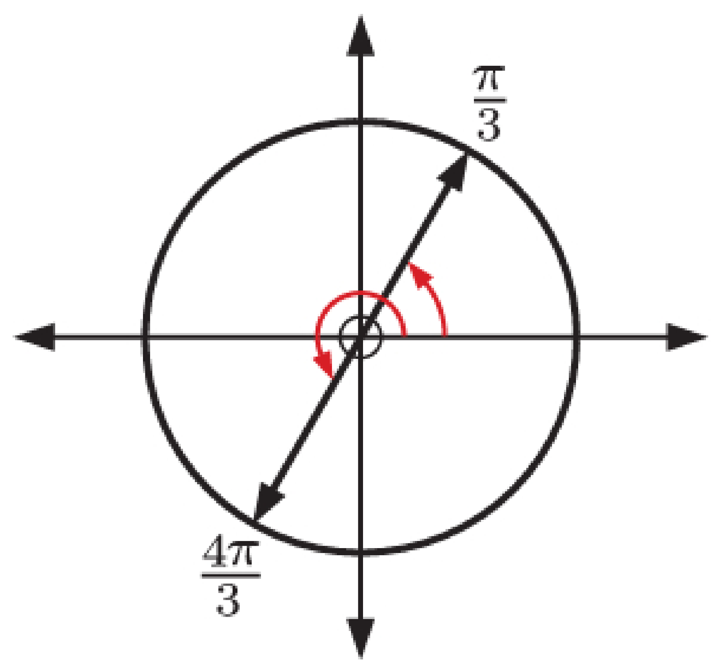
If $0 \leqslant x \leqslant 2\pi$, then $0 \leqslant 2x \leqslant 4\pi$.

The angles between 0 and 4π with tangent $\sqrt{3}$ are

$\frac{\pi}{3}, \frac{4\pi}{3}, \frac{7\pi}{3}$, and $\frac{10\pi}{3}$

$\therefore\ 2x = \frac{\pi}{3}, \frac{4\pi}{3}, \frac{7\pi}{3}$, or $\frac{10\pi}{3}$

$\therefore$ the solutions are $x = \frac{\pi}{6}, \frac{2\pi}{3}, \frac{7\pi}{6}$, or $\frac{5\pi}{3}$.

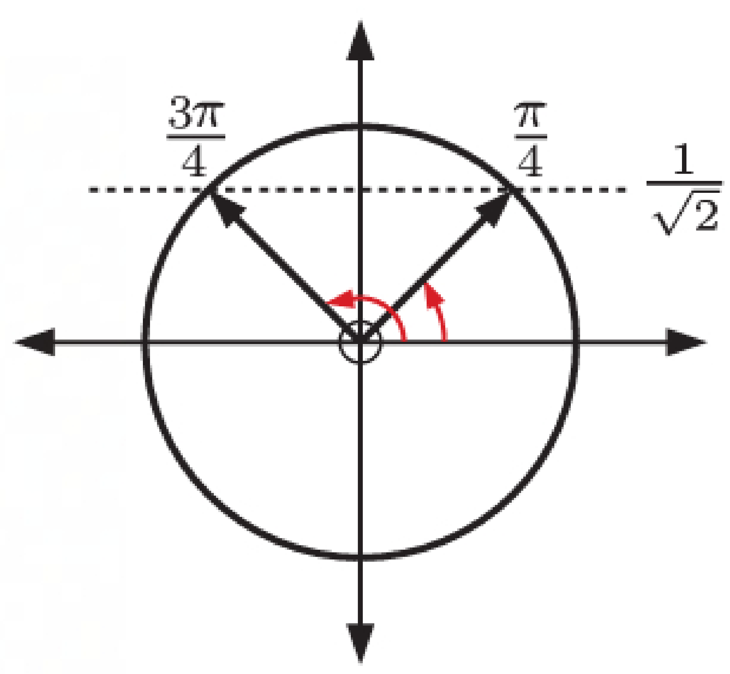


d If $0 \leqslant x \leqslant 2\pi$, then $0 \leqslant \frac{x}{2} \leqslant \pi$.

The angles between 0 and π with sine $\frac{1}{\sqrt{2}}$ are $\frac{\pi}{4}$ and $\frac{3\pi}{4}$.

$\therefore\ \frac{x}{2} = \frac{\pi}{4}$ or $\frac{3\pi}{4}$

$\therefore$ the solutions are $x = \frac{\pi}{2}$ or $\frac{3\pi}{2}$.



e $2\cos\frac{x}{2}+1=0$

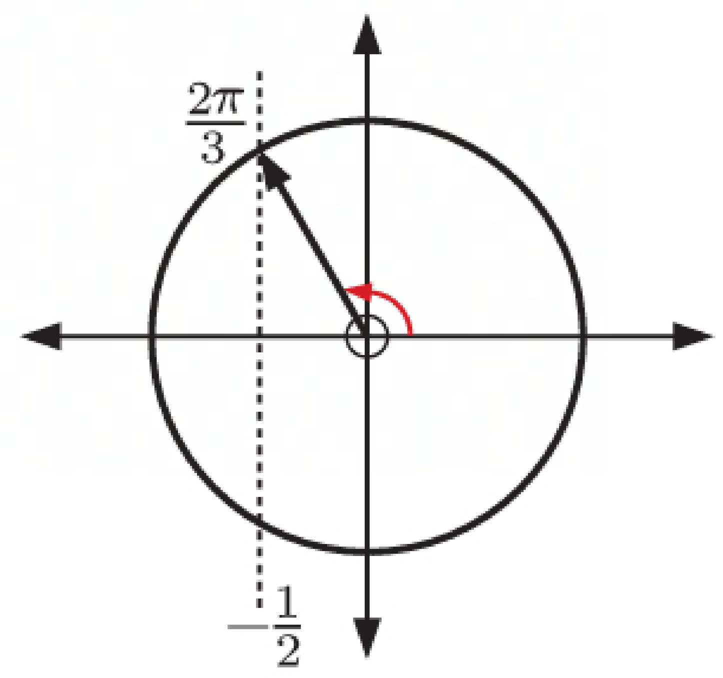
$\therefore\ \cos\frac{x}{2}=-\frac{1}{2}$

If $0 \leqslant x \leqslant 2\pi$, then $0 \leqslant \dfrac{x}{2} \leqslant \pi$.

The angle between 0 and π with cosine $-\frac{1}{2}$ is $\frac{2\pi}{3}$.

$$\therefore\ \frac{x}{2}=\frac{2\pi}{3}$$

$\therefore$ the solution is $x=\frac{4\pi}{3}$.



f $3\tan\frac{x}{3}-3=0$

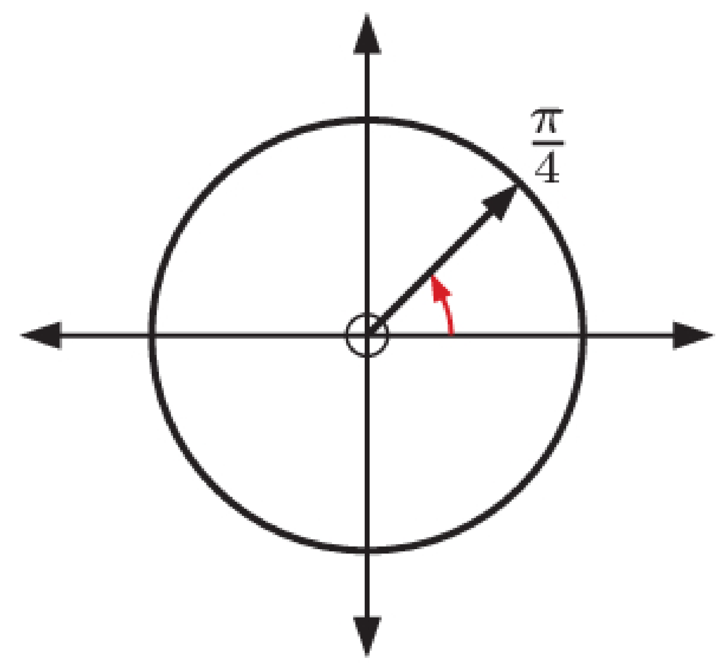
$\therefore\ \tan\frac{x}{3}=\frac{3}{3}=1$

If $0 \leqslant x \leqslant 2\pi$, then $0 \leqslant \dfrac{x}{3} \leqslant \frac{2\pi}{3}$.

The angle between 0 and $\frac{2\pi}{3}$ with tangent 1 is $\frac{\pi}{4}$.

$$\therefore\ \frac{x}{3}=\frac{\pi}{4}$$

$\therefore$ the solution is $x=\frac{3\pi}{4}$.

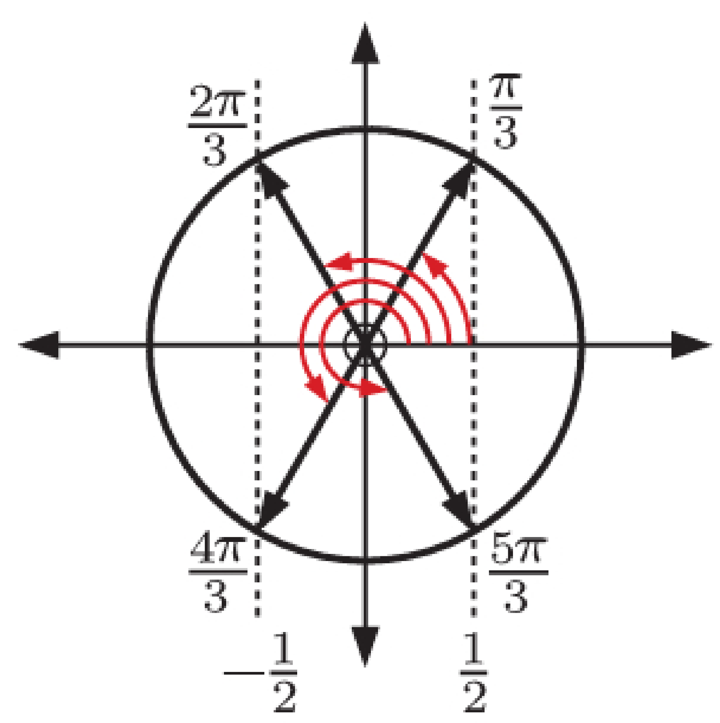


10 **a** $\cos^2 3x=\frac{1}{4}$

$\therefore\ \cos 3x=\pm\frac{1}{2}$

If $0 \leqslant x \leqslant 2\pi$, then $0 \leqslant 3x \leqslant 6\pi$.

On $0 \leqslant 3x \leqslant 6\pi$, the angles with cosine $\frac{1}{2}$ are $\frac{\pi}{3}$, $\frac{5\pi}{3}$, $\frac{7\pi}{3}$, $\frac{11\pi}{3}$, $\frac{13\pi}{3}$, and $\frac{17\pi}{3}$, and the angles with cosine $-\frac{1}{2}$ are $\frac{2\pi}{3}$, $\frac{4\pi}{3}$, $\frac{8\pi}{3}$, $\frac{10\pi}{3}$, $\frac{14\pi}{3}$, and $\frac{16\pi}{3}$.



$$\therefore\ 3x=\frac{\pi}{3},\ \frac{2\pi}{3},\ \frac{4\pi}{3},\ \frac{5\pi}{3},\ \frac{7\pi}{3},\ \frac{8\pi}{3},\ \frac{10\pi}{3},\ \frac{11\pi}{3},\ \frac{13\pi}{3},\ \frac{14\pi}{3},\ \frac{16\pi}{3},\ \text{or}\ \frac{17\pi}{3}$$

$\therefore$ the solutions are $x=\frac{\pi}{9},\ \frac{2\pi}{9},\ \frac{4\pi}{9},\ \frac{5\pi}{9},\ \frac{7\pi}{9},\ \frac{8\pi}{9},\ \frac{10\pi}{9},\ \frac{11\pi}{9},\ \frac{13\pi}{9},\ \frac{14\pi}{9},\ \frac{16\pi}{9},\ \text{or}\ \frac{17\pi}{9}$.

b $\sin^2 2x=1$

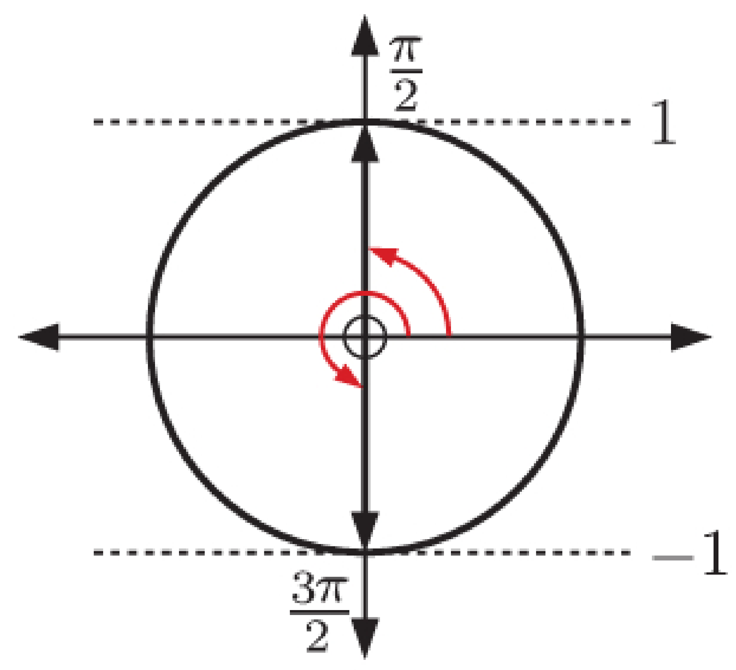
$\therefore\ \sin 2x=\pm 1$

If $0 \leqslant x \leqslant 2\pi$, then $0 \leqslant 2x \leqslant 4\pi$.

On $0 \leqslant 2x \leqslant 4\pi$, the angles with sine 1 are $\frac{\pi}{2}$ and $\frac{5\pi}{2}$, and the angles with sine -1 are $\frac{3\pi}{2}$ and $\frac{7\pi}{2}$.

$$\therefore\ 2x=\frac{\pi}{2},\ \frac{3\pi}{2},\ \frac{5\pi}{2},\ \text{or}\ \frac{7\pi}{2}$$

$\therefore$ the solutions are $x=\frac{\pi}{4},\ \frac{3\pi}{4},\ \frac{5\pi}{4},\ \text{or}\ \frac{7\pi}{4}$.

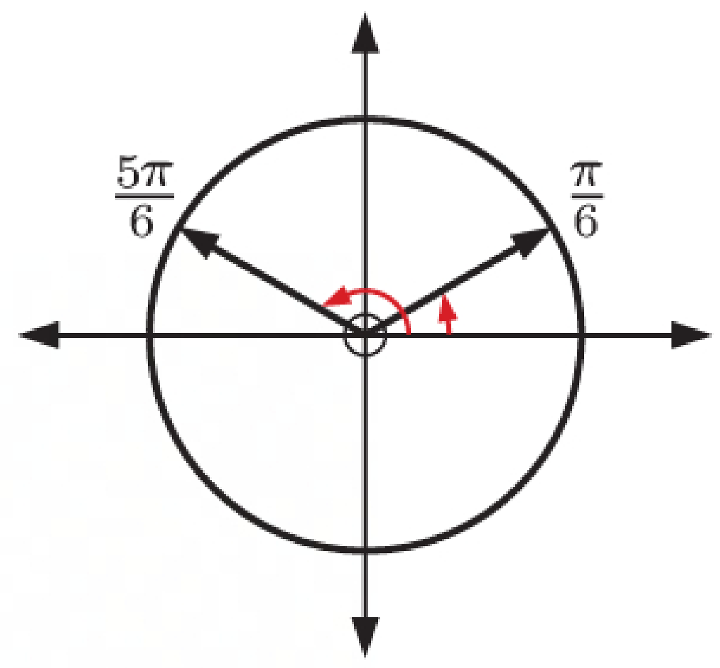


c $\tan^2\left(\frac{x}{2}\right)=\frac{1}{3}$

$\therefore\ \tan\frac{x}{2}=\pm\frac{1}{\sqrt{3}}$

If $0 \leqslant x \leqslant 2\pi$, then $0 \leqslant \dfrac{x}{2} \leqslant \pi$.

On $0 \leqslant \dfrac{x}{2} \leqslant \pi$, the angle with tangent $\frac{1}{\sqrt{3}}$ is $\frac{\pi}{6}$, and the angle with tangent $-\frac{1}{\sqrt{3}}$ is $\frac{5\pi}{6}$.



$$\therefore\ \frac{x}{2}=\frac{\pi}{6}\ \text{or}\ \frac{5\pi}{6}$$

$\therefore$ the solutions are $x=\frac{\pi}{3}$ or $\frac{5\pi}{3}$.

11 **a**

$$\sin x = -\cos x$$

$$\therefore \frac{\sin x}{\cos x} = -1$$

$$\therefore \tan x = -1$$

$$\therefore x = \tfrac{3\pi}{4} \text{ or } \tfrac{7\pi}{4}$$

b $0 \leqslant x \leqslant 2\pi \quad \therefore \quad 0 \leqslant 3x \leqslant 6\pi$

$$\sin 3x = \cos 3x$$

$$\therefore \frac{\sin 3x}{\cos 3x} = 1$$

$$\therefore \tan 3x = 1$$

$$\therefore 3x = \tfrac{\pi}{4}, \tfrac{5\pi}{4}, \tfrac{9\pi}{4}, \tfrac{13\pi}{4}, \tfrac{17\pi}{4}, \text{ or } \tfrac{21\pi}{4}$$

$$\therefore x = \tfrac{\pi}{12}, \tfrac{5\pi}{12}, \tfrac{3\pi}{4}, \tfrac{13\pi}{12}, \tfrac{17\pi}{12}, \text{ or } \tfrac{7\pi}{4}$$

c $0 \leqslant x \leqslant 2\pi \quad \therefore \quad 0 \leqslant 2x \leqslant 4\pi$

$$\sin 2x = \sqrt{3}\cos 2x$$

$$\therefore \frac{\sin 2x}{\cos 2x} = \sqrt{3}$$

$$\therefore \tan 2x = \sqrt{3}$$

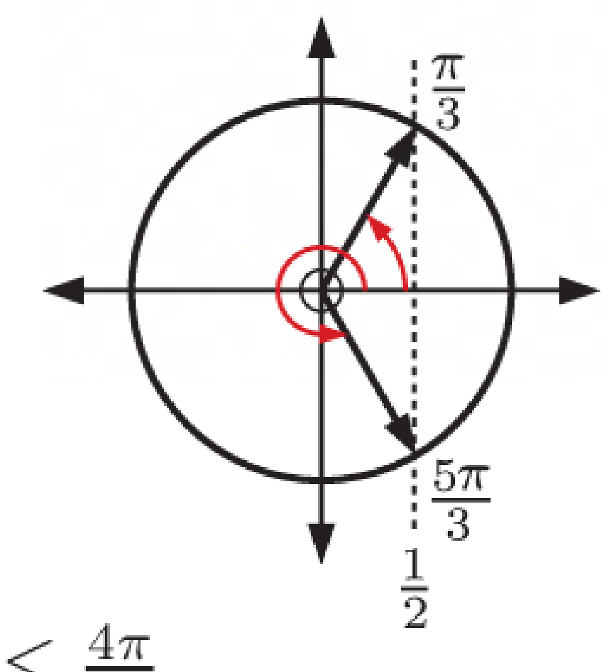
$$\therefore 2x = \tfrac{\pi}{3}, \tfrac{4\pi}{3}, \tfrac{7\pi}{3}, \text{ or } \tfrac{10\pi}{3}$$

$$\therefore x = \tfrac{\pi}{6}, \tfrac{2\pi}{3}, \tfrac{7\pi}{6}, \text{ or } \tfrac{5\pi}{3}$$

12 **a** $\cos\left(x - \frac{2\pi}{3}\right) = \frac{1}{2}, \quad -2\pi \leqslant x \leqslant 2\pi$

There are two points on the unit circle with cosine $\frac{1}{2}$.

They correspond to angles $\frac{\pi}{3}$ and $\frac{5\pi}{3}$.



Since $\quad -2\pi \leqslant x \leqslant 2\pi$

$$-\tfrac{8\pi}{3} \leqslant x - \tfrac{2\pi}{3} \leqslant \tfrac{4\pi}{3}$$

So, $\quad x - \frac{2\pi}{3} = -\frac{7\pi}{3}, -\frac{5\pi}{3}, -\frac{\pi}{3}, \text{ or } \frac{\pi}{3}$

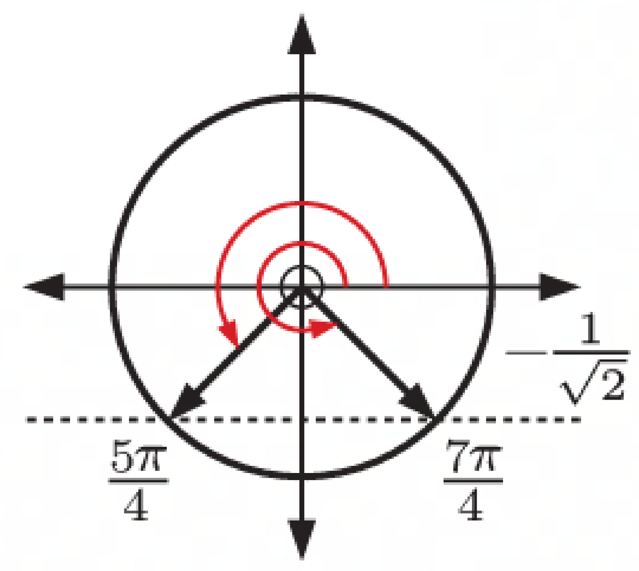
$$\therefore x = -\tfrac{5\pi}{3}, -\pi, \tfrac{\pi}{3}, \text{ or } \pi$$

b $\sqrt{2}\sin\left(x - \frac{\pi}{4}\right) + 1 = 0, \quad 0 \leqslant x \leqslant 3\pi$

$$\therefore \sin\left(x - \tfrac{\pi}{4}\right) = -\tfrac{1}{\sqrt{2}}$$

There are two points on the unit circle with sine $-\frac{1}{\sqrt{2}}$.

They correspond to angles $\frac{5\pi}{4}$ and $\frac{7\pi}{4}$.



Since $\quad 0 \leqslant x \leqslant 3\pi$

$$-\tfrac{\pi}{4} \leqslant x - \tfrac{\pi}{4} \leqslant \tfrac{11\pi}{4}$$

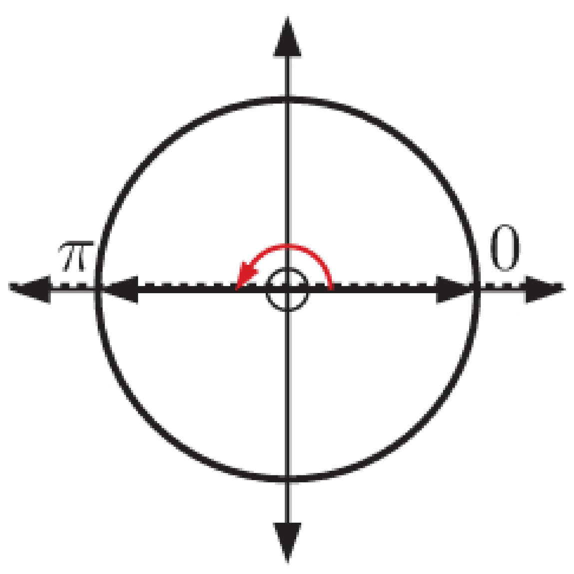
So, $\quad x - \frac{\pi}{4} = -\frac{\pi}{4}, \frac{5\pi}{4}, \text{ or } \frac{7\pi}{4}$

$$\therefore x = 0, \tfrac{3\pi}{2}, \text{ or } 2\pi$$

c $\sin\left(4\left(x - \frac{\pi}{4}\right)\right) = 0, \quad 0 \leqslant x \leqslant \pi$

There are two points on the unit circle with sine 0.

They correspond to angles 0 and π.



Since $\quad 0 \leqslant x \leqslant \pi$

$$-\tfrac{\pi}{4} \leqslant x - \tfrac{\pi}{4} \leqslant \tfrac{3\pi}{4}$$

$$\therefore -\pi \leqslant 4\left(x - \tfrac{\pi}{4}\right) \leqslant 3\pi$$

So, $\quad 4\left(x - \frac{\pi}{4}\right) = -\pi, 0, \pi, 2\pi, \text{ or } 3\pi$

$$\therefore x - \tfrac{\pi}{4} = -\tfrac{\pi}{4}, 0, \tfrac{\pi}{4}, \tfrac{\pi}{2}, \text{ or } \tfrac{3\pi}{4}$$

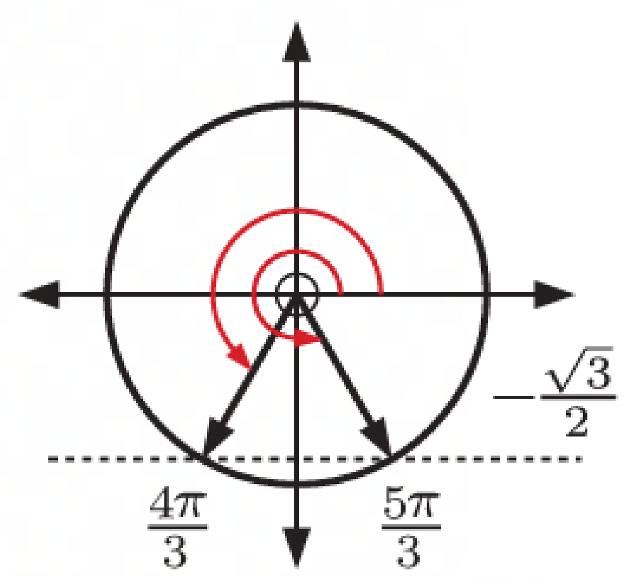
$$\therefore x = 0, \tfrac{\pi}{4}, \tfrac{\pi}{2}, \tfrac{3\pi}{4}, \text{ or } \pi$$

d $2\sin\left(2\left(x - \frac{\pi}{3}\right)\right) = -\sqrt{3}, \quad 0 \leqslant x \leqslant 2\pi$

$$\therefore \sin\left(2\left(x - \tfrac{\pi}{3}\right)\right) = -\tfrac{\sqrt{3}}{2}$$

There are two points on the unit circle with sine $-\frac{\sqrt{3}}{2}$.

They correspond to angles $\frac{4\pi}{3}$ and $\frac{5\pi}{3}$.



Since $\quad 0 \leqslant x \leqslant 2\pi$

$$-\tfrac{\pi}{3} \leqslant x - \tfrac{\pi}{3} \leqslant \tfrac{5\pi}{3}$$

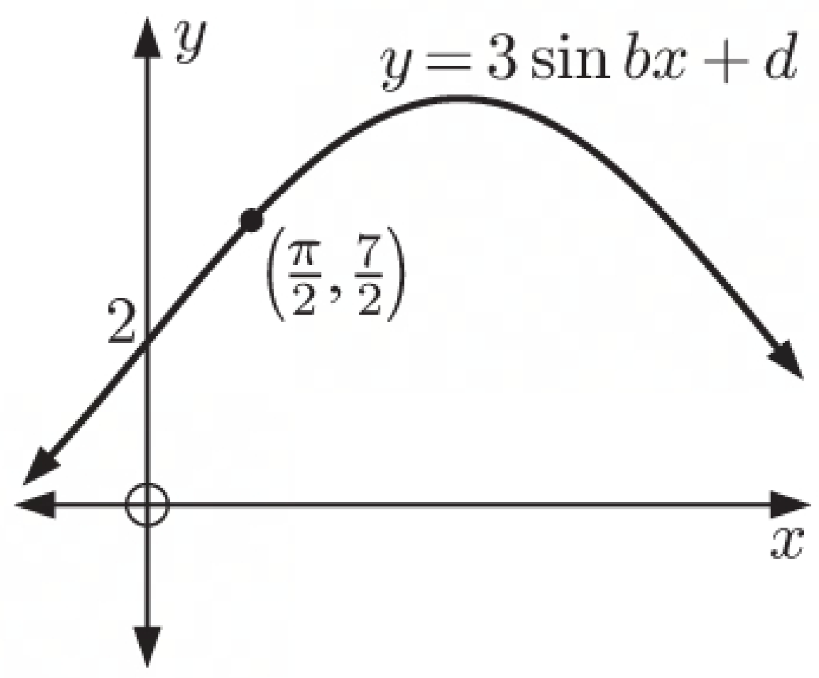
$$\therefore -\tfrac{2\pi}{3} \leqslant 2\left(x - \tfrac{\pi}{3}\right) \leqslant \tfrac{10\pi}{3}$$

So, $\quad 2\left(x - \frac{\pi}{3}\right) = -\frac{2\pi}{3}, -\frac{\pi}{3}, \frac{4\pi}{3}, \frac{5\pi}{3}, \text{ or } \frac{10\pi}{3}$

$$\therefore x - \tfrac{\pi}{3} = -\tfrac{\pi}{3}, -\tfrac{\pi}{6}, \tfrac{2\pi}{3}, \tfrac{5\pi}{6}, \text{ or } \tfrac{5\pi}{3}$$

$$\therefore x = 0, \tfrac{\pi}{6}, \pi, \tfrac{7\pi}{6}, \text{ or } 2\pi$$

13 **a** $y = 3\sin bx + d$



When $x = 0$, $y = 2$

$\therefore\ 2 = 3\sin 0 + d$

$\therefore\ d = 2$

When $x = \frac{\pi}{2}$, $y = \frac{7}{2}$

$\therefore\ \frac{7}{2} = 3\sin\left(b \times \frac{\pi}{2}\right) + 2$

$\therefore\ \frac{3}{2} = 3\sin\frac{b\pi}{2}$

$\therefore\ \sin\frac{b\pi}{2} = \frac{1}{2}$

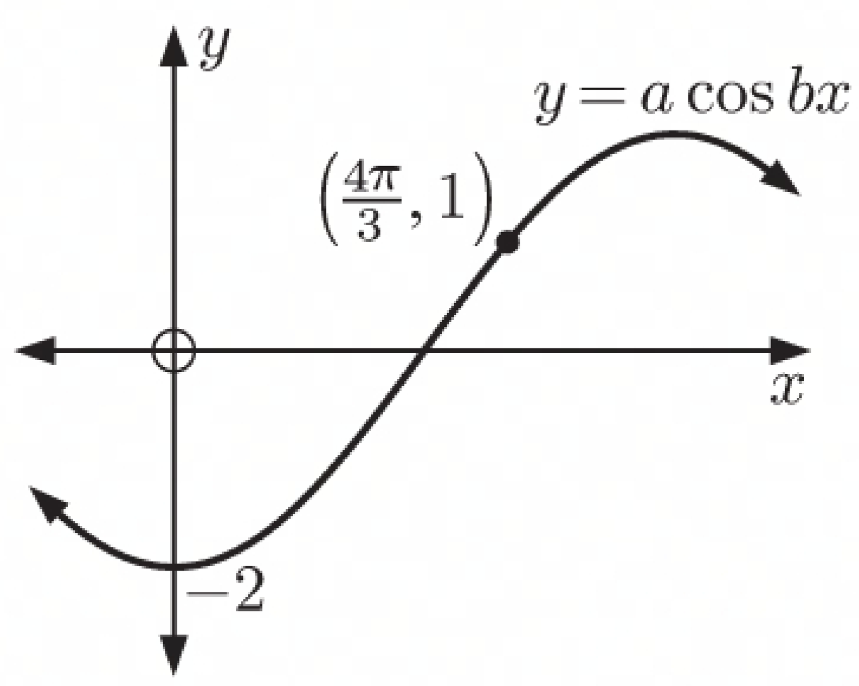
$\therefore\ \dfrac{b\pi}{2} = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{11\pi}{6}, ...$

$\therefore\ b = \frac{1}{3}, \frac{5}{3}, \frac{7}{3}, \frac{11}{3}, ...$

From the graph, we require the value of b which gives the largest period, because $x = \frac{\pi}{2}$ is the *first* value for which $y = \frac{7}{2}$. This is the smallest value of b.

So, $b = \frac{1}{3}$, $d = 2$.

b $y = a\cos bx$



When $x = 0$, $y = -2$

$\therefore\ -2 = a\cos 0$

$\therefore\ a = -2$

When $x = \frac{4\pi}{3}$, $y = 1$

$\therefore\ 1 = -2\cos\left(b \times \frac{4\pi}{3}\right)$

$\therefore\ \cos\frac{4b\pi}{3} = -\frac{1}{2}$

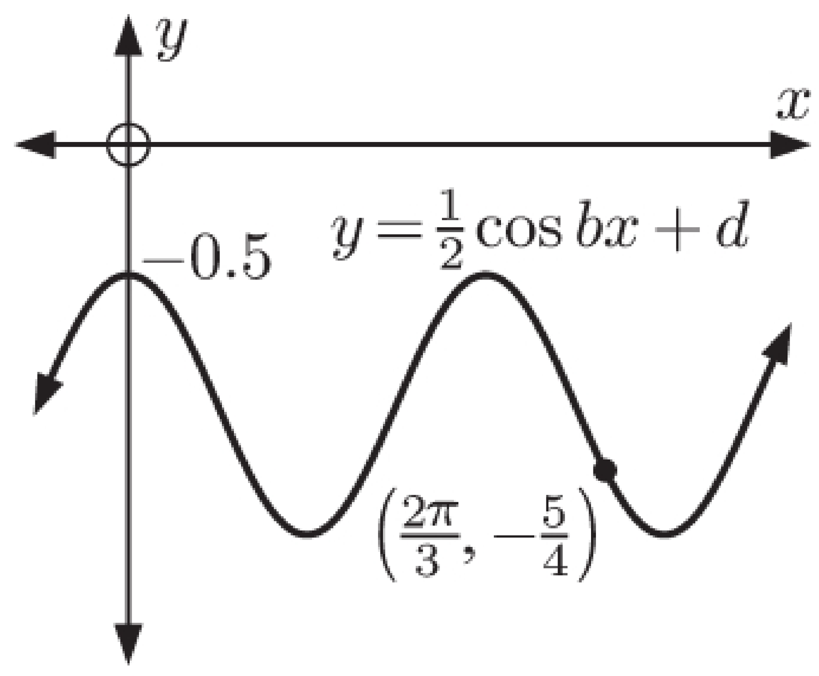
$\therefore\ \dfrac{4b\pi}{3} = \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{8\pi}{3}, \frac{10\pi}{3}, ...$

$\therefore\ b = \frac{1}{2}, 1, 2, \frac{5}{2}, ...$

From the graph, we require the value of b which gives the largest period, because $x = \frac{4\pi}{3}$ is the *first* value for which $y = 1$. This is the smallest value of b.

So, $a = -2$, $b = \frac{1}{2}$.

c $y = \frac{1}{2}\cos bx + d$



When $x = 0$, $y = -0.5$

$\therefore\ -\frac{1}{2} = \frac{1}{2}\cos 0 + d$

$\therefore\ d = -1$

When $x = \frac{2\pi}{3}$, $y = -\frac{5}{4}$

$\therefore\ -\frac{5}{4} = \frac{1}{2}\cos\left(b \times \frac{2\pi}{3}\right) - 1$

$\therefore\ -\frac{1}{4} = \frac{1}{2}\cos\frac{2b\pi}{3}$

$\therefore\ \cos\frac{2b\pi}{3} = -\frac{1}{2}$

$\therefore\ \dfrac{2b\pi}{3} = \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{8\pi}{3}, \frac{10\pi}{3}, ...$

$\therefore\ b = 1, 2, 4, 5, ...$

From the graph, we require the value of b which gives the 3rd largest period, because $x = \frac{2\pi}{3}$ is the *third* value for which $y = -\frac{5}{4}$. This is the 3rd smallest value of b.

So, $b = 4$, $d = -1$.

d $y = 5\cos\left(b\left(x - \frac{\pi}{4}\right)\right) + d$

When $x = \frac{\pi}{4}$, $y = 1$

$\therefore\ 1 = 5\cos(b \times 0) + d$

$\therefore\ d = -4$

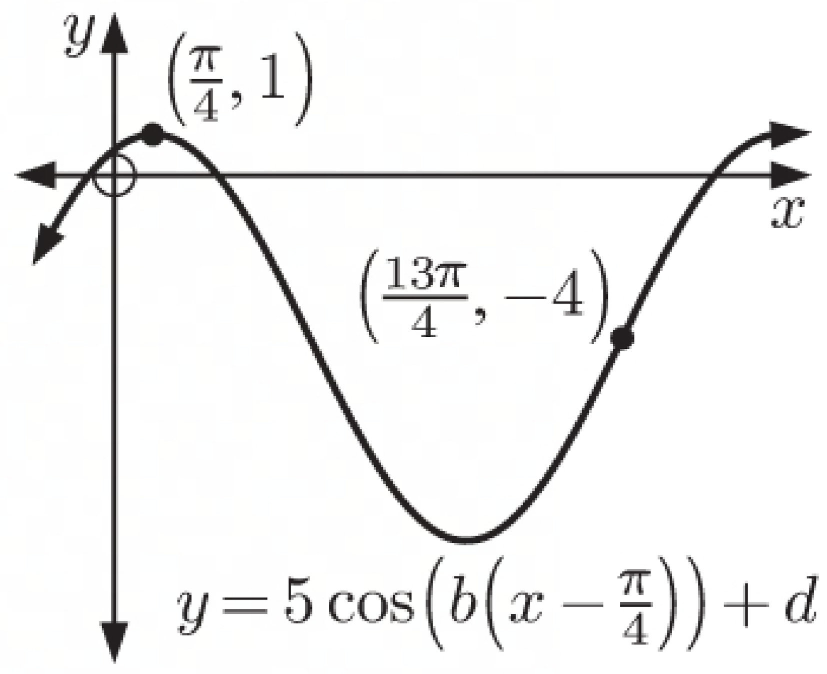
When $x = \frac{13\pi}{4}$, $y = -4$

$\therefore\ -4 = 5\cos(b \times 3\pi) - 4$

$\therefore\ \cos 3b\pi = 0$

$\therefore\ 3b\pi = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \frac{7\pi}{2}, \ldots$

$\therefore\ b = \frac{1}{6}, \frac{1}{2}, \frac{5}{6}, \frac{7}{6}, \ldots$



From the graph, we require the value of b which gives the 2nd largest period, because $x = \frac{13\pi}{4}$ is the *second* value for which $y = -4$. This is the 2nd smallest value of b.

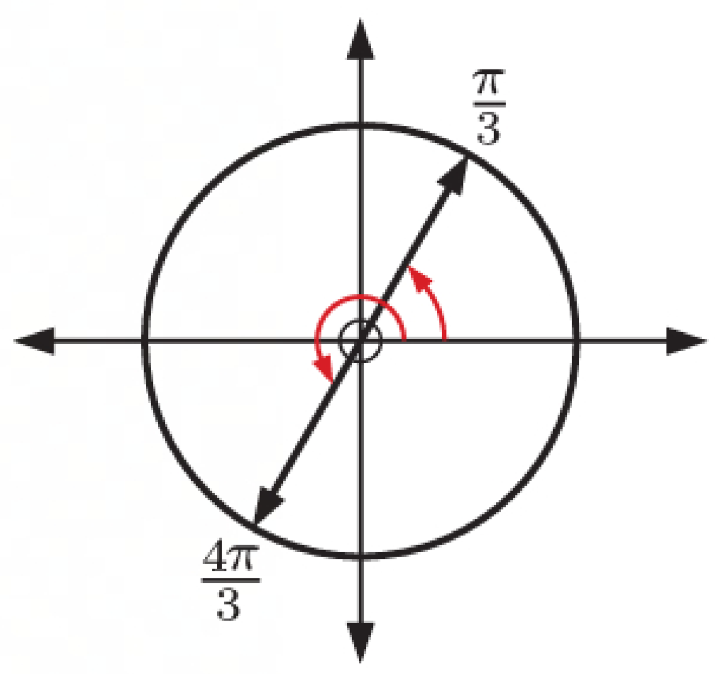
So, $b = \frac{1}{2}$, $d = -4$.

14 $\tan x = \sqrt{3}$, $0 \leqslant x \leqslant 2\pi$

There are two points on the unit circle with tangent $\sqrt{3}$.

They correspond to angles $\frac{\pi}{3}$ and $\frac{4\pi}{3}$.

$\therefore\ x = \frac{\pi}{3}$ or $\frac{4\pi}{3}$



a Since $0 \leqslant x \leqslant 2\pi$

$-\frac{\pi}{6} \leqslant x - \frac{\pi}{6} \leqslant \frac{11\pi}{6}$

So, $x - \frac{\pi}{6} = \frac{\pi}{3}$ or $\frac{4\pi}{3}$

$\therefore\ x = \frac{\pi}{2}$ or $\frac{3\pi}{2}$

b Since $0 \leqslant x \leqslant 2\pi$

$0 \leqslant 4x \leqslant 8\pi$

So, $4x = \frac{\pi}{3}, \frac{4\pi}{3}, \frac{7\pi}{3}, \frac{10\pi}{3}, \frac{13\pi}{3}, \frac{16\pi}{3}, \frac{19\pi}{3}$, or $\frac{22\pi}{3}$

$\therefore\ x = \frac{\pi}{12}, \frac{\pi}{3}, \frac{7\pi}{12}, \frac{5\pi}{6}, \frac{13\pi}{12}, \frac{4\pi}{3}, \frac{19\pi}{12}$, or $\frac{11\pi}{6}$

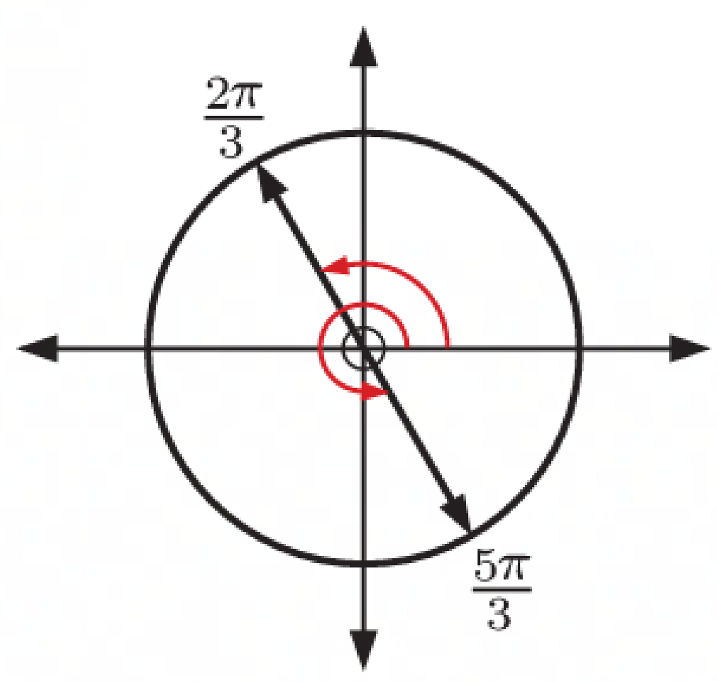
c $\tan^2 x = 3$, $0 \leqslant x \leqslant 2\pi$

$\therefore\ \tan x = \pm\sqrt{3}$

There are two points on the unit circle with tangent $-\sqrt{3}$.

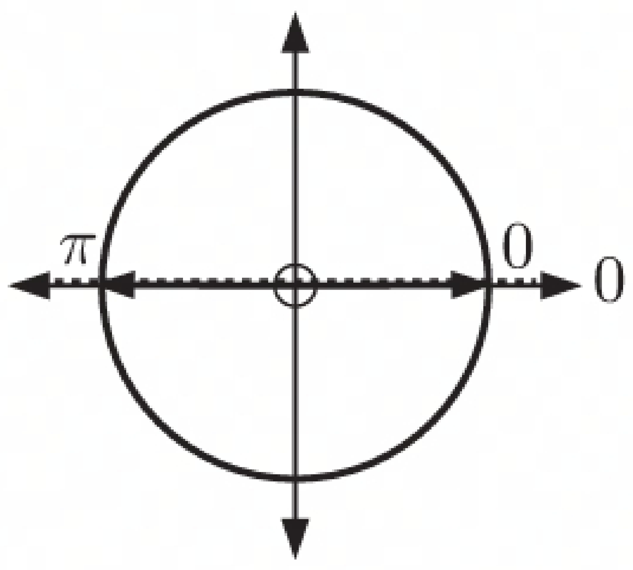
They correspond to angles $\frac{2\pi}{3}$ and $\frac{5\pi}{3}$.

So, $x = \frac{\pi}{3}, \frac{2\pi}{3}, \frac{4\pi}{3}$, or $\frac{5\pi}{3}$

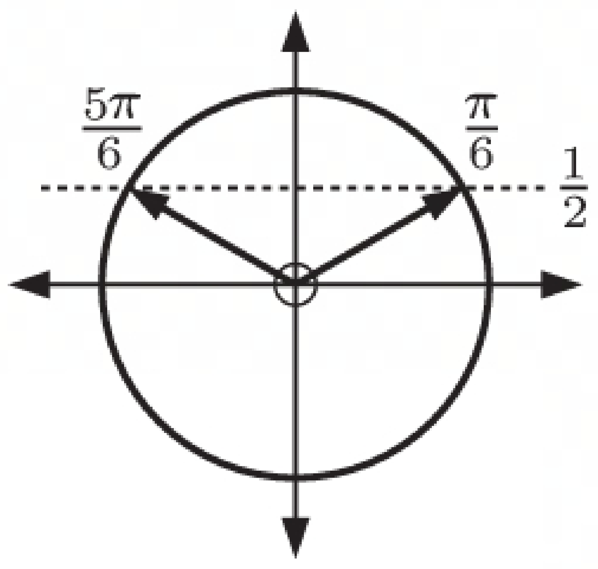


15 **a**

$$2\sin^2 x - \sin x = 0$$
$$\therefore \quad \sin x(2\sin x - 1) = 0$$
$$\therefore \quad \sin x = 0 \text{ or } \tfrac{1}{2}$$



$\sin x = 0$ when
$x = 0,\ \pi,$ or 2π
$\{0 \leqslant x \leqslant 2\pi\}$

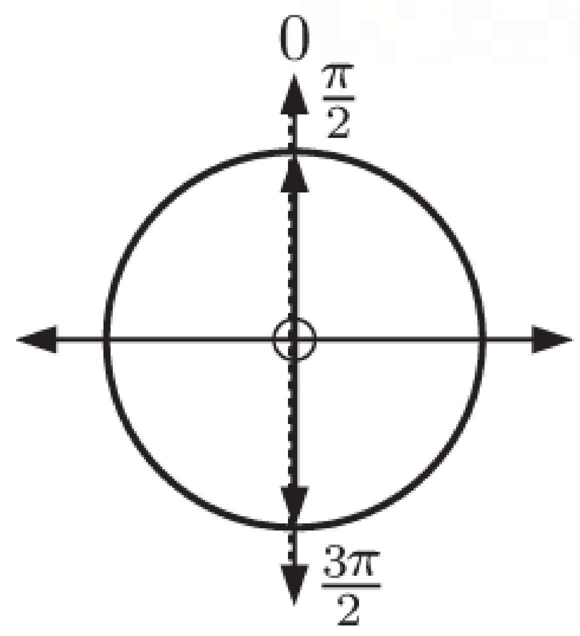


$\sin x = \frac{1}{2}$ when
$x = \frac{\pi}{6}$ or $\frac{5\pi}{6}$
$\{0 \leqslant x \leqslant 2\pi\}$

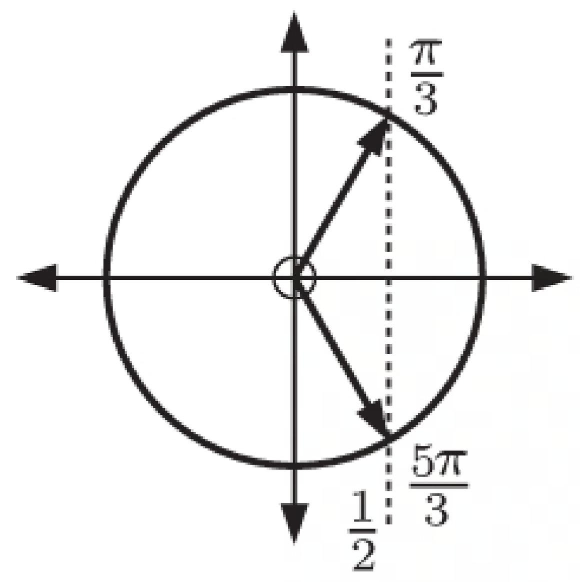
The solutions are:
$x = 0,\ \frac{\pi}{6},\ \frac{5\pi}{6},\ \pi,$ or 2π.

b

$$2\cos^2 x = \cos x$$
$$\therefore \quad 2\cos^2 x - \cos x = 0$$
$$\therefore \quad \cos x(2\cos x - 1) = 0$$
$$\therefore \quad \cos x = 0 \text{ or } \tfrac{1}{2}$$



$\cos x = 0$ when
$x = \frac{\pi}{2}$ or $\frac{3\pi}{2}$
$\{0 \leqslant x \leqslant 2\pi\}$

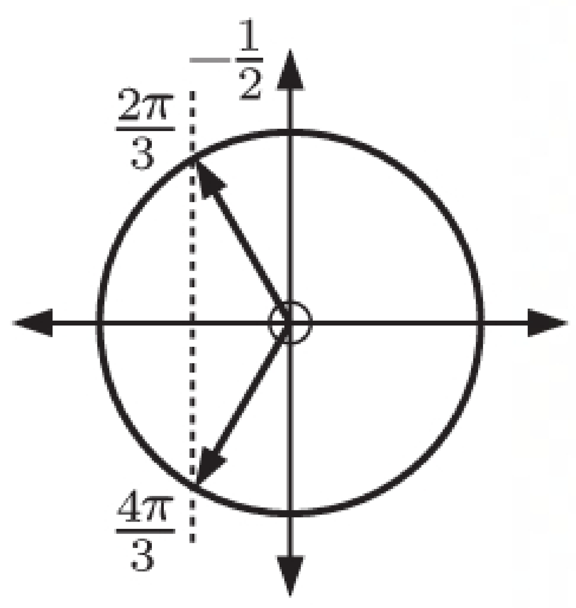


$\cos x = \frac{1}{2}$ when
$x = \frac{\pi}{3}$ or $\frac{5\pi}{3}$
$\{0 \leqslant x \leqslant 2\pi\}$

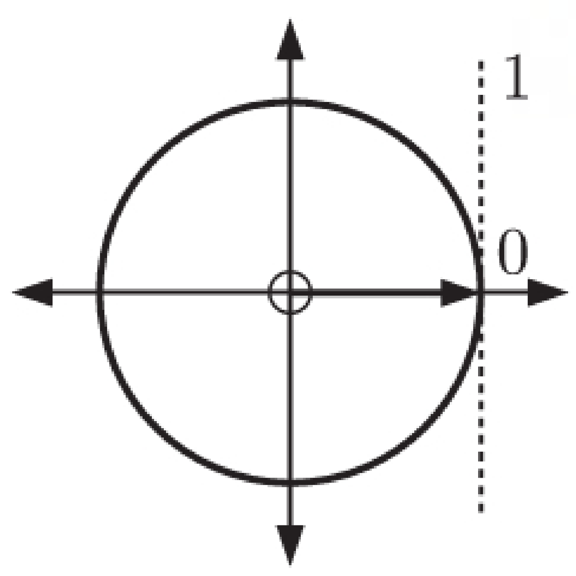
The solutions are:
$x = \frac{\pi}{3},\ \frac{\pi}{2},\ \frac{3\pi}{2},$ or $\frac{5\pi}{3}$.

c

$$2\cos^2 x - \cos x - 1 = 0$$
$$\therefore \quad (2\cos x + 1)(\cos x - 1) = 0$$
$$\therefore \quad \cos x = -\tfrac{1}{2} \text{ or } 1$$



$\cos x = -\frac{1}{2}$ when
$x = \frac{2\pi}{3}$ or $\frac{4\pi}{3}$
$\{0 \leqslant x \leqslant 2\pi\}$

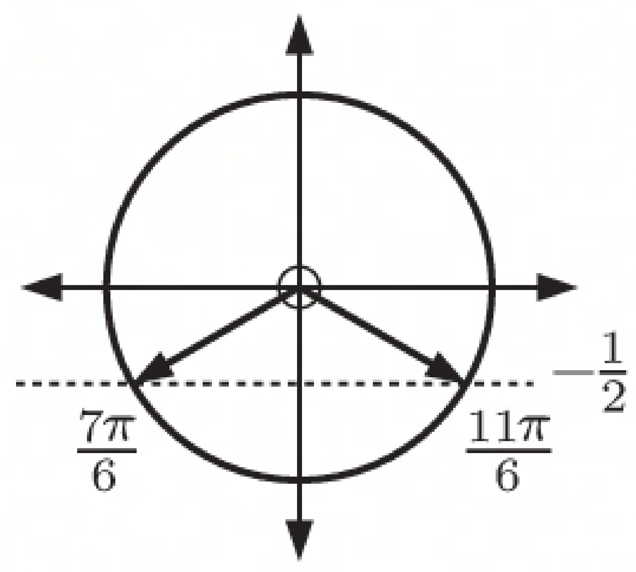


$\cos x = 1$ when
$x = 0$ or 2π
$\{0 \leqslant x \leqslant 2\pi\}$

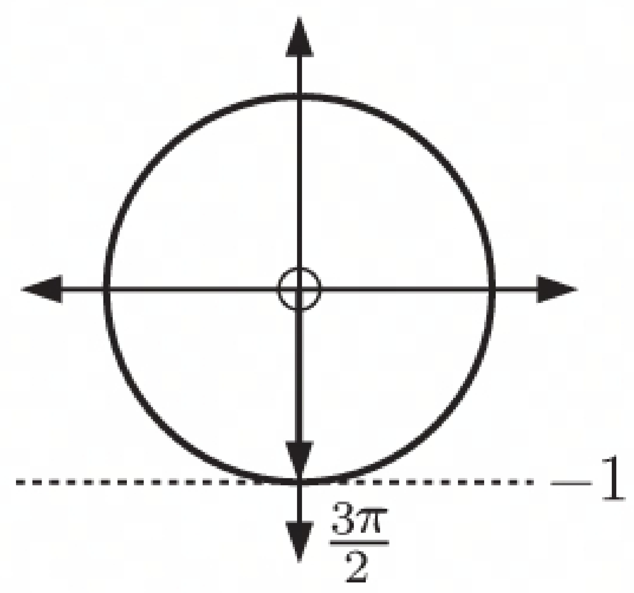
The solutions are:
$x = 0,\ \frac{2\pi}{3},\ \frac{4\pi}{3},$ or 2π.

d

$$2\sin^2 x + 3\sin x + 1 = 0$$
$$\therefore \quad (2\sin x + 1)(\sin x + 1) = 0$$
$$\therefore \quad \sin x = -\tfrac{1}{2} \text{ or } -1$$



$\sin x = -\frac{1}{2}$ when
$x = \frac{7\pi}{6}$ or $\frac{11\pi}{6}$
$\{0 \leqslant x \leqslant 2\pi\}$



$\sin x = -1$ when
$x = \frac{3\pi}{2}$
$\{0 \leqslant x \leqslant 2\pi\}$

The solutions are:
$x = \frac{7\pi}{6},\ \frac{3\pi}{2},$ or $\frac{11\pi}{6}$.

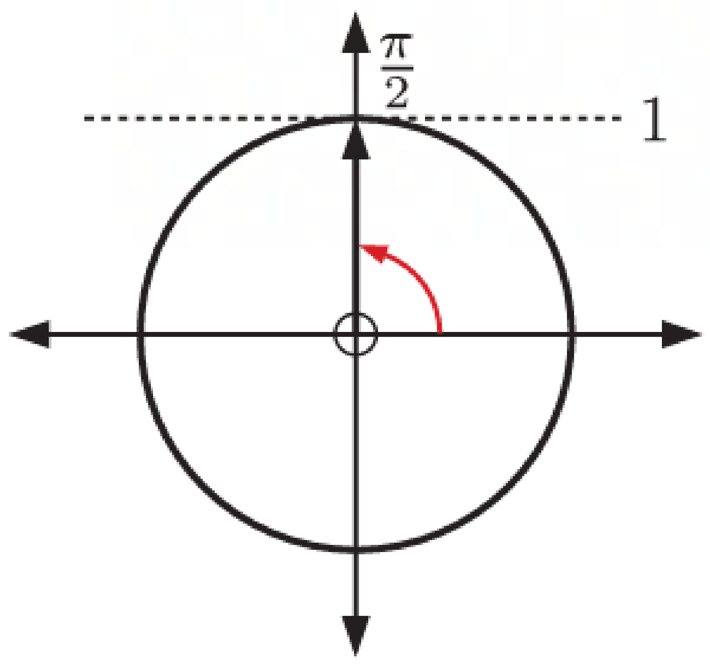
EXERCISE 17H

1 $P(t) = 7500 + 3000\sin \frac{\pi t}{8}, \quad 0 \leqslant t \leqslant 12$

a **i** $P(0) = 7500 + 3000\sin 0$
$= 7500 + 0$
$= 7500$ grasshoppers

ii $P(5) = 7500 + 3000\sin\frac{5\pi}{8}$
$\approx 10\,271.6$
$\approx 10\,300$ grasshoppers

b The greatest value of $P(t)$ occurs when $\sin \frac{\pi t}{8} = 1$, so the greatest population is $7500 + 3000 = 10\,500$ grasshoppers.



The point on the unit circle with sine 1 corresponds to angle $\frac{\pi}{2}$.

$0 \leqslant t \leqslant 12 \quad \therefore \quad 0 \leqslant \frac{\pi t}{8} \leqslant \frac{3\pi}{2}$

So $\frac{\pi t}{8} = \frac{\pi}{2}$

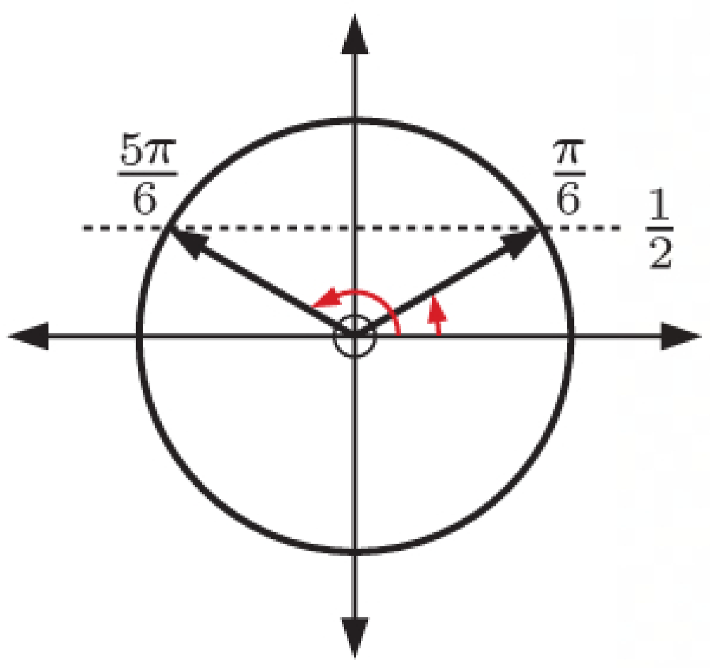
$\therefore \quad t = 4$

So the greatest population occurs after 4 weeks.

c **i** When $P(t) = 9000$, $7500 + 3000 \sin \frac{\pi t}{8} = 9000$

$\therefore \quad 3000 \sin \frac{\pi t}{8} = 1500$

$\therefore \quad \sin \frac{\pi t}{8} = \frac{1}{2}$



The points on the unit circle with sine $\frac{1}{2}$ correspond to angles $\frac{\pi}{6}$ and $\frac{5\pi}{6}$.

$0 \leqslant t \leqslant 12 \quad \therefore \quad 0 \leqslant \frac{\pi t}{8} \leqslant \frac{3\pi}{2}$

So $\frac{\pi t}{8} = \frac{\pi}{6}, \frac{5\pi}{6}$

$\therefore \quad t = \frac{8}{6}, \frac{40}{6}$

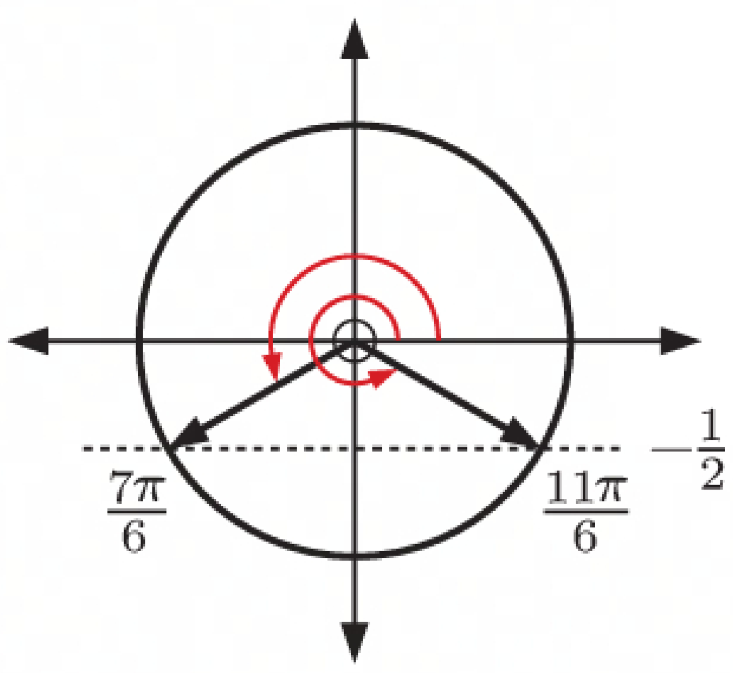
$\therefore \quad t = 1\frac{1}{3}$ or $6\frac{2}{3}$

So, the population is 9000 at $1\frac{1}{3}$ weeks and $6\frac{2}{3}$ weeks.

ii When $P(t) = 6000$, $7500 + 3000 \sin \frac{\pi t}{8} = 6000$

$\therefore \quad 3000 \sin \frac{\pi t}{8} = -1500$

$\therefore \quad \sin \frac{\pi t}{8} = -\frac{1}{2}$



The points on the unit circle with sine $-\frac{1}{2}$ correspond to angles $\frac{7\pi}{6}$ and $\frac{11\pi}{6}$.

$0 \leqslant t \leqslant 12 \quad \therefore \quad 0 \leqslant \frac{\pi t}{8} \leqslant \frac{3\pi}{2}$

So $\frac{\pi t}{8} = \frac{7\pi}{6}$

$\therefore \quad t = \frac{56}{6}$

$\therefore \quad t = 9\frac{1}{3}$

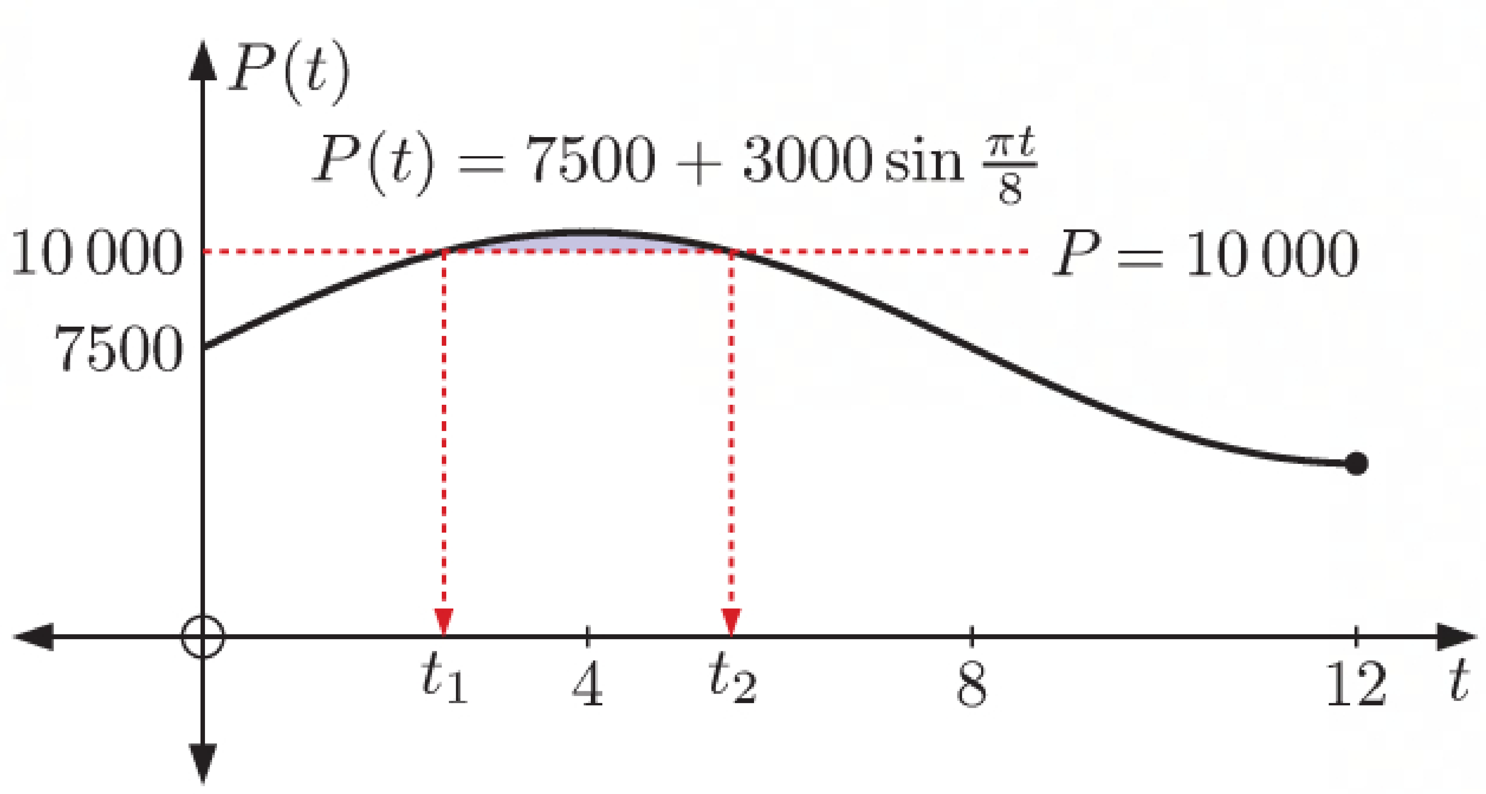
So, the population is 6000 at $9\frac{1}{3}$ weeks.

d We need to solve $P(t) = 10\,000$

$\therefore \quad 7500 + 3000 \sin \frac{\pi t}{8} = 10\,000$

Using technology, we obtain $t_1 \approx 2.51$, $t_2 \approx 5.49$.

So, the population exceeds 10 000 for $2.51 \leqslant t \leqslant 5.49$ weeks.



2 $H(t) = 20 - 19\cos\frac{2\pi t}{3}$

a $H(0) = 20 - 19\cos 0$
$= 20 - 19$
$= 1$ m

So, at time $t = 0$, the passenger is 1 m above the ground.

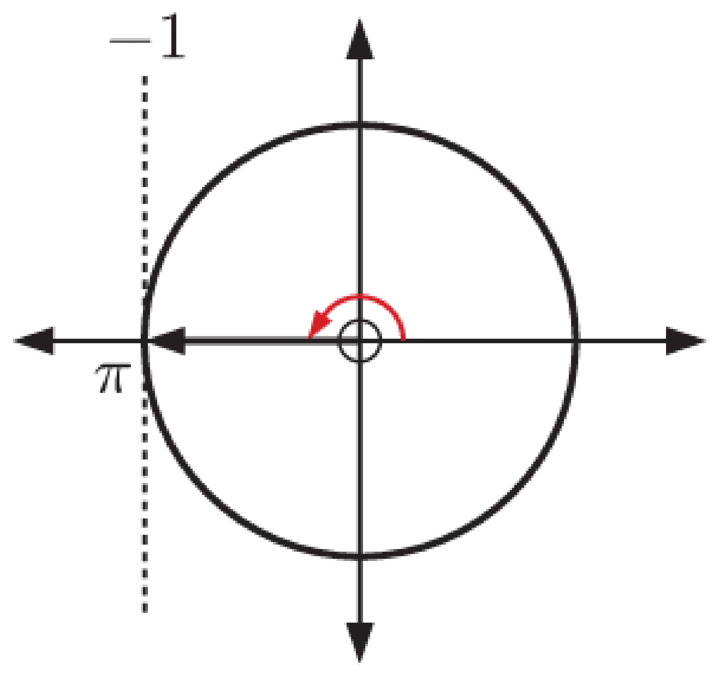
b H is a maximum when $\cos\frac{2\pi t}{3} = -1$

$\therefore \frac{2\pi t}{3} = \pi + k2\pi$

$\therefore \frac{2t}{3} = 1 + k(2)$

$\therefore t = \frac{3}{2} + 3k$

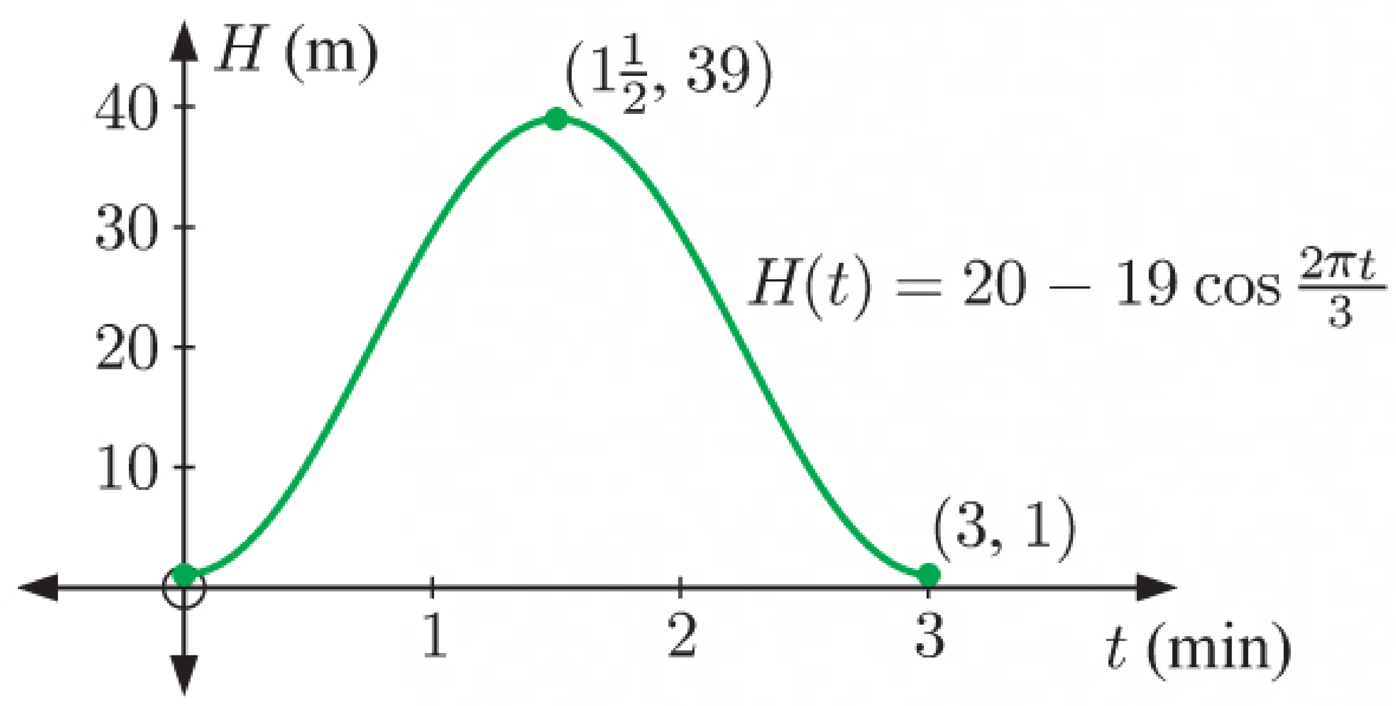
$\therefore t = 1\frac{1}{2}$ min $\{$as $k = 0\}$



c period $= \dfrac{2\pi}{\frac{2\pi}{3}} = 3$ min

$\therefore$ one revolution takes 3 minutes.

d

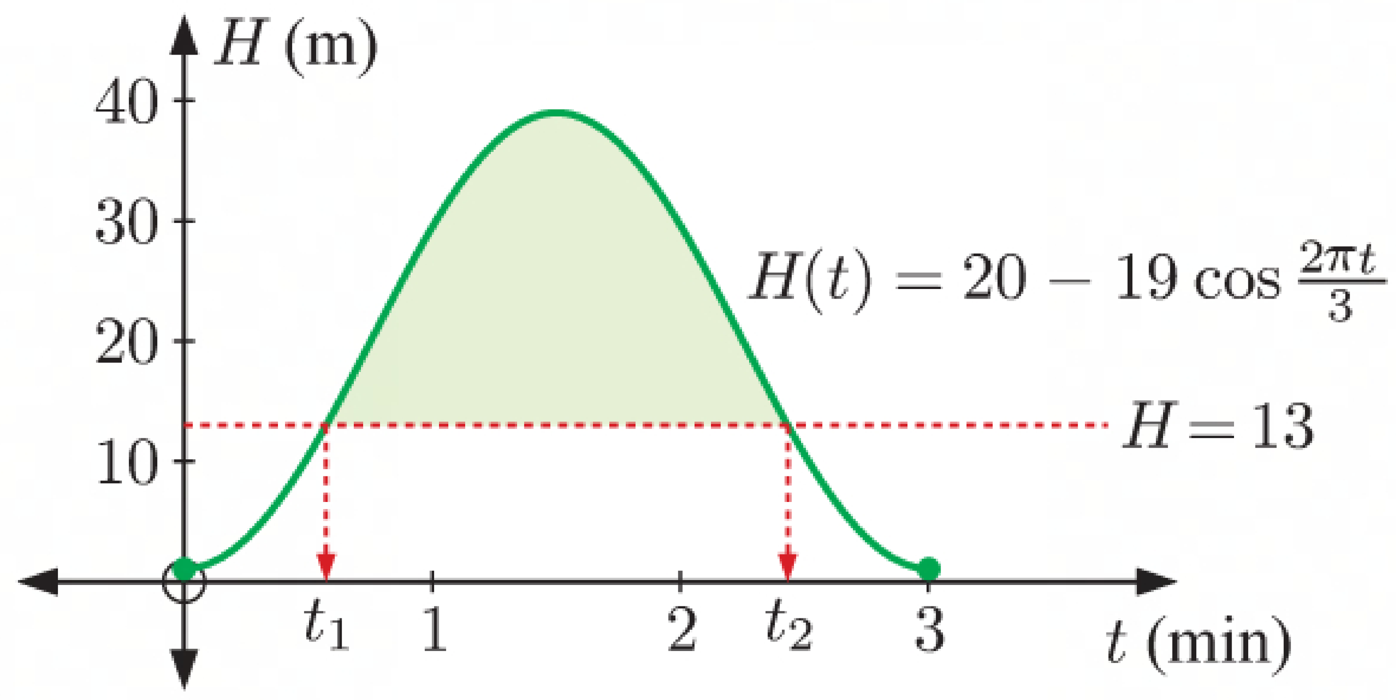


e We need to solve $H(t) = 13$

$\therefore 20 - 19\cos\frac{2\pi t}{3} = 13$

Using technology, we obtain $t_1 \approx 0.570$, $t_2 \approx 2.43$.

So, the passenger can see his friend for $0.570 \leqslant t \leqslant 2.43$ minutes.



3 $P(t) = 400 + 250\sin\frac{\pi t}{2}$

a $P(0) = 400 + 250\sin 0$
$= 400 + 250(0)$
$= 400$ water buffalo

b **i** 6 months $= \frac{1}{2}$ year

$P(\frac{1}{2}) = 400 + 250\sin\left(\dfrac{\pi(\frac{1}{2})}{2}\right)$

$= 400 + 250\sin\frac{\pi}{4}$

$= 400 + 250 \times \frac{1}{\sqrt{2}}$

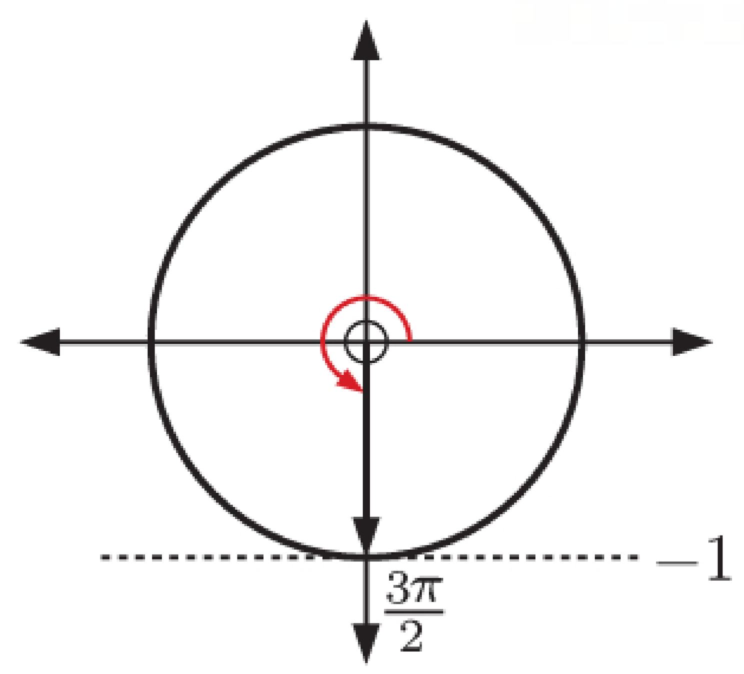
≈ 577 water buffalo

ii $P(2) = 400 + 250\sin\pi$
$= 400 + 250(0)$
$= 400$ water buffalo

c $P(1) = 400 + 250\sin\frac{\pi}{2}$
$= 400 + 250 \times 1$
$= 650$ water buffalo

This is the maximum herd size.

d $P(t)$ is smallest when $\sin\frac{\pi t}{2} = -1$
and is $400 - 250 = 150$ water buffalo.



It occurs when $\dfrac{\pi t}{2} = \dfrac{3\pi}{2} + k2\pi$

$\therefore \dfrac{t}{2} = \frac{3}{2} + k(2)$

$\therefore t = 3 + 4k$

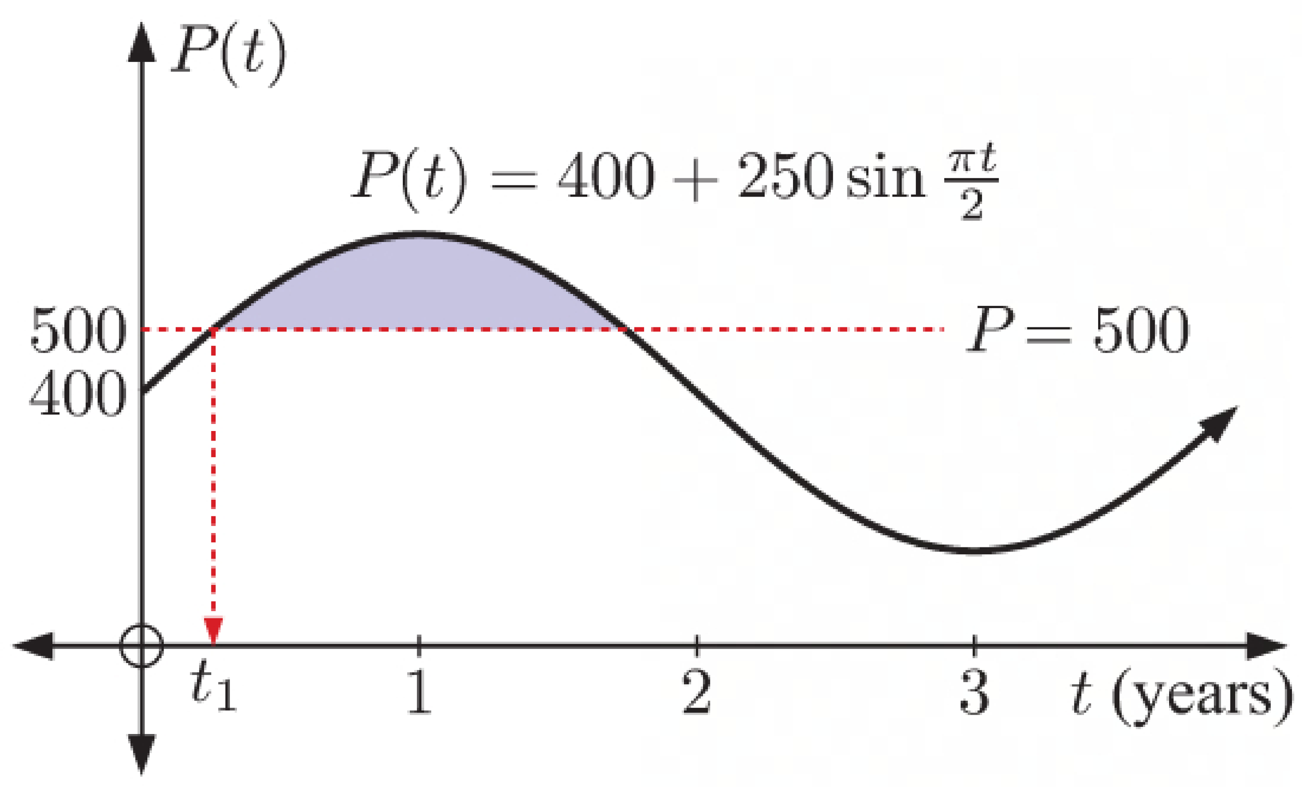
So, the first time is after 3 years.

e We need to solve $P(t) = 500$

$\therefore 400 + 250\sin\frac{\pi t}{2} = 500$

Using technology, we obtain $t_1 \approx 0.262$.

$\therefore$ the herd first exceeded 500 when $t \approx 0.262$ years.



4 $C(t) = 9.2\sin\left(\frac{\pi}{7}(t-4)\right) + 107.8$ cents L^{-1}

a **i** 107.8 is the median value.

Values are between $\underset{\text{min.}}{\underset{\uparrow}{107.8 - 9.2 = 98.6}}$ cents L^{-1} $\{$when $\sin\left(\frac{\pi}{7}(t-4)\right) = -1\}$

and $\underset{\text{max.}}{\underset{\uparrow}{107.8 + 9.2 = 117.0}}$ cents L^{-1} $\{$when $\sin\left(\frac{\pi}{7}(t-4)\right) = 1\}$

$\therefore$ the statement is true.

ii period $= \dfrac{2\pi}{\frac{\pi}{7}} = 14$ days $\therefore$ true

b $C(7) = 9.2\sin\left(\frac{\pi}{7}(3)\right) + 107.8$
≈ 116.8 cents L^{-1}

c When $C(t) = \$1.10\ \text{L}^{-1}$

then $9.2\sin\left(\frac{\pi}{7}(t-4)\right) + 107.8 = 110$

$\therefore \sin\left(\frac{\pi}{7}(t-4)\right) = \dfrac{2.2}{9.2}$

$\therefore t \approx 4.538, 10.462, 18.538, 24.462$ using technology

So, the price was \$1.10 per litre on the 5th, 11th, 19th, and 25th days.

d The minimum cost per litre is $-9.2 + 107.8 = 98.6$ cents L^{-1}

when $\sin\left(\frac{\pi}{7}(t-4)\right) = -1$

$\therefore \frac{\pi}{7}(t-4) = \frac{3\pi}{2}$

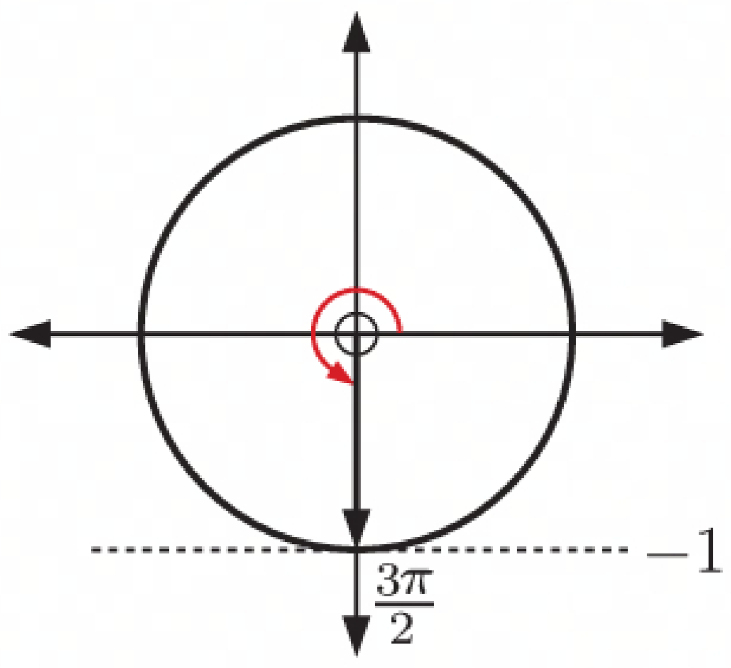
$\therefore \frac{t-4}{7} = \frac{3}{2}$

$\therefore 2t - 8 = 21$

$\therefore 2t = 29$

$\therefore t = 14.5 \pm 14k$ {period is 14 days}

So, the minimum occurred on the 1st day and the 15th day.



5 **a** For $H(t) = a\cos(b(t+c)) + d$:

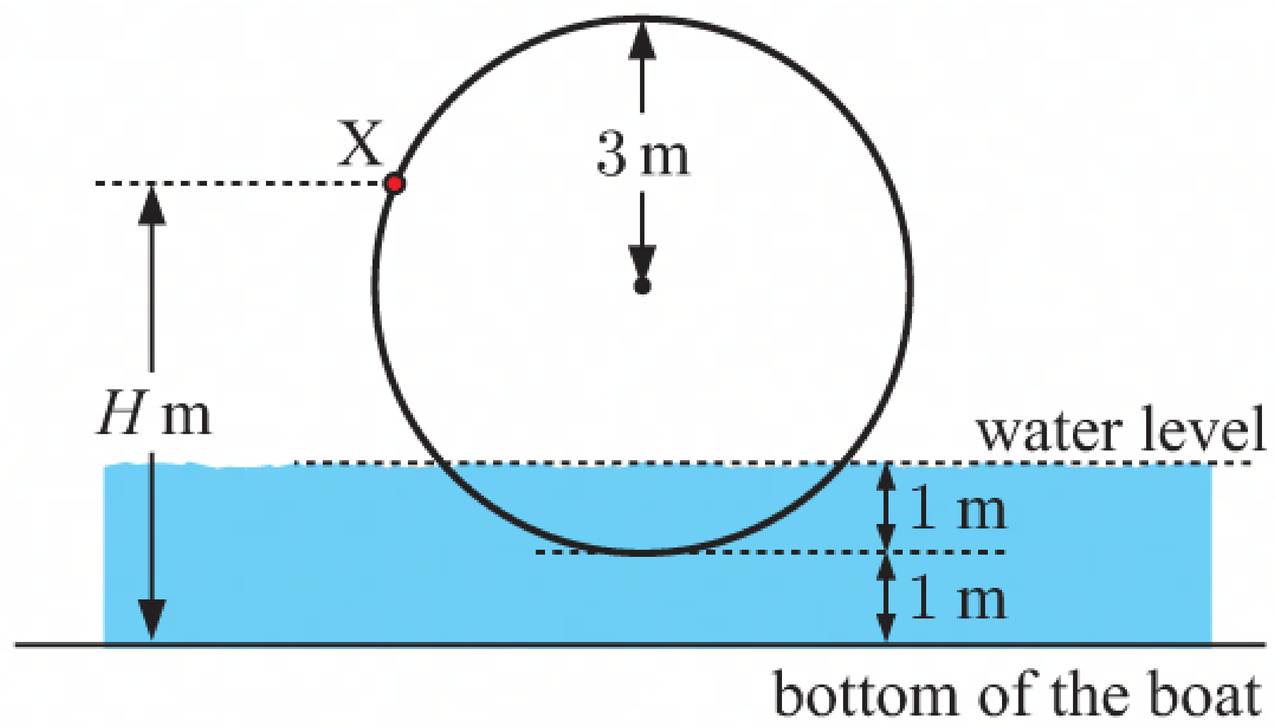
The period is $\frac{2\pi}{b} = 4$ s $\therefore b = \frac{\pi}{2}$

The amplitude $a = 3$

There is a vertical translation of $+4$ $\therefore d = 4$

There is no horizontal translation $\therefore c = 0$

$\therefore H(t) = 3\cos\left(\frac{\pi}{2}t\right) + 4$

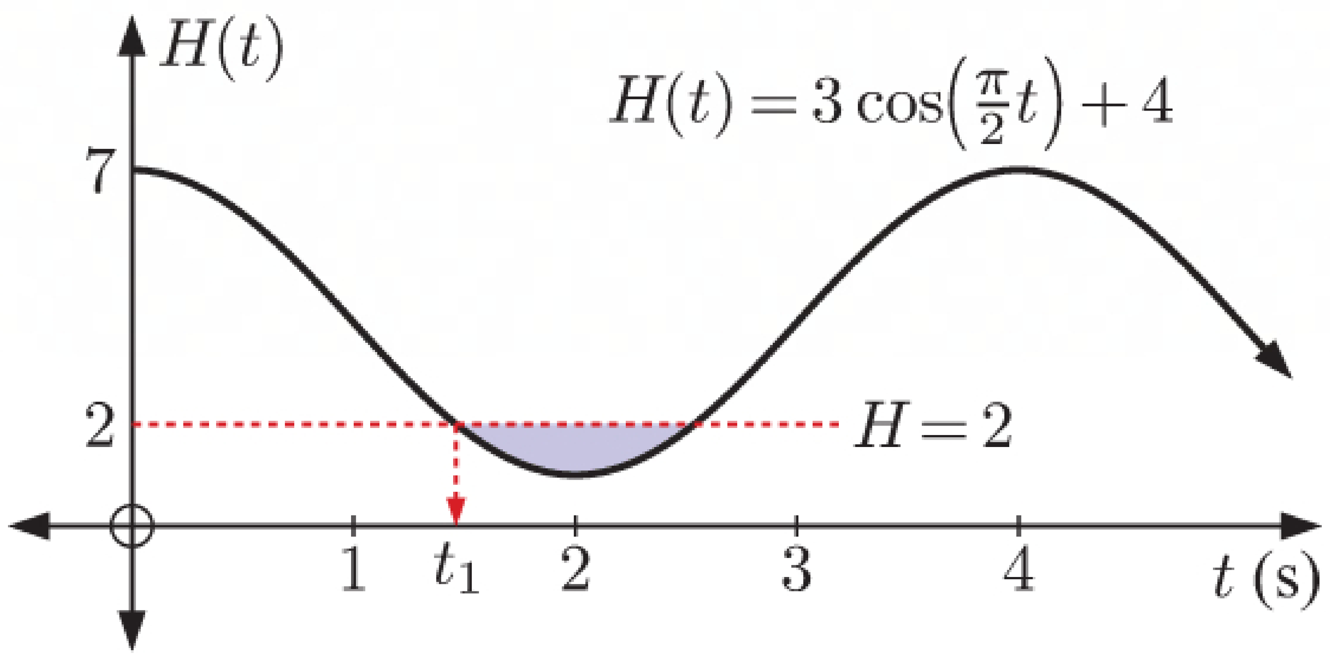


b X first enters the water when $H(t) = 2$

$\therefore 3\cos\left(\frac{\pi}{2}t\right) + 4 = 2$

Using technology, we obtain $t_1 \approx 1.46$.

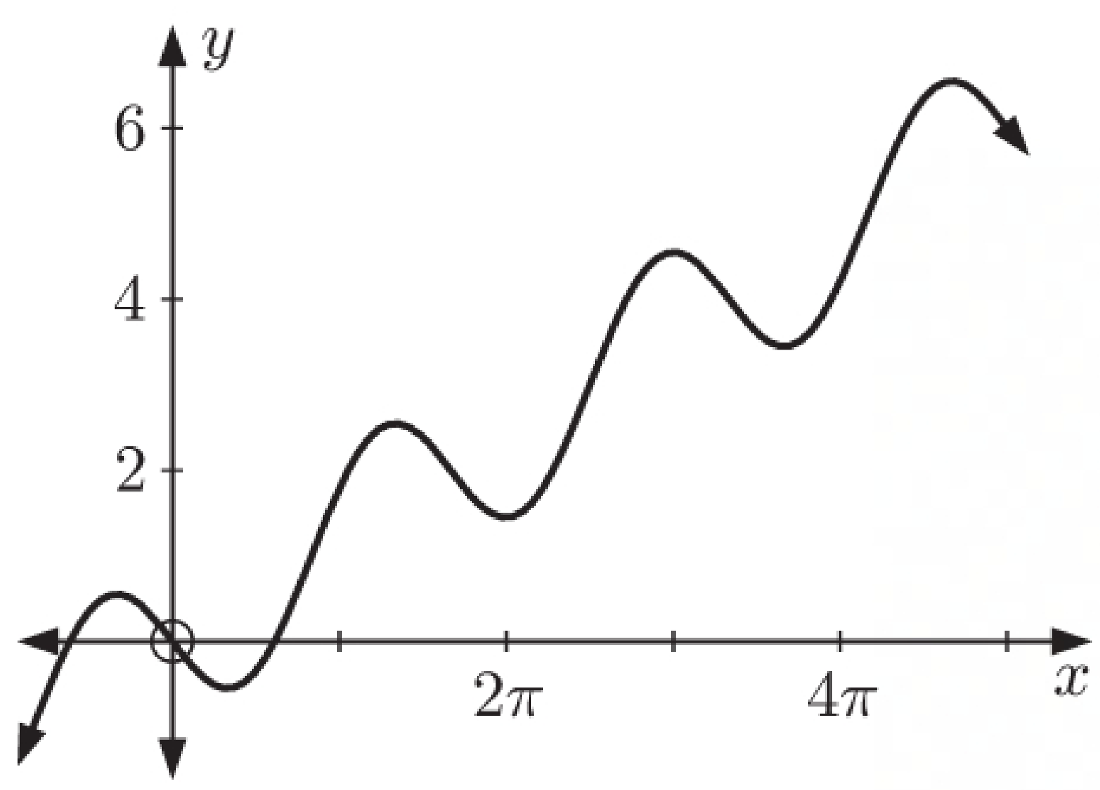
X first enters the water after about 1.46 seconds.



REVIEW SET 17A

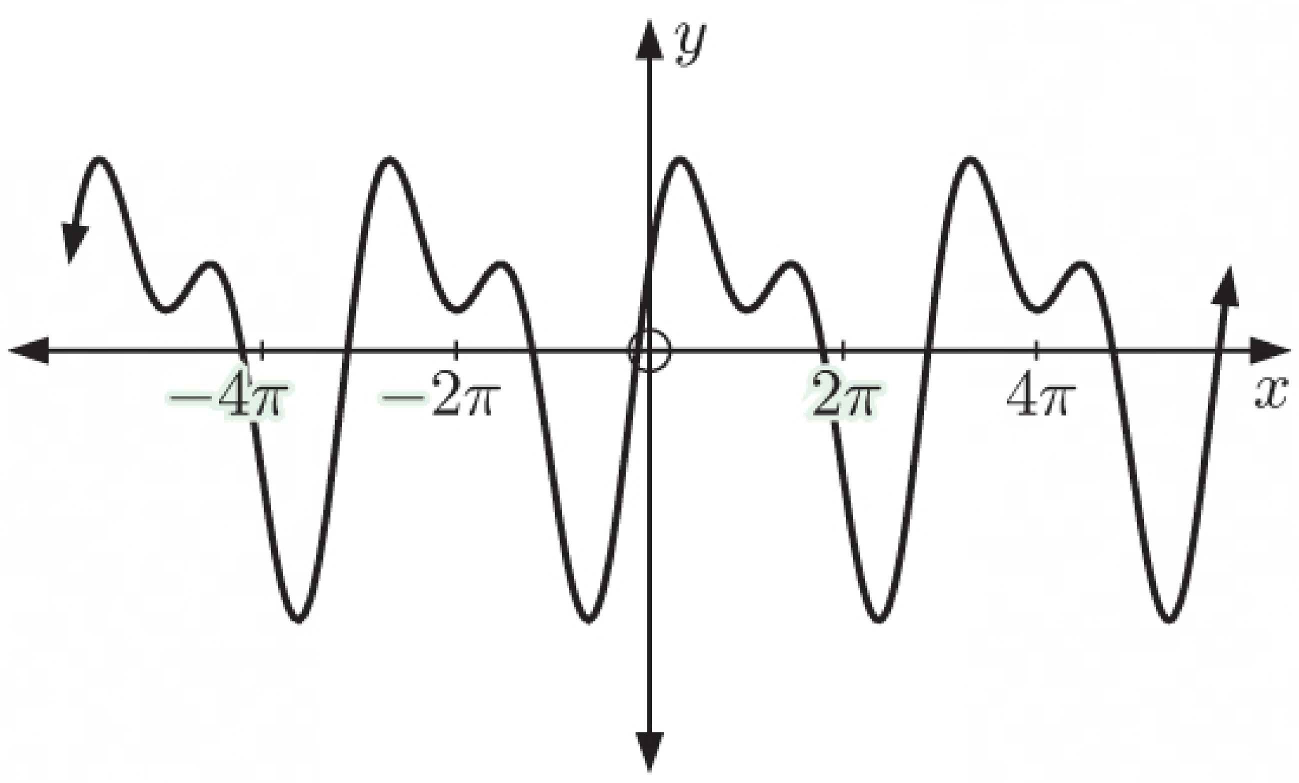
1 **a** This graph repeats itself over and over in intervals of the same length, but not in a horizontal direction.

$\therefore$ this graph does not show periodic behaviour.



b This graph repeats itself over and over in a horizontal direction, in intervals of the same length.

$\therefore$ this graph shows periodic behaviour.



2 **a** $1+\sin x$ has minimum value $1+(-1)=0$ {when $\sin x=-1$}
and maximum value $1+1=2$ {when $\sin x=1$}

b $-2\cos 3x$ has minimum value $-2(1)=-2$ {when $\cos 3x=1$}
and maximum value $-2(-1)=2$ {when $\cos 3x=-1$}

3 **a** $y=4\sin\frac{x}{5}$ has period $\frac{2\pi}{b}=\frac{2\pi}{(\frac{1}{5})}$
$=10\pi$

b $y=-2\cos 4x$ has period $\frac{2\pi}{b}=\frac{2\pi}{4}$
$=\frac{\pi}{2}$

c $y=4\cos\frac{x}{2}+4$ has period $\frac{2\pi}{b}=\frac{2\pi}{(\frac{1}{2})}$
$=4\pi$

d $y=\frac{1}{2}\tan 3x$ has period $\frac{\pi}{b}=\frac{\pi}{3}$

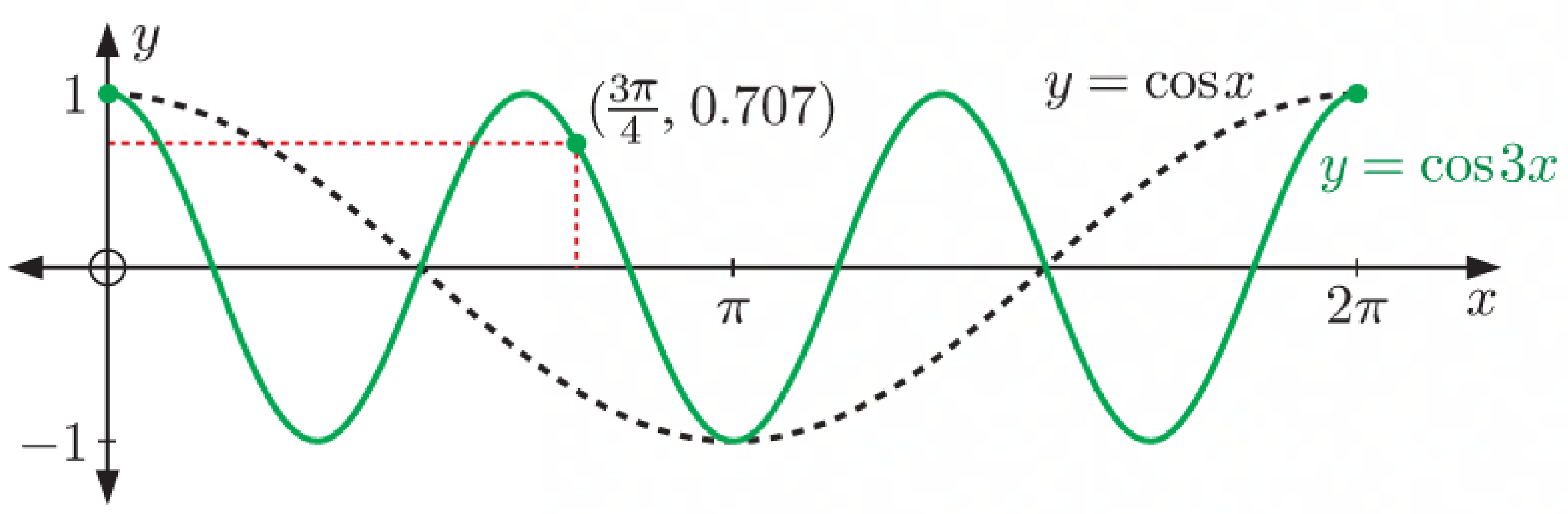
4 $y=-3\sin\frac{x}{4}+1$ has period $=\frac{2\pi}{b}=\frac{2\pi}{(\frac{1}{4})}=8\pi$
amplitude $=|-3|=3$
maximum value $=-3(-1)+1=4$ {when $\sin\frac{x}{4}=-1$}
and minimum value $=-3(1)+1=-2$ {when $\sin\frac{x}{4}=1$}

$y=3\cos\pi x$ has period $=\frac{2\pi}{b}=\frac{2\pi}{\pi}=2$
amplitude $=3$
maximum value $=3(1)=3$ {when $\cos\pi x=1$}
and minimum value $=3(-1)=-3$ {when $\cos\pi x=-1$}

So,

Function	*Period*	*Amplitude*	*Range*
$y=-3\sin\frac{x}{4}+1$	8π	3	$-2\leqslant y\leqslant 4$
$y=3\cos\pi x$	2	3	$-3\leqslant y\leqslant 3$

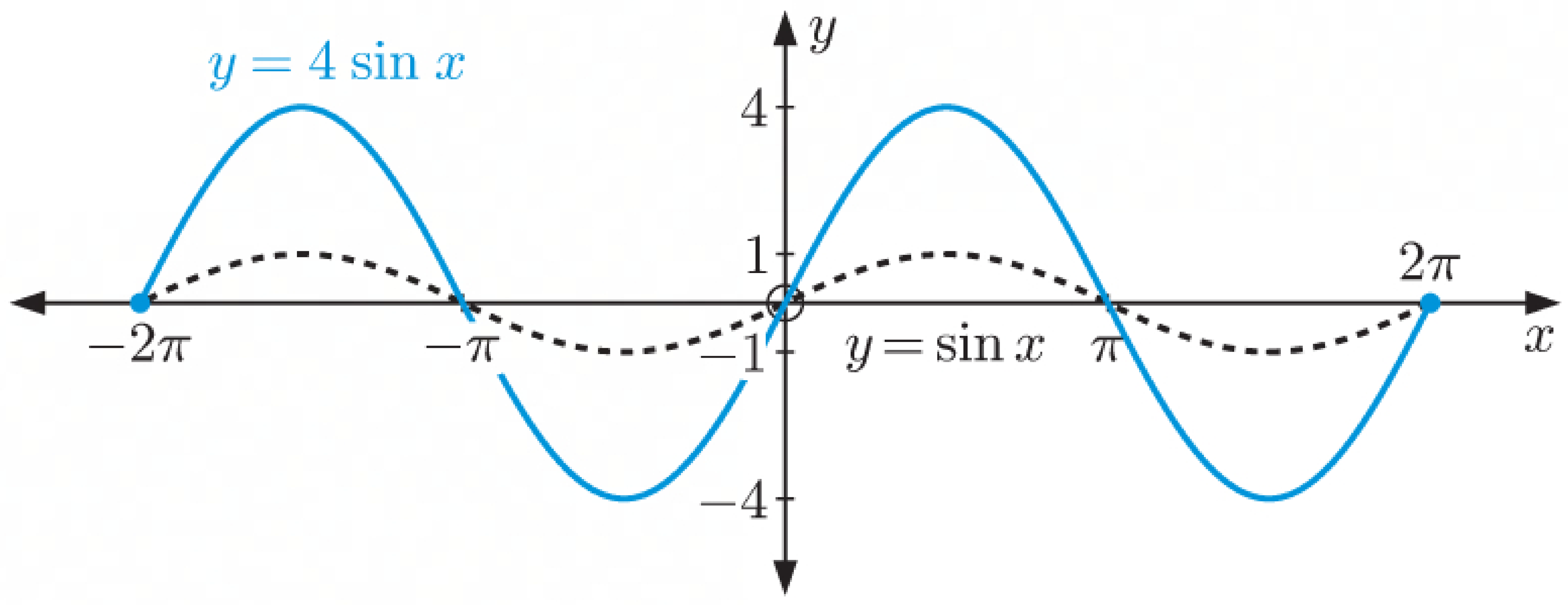
5 **a** $y=\cos 3x$ is a horizontal stretch of $y=\cos x$ with scale factor $\frac{1}{3}$.



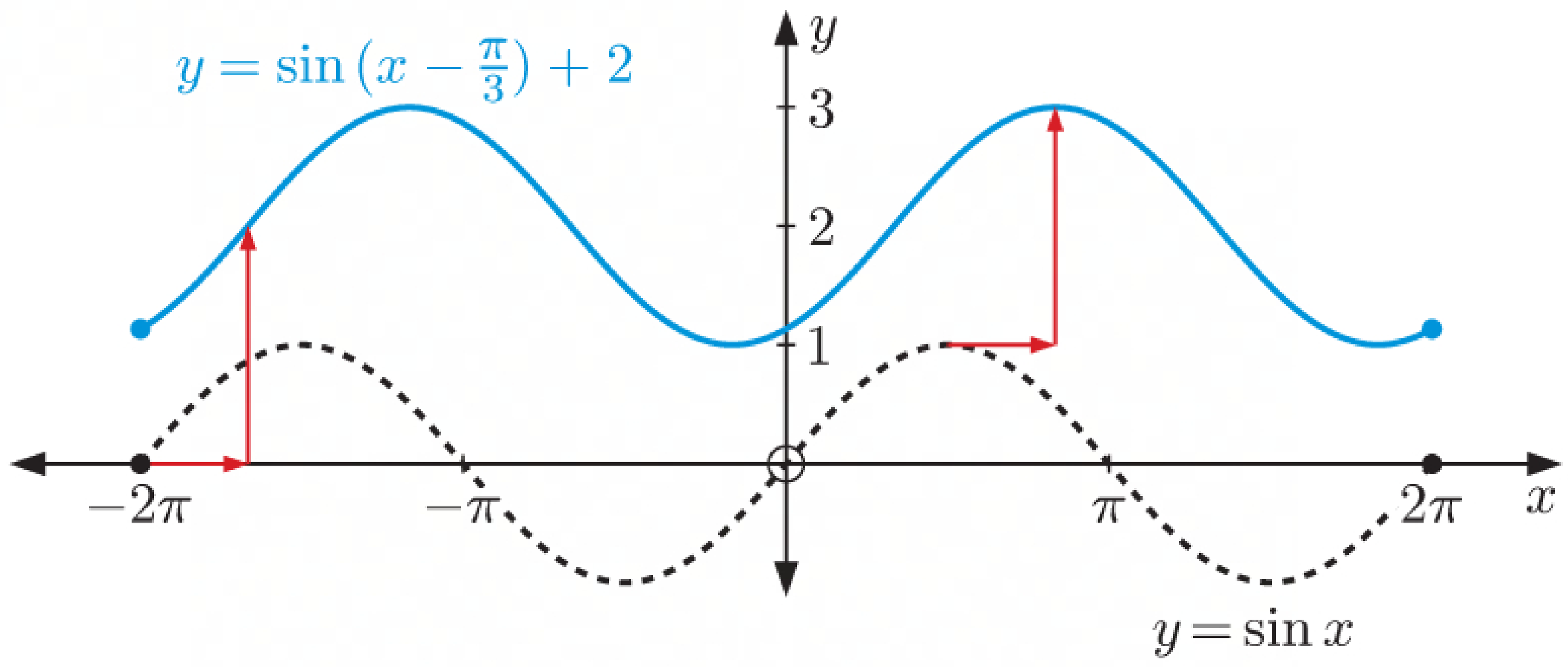
b When $x=\frac{3\pi}{4}$, $y=\cos 3\left(\frac{3\pi}{4}\right)$
$=\cos\frac{9\pi}{4}$
$=\cos\left(\frac{\pi}{4}+2\pi\right)$
$=\cos\frac{\pi}{4}$ {$\cos(\theta+2\pi)=\cos\theta$}
$=\frac{1}{\sqrt{2}}\approx 0.707$

6 **a** $a = 4$, so the amplitude is $|4| = 4$.

We stretch $y = \sin x$ vertically with scale factor 4 to give $y = 4\sin x$.

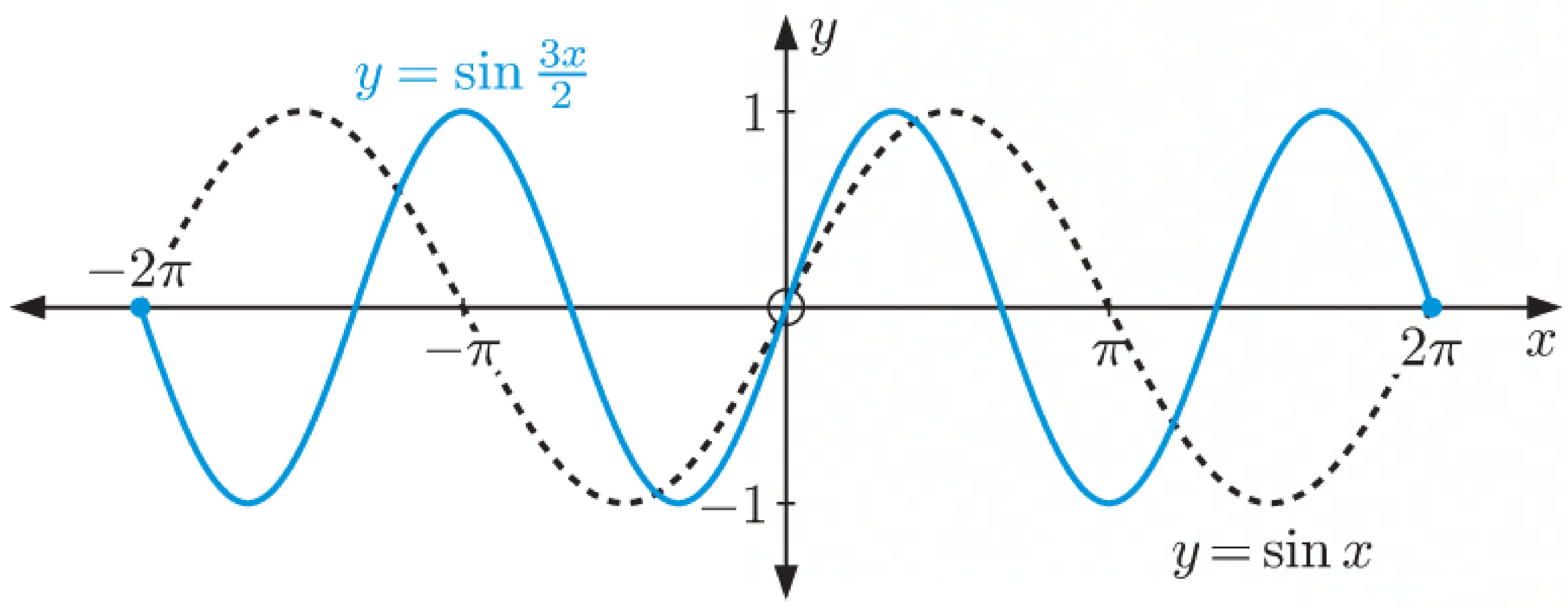


b We translate $y = \sin x$ $\frac{\pi}{3}$ units to the right and 2 units upwards to give $y = \sin\left(x - \frac{\pi}{3}\right) + 2$.

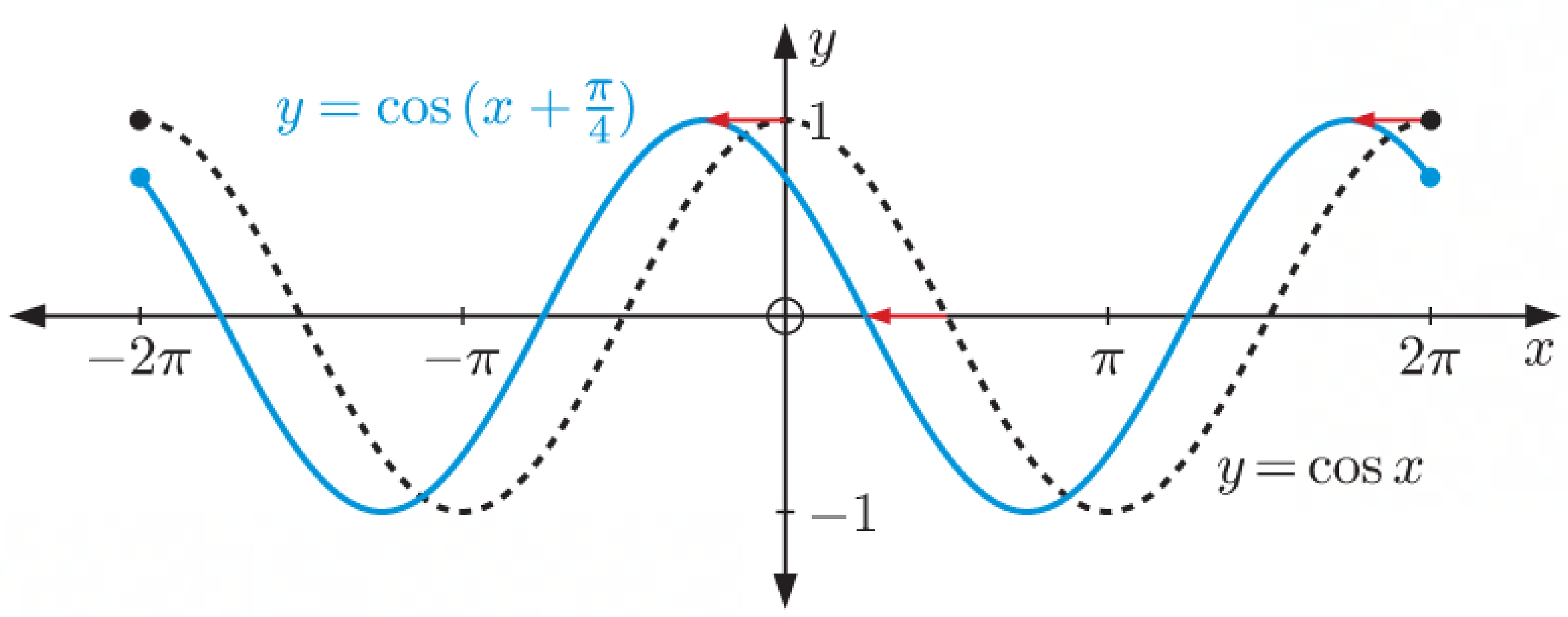


c $b = \frac{3}{2}$, so the period is $\frac{2\pi}{b} = \frac{2\pi}{\left(\frac{3}{2}\right)} = \frac{4\pi}{3}$.

We stretch $y = \sin x$ horizontally with scale factor $\frac{2}{3}$ to give $y = \sin\frac{3x}{2}$.

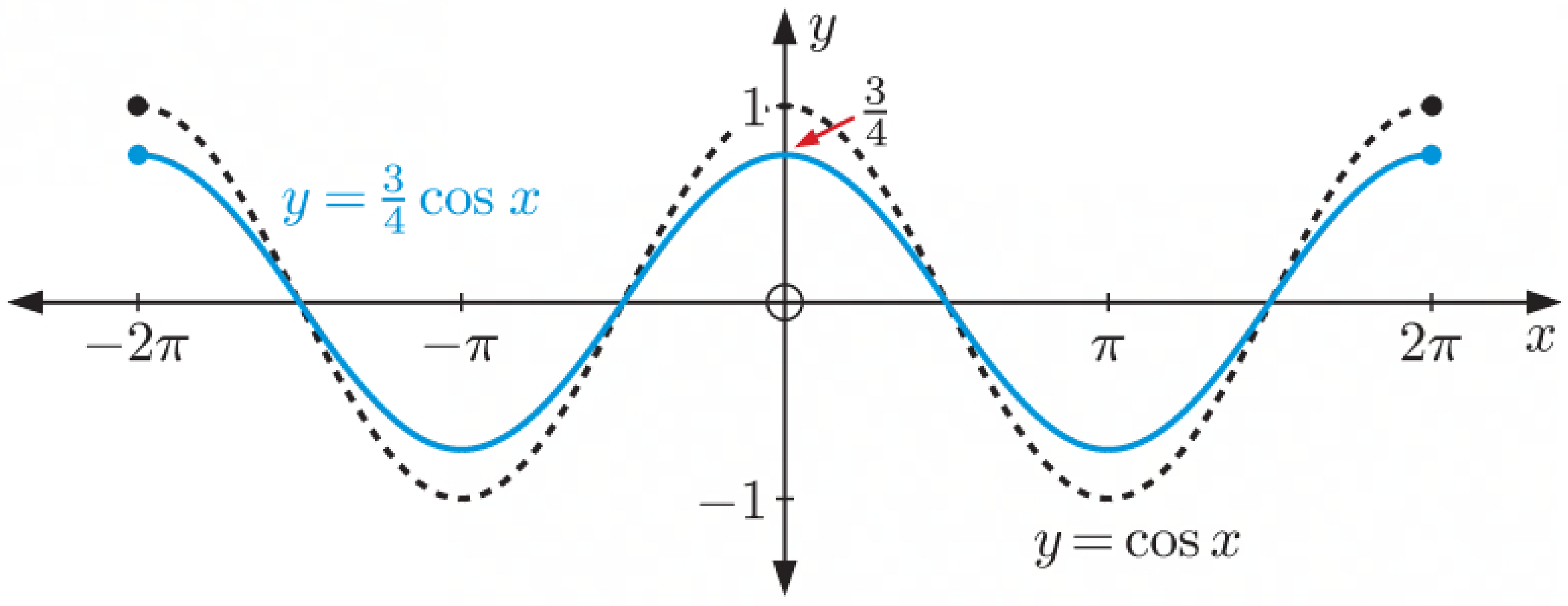


d We translate $y = \cos x$ $\frac{\pi}{4}$ units to the left to give $y = \cos\left(x + \frac{\pi}{4}\right)$.



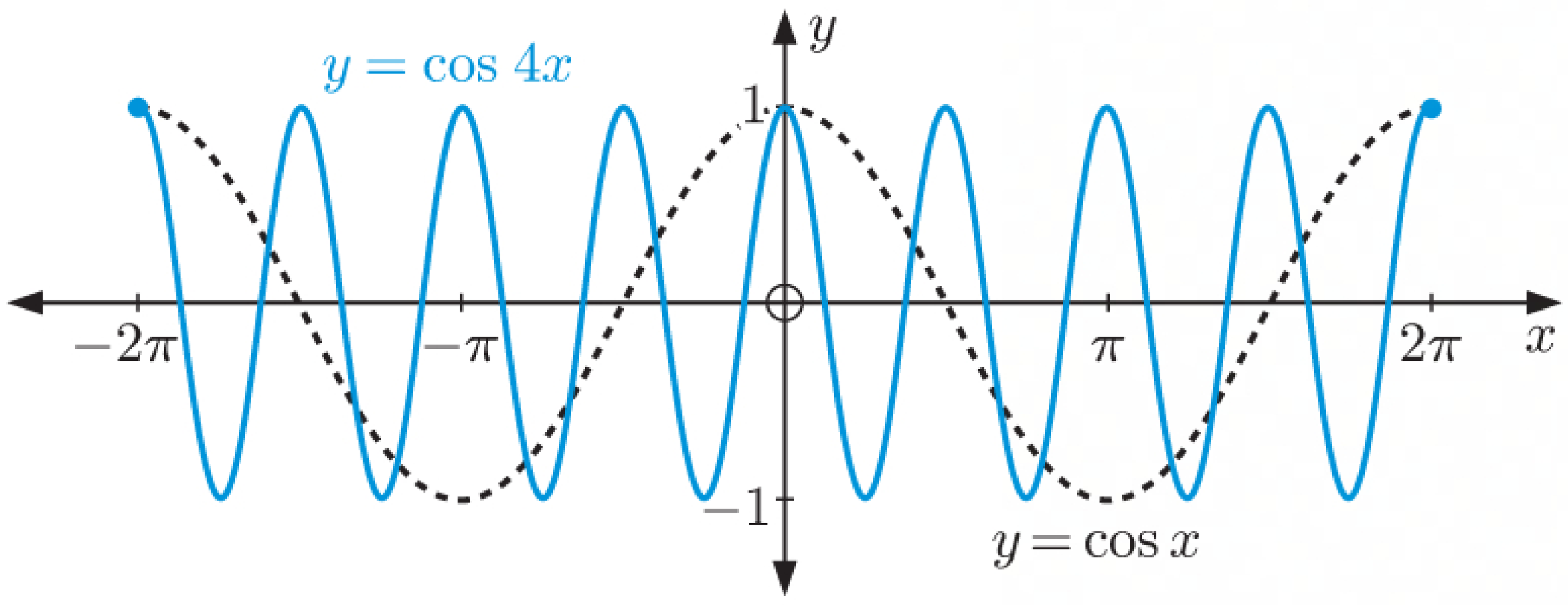
e $a = \frac{3}{4}$, so the amplitude is $\left|\frac{3}{4}\right| = \frac{3}{4}$.

We stretch $y = \cos x$ vertically with scale factor $\frac{3}{4}$ to give $y = \frac{3}{4}\cos x$.



f $b = 4$, so the period is $\frac{2\pi}{b} = \frac{2\pi}{4} = \frac{\pi}{2}$.

We stretch $y = \cos x$ horizontally with scale factor $\frac{1}{4}$ to give $y = \cos 4x$.



7 **a** $\sin x \xrightarrow[\text{scale factor } 3]{\text{vertical stretch}} 3\sin x \xrightarrow[\text{scale factor } \frac{1}{2}]{\text{horizontal stretch}} 3\sin 2x$

A vertical stretch with scale factor 3, then a horizontal stretch with scale factor $\frac{1}{2}$ maps $y = \sin x$ onto $y = 3\sin 2x$.

b $\cos x \xrightarrow{\text{translation} \begin{pmatrix} \frac{\pi}{3} \\ -1 \end{pmatrix}} \cos\left(x - \frac{\pi}{3}\right) - 1$

A translation $\frac{\pi}{3}$ units right and 1 unit downwards maps $y = \cos x$ onto $y = \cos\left(x - \frac{\pi}{3}\right) - 1$.

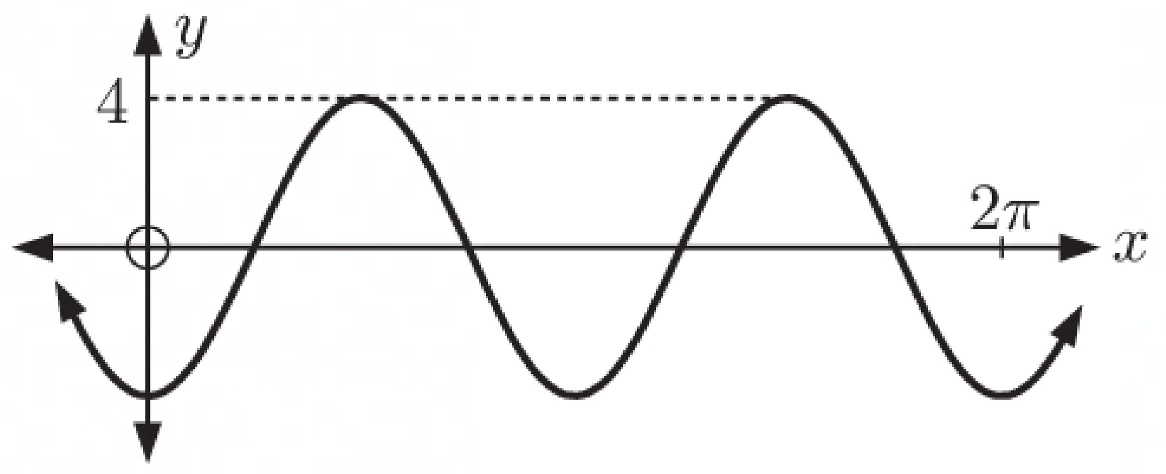
c $\tan x \xrightarrow[\text{in } x\text{-axis}]{\text{reflection}} -\tan x \xrightarrow[\text{scale factor } \frac{1}{2}]{\text{horizontal stretch}} -\tan 2x$

A reflection in the x-axis, then a horizontal stretch with scale factor $\frac{1}{2}$ maps $y = \tan x$ onto $y = -\tan 2x$.

d $\sin x \xrightarrow[\text{scale factor } 2]{\text{vertical stretch}} 2\sin x \xrightarrow{\text{translation} \begin{pmatrix} \frac{\pi}{4} \\ \frac{1}{2} \end{pmatrix}} 2\sin\left(x - \frac{\pi}{4}\right) + \frac{1}{2} \xrightarrow[\text{scale factor } 2]{\text{horizontal stretch}} 2\sin\left(\frac{x}{2} - \frac{\pi}{4}\right) + \frac{1}{2}$

A vertical stretch with scale factor 2, then a translation $\frac{\pi}{4}$ units right and $\frac{1}{2}$ unit upwards, then a horizontal stretch with scale factor 2 maps $y = \sin x$ onto $y = 2\sin\left(\frac{x}{2} - \frac{\pi}{4}\right) + \frac{1}{2}$.

8 **a**



The amplitude is 4, and $y < 0$ when $x = 0$, so $a = -4$.

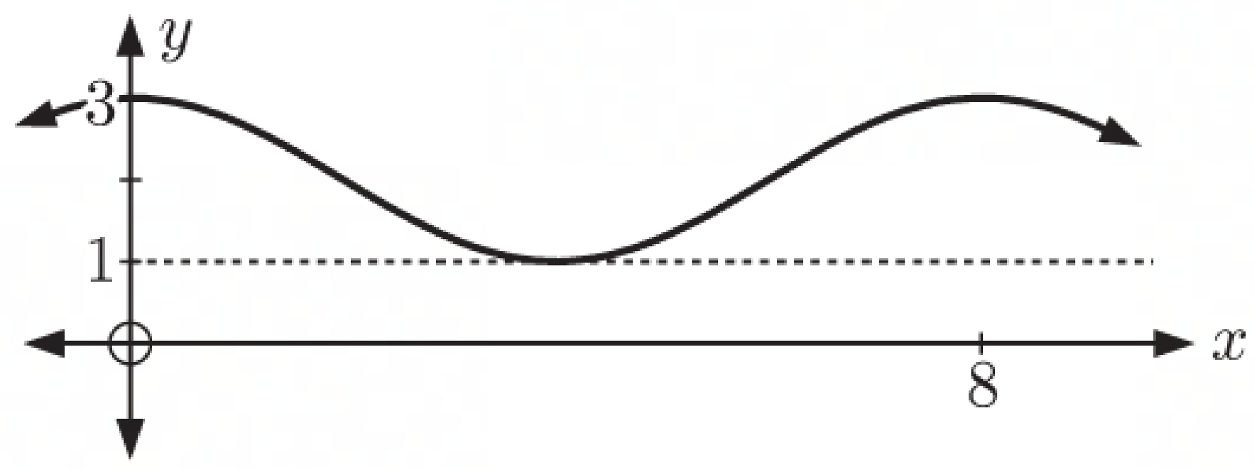
The period is π, so $\dfrac{2\pi}{b} = \pi$ and $\therefore$ $b = 2$.

There is no horizontal translation, so $c = 0$.

The principal axis is $y = 0$, so $d = 0$.

$\therefore$ the equation of the function is $y = -4\cos 2x$.

b



The amplitude is 1, so $a = 1$.

The period is 8, so $\dfrac{2\pi}{b} = 8$ and $\therefore$ $b = \frac{\pi}{4}$.

There is no horizontal translation, so $c = 0$.

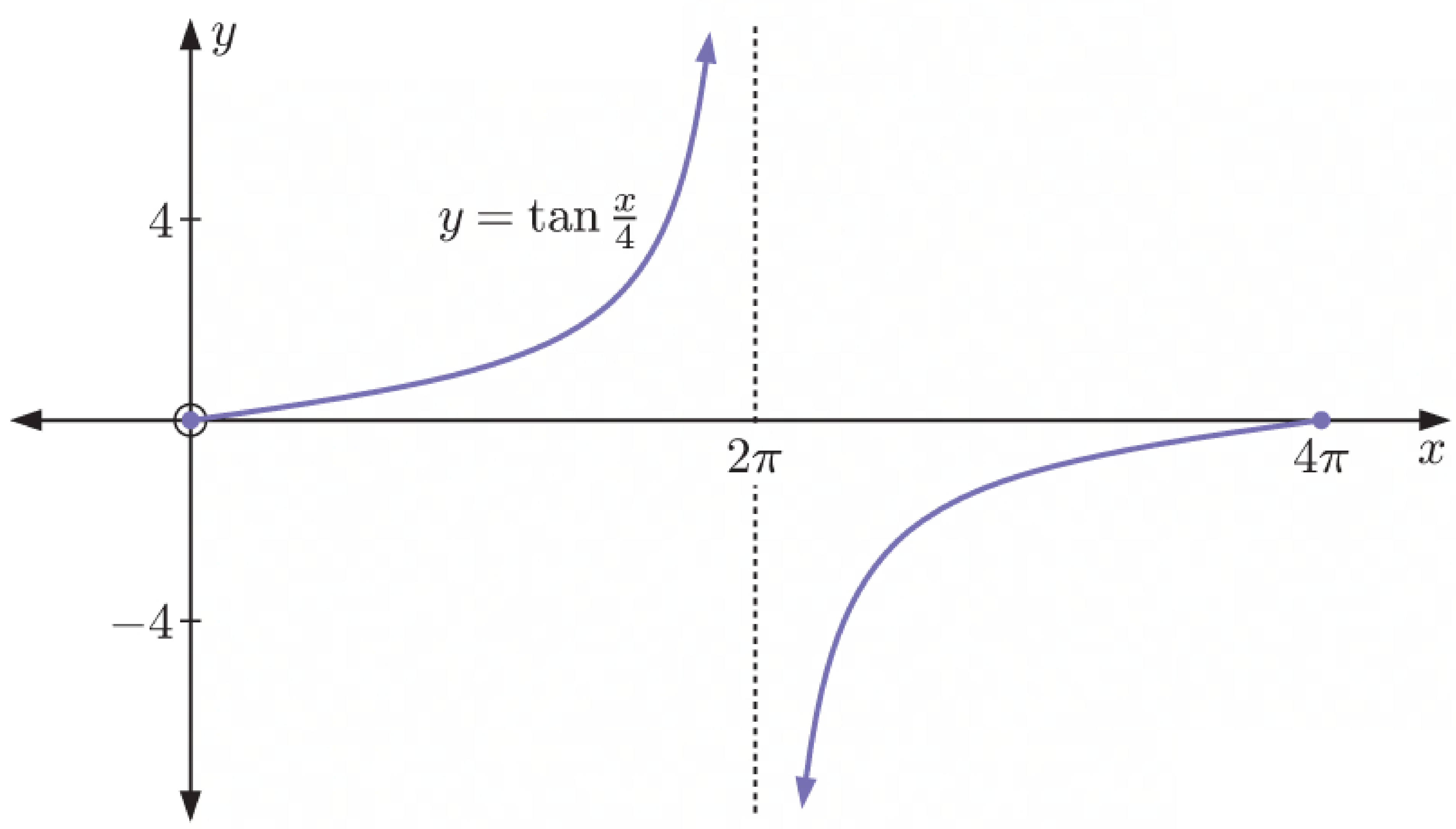
The principal axis is $y = 2$, so $d = 2$.

$\therefore$ the equation of the function is $y = \cos \frac{\pi x}{4} + 2$.

9 **a** $y = \tan \frac{x}{4}$ is a horizontal stretch of $y = \tan x$ with scale factor 4.

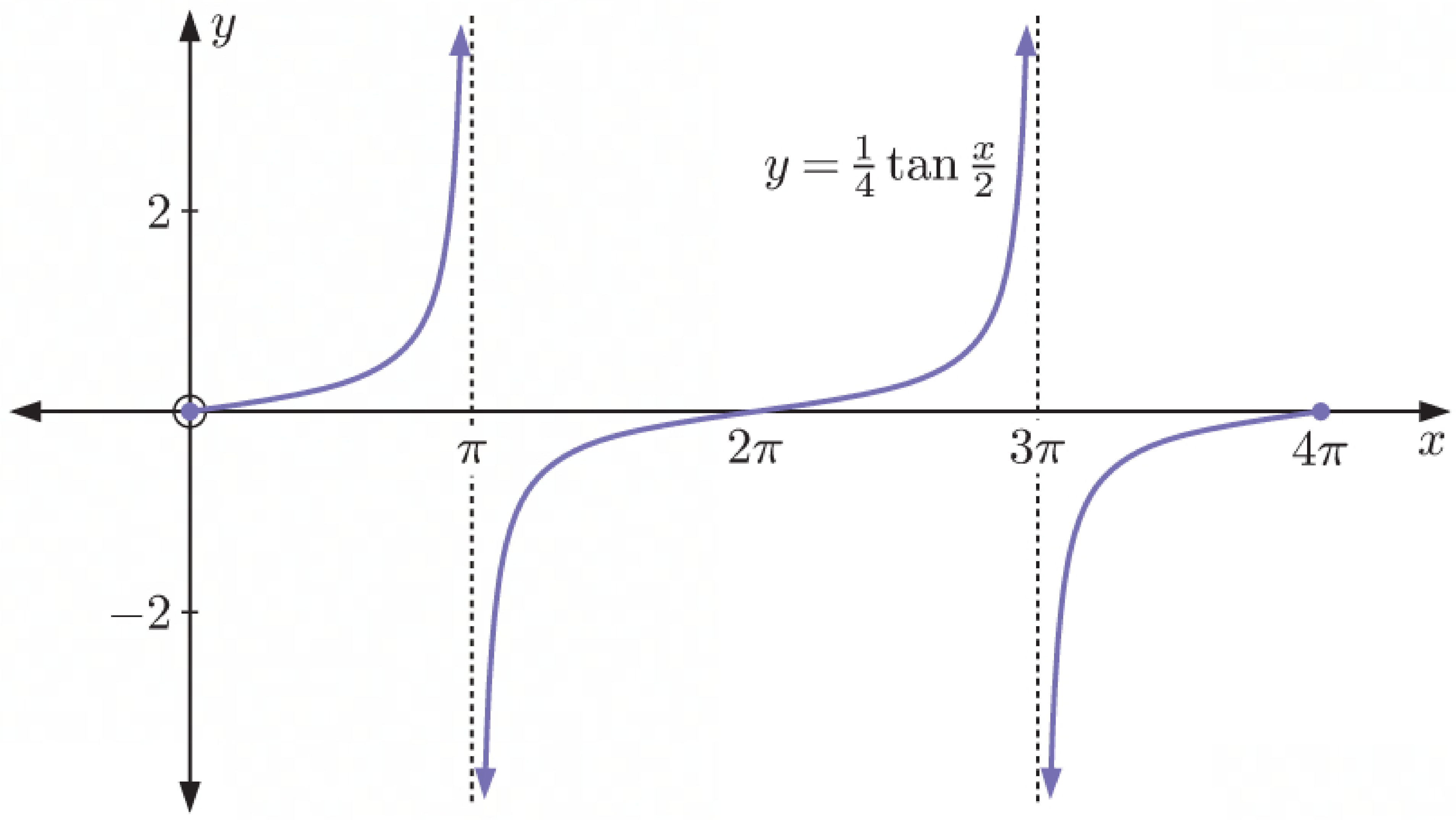
$y = \tan x$ has vertical asymptotes $x = \frac{\pi}{2}$, $x = \frac{3\pi}{2}$, $x = \frac{5\pi}{2}$, $x = \frac{7\pi}{2}$, and x-intercepts 0, π, 2π, 3π, and 4π.

$\therefore$ $y = \tan \frac{x}{4}$ has vertical asymptote $x = 2\pi$, and x-intercepts 0 and 4π.



b $y = \frac{1}{4}\tan\frac{x}{2}$ is a horizontal stretch of $y = \tan x$ with scale factor 2, followed by a vertical stretch with scale factor $\frac{1}{4}$.

$y = \frac{1}{4}\tan\frac{x}{2}$ has vertical asymptotes $x = \pi$, $x = 3\pi$, and x-intercepts 0, 2π, and 4π.

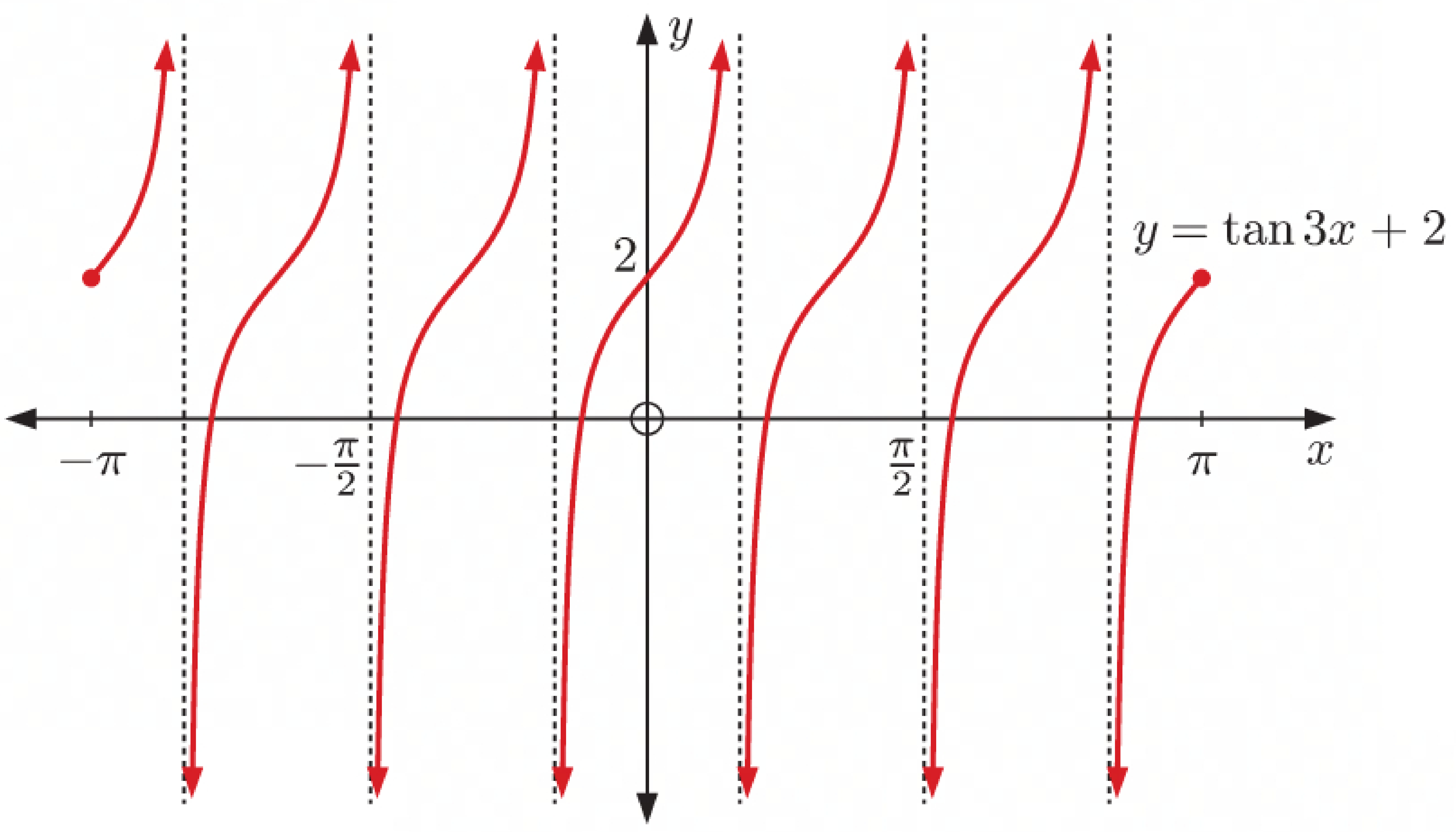


10 **a** $\tan x \xrightarrow[\text{scale factor } \frac{1}{3}]{\text{horizontal stretch}} \tan 3x \xrightarrow{\text{translation } \binom{0}{2}} \tan 3x + 2$

A horizontal stretch with scale factor $\frac{1}{3}$, then a vertical translation 2 units upwards maps $y = \tan x$ onto $y = \tan 3x + 2$.

b $y = \tan 3x + 2$ has period $\frac{\pi}{b} = \frac{\pi}{3}$

c



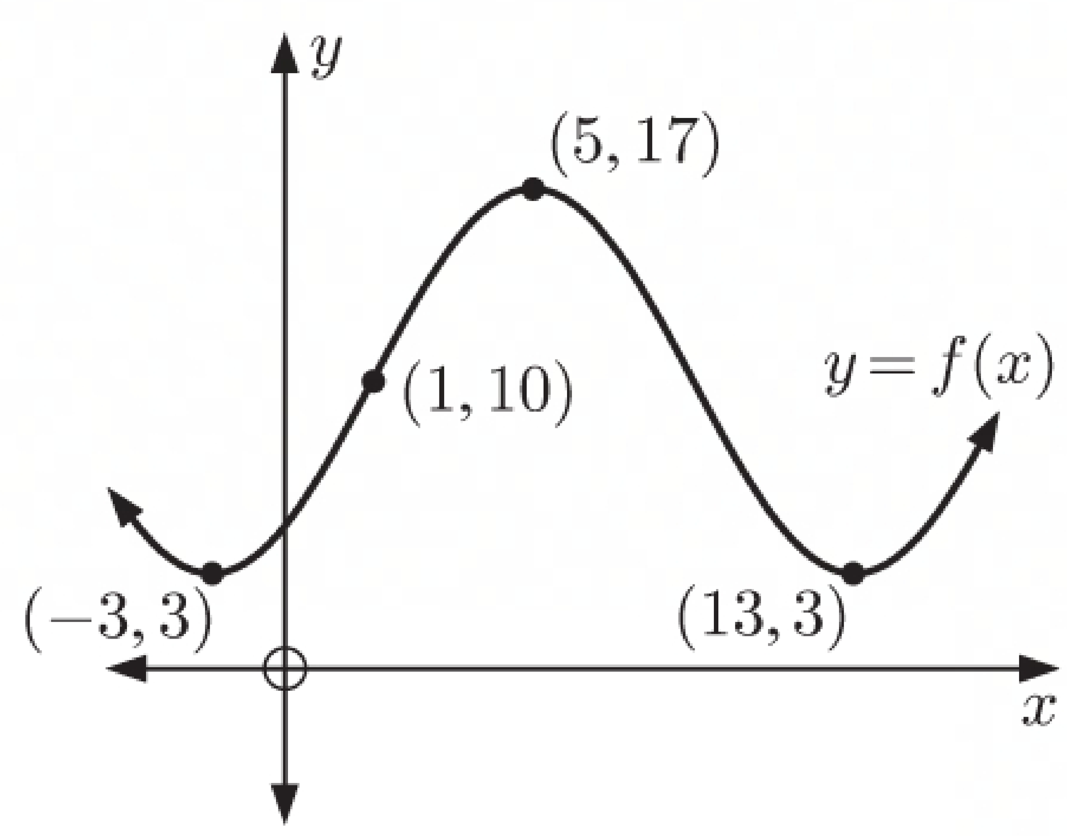
11 **a** The amplitude $= \dfrac{\max - \min}{2} = \dfrac{17-3}{2} = 7$, so $a = 7$.

The period is $\dfrac{2\pi}{b} = 13 - (-3)$

$\therefore \dfrac{2\pi}{b} = 16$

$\therefore \dfrac{b}{2\pi} = \dfrac{1}{16}$

$\therefore b = \frac{\pi}{8}$



The principal axis is $y = \dfrac{\max + \min}{2} = \dfrac{17+3}{2} = 10$,
so $d = 10$.

The point midway between $(-3, 3)$ and $(5, 17)$ is $(1, 10)$, so there is a horizontal translation of 1 unit to the right, and thus $c = 1$.

So, $a = 7$, $b = \frac{\pi}{8}$, $c = 1$, and $d = 10$.

b $f(x) = 7\sin\left(\frac{\pi}{8}(x-1)\right) + 10$

$$f(x) \xrightarrow{\text{translation } \begin{pmatrix} 2 \\ -3 \end{pmatrix}} f(x-2)-3 \xrightarrow[\text{scale factor } 2]{\text{vertical stretch}} 2[f(x-2)-3]$$

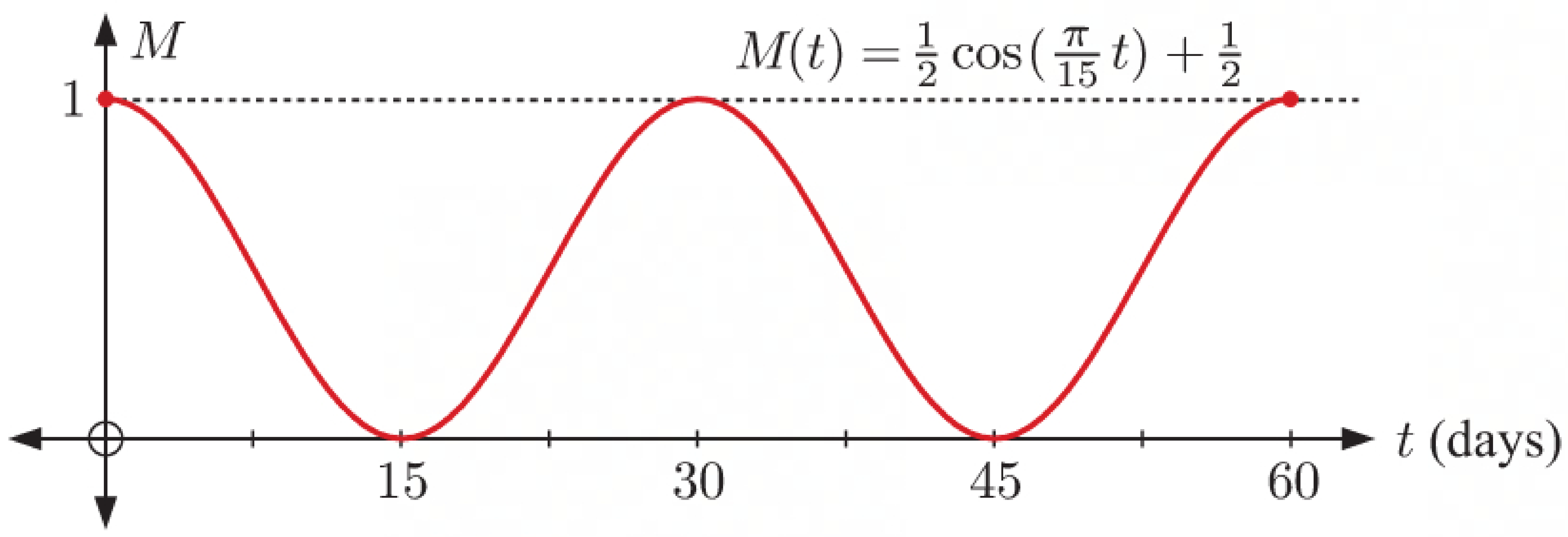
So, $g(x) = 2[f(x-2)-3]$

$= 2[7\sin\left(\frac{\pi}{8}((x-2)-1)\right) + 10 - 3]$

$= 2[7\sin\left(\frac{\pi}{8}(x-3)\right) + 7]$

$= 14\sin\left(\frac{\pi}{8}(x-3)\right) + 14$

12 **a**



b **i** January 6th is 5 days after January 1st, so $t = 5$.

$M(5) = \frac{1}{2}\cos\left(\frac{\pi}{15} \times 5\right) + \frac{1}{2}$

$= \frac{1}{2}\cos\frac{\pi}{3} + \frac{1}{2}$

$= \frac{1}{2} \times \frac{1}{2} + \frac{1}{2}$

$= \frac{1}{4} + \frac{1}{2}$

$= \frac{3}{4} = 0.75$

$\therefore$ the proportion of the moon illuminated on the night of January 6th is 0.75.

ii January 21st is 20 days after January 1st, so $t = 20$.

$M(20) = \frac{1}{2}\cos\left(\frac{\pi}{15} \times 20\right) + \frac{1}{2}$

$= \frac{1}{2}\cos\frac{4\pi}{3} + \frac{1}{2}$

$= \frac{1}{2} \times \left(-\frac{1}{2}\right) + \frac{1}{2}$

$= -\frac{1}{4} + \frac{1}{2}$

$= \frac{1}{4} = 0.25$

$\therefore$ the proportion of the moon illuminated on the night of January 21st is 0.25.

iii January 27th is 26 days after January 1st, so $t = 26$.

$M(26) = \frac{1}{2}\cos\left(\frac{\pi}{15} \times 26\right) + \frac{1}{2}$

≈ 0.835

$\therefore$ the proportion of the moon illuminated on the night of January 27th is 0.835.

iv February 19th is 49 days after January 1st, so $t = 49$.

$M(49) = \frac{1}{2}\cos\left(\frac{\pi}{15} \times 49\right) + \frac{1}{2}$

≈ 0.165

$\therefore$ the proportion of the moon illuminated on the night of February 19th is 0.165.

c The period is $\frac{2\pi}{b} = \frac{2\pi}{(\frac{\pi}{15})} = 30$ days.

$\therefore$ a full moon occurs once every 30 days.

d $M(t) = 0$ when $\frac{1}{2}\cos\left(\frac{\pi}{15}t\right) + \frac{1}{2} = 0$

$$\therefore \frac{1}{2}\cos\left(\frac{\pi}{15}t\right) = -\frac{1}{2}$$
$$\therefore \cos\left(\frac{\pi}{15}t\right) = -1$$
$$\therefore \frac{\pi}{15}t = \pi, 3\pi, 5\pi, ...$$

which is true when $t = 15$ or 45 $\{0 \leqslant t \leqslant 60\}$

$t = 15$ corresponds to January 16, and $t = 45$ corresponds to February 15.

$\therefore$ the moon is not illuminated at all on January 16th and February 15th.

13 **a** The mean temperature $= \frac{14.1 + 6.7}{2} = 10.4°\text{C}$, so $d = 10.4$.

The amplitude $= \frac{14.1 - 6.7}{2} = 3.7°\text{C}$

$$\therefore a = 3.7$$

The period is 24 hours, so $b = \frac{2\pi}{24} = \frac{\pi}{12}$.

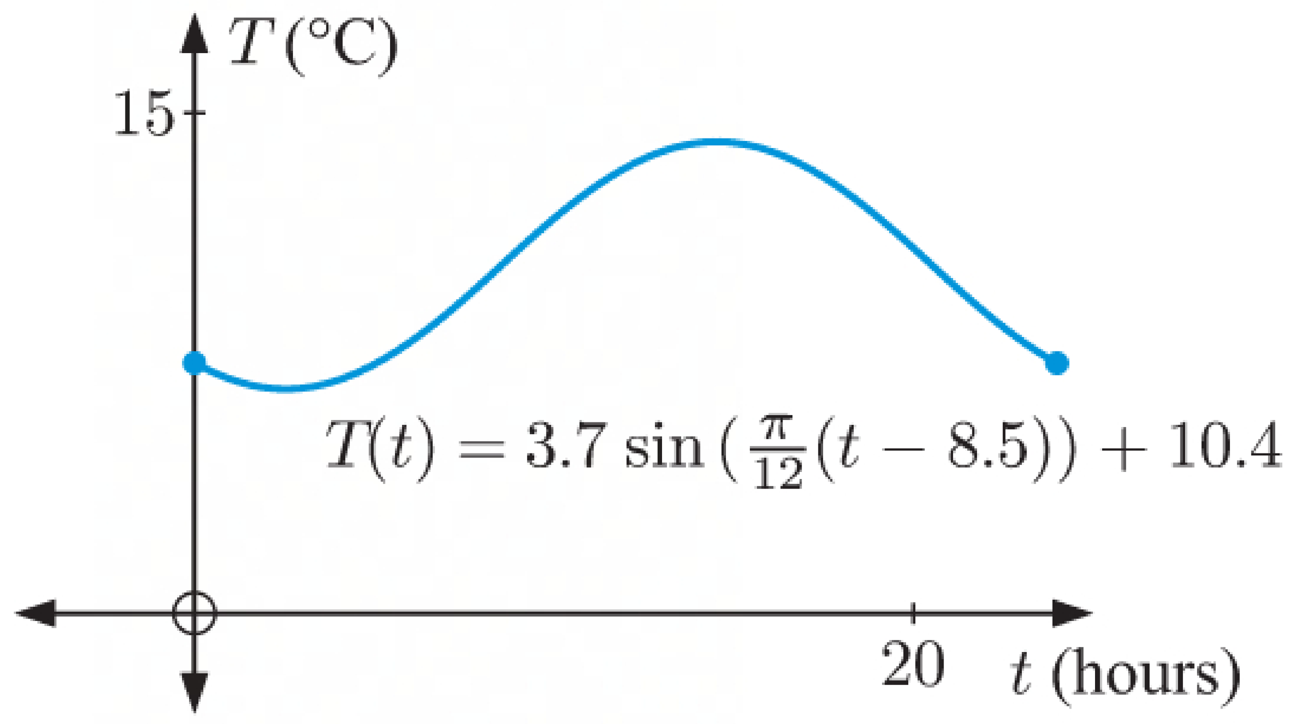
The maximum occurs at 2:30 pm, so we assume the temperature passed its mean value 6 hours earlier, at 8:30 am.

The day begins at midnight, so the function is shifted $8\frac{1}{2}$ hours to the right, thus $c = 8.5$.

If t is the number of hours after midnight, the temperature T is modelled by

$$T(t) = 3.7\sin\left(\frac{\pi}{12}(t - 8.5)\right) + 10.4 \text{ °C}.$$

b



14

Number of Mars days (n)	0	100	200	300	400	500	600	700	800	900	1000	1100	1200	1300
Temp. (°C)	-43	-15	-5	-21	-59	-79	-68	-50	-27	-8	-15	-70	-78	-68

a The maximum temperature recorded was $-5°\text{C}$, the minimum temperature recorded was $-79°\text{C}$.

b The time between maximum values is $900 - 200 = 700$ so we estimate the length of a Mars year to be about 700 Mars days.

c We estimate that the period is 700 Mars days, so $b \approx \frac{2\pi}{700}$.

The amplitude $= \frac{\max - \min}{2} \approx \frac{-5 - (-79)}{2} \approx 37$, so $a \approx 37$.

The principal axis is midway between the maximum and minimum, so

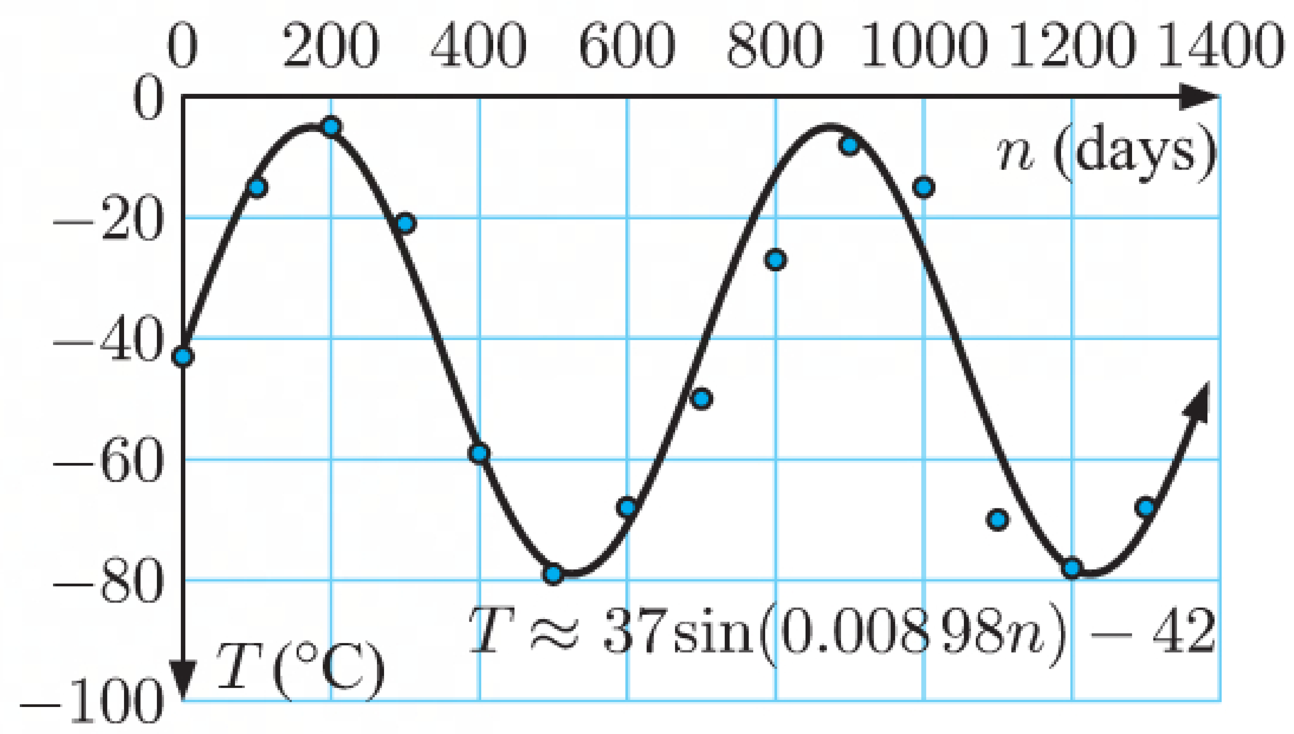
$d \approx \frac{-5 + (-79)}{2} \approx -42$.

The first minimum occurs at $n \approx 500$ and the following maximum occurs at $n \approx 900$. The sine function starts a new period midway between these two points.

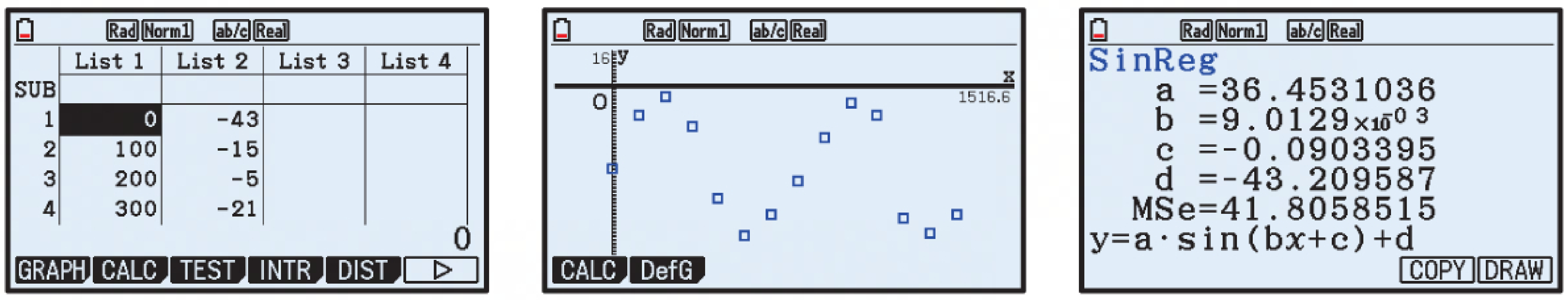
$$\therefore \quad c \approx \frac{500 + 900}{2} \approx 700$$

$$\begin{aligned}\therefore \quad T &\approx 37\sin\left(\tfrac{2\pi}{700}(n - 700)\right) - 42\\ &\approx 37\sin\left(\tfrac{2\pi n}{700} - 2\pi\right) - 42\\ &\approx 37\sin \tfrac{2\pi n}{700} - 42\\ &\approx 37\sin(0.008\,98n) - 42\end{aligned}$$

d



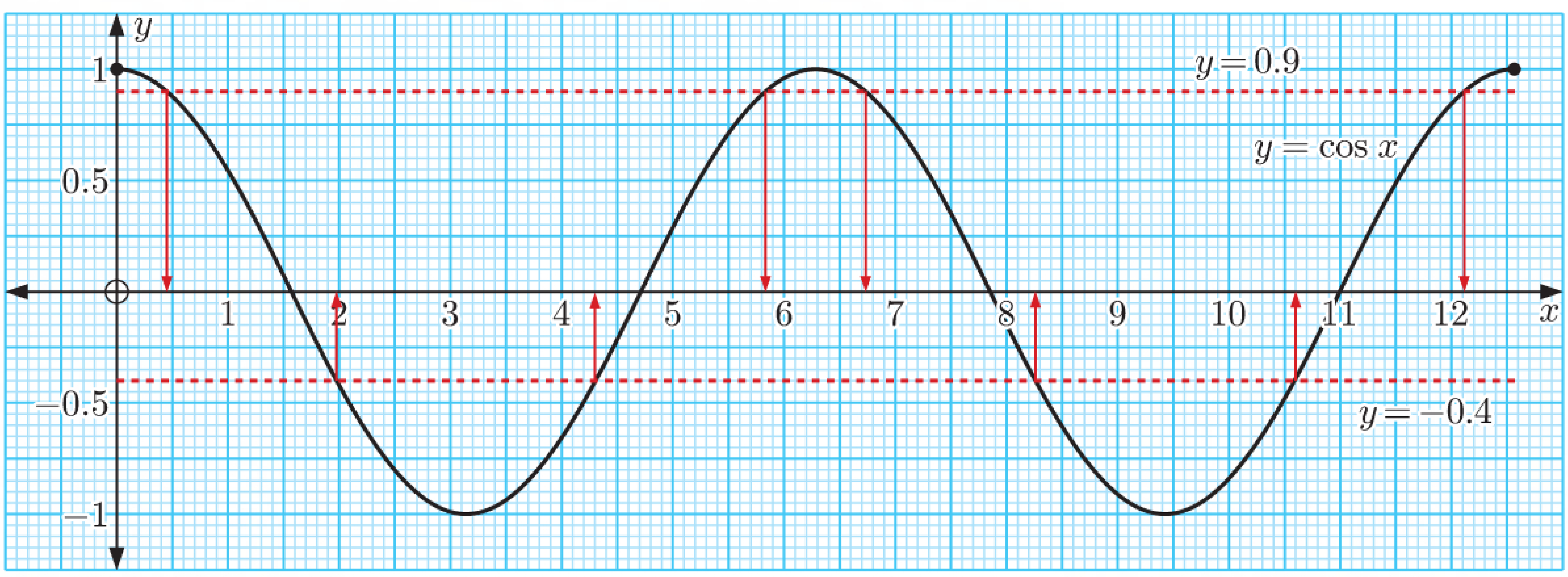
e



Using technology, $T \approx 36.5\sin(0.009\,01n - 0.0903) - 43.2$.

Our model fits the data reasonably well.

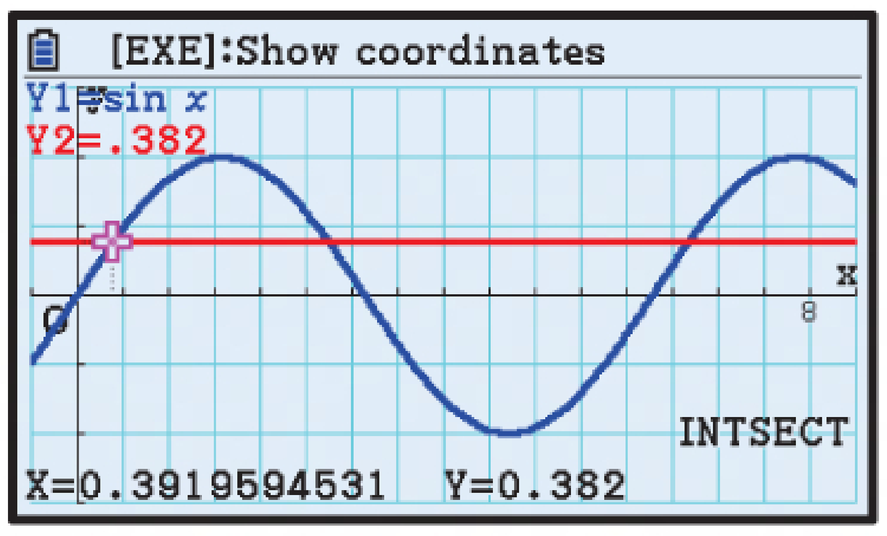
15

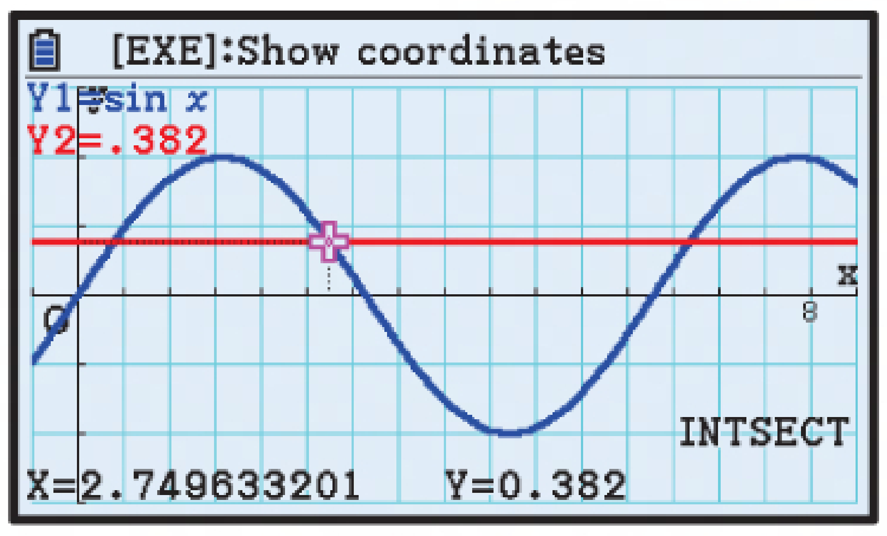


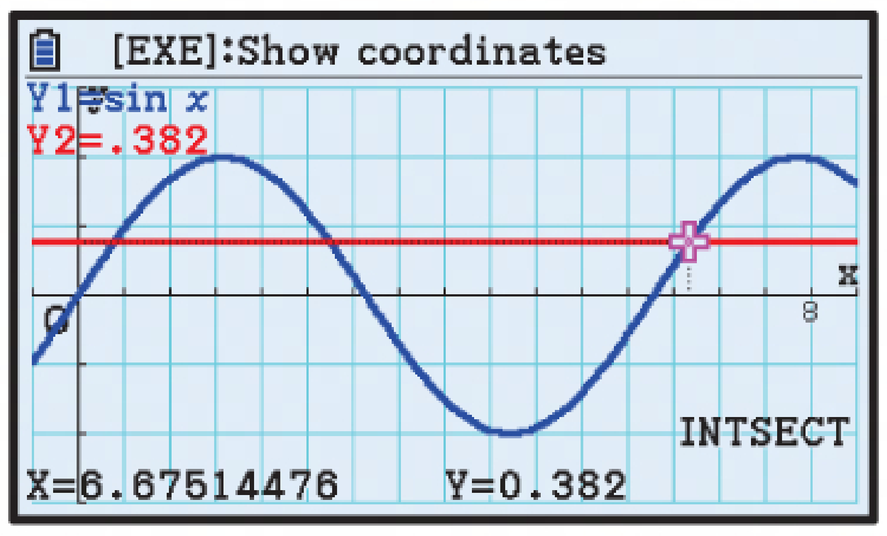
a When $\cos x = -0.4$, $0 \leqslant x \leqslant 4\pi$, $x \approx 2.0, 4.3, 8.3, 10.6$

b When $\cos x = 0.9$, $0 \leqslant x \leqslant 4\pi$, $x \approx 0.5, 5.8, 6.7, 12.1$

16 **a** We graph the functions $Y_1 = \sin X$ and $Y_2 = 0.382$ on the same set of axes.
We use **window** settings just larger than the domain.
In this case, $X\text{min} = -0.5$, $X\text{max} = 8.5$, $X\text{scale} = 0.5$.

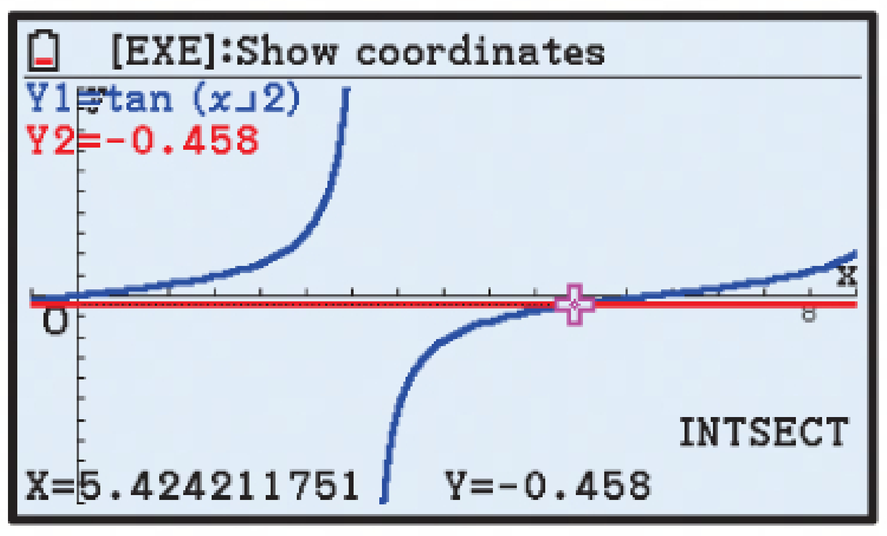






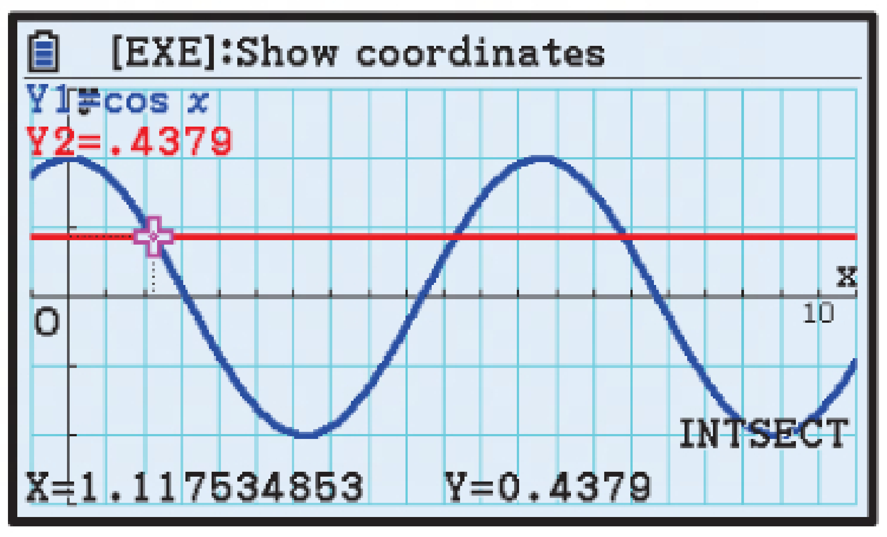
The solutions are $x \approx 0.392,\ 2.75,\ 6.68$.

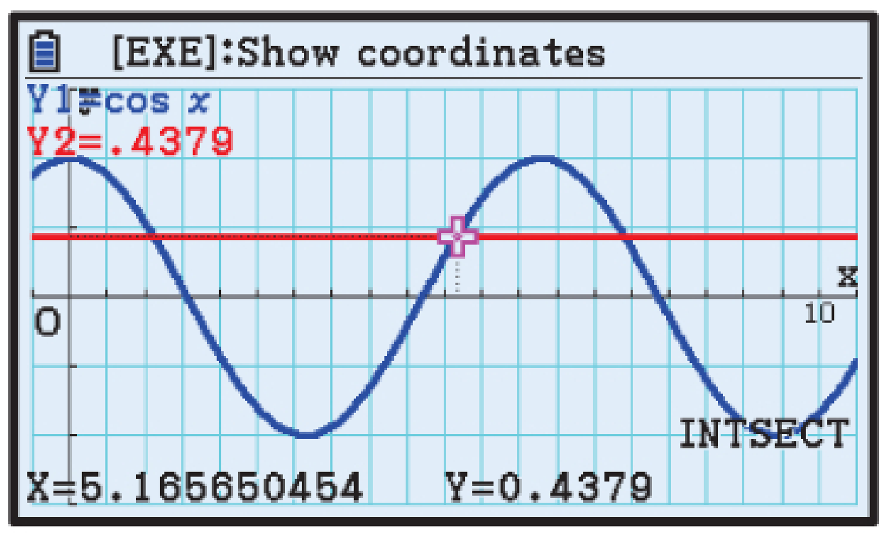
b We graph the functions $Y_1 = \tan \frac{X}{2}$ and $Y_2 = -0.458$ on the same set of axes.
We use **window** settings just larger than the domain.
In this case, $X\text{min} = -0.5$, $X\text{max} = 8.5$, $X\text{scale} = 0.5$.

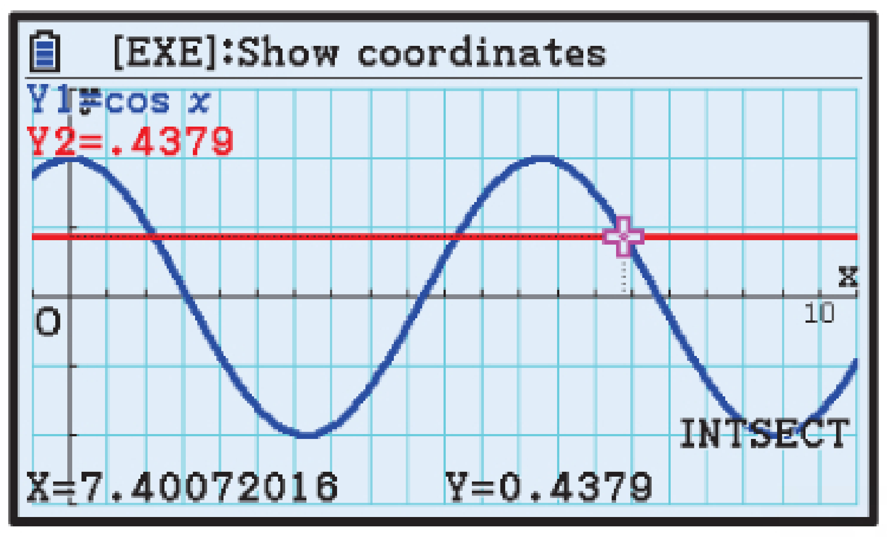


The solution is $x \approx 5.42$.

17 **a** We graph the functions $Y_1 = \cos X$ and $Y_2 = 0.4379$ on the same set of axes.
We need to use **window** settings just larger than the domain.
In this case, $X\text{min} = -0.5$, $X\text{max} = 10.5$, $X\text{scale} = 0.5$.

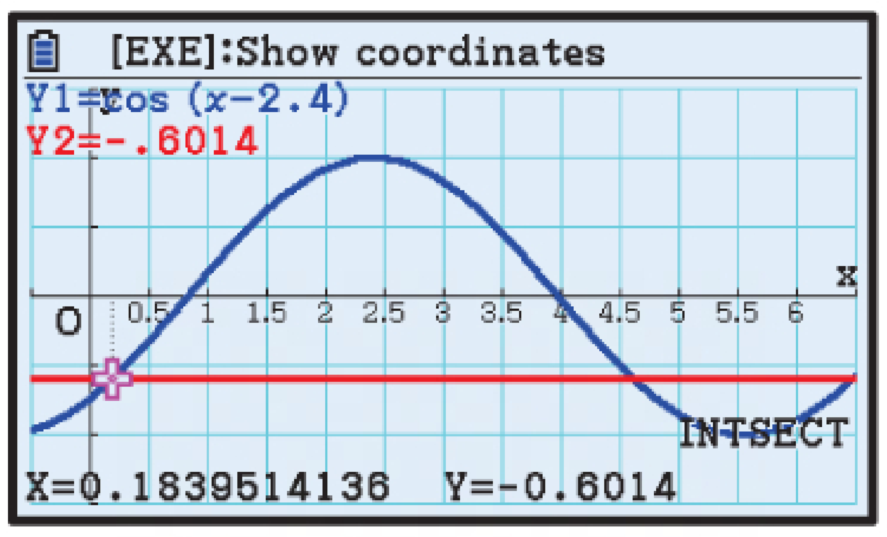


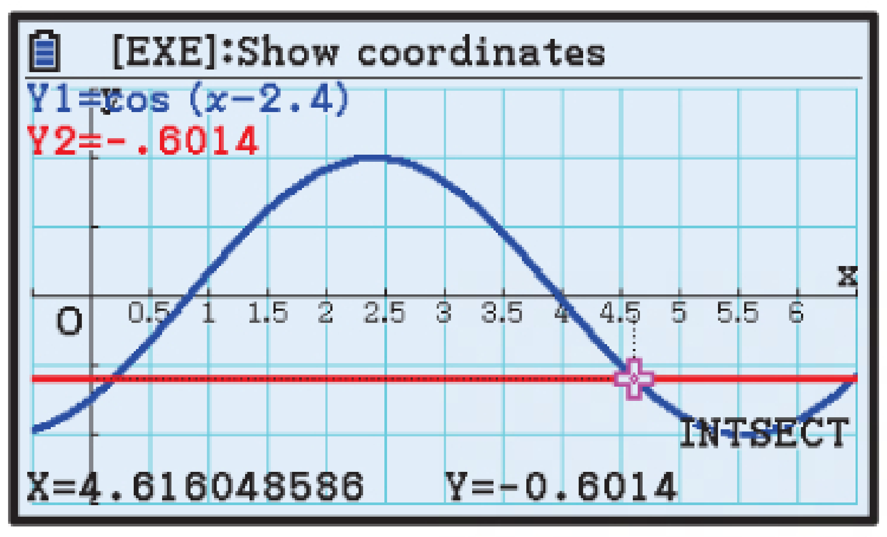




The solutions are $x \approx 1.12,\ 5.17,\ 7.40$.

b We graph the functions $Y_1 = \cos(X - 2.4)$ and $Y_2 = -0.6014$ on the same set of axes.
We need to use **window** settings just larger than the domain.
In this case, $X\text{min} = -0.5$, $X\text{max} = 6.5$, $X\text{scale} = 0.5$.





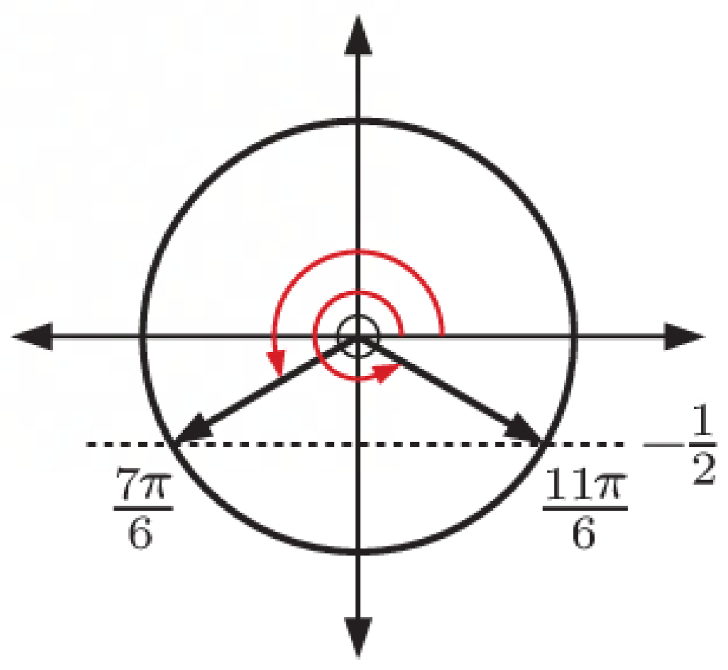
The solutions are $x \approx 0.184,\ 4.62$.

18 **a** $2\sin x = -1$

$\therefore\ \sin x = -\frac{1}{2}$

On $0 \leqslant x \leqslant 2\pi$, the angles with sine $-\frac{1}{2}$ are $\frac{7\pi}{6}$ and $\frac{11\pi}{6}$.

$\therefore$ the solutions are $x = \frac{7\pi}{6}$ or $\frac{11\pi}{6}$.

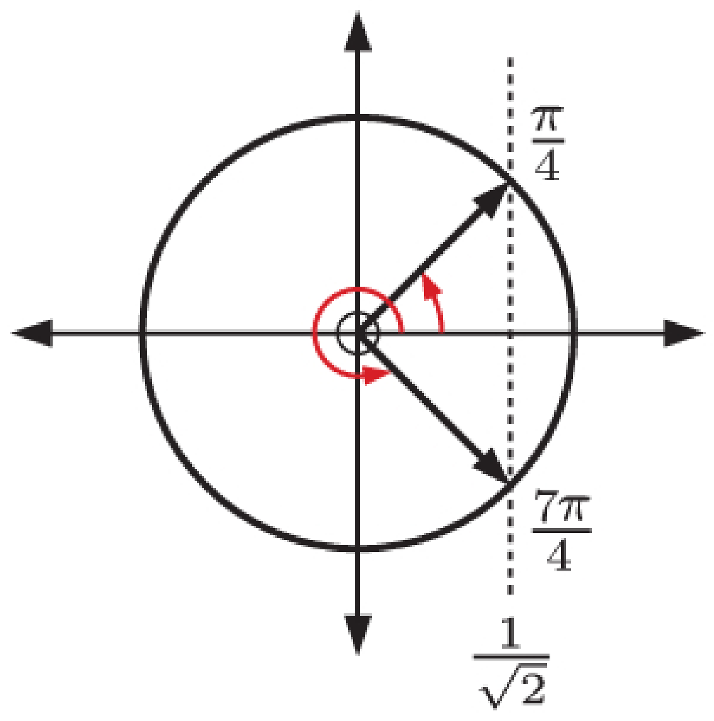


b $\sqrt{2}\cos x - 1 = 0$

$\therefore\ \cos x = \frac{1}{\sqrt{2}}$

On $0 \leqslant x \leqslant 2\pi$, the angles with cosine $\frac{1}{\sqrt{2}}$ are $\frac{\pi}{4}$ and $\frac{7\pi}{4}$.

$\therefore$ the solutions are $x = \frac{\pi}{4}$ or $\frac{7\pi}{4}$.



c $2\cos 2x + 1 = 0$

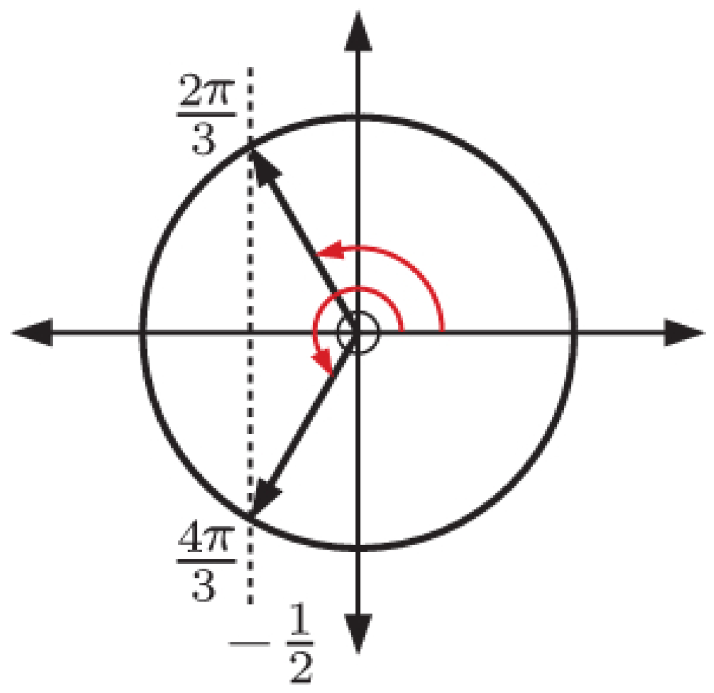
$\therefore\ \cos 2x = -\frac{1}{2}$

If $0 \leqslant x \leqslant 2\pi$, then $0 \leqslant 2x \leqslant 4\pi$.

On $0 \leqslant 2x \leqslant 4\pi$, the angles with cosine $-\frac{1}{2}$ are $\frac{2\pi}{3}$, $\frac{4\pi}{3}$, $\frac{8\pi}{3}$, and $\frac{10\pi}{3}$.

$\therefore$ the solutions are $2x = \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{8\pi}{3}$, or $\frac{10\pi}{3}$

$\therefore\ x = \frac{\pi}{3}, \frac{2\pi}{3}, \frac{4\pi}{3}$, or $\frac{5\pi}{3}$



19 **a** $\tan^2 2x = 1$

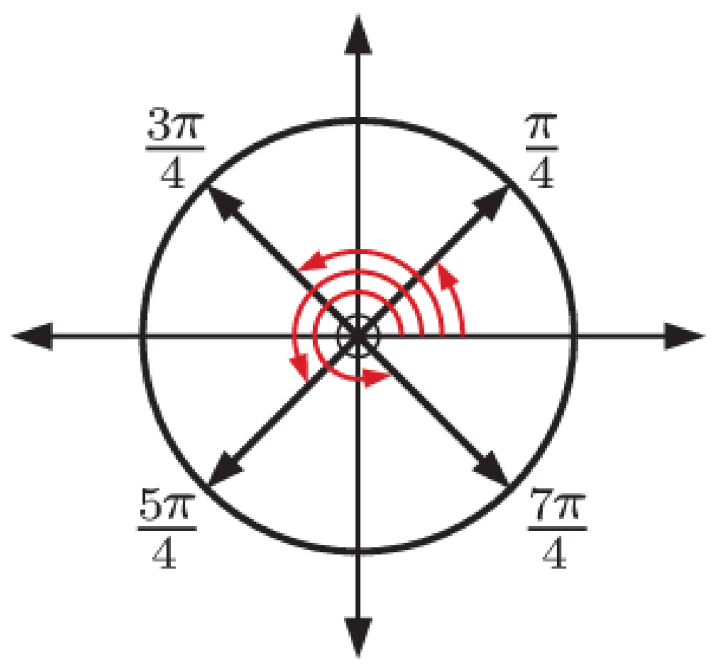
$\therefore\ \tan 2x = \pm 1$

If $0 \leqslant x \leqslant 2\pi$, then $0 \leqslant 2x \leqslant 4\pi$.

On $0 \leqslant 2x \leqslant 4\pi$, the angles with tangent 1 are $\frac{\pi}{4}$, $\frac{5\pi}{4}$, $\frac{9\pi}{4}$, and $\frac{13\pi}{4}$, and the angles with tangent -1 are $\frac{3\pi}{4}$, $\frac{7\pi}{4}$, $\frac{11\pi}{4}$, and $\frac{15\pi}{4}$.

$\therefore\ 2x = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}, \frac{9\pi}{4}, \frac{11\pi}{4}, \frac{13\pi}{4}$, or $\frac{15\pi}{4}$

$\therefore\ x = \frac{\pi}{8}, \frac{3\pi}{8}, \frac{5\pi}{8}, \frac{7\pi}{8}, \frac{9\pi}{8}, \frac{11\pi}{8}, \frac{13\pi}{8}$, or $\frac{15\pi}{8}$



b $\sin^2 x - \sin x - 2 = 0, \quad 0 \leqslant x \leqslant 2\pi$

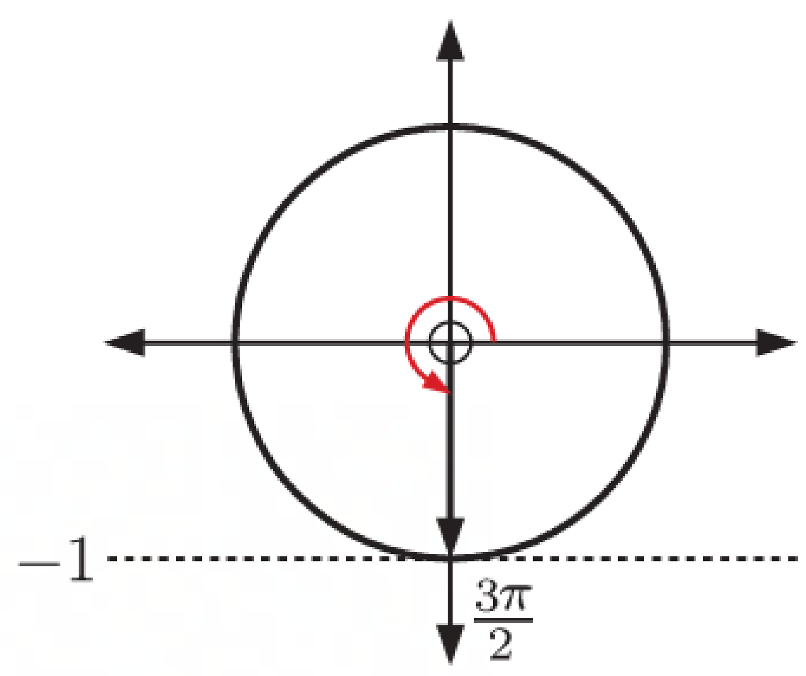
$\therefore\ (\sin x - 2)(\sin x + 1) = 0$

$\therefore\ \sin x = 2$ or -1

But $\sin x$ values lie between -1 and 1 inclusive.

$\therefore\ \sin x = -1$

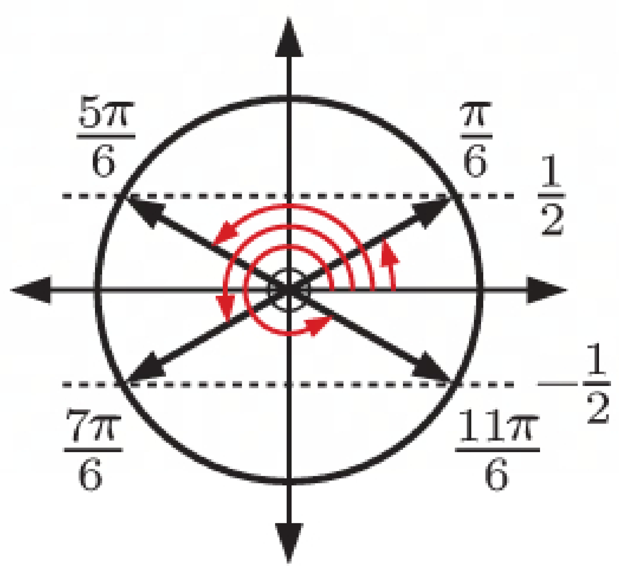
$\therefore\ x = \frac{3\pi}{2}$



c $4\sin^2 x = 1, \quad 0 \leqslant x \leqslant 2\pi$

$\therefore \sin^2 x = \frac{1}{4}$

$\therefore \sin x = \pm\frac{1}{2}$



$\therefore x = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{7\pi}{6}, \text{ or } \frac{11\pi}{6}$

20 **a** $\sqrt{2}\cos\left(x+\frac{\pi}{4}\right) - 1 = 0, \quad 0 \leqslant x \leqslant 4\pi$

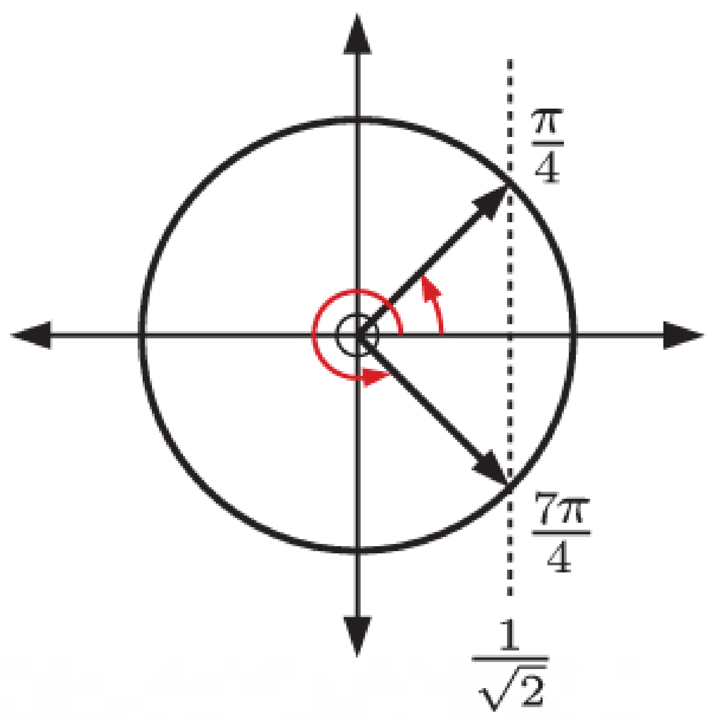
$\therefore \cos\left(x+\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}$

Since $0 \leqslant x \leqslant 4\pi$,

$\frac{\pi}{4} \leqslant x + \frac{\pi}{4} \leqslant \frac{17\pi}{4}$

So, $x + \frac{\pi}{4} = \frac{\pi}{4}, \frac{7\pi}{4}, \frac{9\pi}{4}, \frac{15\pi}{4}, \text{ or } \frac{17\pi}{4}$

$\therefore x = 0, \frac{3\pi}{2}, 2\pi, \frac{7\pi}{2}, \text{ or } 4\pi$



b $\tan 2x - \sqrt{3} = 0, \quad 0 \leqslant x \leqslant 2\pi$

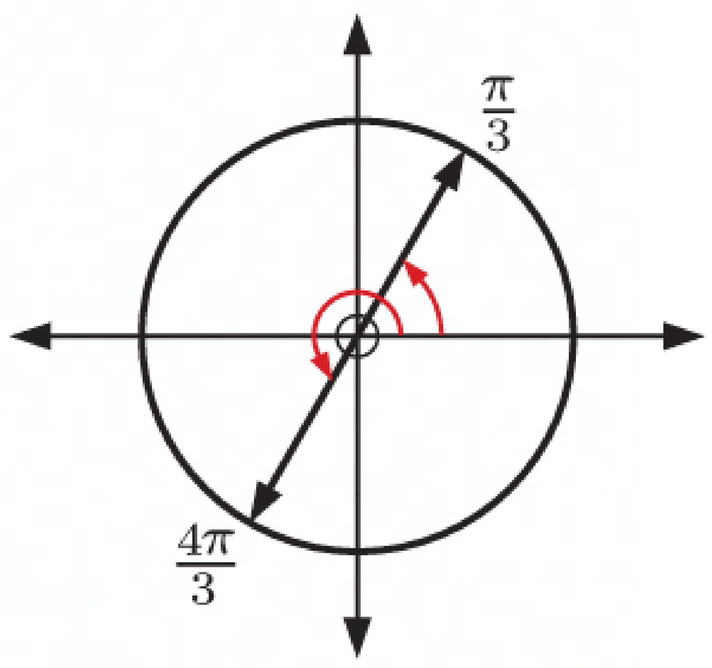
$\therefore \tan 2x = \sqrt{3}$

Since $0 \leqslant x \leqslant 2\pi$,

$0 \leqslant 2x \leqslant 4\pi$

So, $2x = \frac{\pi}{3}, \frac{4\pi}{3}, \frac{7\pi}{3}, \text{ or } \frac{10\pi}{3}$

$\therefore x = \frac{\pi}{6}, \frac{2\pi}{3}, \frac{7\pi}{6}, \text{ or } \frac{5\pi}{3}$



21 $P(t) = 5 + 2\sin\frac{\pi t}{3}, \quad 0 \leqslant t \leqslant 8,$ where $P(t)$ is in thousands of water beetles.

a $P(0) = 5 + 2\sin 0$
$= 5$

The initial population was 5000 water beetles.

b Smallest $P = 5 + 2(-1) = 3$ {when $\sin\frac{\pi t}{3} = -1$}

Largest $P = 5 + 2(1) = 7$ {when $\sin\frac{\pi t}{3} = 1$}

$\therefore$ the smallest population was 3000 water beetles and the largest population was 7000 water beetles.

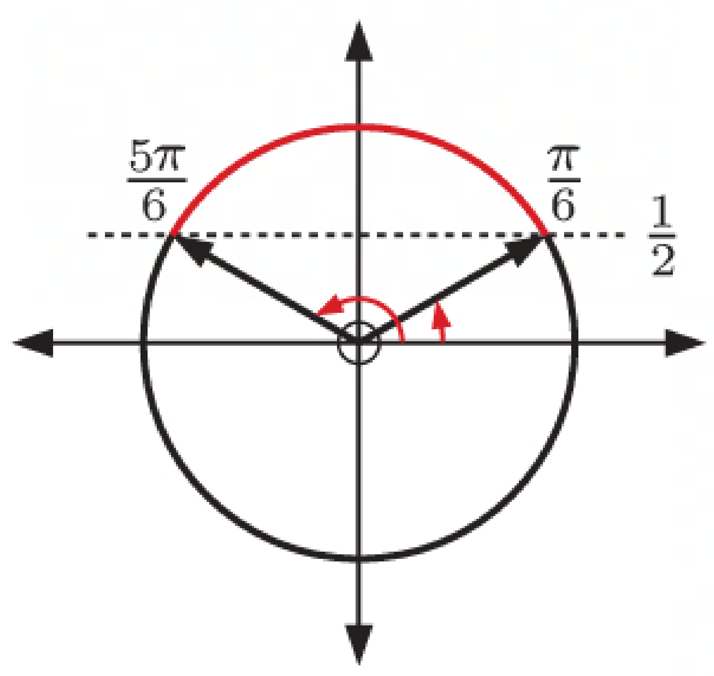
c If the population is > 6000, then $P(t) > 6$

$\therefore 5 + 2\sin\frac{\pi t}{3} > 6$

$\therefore 2\sin\frac{\pi t}{3} > 1$

$\therefore \sin\frac{\pi t}{3} > \frac{1}{2}$

The points on the unit circle with sine $\frac{1}{2}$ correspond to angles $\frac{\pi}{6}$ and $\frac{5\pi}{6}$.



$0 \leqslant t \leqslant 8 \quad \therefore 0 \leqslant \frac{\pi t}{3} \leqslant \frac{8\pi}{3}$

So, $\frac{\pi}{6} < \frac{\pi t}{3} < \frac{5\pi}{6}, \quad \frac{13\pi}{6} < \frac{\pi t}{3} \leqslant \frac{8\pi}{3}$

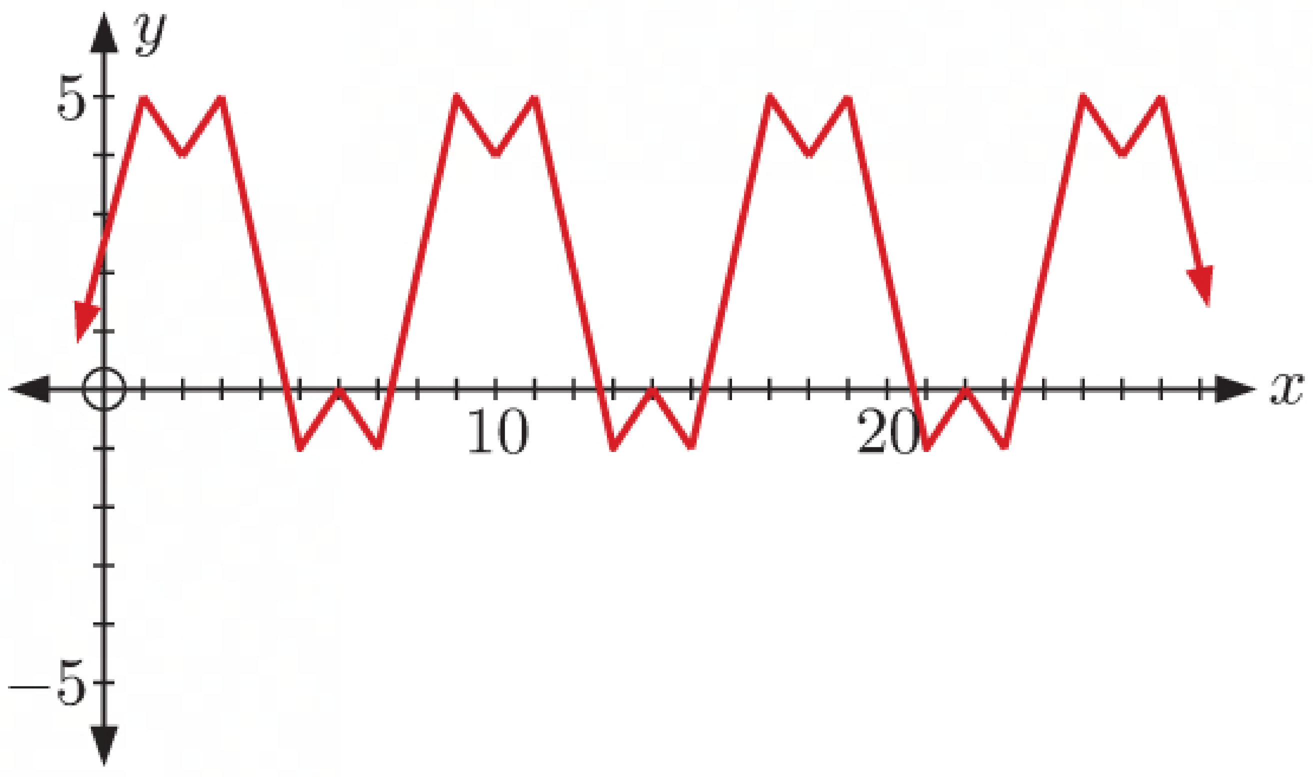
$\therefore \frac{1}{2} < t < \frac{5}{2}, \quad \frac{13}{2} < t \leqslant 8$

$\therefore 0.5 < t < 2.5, \quad 6.5 < t \leqslant 8$

So, the population of water beetles was greater than 6000 when $0.5 < t < 2.5$ weeks, and $6.5 < t \leqslant 8$ weeks.

REVIEW SET 17B

1 **a** The graph is periodic because it repeats itself over and over in a horizontal direction in intervals of the same length.



b **i** period $= 8$ **ii** maximum value $= 5$ **iii** minimum value $= -1$

2 **a** $y = \cos\left(x - \frac{\pi}{3}\right) + 1$ is a horizontal translation to the right by $\frac{\pi}{3}$ units followed by a vertical translation upwards by 1 unit.

So, a translation of $\begin{pmatrix} \frac{\pi}{3} \\ 1 \end{pmatrix}$ will map $y = \cos x$ onto $y = \cos\left(x - \frac{\pi}{3}\right) + 1$.

b $y = \sin 3x$ is a horizontal stretch of $y = \sin x$ with scale factor $\frac{1}{3}$.
So, a horizontal stretch with scale factor $\frac{1}{3}$ will map $y = \sin x$ onto $y = \sin 3x$.

3 **a** $y = 4\sin\frac{x}{3}$ has period $\frac{2\pi}{b} = \frac{2\pi}{\left(\frac{1}{3}\right)}$
$= 6\pi$

b $y = \tan 4x$ has period $\frac{\pi}{b} = \frac{\pi}{4}$

4 $y = \sin bx, \quad b > 0$

a period $= \frac{2\pi}{b} = 6\pi$
$\therefore \quad b = \frac{1}{3}$

b period $= \frac{2\pi}{b} = \frac{\pi}{12}$
$\therefore \quad b = 24$

c period $= \frac{2\pi}{b} = 9$
$\therefore \quad b = \frac{2\pi}{9}$

5 **a** $y = 5\sin x - 3$ has minimum value $5(-1) - 3 = -8$ $\{$when $\sin x = -1\}$
and maximum value $5(1) - 3 = 2$ $\{$when $\sin x = 1\}$

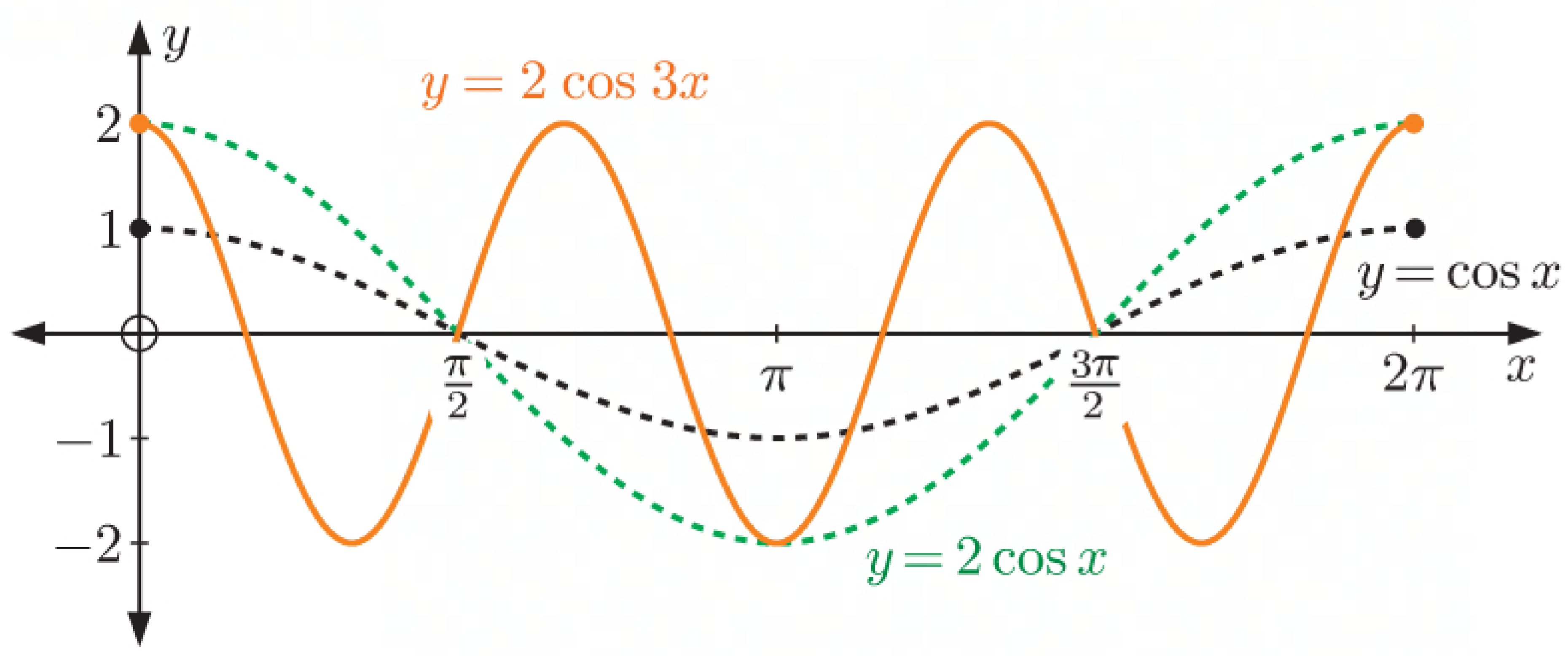
b $y = \frac{1}{3}\cos x + 1$ has minimum value $\frac{1}{3}(-1) + 1 = \frac{2}{3}$ $\{$when $\cos x = -1\}$
and maximum value $\frac{1}{3}(1) + 1 = 1\frac{1}{3}$ $\{$when $\cos x = 1\}$

6 **a** $y = -\frac{1}{3}\sin\left(x - \frac{\pi}{4}\right) + 5$ has principal axis $y = 5$.

b $y = 2\cos\frac{x}{3} - 4$ has principal axis $y = -4$.

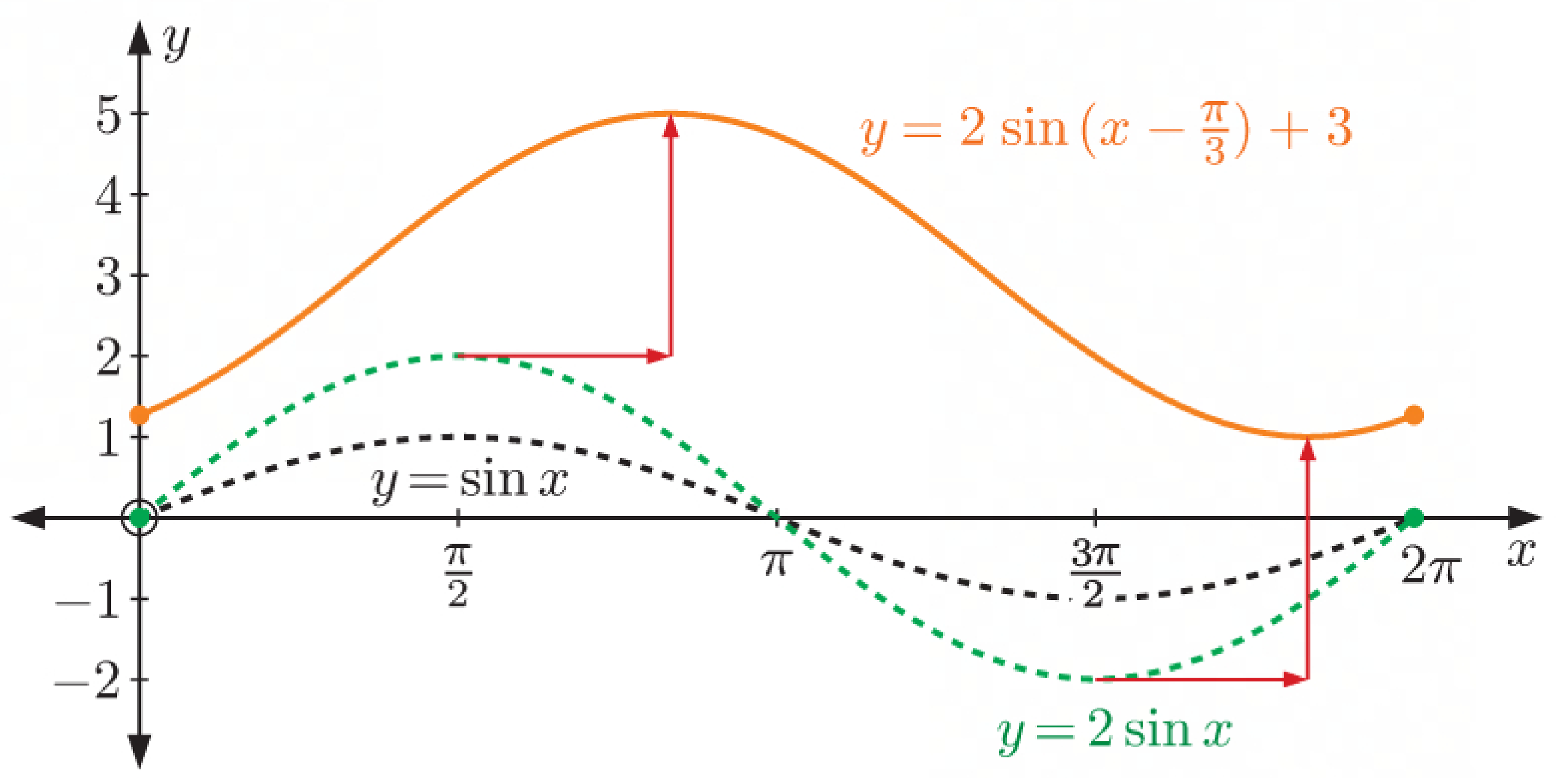
7 **a** $a = 2$, so the amplitude is $|2| = 2$. $b = 3$, so the period is $\frac{2\pi}{b} = \frac{2\pi}{3}$.

We stretch $y = \cos x$ vertically with scale factor 2 to give $y = 2\cos x$, then stretch $y = 2\cos x$ horizontally with scale factor $\frac{1}{3}$ to give $y = 2\cos 3x$.



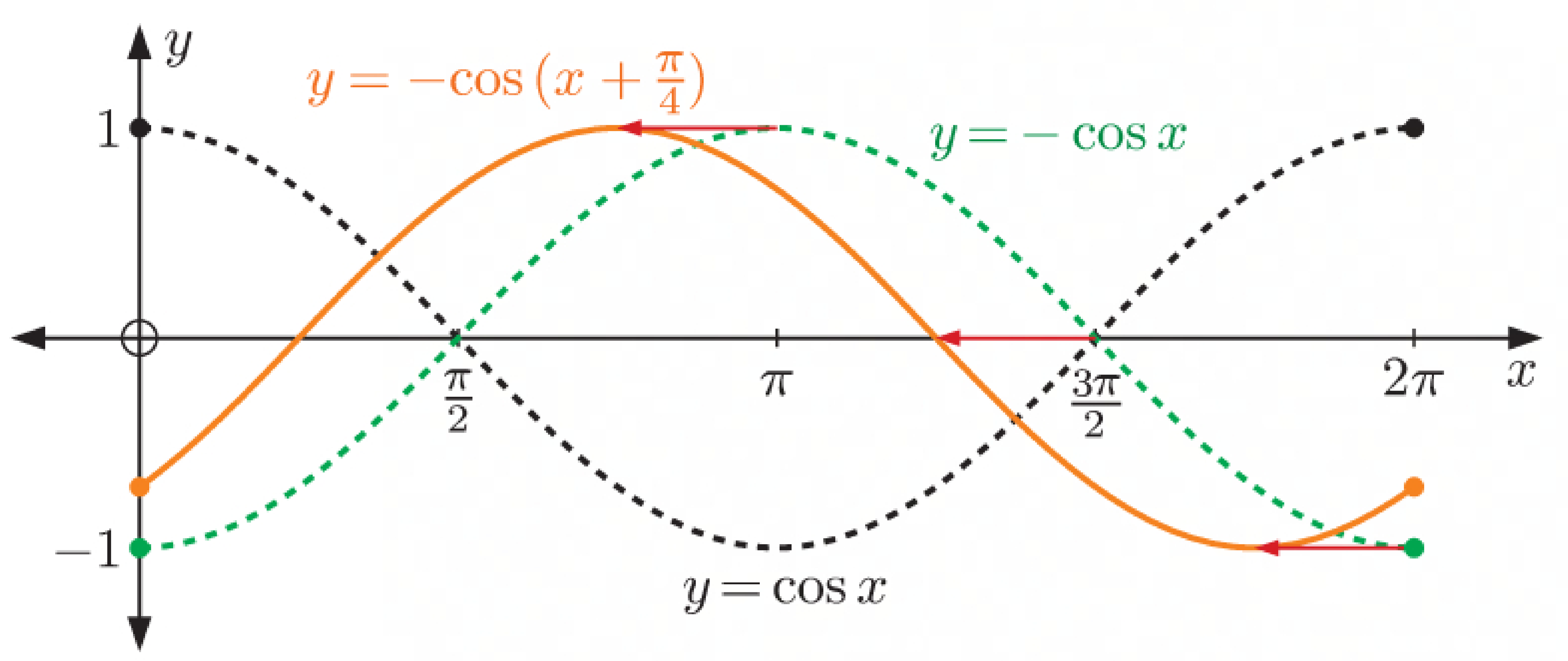
b $a = 2$, so the amplitude is $|2| = 2$.

We stretch $y = \sin x$ vertically with scale factor 2 to give $y = 2\sin x$, then translate $y = 2\sin x$ $\frac{\pi}{3}$ units to the right and 3 units upwards to give $y = 2\sin\left(x - \frac{\pi}{3}\right) + 3$.



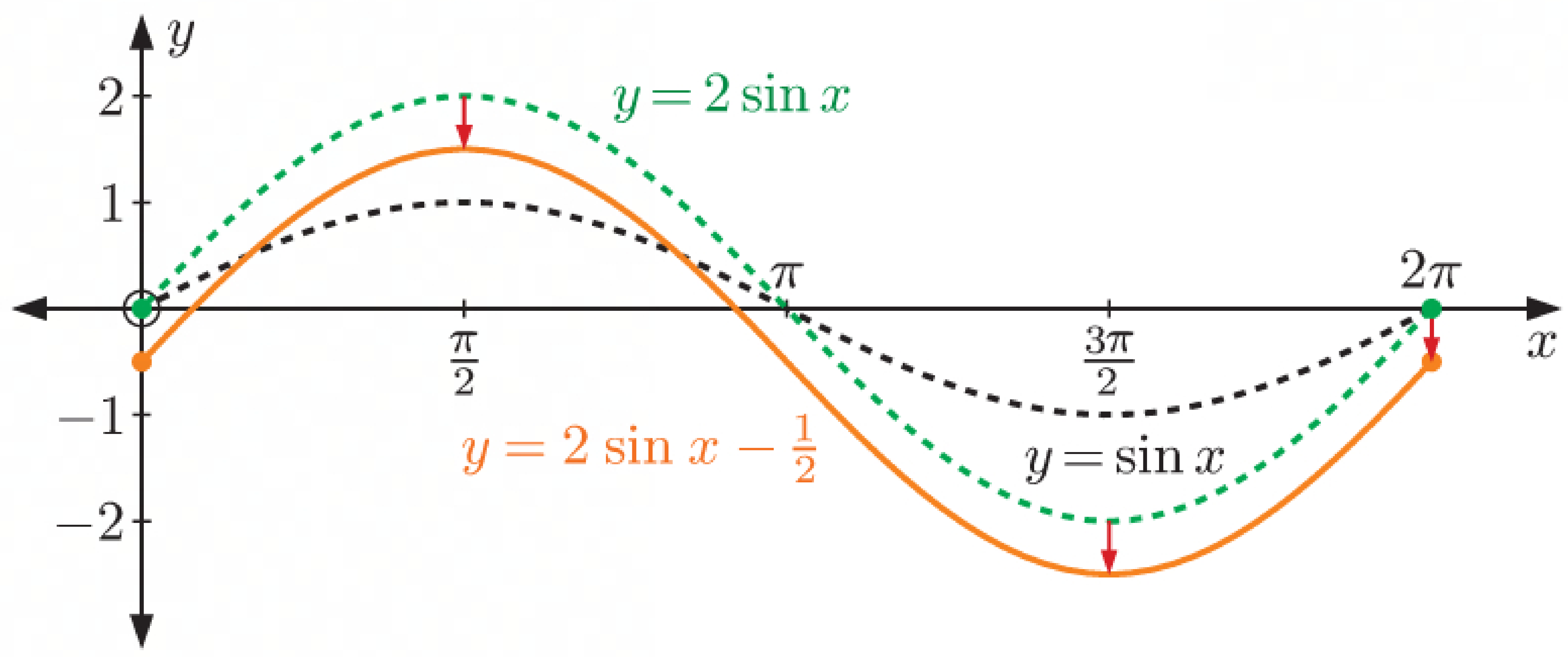
c $a = -1$, so the amplitude is $|-1| = 1$.

We reflect $y = \cos x$ in the x-axis to give $y = -\cos x$, then translate $y = -\cos x$ $\frac{\pi}{4}$ units to the left to give $y = -\cos\left(x + \frac{\pi}{4}\right)$.

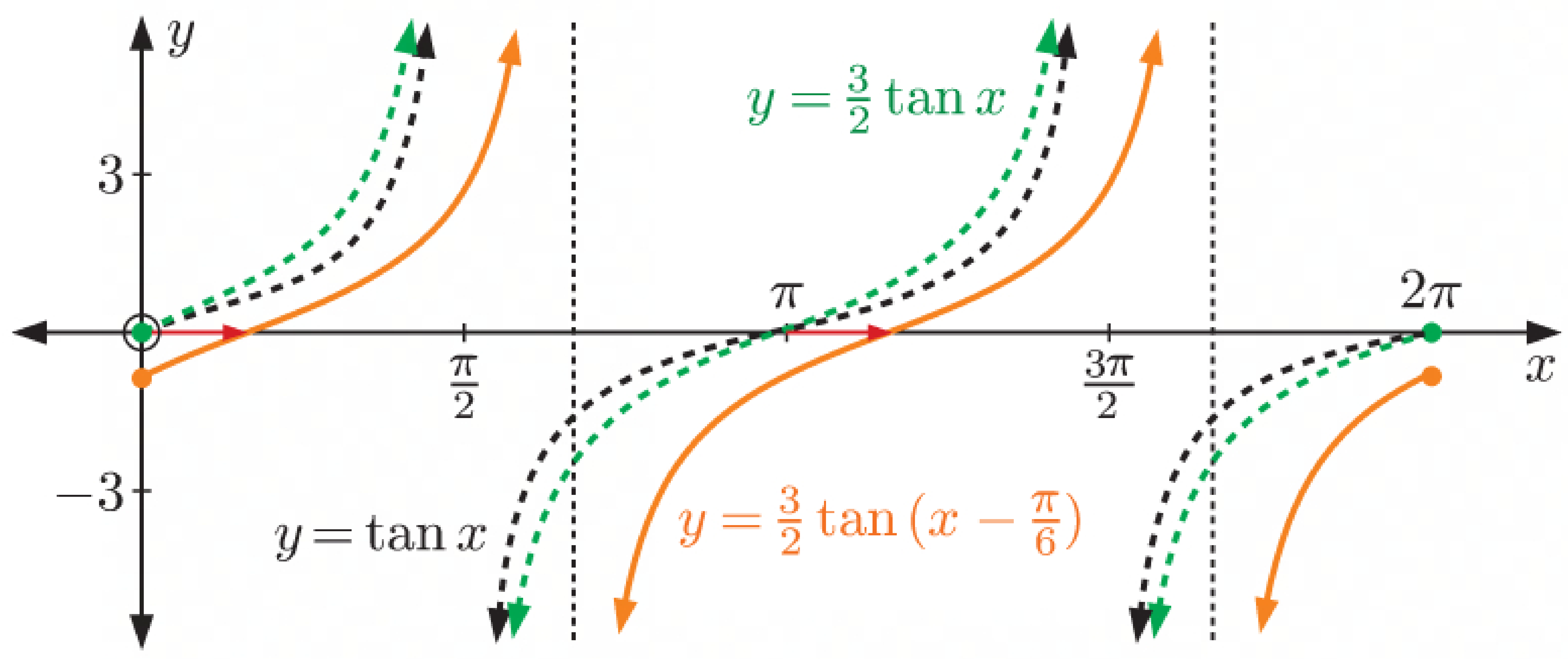


d $a = 2$, so the amplitude is $|2| = 2$.

We stretch $y = \sin x$ vertically with scale factor 2 to give $y = 2\sin x$, then translate $y = 2\sin x$ $\frac{1}{2}$ unit downwards to give $y = 2\sin x - \frac{1}{2}$.

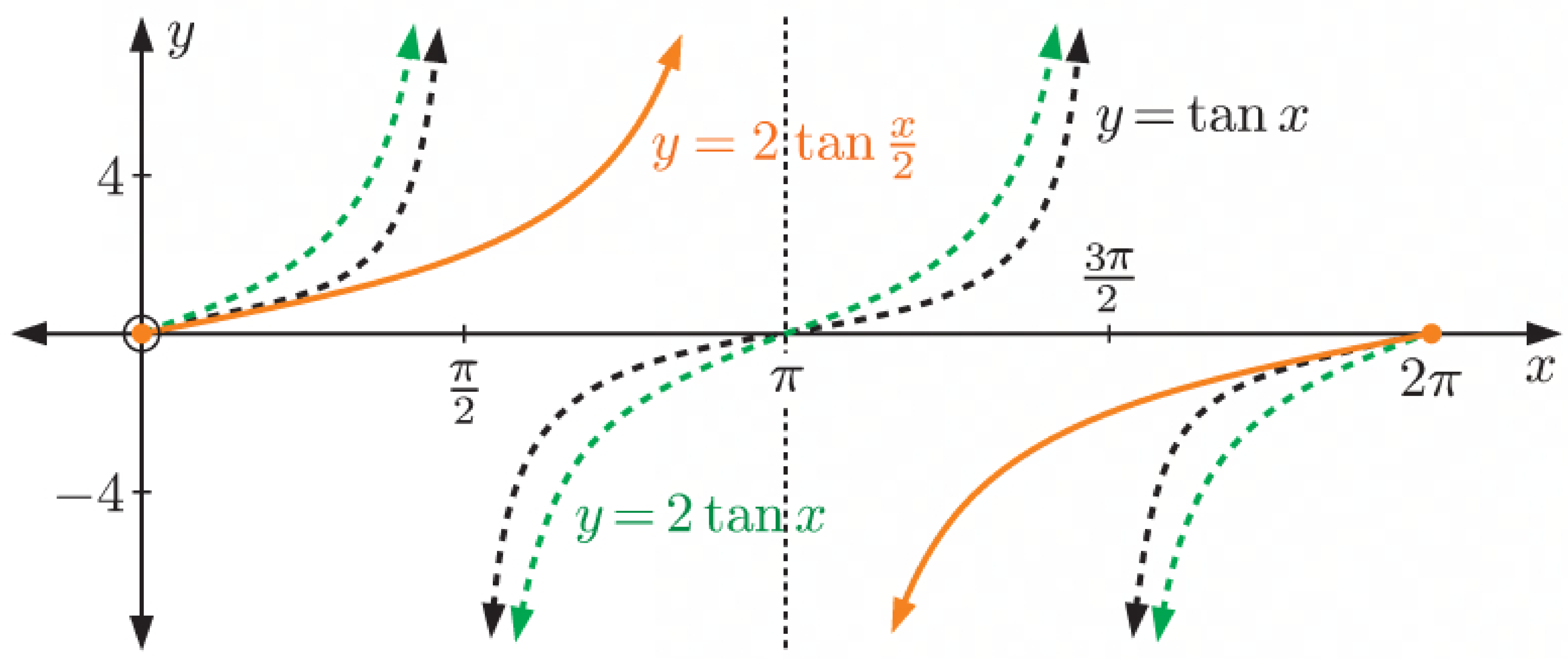


e We stretch $y = \tan x$ vertically with scale factor $\frac{3}{2}$ to give $y = \frac{3}{2}\tan x$, then translate $y = \frac{3}{2}\tan x$ $\frac{\pi}{6}$ units to the right to give $y = \frac{3}{2}\tan\left(x - \frac{\pi}{6}\right)$.



f $b = \frac{1}{2}$, so the period is $\frac{\pi}{b} = \frac{\pi}{\left(\frac{1}{2}\right)} = 2\pi$.

We stretch $y = \tan x$ vertically with scale factor 2 to give $y = 2\tan x$, then stretch $y = 2\tan x$ horizontally with scale factor 2 to give $y = 2\tan\frac{x}{2}$.



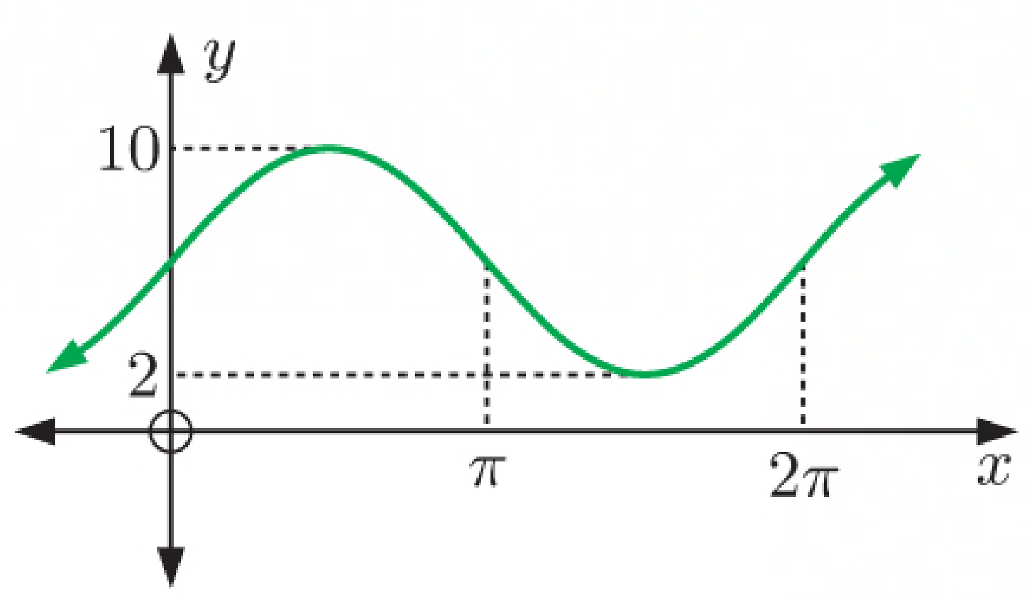
8 **a** The amplitude is 4, so $a = 4$.

The period is 2π, so $\frac{2\pi}{b} = 2\pi$ and $\therefore\ b = 1$.

There is no horizontal translation, so $c = 0$.

The principal axis is $y = 6$, so $d = 6$.

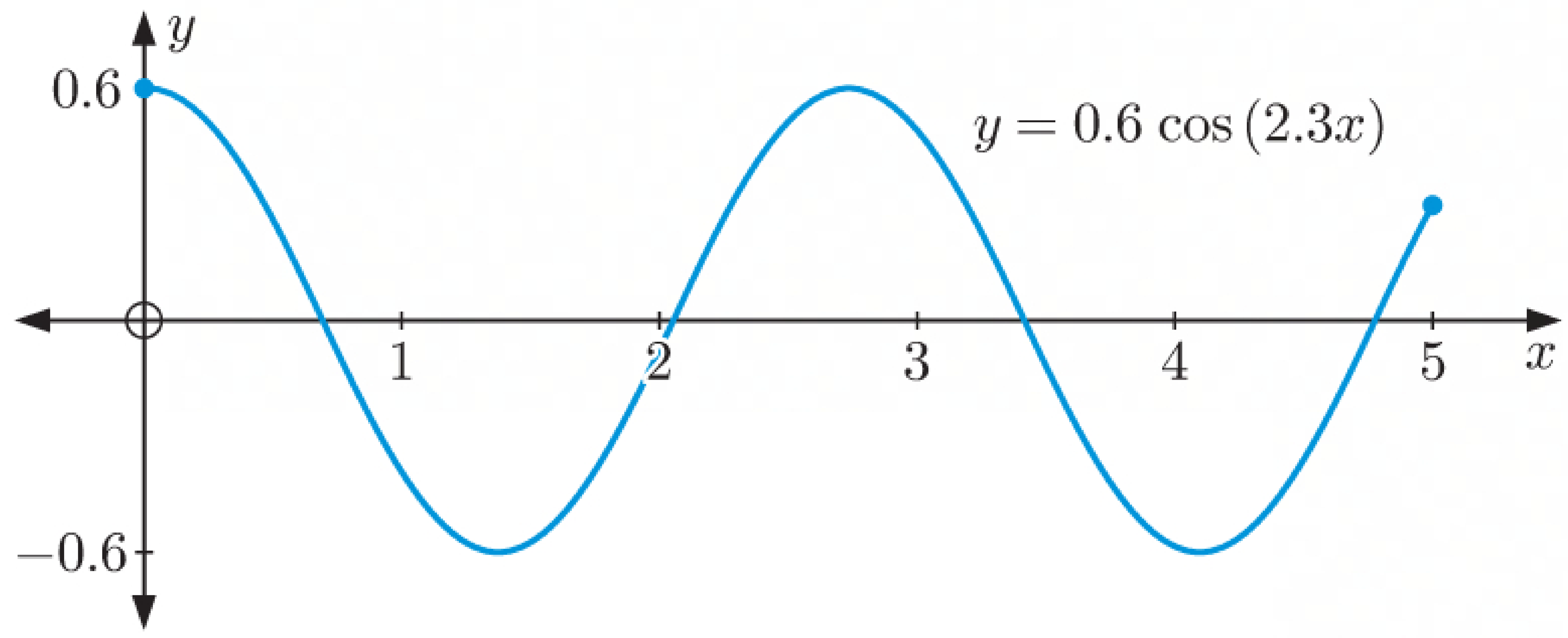
The equation of the function is $y = 4\sin x + 6$.



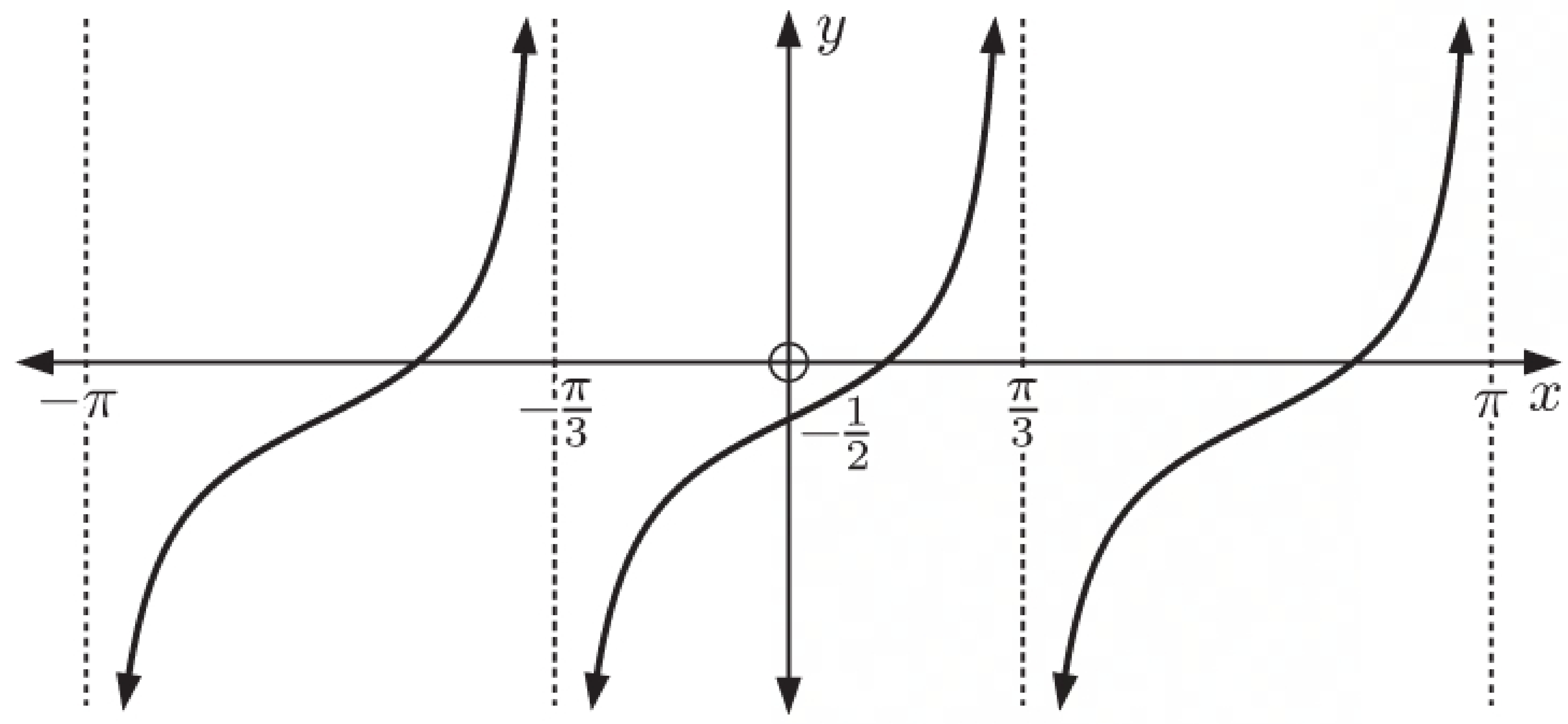
b The amplitude, period, and principal axis are the same for an equivalent cosine function, however there is a horizontal translation of $\frac{\pi}{2}$ units to the right, so $c = \frac{\pi}{2}$.

The equation of the function is $y = 4\cos\left(x - \frac{\pi}{2}\right) + 6$.

9



10



$y = \tan ax + b$ has period $\frac{\pi}{a} = \frac{2\pi}{3}$

$\therefore \frac{a}{\pi} = \frac{3}{2\pi}$

$\therefore a = \frac{3}{2}$

When $x = 0$, $y = -\frac{1}{2}$

$\therefore -\frac{1}{2} = \tan 0 + b$

$\therefore b = -\frac{1}{2}$

So, $a = \frac{3}{2}$, $b = -\frac{1}{2}$

11 **a** $\tan x \xrightarrow{\text{reflection in } x\text{-axis}} -\tan x \xrightarrow{\text{horizontal stretch scale factor } \frac{1}{2}} -\tan 2x$

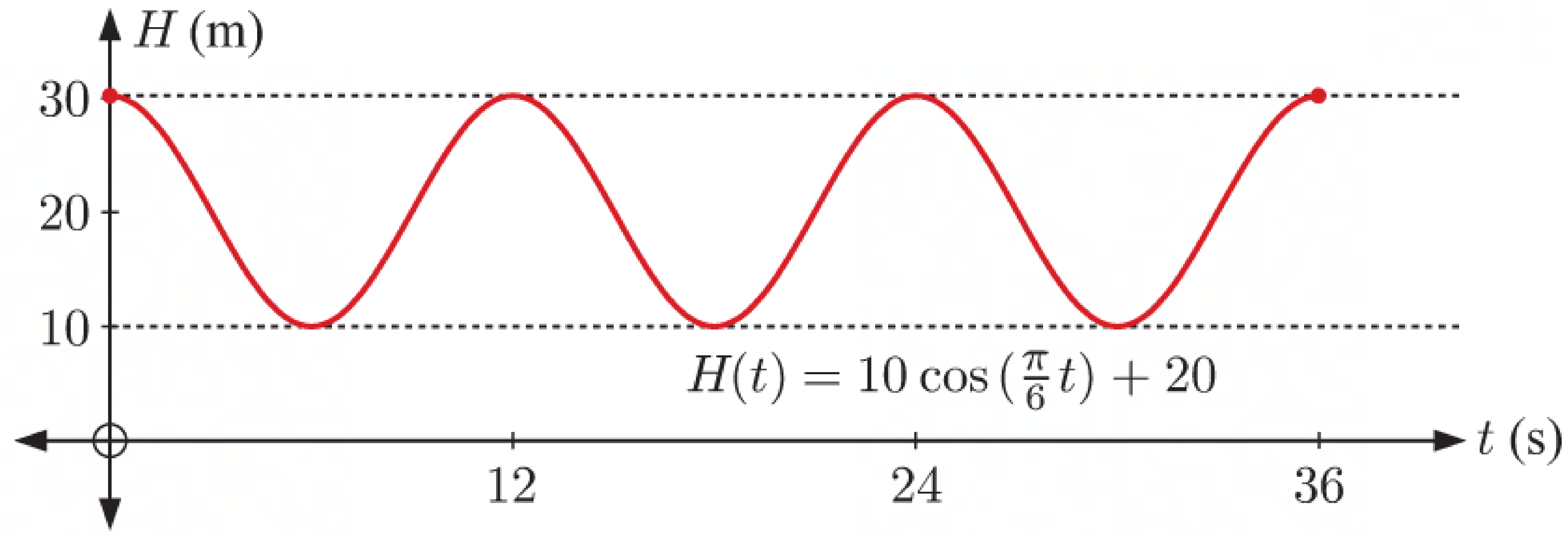
A reflection in the x-axis, then a horizontal stretch with scale factor $\frac{1}{2}$ will map $y = \tan x$ onto $y = -\tan 2x$.

b $\sin x \xrightarrow{\text{vertical stretch scale factor 2}} 2\sin x \xrightarrow{\text{horizontal stretch scale factor 2}} 2\sin \frac{1}{2}x \xrightarrow{\text{translation } \begin{pmatrix} \frac{\pi}{2} \\ \frac{1}{2} \end{pmatrix}} 2\sin\left(\frac{1}{2}(x - \frac{\pi}{2})\right) + \frac{1}{2}$

A vertical stretch with scale factor 2, then a horizontal stretch with scale factor 2, then a translation $\frac{\pi}{2}$ units right and $\frac{1}{2}$ unit upwards will map $y = \sin x$ onto $y = 2\sin\left(\frac{1}{2}\left(x - \frac{\pi}{2}\right)\right) + \frac{1}{2}$.

12 **a** $H(0) = 10\cos 0 + 20$
$= 30$

$H(t) = 10\cos\left(\frac{\pi}{6}t\right) + 20$ has period $\frac{2\pi}{b} = \frac{2\pi}{\frac{\pi}{6}} = 12$ seconds



b After 9 seconds, $t = 9$ and $H(9) = 10\cos\left(\frac{\pi}{6} \times 9\right) + 20$
$= 10\cos\frac{3\pi}{2} + 20$
$= 0 + 20$
$= 20$

So, the height of the blade's tip after 9 seconds is 20 m.

c $H(t) = 10\cos\left(\frac{\pi}{6}t\right) + 20$ has minimum value $10(-1) + 20 = 10$ $\{$when $\cos\left(\frac{\pi}{6}t\right) = -1\}$
So, the minimum height of the blade's tip is 10 m.

d From part **a**, the period is 12 seconds, so it takes 12 seconds for the blade to complete a full revolution.

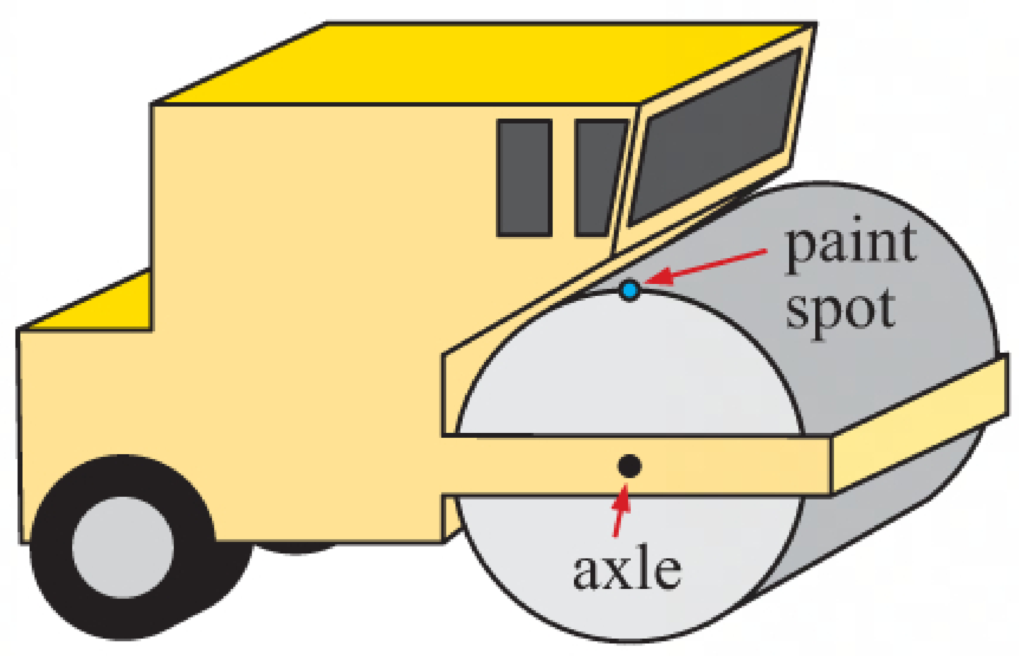
13 **a** The amplitude $a = 1$.

The period is $\frac{2\pi}{b} = 2$ s $\therefore$ $b = \pi$

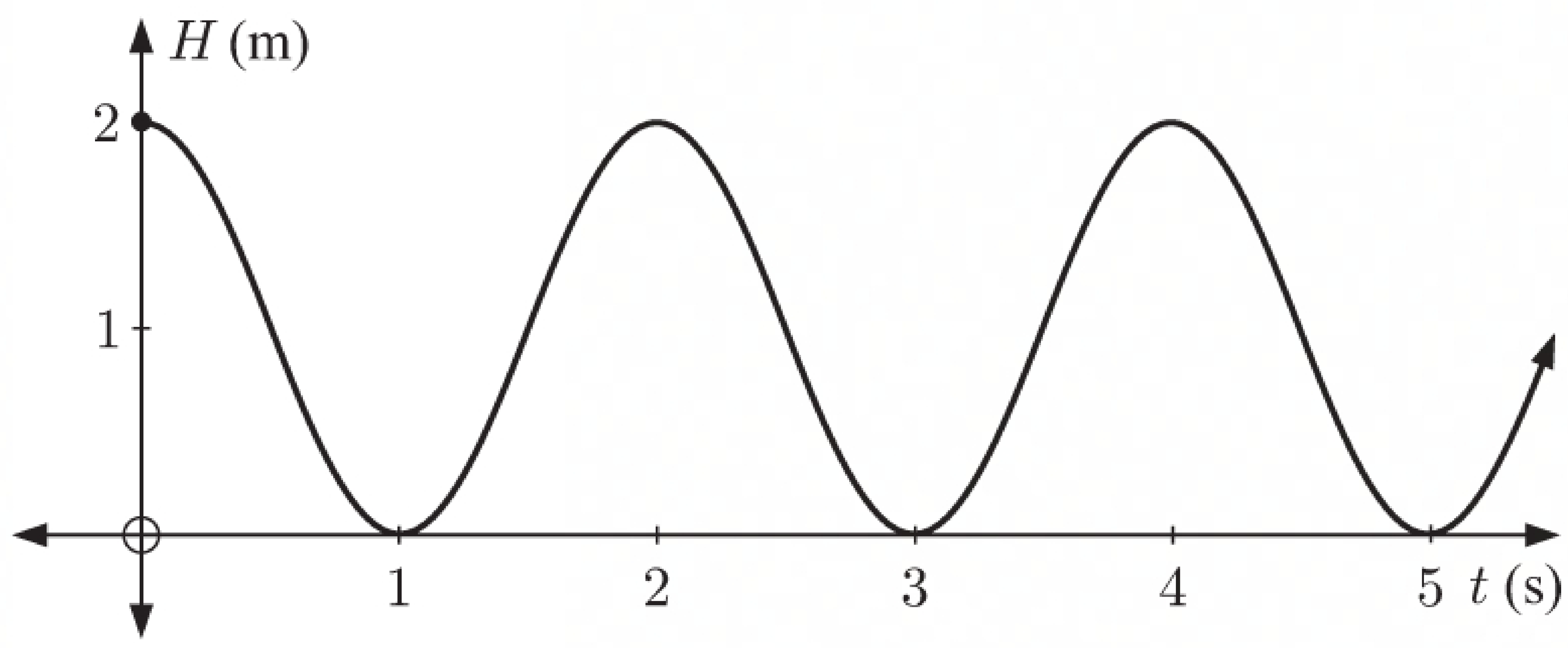
The principal axis is $y = \frac{\max + \min}{2} = \frac{2+0}{2} = 1$

$\therefore$ $d = 1$

Assume that the paint spot is initially at its highest point.



The graph is:



b The graph in **a** is a horizontal translation of $H(t) = \sin \pi t + 1$ 1.5 units to the right.
So, the height of the paint spot at time t is $H(t) = \sin(\pi(t - 1.5)) + 1$ m.
Alternatively, $H(t) = \cos \pi t + 1$ m.

14

Month (t)	Jan	Feb	Mar	Apr	May	Jun	Jul	Aug	Sep	Oct	Nov	Dec
Temperature (°C)	31.5	31.8	29.5	25.4	21.5	18.8	17.7	18.3	20.1	22.4	25.5	28.8

a The period is 12 months, so $\frac{2\pi}{b} = 12$ and $\therefore\ b = \frac{\pi}{6}$.

The amplitude $= \frac{\text{max} - \text{min}}{2} \approx \frac{31.8 - 17.7}{2} \approx 7.05$, so $a \approx 7.05$.

The principal axis is midway between the maximum and minimum, so

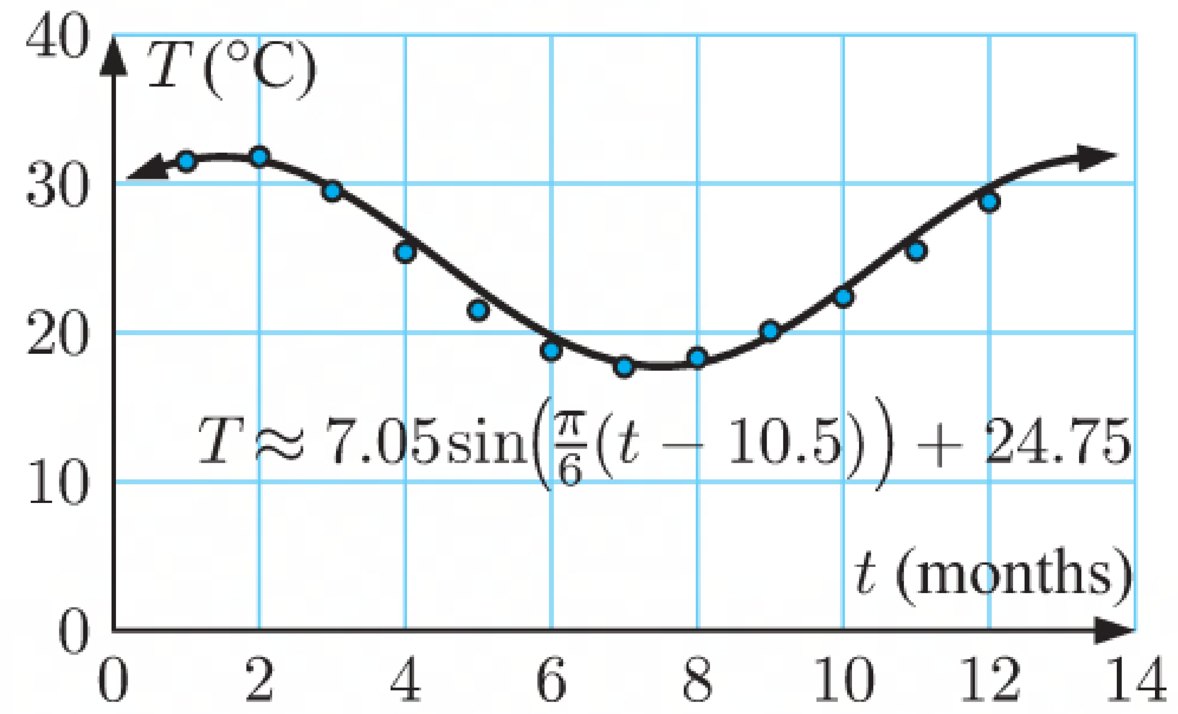
$d \approx \frac{31.8 + 17.7}{2} \approx 24.75$.

The sine function starts a new period between July and the following February (months 7 and 14). We estimate that $c \approx \frac{7+14}{2} \approx 10.5$.

So, $a \approx 7.05$, $b \approx \frac{\pi}{6}$, $c \approx 10.5$, and $d \approx 24.75$.

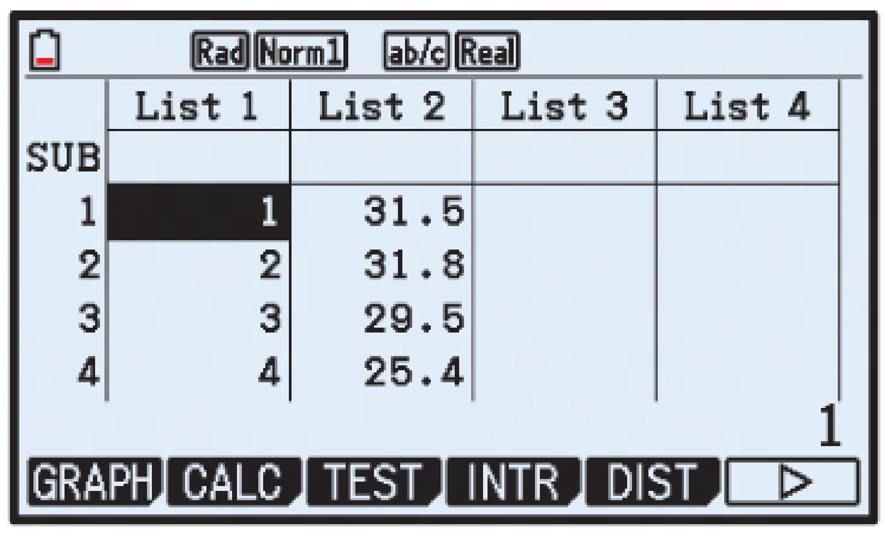
$\therefore\ T \approx 7.05 \sin\left(\frac{\pi}{6}(t - 10.5)\right) + 24.75$

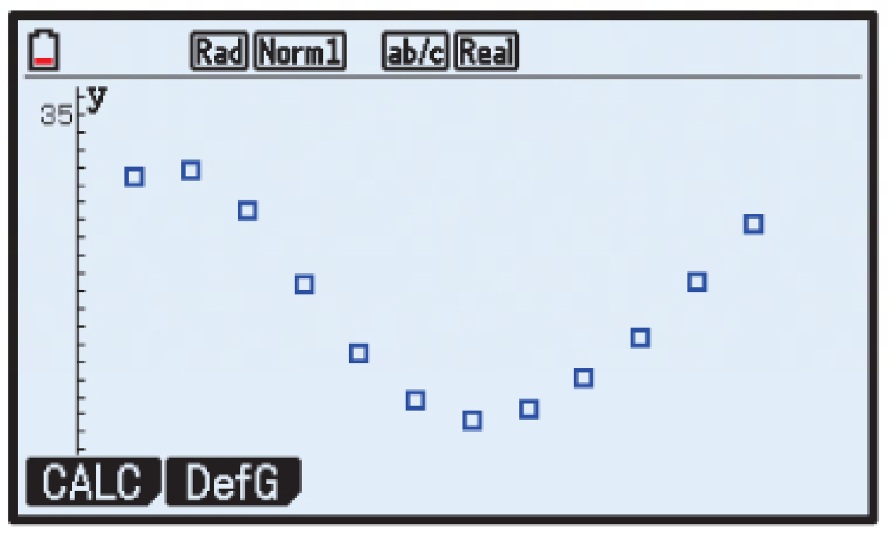
b

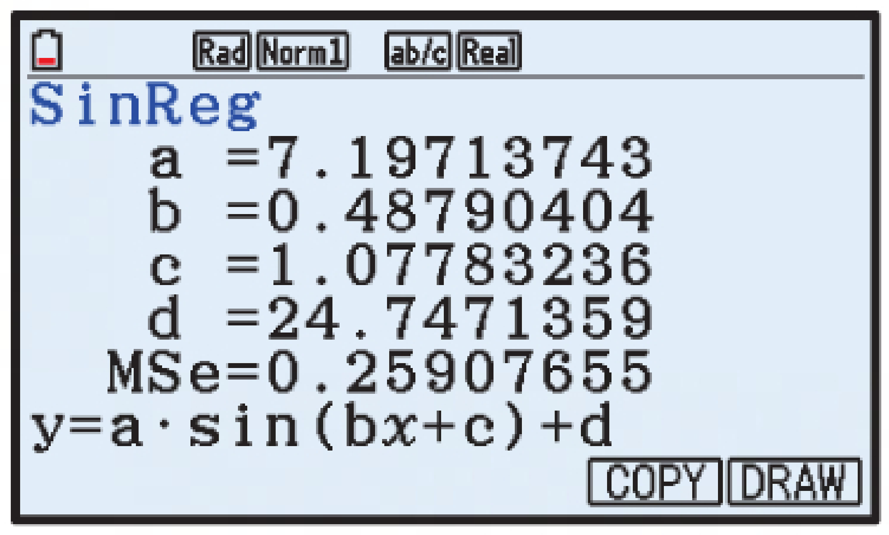


c From **a**, our model is $T \approx 7.05 \sin\left(\frac{\pi}{6}(t - 10.5)\right) + 24.75$

$\approx 7.05 \sin(0.524t - 5.50) + 24.75$







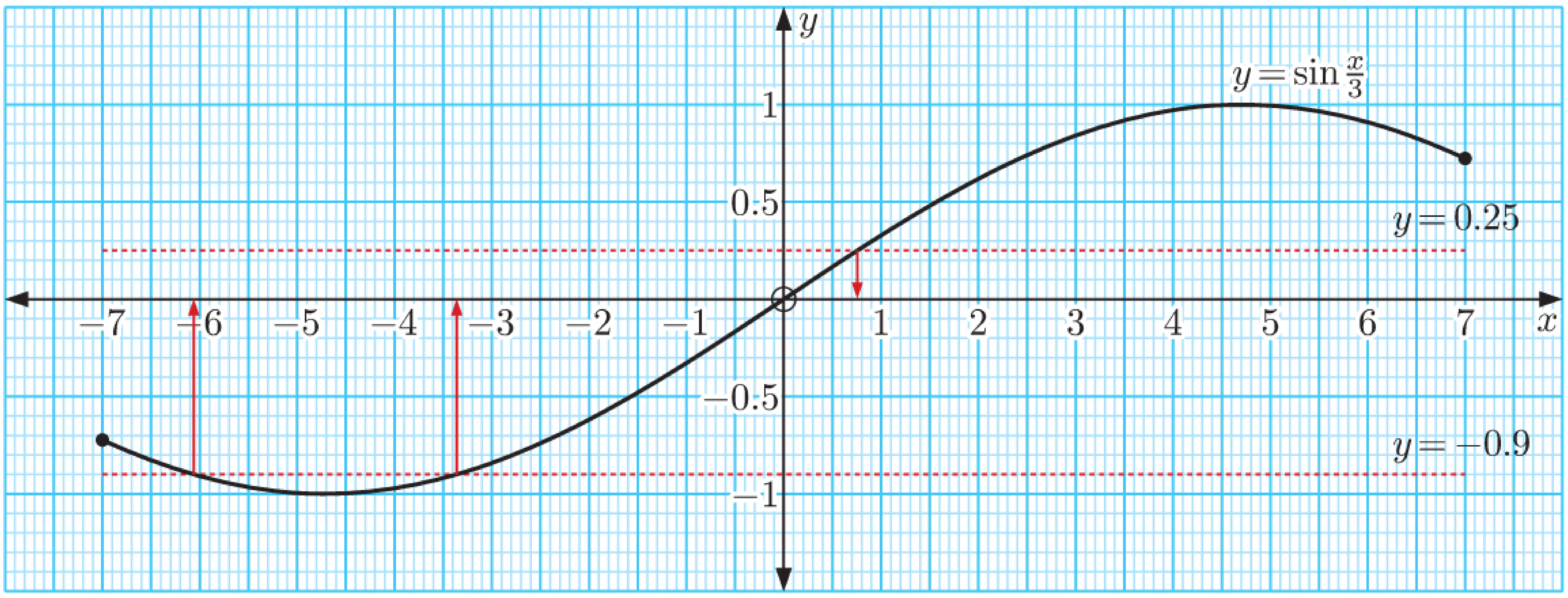
Using technology, $T \approx 7.20 \sin(0.488t + 1.08) + 24.7$

$\approx 7.20 \sin(0.488t + 1.08 - 2\pi) + 24.7 \qquad \{\sin(\theta - 2\pi) = \sin\theta\}$

$\approx 7.20 \sin(0.488t - 5.20) + 24.7$

The model fits reasonably well but not perfectly.

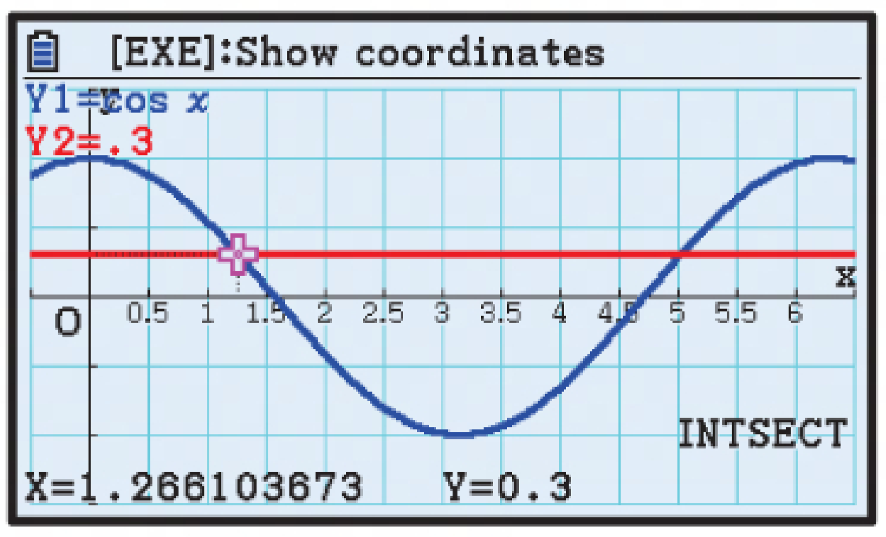
15

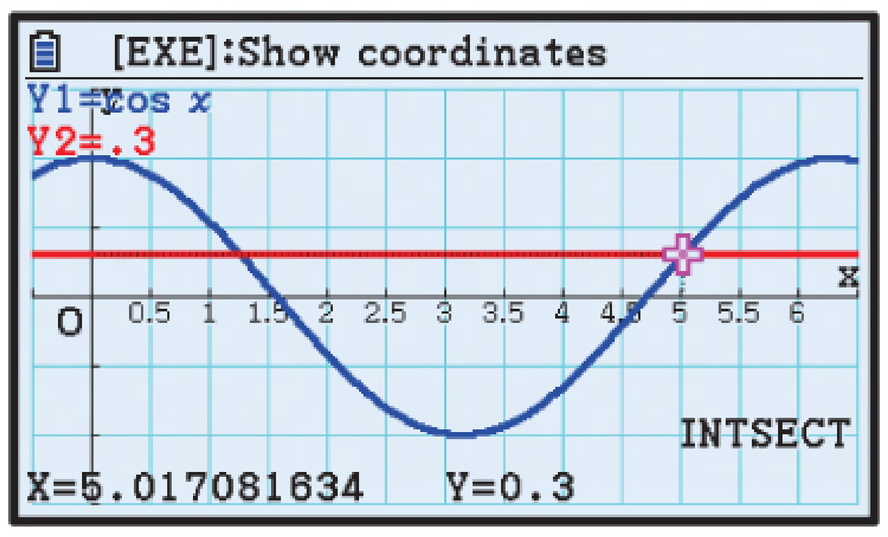


a When $\sin \frac{x}{3} = -0.9$, $-7 \leqslant x \leqslant 7$,
$x \approx -6.1, -3.4$

b When $\sin \frac{x}{3} = \frac{1}{4}$, $-7 \leqslant x \leqslant 7$,
$x \approx 0.8$

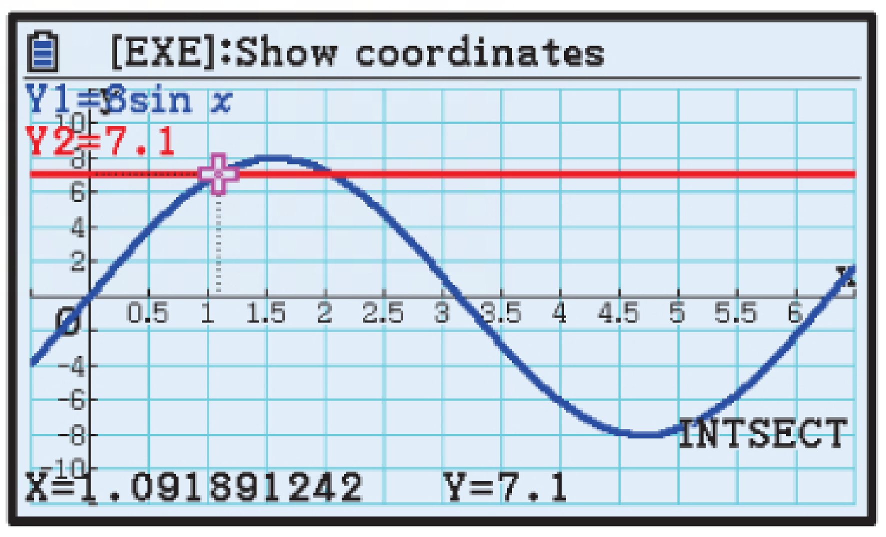
16 **a** We graph the functions $\text{Y}_1 = \cos \text{X}$ and $\text{Y}_2 = 0.3$ on the same set of axes.
We need to use **window** settings just larger than the domain.
In this case, $X\text{min} = -0.5$, $X\text{max} = 6.5$, $X\text{scale} = 0.5$.

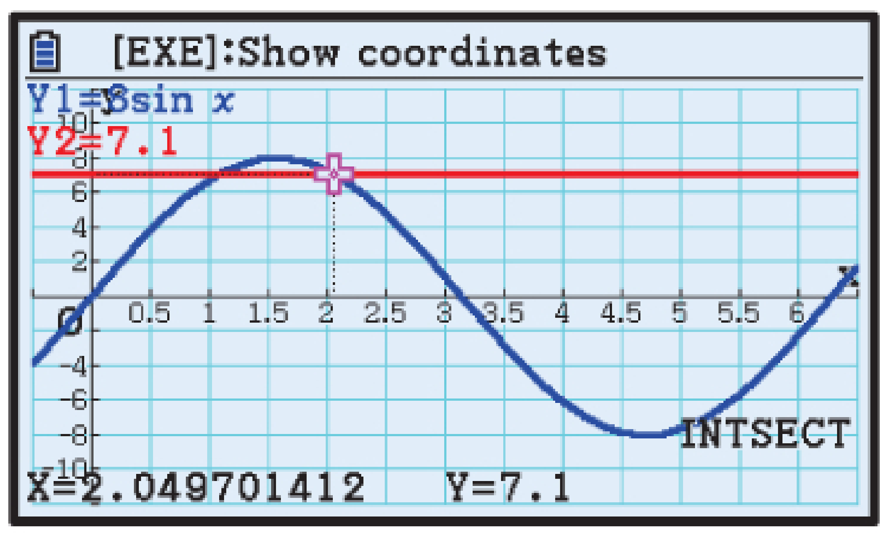




The solutions are $x \approx 1.27,\ 5.02$.

b We graph the functions $\text{Y}_1 = 8 \sin \text{X}$ and $\text{Y}_2 = 7.1$ on the same set of axes.
We need to use the **window** settings just larger than the domain.
In this case, $X\text{min} = -0.5$, $X\text{max} = 6.5$, $X\text{scale} = 0.5$.





The solutions are $x \approx 1.09,\ 2.05$.

17 **a** $\tan x = 4$

$\tan^{-1}(4) \approx 1.33$ {using technology}

$\therefore$ on $0 \leqslant x \leqslant 10$ the solutions are:

$x \approx 1.33,\ 1.33 + \pi$, or $1.33 + 2\pi$ $\{\tan(\theta + \pi) = \tan\theta\}$

$\therefore$ $x \approx 1.33, 4.47$, or 7.61

b $\tan \frac{x}{4} = 4$

$\therefore$ on $0 \leqslant \dfrac{x}{4} \leqslant \frac{5}{2}$ the solution is:

$\dfrac{x}{4} \approx 1.33$ {using **a**}

$\therefore$ $x \approx 5.30$

c $\tan(x - 1.5) = 4$

$\therefore$ on $-1.5 \leqslant x - 1.5 \leqslant 8.5$ the solutions are:

$x - 1.5 \approx 1.33,\ 1.33 + \pi,$ or $1.33 + 2\pi$ {using **a**}

$\therefore\ x - 1.5 \approx 1.33, 4.47,$ or 7.61

$\therefore\ x \approx 2.83, 5.97,$ or 9.11

18 **a** $2\sin 3x = -\sqrt{3},\quad 0 \leqslant x \leqslant 2\pi$

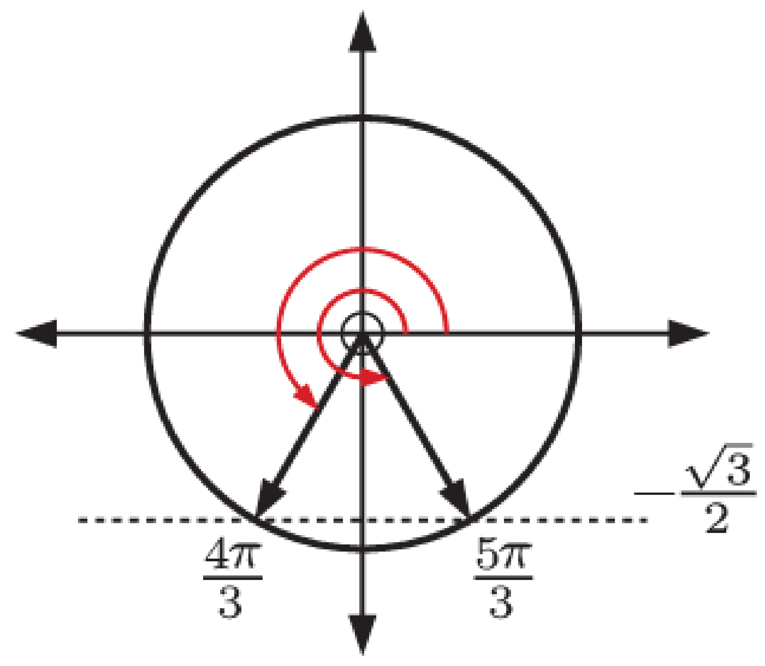
$\therefore\ \sin 3x = -\frac{\sqrt{3}}{2}$

If $0 \leqslant x \leqslant 2\pi$, then $0 \leqslant 3x \leqslant 6\pi$.

On $0 \leqslant 3x \leqslant 6\pi$, the angles with sine $-\frac{\sqrt{3}}{2}$ are $\frac{4\pi}{3}, \frac{5\pi}{3}, \frac{10\pi}{3}, \frac{11\pi}{3}, \frac{16\pi}{3}$, and $\frac{17\pi}{3}$.

So, $3x = \frac{4\pi}{3}, \frac{5\pi}{3}, \frac{10\pi}{3}, \frac{11\pi}{3}, \frac{16\pi}{3}$, or $\frac{17\pi}{3}$

$\therefore\ x = \frac{4\pi}{9}, \frac{5\pi}{9}, \frac{10\pi}{9}, \frac{11\pi}{9}, \frac{16\pi}{9}$, or $\frac{17\pi}{9}$



b $\sqrt{3}\tan\frac{x}{2} = -1,\quad 0 \leqslant x \leqslant 2\pi$

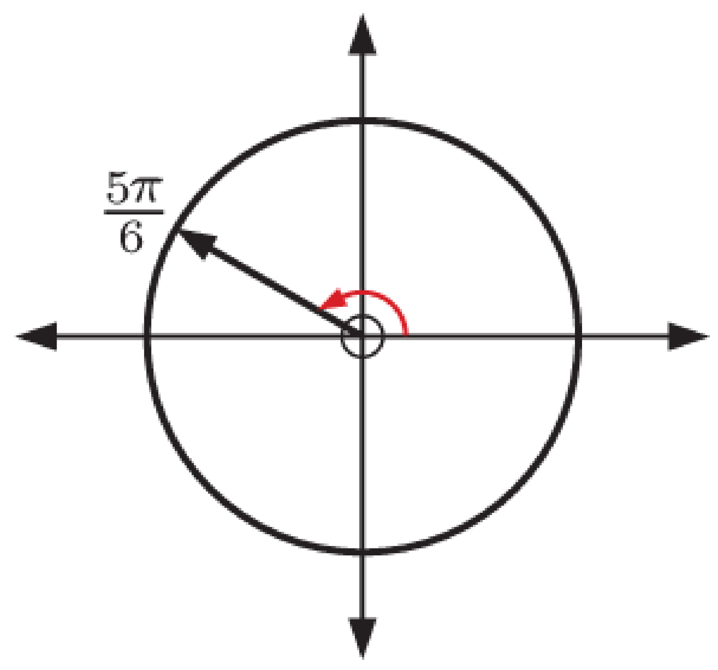
$\therefore\ \tan\frac{x}{2} = -\frac{1}{\sqrt{3}}$

If $0 \leqslant x \leqslant 2\pi$, then $0 \leqslant \dfrac{x}{2} \leqslant \pi$.

On $0 \leqslant \dfrac{x}{2} \leqslant \pi$, the angle with tangent $-\frac{1}{\sqrt{3}}$ is $\frac{5\pi}{6}$.

So, $\dfrac{x}{2} = \frac{5\pi}{6}$

$\therefore\ x = \frac{5\pi}{3}$



c $\cos 2x = \sqrt{3}\sin 2x,\quad 0 \leqslant x \leqslant 2\pi$

$\therefore\ \frac{1}{\sqrt{3}} = \dfrac{\sin 2x}{\cos 2x}$

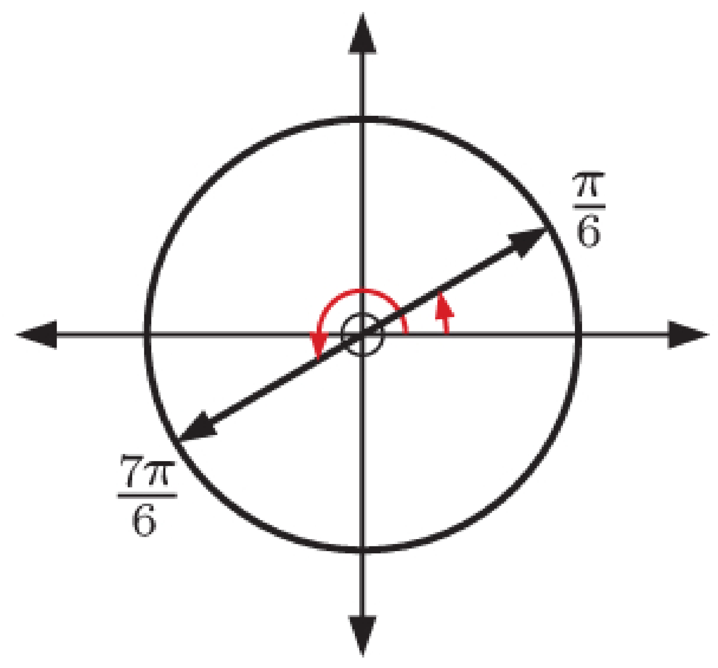
$\therefore\ \tan 2x = \frac{1}{\sqrt{3}}$

If $0 \leqslant x \leqslant 2\pi$, then $0 \leqslant 2x \leqslant 4\pi$.

On $0 \leqslant 2x \leqslant 4\pi$, the angles with tangent $\frac{1}{\sqrt{3}}$ are $\frac{\pi}{6}, \frac{7\pi}{6}, \frac{13\pi}{6}$, and $\frac{19\pi}{6}$.

So, $2x = \frac{\pi}{6}, \frac{7\pi}{6}, \frac{13\pi}{6}$, or $\frac{19\pi}{6}$

$\therefore\ x = \frac{\pi}{12}, \frac{7\pi}{12}, \frac{13\pi}{12}$, or $\frac{19\pi}{12}$



19 $f(x) = \cos x,\ g(x) = 2x,\ 0 \leqslant x \leqslant 2\pi$

a $(f \circ g)(x) = 1$

$\therefore\ f(g(x)) = 1$

$\therefore\ f(2x) = 1$

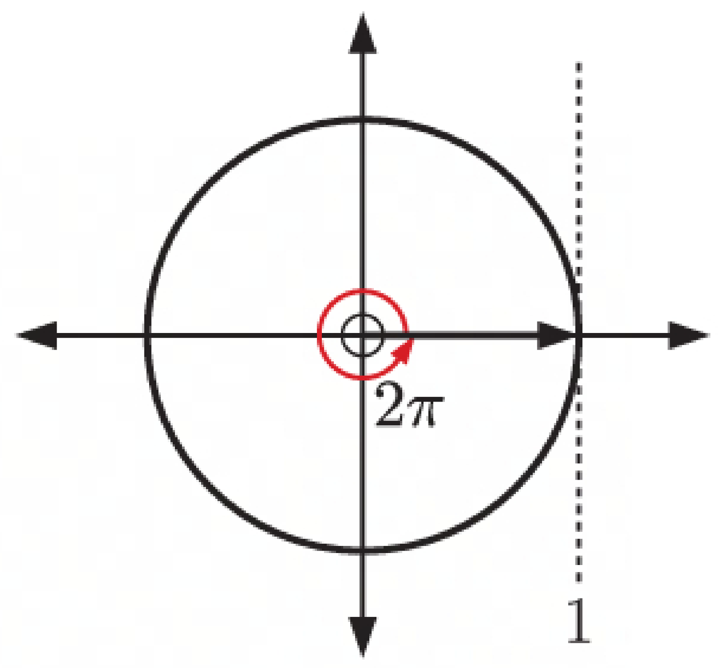
$\therefore\ \cos 2x = 1$

If $0 \leqslant x \leqslant 2\pi$, then $0 \leqslant 2x \leqslant 4\pi$.

On $0 \leqslant 2x \leqslant 4\pi$, the angles with cosine 1 are 0, 2π, and 4π.

So, $2x = 0, 2\pi$, or 4π

$\therefore\ x = 0, \pi$, or 2π



b $(g \circ f)(x) = 1$

$\therefore\ g(f(x)) = 1$

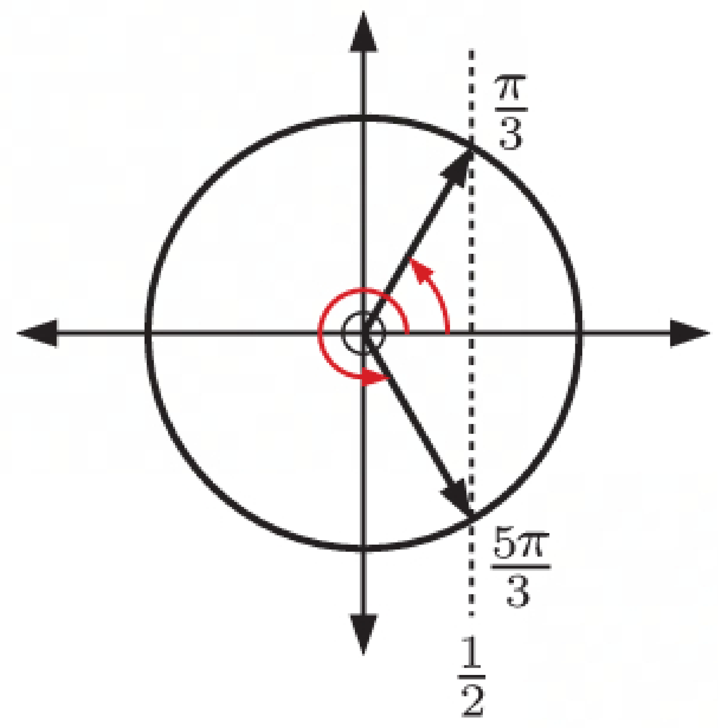
$\therefore\ g(\cos x) = 1$

$\therefore\ 2\cos x = 1$

$\therefore\ \cos x = \frac{1}{2}$

On $0 \leqslant x \leqslant 2\pi$, the angles with cosine $\frac{1}{2}$ are $\frac{\pi}{3}$ and $\frac{5\pi}{3}$.

$\therefore\ x = \frac{\pi}{3}$ or $\frac{5\pi}{3}$



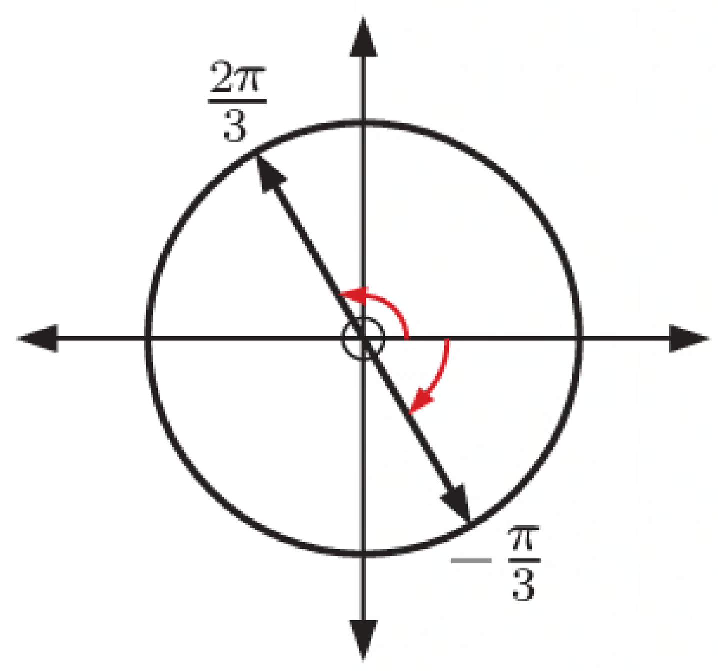
20 **a** $\tan\left(x + \frac{\pi}{6}\right) = -\sqrt{3}, \quad -\pi \leqslant x \leqslant \pi$

If $-\pi \leqslant x \leqslant \pi$, then $-\frac{5\pi}{6} \leqslant x + \frac{\pi}{6} \leqslant \frac{7\pi}{6}$.

On $-\frac{5\pi}{6} \leqslant x + \frac{\pi}{6} \leqslant \frac{7\pi}{6}$, the angles with tangent $-\sqrt{3}$ are $-\frac{\pi}{3}$ and $\frac{2\pi}{3}$.

So, $x + \frac{\pi}{6} = -\frac{\pi}{3}$ or $\frac{2\pi}{3}$

$\therefore\ x = -\frac{\pi}{2}$ or $\frac{\pi}{2}$



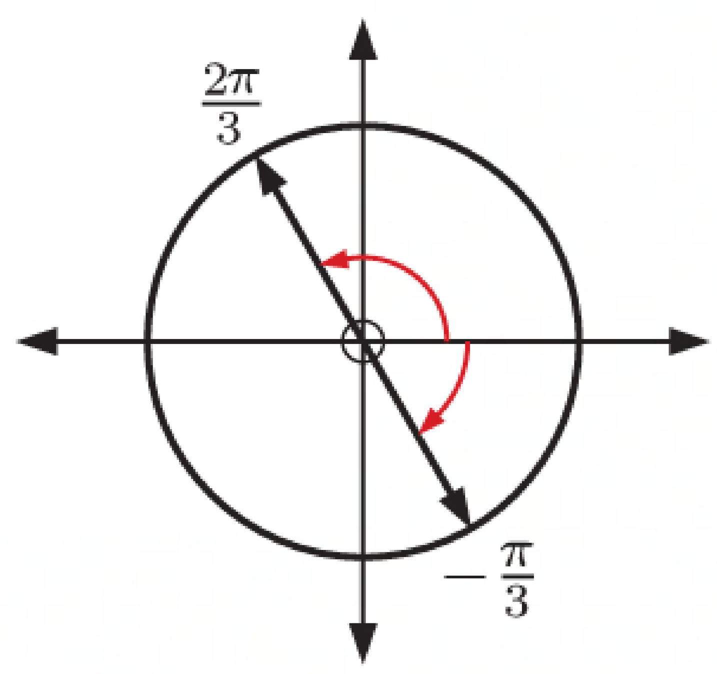
b $\tan 2x = -\sqrt{3}, \quad -\pi \leqslant x \leqslant \pi$

If $-\pi \leqslant x \leqslant \pi$, then $-2\pi \leqslant 2x \leqslant 2\pi$.

On $-2\pi \leqslant 2x \leqslant 2\pi$, the angles with tangent $-\sqrt{3}$ are $-\frac{4\pi}{3}, -\frac{\pi}{3}, \frac{2\pi}{3}$, and $\frac{5\pi}{3}$.

So, $2x = -\frac{4\pi}{3}, -\frac{\pi}{3}, \frac{2\pi}{3}$, or $\frac{5\pi}{3}$

$\therefore\ x = -\frac{2\pi}{3}, -\frac{\pi}{6}, \frac{\pi}{3}$, or $\frac{5\pi}{6}$



c $\tan^2 x - 3 = 0, \quad -\pi \leqslant x \leqslant \pi$

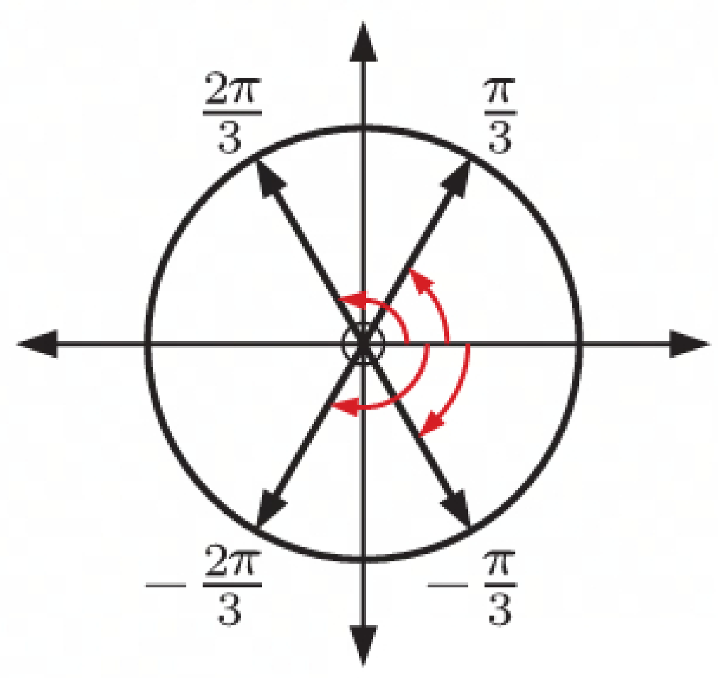
$\therefore\ \tan^2 x = 3$

$\therefore\ \tan x = \pm\sqrt{3}$

On $-\pi \leqslant x \leqslant \pi$, the angles with tangent $\sqrt{3}$ are $-\frac{2\pi}{3}$ and $\frac{\pi}{3}$.

The angles with tangent $-\sqrt{3}$ are $-\frac{\pi}{3}$ and $\frac{2\pi}{3}$.

$\therefore\ x = -\frac{2\pi}{3}, -\frac{\pi}{3}, \frac{\pi}{3}$, or $\frac{2\pi}{3}$



21 **a** $y = 2\sin 3x + \sqrt{3}, \quad 0 \leqslant x \leqslant 2\pi$

The x-intercepts are the values of x such that

$2\sin 3x + \sqrt{3} = 0$

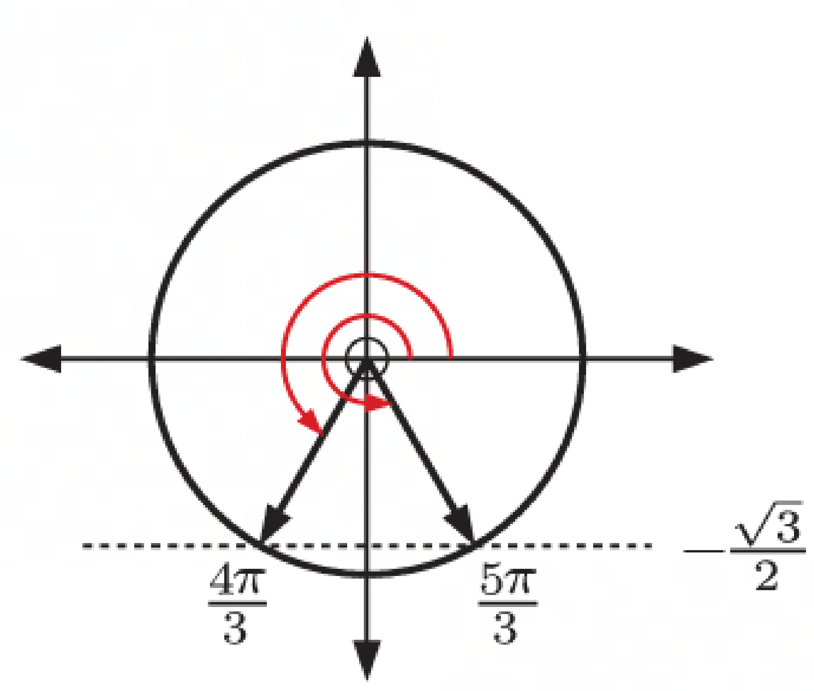
$\therefore\ \sin 3x = -\frac{\sqrt{3}}{2}$

If $0 \leqslant x \leqslant 2\pi$, then $0 \leqslant 3x \leqslant 6\pi$.

On $0 \leqslant 3x \leqslant 6\pi$, the angles with sine $-\frac{\sqrt{3}}{2}$ are $\frac{4\pi}{3}, \frac{5\pi}{3}, \frac{10\pi}{3}, \frac{11\pi}{3}, \frac{16\pi}{3}$, and $\frac{17\pi}{3}$.

So, $3x = \frac{4\pi}{3}, \frac{5\pi}{3}, \frac{10\pi}{3}, \frac{11\pi}{3}, \frac{16\pi}{3}$, or $\frac{17\pi}{3}$

$\therefore\ x = \frac{4\pi}{9}, \frac{5\pi}{9}, \frac{10\pi}{9}, \frac{11\pi}{9}, \frac{16\pi}{9}$, or $\frac{17\pi}{9}$



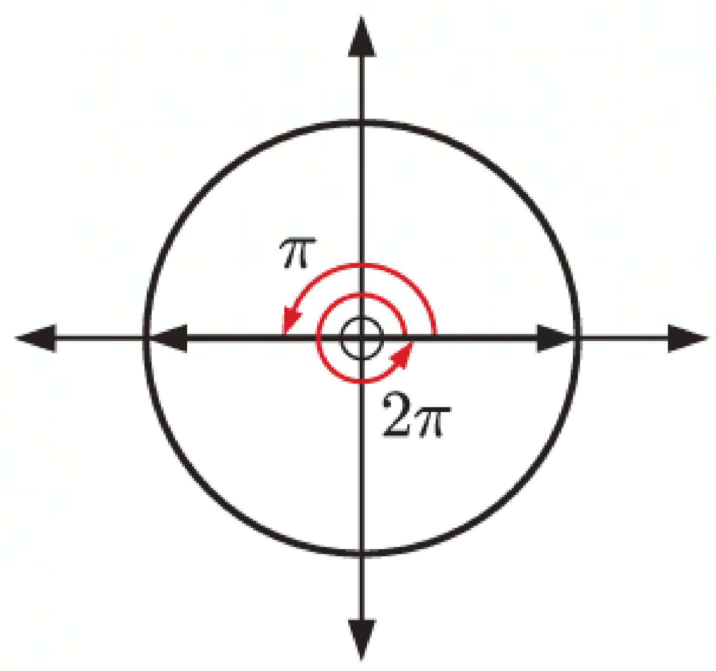
$\therefore$ the x-intercepts of $y = 2\sin 3x + \sqrt{3},\ 0 \leqslant x \leqslant 2\pi$ are $\frac{4\pi}{9}, \frac{5\pi}{9}, \frac{10\pi}{9}, \frac{11\pi}{9}, \frac{16\pi}{9}$, and $\frac{17\pi}{9}$.

b $y = \sqrt{2}\sin\left(x + \frac{\pi}{4}\right), \quad 0 \leqslant x \leqslant 3\pi$

The x-intercepts are the values of x such that

$\sqrt{2}\sin\left(x + \frac{\pi}{4}\right) = 0$

$\therefore \ \sin\left(x + \frac{\pi}{4}\right) = 0$



If $0 \leqslant x \leqslant 3\pi$, then $\frac{\pi}{4} \leqslant x + \frac{\pi}{4} \leqslant \frac{13\pi}{4}$.

On $\frac{\pi}{4} \leqslant x + \frac{\pi}{4} \leqslant \frac{13\pi}{4}$, the angles with sine 0 are π, 2π, and 3π.

So, $x + \frac{\pi}{4} = \pi$, 2π, or 3π

$\therefore \ x = \frac{3\pi}{4}, \frac{7\pi}{4}$, or $\frac{11\pi}{4}$

$\therefore$ the x-intercepts of $y = \sqrt{2}\sin\left(x + \frac{\pi}{4}\right)$, $0 \leqslant x \leqslant 3\pi$ are $\frac{3\pi}{4}$, $\frac{7\pi}{4}$, and $\frac{11\pi}{4}$.

22 $P(t) = 40 + 12\sin\left(\frac{2\pi}{7}\left(t - \frac{37}{12}\right)\right)$ mg

a $P(t)$ has a minimum of $40 + 12(-1) = 28$ mg per m^3 $\{$when $\sin\left(\frac{2\pi}{7}\left(t - \frac{37}{12}\right)\right) = -1\}$

b The minimum level of pollution occurs when

$\sin\left(\frac{2\pi}{7}\left(t - \frac{37}{12}\right)\right) = -1$

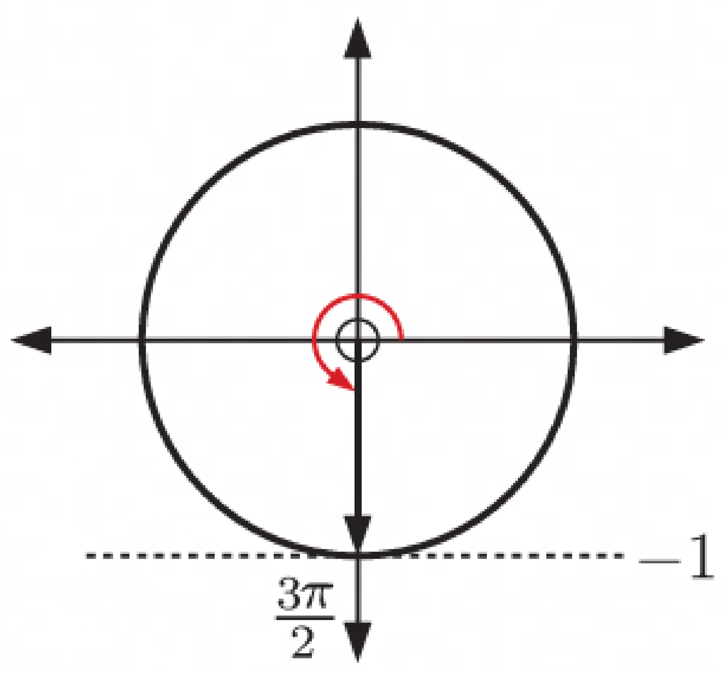
$\therefore \ \frac{2\pi}{7}\left(t - \frac{37}{12}\right) = \frac{3\pi}{2} + k2\pi$

$\therefore \ \frac{2}{7}\left(t - \frac{37}{12}\right) = \frac{3}{2} + k(2)$

So, $t - \frac{37}{12} = \frac{21}{4} + k(7)$

$\therefore \ t = 8\frac{1}{3} + k(7)$

$\therefore \ t = 1\frac{1}{3}, 8\frac{1}{3}, 15\frac{1}{3}$, and so on



$\therefore$ it occurs on Mondays at 8:00 am. $\{1\frac{1}{3}$ days after midnight on Saturday$\}$